REGULARITY OF WEAK SOLUTION OF VARIATIONAL PROBLEMS MODELING THE COSSERAT MICROPOLAR ELASTICITY

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ABSTRACT. In this paper, we consider weak solutions of the Euler-Lagrange equation to a variational energy functional modeling the geometrically nonlinear Cosserat micropolar elasticity of continua in dimension three, which is a system coupling between the Poisson equation and the equation of p-harmonic maps ($2 \leq p \leq 3$). We show that if a weak solutions is stationary, then its singular set is discrete for $2 < p < 3$ and has zero 1-dimensional Hausdorff measure for $p = 2$. If, in addition, it is a stable-stationary weak solution, then it is regular everywhere when $p \in [2, \frac{32}{15}]$.

1. Introduction

General continuum models involving independent rotations were introduced by the Eugene and Francois Cosserat brothers in 1909 [1], and were later rediscovered in 1960's (see Eringen [2]). The micromorphic balance equations were derived by Eringen [2]. The major difficulty of mathematical treatment in the finite strain case comes from the geometrically exact formulation of the theory and the appearance of nonlinear manifolds that are necessary to describe the microstructure. The geometrically nonlinear Cosserat framework is also encountered in the modeling of thin-structures [5]. Among many variants and vast body of results of Cosserat theory available in the literature, P. Neff [3, 4, 7] has made some systematical analysis of the Cosserat theory for micropolar elastic bodies by establishing the existence of minimizers in the framework of calculus of variations. Very recently, in an interesting article [8], Gastel has shown a partial regularity theorem of minimizer of a Cosserat energy functional for microplar elastic bodies.

The elastic body $\Omega \subset \mathbb{R}^3$ is assumed to be a bounded Lipschitz domain. The elastic body can be deformed by a translation mapping $\phi : \Omega \to \mathbb{R}^3$, and $\phi(x) - x$ denotes the displacement of the point $x \in \Omega$. Furthermore, the micropolar structure of the material associates each point $x \in \Omega$ with an orthonormal frame that is free to rotate in $\mathbb{R}^3$ by an orthogonal matrix $R(x) \in \mathrm{SO}(3)$. Both translations and rotations induce material stresses that are given by $R^t\nabla\phi - I_3$ and $R^t\nabla R$ respectively. We point out that since $R^T\mathrm{Curl}R$ is isomorphic to $R^T\nabla R$ (cf. [6]), it seems possible to measure the material stress induced by rotation by $R^T\mathrm{Curl}R$. The Cosserat energy functional stored in the elastic body Ω consists of the contributions by both translations and rotations. For a pair of translation and rotation maps $(\phi, R) : \Omega \to \mathbb{R}^3 \times \mathrm{SO}(3)$, the contribution of rotational stresses to the Cosserat energy is given by

$$l_c \int_\Omega |R^t\nabla R|^p \, dx \ (= l_c \int_\Omega |\nabla R|^p \, dx)$$

for some $l_c > 0$ and $2 \le p \le 3$, while the contribution of translational stresses is given by

$$\int_\Omega \left|\mathbb{P}(R^t\nabla\phi - I_3)\right|^2 dx,$$

where $\mathbb{P} : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ is the linear map defined by

$$\mathbb{P}(A) = \sqrt{\mu}\ \text{devsym}\ A + \sqrt{\mu_c}\ \text{skew}\ A + \frac{\sqrt{\mu_2}}{3}(\text{tr}A)I_3,\ \ A \in \mathbb{R}^{3\times3},$$

and

$$\text{devsym}\ A = \frac{1}{2}(A + A^t) - \frac{1}{3}(\text{tr}A)I_3, \quad \text{skew}\ A = \frac{1}{2}(A - A^t),$$

denotes the deviatoric symmetric part of A and the skew-symmetric part of A respectively. The constants $\mu > 0, \mu_c \ge 0$, and μ_2 represent the shear modulus, the Cosserat couple modulus, and the bulk modulus for isotropic response.

The elastic body Ω may be subject to external forces, such as gravity or electromagnetic forces, that can be modeled by

$$\int_\Omega \langle \phi - x, f\rangle\, dx + \int_\Omega \langle R, M\rangle\, dx,$$

where $f : \Omega \to \mathbb{R}^3$ stands for the potential of external volume forces applied to the displacement of the material and $M : \Omega \to \mathbb{R}^{3\times3}$ is a matrix-valued function representing the role of the potential of external volume couples applied to the microscopic rotation of the material, both of which are assumed to be given in this paper.

Collecting together all these terms, the Cosserat energy functional[1] is given by

$$\text{Coss}(\phi, R) = \int_\Omega \big(|\mathbb{P}(R^t\nabla\phi - I_3)|^2 + |\nabla R|^p + \langle \phi - x, f\rangle + \langle R, M\rangle\big)\, dx. \tag{1.1}$$

Recall that $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, \text{SO}(3))$ is a minimizer of the Cosserat energy functional, if

$$\text{Coss}(\phi, R) \le \text{Coss}(\widetilde{\phi}, \widetilde{R}),$$

holds for any $(\widetilde{\phi}, \widetilde{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, \text{SO}(3))$ that belongs to the same class of deformation configurations (for example, $(\widetilde{\phi}, \widetilde{R}) = (\phi, R)$ on a given part $\Gamma \subset \partial\Omega$, while there is no restriction imposed on $(\widetilde{\phi}, \widetilde{R})$ over the free boundary portion $\partial\Omega \setminus \Gamma$). The existence of minimizers of $\text{Coss}(\phi, R)$ in the Sobolev spaces, under various boundary conditions including either the Dirichlet or some other natural mixed boundary conditions, has been obtained by Neff [4]. We point out that since we only consider the interior partial regularity of certain weak solutions of the Cosserat equation, the boundary condition does not play a role in this analysis. By direct calculations, any minimizer (ϕ, R) of $\text{Coss}(\phi, R)$ solves the Euler-Lagrange equation, called as the Cosserat equation:

$$\begin{cases} \text{div}\big(R\mathbb{P}^t\mathbb{P}(R^t\nabla\phi - I_3)\big) = \frac{1}{2}f, \\ \Big\{\text{div}(|\nabla R|^{p-2}\nabla R) - \frac{2}{p}\nabla\phi\big(\mathbb{P}^t\mathbb{P}(R^t\nabla\phi - I_3)\big)^t - \frac{1}{p}M\Big\} \perp T_R\text{SO}(3). \end{cases} \tag{1.2}$$

[1]We may consider a more general Cosserat energy functional by replacing the translational material stress density $|\mathbb{P}(R^t\nabla\phi - I_3)|^2$ in (1.1) by a non-Hilbert type form $|\mathbb{P}(R^t\nabla\phi - I_3)|^q$ for some $2 \le q \le p \le 3$. It seems possible that the arguments in this paper can be extended to show the main results remain to be true for all $2 \le q \le p$. However, to make this paper slightly less technical we only consider the power $q = 2$ for the translational material stress density function.

Here $T_R\mathrm{SO}(3)$ denotes the tangent space of $\mathrm{SO}(3)$, at $R \in \mathrm{SO}(3)$, that is given by

$$T_R\mathrm{SO}(3) = \Big\{X \in \mathbb{R}^{3\times 3} \ \big|\ R^t X + X^t R = 0\Big\},$$

and $\mathbb{P}^t : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ is the adjoint map of $\mathbb{P}$.

When $\mu_1 = \mu_2 = \mu_c = 1$, we have that $\mathbb{P} = \mathbb{P}^t = \mathrm{Id}$ is the identity map. Hence

$$|\mathbb{P}(R^t\nabla\phi - I_3)|^2 = |\nabla\phi|^2 - 2\langle R, \nabla\phi\rangle + 3,$$

and the Cosserat equation (1.2) reduces to the following simplified form:

$$\begin{cases} \Delta\phi = \mathrm{div}R + \frac{1}{2}f, \\ \Big(\mathrm{div}(|\nabla R|^{p-2}\nabla R) + \frac{2}{p}\nabla\phi - \frac{1}{p}M\Big) \perp T_R\mathrm{SO}(3). \end{cases} \tag{1.3}$$

We would like to remark that the system (1.2) and (1.3) are systems coupling between the Poisson equation for the macroscopic translational deformation variable $\phi : \Omega \to \mathbb{R}^3$ and the (nonlinear) p-harmonic map equation for the microscopic rotational deformation variable $R : \Omega \to \mathrm{SO}(3)$.

By extending the techniques in the study of minimizing p-harmonic maps by Schoen-Uhlenbeck [13], Hardt-Lin [9], Fuchs [10], and especially Luckhaus [11], Gastel has recently shown in an interesting article [8] that any minimizer $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, \mathrm{SO}(3))$ of the Cosserat energy functional $\mathrm{Coss}(\phi, R)$ of the Cosserat functional (1.1) belongs to $C^{1,\alpha} \times C^\alpha$ in Ω away from a singular set Σ of isolated points for all $2 \le p < 3$. Moreover, Σ is shown to be an empty set when $p \in [2, \frac{32}{15}]$ by extending stability inequality arguments by Schoen-Uhlenbeck [14], Xin-Yang [15], and Chang-Chen-Wei [16].

An interesting question to ask is whether the regularity result on minimizers of the Cosserat functional in [8] remains to hold for certain classes of weak solutions to the Cosserat equation (1.2). In this paper, we will answer this question affirmatively. To address it, we first need to introduce a few definitions.

For $1 \le p < \infty$, recall the Sobolev space

$$W^{1,p}\big(\Omega, \mathrm{SO}(3)\big) = \Big\{R \in W^{1,p}(\Omega, \mathbb{R}^{3\times 3}) \ \big|\ R(x) \in \mathrm{SO}(3),\ \text{a.e. } x \in \Omega\Big\}.$$

Definition 1.1. *For $2 \le p \le 3$, given $f \in H^{-1}(\Omega, \mathbb{R}^3)$ and $M \in W^{-1,\frac{p}{p-1}}(\Omega, \mathbb{R}^{3\times 3})$, a pair of maps $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, \mathrm{SO}(3))$ is a weak solution to the Cosserat equation* (1.2)*, if it satisfies* (1.2) *in the sense of distributions, i.e.,*

$$\begin{cases} \displaystyle\int_\Omega (\langle \mathbb{P}(R^t\nabla\phi - I_3), \mathbb{P}R^t\nabla\psi_1\rangle + \frac{1}{2}\langle f, \psi_1\rangle)\, dx = 0, \\ \displaystyle\int_\Omega \big(\langle |\nabla R|^{p-2}\nabla R, \nabla\psi_2\rangle + \frac{2}{p}\langle \mathbb{P}(R^t\nabla\phi - I_3), \mathbb{P}\psi_2^t\nabla\phi\rangle + \frac{1}{p}\langle M, \psi_2\rangle\big)\, dx = 0, \end{cases}$$

hold for any $\psi_1 \in H_0^1(\Omega, \mathbb{R}^3)$ and $\psi_2 \in W_0^{1,p}(\Omega, T_R\mathrm{SO}(3)) \cap L^\infty(\Omega, \mathbb{R}^{3\times 3})$.

It is readily seen that any minimizer (ϕ, R) of the Cosserat energy functional (1.1) is a weak solution of the Cosserat equation (1.2). A restricted class of weak solutions of (1.2) is the class of stationary weak solutions, which is defined as follows.

Definition 1.2. *For $2 \le p \le 3$, $f \in H^{-1}(\Omega, \mathbb{R}^3)$, and $M \in W^{-1,\frac{p}{p-1}}(\Omega, \mathbb{R}^{3\times 3})$, a weak solution $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, \mathrm{SO}(3))$ to the Cosserat equation* (1.2) *is called a* *stationary weak solution, if, in addition,* (ϕ, R) *is a critical point of the Cosserat energy functional* (1.1) *with respect to the domain variations, i.e,*

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Coss}(\phi_t, R_t) = 0, \tag{1.4}$$

where $(\phi_t(x), R_t(x)) = (\phi(x + tY(x)), R(x + tY(x)))$ *for* $x \in \Omega$, *and* $Y \in C_0^\infty(\Omega, \mathbb{R}^3)$.

It is easy to check that any minimizer (ϕ, R) of the Cosserat energy functional (1.1) is a stationary weak solution of the Cosserat equation (1.2). It can also be shown by a Pohozaev argument that any regular solution $(\phi, R) \in C^{1,\alpha}(\Omega, \mathbb{R}^3 \times \mathrm{SO}(3))$ of the Cosserat equation (1.2) is a stationary weak solution.

In section 2 below, we will show that when $\mu_1 = \mu_c = \mu_2 = 1$, any stationary weak solution (ϕ, R) of Cosserat equation (1.3) satisfies the following stationarity identity: for any $Y \in C_0^\infty(\Omega, \mathbb{R}^3)$, it holds that

$$\begin{aligned}
&\int_\Omega \big(|\nabla\phi|^2 - 2\langle R, \nabla\phi\rangle + |\nabla R|^p\big)(-\mathrm{div}Y)\,dx + \int_\Omega (\langle f, Y\cdot\nabla\phi\rangle + \langle M, Y\cdot\nabla R\rangle)\,dx \\
&+ \int_\Omega \Big(2\nabla\phi\otimes\nabla\phi : \nabla Y - 2R_{ij}\frac{\partial\phi^i}{\partial x_k}\frac{\partial Y^k}{\partial x_j} + p|\nabla R|^{p-2}\nabla R\otimes\nabla R : \nabla Y\Big)\,dx = 0.
\end{aligned} \tag{1.5}$$

As a direct consequence of (1.5), we will establish an almost energy monotonicity inequality for stationary weak solutions to (1.3) when $\mu_1 = \mu_c = \mu_2 = 1$ holds. This, combined with the symmetry of SO(3), enables us to extend the compensated regularity technique by Hélein [19], Evans [20], and Toro-Wang [21] to show the following partial regularity.

Theorem 1.3. *For* $2 \le p < 3$, $f \in L^\infty(\Omega, \mathbb{R}^3)$ *and* $M \in L^\infty(\Omega, \mathbb{R}^{3\times3})$, *if* $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, \mathrm{SO}(3))$ *is a stationary weak solution to the Cosserat equation* (1.3), *then there exist* $\alpha \in (0, 1)$ *and a closed set* $\Sigma \subset \Omega$ *such that* $(\phi, R) \in C^{1,\alpha}(\Omega \setminus \Sigma, \mathbb{R}^3) \times C^\alpha(\Omega \setminus \Sigma, \mathrm{SO}(3))$. *Here* Σ *is a discrete set when* $p \in (2, 3)$, *and has zero* 1*-dimensional Hausdorff measure when* $p = 2$.

We would like to point out that the discreteness of singular set Σ for $2 < p < 3$ is a corollary of $H^1 \times W^{1,p}$-compactness property of weakly convergent stationary weak solutions of the Cosserat equation (1.3), which is a consequence of monotonicity inequality (2.3) and the Marstrand Theorem (see [23]).

To further improve the estimate of the singular set Σ for a stationary weak solution (ϕ, R) of the Cosserat equation (1.2) both for $p = 2$ and $2 < p < 3$, we restrict our attention to a subclass of stationary weak solutions that are stable.

Definition 1.4. *For* $2 \le p < 3$, $f \in H^{-1}(\Omega, \mathbb{R}^3)$, *and* $M \in W^{-1,\frac{p}{p-1}}(\Omega, \mathbb{R}^{3\times3})$, *a weak solution* $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, \mathrm{SO}(3))$ *to the Cosserat equation* (1.2) *is called a stable weak solution, if, in addition, the second order variation of the Cosserat energy functional at* (ϕ, R) *is nonnegative, i.e.,*

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Coss}(\phi_t, R_t) \ge 0, \tag{1.6}$$

where $(\phi_t, R_t) \in C^2\big((-\delta, \delta), H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, \mathrm{SO}(3))\big)$ *for some* $\delta > 0$, *satisfying* $(\phi_0, R_0) = (\phi, R)$, *is a variation of* (ϕ, R) *in the target space* $\mathbb{R}^3 \times \mathrm{SO}(3)$.

From the definition, any minimizer $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, \mathrm{SO}(3))$ of the Cosserat energy functional $\mathrm{Coss}(\cdot, \cdot)$ is a stable weak solution of the Cosserat equation (1.2).

In section 3, we will establish in the stability Lemma 3.2 that any stable weak solutions (ϕ, R) of Cosserat equation satisfies the following stability inequality:

$$\int_\Omega \left((p+1)|\nabla R|^{p-2}|\nabla \psi|^2 - 2|\nabla R|^p|\psi|^2\right) dx \geq 0, \ \forall \psi \in C_0^\infty(\Omega). \tag{1.7}$$

Utilizing the stability inequality (1.7), we can extend the ideas by Hong-Wang [17] and Lin-Wang [18] to establish a pre-compactness property of stable-stationary weak solutions of the Cosserat equation for $p = 2$, which can be employed to improve the estimate of singular set Σ. Moreover, by applying the non-existence theorem on stable p-harmonic maps from $\mathbb{S}^2$ to $\mathrm{SO}(3)$ for $p \in [2, \frac{32}{15}]$ that was established by Schoen-Uhlenbeck [14], Xin-Yang [15], and Chang-Chen-Wei [16], we prove a complete regularity result for stable stationary weak solutions to the Cosserat equation (1.3) when p belongs to the range $[2, \frac{32}{15}]$. More precisely, we have

Theorem 1.5. *For $p \in [2, \frac{32}{15}]$, $f \in L^\infty(\overline{\Omega}, \mathbb{R}^3)$, and $M \in L^\infty(\overline{\Omega}, \mathbb{R}^{3\times 3})$, if $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, \mathrm{SO}(3))$ is a stable stationary weak solution to the Cosserat equation* (1.3)*, then there exists $\alpha \in (0, 1)$ such that $(\phi, R) \in C^{1,\alpha}(\Omega, \mathbb{R}^3) \times C^\alpha(\Omega, \mathrm{SO}(3))$.*

Now we would like to mention a couple of questions.

Remark 1.6. 1) *It remains to be an open question whether Theorem 1.5 remains to be true when $\frac{32}{15} < p < 3$. The main difficulty arises from that we can't rule out the existence of nontrivial stable p-harmonic maps from $\mathbb{S}^2$ to $\mathrm{SO}(3)$ when p lies in the interval $(\frac{32}{15}, 3)$.*
2) *It is an open question whether Theorems 1.3 and 1.5 hold true when the positive constants μ_1, μ_c, μ_2 are not necessarily equal. Note that Neff* [4] *has shown the existence of minimizers of the Cosserat energy functional even when the Cosserat couple modulus $\mu_c = 0$. The main difficulty is that it is unknown whether an almost energy monotonicity inequality holds for stationary weak solutions (ϕ, R) of the Cosserat equation* (1.2) *when $\mathbb{P}$ is not an identity map.*
3) *It is also an interesting question to ask whether the regularity theorems remain to hold, if we replace the rotational stress energy density $|R^T \nabla R|^p$ by $|R^T \mathrm{Curl} R|^p$ in the Cosserat energy functional.*

The paper is organized as follows. In section 2, we will derive both stationarity identity and an almost energy monotonicity inequality for stationary weak solutions (ϕ, R) of the Cosserat equation (1.2). In section 3, we will rewrite the Cosserat equation (1.3) into a form in which the nonlinearity exhibits div-curl structures. In section 4, we will prove an ϵ_0-regularity theorem for stationary weak solutions (ϕ, R) of the Cosserat equation (1.3), and apply Marstrand's theorem to obtain a refined estimate of the singular set when $2 < p < 3$. In section 5, we will derive the stability inequality for stable weak solutions and obtain the full regularity for stable stationary weak solutions (ϕ, R) of the Cosserat equation (1.3) when $p \in [2, \frac{32}{15}]$.

2. Stationarity identity and almost monotonicity inequality

This section is devoted to the derivation of stationarity identity and almost energy monotonicity inequality for stationary weak solutions to the Cosserat equation (1.3).

Lemma 2.1. *For $2 \le p < 3$, assume $\mu_1 = \mu_c = \mu_2 = 1$, $f \in L^2(\Omega, \mathbb{R}^3)$, and $M \in L^{\frac{p}{p-1}}(\Omega, \mathrm{SO}(3))$. If $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, \mathrm{SO}(3))$ is a stationary weak solution of the Cosserat equation (1.3), then for any $Y \in C_0^\infty(\Omega, \mathbb{R}^3)$ it holds that*

$$\begin{aligned}&\int_\Omega \Big(2\nabla\phi \otimes \nabla\phi : \nabla Y + p|\nabla R|^{p-2}\nabla R \otimes \nabla R : \nabla Y - 2R^{ik}\frac{\partial \phi^k}{\partial x_j}\frac{\partial Y^j}{\partial x_i}\Big)\,dx \\ &= \int_\Omega \big[\big(|\nabla\phi|^2 - 2\langle R, \nabla\phi\rangle + |\nabla R|^p\big)\mathrm{div}Y - \langle Y(x)\cdot\nabla\phi, f\rangle - \langle Y(x)\cdot\nabla R, M\rangle\big]\,dx.\end{aligned} \tag{2.1}$$

Proof. For $Y \in C_0^\infty(\Omega, \mathbb{R}^3)$, there is a sufficiently small $\delta > 0$ so that $\mathrm{dist}(\mathrm{supp}(Y), \partial\Omega) > \delta$. Define $(\phi_t, R_t)(x) = (\phi, R)(x + tY(x))$ for $x \in \Omega$ and $t \in (-\delta, \delta)$. Since (ϕ, R) is a stationary weak solution of (1.3), we have that

$$0 = \frac{d}{dt}\Big|_{t=0}\int_\Omega \big(|\nabla\phi_t|^2 - 2\langle R_t, \nabla\phi_t\rangle + |\nabla R_t|^p + \langle \phi_t - x, f\rangle + \langle R_t, M\rangle\big)\,dx.$$

Applying change of variables and direct calculations, it is not hard to see that

$$\begin{aligned}&\int_\Omega \big(|\nabla\phi|^2 - 2\langle R, \nabla\phi\rangle + |\nabla R|^p\big)(-\mathrm{div}Y)\,dx + \int_\Omega \big(\langle Y\cdot\nabla\phi, f\rangle + \langle Y\cdot\nabla R, M\rangle\big)\,dx \\ &+ \int_\Omega \Big(2\nabla\phi\otimes\nabla\phi : \nabla Y - 2R_{\alpha\beta}\frac{\partial\phi^\beta}{\partial x_\gamma}\frac{\partial Y^\gamma}{\partial x_\alpha} + p|\nabla R|^{p-2}\nabla R\otimes\nabla R : \nabla Y\Big)\,dx = 0.\end{aligned} \tag{2.2}$$

This yields (2.1). □

By choosing suitable test variation fields $Y \in C_0^\infty(\Omega, \mathbb{R}^3)$, we will obtain an almost energy monotonicity inequality for stationary weak solutions to the Cosserat equation (1.3).

Corollary 2.2. *For $2 \le p < 3$, assume $\mu_1 = \mu_c = \mu_2 = 1$, $f \in L^\infty(\Omega, \mathbb{R}^3)$ and $M \in L^\infty(\Omega, \mathrm{SO}(3))$. If $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, \mathrm{SO}(3))$ is a stationary weak solution of the Cosserat equation (1.3), then for any $x \in \Omega$ and $0 < r_1 \le r_2 < \mathrm{dist}(x, \partial\Omega)$, it holds that*

$$\begin{aligned}&\mathrm{Coss}_x((\phi, R), r_1) + \int_{r_1}^{r_2} r^{p-3}\int_{\partial B_r}\big(p|\nabla R|^{p-2}|\frac{\partial R}{\partial r}|^2 + |\frac{\partial\phi}{\partial r}|^2\big)\,dH^2 dr \\ &\le \mathrm{Coss}_x((\phi, R), r_2),\end{aligned} \tag{2.3}$$

where $\mathrm{Coss}_x((\phi, R), r)$ is the modified renormalized Cosserat energy defined by

$$\mathrm{Coss}_x((\phi, R), r) := e^{Cr^2} r^{p-3}\int_{B_r(x)}\big(|\nabla R|^p + |\nabla\phi|^2\big)\,dx + Cr^3, \tag{2.4}$$

where $C > 0$ depends on p, $\|f\|_{L^\infty(\Omega)}$, and $\|M\|_{L^\infty(\Omega)}$.

Proof. For simplicity, assume $x = 0 \in \Omega$ and $0 < r < \mathrm{dist}(0, \partial\Omega)$ and write $B_r = B_r(0)$. Let $Y(x) = x\eta_\epsilon(|x|)$, where $\eta_\epsilon \in C_0^\infty(B_r)$ is chosen such that $\eta_\epsilon \to \chi_{B_r}$ as $\epsilon \to 0$. Plugging

Y into (2.1) and sending ϵ to 0, we obtain that

$$
\begin{aligned}
&(p-3)\int_{B_r}|\nabla R|^p\,dx + r\int_{\partial B_r}|\nabla R|^p\,dH^2 - \int_{B_r}|\nabla\phi|^2\,dx + r\int_{\partial B_r}|\nabla\phi|^2\,dH^2\\
&= -4\int_{B_r}\langle R,\nabla\phi\rangle\,dx + 2r\int_{\partial B_r}\langle R,\nabla\phi\rangle\,dH^2 - \int_{B_r}\left(\langle x\cdot\nabla\phi, f\rangle + \langle x\cdot\nabla R, M\rangle\right)dx\\
&\quad + pr\int_{\partial B_r}|\nabla R|^{p-2}|\frac{\partial R}{\partial r}|^2\,dH^2 + 2r\int_{\partial B_r}|\frac{\partial\phi}{\partial r}|^2\,dH^2 - 2\int_{\partial B_r}x^iR^{ik}\frac{\partial\phi^k}{\partial r}\,dH^2. \qquad (2.5)
\end{aligned}
$$

It is easy to estimate

$$
\begin{aligned}
&\big|2r\int_{\partial B_r}\langle R,\nabla\phi\rangle\,dx\big| \le Cr^2\int_{\partial B_r}|\nabla\phi|^2\,dH^2 + Cr^2,\\
&\big|-4\int_{B_r}\langle R,\nabla\phi\rangle\,dx\big| \le Cr\int_{B_r}|\nabla\phi|^2\,dx + Cr^2,\\
&\big|-2\int_{\partial B_r}R : x\otimes\frac{\partial\phi}{\partial r}\,dH^2\big| \le r\int_{\partial B_r}|\frac{\partial\phi}{\partial r}|^2\,dH^2 + Cr^3,\\
&\big|-\int_{B_r}\langle x\cdot\nabla\phi, f\rangle\,dx\big| \le Cr\int_{B_r}|\nabla\phi|^2\,dx + C\|f\|^2_{L^\infty(\Omega)}r^4,\\
&\big|-\int_{B_r}\langle x\cdot\nabla R, M\rangle\,dx\big| \le Cr\int_{B_r}|\nabla R|^p\,dx + C\|M\|^{\frac{p}{p-1}}_{L^\infty(\Omega)}r^4.
\end{aligned}
$$

Substituting these estimates into (2.5) yields

$$
\begin{aligned}
&(p-3)\int_{B_r}|\nabla R|^p\,dx + r\int_{\partial B_r}|\nabla R|^p\,dH^2 - \int_{B_r}|\nabla\phi|^2\,dx + r\int_{\partial B_r}|\nabla\phi|^2\,dH^2\\
&\ge pr\int_{\partial B_r}|\nabla R|^{p-2}|\frac{\partial R}{\partial r}|^2\,dH^2 + r\int_{\partial B_r}|\frac{\partial\phi}{\partial r}|^2\,dH^2\\
&\quad - Cr\int_{B_r}|\nabla R|^p\,dx - Cr\int_{B_r}|\nabla\phi|^2\,dx - Cr^2\int_{\partial B_r}|\nabla\phi|^2\,dH^2\\
&\quad - C\big(1 + \|f\|^2_{L^\infty(\Omega)} + \|M\|^{\frac{p}{p-1}}_{L^\infty(\Omega)}\big)r^2. \qquad (2.6)
\end{aligned}
$$

Hence we obtain for $0 < r \le \min\big\{1, \mathrm{dist}(0,\partial\Omega)\big\}$,

$$
\begin{aligned}
&\frac{d}{dr}\Big\{e^{Cr^2}r^{p-3}\int_{B_r}\big(|\nabla R|^p + |\nabla\phi|^2\big)\,dx\Big\}\\
&\ge e^{Cr^2}r^{p-3}\int_{\partial B_r}\big(p|\nabla R|^{p-2}|\frac{\partial R}{\partial r}|^2 + 2|\frac{\partial\phi}{\partial r}|^2\big)\,dH^2 + (p-2)e^{Cr^2}r^{p-4}\int_{B_r}|\nabla\phi|^2\,dx\\
&\quad - C\big(1 + \|f\|^2_{L^\infty(\Omega)} + \|M\|^{\frac{p}{p-1}}_{L^\infty(\Omega)}\big)e^{Cr^2}r^2\\
&\ge r^{p-3}\int_{\partial B_r}\big(p|\nabla R|^{p-2}|\frac{\partial R}{\partial r}|^2 + 2|\frac{\partial\phi}{\partial r}|^2\big)\,dH^2\\
&\quad - C\big(1 + \|f\|^2_{L^\infty(\Omega)} + \|M\|^{\frac{p}{p-1}}_{L^\infty(\Omega)}\big)r^2. \qquad (2.7)
\end{aligned}
$$

Integrating from $0 < r_1 \le r_2 \le \min\{1, \mathrm{dist}(0, \partial\Omega)\}$, we obtain that the following monotonicity inequality:

$$
\begin{aligned}
& e^{Cr_2^2} {r_2}^{p-3} \int_{B_{r_2}} \left(|\nabla R|^p + |\nabla\phi|^2\right) dx + Cr_2^3 \\
& \ge e^{Cr_1^2} {r_1}^{p-3} \int_{B_{r_1}} \left(|\nabla R|^p + |\nabla\phi|^2\right) dx + Cr_1^3 \\
& \quad + \int_{r_1}^{r_2} r^{p-3} \int_{\partial B_r} \left(p|\nabla R|^{p-2}|\frac{\partial R}{\partial r}|^2 + |\frac{\partial\phi}{\partial r}|^2\right) dH^2 dr, \qquad (2.8)
\end{aligned}
$$

where $C > 0$ depends on p, $\|f\|_{L^\infty(\Omega)}$, and $\|M\|_{L^\infty(\Omega)}$. This completes the proof of (2.3). □

3. Div-curl structure of the Cosserat equation (1.3)

This section is devoted to rewriting of the Cosserat equation $(1.3)_2$ into a form where the nonlinearity exhibits algebraic structures similar to that of p-harmonic maps into symmetric manifolds given by Hélein [19] and Toro-Wang [21].

Let $\mathfrak{so}(3)$ be the Lie algebra of SO(3) or equivalently the tangent space of SO(3) at I_3. Recall that a standard orthonormal base of $\mathfrak{so}(3)$ is given by

$$
\mathbf{a}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ \mathbf{a}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ \mathbf{a}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

For any $R \in \mathrm{SO}(3)$,

$$
\left\{\mathbf{V}_1(R) = \mathbf{a}_1 R,\ \mathbf{V}_2(R) = \mathbf{a}_2 R,\ \mathbf{V}_3(R) = \mathbf{a}_3 R\right\}
$$

forms an orthonormal base of $T_R\mathrm{SO}(3)$, the tangent space of SO(3) at R.

From $(1.3)_2$ we have that for $i = 1, 2, 3$,

$$
\langle \mathrm{div}(|\nabla R|^{p-2}\nabla R), \mathbf{V}_i(R)\rangle = -\frac{2}{p}\langle \nabla\phi, \mathbf{V}_i(R)\rangle + \frac{1}{p}\langle M, \mathbf{V}_i(R)\rangle. \qquad (3.1)
$$

For $i = 1, 2, 3$, since $\mathbf{a}_i$ is skew-symmetric, we have that

$$
\langle |\nabla R|^{p-2}\nabla R, \nabla(\mathbf{V}_i(R))\rangle = \langle |\nabla R|^{p-2}\nabla R, \mathbf{a}_i \nabla R\rangle = 0.
$$

Thus we can rewrite the Cosserat equation $(1.3)_2$ as follows.

$$
\begin{aligned}
\mathrm{div}(|\nabla R|^{p-2}\nabla R) &= \sum_{i=1}^{3} \mathrm{div}\big(\langle |\nabla R|^{p-2}\nabla R, \mathbf{V}_i(R)\rangle \mathbf{V}_i(R)\big) \\
&= \sum_{i=1}^{3} \left[\langle \mathrm{div}(|\nabla R|^{p-2}\nabla R), \mathbf{V}_i(R)\rangle + \langle |\nabla R|^{p-2}\nabla R, \nabla(\mathbf{V}_i(R))\rangle\right] \mathbf{V}_i(R) \\
&\quad + \sum_{i=1}^{3} \langle |\nabla R|^{p-2}\nabla R, \mathbf{V}_i(R)\rangle \nabla(\mathbf{V}_i(R)) \qquad (3.2) \\
&= \sum_{i=1}^{3} \Big[\big(-\frac{2}{p}\langle \nabla\phi, \mathbf{V}_i(R)\rangle + \frac{1}{p}\langle M, \mathbf{V}_i(R)\rangle\big)\mathbf{V}_i(R) + \langle |\nabla R|^{p-2}\nabla R, \mathbf{V}_i(R)\rangle \nabla(\mathbf{V}_i(R))\Big].
\end{aligned}
$$

From the above derivation, we see that for $i = 1, 2, 3$,

$$\operatorname{div}\big(\langle |\nabla R|^{p-2}\nabla R, \mathbf{V}_i(R)\rangle\big) = -\frac{2}{p}\langle \nabla\phi, \mathbf{V}_i(R)\rangle + \frac{1}{p}\langle M, \mathbf{V}_i(R)\rangle. \tag{3.3}$$

For $i = 1, 2, 3$, let $Y_i : \Omega \to \mathbb{R}$ solve the auxiliary equation

$$\Delta Y_i = \frac{2}{p}\langle \nabla\phi, \mathbf{V}_i(R)\rangle - \frac{1}{p}\langle M, \mathbf{V}_i(R)\rangle, \tag{3.4}$$

so that

$$\operatorname{div}\big(\langle |\nabla R|^{p-2}\nabla R, \mathbf{V}_i(R)\rangle + \nabla Y_i\big) = 0. \tag{3.5}$$

Putting (3.2), (3.3), (3.4), (3.5) together, we obtain an equivalent form of $(1.3)_2$:

$$\begin{aligned}
&\operatorname{div}(|\nabla R|^{p-2}\nabla R)\\
&= \sum_{i=1}^{3} \big(\langle |\nabla R|^{p-2}\nabla R, \mathbf{V}_i(R)\rangle + \nabla Y_i\big)\nabla(\mathbf{V}_i(R))\\
&\quad - \sum_{i=1}^{3} \nabla Y_i \cdot \nabla(\mathbf{V}_i(R)) + \sum_{i=1}^{3}\big(-\frac{2}{p}\langle \nabla\phi, \mathbf{V}_i(R)\rangle + \frac{1}{p}\langle M, \mathbf{V}_i(R)\rangle\big)\mathbf{V}_i(R).
\end{aligned} \tag{3.6}$$

It is readily seen that as the leading order term of nonlinearity in the right hand side of the equation (3.6), $\big(\langle |\nabla R|^{p-2}\nabla R, \mathbf{V}_i(R)\rangle + \nabla Y_i\big)\nabla(\mathbf{V}_i(R))$ is the inner product of a divergence free vector field $\big(\langle |\nabla R|^{p-2}\nabla R, \mathbf{V}_i(R)\rangle + \nabla Y_i\big)$ and a curl free vector field $\nabla(\mathbf{V}_i(R))$.

4. ϵ_0-regularity of stationary solutions of the Cosserat equation

In this section, we will establish an ϵ_0-regularity estimate and a partial regularity of stationary weak solutions of the Cosserat equation (1.3) and give a proof of Theorem 1.3. The key ingredient is the following energy decay lemma, under the smallness condition.

Lemma 4.1. *For any $2 \le p < 3$, $\mu_1 = \mu_c = \mu_2 = 1$, $f \in L^\infty(\Omega, \mathbb{R}^3)$ and $M \in L^\infty(\Omega, \mathrm{SO}(3))$, there exist $\epsilon_0 > 0$ and $\theta_0 \in (0, \frac{1}{2})$ depending on p, $\|f\|_{L^\infty(\Omega)}$, and $\|M\|_{L^\infty(\Omega)}$ such that if (ϕ, R) is a stationary weak solution of the Cosserat equation* (1.3)*, and satisfies, for $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$,*

$$r^{p-3}\int_{B_r(x)} \big(|\nabla R|^p + |\nabla\phi|^2\big)\,dx \le \epsilon_0^p, \tag{4.1}$$

then

$$\begin{aligned}
&(\theta_0 r)^{p-3}\int_{B_{\theta_0 r}(x)} \big(|\nabla R|^p + |\nabla\phi|^2\big)\,dx\\
&\le \frac{1}{2}\max\Big\{r^{p-3}\int_{B_r(x)} \big(|\nabla R|^p + |\nabla\phi|^2\big)\,dx,\ r^p\Big\}.
\end{aligned} \tag{4.2}$$

Proof. We argue it by contradiction. Suppose that the conclusion were false. Then for any $L > 0$ with $\|f\|_{L^\infty(\Omega)} + \|M\|_{L^\infty(\Omega)} \le L$ and $\theta \in (0, \frac{1}{2})$, there exist $\epsilon_k \to 0$, $x_k \in \Omega$, and $r_k \to 0$ such that

$$r_k^{p-3}\int_{B_{r_k}(x_k)} \big(|\nabla R|^p + |\nabla\phi|^2\big)\,dx \le \epsilon_k^p, \tag{4.3}$$

but

$$\begin{aligned}&(\theta r_k)^{p-3}\int_{B_{\theta r_k}(x_k)}\left(|\nabla R|^p+|\nabla\phi|^2\right)dx\\&>\frac{1}{2}\max\Big\{r_k^{p-3}\int_{B_{r_k}(x_k)}\left(|\nabla R|^p+|\nabla\phi|^2\right)dx,\ r_k^p\Big\}.\end{aligned}\tag{4.4}$$

Define the rescaling maps

$$\begin{cases}R_k(x)=R(x_k+r_kx),\\ \phi_k(x)=r_k^{\frac{p-2}{2}}\phi(x_k+r_kx),\\ f_k(x)=r_k^{\frac{p+2}{2}}f(x_k+r_kx),\\ M_k(x)=r_k^pM(x_k+r_kx),\end{cases}\quad \forall x\in B_1.$$

Then (ϕ_k, R_k) solves in B_1

$$\begin{cases}\Delta\phi_k=r_k^{\frac{p}{2}}\operatorname{div}(R_k)+\frac{1}{2}f_k,\\ \operatorname{div}(|\nabla R_k|^{p-2}\nabla R_k)=\sum_{\alpha=1}^3\langle|\nabla R_k|^{p-2}\nabla R_k,\mathbf{V}_\alpha(R_k)\rangle\nabla(\mathbf{V}_\alpha(R_k))\\ -\frac{1}{p}\sum_{\alpha=1}^3\left[2r_k^{\frac{p}{2}}\langle\nabla\phi_k,\mathbf{V}_\alpha(R_k)\rangle-\langle M_k,\mathbf{V}_\alpha(R_k)\rangle\right]\mathbf{V}_\alpha(R_k).\end{cases}\tag{4.5}$$

Moreover, it holds that

$$\int_{B_1}\left(|\nabla R_k|^p+|\nabla\phi_k|^2\right)dx=r_k^{p-3}\int_{B_{r_k}(x_k)}\left(|\nabla R|^p+|\nabla\phi|^2\right)dx=\epsilon_k^p,\tag{4.6}$$

and

$$\theta^{p-3}\int_{B_\theta}\left(|\nabla R_k|^p+|\nabla\phi_k|^2\right)dx>\frac{1}{2}\max\Big\{\int_{B_1}\left(|\nabla R_k|^p+|\nabla\phi_k|^2\right)dx,\ r_k^p\Big\}.\tag{4.7}$$

Now we define the blow-up sequence:

$$\begin{cases}\widehat{R_k}(x)=\dfrac{R_k(x)-\overline{R_k}}{\epsilon_k},\\ \widehat{\phi_k}(x)=\dfrac{\phi_k(x)-\overline{\phi_k}}{\epsilon_k^{\frac{p}{2}}},\end{cases}\quad\forall x\in B_1,$$

where $\overline{f}=\frac{1}{|B_1|}\int_{B_1}f$ denotes the average of f over B_1. Then $(\widehat{\phi_k},\widehat{R_k})$ solves, in B_1,

$$\begin{cases}\Delta\widehat{\phi_k}=r_k^{\frac{p}{2}}\epsilon_k^{1-\frac{p}{2}}\operatorname{div}(\widehat{R_k})+\frac{1}{2}\epsilon_k^{-\frac{p}{2}}f_k,\\ \operatorname{div}(|\nabla\widehat{R_k}|^{p-2}\nabla\widehat{R_k})=\epsilon_k\sum_{\alpha=1}^3\langle|\nabla\widehat{R_k}|^{p-2}\nabla\widehat{R_k},\mathbf{V}_\alpha(R_k)\rangle\nabla(\mathbf{V}_\alpha(\widehat{R_k}))\\ -\frac{1}{p}\sum_{\alpha=1}^3\left[2r_k^{\frac{p}{2}}\epsilon_k^{1-\frac{p}{2}}\langle\nabla\widehat{\phi_k},\mathbf{V}_\alpha(R_k)\rangle-\epsilon_k^{1-p}\langle M_k,\mathbf{V}_\alpha(R_k)\rangle\right]\mathbf{V}_\alpha(R_k),\end{cases}\tag{4.8}$$

satisfies

$$\int_{B_1} \widehat{R_k}\,dx = 0,\ \int_{B_1} \widehat{\phi_k}\,dx = 0,\ \int_{B_1} \big(|\nabla \widehat{R_k}|^p + |\nabla \widehat{\phi_k}|^2\big)\,dx = 1, \tag{4.9}$$

and

$$\theta^{p-3}\int_{B_\theta} \big(|\nabla \widehat{R_k}|^p + |\nabla \widehat{\phi_k}|^2\big)\,dx > \frac{1}{2}\max\Big\{1,\ \frac{r_k^p}{\epsilon_k^p}\Big\}. \tag{4.10}$$

In particular, we have

$$\frac{r_k^p}{\epsilon_k^p} \le 2\theta^{p-3}\int_{B_\theta} \big(|\nabla \widehat{R_k}|^p + |\nabla \widehat{\phi_k}|^2\big)\,dx \le 2\theta^{p-3}. \tag{4.11}$$

This implies that

$$r_k \le C\epsilon_k. \tag{4.12}$$

We may assume that there exist $\phi_\infty \in H^1(B_1, \mathbb{R}^3)$, $R_\infty \in W^{1,p}(B_1, \mathrm{SO}(3))$ such that, after passing to a subsequence,

$$(\widehat{\phi_k}, \widehat{R_k}) \rightharpoonup (\phi_\infty, R_\infty) \text{ in } H^1(B_1)\times W^{1,p}(B_1),\ (\widehat{\phi_k}, \widehat{R_k}) \to (\phi_\infty, R_\infty) \text{ in } L^2(B_1)\times L^p(B_1).$$

Then (ϕ_∞, R_∞) satisfies

$$\begin{cases} \overline{\phi_\infty} = 0, \\ \overline{R_\infty} = 0, \\ \displaystyle\int_{B_1} \big(|\nabla R_\infty|^p + |\nabla \phi_\infty|^2\big)\,dx \le 1. \end{cases}$$

Moreover, it follows from (4.11) that

$$\big\|\epsilon_k^{-\frac{p}{2}} f_k\big\|_{L^\infty(B_1)} \le C\epsilon_k^{-\frac{p}{2}} r_k^{\frac{p+2}{2}} \le Cr_k \to 0,$$

$$\big\|\epsilon_k^{1-p} M_k\big\|_{L^\infty(B_1)} \le C\epsilon_k^{1-p} r_k^p \le C\epsilon_k \to 0,$$

and

$$\big\|r_k^{\frac{p}{2}}\epsilon_k^{1-\frac{p}{2}}\mathrm{div}(\widehat{R_k})\big\|_{L^p(B_1)} + \|\big\|r_k^{\frac{p}{2}}\epsilon_k^{2-p}\nabla\widehat{\phi_k}\big\|_{L^2(B_1)} \le C\epsilon_k^{1-\frac{p}{2}} r_k^{\frac{p}{2}} \le C\epsilon_k \to 0.$$

Hence, after sending $k \to \infty$ in the equation (4.8), we conclude that ϕ_∞ is a harmonic function and R_∞ is a p-harmonic function, i.e.,

$$\begin{cases} \Delta\phi_\infty = 0 \\ \mathrm{div}\big(|\nabla R_\infty|^{p-2}\nabla R_\infty\big) = 0, \end{cases} \quad \text{in } B_1. \tag{4.13}$$

Hence we have that for $0 < \theta < \frac{1}{2}$,

$$\begin{aligned} &\theta^{p-3}\int_{B_\theta} (|\nabla R_\infty|^p + |\nabla\phi_\infty|^2)\,dx \\ &\le C\theta^p\big(\|\nabla R_\infty\|^p_{L^\infty(B_{\frac{1}{2}})} + \|\nabla\phi_\infty\|^2_{L^\infty(B_{\frac{1}{2}})}\big) \\ &\le C\theta^p\int_{B_1} (|\nabla R_\infty|^p + |\nabla\phi_\infty|^2)\,dx \le C\theta^p. \end{aligned} \tag{4.14}$$

Next we need to show that $(\widehat{\phi_k}, \widehat{R_k})$ converges strongly to (ϕ_∞, R_∞) in $H^1(B_{\frac12}) \times W^{1,p}(B_{\frac12})$, which is based on the duality between the Hardy space and the BMO space. Let $\eta : \mathbb{R}^3 \to \mathbb{R}$ be a smooth cutoff function satisfying

$$0 \le \eta \le 1, \quad \eta = 1 \ \text{ on } B_{\frac14}, \quad \eta = 0 \ \text{ on } \ \mathbb{R}^3 \backslash B_{\frac38}.$$

Then we have the following lemma, whose proof is based on the energy monotonicity inequality (2.3) and is similar to that by [20] and [21]. Denote by $\mathrm{BMO}(\mathbb{R}^3)$ the space of functions of bounded mean oscillations in $\mathbb{R}^3$.

Lemma 4.2. *The sequence* $\{\eta \widehat{R_k}\}_{k\ge1}$ *is bounded in* $\mathrm{BMO}(\mathbb{R}^3)$.

Proof. This is a well-known fact. We skip the details and refer the readers to either [20] or [21] for a proof. □

Lemma 4.3. $\nabla \widehat{R_k}$ *converge strongly to* ∇R_∞ *in* $L^p(B_{\frac14})$*, and* $\nabla\widehat{\phi_k}$ *converge strongly to* $\nabla \phi_\infty$ *in* $L^2(B_{\frac14})$.

Proof. First notice that scalings of the equation (3.3) imply that for $i = 1, 2, 3$,

$$\operatorname{div}\big(\langle |\nabla \widehat{R_k}|^{p-2}\nabla \widehat{R_k}, \mathbf{V}_i(R_k)\rangle\big) = -\frac{2}{p} r_k^{\frac p2} \epsilon_k^{1-\frac p2} \langle \nabla \widehat{\phi_k}, \mathbf{V}_i(R_k)\rangle + \frac1p \epsilon_k^{1-p}\langle M_k, \mathbf{V}_i(R_k)\rangle. \tag{4.15}$$

As in (3.4), let $Y_k^i : B_1 \to \mathbb{R}$ solve

$$\begin{cases} \Delta Y_k^i = \dfrac{2}{p} r_k^{\frac p2} \epsilon_k^{1-\frac p2} \langle \nabla \widehat{\phi_k}, \mathbf{V}_i(R_k)\rangle - \dfrac1p \epsilon_k^{1-p}\langle M_k, \mathbf{V}_i(R_k)\rangle & \text{in } B_1, \\ \ \ Y_k^i = 0 & \text{on } \partial B_1. \end{cases} \tag{4.16}$$

It is easy to see that by $W^{2,2}$-theory, Y_k^i satisfies

$$\begin{aligned} \big\|\nabla Y_k^i\big\|_{L^2(B_1)} + \big\|\nabla^2 Y_k^i\big\|_{L^2(B_1)} &\le C r_k^{\frac p2} \epsilon_k^{1-\frac p2} \big\|\nabla \widehat{\phi_k}\big\|_{L^2(B_1)} + C\epsilon_k^{1-p}\big\|M_k\big\|_{L^2(B_1)} \\ &\le C\big(r_k^{\frac p2}\epsilon_k^{\frac{2-p}{2}} + r_k^p \epsilon_k^{1-p}\big) \le C\epsilon_k, \end{aligned} \tag{4.17}$$

where we have used (4.11) in the last step.

Adding the equations (4.15) and (4.16), we have that

$$\operatorname{div}\big(\langle |\nabla \widehat{R_k}|^{p-2}\nabla \widehat{R_k}, \mathbf{V}_i(R_k)\rangle + \nabla Y_k^i\big) = 0 \ \text{ in } \ B_1, \tag{4.18}$$

and the blowup equation $(4.8)_2$ becomes

$$\begin{cases} \operatorname{div}(|\nabla \widehat{R_k}|^{p-2}\nabla \widehat{R_k}) \\ = \epsilon_k \displaystyle\sum_{i=1}^3 \big(\langle |\nabla \widehat{R_k}|^{p-2}\nabla \widehat{R_k}, \mathbf{V}_i(R_k)\rangle + \nabla Y_k^i\big)\cdot \nabla(\mathbf{V}_i(\widehat{R_k})) \\ \quad -\dfrac1p \displaystyle\sum_{i=1}^3 \big[2 r_k^{\frac p2}\epsilon_k^{1-\frac p2}\langle \nabla\widehat{\phi_k}, \mathbf{V}_i(R_k)\rangle - \epsilon_k^{1-p}\langle M_k, \mathbf{V}_i(R_k)\rangle\big]\mathbf{V}_i(R_k) & \text{in } \ B_1. \\ \quad -\epsilon_k \displaystyle\sum_{i=1}^3 \nabla Y_k^i \cdot \nabla(\mathbf{V}_i(\widehat{R_k})) \end{cases} \tag{4.19}$$

Define

$$H_k^i := \big(\langle |\nabla \widehat{R_k}|^{p-2}\nabla \widehat{R_k}, \mathbf{V}_i(R_k)\rangle + \nabla Y_k^i\big)\cdot\nabla(\mathbf{V}_i(\widehat{R_k})).$$

Then it follows from (4.18) that $H_k^i \in \mathcal{H}^1_{\rm loc}(B_1)$, the local Hardy space (see [19] and [20] for some basic properties of Hardy spaces). For any compact $K \subset B_1$ and $i = 1, 2, 3$, we can use $\frac{3}{2} < \frac{p}{p-1} \le 2$ and (4.17) to estimate

$$
\begin{aligned}
\big\|H_k^i\big\|_{\mathcal{H}^1(K)} &\le C\big\|\langle|\nabla\widehat{R_k}|^{p-2}\nabla\widehat{R_k}, \mathbf{V}_i(R_k)\rangle + \nabla Y_k^i\big\|_{L^{\frac{p}{p-1}}(B_1)}\big\|\nabla(\mathbf{V}_i(\widehat{R_k})\big\|_{L^p(B_1)}\\
&\le C\big[\|\nabla\widehat{R_k}\|_{L^p(B_1)}^{p-1} + \|\nabla Y_k^i\|_{L^{\frac{p}{p-1}}(B_1)}\big]\big\|\nabla(\mathbf{V}_i(\widehat{R_k}))\|_{L^p(B_1)}\\
&\le C, \ \forall k \ge 1.
\end{aligned}
$$

and

$$
\begin{aligned}
\big\|H_k^i\big\|_{L^1(B_1)} &\le C\big[\|\nabla\widehat{R_k}\|_{L^p(B_1)}^{p-1} + \|\nabla Y_k^i\|_{L^{\frac{p}{p-1}}(B_1)}\big]\|\nabla(\mathbf{V}_i(\widehat{R_k}))\|_{L^p(B_1)}\\
&\le C, \ \forall k \ge 1.
\end{aligned}
$$

Assume $\displaystyle\int_{\mathbb{R}^3} \eta\,dx \ne 0$. For $i = 1, 2, 3$, set

$$
\mu_k^i = \frac{\int_{\mathbb{R}^3} H_k^i \eta\,dx}{\int_{\mathbb{R}^3}\eta\,dx}, \ \forall k \ge 1.
$$

Then we have that

$$
\sup_{k\ge1}\big\|\eta(H_k^i - \mu_k^i)\big\|_{\mathcal{H}^1(\mathbb{R}^3)} \le C\sup_{k\ge1}\big(\|H_k^i\|_{\mathcal{H}^1(\mathrm{supp}\eta)} + \|H_k^i\|_{L^1(B_1)}\big) \le C, \tag{4.20}
$$

and

$$
|\mu_k^i| \le C\|H_k^i\|_{L^1(B_1)} \le C. \tag{4.21}
$$

Observe that

$$
\begin{aligned}
&\mathrm{div}\big(|\nabla\widehat{R_k}|^{p-2}\nabla\widehat{R_k} - |\nabla R_\infty|^{p-2}\nabla R_\infty\big)\\
&= \epsilon_k\sum_{i=1}^3 H_k^i - \epsilon_k\sum_{i=1}^3 \nabla Y_k^i\cdot\nabla(\mathbf{V}_i(\widehat{R_k}))\\
&\quad - \frac{1}{p}\sum_{i=1}^3\big[2r_k^{\frac{p}{2}}\epsilon_k^{1-\frac{p}{2}}\langle\nabla\widehat{\phi_k}, \mathbf{V}_i(R_k)\rangle - \epsilon_k^{1-p}\langle M_k, \mathbf{V}_i(R_k)\rangle\big]\mathbf{V}_i(R_k).
\end{aligned}
$$

Multiplying this equation by $\eta^2(\widehat{R_k} - R_\infty)$ and integrating it over $\mathbb{R}^3$, we obtain that

$$
\begin{aligned}
&\int_{B_1}\eta^2(|\nabla\widehat{R_k}|^{p-2}\nabla\widehat{R_k} - |\nabla R_\infty|^{p-2}\nabla R_\infty) : \nabla(\widehat{R_k} - R_\infty)\,dx\\
&+ 2\int_{B_1}\eta(|\nabla\widehat{R_k}|^{p-2}\nabla\widehat{R_k} - |\nabla R_\infty|^{p-2}\nabla R_\infty) : \nabla\eta\otimes(\widehat{R_k} - R_\infty)\,dx\\
&= \epsilon_k\int_{B_1}\big[\nabla Y_k^i\cdot\nabla(\mathbf{V}_i(\widehat{R_k})) - H_k^i\big]\eta^2(\widehat{R_k} - R_\infty)\,dx\\
&+ \frac{1}{p}\sum_{i=1}^3\int_{B_1}\big[2r_k^{\frac{p}{2}}\epsilon_k^{1-\frac{p}{2}}\langle\nabla\widehat{\phi_k}, \mathbf{V}_i(R_k)\rangle - \epsilon_k^{1-p}\langle M_k, \mathbf{V}_i(R_k)\rangle\big]\mathbf{V}_i(R_k)\eta^2(\widehat{R_k} - R_\infty)\,dx.
\end{aligned}
$$

It is not hard to see that

$$
\begin{aligned}
&\int_{B_1} \eta^2|\nabla\widehat{R_k}-\nabla R_\infty|^p\,dx\\
&\le C\int_{B_1}\eta|(|\nabla\widehat{R_k}|^{p-2}\nabla\widehat{R_k}-|\nabla R_\infty|^{p-2}\nabla R_\infty)||\nabla\eta||\widehat{R_k}-R_\infty|\,dx\\
&\ + C\epsilon_k\Big|\int_{B_1} H_k^i\cdot\eta^2(\widehat{R_k}-R_\infty)\,dx\Big| + C\epsilon_k\int_{B_1}\eta^2|\nabla Y_k^i||\nabla\widehat{R_k}||\widehat{R_k}-R_\infty|\,dx\\
&\ + C\int_{B_1}\big[2r_k^{\frac{p}{2}}\epsilon_k^{1-\frac{p}{2}}|\nabla\widehat{\phi_k}|+\epsilon_k^{1-p}|M_k|\big]\eta^2|\widehat{R_k}-R_\infty|\,dx\\
&= I_k+II_k+III_k+IV_k.
\end{aligned}
$$

Since

$$
|\nabla\widehat{R_k}|^{p-2}\nabla\widehat{R_k}\rightharpoonup|\nabla R_\infty|^{p-2}\nabla R_\infty \ \text{ in } \ L^{\frac{p}{p-1}}(B_1),\quad \widehat{R_k}\to R_\infty \ \text{ in } \ L^p(B_1),
$$

we conclude that

$$
I_k\to 0.
$$

For III_k, we have

$$
\begin{aligned}
|III_k| &\le C\epsilon_k\|\nabla Y_k^i\|_{L^6(B_1)}\|\nabla\widehat{R_k}\|_{L^2(B_1)}\|\widehat{R_k}-R_\infty\|_{L^3(B_1)}\\
&\le C\epsilon_k\|\nabla Y_k^i\|_{H^1(B_1)}\|\nabla\widehat{R_k}\|_{L^2(B_1)}\|\widehat{R_k}-R_\infty\|_{L^3(B_1)}\\
&\le C\epsilon_k\to 0.
\end{aligned}
$$

We can apply (4.11) to estimate IV_k by

$$
\begin{aligned}
|IV_k| &\le Cr_k^{\frac{p}{2}}\epsilon_k^{1-\frac{p}{2}}\|\nabla\widehat{\phi_k}\|_{L^2(B_1)}\|\widehat{R_k}-R_\infty\|_{L^2(B_1)}+Cr_k^p\epsilon_k^{1-p}\|\widehat{R_k}-R_\infty\|_{L^1(B_1)}\\
&\le Cr_k^{\frac{p}{2}}\epsilon_k^{\frac{2-p}{2}}\|\widehat{R_k}-R_\infty\|_{L^2(B_1)}+Cr_k^p\epsilon_k^{1-p}\|\widehat{R_k}-R_\infty\|_{L^1(B_1)}\\
&\le C\epsilon_k\|\widehat{R_k}-R_\infty\|_{L^2(B_1)}\to 0.
\end{aligned}
$$

While the most difficult term II_k can be estimated by employing the duality between $\mathcal{H}^1(\mathbb{R}^3)$ and $\mathrm{BMO}(\mathbb{R}^3)$ as follows.

$$
\begin{aligned}
&\int_{B_1} H_k^i\eta^2(\widehat{R_k}-R_\infty)\,dx\\
&=\int_{B_1(0)}\eta(H_k^i-\mu_k^i)\eta(\widehat{R_k}-R_\infty)\,dx+\mu_k^i\int_{B_1}\eta^2(\widehat{R_k}-R_\infty)\,dx\\
&=V_k+VI_k.
\end{aligned}
$$

It is easy to estimate

$$
|VI_k|\le C|\mu_k^i|\int_{B_1}|\widehat{R_k}-R_\infty|\,dx\le C\|H_k^i\|_{L^1(B_1)}\int_{B_1}|\widehat{R_k}-R_\infty|\,dx\to 0.
$$

We can apply Lemma 4.2 and (4.20) and (4.21) to estimate V_k by

$$
\begin{aligned}
|V_k| &= \Big|\int_{B_1}\eta(H_k^i-\mu_k^i)\eta(\widehat{R_k}-R_\infty)\,dx\Big|\\
&\le C\big\|\eta(H_k^i-\mu_k^i)\big\|_{\mathcal{H}^1(\mathbb{R}^3)}\big\|\eta(\widehat{R_k}-R_\infty)\|_{\mathrm{BMO}(\mathbb{R}^3)}\le C.
\end{aligned}
$$

Therefore we obtain that

$$|II_k| \le C\epsilon_k(|V_k| + |VI_k|) \le C\epsilon_k \to 0.$$

Putting all the estimates of I_k, II_k, III_k, IV_k together, we arrive that

$$\int_{B_{\frac{1}{4}}} |\nabla(\widehat{R_k} - R_\infty)|^p \, dx \to 0.$$

Next, we are going to prove that

$$\nabla\widehat{\phi_k} \longrightarrow \nabla\phi_\infty \quad \text{in} \quad L^2(B_{\frac{1}{4}}).$$

Since

$$-\Delta\widehat{\phi_k} = r_k^{\frac{p}{2}}\epsilon_k^{1-\frac{p}{2}}\operatorname{div}(\widehat{R_k}) + \frac{1}{2}\epsilon_k^{-\frac{p}{2}} f_k \quad \text{in} \quad B_1,$$

and

$$-\Delta\phi_\infty = 0 \quad \text{in} \quad B_1,$$

multiplying both equations by $\eta^2(\widehat{\phi_k} - \phi_\infty)$, subtracting the resulting equations, and integrating over $\mathbb{R}^3$, we obtain that

$$\begin{aligned}
&\int_{B_1} \eta^2|\nabla(\widehat{\phi_k} - \phi_\infty)|^2 \, dx + 2\int_{B_1} \eta\nabla(\widehat{\phi_k} - \phi_\infty)\nabla\eta(\widehat{\phi_k} - \phi_\infty) \, dx \\
&= \int_{B_1} \big[r_k^{\frac{p}{2}}\epsilon_k^{1-\frac{p}{2}}\operatorname{div}(\widehat{R_k}) + \frac{1}{2}\epsilon_k^{-\frac{p}{2}} f_k\big]\eta^2(\widehat{\phi_k} - \phi_\infty) \, dx.
\end{aligned}$$

Since $\widehat{\phi_k} \longrightarrow \phi_\infty$ and $\nabla\widehat{\phi_k} \rightharpoonup \nabla\phi_\infty$ in $L^2(B_{\frac{1}{4}})$, we conclude that

$$2\int_{B_1} \eta\nabla(\widehat{\phi_k} - \phi_\infty)\nabla\eta(\widehat{\phi_k} - \phi_\infty) \, dx \to 0.$$

Also, since

$$\begin{aligned}
&\big\|r_k^{\frac{p}{2}}\epsilon_k^{1-\frac{p}{2}}\operatorname{div}(\widehat{R_k}) + \frac{1}{2}\epsilon_k^{-\frac{p}{2}} f_k\big\|_{L^2(B_1)} \\
&\le Cr_k^{\frac{p}{2}}\epsilon_k^{1-\frac{p}{2}}\|\nabla\widehat{R_k}\|_{L^2(B_1)} + C\epsilon_k^{-\frac{p}{2}} r_k^{\frac{p+2}{2}} \le C\epsilon_k \to 0,
\end{aligned}$$

we conclude that

$$\int_{B_1} \big[r_k^{\frac{p}{2}}\epsilon_k^{1-\frac{p}{2}}\operatorname{div}(\widehat{R_k}) + \frac{1}{2}\epsilon_k^{-\frac{p}{2}} f_k\big]\eta^2(\widehat{\phi_k} - \phi_\infty) \, dx \to 0.$$

Thus we obtain that

$$\int_{B_{\frac{1}{4}}} |\nabla(\widehat{\phi_k} - \phi_\infty)|^2 \, dx \to 0.$$

This completes the proof of Lemma 4.3. □

Now we return to the proof of Lemma 4.1. It follows from Lemma 4.3 and the estimate (4.14) that for sufficiently large $k > 1$, it holds that

$$\theta^{p-3}\int_{B_\theta} \big(|\nabla\widehat{R_k}|^p + |\nabla\widehat{\phi_k}|^2\big) \le C\theta^p + o(1) \le \frac{1}{2}\max\big\{1, \frac{r_k^p}{\epsilon_k^p}\big\},$$

provided that $0 < \theta < \frac{1}{4}$ is chosen to be sufficiently small. This contradicts to the assumed inequality (4.10). Hence the proof of Lemma 4.1 is complete. □

Next we apply Lemma 4.1 and the Marstrand Theorem to give a proof of Theorem 1.3.

Proof of Theorem 1.3. First, define the concentration set Σ of (ϕ, R) by

$$\Sigma = \Big\{x \in \Omega \ \Big|\ \Theta^{3-p}((\phi, R), x) \equiv \lim_{r\to 0} \mathrm{Coss}_x\big((\phi, R), r\big) \geq \frac{1}{2}\epsilon_0^p\Big\}.$$

Here $\mathrm{Coss}_x\big((\phi, R), r\big)$ denotes the modified renormalized Cosserat energy of (ϕ, R) in $B_r(x)$ defined by (2.4), which is monotonically increasing with resepct to $r > 0$ by Corollary 2.2. Hence the density function

$$\Theta^{3-p}((\phi, R), x) = \lim_{r\to 0} \mathrm{Coss}_x\big((\phi, R), r\big)$$

exists for any $x \in \Omega$ and is upper semicontinuous in Ω. From a simple covering argument (see [22]), we know that the $(3-p)$-dimensional Hausdorff measure of Σ

$$\mathcal{H}^{3-p}(\Sigma) = 0.$$

For any $x_0 \in \Omega \setminus \Sigma$, there exists $r_0 > 0$ such that $B_{r_0}(x_0) \subset \Omega$, and

$$\mathrm{Coss}_{x_1}((\phi, R), \frac{r_0}{2}) = e^{Cr_0^2}(\frac{r_0}{2})^{p-3}\int_{B_{\frac{r_0}{2}}(x_1)} \big(|\nabla R|^p + |\nabla\phi|^2\big)\,dx + C(\frac{r_0}{2})^3 \leq \epsilon_0^p$$

holds for all $x_1 \in B_{\frac{r_0}{2}}(x_0)$.

Applying Lemma 4.1 repeatedly, we would obtain that there exists $\theta_0 \in (0, \frac{1}{2})$ such that

$$\begin{aligned}
&(\theta_0^l r_0)^{p-3}\int_{B_{\theta_0^l r_0}(x_1)} (|\nabla R|^p + |\nabla\phi|^2)\,dx \\
&\leq 2^{-l}\max\Big\{r_0^{p-3}\int_{B_{r_0}(x_0)} (|\nabla R|^p + |\nabla\phi|^2)\,dx,\ \ \frac{Cr_0^p}{1-2\theta_0^p}\Big\}
\end{aligned}\tag{4.22}$$

holds for all $x_1 \in B_{\frac{r_0}{2}}(x_0)$ and $l \geq 1$.

It follows from (4.22) that there exists $\alpha_0 \in (0, 1)$ such that

$$\begin{aligned}
&r^{p-3}\int_{B_r(x_1)} (|\nabla R|^p + |\nabla\phi|^2)\,dx \\
&\leq (\frac{r}{r_0})^{p\alpha_0}\max\Big\{r_0^{p-3}\int_{B_{r_0}(x_0)} (|\nabla R|^p + |\nabla\phi|^2)\,dx,\ \ \frac{Cr_0^p}{1-2\theta_0^p}\Big\} \\
&\leq C(\epsilon_0)(\frac{r}{r_0})^{p\alpha_0}
\end{aligned}\tag{4.23}$$

holds for all $x_1 \in B_{\frac{r_0}{2}}(x_0)$ and $0 < r \leq \frac{r_0}{2}$. Thus, by Morrey's decay Lemma [22], we conclude that $(\phi, R) \in C^{\alpha_0}(B_{\frac{r_0}{2}}(x_0))$. Since

$$\Delta\phi = \mathrm{div}(R) + \frac{1}{2}f \ \text{ in } \ B_{r_0}(x_0),$$

the higher order regularity theory of Poisson equation implies that $\phi \in C^{1,\alpha_0}(B_{\frac{r_0}{2}}(x_0))$. Since $x_0 \in \Omega \setminus \Sigma$ is arbitrary, we obtain that $(\phi, R) \in C^{1,\alpha_0}(\Omega \setminus \Sigma) \times C^{\alpha_0}(\Omega \setminus \Sigma)$.

Next we will employ the Marstrand Theorem [23] to show that the singular set Σ is discrete for $2 < p < 3$. We argue it by contradiction. Suppose Σ is not discrete. Then there exist a sequence of points $\{x_k\} \subset \Sigma$ and $x_0 \in \Sigma$ such that $x_k \to x_0$. Set $r_k = |x_k - x_0| \to 0$ and define

$$(\phi_k,\ R_k,\ f_k,\ M_k)(x) = (r_k^{\frac{p-2}{2}}\phi,\ R,\ r_k^{\frac{p+2}{2}}f,\ r_k^p M)(x_0 + r_k x),\ \forall x \in B_2.$$

It is readily seen that (ϕ_k, R_k) is singular at 0 and $y_k = \frac{x_k - x_0}{r_k} \in \mathbb{S}^2$. Moreover, similar to (4.5), (ϕ_k, R_k) solves

$$\begin{cases} \Delta\phi_k = r_k^{\frac{p}{2}}\operatorname{div}(R_k) + \frac{1}{2}f_k, \\ \operatorname{div}(|\nabla R_k|^{p-2}\nabla R_k) = \displaystyle\sum_{i=1}^{3}\langle|\nabla R_k|^{p-2}\nabla R_k, \mathbf{V}_i(R_k)\rangle\nabla(\mathbf{V}_i(R_k)) & \text{in } \ B_2. \\ -\displaystyle\frac{1}{p}\sum_{i=1}^{3}\left[2r_k^{\frac{p}{2}}\langle\nabla\phi_k, \mathbf{V}_i(R_k)\rangle - \langle M_k, \mathbf{V}_i(R_k)\rangle\right]\mathbf{V}_i(R_k) \end{cases} \tag{4.24}$$

It follows from the monotonicity inequality (2.3) for (ϕ, R) and the scaling argument that (ϕ_k, R_k) also enjoys the following monotonicity inequality, i.e., for $0 < r_1 < r_2 \le 2$

$$\begin{aligned} & e^{Cr_1^2}r_1^{p-3}\int_{B_{r_1}}(|\nabla R_k|^p + |\nabla\phi_k|^2)\,dx + Cr_1^3 \\ & + \int_{r_1}^{r_2} r^{p-3}\int_{\partial B_r}\big(p|\nabla R_k|^{p-2}\big|\frac{\partial R_k}{\partial r}\big|^2 + \big|\frac{\partial\phi_k}{\partial r}\big|^2\big)\,dH^2 dr \\ & \le e^{Cr_2^2}r_2^{p-3}\int_{B_{r_2}}\big(|\nabla R_k|^p + |\nabla\phi_k|^2\big)\,dx + Cr_2^3. \end{aligned} \tag{4.25}$$

Moreover, for $k > 1$ sufficiently large,

$$\begin{aligned} \frac{1}{4}\epsilon_0^p &\le 2^{p-3}\int_{B_2}(|\nabla R_k|^p + |\nabla\phi_k|^2)\,dx \\ &= (2r_k)^{p-3}\int_{B_{2r_k}(x_0)}(|\nabla R|^p + |\nabla\phi|^2)\,dx \le C. \end{aligned} \tag{4.26}$$

Hence

$$\int_{B_2}(|\nabla R_k|^p + |\nabla\phi_k|^2)\,dx \quad \text{is uniformly bounded above and below.}$$

Then there exists $(\phi_\infty, R_\infty) \in H^1(B_2, \mathbb{R}^3) \times W^{1,p}(B_2, SO(3))$ such that, after passing to a subsequence,

$$(\phi_k, R_k) \rightharpoonup (\phi_\infty, R_\infty) \quad \text{in} \quad H^1(B_2) \times W^{1,p}(B_2).$$

Now we want to show

Claim 0. ϕ_∞ is a harmonic function in B_2, and R_∞ is a p-harmonic map from B_2 to SO(3).

In fact, since

$$\begin{aligned} \big\|r_k^{\frac{p}{2}}\operatorname{div}(R_k) + \frac{1}{2}f_k\big\|_{L^p(B_2)} &\le C\big(r_k^{\frac{p}{2}}\|\nabla R_k\|_{L^p(B_2)} + \|f_k\|_{L^\infty(B_2)}\big) \\ &\le Cr_k^{\frac{p}{2}}\big(\|\nabla R_k\|_{L^p(B_2)} + \|f\|_{L^\infty(\Omega)}\big) \le Cr_k^{\frac{p}{2}} \to 0, \end{aligned}$$

after passing to the limit in (4.24) we have that $\Delta\phi_\infty = 0$ in B_2. For R_k, observe that

$$\begin{aligned}&\operatorname{div}\langle|\nabla R_k|^{p-2}\nabla R_k, \mathbf{V}_i(R_k)\rangle\\ &= \langle|\nabla R_k|^{p-2}\nabla R_k, \mathbf{V}_i(\nabla R_k)\rangle + \langle\operatorname{div}(|\nabla R_k|^{p-2}\nabla R_k), \mathbf{V}_i(R_k)\rangle\\ &= -\frac{1}{p}\langle 2r_k^{\frac{p}{2}}\nabla\phi_k - M_k, \mathbf{V}_i(\nabla R_k)\rangle := g_k,\end{aligned}$$

where we have used the skew-symmetry of $\mathbf{V}_i$, and the equation of R_k in the last step. Observe that

$$\|g_k\|_{L^2(B_2)} \le C\big(r_k^{\frac{p}{2}}\|\nabla\phi_k\|_{L^2(B_2)} + r_k^{\frac{p}{2}}\|M\|_{L^\infty(\Omega)}\big) \to 0,$$

we can apply the Div-Curl Lemma to conclude that

$$\langle|\nabla R_k|^{p-2}\nabla R_k, \mathbf{V}_i(R_k)\rangle\nabla(\mathbf{V}_i(R_k)) \to \langle|\nabla R_\infty|^{p-2}\nabla R_\infty, \mathbf{V}_i(R_\infty)\rangle\nabla(\mathbf{V}_i(R_\infty))$$

in $L^1(B_2)$. Hence by passing to the limit in the equation (4.24), we see that R_∞ solves the equation

$$\operatorname{div}(|\nabla R_\infty|^{p-2}\nabla R_\infty) = \sum_{i=1}^{3}\langle|\nabla R_\infty|^{p-2}\nabla R_\infty, \mathbf{V}_i(R_\infty)\rangle\nabla(\mathbf{V}_i(R_\infty)),$$

or equivalently R_∞ is a p-harmonic map in B_2. This shows *Claim 0*.

Moreover, it follows from the lower semicontinuity and the monotonicity inequality (4.25) that for any $0 < s \le 2$, it holds

$$\int_s^2 r^{p-3}\int_{\partial B_r}\big(p|\nabla R_\infty|^{p-2}\big|\frac{\partial R_\infty}{\partial r}\big|^2 + \big|\frac{\partial\phi_\infty}{\partial r}\big|^2\big)\, dH^2 dr = 0,$$

this follows from the fact that for any fixed $0 < s \le 2$,

$$e^{Cs^2}s^{p-3}\int_{B_s}(|\nabla R_k|^p + |\nabla\phi_k|^2)\,dx + Cs^3 \to \Theta^{3-p}\big((\phi, R), x_0\big), \text{ as } k \to \infty.$$

Therefore we must have that

$$\Big(\frac{\partial\phi_\infty}{\partial r},\ \frac{\partial R_\infty}{\partial r}\Big) = (0,\ 0),$$

or equivalently (ϕ_∞, R_∞) is homogeneous of degree zero:

$$\big(\phi_\infty(x), R_\infty(x)\big) = \big(\phi_\infty(\frac{x}{|x|}), R_\infty(\frac{x}{|x|})\big),\ x \in B_2. \tag{4.27}$$

Since ϕ_∞ is a smooth harmonic function with homogeneous degree zero, it follows that ϕ_∞ is a constant.

Next we need to show

Claim 1.

$$(\phi_k, R_k) \longrightarrow (\phi_\infty, R_\infty) \quad \text{in} \quad H^1(B_1)\times W^{1,p}(B_1).$$

Assume the claim for the moment. Then it follows from (4.26) and ϕ_∞ = constant that $R_\infty : B_2 \to \mathrm{SO}(3)$ is a nontrivial stationary p-harmonic map, which has at least two singular points 0 and $y_\infty \in \mathbb{S}^2$ given by

$$y_\infty = \lim_{k\to\infty}\frac{x_k - x_0}{|x_k - x_0|}.$$

The singular set of R_∞ contains the line segement $[0y_\infty]$ so that $\mathcal{H}^1(\mathrm{Sing}(R_\infty)) > 0$, which is impossible. Thus Σ is a discrete set.

Finally, we would like to apply Marstrand theorem to prove *Claim 1.* To do it, we consider a sequence of Radon measures

$$\mu_k = (|\nabla R_k|^p + |\nabla \phi_k|^2)\, dx.$$

Since $\mu_k(B_2)$ is uniformly bounded, we may assume that there is a nonnegative Radon measure μ in B_2 such that after passing to a subsequence,

$$\mu_k \to \mu$$

as convergence of Radon measures. By Fatou's lemma, we can decompose μ into

$$\mu = (|\nabla R_\infty|^p + |\nabla \phi_\infty|^2)\, dx + \nu$$

for a nonnegative Radon measure ν, called a defect measure. The monotonicity inequality (4.25) for (ϕ_k, R_k) implies that μ is a monotone measure in the following sense: for $x \in B_1$, $0 < r_1 < r_2 < \mathrm{dist}(x, \partial B_2)$,

$$e^{Cr_1^2} r_1^{p-3} \mu(B_{r_1}(x)) + Cr_1^3 \le e^{Cr_2^2} r_2^{p-3} \mu(B_{r_2}(x)) + Cr_2^3.$$

In particular, for any $x \in B_1$, the density function

$$\Theta^{3-p}(\mu, x) = \lim_{r\to 0} r^{p-3} \mu(B_r(x))$$

exists and is upper semicontinuous in B_1. Define the concentration set

$$\boldsymbol{S} := \bigcap_{r>0} \left\{ x \in B_1 \;\Big|\; \liminf_{k\to\infty} r^{p-3} \int_{B_r(x)} (|\nabla R_k|^p + |\nabla \phi_k|^2)\, dx \ge \frac{1}{2}\epsilon_0^p \right\}.$$

We claim that $\boldsymbol{S}$ is a closed subset of B_1. In fact, let $\{x_k\}$ be a sequence of points in $\boldsymbol{S}$ such that $x_k \to x_0 \in B_1$. If $x_0 \notin \boldsymbol{S}$, then there exists $r_0 > 0$ and $\delta_0 > 0$ such that for $k > 1$ sufficiently large it holds that

$$r_0^{p-3} \int_{B_{r_0}(x_0)} (|\nabla R_k|^p + |\nabla \phi_k|^2)\, dx \le \frac{1}{2}\epsilon_0^p - \delta_0.$$

Taking k large enough so that $|x_k - x_0| < \frac{r_0}{2}$ and applying the monotonicity inequality to each (ϕ_k, R_k), we have

$$\begin{aligned}
& e^{C(\frac{r_0}{2})^2} \Big(\frac{r_0}{2}\Big)^{p-3} \int_{B_{\frac{r_0}{2}}(x_k)} (|\nabla R_k|^p + |\nabla \phi_k|^2)\, dx + C\left(\frac{r_0}{2}\right)^3 \\
&\le e^{C(r_0 - |x_k - x_0|)^2} (r_0 - |x_k - x_0|)^{p-3} \int_{B_{r_0 - |x_k - x_0|}(x_k)} (|\nabla R_k|^p + |\nabla \phi_k|^2)\, dx \\
&\quad + C\left((r_0 - |x_k - x_0|\right)^3 \\
&\le e^{C(r_0 - |x_k - x_0|)^2} \Big(\frac{r_0}{(r_0 - |x_k - x_0|)}\Big)^{3-p} r_0^{p-3} \int_{B_{r_0}(x_0)} |\nabla R_k|^p + |\nabla \phi_k|^2)\, dx \\
&\quad + C\left((r_0 - |x_k - x_0|\right)^3 \\
&\le e^{Cr_0^2} \Big(\frac{r_0}{(r_0 - |x_k - x_0|)}\Big)^{3-p} \Big(\frac{1}{2}\epsilon_0^p - \delta_0\Big) + C\left((r_0 - |x_k - x_0|\right)^3 \\
&\le \frac{1}{2}\epsilon_0^p,
\end{aligned}$$

provided that k large enough and r_0 is chosen sufficiently small. This contradicts to the fact $x_k \in \boldsymbol{S}$. Hence $\boldsymbol{S}$ is a closed subset.

Suppose $x_* \in B_1 \setminus \boldsymbol{S}$. Then there exists $r_* > 0$ such that

$$\liminf_{k\to\infty} {r_*}^{p-3} \int_{B_{r_*}(x_*)} (|\nabla R_k|^p + |\nabla \phi_k|^2)\, dx < \frac{1}{2}\epsilon_0^p.$$

Applying the ϵ_0-regularity Theorem 1.3, we may conclude that after passing to another subsequence,

$$R_k \longrightarrow R_\infty \quad \text{in} \quad C^1_{\text{loc}} \cap W^{1,p}_{\text{loc}}(B_1 \setminus \boldsymbol{S}),$$

and

$$\phi_k \longrightarrow R_\infty \quad \text{in} \quad C^1_{\text{loc}} \cap H^1_{\text{loc}}(B_1 \setminus \boldsymbol{S}).$$

If we denote by $\text{Sing}(\phi_\infty, R_\infty)$ the set of discontinuity of (ϕ_∞, R_∞), and $\text{supp}(\nu)$ the support of the defect measure ν. Then the above convergence implies that

$$\text{Sing}(\phi_\infty, R_\infty) \cup \text{supp}(\nu) \subset \boldsymbol{S}.$$

On the other hand, if $\hat{x} \in \boldsymbol{S}$, then after sending $k \to \infty$, we have that

$$\frac{\mu(B_r(\hat{x}))}{r^{3-p}} \geq \frac{1}{2}\epsilon_0^p, \ \forall r > 0.$$

If $\hat{x} \notin \text{Sing}(\phi_\infty, R_\infty)$, then (ϕ_∞, R_∞) is regular near $\hat{x}$ and hence for r sufficiently small,

$$r^{p-3} \int_{B_r(\hat{x})} (|\nabla R_\infty|^p + |\nabla \phi_\infty|^2) dx \leq \frac{1}{4}\epsilon_0^p,$$

this implies that for small $r > 0$,

$$\frac{\nu(B_r(\hat{x}))}{r^{3-p}} \geq \frac{1}{4}\epsilon_0^p,$$

and hence $\hat{x} \in \text{supp}(\nu)$. Therefore, we conclude that

Lemma 4.4.

$$\text{Sing}(\phi_\infty, R_\infty) \cup \text{supp}(\nu) = \boldsymbol{S}.$$

Notice that if $x \in \boldsymbol{S}$, then

$$\Theta^{3-p}(\mu, x) = \lim_{r\to 0} r^{3-p} \mu(B_r(x)) \geq \frac{1}{2}\epsilon_0^p.$$

Moreover, for any compact subset $K \subset\subset B_1$, and any $x \in \boldsymbol{S} \cap K$,

$$\frac{1}{2}\epsilon_0^p \leq \Theta^{3-p}(\mu, x) \leq r_K^{p-3} \mu(B_2) \leq r_K^{p-3} E_0,$$

where $r_K = \frac{1}{2}\text{dist}(K, \partial B_2) > 0$, and $E_0 = \sup_k \int_{B_2} (|\nabla R_k|^p + |\nabla \phi_k|^2)\, dx$. Recall that by Federer-Ziemer theorem (see [22])

$$\lim_{r\to 0} r^{p-3} \int_{B_r(x)} (|\nabla R_\infty|^p + |\nabla \phi_\infty|^2)\, dy = 0$$

holds for H^{3-p} a.e. $x \in B_2$. Thus we obtain that

Lemma 4.5. *For any compact $K \subset B_1$, if $x \in \boldsymbol{S} \cap K$, then*

$$\frac{1}{2}\epsilon_0^p \leq \Theta^{3-p}(\mu, x) < C(K) < \infty.$$

For H^{3-p} a.e. $x \in \boldsymbol{S}$,

$$\Theta^{3-p}(\mu, x) = \Theta^{3-p}(\nu, x).$$

It follows from Lemma 4.5 and standard covering arguments that for any compact set $K \subset B_1$

$$\epsilon^p \mathcal{H}^{3-p}(\boldsymbol{S} \cap K) \leq \nu(\boldsymbol{S} \cap K) \leq C\mathcal{H}^{3-p}(\boldsymbol{S} \cap K).$$

Therefore,

$$\nu(\boldsymbol{S}) = 0 \iff \mathcal{H}^{3-p}(\boldsymbol{S}) = 0.$$

In particular, we have that

Lemma 4.6. *$(\phi_k, R_k) \nrightarrow (\phi_\infty, R_\infty)$ strongly in $H^1(B_1) \times W^{1,p}(B_1)$ if and only if $\nu(B_1) > 0$ if and only if $\mathcal{H}^{3-p}(\boldsymbol{S}) > 0$.*

Return to the proof of *Claim 1.* For $2 < p < 3$, if

$$(\phi_k, R_k) \nrightarrow (\phi_\infty, R_\infty) \text{ in } H^1(B_1) \times W^{1,p}_{loc}(B_1),$$

then by Lemma 4.6, we must have $\mathcal{H}^{3-p}(\boldsymbol{S}) > 0$. Hence by Lemma 4.4 we have for $\mathcal{H}^{3-p}$ a.e. $x \in \boldsymbol{S}$,

$$0 < \Theta^{3-p}(\nu, x) < \infty.$$

Now we recall the following theorem due to Marstrand [23].

Theorem 4.7. *Let s be a positive number. Suppose that $\widehat{\nu}$ is a Radon measure on $\mathbb{R}^n$ such that the density $\Theta^s(\widehat{\nu}, a)$ exists and is positive and finite in a set S of positive $\widehat{\nu}$ measure. Then s is an integer.*

Applying Theorem 4.7 to $s = 3 - p$, $\widehat{\nu} = \nu$ and $S = \boldsymbol{S}$, we conclude that $3 - p$ must be an integer. This is impossible. Hence *Claim 1* is true. This completes the proof of Theorem 1.3. $\square$

5. Stable-stationary solutions of the Cosserat equation

This section is devoted to the proof of Theorem 1.5. More precisely, we will show that if (ϕ, R) is a stable stationary solution to the Cosserat equation (1.3). Then the singular set is empty for p belonging to the range $[2, \frac{32}{15}]$.

It is well-known that $\mathbb{S}^3$ is the universal cover of SO(3), and we can choose a matrix norm on SO(3) such that $\mathbb{S}^3$ is locally isometric to SO(3). In fact, a canonical, locally isometric 2-to-1 covering map $\pi : \mathbb{S}^3 \to \mathrm{SO}(3)$ is given by

$$\pi(w, x, y, z) = \begin{pmatrix} 1 - 2y^2 - 2z^2 & 2xy - 2zw & 2xz + 2yw \\ 2xy + 2zw & 1 - 2x^2 - 2z^2 & 2yz - 2xw \\ 2xz - 2yw & 2yz + 2xw & 1 - 2x^2 - 2y^2 \end{pmatrix}, \ \forall (w, x, y, z) \in \mathbb{S}^3.$$

In particular, the curvature operator of SO(3), $R_{\mathrm{SO}(3)}$, satisfies

$$\langle R_{\mathrm{SO}(3)}(v, w)v, w\rangle = |v|^2|w|^2 - \langle v, w\rangle^2, \ v, w \in T_R\mathrm{SO}(3).$$

For $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, \mathrm{SO}(3))$, let

$$(\phi_t, R_t) \in C^2((-\delta, \delta), H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, \mathrm{SO}(3)))$$

be a family of variations of (ϕ, R). Denote by

$$\eta = \frac{d}{dt}|_{t=0}\phi_t, \ \hat{\eta} = \frac{d^2}{dt^2}|_{t=0}\phi_t,$$

and

$$v = \frac{\partial R_t}{\partial t}\Big|_{t=0}, \ \hat{v} = \nabla_{\frac{\partial}{\partial t}} \frac{\partial R_t}{\partial t}\Big|_{t=0}.$$

Applying the equation (1.3) for (ϕ, R) and direct calculations as in Smith [12], we obtain that

$$\begin{aligned}
&\frac{d^2}{dt^2}\Big|_{t=0}\mathrm{Coss}(\phi_t, R_t) \\
&= \frac{d^2}{dt^2}\Big|_{t=0}\int_\Omega \big(|\nabla\phi_t|^2 - 2\langle R_t, \nabla\phi_t\rangle + |\nabla R_t|^p + (\phi_t - x)\cdot f + \langle R_t, M\rangle\big)\, dx \\
&= \int_\Omega \big(2|\nabla\eta|^2 - 4\langle v, \nabla\eta\rangle + p|\nabla R|^{p-2}(|\nabla v|^2 - \mathrm{tr}\langle R_{\mathrm{SO}(3)}(v, \nabla R)v, \nabla R\rangle) \\
&\qquad + p(p-2)|\nabla R|^{p-4}\langle \nabla R, \nabla v\rangle^2\big)\, dx \\
&= \int_\Omega \big(2|\nabla\eta|^2 - 4\langle v, \nabla\eta\rangle + p|\nabla R|^{p-2}(|\nabla v|^2 - |\nabla R|^2|v|^2) \\
&\qquad + p(p-2)|\nabla R|^{p-4}\langle \nabla R, \nabla v\rangle^2\big)\, dx
\end{aligned}$$

holds for any $\eta \in H_0^1(\Omega, \mathbb{R}^3)$ and $v \in H_0^1 \cap L^\infty(\Omega, T_R\mathrm{SO}(3))$.

Definition 5.1. *For $2 \le p < 3$, $\mu_1 = \mu_c = \mu_2 = 1$, $f \in L^\infty(\Omega, \mathbb{R}^3)$ and $M \in L^\infty(\Omega, \mathrm{SO}(3))$, a stationary weak solution (ϕ, R) of the Cosserat equation (1.3) is called a stable, stationary weak solution of the Cosserat equation (1.3) if, in addition,*

$$\frac{d^2}{dt^2}\Big|_{t=0}\mathrm{Coss}(\phi_t, R_t) \ge 0,$$

or, equivalently,

$$\begin{aligned}
&\int_\Omega \big(2|\nabla\eta|^2 - 4\langle v, \nabla\eta\rangle + p|\nabla R|^{p-2}(|\nabla v|^2 - |\nabla R|^2|v|^2) \\
&\qquad + p(p-2)|\nabla R|^{p-4}\langle \nabla R, \nabla v\rangle^2\big)\, dx \ge 0
\end{aligned} \tag{5.1}$$

holds for any $\eta \in C_0^\infty(\Omega, \mathbb{R}^3)$ and $v \in H_0^1(\Omega, T_R\mathrm{SO}(3))$.

Lemma 5.2. *For $2 \le p < 3$, $\mu_1 = \mu_c = \mu_2 = 1$, $f \in L^\infty(\Omega, \mathbb{R}^3)$ and $M \in L^\infty(\Omega, \mathrm{SO}(3))$, if (ϕ, R) is a stable, stationary weak solution of the Cosserat equation (1.3), then*

$$\int_\Omega \big(6|\nabla\omega|^2 - 4\sum_{i=1}^3 \psi\langle \mathbf{a}_i R, \nabla\omega \otimes \mathbf{e}^i\rangle + p(p+1)|\nabla R|^{p-2}|\nabla\psi|^2 - 2p|\nabla R|^p|\psi|^2\big)\, dx \ge 0 \tag{5.2}$$

holds for any $\omega \in C_0^\infty(\Omega)$ and $\psi \in C_0^\infty(\Omega)$. Here $(\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3)$ is the standard base of $\mathbb{R}^3$. In particular,

$$\int_\Omega \big((p+1)|\nabla R|^{p-2}|\nabla\psi|^2 - 2|\nabla R|^p|\psi|^2\big)\, dx \ge 0 \tag{5.3}$$

holds for any $\psi \in C_0^\infty(\Omega)$.

Proof. It is readily seen that (5.3) follows immediately from (5.2) by taking $\omega = 0$. Thus it suffices to show (5.2). For any $\omega \in C_0^\infty(\Omega)$ and $\psi \in C_0^\infty(\Omega)$, let $\eta = \omega \mathbf{e}^i$ and $v = \psi \mathbf{a}_i R$ and substitute them into (5.1) and then take summation over $i = 1, 2, 3$, we obtain that

$$\int_\Omega \big(2\sum_{i=1}^3 |\nabla(\omega\mathbf{e}^i)|^2 - 4\sum_{i=1}^3 \psi\langle \mathbf{a}_i R, \nabla\omega\otimes \mathbf{e}^i\rangle + p(p-2)|\nabla R|^{p-4}\sum_{i=1}^3 \langle \nabla R, \nabla(\psi\mathbf{a}_i R)\rangle^2$$
$$+p|\nabla R|^{p-2}\sum_{i=1}^3 (|\nabla(\psi\mathbf{a}_i R)|^2 - |\nabla R|^2|\psi\mathbf{a}_i R|^2)\big)\,dx \ge 0. \tag{5.4}$$

Observe that

$$\begin{aligned}
&\sum_{i=1}^3 \langle \nabla R, \nabla(\psi\mathbf{a}_i R)\rangle^2 = \sum_{i=1}^3 \big[\sum_{j=1}^3 \nabla_j\psi\langle\nabla_j R, \mathbf{a}_i R\rangle + \psi\langle \nabla R, \mathbf{a}_i\nabla R\rangle\big]^2\\
&= \sum_{i=1}^3 \langle \nabla\psi\cdot\nabla R, \mathbf{a}_i R\rangle^2 = |\nabla\psi\cdot\nabla R|^2 \le |\nabla\psi|^2|\nabla R|^2,\\
&\sum_{i=1}^3 |\nabla(\omega\mathbf{e}^i)|^2 = 3|\nabla\omega|^2, \quad \sum_{i=1}^3 |\nabla R|^2|\psi\mathbf{a}_i R|^2 = 3|\nabla R|^2|\psi|^2,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^3 |\nabla(\psi\mathbf{a}_i R)|^2\\
&= |\nabla\psi|^2\sum_{i=1}^3 |\mathbf{a}_i R|^2 + 2\psi\nabla\psi\sum_{i=1}^3 \langle \mathbf{a}_i R, \mathbf{a}_i\nabla R\rangle + |\psi|^2\sum_{i=1}^3 \langle \mathbf{a}_i\nabla R, \mathbf{a}_i\nabla R\rangle\\
&= 3|\nabla\psi|^2 + \psi\nabla\psi\mathrm{tr}(R^T\mathbf{a}_i^T\mathbf{a}_i\nabla R + \nabla R^T\mathbf{a}_i^T\mathbf{a}_i R) + |\psi|^2\mathrm{tr}\big(\nabla R^T\nabla R(\sum_{i=1}^3 \mathbf{a}_i^T\mathbf{a_i})\big)\\
&= 3|\nabla\psi|^2 + \psi\nabla\psi\mathrm{tr}[(R^T\nabla R + \nabla R^T R)(\mathbf{a}_i^T\mathbf{a}_i)] + |\psi|^2\mathrm{tr}\big(\nabla R^T\nabla R(\sum_{i=1}^3 \mathbf{a}_i^T\mathbf{a_i})\big)\\
&= 3|\nabla\psi|^2 + |\nabla R|^2|\psi|^2,
\end{aligned}$$

where we have used

$$\mathbf{a}_1^T\mathbf{a}_1 = \mathrm{diag}(0, \frac{1}{2}, \frac{1}{2}),\ \mathbf{a}_2^T\mathbf{a}_2 = \mathrm{diag}(\frac{1}{2}, 0, \frac{1}{2}),\ \mathbf{a}_3^T\mathbf{a}_3 = \mathrm{diag}(\frac{1}{2}, \frac{1}{2}, 0),$$

and

$$\langle R, \mathbf{a}_i\nabla R\rangle = 0,\ R^T\nabla R + \nabla R^T R = 0.$$

Plugging these identities into (5.4), we obtain (5.2). □

Now we can extend the partial regularity theorem for stationary weak solutions of the Cosserat euqation (1.3) obtained in the previous section to the class of stable weak solutions of the Cosserat euqation (1.3). First, we consider Theorem 1.5 in the case that $p = 2$. Namely, we will show that

Theorem 5.3. *For $f \in L^\infty(\Omega, \mathbb{R}^3)$ and $M \in L^\infty(\Omega, \mathrm{SO}(3))$, and $\mu_1 = \mu_c = \mu_2 = 1$, assume that $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times H^1(\Omega, \mathrm{SO}(3))$ is a stable, stationary weak solution of the Cosserat equation* (1.3) *for $p = 2$. Then $(\phi, R) \in C^{1,\alpha}(\Omega, \mathbb{R}^3) \times C^\alpha(\Omega, \mathrm{SO}(3))$ for some $\alpha \in (0, 1)$.*

Proof. From the small energy regularity theorem obtained in the previous section, we know that there exists a closed singular set $\Sigma \subset \Omega$, with $H^1(\Sigma) = 0$, such that $(\phi, R) \in C^{1,\alpha}(\Omega \setminus \Sigma) \times C^\alpha(\Omega \setminus \Sigma)$ for some $0 < \alpha < 1$.

Now we want to show $\Sigma = \emptyset$. For, otherwise, there exists $x_0 \in \Sigma$ such that

$$\Theta^1((\phi, R), x_0) \equiv \lim_{r \downarrow 0} r^{-1} \int_{B_r(x_0)} (|\nabla R|^2 + |\nabla \phi|^2)\, dx \geq \epsilon_0^2 > 0.$$

For any sequence of radius $r_i \downarrow 0$, define the blow up sequence

$$(\phi_i, R_i, f_i, M_i)(x) = (\phi, R, r_i^2 f, r_i^2 M)(x_0 + r_i x), \ \forall x \in B_2.$$

Then

$$\lim_{i \to \infty} 2^{-1} \int_{B_2} (|\nabla R_i|^2 + |\nabla \phi_i|^2)\, dx = \Theta^1((\phi, R), x_0) \geq \epsilon_0^2.$$

Thus there exists $(\phi_0, R_0) \in H^1(B_2, \mathbb{R}^3) \times H^1(B_2, \mathrm{SO}(3))$ such that after passing to a subsequence,

$$(\phi_i, R_i) \rightharpoonup (\phi_0, R_0) \text{ in } H^1(B_2, \mathbb{R}^3) \times H^1(B_2, \mathrm{SO}(3)).$$

Since (ϕ_i, R_i) satisfies

$$\begin{cases} \Delta \phi_i = r_i \mathrm{div} R_i + \frac{1}{2} f_i \\ \Delta R_i + r_i \nabla \phi_i - \frac{1}{2} M_i \perp T_{R_i} \mathrm{SO}(3), \end{cases} \tag{5.5}$$

it follows the same argument as in *Claim 0* that after sending $i \to \infty$, ϕ_0 is a harmonic function and R_0 is a harmonic map into SO(3) in B_2. We now need

Claim 2: $(\phi_i, R_i) \to (\phi_0, R_0)$ in $H^1(B_1, \mathbb{R}^3) \times H^1(B_1, \mathrm{SO}(3))$.

We will apply the technique of potential theory by Hong-Wang [17] and Lin-Wang [18] to prove this claim. Let $\nu \geq 0$ be a Radon measure in B_2 such that

$$\mu_i \equiv (|\nabla R_i|^2 + |\nabla \phi_i|^2)\, dx \rightharpoonup \mu \equiv (|\nabla R_0|^2 + |\nabla \phi_0|^2)\, dx + \nu$$

as convergence of measures in B_2. It suffices to show $\nu \equiv 0$ in B_1. Notice that (ϕ_i, R_i), solving (5.5), is indeed a stationary weak solution of the Euler-Lagrange equation of critical point of the Cosserart energy functional

$$E_i(\widehat{\phi}, \widehat{R}) = \int_{B_2} (|\nabla \widehat{R}|^2 + |\nabla \widehat{\phi}|^2 - 2r_i \langle \widehat{R}, \nabla \widehat{\phi} \rangle + (\widehat{\phi} - x) \cdot f_i + \langle \widehat{R}, M_i \rangle)\, dx.$$

In particular, the ϵ_0-regularity theorem is applicable to (ϕ_i, R_i) and we conclude that if we define

$$\begin{aligned} \mathcal{S} &= \bigcap_{r>0} \Big\{ y \in B_{\frac{3}{2}} : \lim_{i \to \infty} r^{-1} \int_{B_r(y)} (|\nabla R_i|^2 + |\nabla \phi_i|^2)\, dx \geq \epsilon_0^2 \Big\} \\ &= \Big\{ y \in B_{\frac{3}{2}} : \Theta^1(\mu, y) = \lim_{r \to 0} r^{-1} \mu(B_r(y)) \geq \epsilon_0^2 \Big\}. \end{aligned}$$

Then the following statements hold:
(i) $\mathcal{S}$ is closed with $\mathcal{H}^1(\mathcal{S}) < \infty$, $\mathrm{supp}(\nu) \subset \mathcal{S}$ and $\Theta^1(\nu, y) = \Theta^1(\mu, y) \geq \epsilon_0^2$ for H^1 a.e.

$y \in \mathcal{S}$.
(ii) There exists $\alpha \in (0,1)$ such that

$$(\phi_i, R_i) \to (\phi_0, R_0) \text{ in } (C^{\alpha}_{\mathrm{loc}} \cap H^1_{\mathrm{loc}})(B_{\frac{3}{2}} \setminus \mathcal{S}).$$

(iii)

$$C_1(\epsilon_0)\mathcal{H}^1(\mathcal{S}) \le \nu(B_{\frac{3}{2}}) \le C_2(\epsilon_0)\mathcal{H}^1(\mathcal{S}).$$

In particular, $\nu \equiv 0$ if and only if $\mathcal{H}^1(\mathcal{S}) = 0$. It follows from $\mathcal{H}^1(\mathcal{S}) < +\infty$ that $Cap_2(\mathcal{S}) = 0$. Hence for any $\delta > 0$, there exists $\omega_\delta \in C_0^\infty(B_2)$ such that

$$\mathcal{S} \subset \mathrm{int}(\{\omega_\delta = 1\}),$$

and

$$\int_{B_2} |\nabla \omega_\delta|^2 \, dx \le \delta. \tag{5.6}$$

Hence for any $a \in \mathcal{S}$, there exists $0 < r_a < \delta^2$ such that

$$\omega_\delta \ge \frac{1}{2} \text{ on } B_{r_a}(a).$$

From the compactness of $\mathcal{S}$ and Vitali's covering lemma, there exist $1 \le l < \infty$ and $\{a_m\}_{m=1}^l \subset \mathcal{S}$ such that $\{B_{\frac{r_{a_m}}{5}}(a_m)\}_{m=1}^l$ are mutually disjoint, and

$$\mathcal{S} \subset \bigcup_{m=1}^{l} B_{r_{a_m}}(a_m).$$

From the definition of $\mathcal{S}$, there exists a sufficiently large $i_l > 0$ such that

$$\frac{\epsilon_0^2}{2} \le \big(\frac{r_{a_m}}{5}\big)^{-1} \int_{B_{\frac{r_{a_m}}{5}}(a_m)} (|\nabla R_i|^2 + |\nabla \phi_i|^2) \, dx, \ \forall i \ge i_l, \ m = 1, \cdots, l. \tag{5.7}$$

By the $W^{1,q}$-estimate on ϕ, we know that

$$\|\nabla \phi\|_{L^q(K)} \le C(q, K)$$

holds for any compact set $K \Subset \Omega$ and $1 < q < \infty$. Hence for any $i \ge i_l$ and $m = 1, \cdots, l$, it follows from Hölder's inequality that

$$\begin{aligned} \big(\frac{r_{a_m}}{5}\big)^{-1} \int_{B_{\frac{r_{a_m}}{5}}(a_m)} |\nabla \phi_i|^2 \, dx &\le C\big(r_i r_{a_m}\big)^{-1} \int_{B_{r_i r_{a_m}}(x_0 + r_i a_m)} |\nabla \phi|^2 \, dx \\ &\le C(q)(r_i r_{a_m})^{2 - \frac{6}{q}} \le C\delta^{\frac{3}{2}} \le \frac{1}{4}\epsilon_0^2, \end{aligned}$$

provided we choose $q = 12$ and $\delta \le \big(\frac{\epsilon_0^2}{4C}\big)^{\frac{2}{3}}$ in the last step. Substituting this estimate into (5.7), we obtain that

$$\frac{1}{4}\epsilon_0^2 \le \big(\frac{r_{a_m}}{5}\big)^{-1} \int_{B_{\frac{r_{a_m}}{5}}(a_m)} |\nabla R_i|^2 \, dx, \ \forall i \ge i_l, \ m = 1, \cdots, l. \tag{5.8}$$

Therefore for all $i \geq i_l$, we can bound

$$\begin{aligned}
\mathcal{H}^1_{\delta^2}(\mathcal{S}) &\leq C\sum_{m=1}^{l} r_{a_m} = 5C\sum_{m=1}^{l}\frac{r_{a_m}}{5}\\
&\leq \frac{20C}{\epsilon_0^2}\sum_{m=1}^{l}\int_{B_{\frac{r_{a_m}}{5}}(a_m)}|\nabla R_i|^2\,dx\\
&\leq \frac{80C}{\epsilon_0^2}\int_{\bigcup_{m=1}^{l} B_{\frac{r_{a_m}}{5}}(a_m)}|\nabla R_i|^2\omega_\delta^2\,dx\\
&\leq \frac{80C}{\epsilon_0^2}\int_{B_2}|\nabla R_i|^2\omega_\delta^2\,dx. \qquad (5.9)
\end{aligned}$$

It follows from the stability of (ϕ, R) and a scaling argument that R_i satisfies the stability inequality (5.3) so that

$$\int_{B_2}|\nabla R_i|^2\omega_\delta^2\,dx \leq \frac{3}{2}\int_{B_2}|\nabla\omega_\delta|^2\,dx,\ \forall i \geq i_l. \qquad (5.10)$$

Plugging (5.10) into (5.9) and applying (5.6), we would obtain that

$$\mathcal{H}^1_{\delta^2}(\mathcal{S}) \leq C(\epsilon_0)\delta.$$

This, after sending $\delta \to 0$, would yield $\mathcal{H}^1(\mathcal{S}) = 0$ and hence *Claim 2* is true.

It follows from the H^1-strong convergence of (ϕ_i, R_i) to (ϕ_0, R_0) and the energy monotonicity inequality (2.8), we conclude that

$$(\phi_0, R_0)(x) = (\phi_0, R_0)(\frac{x}{|x|}),\ \forall x \in B_2,$$

is homogeneous of degree zero. Since ϕ_0 is a harmonic function in B_2, it follows that ϕ_0 is a constant. Thus

$$\int_{\mathbb{S}^2}|\nabla_{\mathbb{S}^2}R_0|^2\,dH^2 = \Theta^1((\phi,R),x_0) \geq \epsilon_0^2,$$

and $R_0 \in C^\infty(\mathbb{S}^2, SO(3))$ is a nontrivial harmonic map. Since $\Pi_1(\mathbb{S}^3) = \{0\}$, it follows that there exists a nontrivial harmonic map $\widehat{R_0} \in C^\infty(\mathbb{S}^2, \mathbb{S}^3)$ such that $R_0 = \pi \circ \widehat{R_0}$. Moreover, it follows from the stability inequality (5.1) that $\widehat{R_0}$ is a stable harmonic map from $\mathbb{S}^2$ to $\mathbb{S}^3$, i.e.

$$\int_{\mathbb{S}^2}\left(|\nabla_{\mathbb{S}^2}\omega|^2 - |\nabla\widehat{R_0}|^2|\omega|^2\right)dH^2 \geq 0 \qquad (5.11)$$

for any $\omega \in C^\infty(\mathbb{S}^2, T_{\widehat{R_0}}\mathbb{S}^3)$. However it follows from Schoen-Uhlenbeck [14] that there is no nontrivial stable harmonic map from $\mathbb{S}^2$ to $\mathbb{S}^3$. We get a desired contradiction. Thus the singular set Σ of (ϕ, R) is empty. □

Theorem 1.5 for the cases that $p > 2$ can be summarized into the following theorem.

Theorem 5.4. *For $f \in L^\infty(\Omega, \mathbb{R}^3)$ and $M \in L^\infty(\Omega, \mathrm{SO}(3))$, and $\mu_1 = \mu_c = \mu_2 = 1$, if $p \in (2, \frac{32}{15}]$ and $(\phi, R) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, \mathrm{SO}(3))$ is a stable, stationary weak solution of the Cosserat equation* (1.3)*, then there exists $\alpha \in (0,1)$ such that $(\phi, R) \in C^{1,\alpha}(\Omega, \mathbb{R}^3) \times C^\alpha(\Omega, \mathrm{SO}(3))$.*

Proof. It follows from $2 < p < 3$ and Theorem 1.3 that $\mathrm{Sing}(\phi, R)$ is discrete. Suppose $\mathrm{Sing}(\phi, R) \neq \emptyset$. Then there exist $x_0 \in \mathrm{Sing}(\phi, R)$ and $r_0 > 0$ such that $\mathrm{Sing}(\phi, R) \cap B_{r_0}(x_0) = \{x_0\}$. For $r_k \to 0$, define $(\phi_k, R_k)(x) = (\phi, R)(x_0 + r_k x)$ for $x \in B_2$. As in the proof of Theorem 1.3, we can apply the monotonicity inequality (2.3), Lemma 4.1, and Marstrand theorem to show that there exists a nontrivial $(\phi_0, R_0) \in H^1(B_1, \mathbb{R}^3) \times W^{1,p}(B_1, \mathrm{SO}(3))$ such that, after passing to a subsequence, $(\phi_k, R_2) \to (\phi_0, R_0)$ strongly in $H^1(B_1, \mathbb{R}^3) \times W^{1,p}(B_1, \mathrm{SO}(3))$. Hence (ϕ_0, R_0) is of homogeneous degree zero, ϕ_0 is constant and $R_0 \in C^{1,\alpha}(B_1 \backslash \{0\}, \mathrm{SO}(3))$ is a nontrivial, stable, stationary p-harmonic map. However, it follows from the stability Lemma 6.3 and Proposition 6.4 in Gastel [8] that for $p \in (2, \frac{32}{15})$, any stable stationary p-harmonic map $R(x) = R(\frac{x}{|x|}) \in C^{1,\alpha}(B_1 \setminus \{0\}, \mathrm{SO}(3))$ must be constant. We get a desired contradiction. Hence $\mathrm{Sing}(\phi, R) = \emptyset$ when $p \in (2, \frac{32}{15}]$. This completes the proof. □

Finally we would like to point out that Theorem 1.5 follows from Theorem 5.3 and Theorem 5.4.

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