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Threshold solutions in the focusing 3D cubic NLS equation outside a strictly convex obstacle

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ABSTRACT

We study the dynamics of the focusing $3d$ cubic nonlinear Schrödinger equation in the exterior of a strictly convex obstacle at the mass-energy threshold, namely, when $E_{\Omega}[u_0]M_{\Omega}[u_0] = E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q]$ and $\|\nabla u_0\|_{L^2(\Omega)} \|u_0\|_{L^2(\Omega)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)} \|Q\|_{L^2(\mathbb{R}^3)}$, where $u_0 \in H_0^1(\Omega)$ is the initial data, Q is the ground state on the Euclidean space, E is the energy and M is the mass. In the whole Euclidean space Duyckaerts and Roudenko (following the work of Duyckaerts and Merle on the energy-critical problem) have proved the existence of a specific global solution that scatters for negative times and converges to the soliton in positive times. We prove that these heteroclinic orbits do not exist for the problem in the exterior domain and that all solutions at the threshold are globally defined and scatter. This is the first step in the study of the global dynamics of the equation above the ground-state threshold. The main difficulty is to control the position of the center of mass of the solution for large time without the

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momentum conservation law and the Galilean transformation which are not available for this equation.

Contents

1. Introduction

We consider the focusing nonlinear Schrödinger equation in the exterior of a smooth compact strictly convex obstacle $\Theta \subset \mathbb{R}^3$ with Dirichlet boundary conditions:

$$\begin{cases} i\partial_t u + \Delta_\Omega u = -|u|^2 u & (t,x) \in \mathbb{R} \times \Omega, \\ u(t_0, x) = u_0(x) & x \in \Omega, \\ u(t,x) = 0 & (t,x) \in \mathbb{R} \times \partial\Omega, \end{cases} \tag{NLS$_\Omega$}$$

where $\Omega = \mathbb{R}^3 \setminus \Theta$, Δ_Ω is the Dirichlet Laplace operator on Ω and $t_0 \in \mathbb{R}$ is the initial time. Here, u is a complex-valued function,

$$u : \mathbb{R} \times \Omega \longrightarrow \mathbb{C}$$
$$(t,x) \longmapsto u(t,x).$$

We take the initial data $u_0 \in H_0^1(\Omega)$, where $H_0^1(\Omega)$ is the Sobolev space

$$\{u \in L^2(\Omega) \text{ such that } |\nabla u| \in L^2(\Omega) \text{ and } u_{|\partial\Omega} = 0\}.$$

The NLS_Ω equation is locally wellposed in $H_0^1(\Omega)$, see [1], [33], [16] and [3]. The solutions of the NLS_Ω equation satisfy the mass and energy conservation laws:

$$M_\Omega[u(t)] := \int\limits_\Omega |u(t,x)|^2 dx = M[u_0],$$

$$E_\Omega[u(t)] := \frac{1}{2}\int\limits_\Omega |\nabla u(t,x)|^2 dx - \frac{1}{4}\int\limits_\Omega |u(t,x)|^4 dx = E[u_0].$$

Unlike the nonlinear Schrödinger equation $\mathrm{NLS}_{\mathbb{R}^3}$ posed on the whole Euclidean space $\mathbb{R}^3$, the NLS_Ω equation does not have the momentum conservation.

The $\mathrm{NLS}_{\mathbb{R}^3}$ equation is invariant by the scaling transformation, that is,

$$u(t,x) \longmapsto \lambda u(\lambda x, \lambda^2 t) \quad \text{for } \lambda > 0.$$

This scaling identifies the critical Sobolev space $\dot{H}_x^{\frac{1}{2}}$. Since the presence of an obstacle does not change the intrinsic dimensionality of the problem, we regard the NLS_Ω equation as having the same criticality, and thus as an energy-subcritical, mass-supercritical equation.

In this paper, we study the global well-posedness and scattering of solutions to the NLS_Ω equation. We start recalling earlier results on global existence and scattering ([33], [21]): if u has a finite Strichartz norm (Cf. Theorem 2.7), then u scatters in $H_0^1(\Omega)$, i.e.,

$$\exists u_\pm \in H_0^1(\Omega) \quad \text{such that} \quad \lim_{t\longrightarrow \pm\infty} \left\| u(t) - e^{it\Delta_\Omega} u_\pm \right\|_{H_0^1(\Omega)} = 0.$$

This holds in particular if the initial data is sufficiently small in $H_0^1(\Omega)$.

Global existence and scattering for large data was studied for the $\mathrm{NLS}_{\mathbb{R}^3}$ equation, posed on the whole Euclidean space $\mathbb{R}^3$, in several articles in different contexts. The $\mathrm{NLS}_{\mathbb{R}^3}$ equation has solutions of the form $e^{it\Delta_{\mathbb{R}^3}}Q$, where Q solves the following nonlinear elliptic equation

$$\begin{cases} -Q + \Delta Q + |Q|^2 Q = 0, \\ Q \in H^1(\mathbb{R}^3). \end{cases} \tag{1.1}$$

In this paper, we denote by Q the ground state solution, that is, the unique radial, vanishing at infinity, positive solution of (1.1). Such Q is smooth, exponentially decaying at infinity, and characterized as the unique minimizer for the Gagliardo-Nirenberg inequality up to scaling, space translation and phase shift, see [23].

In [14], the authors have studied the global existence and scattering[1] for large initial data of the radial solutions of the cubic $\mathrm{NLS}_{\mathbb{R}^3}$ equation on $\mathbb{R}^3$, below a threshold given by the ground state. This result was later extended to the non-radial case in [6] and to arbitrary space dimensions and focusing intercritical power nonlinearities in [10] and [13]. This was generalized to the cubic NLS_Ω equation outside a strictly convex obstacle in [21] (see also [37] for $1 < p < 5$).

[1] also, blow-up, however, we do not need it in this paper.

Theorem A. *Let $u_0 \in H^1_0(\Omega)$ satisfy*

$$\|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} < \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}, \tag{1.2}$$

$$M_\Omega[u_0]E_\Omega[u_0] < M_{\mathbb{R}^3}[Q]E_{\mathbb{R}^3}[Q]. \tag{1.3}$$

Then u scatters in $H^1_0(\Omega)$, in both time directions.

Note that in the case $\Omega = \mathbb{R}^3$, the criteria (1.2) and (1.3) are expressed in terms of the scale-invariant quantities $\|\nabla u_0\|_{L^2} \|u_0\|_{L^2}$ and $M[u_0]E[u_0]$.

The purpose of this paper is to study the behavior of solutions to the NLS_Ω equation exactly at the mass-energy threshold, i.e., when

$$E_\Omega[u_0]M_\Omega[u_0] = E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q], \tag{1.4}$$

$$\|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} < \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}. \tag{1.5}$$

In [8] T. Duyckaerts and S. Roudenko described the behavior of the solutions of the $\text{NLS}_{\mathbb{R}^3}$ equation at the mass-energy threshold. At this mass-energy level, the $\text{NLS}_{\mathbb{R}^3}$ equation has a richer dynamics for the long time behavior of the solutions compared to the result mentioned above. The authors proved the existence of special solutions, denoted by Q^+ and Q^-. These special solutions have the same mass-energy of the soliton, $M_{\mathbb{R}^3}[Q^\pm]E_{\mathbb{R}^3}[Q^\pm] = M_{\mathbb{R}^3}[Q]E_{\mathbb{R}^3}[Q]$, however, $\|\nabla Q^-(t)\|_{L^2(\mathbb{R}^3)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)}$ and $\|\nabla Q^+(t)\|_{L^2(\mathbb{R}^3)} > \|\nabla Q\|_{L^2(\mathbb{R}^3)}$, for all t in the interval of existence of $Q^\pm$. Only the solution Q^- is relevant in the study of the global existence and scattering. This solution Q^- scatters for negative time and approach the soliton, up to symmetries, for positive time direction: there exists $e_0 > 0$ such that

$$\left\|Q^- - e^{it}Q\right\|_{H^1(\mathbb{R}^3)} \leq ce^{-e_0 t} \quad \text{for } t \geq 0. \tag{1.6}$$

Furthermore, if we consider initial data $u_0 \in H^1(\mathbb{R}^3)$ such that (1.4) and (1.5) hold on $\mathbb{R}^3$ then the corresponding solution $u(t)$ of the $\text{NLS}_{\mathbb{R}^3}$ equation is global and either scatters in $H^1(\mathbb{R}^3)$ or $u \equiv Q^-$, up to the symmetries.

Note that for the NLS_Ω equation, there do not exist analogs of the solutions $e^{it}Q$, Q^- at the threshold $M_\Omega[u]E_\Omega[u] = M_{\mathbb{R}^3}[Q]E_{\mathbb{R}^3}[Q]$. Indeed there is no function $u_0 \in H^1_0(\Omega)$ satisfying (1.4) and $\|\nabla u_0\|_{L^2(\Omega)} \|u_0\|_{L^2(\Omega)} = \|\nabla Q\|_{L^2(\mathbb{R}^3)} \|Q\|_{L^2(\mathbb{R}^3)}$. By extending u_0 with 0 on the obstacle, the solution u_0 must be equal to Q, up to the symmetries, which would not satisfy Dirichlet boundary conditions. Similarly, in the presence of the obstacle there is no function in $H^1_0(\Omega)$ such that (1.6) holds, since such a solution has to converge to Q for the sequence of times $t_n = 2\pi n$, contradicting the fact that Q does not satisfy Dirichlet boundary conditions.

We now state the main result of this paper.

Theorem 1. *Let* $u_0 \in H_0^1(\Omega)$ *and let* $u(t)$ *be the corresponding solution to* (NLS_Ω) *such that* u_0 *satisfy*

$$M_\Omega[u]E_\Omega[u] = M_{\mathbb{R}^3}[Q]E_{\mathbb{R}^3}[Q], \tag{1.7}$$

$$\|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} < \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}. \tag{1.8}$$

Then u *scatters in* $H_0^1(\Omega)$ *in both time directions.*

Remark 1.1. The existence of initial data that satisfy (1.7) and (1.8) can be obtained using the variational characterization of the ground state Q. Indeed, let $\lambda > 0$, $\varphi \in H_0^1(\Omega)\backslash\{0\}$ and let $u_\lambda(t)$ be the solution of the NLS_Ω equation with initial data $u_\lambda(t_0) := u_{0,\lambda} = \lambda\varphi$. Then, there exists a unique $\lambda_1 > 0$, such that $M_\Omega[u_{0,\lambda_1}]E_\Omega[u_{0,\lambda_1}] = M_{\mathbb{R}^3}[Q]E_{\mathbb{R}^3}[Q]$ and $\|u_{0,\lambda_1}\|_{L^2(\Omega)} \|\nabla u_{0,\lambda_1}\|_{L^2(\Omega)} < \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}$. (Cf. Appendix A for more details).

The proof of Theorem 1 is based on the approach of the Euclidean setting results in [7] and [8]. The first step is similar to the proof of the compactness of the critical solution developed by C. Kenig and F. Merle in [18] in the energy-critical setting and adapted to the energy-subcritical case in [14] and [6]. It uses a concentration-compactness argument that requires a profile decomposition as in the works of F. Merle and L. Vega [29], P. Gérard [11], and S. Keraani [19], adapted by R. Killip, M. Visan and X. Zhang for the problem in the exterior of a convex obstacle in [22] (in the energy-critical case) and in [21] (in the energy-subcritical case). The second step of the proof is a careful study of the space translation and phase parameters for a solution of NLS_Ω that is close to Q, up to the transformations. The presence of the obstacle brings significant difficulties. One of them (that we tackle with the techniques developed in [25] by the second author) is that we must linearize around a space translation of the solitary wave $e^{it}Q$, which is not an exact solution of (NLS_Ω). Another difficulty is the fact that the momentum conservation law and Galilean transformation, which were used in [8] to control the space translation of the solution, are not available for the equation outside an obstacle. This control is achieved through a new intricate compactness argument for solutions escaping at infinity, that relies among other things on the uniqueness theorem in [6].

In [24], the second author has proved that when the obstacle is the Euclidean ball of $\mathbb{R}^3$, solutions such that $M_\Omega[u]E_\Omega[u] < M_{\mathbb{R}^3}[Q]E_{\mathbb{R}^3}[Q]$ and $\|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} > \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}$ with a finite variance and a certain symmetry blow up in finite time. In view of the known results on $\mathbb{R}^3$, one should expect blow-up in finite or infinite time for all solutions of this type, however, the blow-up for the NLS_Ω equation is a delicate issue. One of the difficulties is the appearance of boundary terms with the wrong sign in the virial identity that is used to prove blow-up on $\mathbb{R}^3$. Blow-up is also expected in the threshold case $M_\Omega[u]E_\Omega[u] = M_{\mathbb{R}^3}[Q]E_{\mathbb{R}^3}[Q]$ and $\|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} > \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}$, which is an open question. Let us mention however that linear scattering is precluded for these solutions. Indeed, if u is such a solution, then by the bound $\|u(t)\|_{L^2(\Omega)} \|\nabla u(t)\|_{L^2(\Omega)} > \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}$ (which is valid on the domain of existence of u), we have

$$\lim_{t \to T_+(u)} \|u(t)\|^2_{L^2(\Omega)} \|\nabla u(t)\|^2_{L^2(\Omega)} \geq \|Q\|^2_{L^2(\mathbb{R}^3)} \|\nabla Q\|^2_{L^2(\mathbb{R}^3)} = 6\|Q\|^2_{L^2(\mathbb{R}^3)} E_{\mathbb{R}^3}(Q) \quad (1.9)$$

(where we have used Pohozaev's identity, see (2.4) below). However, if u is a scattering solution with $M_\Omega[u]E_\Omega[u] = M_{\mathbb{R}^3}[Q]E_{\mathbb{R}^3}[Q]$, we have $T_+(u) = +\infty$ and (using the conservation of mass and that $\lim_{t\to\infty} \|u(t)\|_{L^4(\Omega)} = 0$),

$$\lim_{t\to\infty} \|u(t)\|^2_{L^2(\Omega)} \|\nabla u(t)\|^2_{L^2(\Omega)} = 2M_\Omega[u]E_\Omega[u] = 2\|Q\|^2_{L^2(\mathbb{R}^3)} E_{\mathbb{R}^3}[Q],$$

contradicting (1.9).

When $\Omega = \mathbb{R}^3$, K. Nakanishi and W. Schlag [32] described the dynamics of solutions slightly above the mass-energy threshold, that is such that $E_{\mathbb{R}^3}[Q]_{\mathbb{R}^3} M[Q] \leq E_{\mathbb{R}^3}[u_0] M_{\mathbb{R}^3}[u_0] < E_{\mathbb{R}^3}[Q]_{\mathbb{R}^3} M[Q] + \varepsilon$ for a small $\varepsilon > 0$, showing that all 9 expected behaviors (any combination of blow-up in finite time, linear scattering or scattering to the ground state solution) do indeed occur. Some sufficient conditions for scattering and blow-up in this regime are given by the first and third authors in [9]. The analog of the result in [32] outside of an obstacle is currently out of reach, due to insufficient understanding of blow-up in finite time. Let us mention however that in this case, the soliton-like behavior is possible. Indeed, the second author in [25] constructed a solution behaving as a traveling wave in $\mathbb{R}^3$ for large t, moving away from the obstacle with an arbitrary small speed v and such that $E[u_0]M[u_0] = E[Q]M[Q] + c|v|^2$ for a constant $c > 0$. See also [26] for numerical investigations in this regime.

The study of the obstacle problem for dispersive equations, motivated by the understanding of the influence of the underlying space geometry on the dynamics of the equation, started long ago. Let us mention some of the works on a wave-type equation in the exterior of an obstacle with Dirichlet or Neuman boundary conditions. In 1959, H. W. Calvin studied the rate of decay of solutions to the linear wave equation outside of a sphere, see [36]. Later, Morawetz extended this result to star-shaped obstacles, see [30] and, with Ralston and Strauss, to non-trapping obstacles, see [31]. The Cauchy theory for the NLS_Ω equation with initial data in $H^1_0(\Omega)$, was initiated in 2004 by N. Burq, P. Gérard and N. Tzvetkov in [4]. Assuming that the obstacle is non-trapping, the authors proved a local existence result for the $3d$ sub-cubic (i.e., $p < 3$) NLS_Ω equation. This was later extended by R. Anton in [1] for the cubic nonlinearity, by F. Planchon and L. Vega in [33] for the energy-subcritical NLS_Ω equation in dimension $d = 3$ (i.e., $1 < p < 5$) and by F. Planchon and O. Ivanovici in [17] for the energy-critical case in dimension $d = 3$ (i.e., $p = 5$), see also [3] and [15], [16], [27] for convex obstacle. The local well-posedness in the critical Sobolev space was first obtained in [17], for $3 + \frac{2}{5} < p < 5$. In [25], the second author extended this result for $\frac{7}{3} < p < 5$, using the fractional chain rule in the exterior of a compact convex obstacle from [20].

The paper is organized as follows: In Section 2, we recall known properties of the ground state and coercivity property associated to the linearized operator under certain orthogonality conditions. There, we also recall Strichartz estimates, stability theory and the profile decomposition for the NLS_Ω equation outside of a strictly convex obstacle. In Section 3, we discuss modulation, in particular, in §3.2 we use the modulation in phase rotation and in space translation parameters near the truncated ground state solution, in order to obtain orthogonality conditions. Section 4 is dedicated to the proof of the main theorem. In §4.1 we use the profile decomposition to prove a compactness property, which yields the existence of a continuous translation parameter $x(t)$ such that the extension of a non-scattering solution $\underline{u}(t, x+x(t))$, that satisfy (1.7) and (1.8), is compact in $H^1(\mathbb{R}^3)$. In §4.2, we control the space translation $x(t)$ by approximating it by auxiliary translation parameter given by modulation on $\mathbb{R}^3$, in [8]. Moreover, we use a local virial identity with estimates from previous sections on the modulation parameter to prove that $x(t)$ is bounded. In §4.3, we prove that the parameter $\delta(t) := |\|\nabla Q\|_{L^2} - \|\nabla u\|_{L^2}|$ converges to 0 in mean. Finally, we conclude the proof of Theorem 1 using the compactness properties with the control of the space translation parameter $x(t)$ and the convergence in mean. In Appendix A, we prove the existence of an initial data in $H_0^1(\Omega)$ that satisfies the mass-energy threshold.

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Notation. Define Ψ as a C^∞ function such that

$$\Psi = \begin{cases} 0 & \text{near } \Theta, \\ 1 & \text{if } |x| \gg 1. \end{cases} \tag{1.10}$$

We write $a = O(b)$, when a and b are two quantities, and there exists a positive constant C independent of parameters, such that $|a| \leq C\,b$, and $a \approx b$, when $a = O(b)$ and $b = O(a)$. For $h \in \mathbb{C}$, we denote $h_1 = \operatorname{Re} h$ and $h_2 = \operatorname{Im} h$. Throughout this paper, C denotes a large positive constant and c is a small positive constant, that may change from line to line; both do not depend on parameters. We denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^3$. For simplicity, we write $\Delta = \Delta_\Omega$. The real L^2-scalar product $(\cdot, \cdot)$ means

$$(f, g) = \operatorname{Re} \int f\, \overline{g} = \int \operatorname{Re} g\, \operatorname{Re} f + \int \operatorname{Im} g\, \operatorname{Im} f\, .$$

2. Preliminaries

2.1. Properties of the ground state

We recall here some well-known properties of the ground state. We refer the reader to [35], [23], [34, Appendix B] for a general setting and [14] for the 3d cubic $\mathrm{NLS}_{\mathbb{R}^3}$ case, for more details. Consider the following nonlinear elliptic equation on $\mathbb{R}^3$

$$-Q + \Delta Q + |Q|^2 Q = 0. \tag{2.1}$$

We are interested in a positive, decaying at infinity, solution $Q \in H^1(\mathbb{R}^3)$. The ground state solution is the unique positive, radial, vanishing at infinity, smooth solution of (2.1). It is also (up to standard transformations) the unique minimizer of the Gagliardo-Nirenberg inequality: if $u \in H^1(\mathbb{R}^3)$, then

$$\|u\|^4_{L^4(\mathbb{R}^3)} \leq C_{GN} \|\nabla u\|^3_{L^2(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)}, \quad \|Q\|^4_{L^4(\mathbb{R}^3)} = C_{GN} \|\nabla Q\|^3_{L^2(\mathbb{R}^3)} \|Q\|_{L^2(\mathbb{R}^3)}. \tag{2.2}$$

Moreover,

$$\|u\|^4_{L^4(\mathbb{R}^3)} = C_{GN} \|\nabla u\|^3_{L^2(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)} \Longrightarrow \exists \lambda_0 \in \mathbb{C}, \exists \mu_0 \in \mathbb{R}, \exists x_0 \in \mathbb{R}^3 : u(x) = \lambda_0 Q(\mu_0(x + x_0)). \tag{2.3}$$

We also have the Pohozhaev identities

$$\|Q\|^4_{L^4(\mathbb{R}^3)} = 4 \|Q\|^2_{L^2(\mathbb{R}^3)} \quad \text{and} \quad \|\nabla Q\|^2_{L^2(\mathbb{R}^3)} = 3 \|Q\|^2_{L^2(\mathbb{R}^3)}. \tag{2.4}$$

As a consequence of (2.2), (2.3) and the concentration-compactness principle [28] one has

Proposition 2.1. *There exists a function $\varepsilon(\eta)$, defined for small $\eta > 0$, such that $\lim_{\eta \to 0} \varepsilon(\eta) = 0$ and*

$$\forall u \in H^1(\mathbb{R}^3), \quad \left| \|u\|_{L^4(\mathbb{R}^3)} - \|Q\|_{L^4(\mathbb{R}^3)} \right| + \left| \|u\|_{L^2(\mathbb{R}^3)} - \|Q\|_{L^2(\mathbb{R}^3)} \right| + \left| \|\nabla u\|_{L^2(\mathbb{R}^3)} - \|\nabla Q\|_{L^2(\mathbb{R}^3)} \right| \leq \eta \Longrightarrow \exists\, \theta_0 \in \mathbb{R} \text{ and } \exists\, x_0 \in \mathbb{R}^3 : \left\| u - e^{i\theta_0} Q(\cdot - x_0) \right\|_{H^1(\mathbb{R}^3)} \leq \varepsilon(\eta). \tag{2.5}$$

Next, we recall some known properties on the decay of Q, see [12], [2] and [5, Chapter 8].

Proposition 2.2 *(Exponential decay of Q). Let Q be the ground state solution of (2.1), then there exist $a, C > 0$ such that for $|x| > 1$,*

$$\left|Q(x) - \frac{a}{|x|}e^{-|x|}\right| \leq C\frac{e^{-|x|}}{|x|^{3/2}}.$$

Moreover,

$$\left|\nabla Q(x) + \nabla^2 Q(x)\right| \leq C\frac{e^{-|x|}}{|x|}.$$

Lemma 2.3. *Let Q be the ground state solution of (2.1), $M > 0$ large, $X \in \mathbb{R}^3$ and let g be an L^1-function. Then for $k > 0$, we have*

$$|X| \geq 2M \implies \int\limits_{|x|\leq M} \left(Q^k(x-X) + |\nabla Q(x-X)|^k\right) g(x)\,dx = O\left(\frac{e^{-k|X|}}{|X|^k}\right), \qquad (2.6)$$

where $O(\cdot)$ depends on k, g and M.

Furthermore, there exists $c_M > 0$ such that

$$\int\limits_{|x|\leq M} Q^k(x-X)\,dx \geq c_M\,\frac{e^{-k|X|}}{|X|^k}. \qquad (2.7)$$

Proof. First, note that

$$\frac{1}{2}|X| < |X| - M < |x - X|, \quad \text{and} \ |X| \geq 2M.$$

This implies that, for $|X| \geq 2M$ we have

$$e^{-|x-X|} \leq e^M e^{-|X|} \quad \text{and} \quad \frac{1}{2|x-X|} \leq \frac{1}{|X|}.$$

Using the exponential decay of Q from Proposition 2.2, we obtain,

$$\int\limits_{|x|\leq M} Q^k(x-X)g(x)\,dx = O\left(\frac{e^{-k|X|}}{|X|^k}\right), \quad \text{for} \ \ k > 0.$$

Similarly, we get

$$\int\limits_{|x|\leq M} |\nabla Q(x-X)|^k\, g(x)\,dx = O\left(\frac{e^{-k|X|}}{|X|^k}\right), \quad \text{for} \ \ k > 0.$$

The proof of (2.7) is similar by applying again Proposition 2.2 and we omit it. □

Let $u \in H_0^1(\Omega)$ and define $\underline{u} \in H^1(\mathbb{R}^3)$ such that

$$\underline{u}(x) = \begin{cases} u(x) & \forall x \in \Omega, \\ 0 & \forall x \in \Omega^c. \end{cases} \tag{2.8}$$

Remark 2.4. We denote by $M_{\mathbb{R}^3}[\underline{u}] = \|\underline{u}\|^2_{L^2(\mathbb{R}^3)}$ and $E_{\mathbb{R}^3}[\underline{u}] = \frac{1}{2}\|\nabla \underline{u}\|^2_{L^2(\mathbb{R}^3)} - \frac{1}{p+1}\|\underline{u}\|^{p+1}_{L^{p+1}(\mathbb{R}^3)}$. Note that, we have $M_\Omega[u] = M_{\mathbb{R}^3}[\underline{u}]$ and $E_\Omega[u] = E_{\mathbb{R}^3}[\underline{u}]$. To simplify notations in what follows we drop the index Ω in the mass and the energy of the NLS_Ω equation, so that we just write $M[u]$ and $E[u]$ instead of $M_\Omega[u]$ and $E_\Omega[u]$.

Assume that $\underline{u}$ satisfies the left-hand side of (2.5). Then there exists $x_0 \in \mathbb{R}^3$ and $\theta_0 \in \mathbb{R}$ such that

$$\left\| \underline{u} - e^{i\theta_0} Q(\cdot - x_0) \right\|_{H^1(\mathbb{R}^3)} \leq \varepsilon(\eta),$$

which yields, by Proposition 2.2 and (2.7),

$$\frac{1}{C} \frac{e^{-|x_0|}}{|x_0|} \leq \|Q(x - x_0)\|_{H^1(\Omega^c)} \leq \varepsilon(\eta). \tag{2.9}$$

This implies that $|x_0|$ is large when η is small.

2.2. Coercivity property

We next recall some known properties of the linearized operator on $\mathbb{R}^3$. Consider a solution u of $\text{NLS}_{\mathbb{R}^3}$ close to $e^{it}Q$ and write $u(t)$ as

$$u(t,x) = e^{it}\left(Q(x) + \hbar(t,x)\right).$$

Note that $\hbar$ is the solution of the equation

$$\partial_t \hbar + \mathcal{L}\hbar = \mathcal{R}(\hbar), \qquad \mathcal{L}\hbar = -\mathcal{L}_- \hbar_2 + i\mathcal{L}_+ \hbar_1,$$

where

$$\begin{aligned} \mathcal{L}_+ \hbar_1 &= -\Delta \hbar_1 + \hbar_1 - 3Q^2 \hbar_1, \\ \mathcal{L}_- \hbar_1 &= -\Delta \hbar_2 + \hbar_2 - Q^2 \hbar_2, \\ \mathcal{R}(\hbar) &= iQ(2|\hbar|^2 + \hbar^2) + i|\hbar|^2 \hbar. \end{aligned}$$

Define $\Phi(\hbar)$, a linearized energy on $\mathbb{R}^3$, by

$$\Phi(\hbar) := \frac{1}{2}\int_{\mathbb{R}^3} |\hbar|^2 + \int_{\mathbb{R}^3} \frac{1}{2}|\nabla \hbar|^2 - \frac{1}{2}\int_{\mathbb{R}^3} Q^2(3\hbar_1^2 + \hbar_2^2). \tag{2.10}$$

We next define a subspace of $H^1(\mathbb{R}^3)$, on which Φ is positive

$$\mathcal{G} := \left\{ \hbar \in H^1(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} \partial_{x_j} Q \hbar_1 = 0, \int_{\mathbb{R}^3} Q\hbar_2 = 0, \; j = 1,2,3 \right\}.$$

Then by [8], there exists $c > 0$ such that

$$\forall \hbar \in \mathcal{G}, \quad \Phi(\hbar) \geq c \, \|\hbar\|^2_{H^1(\mathbb{R}^3)} . \tag{2.11}$$

Let $h \in H^1(\mathbb{R}^3)$. Define

$$\Phi_X(h) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla h|^2 - \frac{1}{2} \int_{\mathbb{R}^3} Q^2 \Psi^2(\cdot + X)(3h_1^2 + h_2^2) + \frac{1}{2} \int_{\mathbb{R}^3} |h|^2, \tag{2.12}$$

where Ψ is defined in (1.10).

Lemma 2.5. *There exist $c > 0$ such that for all $h \in H^1(\mathbb{R}^3)$, if the following orthogonality relations hold for all $X \in \mathbb{R}^3$ with $|X|$ large*

$$\operatorname{Re} \int_{\mathbb{R}^3} \Delta(Q(x)\Psi(x + X))h(x + X)\, dx = 0, \quad \operatorname{Im} \int_{\mathbb{R}^3} Q(x)\Psi(x + X)h(x + X)\, dx = 0, \tag{2.13}$$

$$\operatorname{Re} \int_{\mathbb{R}^3} \partial_{x_k}(Q(x)\Psi(x + X))h(x + X)\, dx = 0, \quad k = 1,2,3, \tag{2.14}$$

then

$$\Phi_X(h(\cdot + X)) \geq c \, \|h\|^2_{H^1(\mathbb{R}^3)} . \tag{2.15}$$

Proof. Define

$$\mathcal{A} = \left\{ f \in H^1(\mathbb{R}^3) : \; \operatorname{Re} \int_{\mathbb{R}^3} \Delta Q \, f = \operatorname{Im} \int_{\mathbb{R}^3} Qf = \operatorname{Re} \int_{\mathbb{R}^3} \partial_{x_k} Q \, f = 0, \; k = 1,2,3 \right\},$$

$$\mathcal{B} = \operatorname{span}\{iQ, \Delta Q, \partial_{x_1} Q, \partial_{x_2} Q, \partial_{x_3} Q\},$$

then we write $h(\cdot + X) = \tilde{h}(\cdot + X) + r(\cdot + X)$ with $\tilde{h}(\cdot + X) \in \mathcal{A}$ and $r(\cdot + X) \in \mathcal{B}$.

By (2.10) and (2.11), we have

$$\Phi(\tilde{h}(\cdot + X)) \geq c\|\tilde{h}\|^2_{H^1(\mathbb{R}^3)}.$$

Since $r(\cdot + X) \in \mathcal{B}$, we can write r as

$$r(\cdot + X) = \sum_{k=1}^{3} \alpha_k \partial_{x_k} Q + \beta i Q + \gamma \Delta Q.$$

Taking the real L^2-scalar product in $\mathbb{R}^3$ of r with iQ and using the fact that Q is radial, we get

$$\begin{aligned}\beta &= \frac{1}{\|Q\|^2_{L^2(\mathbb{R}^3)}}(r(\cdot+X), iQ) = \frac{1}{\|Q\|^2_{L^2(\mathbb{R}^3)}}((h(\cdot+X) - \tilde{h}(\cdot+X), iQ) \\ &= \frac{1}{\|Q\|^2_{L^2(\mathbb{R}^3)}}\left(\operatorname{Im}\int_{\mathbb{R}^3} h(x+X)Q(x)\,dx - \operatorname{Im}\int_{\mathbb{R}^3} \tilde{h}(x+X)Q(x)\,dx\right).\end{aligned}$$

By the definition of $\tilde{h}$, we have $\operatorname{Im}\int \tilde{h}(x+X)Q(x)dx = 0$. Using the orthogonality conditions in Lemma 2.5 and the exponential decay of Q from Lemma 2.3, we obtain

$$\begin{aligned}\beta &= \frac{1}{\|Q\|^2_{L^2(\mathbb{R}^3)}}\operatorname{Im}\int_{\mathbb{R}^3} h(x+X)Q(x)dx \\ &= \frac{1}{\|Q\|^2_{L^2}}\operatorname{Im}\int_{\mathbb{R}^3} h(x+X)Q(x)\Psi(x+X)\,dx \\ &- \frac{1}{\|Q\|^2_{L^2(\mathbb{R}^3)}}\operatorname{Im}\int_{\mathbb{R}^3} h(x+X)Q(x)(\Psi(x+X)-1)dx \\ &= O(e^{-|X|}\|h\|_{H^1(\mathbb{R}^3)}).\end{aligned}$$

Similarly, by taking the scalar product of r with ΔQ and $\partial_{x_k} Q$ and using the fact that Q is radial, the orthogonality condition in Lemma 2.5 and the exponential decay of Q from Lemma 2.3, we obtain $\gamma = \alpha_k = O(e^{-|X|}\|h\|_{H^1(\mathbb{R}^3)})$.

Thus,

$$\begin{aligned}\|r\|_{H^1(\mathbb{R}^3)} &\le Ce^{-|X|}\|h\|_{H^1(\mathbb{R}^3)}, \\ |\Phi_X(r(\cdot+X))| &\le e^{-2|X|}\|h\|^2_{H^1(\mathbb{R}^3)}.\end{aligned}$$

We now have

$$\Phi_X(h(\cdot+X)) = \Phi_X(\tilde{h}(\cdot+X)) + \Phi_X(r(\cdot+X)) + 2B_X(\tilde{h}(\cdot+X), r(\cdot+X)),$$

where the bilinear form B_X is defined as

$$\begin{aligned}B_X(f,g) := &\frac{1}{2}\int\Big(\nabla f_1(x)\nabla g_1(x) + f_1(x)\,g_1(x) - 3Q^2(x)\Psi^2(x+X)f_1(x)\;g_1(x)\Big)\,dx \\ &+ \frac{1}{2}\int\Big(\nabla f_2(x)\nabla g_2(x) + f_2(x)g_2(x) - Q^2(x)\Psi^2(x+X)f_2(x))\;g_2(x)\Big)\,dx.\end{aligned}$$

Note that

$$\left|B_X(\tilde{h}(\cdot + X), r(\cdot + X))\right| \leq e^{-|X|} \left\|h\right\|_{H^1(\mathbb{R}^3)} .$$

Then,

$$\Phi_X(h(\cdot + X)) = \Phi(\tilde{h}(\cdot + X)) + O\left(e^{-|X|} \left\|h\right\|_{H^1(\mathbb{R}^3)}\right) \geq c \left\|h\right\|^2_{H^1(\mathbb{R}^3)} .$$

This implies that there exists $c, R > 0$ such that for $|X| > R$

$$\Phi_X(h(\cdot + X)) \geq c \left\|h\right\|^2_{H^1(\mathbb{R}^3)} . \quad \square$$

2.3. Cauchy theory and profile decomposition

Next, we review tools needed in Section 4.1 to prove the compactness property, up to space translation, of a critical solution of the NLS_Ω equation, using a profile decomposition. We use the same notations as in [21]. Without loss of generality, we assume that $0 \in \Theta = \Omega^c$ and $\Theta \subset B(0,1)$. We define χ to be a smooth cutoff function in $\mathbb{R}^3$

$$\chi(x) = \begin{cases} 1 & |x| \leq \frac{1}{4}, \\ 0 & |x| > \frac{1}{2}. \end{cases}$$

We define spaces $S^k(I),\ k = 0, 1$, as follows

$$S^0(I) = L^\infty_t L^2_x(I \times \Omega) \cap L^{\frac{5}{2}}_t L^{\frac{30}{7}}_x(I \times \Omega),$$
$$S^1(I) = \{u : I \times \Omega \longrightarrow \mathbb{C} \mid\ u \text{ and } (-\Delta_\Omega)^{\frac{1}{2}} u \in S^0(I)\}.$$

Remark 2.6. In order to avoid the endpoints in Strichartz estimates for an exterior domain, see Theorem 2.7 below, we take a specific pair $(\frac{5}{2}, \frac{30}{7})$, for simplicity. However, one could use another pair (p, q) with $p = 2 + \varepsilon$ and $q = \frac{6(2+\varepsilon)}{2+3\varepsilon}$ instead of $(\frac{5}{2}, \frac{30}{7})$, where $\varepsilon > 0$ is small enough.

By interpolation,

$$\left\|u\right\|_{L^q_t L^r_x(I\times\Omega)} \leq \left\|u\right\|_{S^0(I)} , \qquad \text{for all } \frac{2}{q} + \frac{3}{r} = \frac{3}{2} \text{ with } \frac{5}{2} \leq q \leq \infty.$$

Similar estimates hold for $S^1(I)$. We will, in particular, use (q, r) equal to $(5, \frac{30}{11})$ and $(\infty, 2)$.

One particular Strichartz space we use is

$$X^1(I) := L^5_t H^{1,\frac{30}{11}}_0(I \times \Omega).$$

Note that, $S^1(I) \subset X^1(I)$ and by Sobolev embedding, there exists $C > 0$ such that $\|f\|_{L^5_{t,x}(I\times\Omega)} \leq C\,\|f\|_{X^1(I)}$.

We next define $N^0(I)$ as the corresponding dual of $S^0(I)$ and

$$N^1(I) = \{u : I \times \Omega \longrightarrow \mathbb{C} \mid\ u \text{ and } (-\Delta_\Omega)^{\frac{1}{2}} u \in N^0(I)\}. \tag{2.16}$$

Then, we have

$$\|u\|_{N^0(I)} \leq \|u\|_{L_t^{q'} L_x^{r'}(I\times\Omega)} \qquad \text{for all} \quad \frac{2}{q} + \frac{3}{r} = \frac{3}{2} \text{ with } \frac{5}{2} \leq q \leq \infty, \tag{2.17}$$
$$\text{where} \quad \frac{1}{q} + \frac{1}{q'} = 1 \quad \text{and} \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

In particular, we will use $(q', r') = (\frac{5}{3}, \frac{30}{23})$, the Hölder dual to the Strichartz pair $(q, r) = (\frac{5}{2}, \frac{30}{7})$. One can get a similar estimate to (2.17) for $N^1(I)$ using the same pair, see Theorem 2.7.

Next, we state the Strichartz estimates using the above pairs and other necessary results from [21].

Theorem 2.7 *(Strichartz estimates, [16]). Let I be a time interval and $t_0 \in I$. Let $u_0 \in H_0^1(\Omega)$, then there exists a constant $C > 0$ such that the solution $u(t,x)$ to the nonlinear Schrödinger equation on $\mathbb{R} \times \Omega$ with Dirichlet boundary condition*

$$\begin{cases} i\partial_t u + \Delta_\Omega u = f \quad \text{on } \mathbb{R} \times \Omega \\ u(0,x) = u_0(x) \\ u_{|\partial\Omega} = 0 \end{cases}$$

satisfies

$$\|u\|_{S^0(I)} \leq C\left(\|u_0\|_{L^2(\Omega)} + \|f\|_{N^0(I)}\right),$$

and

$$\|u\|_{S^1(I)} \leq C\left(\|u_0\|_{H_0^1(\Omega)} + \|f\|_{N^1(I)}\right). \tag{2.18}$$

In particular,

$$\|u\|_{X^1(I\times\Omega)} \leq C\left(\|u_0\|_{H_0^1(\Omega)} + \|f\|_{L^{\frac{5}{3}} H_0^{1,\frac{30}{23}}(I\times\Omega)}\right).$$

Proposition 2.8 *(Local smoothing, [22, Corollary 2.14]). Given $\omega_0 \in H_0^1(\Omega)$, we have*

$$\left\|\nabla e^{it\Delta_\Omega}\omega_0\right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}(|t-\tau|\leq T,\, |x-z|\leq R)} \leq R^{\frac{31}{60}} T^{\frac{1}{5}} \left\|e^{it\Delta_\Omega}\omega_0\right\|^{\frac{1}{6}}_{L^5_{t,x}(R\times\Omega)} \|\omega_0\|^{\frac{5}{6}}_{H_0^1(\Omega)},$$

uniformly in ω_0 *and the parameters* $R, T > 0$, $z \in \mathbb{R}^3$ *and* $\tau \in \mathbb{R}$.

Lemma 2.9 *(Stability, [21]). Let* $I \subset \mathbb{R}$ *be a time interval and let* $\tilde{u}$ *be an approximate solution to (*NLS_Ω*) on* $I \times \Omega$ *in the sense that*

$$i\partial_t \tilde{u} + \Delta_\Omega \tilde{u} = -|\tilde{u}|^2 \tilde{u} + e \quad \text{for some function } e.$$

Assume that

$$\|\tilde{u}\|_{L^\infty H_0^1(I\times\Omega)} \leq \mathcal{E} \quad \text{and} \quad \|\tilde{u}\|_{L^5_{t,x}(I\times\Omega)} \leq L$$

for some positive constants $\mathcal{E}$ *and* L. *Let* $t_0 \in I$ *and* $u_0 \in H_0^1(\Omega)$ *and assume the smallness conditions*

$$\|\tilde{u}(t_0) - u_0\|_{H_0^1(\Omega)} \leq \varepsilon \quad \text{and} \quad \|e\|_{N^1(I)} \leq \varepsilon$$

for some $0 < \varepsilon < \varepsilon_1 = \varepsilon_1(\mathcal{E}, L)$. *Then there exists a unique solution* $u : I \times \Omega \longrightarrow \mathbb{C}$ *to (*NLS_Ω*) with initial data* $u(t_0) = u_0$ *satisfying*

$$\|u - \tilde{u}\|_{X^1(I\times\Omega)} \leq C(\mathcal{E}, L)\varepsilon.$$

Theorem 2.10 *(Linear profile decomposition in* $H_0^1(\Omega)$*, [21, Theorem 3.2]). Let* $\{f_n\}$ *be a bounded sequence in* $H_0^1(\Omega)$. *After passing to a subsequence, there exist* $J^* \in \{0, 1, 2, ..., \infty\}$, $\{\phi_n^j\}_{j=1}^{J^*} \subset H_0^1(\Omega) \setminus \{0\}$, $\{t_n^j\}_{j=1}^{J^*} \subset \mathbb{R}$ *such that, for each* j *either* $t_n^j \equiv 0$ *or* $t_n^j \to \pm\infty$ *and* $\{x_n^j\}_{j=1}^{J^*} \subset \Omega$ *conforming to one of the following two cases for each* j*:*
Case 1: $x_n^j = 0$ *and there exists* $\phi^j \in H_0^1(\Omega)$ *so that* $\phi_n^j := e^{it_n^j \Delta_\Omega}\phi^j$.
Case 2: $|x_n^j| \to \infty$ *and there exists* $\phi^j \in H^1(\mathbb{R}^3)$ *so that*

$$\phi_n^j := e^{it_n^j\Delta_\Omega}[(\chi_n^j \phi^j)(x - x_n^j)] \quad \text{with} \quad \chi_n^j(x) := \chi\left(\frac{x}{|x_n^j|}\right).$$

Moreover, for any finite $0 \leq J \leq J^*$ *we have the decomposition*

$$f_n = \sum_{j=1}^{J} \phi_n^j + \omega_n^J$$

with the remainder $\omega_n^J \in H_0^1(\Omega)$ *satisfying*

$$\lim_{J\to J^*} \limsup_{n\to\infty} \left\|e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5_{t,x}(\mathbb{R}\times\Omega)} = 0, \tag{2.19}$$

$$\forall J \geq 1, \quad \lim_{n\to\infty} \left\{ M[f_n] - \sum_{j=1}^{J} M[\phi_n^j] - M[\omega_n^J] \right\} = 0, \tag{2.20}$$

$$\forall J \geq 1, \qquad \lim_{n\to\infty} \left\{ E[f_n] - \sum_{j=1}^{J} E[\phi_n^j] - E[\omega_n^J] \right\} = 0, \tag{2.21}$$

$$\lim_{n\to\infty} \left|x_n^j - x_n^k\right| + \left|t_n^j - t_n^k\right| = \infty \quad \textit{for each } j \neq k. \tag{2.22}$$

Theorem 2.11 *([21, Theorem 4.1]). Let $\{t_n\} \subset \mathbb{R}$ be such that $t_n \equiv 0$ or $t_n \to \pm\infty$. Let $\{x_n\} \subset \Omega$ be such that $|x_n|$ tends to ∞, as n goes to ∞. Assume $\phi \in H^1(\mathbb{R}^3)$ satisfies*

$$\|\nabla\phi\|_{L^2(\mathbb{R}^3)} \|\phi\|_{L^2(\mathbb{R}^3)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)} \|Q\|_{L^2(\mathbb{R}^3)}, \tag{2.23}$$

$$M_{\mathbb{R}^3}[\phi] E_{\mathbb{R}^3}[\phi] < M_{\mathbb{R}^3}[Q] E_{\mathbb{R}^3}[Q]. \tag{2.24}$$

Define

$$\phi_n := e^{it_n\Delta_\Omega}\left[(\chi_n\phi)(x - x_n)\right] \quad \textit{with} \quad \chi_n(x) := \chi\left(\frac{x}{|x_n|}\right).$$

Then, for n sufficiently large, there exists a global solution v_n to (NLS$_\Omega$) *with initial data $v_n(0) := \phi_n$, which satisfies*

$$\|v_n\|_{L^5_{t,x}(\mathbb{R}\times\Omega)} \leq C(\|\phi\|_{H^1(\mathbb{R}^3)}).$$

Furthermore, for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ and $\psi_\varepsilon \in C_c(\mathbb{R} \times \mathbb{R}^3)$ such that, for all $n \geq N_\varepsilon$,

$$\left\|\underline{v}_n(t - t_n, x + x_n) - \psi_\varepsilon(t, x)\right\|_{L^5 H^{1,\frac{30}{11}}(\mathbb{R}\times\mathbb{R}^3)} < \varepsilon. \tag{2.25}$$

Remark 2.12. Note that, we have made a slight modification in the notation of the above Theorem 2.11, in order to keep the consistent notation in this paper. We denote $\underline{v}_n$ the extension of the solution v_n by 0 on Ω^c such that $\underline{v}_n \in H^1(\mathbb{R}^3)$. Let us mention that ϕ_n is well defined in $H_0^1(\Omega)$. Indeed, by the definition of χ_n and as $|x_n| \to \infty$, we have

$$x \in \partial\Omega \quad \Longrightarrow \quad \chi_n(x) = 0 \quad \text{ as } n \to +\infty.$$

Moreover, one can check that the energy-mass assumption (2.24) is equivalent to the one given in [21, Theorem 4.1] using the following identity:

$$\left\{u_0 \in H^1(\mathbb{R}^3) : E_{\mathbb{R}^3}[u_0] M_{\mathbb{R}^3}[u_0] < E_{\mathbb{R}^3}[Q] M_{\mathbb{R}^3}[Q]\right\}$$
$$= \bigcup_{0<\lambda<\infty} \left\{u_0 \in H^1(\mathbb{R}^3) : E_{\mathbb{R}^3}[u_0] + \lambda M_{\mathbb{R}^3}[u_0] < 2\sqrt{\lambda E_{\mathbb{R}^3}[Q] M_{\mathbb{R}^3}[Q]}\right\},$$

which follows by computing the minimum, of $\lambda \mapsto E_{\mathbb{R}^3}[u_0] + \lambda M_{\mathbb{R}^3}[u_0] - 2\sqrt{\lambda E_{\mathbb{R}^3}[Q] M_{\mathbb{R}^3}[Q]}$.

3. Modulation

Let $u \in H_0^1(\Omega)$ and define

$$\delta(u) = \left| \int_{\mathbb{R}^3} |\nabla Q|^2 - \int_{\Omega} |\nabla u|^2 \right|.$$

In this section and the next one, we will consider a solution u such that $M[u]E[u] = M_{\mathbb{R}^3}[Q]E_{\mathbb{R}^3}[Q]$. We first rescale the solution and the obstacle, letting $\tilde{u}(t,x) = \lambda u(\lambda^2 t, \lambda x)$ and $\widetilde{\Omega} = \lambda^{-1}\Omega$ with $\lambda = \frac{M_\Omega[u]}{M_{\mathbb{R}^3}[Q]} = \frac{E_{\mathbb{R}^3}[Q]}{E_{\Omega[u]}}$. Then $\tilde{u}$ is solution of (NLS$_{\widetilde{\Omega}}$) and satisfies $M_{\widetilde{\Omega}}[u] = M_{\mathbb{R}^3}[Q]$, $E_{\widetilde{\Omega}}[u] = E_{\mathbb{R}^3}[Q]$.

Replacing u by $\tilde{u}$ and Ω by $\widetilde{\Omega}$, we conclude that can assume without loss of generality

$$M[u] = M_{\mathbb{R}^3}[Q] \quad \text{and} \quad E[u] = E_{\mathbb{R}^3}[Q]. \tag{3.1}$$

Lemma 3.1. *Let $u \in H_0^1(\Omega)$ satisfying (3.1) and $\delta(u)$ small enough. Then there exists $X_0 \in \mathbb{R}^3$ large and $\theta_0 \in \mathbb{R}$ such that*

$$e^{-i\theta_0}u(x) = Q(x - X_0)\Psi(x) + h(x) \tag{3.2}$$

with $\|h\|_{H_0^1(\Omega)} \le \tilde{\varepsilon}(\delta(u))$, where $\tilde{\varepsilon}(\delta(u)) \to 0$ as $\delta(u) \to 0$.

Proof. Let $\underline{u} \in H^1(\mathbb{R}^3)$ be defined as above in (2.8) and observe that $\delta(u) = \delta(\underline{u})$. By Proposition 2.1, since

$$M[u] = M_{\mathbb{R}^3}[\underline{u}] = M_{\mathbb{R}^3}[Q], \quad E[u] = E_{\mathbb{R}^3}[\underline{u}] = E_{\mathbb{R}^3}[Q], \tag{3.3}$$

and $\delta(\underline{u})$ being small enough, there exist $\theta_0 \in \mathbb{R}$ and $X_0 \in \mathbb{R}^3$ such that

$$e^{-i\theta_0}\underline{u}(x) = Q(x - X_0) + \tilde{h}(x)$$

with $\|\tilde{h}\|_{H^1(\mathbb{R}^3)} \le \tilde{\varepsilon}(\delta(\underline{u}))$, where $\tilde{\varepsilon}(\delta(\underline{u})) \longrightarrow 0$ as $\delta(\underline{u}) \longrightarrow 0$.

Moreover, if $x \in \Omega^c$, then $\underline{u}(x) = 0$, which implies that

$$x \in \Omega^c \quad \Longrightarrow \quad Q(x - X_0) + \tilde{h}(x) = 0, \tag{3.4}$$

and for $\delta(\underline{u})$ small enough, by (2.9), $|X_0|$ is large such that

$$\frac{e^{-|X_0|}}{|X_0|} \le C\,\tilde{\varepsilon}(\delta(\underline{u})).$$

We write,

$$\begin{aligned} e^{-i\theta_0}\underline{u}(x) &= Q(x-X_0)\Psi(x) + (1-\Psi(x))Q(x-X_0)) + \tilde{h}(x) \\ &= Q(x-X_0)\Psi(x) + h(x). \end{aligned}$$

Using the fact that $(1-\Psi)$ has a compact support, Q having an exponential decay, $|X_0|$ being large, and Lemma 2.3, we get

$$\|h\|_{H^1(\mathbb{R}^3)} \leq \tilde{\varepsilon}(\delta(\underline{u})) + C\frac{e^{-|X_0|}}{|X_0|} \leq \tilde{\varepsilon}(\delta(\underline{u})).$$

By (3.4) and the definition of Ψ in (1.10), we have

$$h(x) = 0, \quad \text{if } x \in \Omega^c.$$

Thus, $h(x) = 0$ on $\partial\Omega$ and $h(x) \in H_0^1(\Omega)$, which concludes the proof. □

Lemma 3.2. *There exists $\delta_0 > 0$ and a positive function $\varepsilon(\delta)$ defined for $0 < \delta \leq \delta_0$, which tends to 0 when $\delta \to 0$, such that for any $u \in H_0^1(\Omega)$ satisfying (3.1) and $\delta(u) < \delta_0$, there exists a couple $(\mu, X) \in \mathbb{R} \times \mathbb{R}^3$ such that the following holds*

$$\left\|u(x) - Q(x-X)\Psi(x)e^{i\mu}\right\|_{H_0^1(\Omega)} \leq \varepsilon(\delta), \tag{3.5}$$

$$\operatorname{Re}\int_\Omega u(x)\, \partial_{x_k}(Q(x-X)\Psi(x))e^{-i\mu}\, dx = 0, \quad k = 1,2,3, \tag{3.6}$$

$$\operatorname{Im}\int_\Omega u(x)\, Q(x-X)\Psi(x)e^{-i\mu} dx = 0. \tag{3.7}$$

The parameters μ and X are unique in $\mathbb{R}/\pi\mathbb{Z} \times \mathbb{R}^3$ and the mapping $u \to (\mu, X)$ is C^1.

Proof. Let

$$\begin{aligned} \Phi : H_0^1(\Omega) \times \mathbb{R}^3 \times \mathbb{R} &\longrightarrow \mathbb{R}^4 \\ (u\ ,\ X\ ,\ \mu) &\longmapsto (\Phi_k(u,X,\mu))_{1\leq k\leq 4}\,, \end{aligned}$$

where

$$\Phi_k(u,X,\mu) := \operatorname{Re}\int_\Omega u(x)\, \partial_{x_k}(Q(x-X)\Psi(x))e^{-i\mu}\, dx,\ k = 1,2,3,$$

$$\Phi_4(u,X,\mu) := \operatorname{Im}\int_\Omega u(x)\, Q(x-X)\Psi(x)\, e^{-i\mu} dx.$$

Let $X_0 \in \mathbb{R}^3$. Note that $\Phi(Q(\cdot - X_0)\Psi, X_0, 0) = 0$. Indeed, integrating by parts, we get

$$\Phi_k(Q(\cdot - X_0)\Psi, X_0, 0) = \operatorname{Re} \int_\Omega Q(x - X_0)\Psi(x)\partial_{x_k}(Q(x - X_0)\Psi(x))\, dx$$
$$= \frac{1}{2} \operatorname{Re} \int_\Omega \partial_{x_k}((Q(x - X_0)\Psi(x))^2)\, dx = 0,$$
$$\Phi_4(Q(\cdot - X_0)\Psi, X_0, 0) = \operatorname{Im} \int_\Omega Q(x - X_0)^2\Psi(x)^2\, dx = 0.$$

- **Step 1:** Computation of $d_{(X,\mu)}\Phi_k$.
  We have

$$\frac{\partial}{\partial X_j}\Phi_k(u, X, \mu) = -\operatorname{Re} \int_\Omega e^{-i\mu}u(x)\partial_{x_k}(\partial_{x_j}Q(x - X)\Psi(x))\, dx.$$

Integrating by parts, we obtain

$$\frac{\partial}{\partial X_j}\Phi_k(Q(\cdot - X_0)\Psi, X_0, 0) = \operatorname{Re} \int_\Omega \partial_{x_j}Q(x - X_0)\Psi(x)\partial_{x_k}(Q(x - X_0)\Psi(x))\, dx.$$

If $k = j$, we have

$$\frac{\partial}{\partial X_j}\Phi_k(Q(\cdot - X_0)\Psi, X_0, 0) = \operatorname{Re} \int_\Omega (\partial_{x_j}Q(x - X_0))^2\, dx$$
$$+ \operatorname{Re} \int_\Omega (\partial_{x_j}Q(x - X_0))^2(\Psi(x)^2 - 1)\, dx$$
$$+ \operatorname{Re} \int_\Omega Q(x - X_0)\partial_{x_j}Q(x - X_0)\Psi(x)\partial_{x_j}\Psi(x)\, dx.$$

Since $\partial_{x_j}\Psi$ and $(\Psi^2 - 1)$ have a compact support and Q has an exponential decay, we deduce

$$\frac{\partial}{\partial X_j}\Phi_k(Q(\cdot - X_0)\Psi, X_0, 0) = \left\|\partial_{x_j}Q\right\|^2_{L^2(\mathbb{R}^3)} + O(e^{-2|X_0|})$$
$$= \frac{1}{3}\left\|\nabla Q\right\|^2_{L^2(\mathbb{R}^3)} + O(e^{-2|X_0|}).$$

If $k \neq j$, then

$$\frac{\partial}{\partial X_j}\Phi_k(Q(\cdot - X_0)\Psi, X_0, 0) = \operatorname{Re} \int_\Omega \partial_{x_j}Q(x - X_0)\Psi(x)\partial_{x_k}(Q(x - X_0)\Psi(x))\, dx$$

$$= \mathrm{Re}\int_{\Omega} \partial_{x_j} Q(x-X_0)\partial_{x_k} Q(x-X_0)dx$$
$$+ \mathrm{Re}\int_{\Omega} \partial_{x_j} Q(x-X_0)\partial_{x_k} Q(x-X_0)(\Psi(x)^2-1)dx$$
$$+ \mathrm{Re}\int_{\Omega} \partial_{x_j} Q(x-X_0)\Psi(x)Q(x-X_0)\partial_{x_k}\Psi(x)dx.$$

Using the same argument as before and the fact that Q is radial ($\int \partial_{x_j} Q \partial_{x_k} Q = 0$, if $k \neq j$), we obtain

$$\frac{\partial}{\partial X_j}\Phi_k(Q(\cdot - X_0)\Psi, X_0, 0) = O(e^{-2|X_0|}).$$

Next, we compute $\frac{\partial}{\partial \mu}\Phi_k(u, X, \mu)$:

$$\frac{\partial}{\partial \mu}\Phi_k(u, X, \mu) = \mathrm{Re}\int_{\Omega} -ie^{-i\mu}u(x)\partial_{x_k}(Q(x-X)\Psi(x))dx,$$
$$\frac{\partial}{\partial \mu}\Phi_k(Q(\cdot - X_0)\Psi, X_0, 0) = \mathrm{Im}\int_{\Omega} Q(x-X_0)\Psi(x)\partial_{x_k}(Q(x-X_0)\Psi(x))\, dx = 0.$$

- **Step 2:** Computation of $d_{(X,\mu)}\Phi_4$.
  We have

$$\frac{\partial}{\partial X_j}\Phi_4(u, X, \mu) = -\mathrm{Im}\int_{\Omega} e^{-i\mu}u(x)(\partial_{x_j} Q(x-X)\Psi(x))\, dx,$$

and thus,

$$\frac{\partial}{\partial X_j}\Phi_4(Q(\cdot - X_0)\Psi, X_0, 0) = -\mathrm{Im}\int_{\Omega} Q(x-X_0)\Psi(x)\partial_{x_j}(Q(x-X_0)\Psi(x))\, dx = 0.$$

Also,

$$\frac{\partial}{\partial \mu}\Phi_4(u, X, \mu) = \mathrm{Im}\int_{\Omega} -ie^{-i\mu}u(x)Q(x-X)\Psi(x)\, dx,$$
$$\frac{\partial}{\partial \mu}\Phi_4(Q(\cdot - X_0)\Psi, X_0, 0) = -\int_{\Omega} Q(x-X_0)^2\Psi(x)^2 = -\|Q\|^2_{L^2(\mathbb{R}^3)} + O(e^{-2|X_0|}).$$

- **Step 3:** Conclusion.

Combining Step 1 and Step 2, we get

$$d_{(X,\mu)}\Phi(Q(\cdot - X_0)\Psi, X_0, 0)$$
$$= \begin{pmatrix} \frac{1}{3}\|\nabla Q\|^2_{L^2(\mathbb{R}^3)} & 0 & 0 & 0 \\ 0 & \frac{1}{3}\|\nabla Q\|^2_{L^2(\mathbb{R}^3)} & 0 & 0 \\ 0 & 0 & \frac{1}{3}\|\nabla Q\|^2_{L^2(\mathbb{R}^3)} & 0 \\ 0 & 0 & 0 & -\|Q\|^2_{L^2(\mathbb{R}^3)} \end{pmatrix}$$
$$+ O(e^{-2|X_0|}).$$

We can deduce that $d_{(X,\mu)}\Phi$ is invertible at $(Q(\cdot - X_0)\Psi(\cdot), X_0, 0)$, if $|X_0|$ is large. Then, by the implicit function theorem there exists $\epsilon_0, \eta_0 > 0$ such that for $u \in H_0^1(\Omega)$, we have

$$\|u(\cdot) - Q(\cdot - X_0)\Psi(\cdot)\|^2_{H_0^1(\Omega)} < \epsilon_0 \Longrightarrow \exists!(X, \mu) : \quad |\mu| + |X - X_0| \le \eta_0 \text{ and}$$
$$\Phi(u, X, \mu) = 0. \quad \square$$

Let $u(t)$ be a solution of (NLS_Ω) satisfying (3.1). In the sequel we write

$$\delta(t) = \delta(u(t)).$$

Let $D_{\delta_0} = \{t \in I : \ \delta(t) < \delta_0\}$, where I is the maximal time interval of existence of u.

By Lemma 3.2, we can define C^1 functions $X(t)$ and $\mu(t)$ for $t \in D_{\delta_0}$. We now work with the parameter $\theta(t) = \mu(t) - t$. Write

$$e^{-i\theta(t)-it}u(t,x) = (1 + \rho(t))Q(x - X(t))\Psi(x) + h(t,x), \tag{3.8}$$

where $h(x) \in H_0^1(\Omega)$ and define

$$\rho(t) = \operatorname{Re} \frac{e^{-i\theta(t)-it}\int \nabla\Big(Q(x - X(t))\Psi(x)\Big) \cdot \nabla \underline{u}(t,x)dx}{\int \left|\nabla\big(Q(x - X(t))\Psi(x)\big)\right|^2 dx} - 1.$$

This implies that

$$e^{-i\theta(t)-it}\underline{u}(t, x + X(t)) = (1 + \rho(t))Q(x)\Psi(x + X(t)) + \underline{h}(t, x + X(t)), \tag{3.9}$$

where $\underline{h}(x) \in H^1(\mathbb{R}^3)$ is defined by

$$\underline{h}(t,x) = \begin{cases} h(t,x) & \forall x \in \Omega, \\ 0 & \forall x \in \Omega^c. \end{cases}$$

One can see that $\rho(t)$ is chosen such that h satisfies the orthogonality condition

$$\operatorname{Re}\int_{\Omega} \Delta(Q(x-X(t))\Psi(x))h(t,x)\,dx$$
$$= \operatorname{Re}\int \Delta(Q(x)\Psi(x+X(t)))\underline{h}(t,x+X((t))\,dx = 0. \quad (3.10)$$

By Lemma 3.2, h also satisfies the orthogonality conditions

$$\operatorname{Im}\int_{\Omega} h(t,x)Q(x-X(t))\Psi(x)\,dx = \operatorname{Im}\int \underline{h}(t,x+X(t))Q(x)\Psi(x+X(t))\,dx = 0, \quad (3.11)$$

and

$$\operatorname{Re}\int_{\Omega} h(t,x)\partial_{x_k}(Q(x-X(t))\Psi(x))\,dx$$
$$= \operatorname{Re}\int \underline{h}(t,x+X(t))\partial_{x_k}(Q(x)\Psi(x+X(t)))\,dx = 0, \quad k=1,2,3. \quad (3.12)$$

In the following lemma, to simplify notation, we denote $f(\cdot + X)$ by $f_X(\cdot)$ for any function f. If f is a complex function, then we denote by $f_{1_X}(\cdot)$ the real part of f_X and by $f_{2_X}(\cdot)$ the imaginary part.

Proposition 3.3. *Let $u(t)$ be a solution of* (NLS_Ω) *satisfying* (3.1)*. Then the following estimates hold for $t \in D_{\delta_0}$*

$$|\rho(t)| + O\left(\frac{e^{-2|X(t)|}}{|X(t)|^2}\right) \approx \left|\int Q\Psi_X \underline{h}_{1_X}\,dx\right| + O\left(\frac{e^{-2|X(t)|}}{|X(t)|^2}\right) \approx \delta(t) + O\left(\frac{e^{-2|X(t)|}}{|X(t)|^2}\right)$$
$$\approx \|h(t)\|_{H_0^1(\Omega)} + O\left(\frac{e^{-|X(t)|}}{|X(t)|}\right). \quad (3.13)$$

Proof. Let $\tilde{\delta}(t) = |\rho(t)| + \|\underline{h}\|_{H^1} + \delta(t)$, which is small, if $\delta(t)$ is small. By the expansion of u in (3.9) we have $e^{-i\theta(t)-it}\underline{u}(t,x+X(t)) = (1+\rho(t))Q(x)\Psi_X(x) + \underline{h}_X(t,x)$, thus, if $x+X(t)\in\Omega$, then $\underline{u}(t,x+X(t)) = u(t,x+X(t))$, otherwise $\underline{u}(t,x+X(t)) = 0$.

- **Step 1:** Approximation of $|\rho|$ using the mass conservation.
  Since $M[u] = M_{\mathbb{R}^3}[\underline{u}] = M_{\mathbb{R}^3}[Q\Psi_X + \rho Q\Psi_X + \underline{h}_X] = M_{\mathbb{R}^3}[Q]$, we have,

$$\int \left(Q^2(\Psi_X^2 - 1) + 2\rho\, Q^2\Psi_X^2 + 2\rho\, Q\Psi_X\underline{h}_{1_X} + \rho^2 Q^2\Psi_X^2 + 2Q\Psi_X\underline{h}_{1_X} + |\underline{h}_X|^2\right)dx = 0. \quad (3.14)$$

Using (3.14) and Lemma 2.3, we obtain

$$
\begin{aligned}
2|\rho|\left|\int Q^2\,\Psi_X^2\right| &= \left|2\int Q\Psi_X \underline{h}_{1_X} + \int Q^2(\Psi_X^2-1) + 2\rho\int Q\Psi_X\underline{h}_{1_X}\right.\\
&\qquad \left. + \rho^2\int Q^2\Psi_X^2 + \int |\underline{h}_X|^2\right|\\
&= 2\left|\int Q\Psi_X\underline{h}_{1_X}\right| + O\left(\tilde{\delta}^2 + \frac{e^{-2|X(t)|}}{|X(t)|^2}\right),
\end{aligned}
$$

which yields

$$
|\rho| = \frac{1}{M[Q]}\left|\int Q\Psi_X\underline{h}_{1_X}\,dx\right| + O\left(\tilde{\delta}^2 + \frac{e^{-2|X(t)|}}{|X(t)|^2}\right). \tag{3.15}
$$

- **Step 2:** Approximation of $|\rho|$ in terms of δ.
  By the definition of $\delta(t)$, we have

$$
\begin{aligned}
\delta(t) &= \left|\int |\nabla(Q\Psi_X + \rho Q\Psi_X + \underline{h}_X)|^2\,dx - \int |\nabla Q|^2\,dx\right|\\
&= \left|\int |\nabla(Q\Psi_X)|^2 + 2\rho\,|\nabla(Q\Psi_X)|^2 + \rho^2\,|\nabla(Q\Psi_X)|^2 + 2\rho\nabla(Q\Psi_X)\cdot\nabla\underline{h}_{1_X}\right.\\
&\quad \left. + 2\nabla(Q\Psi_X)\cdot\nabla\underline{h}_{1_X} + |\nabla\underline{h}_X|^2 - \int |\nabla Q|^2\right|.
\end{aligned}
$$

Using integration by parts and the orthogonality condition (3.10), we get

$$
\begin{aligned}
\delta(t) = \Bigg|\int |\nabla Q|^2\,(\Psi_X^2-1) + 2\,\nabla Q\cdot\nabla\Psi_X Q\Psi_X + Q^2\,|\nabla\Psi_X|^2\\
+ (2\rho+\rho^2)\int |\nabla(Q\Psi_X)|^2 + \int |\nabla\underline{h}_X|^2\Bigg|.
\end{aligned}
$$

Using the fact that (Ψ^2-1) and $\nabla\Psi$ have compact supports and applying Lemma 2.3, we get

$$
|\rho| = \frac{\delta}{2\,\|\nabla Q\|^2_{L^2(\mathbb{R}^3)}} + O\left(\tilde{\delta}^2 + \frac{e^{-2|X(t)|}}{|X(t)|^2}\right). \tag{3.16}
$$

- **Step 3:** Energy and Mass conservation.
  We define: $g = \rho Q\Psi_X + \underline{h}_X$. Since $E_{\mathbb{R}^3}[\underline{u}] = E_{\mathbb{R}^3}[Q\Psi_X + g] = E_{\mathbb{R}^3}[Q]$, we have

$$
\frac{1}{2}\int |\nabla(Q\Psi_X)|^2 - \frac{1}{2}\int |\nabla Q|^2 - \frac{1}{4}\int Q^4\Psi_X^4
$$

$$+\frac{1}{4}\int Q^4 + \int \nabla(Q\Psi_X)\cdot\nabla g_1 - \int Q^3\Psi_X^3\, g_1 \tag{3.17}$$

$$+\frac{1}{2}\int |\nabla g|^2 - \frac{1}{2}\int Q^2\Psi_X^2(3g_1^2+g_2^2) - \int Q\Psi_X|g|^2 g_1 - \frac{1}{4}|g|^4 = 0. \tag{3.18}$$

First, we estimate (3.17). For that we denote

$$A_0 = \frac{1}{2}\int |\nabla(Q\Psi_X)|^2 - \frac{1}{2}\int |\nabla Q|^2 - \frac{1}{4}\int Q^4\Psi_X^4 + \frac{1}{4}\int Q^4,$$

$$A_L(g) = \int \nabla(Q\Psi_X)\cdot\nabla g_1 - \int Q^3\Psi_X^3 g_1.$$

In this step, we estimate A_0 and $A_L(g)$. Using the fact that $\nabla\Psi$, (Ψ^2-1) and (Ψ^4-1) have compact supports and Lemma 2.3, we have

$$A_0 = O\left(\frac{e^{-2|X(t)|}}{|X(t)|^2}\right). \tag{3.19}$$

Next, we show that

$$A_L(g) = \frac{1}{2}\int |g|^2 - 2\int \nabla Q.\nabla\Psi_X\, g_1 - \int Q\Delta\Psi_X\, g_1 - \int Q^3\Psi_X(\Psi_X^2-1)g_1 + O\left(\frac{e^{-2|X(t)|}}{|X(t)|^2}\right). \tag{3.20}$$

Integrating by parts, we obtain

$$\int \nabla(Q\Psi_X)\cdot\nabla g_1 = -\int \Delta(Q\Psi_X)g_1 = -\int \Delta Q\,\Psi_X g_1 - 2\int \nabla Q\cdot\nabla\Psi_X g_1 - \int Q\Delta\Psi_X g_1,$$

$$-\int Q^3\Psi_X^3 g_1 = -\int Q^3\Psi_X\, g_1 - \int Q^3\Psi_X(\Psi_X^2-1)g_1.$$

Using the equation (2.1) for Q, we deduce

$$A_L(g) = -\int Q\Psi_X g_1 - 2\int \nabla Q\cdot\nabla\Psi_X g_1 - \int Q\Delta\Psi_X\, g_1 - \int Q^3\Psi_X(\Psi_X^2-1)g_1.$$

Since $M[u] = M[\underline{u}] = M[Q\Psi_X + g] = M[Q]$, we have

$$\int Q^2(\Psi_X^2-1) + 2\int Q\Psi_X g_1 + \int |g|^2 = 0,$$

$$-\int Q\Psi_X g_1 = \frac{1}{2}\int |g|^2 + O\left(\frac{e^{-2|X(t)|}}{|X(t)|^2}\right).$$

This implies (3.20).

- **Step 4:** Approximation of $\|h\|_{H_0^1(\Omega)}$.
Recall that $g = \rho Q\Psi_X + \underline{h}_X$. In this step we prove

$$\|h\|_{H_0^1(\Omega)} = O\left(|\rho| + \tilde{\delta}^{\frac{3}{2}} + \frac{e^{-|X(t)|}}{|X(t)|}\right).$$

Summing up all terms (3.18), (3.19) and (3.20), we obtain

$$\begin{aligned}
&\frac{1}{2}\int |\rho Q\Psi_X + \underline{h}_X|^2 - 2\int \nabla Q \cdot \nabla\Psi_X(\rho Q\Psi_X + \underline{h}_{1_X}) - \int Q\Delta\Psi_X(\rho Q\Psi_X + \underline{h}_{1_X})\\
&- \int Q^3\Psi_X(\Psi_X^2 - 1)(\rho Q\Psi_X + \underline{h}_{1_X}) + \frac{1}{2}\int |\nabla(\rho Q\Psi_X + \underline{h}_X)|^2\\
&\quad - \frac{1}{2}\int Q^2\Psi_X^2(3(\rho Q\Psi_X + \underline{h}_{1_X})^2 + \underline{h}_{2_X}^2)\\
&- \int Q\Psi_X|\rho Q\Psi_X + \underline{h}_X|^2(\rho Q\Psi_X + \underline{h}_{1_X}) - \frac{1}{4}\int |\rho Q\Psi_X + \underline{h}_X|^4 = O\left(\frac{e^{-2|X(t)|}}{|X(t)|^2}\right).
\end{aligned}$$

Denote

$$\begin{aligned}
B_L(\underline{h}) &= -2\int \nabla Q.\nabla\Psi_X(\rho Q\Psi_X + \underline{h}_{1_X}) - \int Q\Delta\Psi_X(\rho Q\Psi_X + \underline{h}_{1_X})\\
&\quad - \int Q^3\Psi_X(\Psi_X^2 - 1)(\rho Q\Psi_X + \underline{h}_{1_X}),\\
B_{NL}^1(\underline{h}) &= \frac{1}{2}\int |\rho Q\Psi_X + \underline{h}_X|^2 + \frac{1}{2}\int |\nabla(\rho Q\Psi_X + \underline{h}_X)|^2,\\
B_{NL}^2(\underline{h}) &= -\frac{1}{2}\int Q^2\Psi_X^2(3(\rho Q\Psi_X + \underline{h}_{1_X})^2 + \underline{h}_{2_X}^2)\\
&\quad - \int Q\Psi_X|\rho Q\Psi_X + \underline{h}_X|^2(\rho Q\Psi_X + \underline{h}_{1_X}) - \frac{1}{4}\int |\rho Q\Psi_X + \underline{h}_X|^4.
\end{aligned}$$

Next, we estimate each term. Using the fact that $\nabla\Psi$, $\Delta\Psi$ and $(\Psi^2 - 1)$ have compact supports and Lemma 2.3, we obtain

$$\begin{aligned}
B_L(\underline{h}) &= -\int (2\nabla Q \cdot \nabla\Psi_X + Q\Delta\Psi_X)(\rho Q\Psi_X + \underline{h}_{1_X})\\
&\quad - \int Q^3\Psi_X(\Psi_X^2 - 1)(\rho Q\Psi_X + \underline{h}_{1_X})\\
&= O\left(|\rho|\frac{e^{-2|X(t)|}}{|X(t)|^2} + \frac{e^{-|X(t)|}}{|X(t)|}\|\underline{h}\|_{H^1}\right) + O\left(|\rho|\frac{e^{-4|X(t)|}}{|X(t)|^4} + \|\underline{h}\|_{H^1}\frac{e^{-3|X(t)|}}{|X(t)|^3}\right).
\end{aligned}$$

Using the orthogonality condition (3.10), we get

$$
\begin{aligned}
B^1_{NL}(\underline{h}) &= \frac{\rho^2}{2}\int Q^2\Psi_X^2 + \rho\int Q\Psi_X\,\underline{h}_{1_X} + \frac{1}{2}\int |\underline{h}_X|^2 + \frac{\rho^2}{2}\int |\nabla(Q\Psi_X)|^2 \\
&\quad + \rho\int \nabla(Q\Psi_X)\cdot\nabla\underline{h}_{1_X} + \frac{1}{2}\int |\nabla\underline{h}_X|^2 \\
&= \rho\int Q\Psi_X\underline{h}_{1_X} + \frac{1}{2}\int |\underline{h}|^2 + \frac{1}{2}\int |\nabla\underline{h}|^2 + O(|\rho|^2),
\end{aligned}
$$

$$
\begin{aligned}
B^2_{NL}(\underline{h}) &= -\frac{1}{2}\int Q^2\Psi_X^2(3\underline{h}^2_{1_X} + \underline{h}^2_{2_X}) - \frac{1}{4}\int |\underline{h}_X|^4 - \rho\int Q\Psi_X|\underline{h}_X|^2\underline{h}_{1_X} \\
&\quad - \int Q\Psi_X|\underline{h}_X|^2\underline{h}_{1_X} - \frac{\rho^2}{2}\int Q^2\Psi_X^2(3\underline{h}^2_{1_X} + \underline{h}^2_{2_X}) - \rho\int Q^2\Psi_X^2|\underline{h}_X|^2 \\
&\quad - 2\rho\int Q^2\Psi_X^2\,\underline{h}^2_{1_X} - \rho^3\int Q^3\Psi_X^3\,\underline{h}_{1_X} - 3\rho^2\int Q^3\Psi_X^3\,\underline{h}_{1_X} \\
&\quad - 3\rho\int Q^3\Psi_X^3\,\underline{h}_{1_X} - \frac{\rho^4}{4}\int Q^4\Psi_X^4 - \rho^3\int Q^4\Psi_X^4 - \frac{3\rho^2}{2}\int Q^4\Psi_X^4.
\end{aligned}
$$

By the equation (1.1) and using again the orthogonality condition (3.10), we have

$$
\begin{aligned}
-3\rho\int Q^3\Psi_X^3\,\underline{h}_{1_X} &= -3\rho\int Q\Psi_X\underline{h}_{1_X} - 3\rho\int (Q-\Delta Q)\,\Psi_X^2(\Psi_X - 1)\underline{h}_{1_X} \\
&\quad - 6\rho\int \nabla Q.\nabla\Psi_X\,\underline{h}_{1_X} - 3\rho\int \Delta\Psi_X Q\,\underline{h}_{1_X} \\
&= -3\rho\int Q\Psi_X\,\underline{h}_{1_X} + O\left(|\rho|\frac{e^{-|X(t)|}}{|X(t)|}\,\|\underline{h}\|_{H^1}\right).
\end{aligned}
$$

Using the facts that

$$
\begin{aligned}
&\rho\int Q\Psi_X|\underline{h}_X|^2\underline{h}_{1_X} = O(|\rho|\,\|\underline{h}\|^3_{H^1}), \\
&\frac{\rho^2}{2}\int Q^2\Psi_X^2(3\underline{h}^2_{1_X} + \underline{h}^2_{2_X}) - \rho\int Q^2\Psi_X^2|\underline{h}_X|^2 - 2\rho\int Q^2\Psi_X^2\underline{h}^2_{1_X} \\
&\quad = O(|\rho|^2\,\|\underline{h}\|^2_{H^1} + |\rho|\,\|\underline{h}\|^2_{H^1}), \\
&-\rho^3\int Q^3\Psi_X^3\underline{h}_{1_X} - 3\rho^2\int Q^3\Psi_X^3\underline{h}_{1_X} = O(|\rho|^3\,\|\underline{h}\|_{H^1} + |\rho|^2\,\|\underline{h}\|_{H^1}),
\end{aligned}
$$

and

$$
-\frac{\rho^4}{4}\int Q^4\Psi_X^4 - \rho^3\int Q^4\Psi_X^4 - \frac{3\rho^2}{2}\int Q^4\Psi_X^4 = O(|\rho|^4 + |\rho^2|),
$$

we obtain

$$
B^2_{NL} = -\frac{1}{2}\int Q^2\Psi_X^2(3\underline{h}^2_{1_X} + \underline{h}^2_{2_X}) - \int Q\Psi_X|\underline{h}_X|^2\underline{h}_{1_X} - \frac{1}{4}\int |\underline{h}|^4 - 3\rho\int Q\Psi_X\underline{h}_{1_X}
$$

$$+ O\left(|\rho| \, \|\underline{h}\|_{H^1}^2 + + |\rho| \frac{e^{-|X(t)|}}{|X(t)|} \, \|\underline{h}\|_{H^1} + |\rho|^2\right).$$

Thus,

$$\begin{aligned} B_L(\underline{h}) + B^1_{NL}(\underline{h}) + B^2_{NL}(\underline{h}) &= \frac{1}{2}\int |\underline{h}|^2 - \frac{1}{2}\int Q^2\Psi_X^2(3\underline{h}_{1_X}^2 + \underline{h}_{2_X}^2) + \frac{1}{2}\int |\nabla \underline{h}|^2 \\ &\quad - \frac{1}{4}\int |\underline{h}|^4 - \int Q\Psi_X |\underline{h}_X|^2 \underline{h}_{1_X} - 2\rho \int Q\Psi_X \underline{h}_{1_X} \\ &= O\left(|\rho| \, \|\underline{h}\|_{H^1}^2 + |\rho|^2 + \frac{e^{-2|X(t)|}}{|X(t)|^2} + \frac{e^{-|X(t)|}}{|X(t)|} \, \|\underline{h}\|_{H^1}\right). \end{aligned} \tag{3.21}$$

Recall that, from (2.12) we have

$$\Phi_X(\underline{h}) = \frac{1}{2}\int |\nabla \underline{h}|^2 - \frac{1}{2}\int Q^2\Psi_X^2(3\underline{h}_1^2 + \underline{h}_2^2) + \frac{1}{2}\int |\underline{h}|^2.$$

By (3.21), one can see that,

$$\begin{aligned} \Phi_X(\underline{h}_X) &= \frac{1}{4}\int |\underline{h}|^4 + \int Q\Psi_X |\underline{h}_X|^2 \underline{h}_{1_X} + 2\rho \int Q\Psi_X \underline{h}_{1_X} \\ &\quad + O\left(|\rho| \, \|\underline{h}\|_{H^1}^2 + |\rho|^2 + \frac{e^{-2|X(t)|}}{|X(t)|^2} + \frac{e^{-|X(t)|}}{|X(t)|} \, \|\underline{h}\|_{H^1}\right). \end{aligned}$$

Thus,

$$|\Phi_X(\underline{h}_X)| \leq C\left(\|\underline{h}\|_{H^1}^3 + 2|\rho| \left|\int Q\Psi_X \underline{h}_{1_X}\right| + |\rho|^2 + \frac{e^{-2|X(t)|}}{|X(t)|^2} + \frac{e^{-|X(t)|}}{|X(t)|} \, \|\underline{h}\|_{H^1}\right).$$

By the coercivity property (2.15), we obtain

$$\|\underline{h}\|_{H^1} = O\left(|\rho| + \tilde{\delta}^{\frac{3}{2}} + \frac{e^{-|X(t)|}}{|X(t)|} + \left|\int Q\Psi_X \underline{h}_{1_X}\right|\right).$$

By (3.15), we deduce

$$\|h\|_{H_0^1(\Omega)} = \|\underline{h}\|_{H^1(\mathbb{R}^3)} = O\left(|\rho| + \tilde{\delta}^{\frac{3}{2}} + \frac{e^{-|X(t)|}}{|X(t)|}\right), \tag{3.22}$$

and thus, by (3.16), we get

$$\tilde{\delta} = O\left(|\rho| + \frac{e^{-|X(t)|}}{|X(t)|}\right),$$

which implies (3.13) and concludes the proof of Proposition 3.3. □

Lemma 3.4. *Under the assumptions of Proposition 3.3, for all $t \in D_{\delta_0}$, we have*

$$|\rho'(t)| + |X'(t)| + |\theta'(t)| = O\left(\delta + \frac{e^{-|X(t)|}}{|X(t)|}\right). \tag{3.23}$$

Proof. Let $\delta^*(t) := \delta(t) + |\rho'(t)| + |X'(t)| + |\theta'(t)|$. Using the NLS_Ω equation, Lemma 2.3, Proposition 3.3 and the Sobolev embedding $H_0^1(\Omega) \subset L^6(\Omega)$, we obtain

$$\begin{aligned} &i\partial_t h + \Delta h + i\rho' Q_{-X}\Psi - iX' \cdot \nabla Q_{-X}\Psi - \theta' Q_{-X}\Psi \\ &\qquad = O\left(\delta + \frac{e^{-|X(t)|}}{|X(t)|} + \delta^*(\delta + \frac{e^{-|X(t)|}}{|X(t)|})\right) \text{ in } L^2. \end{aligned} \tag{3.24}$$

By the orthogonality conditions (3.10), (3.11), (3.12) and Proposition 3.3, we have

$$\operatorname{Im} \int_\Omega \partial_t h\, Q_{-X}\Psi dx = \operatorname{Im} \int_\Omega h\, X' \cdot \nabla Q_{-X}\Psi\, dx = O\left(\delta^*(\delta + \frac{e^{-|X(t)|}}{|X(t)|})\right), \tag{3.25}$$

$$\begin{aligned} \operatorname{Re} \int_\Omega \partial_t h\, \partial_{x_k}(Q_{-X}\Psi) dx &= \sum_{j=1}^{3} \operatorname{Re} \int_\Omega h\, X_j'(\partial_{x_k}(\partial_{x_j} Q\Psi))\, dx \\ &= O\left(\delta^*(\delta + \frac{e^{-|X(t)|}}{|X(t)|})\right), \quad k = 1,2,3, \end{aligned} \tag{3.26}$$

$$\operatorname{Re} \int_\Omega \partial_t h\, \Delta(Q_{-X}\Psi) dx = \sum_{j=1}^{3} \operatorname{Re} \int_\Omega h\, X_j' \Delta(\partial_{x_j} Q_{-X}\Psi)\, dx = O\left(\delta^*(\delta + \frac{e^{-|X(t)|}}{|X(t)|})\right). \tag{3.27}$$

Multiplying (3.24) by $Q_{-X}\Psi$, integrating the real part, using (3.25) and then integrating by parts, we get

$$|\theta'| = O\left(\delta + \frac{e^{-|X(t)|}}{|X(t)|} + \delta^*(\delta + \frac{e^{-|X(t)|}}{|X(t)|})\right). \tag{3.28}$$

Similarly, multiplying (3.24) by $\partial_{x_j}(Q_{-X}\Psi)$, $j \in 1, 2, 3$, integrating the imaginary part, using (3.26) and Proposition 3.3, we obtain

$$|X_j'(t)| = O\left(\delta + \frac{e^{-|X(t)|}}{|X(t)|} + \delta^*(\delta + \frac{e^{-|X(t)|}}{|X(t)|})\right), \qquad j = 1,2,3. \tag{3.29}$$

Multiplying (3.24) by $\Delta(Q_{-X}\Psi)$, integrating the imaginary part, and using (3.27) and Proposition 3.3, we get

$$|\rho'| = O\left(\delta + \frac{e^{-|X(t)|}}{|X(t)|} + \delta^*(\delta + \frac{e^{-|X(t)|}}{|X(t)|})\right). \tag{3.30}$$

Summing up (3.28), (3.29) and (3.30), we obtain

$$\delta^* = O\left(\delta + \frac{e^{-|X(t)|}}{|X(t)|} + \delta^*(\delta + \frac{e^{-|X(t)|}}{|X(t)|})\right),$$

which conclude the proof by choosing δ_0 sufficiently small. □

4. Scattering

In this section, we prove Theorem 1. We start by proving, in §4.1, that the extension $\underline{u}$ of a non-scattering solution $u(t)$ to the NLS_Ω equation, satisfying (1.2) and (1.3), is compact in $H^1(\mathbb{R}^3)$, up to a spatial translation parameter $x(t)$. In §4.2, we prove that $x(t)$ is bounded using an auxiliary translation parameter (obtained by ignoring the obstacle), a local virial identity and the estimates from Section 3 for the modulation parameters. In §4.3, we prove that the parameter $\delta(t)$ converges to 0 in mean. Finally, combining the compactness properties with the control of the space translation parameter $x(t)$ and the convergence in mean, we obtain a contradiction from the existence of a non-scattering solution, thus, concluding the proof of Theorem 1.

4.1. Compactness properties

Proposition 4.1. *Let $u(t)$ be a solution of (NLS_Ω) such that*

$$M[u] = M_{\mathbb{R}^3}[Q], \quad E[u] = E_{\mathbb{R}^3}[Q] \quad \text{and} \quad \|u_0\|_{L^2(\Omega)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)}, \tag{4.1}$$

which does not scatters in positive time. Then there exists a continuous function $x(t)$ such that

$$K = \{\underline{u}(x + x(t), t),\, t \in [0, +\infty)\} \tag{4.2}$$

has a compact closure in $H^1(\mathbb{R}^3)$.

Proof. We first recall that it is sufficient to show that for every sequence of time $\tau_n \geq 0$, there exists (extracting if necessary) a sequence $(x_n)_n$ such that $u(x+x_n, \tau_n)$ has a limit in $H^1_0(\Omega)$. This fact is proved in the case $\Omega = \mathbb{R}^3$ in the appendix of [6]. We give a proof in Appendix B for the sake of completeness.

By the profile decomposition in Theorem 2.10, we have

$$u_n := u(x, \tau_n) = \sum_{j=1}^{J} \phi_n^j(x) + \omega_n^J(x), \tag{4.3}$$

where ϕ_n^j are defined in Theorem 2.10, and ω_n^J satisfies (2.19). We need to show that $J^* = 1, \omega_n^1 \to 0$ in $H^1_0(\Omega)$, and $t_n^j \equiv 0$. By the Pythagorean expansion properties of the profile decomposition we have

$$\sum_{j=1}^{J} \lim_{n\to\infty} M[\phi_n^j] + \lim_{n\to\infty} M[\omega_n^J] = \lim_{n\to\infty} M[u_n] = M[Q], \tag{4.4}$$

$$\sum_{j=1}^{J} \lim_{n\to\infty} E[\phi_n^j] + \lim_{n\to\infty} E[\omega_n^J] = \lim_{n\to\infty} E[u_n] = E[Q]. \tag{4.5}$$

We consider two possibilities.
Scenario I: More than one profile are nonzero, i.e., $J^* \geq 2$. Thus, there exists an $\varepsilon > 0$ such that for all j,

$$M[\phi_n^j]E[\phi_n^j] \leq M_{\mathbb{R}^3}[Q]E_{\mathbb{R}^3}[Q] - \varepsilon, \tag{4.6}$$

$$\left\|\phi_n^j\right\|_{L^2(\Omega)} \left\|\nabla\phi_n^j\right\|_{L^2(\Omega)} \leq \|Q\|_{L^2(\mathbb{R}^3} \|\nabla Q\|_{L^2(\mathbb{R}^3)} - \varepsilon. \tag{4.7}$$

Recall that by [21, Theorem 3.2], if $v_0 \in H_0^1(\Omega)$ satisfies

$$\|v_0\|_{L^2(\Omega)} \|\nabla v_0\|_{L^2(\Omega)} < \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}, \tag{4.8}$$

$$M[v_0]E[v_0] < M_{\mathbb{R}^3}[Q]E_{\mathbb{R}^3}[Q], \tag{4.9}$$

then the corresponding solution $v(t)$ of (NLS_Ω) scatters in both time directions.

• Suppose j is as in Case 1 (Theorem 2.10), i.e., $x_n^j = 0$ for all n:
When $t_n^j \equiv 0$, we define v^j as the solution to (NLS_Ω) with initial data $v^j(0) = \phi^j$.
When $t_n^j \to \pm\infty$, we define v^j as the solution to (NLS_Ω), which scatters to $e^{it\Delta_\Omega}\phi^j$ as $t \to \pm\infty$:

$$\lim_{t\to\pm\infty} \left\|v^j(t) - e^{it\Delta_\Omega}\phi^j\right\|_{H_0^1(\Omega)} = 0.$$

In both cases, we have

$$\lim_{n\to\infty} \left\|v^j(t_n^j) - \phi_n^j\right\|_{H_0^1(\Omega)} = 0. \tag{4.10}$$

Thus, by (4.6) and (4.7), v^j satisfies (4.8) and (4.9), and we see that v^j is a global solution with finite scattering size. Therefore, we can approximate v^j in $L^5H^{1,\frac{30}{11}}(\mathbb{R}\times\Omega)$ by $C_c^\infty(\mathbb{R}\times\mathbb{R}^3)$ functions. More precisely, for any $\varepsilon > 0$, there exists $\psi_\varepsilon^j \in C_c^\infty(\mathbb{R}\times\mathbb{R}^3)$ such that

$$\left\|v^j - \psi_\varepsilon^j\right\|_{L^5H^{1,\frac{30}{11}}(\mathbb{R}\times\Omega)} \leq \frac{\varepsilon}{2}.$$

Let $v_n^j(t,x) = v^j(t + t_n^j, x)$. Then from above v_n^j is a global and scattering solution and by changing variables in time, for any $\varepsilon > 0$, there exists $\psi_\varepsilon^j \in C_c^\infty(\mathbb{R}\times\mathbb{R}^3)$ such that, for n sufficiently large, we have

$$\left\|v_n^j(t,x) - \psi_\varepsilon^j(t + t_n^j, x)\right\|_{L^5H^{1,\frac{30}{11}}(\mathbb{R}\times\Omega)} < \varepsilon. \tag{4.11}$$

• Suppose j is as in Case 2 (Theorem 2.10):

We apply Theorem 2.11 to obtain a global solution v_n^j with $v_n^j(0) = \phi_n^j$. Furthermore, this solution has finite scattering size and satisfies, for n sufficiently large,

$$\left\| \underline{v}_n^j(t,x) - \psi_\varepsilon^j(t + t_n^j, x - x_n^j) \right\|_{L^5 H^{1,\frac{30}{11}}(\mathbb{R}\times\mathbb{R}^3)} < \varepsilon. \tag{4.12}$$

In all cases, we can find $\psi_\varepsilon^j \in C_c^\infty$ such that (4.12) holds, and there exists $C_j > 0$, independent of n, such that

$$\|v_n^j\|_{X^1(\mathbb{R}\times\Omega)} \leq C_j\,. \tag{4.13}$$

Note that for large j, by the small data theory, we have $\|v_n^j\|_{X^1(\mathbb{R}\times\Omega)} \lesssim \|\phi_n^j\|_{H_0^1(\Omega)}$.

Combining this with (4.4), (4.5), we deduce

$$\limsup_{n\to+\infty} \sum_{j=1}^{J} \left\| v_n^j \right\|^2_{X^1(\mathbb{R}\times\Omega)} \leq C \quad \text{uniformly for finite } J \leq J^\star. \tag{4.14}$$

We first prove the asymptotic decoupling of the nonlinear profile, using the orthogonality properties (2.22).

Lemma 4.2 *(Decoupling of nonlinear profiles). For $k \neq j$, we have*

$$\lim_{n\to+\infty} \left\| v_n^j v_n^k \right\|_{L^{\frac{5}{2}} H_0^{1,\frac{15}{11}}(\mathbb{R}\times\Omega)} + \left\| \nabla v_n^j \nabla v_n^k \right\|_{L^{\frac{5}{2}} L^{\frac{15}{11}}(\mathbb{R}\times\Omega)} + \left\| v_n^j v_n^k \right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}(\mathbb{R}\times\Omega)} + \left\| \nabla v_n^j v_n^k \right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}(\mathbb{R}\times\Omega)} = 0. \tag{4.15}$$

Proof. We only prove $\|v_n^j v_n^k\|_{L^{\frac{5}{2}} H_0^{1,\frac{15}{11}}(\mathbb{R}\times\Omega)} + \|v_n^j v_n^k\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}(\mathbb{R}\times\Omega)} = o_n(1)$. The other proofs are analogous. Recall that by (4.12), for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ and $\psi_\varepsilon^j, \psi_\varepsilon^k \in C_c^\infty(\mathbb{R}\times\mathbb{R}^3)$ such that for all $n \geq \mathrm{N}_\varepsilon$ we have

$$\left\| \underline{v}_n^k(t,x) - \psi_\varepsilon^k(t + t_n^k, x - x_n^k) \right\|_{L^5 H^{1,\frac{30}{11}}(\mathbb{R}\times\mathbb{R}^3)} + \left\| \underline{v}_n^j(t,x) - \psi_\varepsilon^j(t + t_n^j, x - x_n^j) \right\|_{L^5 H^{1,\frac{30}{11}}(\mathbb{R}\times\mathbb{R}^3)} < \varepsilon. \tag{4.16}$$

Using (2.22), one can see that the supports of $\psi_\varepsilon^j(t,x)$ and $\psi_\varepsilon^k(\cdot + t_n^k - t_n^j, \cdot - x_n^k + x_n^j)$ are disjoint for n sufficiently large (if j,k as in Case 1, then $\psi_\varepsilon^j(\cdot,\cdot)$ and $\psi_\varepsilon^k(\cdot + t_n^k - t_n^j, \cdot)$ have disjoint time supports), and similarly, for the derivatives. Hence,

$$\lim_{n\to+\infty} \left\| \psi_\varepsilon^j(t,x)\, \psi_\varepsilon^k(\cdot + t_n^k - t_n^j, \cdot - x_n^k + x_n^j) \right\|_{L^{\frac{5}{2}} H^{1,\frac{15}{11}}(\mathbb{R}\times\mathbb{R}^3)} = 0, \tag{4.17}$$

$$\lim_{n\to+\infty} \left\| \psi_\varepsilon^j(t,x)\, \psi_\varepsilon^k(\cdot + t_n^k - t_n^j, \cdot - x_n^k + x_n^j) \right\|_{L^{\frac{5}{2}} H^{1,\frac{30}{17}}(\mathbb{R}\times\mathbb{R}^3)} = 0. \tag{4.18}$$

Combining (4.16), (4.17) and (4.13), we have

$$\begin{aligned}\left\|v_n^j v_n^k\right\|_{L^{\frac{5}{2}} H_0^{1,\frac{15}{11}}(\mathbb{R}\times\Omega)} &\leq \left\|\underline{v}_n^j - \psi_\varepsilon^j(\cdot + t_n^j, \cdot - x_n^j)\right\|_{L^5 H^{1,\frac{30}{11}}(\mathbb{R}\times\mathbb{R}^3)} \left\|\underline{v}_n^k\right\|_{L^5 H^{1,\frac{30}{11}}(\mathbb{R}\times\mathbb{R}^3)} \\ &+ \left\|\psi_\varepsilon^j\right\|_{L^5 H^{1,\frac{30}{11}}(\mathbb{R}\times\mathbb{R}^3)} \left\|\underline{v}_n^k - \psi_\varepsilon^k(\cdot + t_n^k, \cdot - x_n^k)\right\|_{L^5 H^{1,\frac{30}{11}}(\mathbb{R}\times\mathbb{R}^3)} \\ &+ \left\|\psi_\varepsilon^j(t,x)\, \psi_\varepsilon^k(\cdot + t_n^k - t_n^j, \cdot - x_n^k + x_n^j)\right\|_{L^{\frac{5}{2}} H^{1,\frac{15}{11}}(\mathbb{R}\times\mathbb{R}^3)} \leq C\varepsilon,\end{aligned}$$

provided n is large enough, since the last term goes to 0 as n goes to infinity.

Next, we estimate $\left\|v_n^j v_n^k\right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}(\mathbb{R}\times\Omega)}$ as follows

$$\begin{aligned}\left\|v_n^j v_n^k\right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}(\mathbb{R}\times\Omega)} &\leq \left\|\underline{v}_n^j - \psi_\varepsilon^j(\cdot + t_n^j, \cdot - x_n^j)\right\|_{L^5_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \left\|v_n^k\right\|_{L^5 L^{\frac{30}{11}}(\mathbb{R}\times\mathbb{R}^3)} \\ &+ \left\|\psi_\varepsilon^j\right\|_{L^5 L^{\frac{30}{11}}} \left\|\underline{v}_n^k - \psi_\varepsilon^k(\cdot + t_n^k, \cdot - x_n^k)\right\|_{L^5_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \\ &+ \left\|\psi_\varepsilon^j(t,x)\, \psi_\varepsilon^k(\cdot + t_n^k - t_n^j, \cdot - x_n^k + x_n^j)\right\|_{L^{\frac{5}{2}} H^{1,\frac{30}{17}}(\mathbb{R}\times\mathbb{R}^3)}.\end{aligned}$$

Using (4.16), (4.18) and (4.13) and Sobolev embedding $\|\cdot\|_{L^5_{t,x}} \leq C \|\cdot\|_{L^5 H^{1,\frac{30}{11}}}$, we obtain that, for large n,

$$\left\|v_n^j v_n^k\right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}(\mathbb{R}\times\Omega)} \leq C\varepsilon,$$

provided n is large enough, which concludes the proof of Lemma 4.2. □

We return to the proof of Proposition 4.1. As a consequence of the asymptotic decoupling of the nonlinear profile in Lemma 4.2, we have

$$\limsup_{n\to\infty} \| \sum_{j=1}^{J} v_n^j \|_{X^1(\mathbb{R}\times\Omega)} \leq C \tag{4.19}$$

uniformly for finite $J \leq J^*$. Indeed, by (4.14) and (4.15) we obtain

$$\begin{aligned}\left\|\sum_{j=1}^{J} v_n^j\right\|^2_{L^5 L^{\frac{30}{11}}(\mathbb{R}\times\Omega)} &= \left\|\left(\sum_{j=1}^{J} v_n^j\right)^2\right\|_{L^{\frac{5}{2}} L^{\frac{15}{11}}(\mathbb{R}\times\Omega)} \\ &\leq \sum_{j=1}^{J} \left\|v_n^j\right\|^2_{L^5_t L_x^{\frac{30}{11}}(\mathbb{R}\times\Omega)} + C(J) \sum_{j\neq k} \left\|v_n^j v_n^k\right\|_{L^{\frac{5}{2}} L^{\frac{15}{11}}(\mathbb{R}\times\Omega)} \\ &\leq C + o_n(1).\end{aligned}$$

Similarly,

$$\left\|\sum_{j=1}^{J}\nabla v_n^j\right\|_{L^5L^{\frac{30}{11}}(\mathbb{R}\times\Omega)}^2=\left\|\left(\sum_{j=1}^{J}\nabla v_n^j\right)^2\right\|_{L^{\frac{5}{2}}L^{\frac{15}{11}}(\mathbb{R}\times\Omega)}$$

$$\leq\sum_{j=1}^{J}\left\|\nabla v_n^j\right\|_{L_t^5L^{\frac{30}{11}}(\mathbb{R}\times\Omega)}^2+C(J)\sum_{j\neq k}\left\|\nabla v_n^j\nabla v_n^k\right\|_{L^{\frac{5}{2}}L^{\frac{15}{11}}(\mathbb{R}\times\Omega)}\leq C.$$

This completes the proof of (4.19). Using similar argument, one can check that for given $\eta>0$, there exists $J':=J'(\eta)$ such that

$$\forall J\geq J',\quad \limsup_{n\to\infty}\|\sum_{j=J'}^{J}v_n^j\|_{X^1(\mathbb{R}\times\Omega)}\leq\eta. \tag{4.20}$$

For each n and J, we define an approximate solution u_n^J to (NLS$_\Omega$) by

$$u_n^J=\sum_{j=1}^{J}v_n^j+e^{it\Delta_\Omega}\omega_n^J. \tag{4.21}$$

Before continuing with the rest of the proof of Proposition 4.1, we claim that the following statements hold true.

Claim 4.3.

$$\lim_{n\to\infty}\left\|u_n^J(0)-u_n(0)\right\|_{H_0^1(\Omega)}=0.$$

Claim 4.4.

$$\exists C>0,\ \forall J,\quad \limsup_{n\to\infty}\left\|u_n^J\right\|_{X^1(\mathbb{R}\times\Omega)}\leq C\,.$$

Claim 4.5.

$$\lim_{J\to J^*}\limsup_{n\to\infty}\left\|i\partial_t u_n^J+\Delta_\Omega u_n^J+\left|u_n^J\right|^2u_n^J\right\|_{N^1(\mathbb{R})}=0,$$

with N^1 defined in (2.16).

Applying Lemma 2.9, we get that u_n is a global solution with finite scattering size, which yields a contradiction by showing that there is only one profile. Hence, Scenario I cannot occur.

Proof of Claim 4.3. Using (4.10), if j is as in Case 1, or the fact that $v_n^j(0)=\phi_n^j$ if j is as in Case 2, together with the decomposition of u_n in (4.3) and u_n^J in (4.21), we obtain

$$\left\|u_n^J(0) - u_n(0)\right\|_{H_0^1(\Omega)} \leq \sum_{j=1}^{J} \left\|v_n^j(0) - \phi_n^j\right\|_{H_0^1(\Omega)} \longrightarrow 0 \text{ as } n \to \infty. \quad \square \tag{4.22}$$

Proof of Claim 4.4. Using (4.19), Strichartz estimate (2.18) with (2.19), we obtain

$$\limsup_{n\to\infty} \left\|u_n^J\right\|_{X^1(\mathbb{R}\times\Omega)} \leq \limsup_{n\to\infty} \Big\|\sum_{j=1}^{J} v_n^j\Big\|_{X^1(\mathbb{R}\times\Omega)} + \limsup_{n\to+\infty} \left\|\omega_n^J\right\|_{H_0^1(\Omega)} \leq C. \quad \square$$

Proof of Claim 4.5. Let $F(z) = -|z|^2 z$, recall $\sum_{j=1}^J v_n^j = u_n^J - e^{it\Delta_\Omega}\omega_n^J$, and write

$$\begin{aligned}(i\partial_t + \Delta_\Omega)u_n^J - F(u_n^J) &= \sum_{j=1}^{J} F(v_n^j) - F(u_n^J)\\ &= \sum_{j=1}^{J} F(v_n^j) - F\Big(\sum_{j=1}^{J} v_n^j\Big) + F(u_n^J - e^{it\Delta_\Omega}\omega_n^J) - F(u_n^J).\end{aligned}$$

We have

$$\left|\sum_{j=1}^{J} F(v_n^j) - F\Big(\sum_{j=1}^{J} v_n^j\Big)\right| \leq C \sum_{j\neq k} |v_n^j|^2 |v_n^k|. \tag{4.23}$$

Taking the derivatives, we get

$$\left|\nabla\left\{\sum_{j=1}^{J} F(v_n^j) - F\Big(\sum_{j=1}^{J} v_n^j\Big)\right\}\right| \leq C \sum_{j\neq k} |\nabla v_n^j||v_n^j||v_n^k| + C\sum_{j\neq k} |v_n^j|^2 |\nabla v_n^k|,$$

which yields

$$\left\|\sum_{j=1}^{J} F(v_n^j) - F\Big(\sum_{j=1}^{J} v_n^j\Big)\right\|_{L^{\frac{5}{3}}L^{\frac{30}{23}}} \leq C\left(\sum_{j\neq k} \left\|v_n^j\right\|_{L^5_{t,x}} \left\|v_n^j v_n^k\right\|_{L^{\frac{5}{2}}L^{\frac{30}{17}}}\right),$$

$$\begin{aligned}&\left\|\nabla\left\{\sum_{j=1}^{J} F(v_n^j) - F\Big(\sum_{j=1}^{J} v_n^j\Big)\right\}\right\|_{L^{\frac{5}{3}}L^{\frac{30}{23}}}\\ &\leq C\Big(\sum_{j\neq k} \left\|v_n^j v_n^k \nabla v_n^j\right\|_{L^{\frac{5}{3}}L^{\frac{30}{23}}} + \sum_{j\neq k} \left\||v_n^j|^2 \nabla v_n^k\right\|_{L^{\frac{5}{3}}L^{\frac{30}{23}}}\Big)\\ &\leq C\sum_{j\neq k} \left\|v_n^j\right\|_{L^5_{t,x}} \Big(\left\|\nabla v_n^j v_n^k\right\|_{L^{\frac{5}{2}}L^{\frac{30}{17}}} + \left\|v_n^j \nabla v_n^k\right\|_{L^{\frac{5}{2}}L^{\frac{30}{17}}}\Big),\end{aligned}$$

which goes to 0 as $n \to \infty$, in view of Lemma 4.2 and (4.13).

In addition,

$$\left\|F(u_n^J - e^{it\Delta_\Omega}\omega_n^J) - F(u_n^J)\right\|_{L^{\frac{5}{3}}H^{1,\frac{30}{23}}} \leq \left\|F(u_n^J - e^{it\Delta_\Omega}\omega_n^J) - F(u_n^J)\right\|_{L^{\frac{5}{3}}L^{\frac{30}{23}}} \tag{4.24}$$

$$+ \left\|\nabla\left(F(u_n^J - e^{it\Delta_\Omega}\omega_n^J) - F(u_n^J)\right)\right\|_{L^{\frac{5}{3}}L^{\frac{30}{23}}}. \tag{4.25}$$

We estimate the differences as

$$\left|F(u_n^J - e^{it\Delta_\Omega}\omega_n^J) - F(u_n^J)\right| \leq C\left(\left|e^{it\Delta_\Omega}\omega_n^J\right|^2\left|e^{it\Delta_\Omega}\omega_n^J\right| + \left|u_n^J\right|^2\left|e^{it\Delta_\Omega}\omega_n^J\right|\right),$$

$$\begin{aligned}\left|\nabla\left\{F(u_n^J - e^{it\Delta_\Omega}\omega_n^J) - F(u_n^J)\right\}\right| \leq C\Big(&\left|e^{it\Delta_\Omega}\omega_n^J\right|^2\left|\nabla e^{it\Delta_\Omega}\omega_n^J\right| + \left|\nabla u_n^J\right|\left|u_n^J\right|\left|e^{it\Delta_\Omega}\omega_n^J\right| \\ &+ \left|\nabla u_n^J\right|\left|\nabla e^{it\Delta_\Omega}\omega_n^J\right|^2 + \left|u_n^J\right|^2\left|\nabla e^{it\Delta_\Omega}\omega_n^J\right|\Big).\end{aligned}$$

Using Claim 4.4, Hölder and Sobolev inequalities, we get

$$\begin{aligned}(4.24) &\leq \left\|e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5_{t,x}}\left[\left\|u_n^J\right\|_{L^5L^{\frac{30}{11}}}\left\|u_n^J\right\|_{L^5_{t,x}} + \left\|e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5L^{\frac{30}{11}}}\left\|e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5_{t,x}}\right] \\ &\leq \left\|e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5_{t,x}}\left[\left\|u_n^J\right\|_{X^1}^2 + \left\|e^{it\Delta_\Omega}\omega_n^J\right\|_{X^1}^2\right] + \left\|e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5_{t,x}}^2\left\|u_n^J\right\|_{X^1} \\ &\leq C\left\|e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5_{t,x}},\end{aligned}$$

which converges to 0 as $n \to \infty$ and $J \to \infty$. Similarly,

$$\begin{aligned}(4.25) \leq\ &\left\|\nabla u_n^J u_n^J\right\|_{L^{\frac{5}{2}}L^{\frac{30}{17}}}\left\|e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5_{t,x}} + \left\|\nabla u_n^J\right\|_{L^5L^{\frac{30}{11}}}\left\|\left|e^{it\Delta_\Omega}\omega_n^J\right|^2\right\|_{L^{\frac{5}{2}}_{t,x}} \\ &+ \left\|\nabla(e^{it\Delta_\Omega}\omega_n^J)\right\|_{L^5L^{\frac{30}{11}}}\left\|\left|e^{it\Delta_\Omega}\omega_n^J\right|^2\right\|_{L^{\frac{5}{2}}_{t,x}} + \left\|u_n^J\right\|_{L^5_{t,x}}\left\|u_n^J\nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^{\frac{5}{2}}L^{\frac{30}{17}}} \\ \leq\ &\left\|\nabla u_n^J\right\|_{L^5L^{\frac{30}{11}}}\left[\left\|u_n^J\right\|_{L^5_{t,x}}\left\|e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5_{t,x}} + \left\|e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5_{t,x}}^2\right] \\ &+ \left\|\nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5L^{\frac{30}{11}}}\left\|e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5_{t,x}}^2 + \left\|u_n^J\right\|_{L^5_{t,x}}\left\|u_n^J\nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^{\frac{5}{2}}L^{\frac{30}{17}}}.\end{aligned}$$

Thus, it remains to show that

$$\lim_{J\to\infty}\limsup_{n\to\infty}\left\|u_n^J\nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^{\frac{5}{2}}L^{\frac{30}{17}}} = 0. \tag{4.26}$$

Recall that $u_n^J = \sum_{j=1}^J v_n^j + e^{it\Delta_\Omega}\omega_n^J$. Then

$$\left\|u_n^J\nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^{\frac{5}{2}}L^{\frac{30}{17}}} \leq \left\|\sum_{j=1}^J v_n^j\nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^{\frac{5}{2}}L^{\frac{30}{17}}} + \left\|e^{it\Delta_\Omega}\omega_n^J\nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^{\frac{5}{2}}L^{\frac{30}{17}}}$$

$$\leq \left\| \sum_{j=1}^{J} v_n^j \nabla e^{it\Delta_\Omega} \omega_n^J \right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}} + \left\| e^{it\Delta_\Omega} \omega_n^J \right\|_{L^5_{t,x}} \left\| \nabla e^{it\Delta_\Omega} \omega_n^J \right\|_{L^5 L^{\frac{30}{11}}}.$$

Hence, Claim 4.5 holds if

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left\| \sum_{j=1}^{J} v_n^j \nabla e^{it\Delta_\Omega} \omega_n^J \right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}} = 0.$$

From (4.20), we have $\forall \eta > 0, \exists J' = J'(\eta)$ such that

$$\forall J \geq J', \quad \limsup_{n\to\infty} \left\| \sum_{j=J'}^{J} v_n^j \right\|_{X^1} < \eta.$$

Thus, we have

$$\limsup_{n\to\infty} \left\| \left(\sum_{j=J'}^{J} v_n^j \right) \nabla e^{it\Delta_\Omega} \omega_n^J \right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}} \leq \limsup_{n\to\infty} \left\| \sum_{j=J'}^{J} v_n^j \right\|_{X^1} \left\| \nabla e^{it\Delta_\Omega} \omega_n^J \right\|_{L^5_{t,x}} \leq \eta,$$

where η is arbitrary and $J' = J'(\eta)$ as in (4.20). Thus, to prove (4.26) it suffices to show that

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left\| v_n^j \nabla e^{it\Delta_\Omega} \omega_n^J \right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}} = 0 \quad \text{for all } 1 \leq j \leq J'. \tag{4.27}$$

We approximate v_n^j by $C_c^\infty(R\times\mathbb{R}^3)$ functions ψ_ε^j obeying (4.12) with support in $[-T,T]\times\{|x| \leq R\}$. From Proposition 2.8 and (2.19), we deduce

$$\begin{aligned}
\left\| v_n^j \nabla e^{it\Delta_\Omega} \omega_n^J \right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}} &\leq \left\| v_n^j - \psi_\varepsilon^j(\cdot + t_n^j, \cdot - x_n^j) \right\|_{L^5_{t,x}} \left\| \nabla e^{it\Delta_\Omega} \omega_n^J \right\|_{L^5 L^{\frac{30}{11}}} \\
&\quad + \left\| \psi_\varepsilon^j \right\|_{L^\infty_{t,x}} \left\| \nabla e^{it\Delta_\Omega} \omega_n^J \right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}(\{|t|\leq T,\, |x|\leq R\})} \\
&\leq C\varepsilon + C R^{\frac{31}{60}} T^{\frac{1}{5}} \left\| e^{it\Delta_\Omega} \omega_n^J \right\|_{L^5_{t,x}}^{\frac{1}{6}} \left\| \omega_n^J \right\|_{H_0^1(\Omega)}^{\frac{5}{6}}.
\end{aligned}$$

By taking the limit and choosing ε small, we obtain (4.26). Hence, Claim 4.5 holds. □

Returning to the proof of the Proposition 4.1, we consider the other possibility.
Scenario II: Only one nonzero profile. By (4.3)

$$u_n := u(x, \tau_n) = \phi_n^1 + \omega_n^1,$$

with

$$\lim_{n\to\infty} \left\| \omega_n^1 \right\|_{H_0^1(\Omega)} = 0. \tag{4.28}$$

If not, there exists $\varepsilon > 0$ such that $\forall n$,

$$E[\phi_n^1]M[\phi_n^1] \leq E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q] - \varepsilon,$$

and one can show by the previous argument that u scatters in $H_0^1(\Omega)$.

It remains to show that t_n^1 is bounded and this will prove the convergence, up to a subsequence.

- If $t_n^1 \to +\infty$ (similarly, $t_n^1 \to -\infty$) and ϕ_n^1 conforms to Case 1, i.e., $\phi_n^1 = e^{it_n^1\Delta_\Omega}\phi^1$,

$$\begin{aligned}\left\|e^{it\Delta_\Omega}u_n\right\|_{L^5_{t,x}([0,+\infty)\times\Omega)} &= \left\|e^{it\Delta_\Omega}\phi_n^1 + e^{it\Delta_\Omega}\omega_n^1\right\|_{L^5_{t,x}([0,+\infty)\times\Omega)} \\ &\leq \left\|e^{i(t+t_n^1)\Delta_\Omega}\phi^1\right\|_{L^5_{t,x}([0,+\infty)\times\Omega)} + \left\|\omega_n^1\right\|_{H_0^1(\Omega)} \\ &\leq \left\|e^{it\Delta_\Omega}\phi^1\right\|_{L^5_{t,x}([t_n^1,+\infty)\times\Omega)} + \left\|\omega_n^1\right\|_{H_0^1(\Omega)},\end{aligned}$$

  which goes to 0 as n goes to ∞, showing that u_n scatters for positive (similarly negative) time, a contradiction.
- If $t_n^1 \to +\infty$ (similarly, $t_n^1 \to -\infty$) and ϕ_n^1 conforms to Case 2, i.e.,

$$\phi_n^1 = e^{it_n^1\Delta_\Omega}[(\chi_n^1\phi^1)(x - x_n^1)], \quad \text{where } \chi_n^1 := \chi\left(\frac{x}{|x_n^1|}\right).$$

  We first prove that

$$\lim_{n\to+\infty}\left\|e^{i\,t\Delta_{\Omega_n}}(\chi_n^1\phi^1) - e^{i\,t\Delta_{\mathbb{R}^3}}(\chi_n^1\phi^1)\right\|_{L^5_{t,x}((0,+\infty)\times\mathbb{R}^3)} = 0, \tag{4.29}$$

  where $\Omega_n := \Omega - \{x_n\}$. Indeed, by a density argument, for any $\varepsilon > 0$, there exist $\psi_\varepsilon \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ such that

$$\left\|\phi^1 - \psi_\varepsilon\right\|_{H^1(\mathbb{R}^3)} \leq \frac{\varepsilon}{4}. \tag{4.30}$$

  By the definition of χ_n, as $|x_n| \longrightarrow +\infty$, for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\forall n \geq N_\varepsilon, \quad \left\|\chi_n^1\phi^1 - \phi^1\right\|_{H^1(\mathbb{R}^3)} \leq \frac{\varepsilon}{4}. \tag{4.31}$$

  Using (4.30) and (4.31), we have

$$\forall n \geq N_\varepsilon, \quad \left\|\chi_n^1\phi^1 - \psi_\varepsilon\right\|_{H^1(\mathbb{R}^3)} \leq \frac{\varepsilon}{2}.$$

  Combining this with the Strichartz inequality, we obtain for large n

$$\left\|e^{it\Delta_{\Omega_n}}\left(\chi_n^1\phi^1 - \psi_\varepsilon\right)\right\|_{L^5_{t,x}((0,+\infty)\times\mathbb{R}^3)} + \left\|e^{it\Delta_{\mathbb{R}^3}}\left(\chi_n^1\phi^1 - \psi_\varepsilon\right)\right\|_{L^5_{t,x}((0,+\infty)\times\mathbb{R}^3)} \leq \frac{\varepsilon}{2}. \tag{4.32}$$

From [21, Proposition 2.13], as $|x_n| \longrightarrow +\infty$, we have for large n

$$\left\| e^{it\Delta_{\Omega_n}} \psi_\varepsilon - e^{it\Delta_{\mathbb{R}^3}} \psi_\varepsilon \right\|_{L^5_{t,x}((0,\infty)\times\mathbb{R}^3)} \leq \frac{\varepsilon}{2}, \tag{4.33}$$

which yields (4.29). We now have

$$\begin{aligned}
\left\| e^{it\Delta_\Omega} u_n \right\|_{L^5_{t,x}([0,+\infty)\times\Omega)} &= \left\| e^{it\Delta_\Omega} \phi_n^1 + e^{it\Delta_\Omega} \omega_n^1 \right\|_{L^5_{t,x}([0,+\infty)\times\Omega)} \\
&\leq \left\| e^{i(t+t_n^1)\Delta_\Omega} (\chi_n^1 \phi^1)(x - x_n^1) \right\|_{L^5_{t,x}([0,+\infty)\times\Omega)} + \left\| \omega_n^1 \right\|_{H_0^1(\Omega)} \\
&\leq \left\| e^{it\Delta_\Omega} (\chi_n^1 \phi^1)(x - x_n^1) \right\|_{L^5_{t,x}([t_n^1,+\infty)\times\Omega)} + \left\| \omega_n^1 \right\|_{H_0^1(\Omega)} \\
&\leq \left\| e^{i\,t\Delta_{\Omega_n}} (\chi_n^1 \phi^1) - e^{i\,t\Delta_{\mathbb{R}^3}} (\chi_n^1 \phi^1) \right\|_{L^5_{t,x}((t_n^1,+\infty)\times\mathbb{R}^3)} \\
&\quad + \left\| e^{i\,t\Delta_{\mathbb{R}^3}} (\chi_n^1 \phi^1) \right\|_{L^5_{t,x}((t_n^1,+\infty)\times\mathbb{R}^3)} + \left\| \omega_n^1 \right\|_{H_0^1(\Omega)},
\end{aligned}$$

which goes to 0 as n goes to ∞, by (4.29) and the monotone convergence theorem, showing that u_n scatters for positive (respectively, negative) time, a contradiction. This completes the proof of Proposition 4.1. □

Corollary 4.6. *Let u be as in Proposition 4.1. Then one can choose the continuous function $x(t)$ such that $X(t) = x(t)$ for all $t \in D_{\delta_0}$, and the set K has a compact closure in $H^1(\mathbb{R}^3)$.*

Proof. Recall that by the definition of D_{δ_0}, the modulation parameters $X(t), \theta(t)$ and $\alpha(t)$ are well defined for all $t \in D_{\delta_0}$. Let $x(t)$ be the translation parameter given by Proposition 4.1. Let $R_0 > 0$. Then by the decomposition of u in (3.9), Proposition 3.3 and the fact $\Psi(x) = 1$ for $|x|$ large, there exists $C_\star > 0$ such that

$$\forall t \in D_{\delta_0}, \qquad \int\limits_{|x|\leq R_0} |\nabla Q|^2 + |Q|^2 - C_\star \left(\delta(t) + \frac{e^{-|X(t)|}}{|X(t)|} \right) \leq \int\limits_{|x - X(t)|\leq R_0} |\nabla \underline{u}|^2 + |\underline{u}|^2.$$

Taking δ_0 small if necessary, there exists $\varepsilon_0 > 0$ such that

$$\forall t \in D_{\delta_0}, \qquad \int\limits_{|x + x(t) - X(t)|\leq R_0} |\nabla \underline{u}(t, x + x(t))|^2 + |\underline{u}(t, x + x(t))|^2 \geq \varepsilon_0 > 0.$$

Using the fact that K has a compact closure in $H^1(\mathbb{R}^3)$, we get that $|x(t) - X(t)|$ is bounded. Thus, one can modify $x(t)$ such that $\overline{K}$ remains compact and for all t in D_{δ_0}, $x(t) = X(t)$. □

4.2. Control of the translation parameters

Proposition 4.7. *Consider a solution u of* (NLS_Ω) *such that*

$$M[u]=M_{\mathbb{R}^3}[Q],\ E[u]=E_{\mathbb{R}^3}[Q],\ \|\nabla u_0\|_{L^2(\Omega)}<\|\nabla Q\|_{L^2(\mathbb{R}^3)} \tag{4.34}$$

and

$$K:=\{\underline{u}(t,x+x(t));t\geq 0\} \tag{4.35}$$

has a compact closure in $H^1(\mathbb{R}^3)$. Then $x(t)$ is bounded.

We start with the following lemma.

Lemma 4.8. *Let u be as in the Proposition 4.7. Let $\{t_n\}$ be a sequence of time, such that $t_n \longrightarrow +\infty$. Then $|x(t_n)| \longrightarrow +\infty$ as $n\to+\infty$, if and only if $\delta(t_n)\longrightarrow 0$ as n goes to $+\infty$.*

Proof. We first prove that $\delta(t_n)\longrightarrow 0$ implies that $|x(t_n)|\longrightarrow +\infty$ as $n\to+\infty$. If not, $x(t_n)$ converges (after extraction) to x_∞ in $\mathbb{R}^3$. By the compactness of the closure of K, $\underline{u}(t_n,\cdot+x(t_n))$ converges in $H^1(\mathbb{R}^3)$ to some $v_0(\cdot-x_\infty)\in H^1(\mathbb{R}^3)$. By the assumption (4.34) and the fact that $\delta(t_n)\longrightarrow 0$, $E_{\mathbb{R}^3}[v_0]=E_{\mathbb{R}^3}[Q], M_{\mathbb{R}^3}[v_0]=M_{\mathbb{R}^3}(Q)$ and $\|\nabla v_0\|_{L^2(\mathbb{R}^3)}=\|\nabla Q\|_{L^2(\mathbb{R}^3)}$. By Proposition 2.1, there exist $\theta_0\in\mathbb{R}$ and $x_0\in\mathbb{R}^3$ such that $v_0=e^{i\theta_0}Q(\cdot-x_0)$. On the other hand, if $x+x(t_n)\in\Omega$, then $u(t_n,x+x(t_n))$ converges in $H^1_0(\Omega)$, as $H^1_0(\Omega)$ is a close subspace of $H^1(\mathbb{R}^3)$. Thus, the restriction of $v_0(\cdot-x_\infty)$ to Ω belongs to $H^1_0(\Omega)$, which contradicts the fact that $e^{i\theta_0}Q(\cdot+x_\infty-x_0)\notin H^1_0(\Omega)$.

Next, we prove that $|x(t_n)|\longrightarrow+\infty$ as $n\to+\infty$ implies that $\delta(t_n)\longrightarrow 0$ as n goes to $+\infty$.

We argue by contradiction, assuming (after extraction) that

$$\delta(t_n)\xrightarrow[n\to+\infty]{}\delta_\infty>0 \ \text{ and } \ t_n\xrightarrow[n\to+\infty]{}t_\infty\in\mathbb{R}\cup\{\pm\infty\}.$$

By the continuity of $x(t)$, using $|x(t_n)|\to+\infty$, we must have $t_\infty\in\{\pm\infty\}$.

Assume, say, $t_\infty=+\infty$, and let $\varphi_\infty=\lim_{n\to+\infty}\underline{u}(t_n,x+x(t_n))$ in $H^1(\mathbb{R}^3)$ (after extraction). We have

$$E_{\mathbb{R}^3}[\varphi_\infty]=E_{\mathbb{R}^3}[Q],\quad M_{\mathbb{R}^3}[\varphi_\infty]=M_{\mathbb{R}^3}[Q],$$
$$\int_{\mathbb{R}^3}|\nabla\varphi_\infty|^2=\int_{\mathbb{R}^3}|\nabla Q|^2-\delta_\infty<\int_{\mathbb{R}^3}|\nabla Q|^2.$$

Let φ be the solution of $(\mathrm{NLS}_{\mathbb{R}^3})$ with the initial datum φ_∞ at $t=0$. By [8], φ is global and one of the following holds:

(1) φ scatters in both time directions.
(2) $\exists\,\tau,\theta\in\mathbb{R}$ and $\varepsilon\in\{\pm 1\}$ such that $\varphi(t)=e^{i\theta}U_-(\varepsilon\,t+\tau)$, where $U_-(t)\xrightarrow[t\to+\infty]{}Q$ and U_- scatters for negative time.

In case (1) or in the case (2) with $\varepsilon=-1$, one can prove by approximation, following the proof of Theorem 4.1 in [21], that u scatters for positive time.

In case (2) with $\varepsilon=+1$, we obtain for large n, with the same argument

$$\|u\|_{S(-\infty,t_n)}\leq C\ \|U_-\|_{S(-\infty,t_\infty)}, \qquad \text{where } C \text{ is a fixed constant.}$$

Letting n go to $+\infty$, we see that u has a finite Strichartz norm, thus, u scatters also in both time directions, which contradicts the fact that u satisfies (4.35) and (4.34). □

Lemma 4.9. *Let $X(t)$ be as in (3.8). Taking a smaller δ_0 if necessary, there exists $C>0$ such that*

$$\frac{e^{-|X(t)|}}{|X(t)|}\leq C\delta(t) \quad \textit{for any } \ t\in D_{\delta_0}. \tag{4.36}$$

Proof. Note that, by Proposition 4.1, taking a smaller δ_0 if necessary, we can assume $|X(t)|\geq C$ for an arbitrarily large constant $C>0$. The proof consists of 3 steps.

- **Step 1:** The estimate of $\delta(t)$ with respect to an auxiliary modulation parameter $X_1(t)$ on $\mathbb{R}^3$. Let $\underline{u}(t)\in H^1(\mathbb{R}^3)$ be the extension of u to $\mathbb{R}^3$ defined as in (2.8), we then have

$$M_{\mathbb{R}^3}[\underline{u}]=M_{\mathbb{R}^3}[Q],\quad E_{\mathbb{R}^3}[\underline{u}]=E_{\mathbb{R}^3}[Q],\quad \text{and}\quad \int_{\mathbb{R}^3}|\nabla\underline{u}|^2<\int_{\mathbb{R}^3}|\nabla Q|^2. \tag{4.37}$$

Arguing as in Section 3, but on the whole space $\mathbb{R}^3$, see [8, Lemma 4.1 and 4.2], there exist $\theta_1(t)$ and $X_1(t)$, C^1 functions of t, such that

$$e^{-i\theta_1(t)-it}\underline{u}(t,x+X_1(t))=(1+\rho_1(t))Q(x)+\widetilde{h}(t,x), \tag{4.38}$$

where

$$\rho_1(t)=\operatorname{Re}\frac{e^{-i\theta_1-it}\int_{\mathbb{R}^3}\nabla\underline{u}(t,x+X_1(t)).\nabla Q(x)dx}{\|\nabla Q\|^2_{L^2(\mathbb{R}^3)}}-1, \tag{4.39}$$

$$|\rho_1(t)|\approx\left|\int_{\mathbb{R}^3}Q\,\widetilde{h}\,dx\right|\approx\left\|\widetilde{h}\right\|_{H^1(\mathbb{R}^3)}\approx\delta(t). \tag{4.40}$$

In this step we prove

$$\frac{e^{-|X_1(t)|}}{|X_1(t)|} \leq C\delta(t). \tag{4.41}$$

By (4.38), $x \in \Omega^c$ implies $(1+\rho_1(t))Q(x-X_1(t)) + \widetilde{h}(t, x-X_1(t)) = 0$, i.e.,

$$\left\| (1+\rho_1(t))Q(x-X_1(t)) + \widetilde{h}(t, x-X_1(t)) \right\|_{L^2(\Omega^c)} = 0.$$

By (4.40), we have

$$\int_{\Omega^c} |Q(x-X_1(t))|^2 \, dx \leq C\, \delta(t)^2. \tag{4.42}$$

By (2.9), one can see that $|X_1(t)|$ is large. For $x \in \Omega^c$, we have

$$\frac{1}{2}|X_1(t)| \leq |x - X_1(t)| \leq 2|X_1(t)|.$$

From Lemma 2.2, we have

$$Q(x) = \frac{e^{-|x|}}{|x|}\left(a + O(\frac{1}{|x|^{\frac{1}{2}}})\right), \quad \text{for some } a > 0.$$

Using (4.42), we obtain (4.41).

- **Step 2:** Comparison of $X(t)$ and $X_1(t)$.
  We prove that there exists $C > 0$ such that

$$|X(t) - X_1(t)| \leq C \quad \forall t \in D_{\delta_0}. \tag{4.43}$$

We fix $t \in D_{\delta_0}$. We can assume

$$|X(t) - X_1(t)| \geq 1, \tag{4.44}$$

or else we are done.
Let $x \in \Omega$, by (4.38) and (3.9), we have

$$\begin{aligned} u(t,x) &= e^{i\theta(t)+it}(1+\rho(t))Q(x-X(t))\Psi(x) + e^{i\theta(t)+it}h(t,x) \\ &= e^{i\theta_1(t)+it}(1+\rho_1(t))Q(x-X_1(t)) + e^{i\theta_1(t)+it}\widetilde{h}(t,x). \end{aligned}$$

Using (4.40) and Proposition 3.3, we have

$$\int_{|x-X(t)|<1} \left| Q(x-X(t))\Psi(x)e^{i\theta(t)} - Q(x-X_1(t))e^{i\theta_1(t)} \right|^2 \leq C\left(\delta^2(t) + \frac{e^{-2|X(t)|}}{|X(t)|^2}\right).$$

Recall that $|X_1(t)|$ and $|X(t)|$ are large and $\Psi(x) = 1$ for large $|x|$.

$$\int_{|x|<1} |Q(x)|^2 dx \leq C \int_{|x-X(t)|<1} |Q(x-X_1(t))|^2 dx + C\delta^2(t) + C\,\frac{e^{-2|X(t)|}}{|X(t)|^2}$$
$$\leq \int_{|x-X(t)|<1} \frac{e^{-2|x-X_1(t)|}}{|x-X_1(t)|^2}\,dx + C\delta^2(t) + C\,\frac{e^{-2|X(t)|}}{|X(t)|^2}.$$

Using the fact that $|x - X_1(t)| \geq |X(t) - X_1(t)| - |x - X(t)| \geq |X(t) - X_1(t)| - 1$, in the support of the integral in the last line, we obtain

$$\int_{|x|<1} |Q(x)|^2 dx \leq C\frac{e^{-2|X(t)-X_1(t)|}}{|X(t)-X_1(t)|^2} + C\,\delta^2(t) + C\,\frac{e^{-2|X(t)|}}{|X(t)|^2}.$$

Recall that, by Lemma 4.8 if $|X(t)|$ is large, then $\delta(t)$ and $\frac{e^{-2|X(t)|}}{|X(t)|^2}$ are small. By (4.44), we get

$$\frac{1}{2}\int_{|x|<1} |Q(x)|^2\,dx \leq C\,\frac{e^{-2|X(t)-X_1(t)|}}{|X(t)-X_1(t)|^2} \leq Ce^{-2|X(t)-X_1(t)|},$$

which yields

$$|X(t) - X_1(t)| \leq C - \log\left(\frac{1}{2}\int_{|x|<1} |Q(x)|^2\,dx\right).$$

Thus, $|X(t) - X_1(t)|$ is bounded.

- **Step 3:** Conclusion of the proof.
  From Step 2 we have $|X(t) - X_1(t)| \leq C$, and since $|X(t)|$ is large, we have

$$\frac{1}{2}|X(t)| \leq |X(t)| - |X(t) - X_1(t)| \leq |X_1(t)| \leq |X_1(t) - X(t)| + |X(t)| \leq 2|X(t)|. \tag{4.45}$$

By Step 1, we get $\delta^2(t) \geq C\,\frac{e^{-2|X_1(t)|}}{|X_1(t)|^2}$, which implies

$$\delta^2(t) \geq C\,\frac{e^{-2|X(t)|}}{|X(t)|^2},$$

concluding the proof of Lemma 4.9. □

Lemma 4.10. *Let u be a solution of* (NLS$_\Omega$) *satisfying the assumptions of the Proposition 4.7. Then there exists a constant $C > 0$ such that if $0 \leq \sigma \leq \tau$*

$$\int_\sigma^\tau \delta(t) \leq C\left[1 + \sup_{t\in[\sigma,\tau]} |x(t)|\right](\delta(\sigma) + \delta(\tau)). \tag{4.46}$$

Proof. Let φ be a smooth radial function such that

$$\varphi(x) := \begin{cases} |x|^2 & \text{if } |x| \ \leq 1, \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

Consider the localized variance,

$$\mathcal{Y}_R(t) = \int_{\Omega} R^2 \varphi\left(\frac{x}{R}\right) |u(t,x)|^2 \, dx, \tag{4.47}$$

where R is large positive constant, to be specified later. Then,

$$\mathcal{Y}_R'(t) = 2R \operatorname{Im} \int_{\Omega} \bar{u} \, \nabla \varphi\left(\frac{x}{R}\right) \cdot \nabla u \, dx, \qquad |\mathcal{Y}_R'(t)| \leq C\, R. \tag{4.48}$$

Furthermore,

$$\mathcal{Y}_R''(t) = 8 \int_{\Omega} |\nabla u|^2 \, dx - 6 \int_{\Omega} |u|^4 \, dx + A_R(u(t)) - 2 \int_{\partial\Omega} |\nabla u|^2 \, x \cdot \vec{n} \, d\sigma(x),$$

where $\vec{n}$ is the outward normal vector and

$$A_R(u(t)) := 4 \sum_{j \neq k} \int_{\Omega} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \left(\frac{x}{R}\right) \frac{\partial u}{\partial x_j} \frac{\partial \bar{u}}{\partial x_k} + 4 \sum_{j} \int_{\Omega} \left(\frac{\partial^2 \varphi}{\partial x_j^2} \left(\frac{x}{R}\right) - 2 \right) \left| \partial_{x_j} u \right|^2 \\ - \frac{1}{R^2} \int_{\Omega} |u|^2 \Delta^2 \varphi\left(\frac{x}{R}\right) - \int_{\Omega} \left(\Delta\varphi\left(\frac{x}{R}\right) - 6 \right) |u|^4. \tag{4.49}$$

As $\partial\Omega$ is convex and $0 \in \Omega$, one can see that $x \cdot \vec{n} \leq 0$, for all $x \in \partial\Omega$. Thus,

$$-2 \int_{\partial\Omega} |\nabla u|^2 \; x \cdot \vec{n} \; d\sigma(x) = 2 \int_{\partial\Omega} |\nabla u|^2 \; |x \cdot \vec{n}| \; d\sigma(x).$$

Using the fact $\|Q\|_{L^4}^4 = \frac{4}{3} \|\nabla u\|_{L^2}^2$ and $E[u] = E_{\mathbb{R}^3}[Q]$, we have $8 \|\nabla u\|_{L^2}^2 - 6 \|u\|_{L^4}^4 = 4\delta(t)$, which yields

$$\mathcal{Y}_R''(t) = 4\delta(t) + A_R(u(t)) + 2 \int_{\partial\Omega} |\nabla u|^2 \, |x \cdot \vec{n}| \; d\sigma(x). \tag{4.50}$$

- **Step 1:** Bound on A_R.
  In this step we prove: for $\varepsilon > 0$, there exists a constant $R_\varepsilon > 0$ such that

$$\forall t \geq 0, \; R \geq R_\varepsilon(1 + |x(t)|) \Longrightarrow |A_R(u(t))| \leq \varepsilon \delta(t). \tag{4.51}$$

We distinguish two cases: δ small or not. In the first case, we will use the estimate on the modulation parameters in Section 3. Consider $\delta_0 > 0$, as in the previous Section, such that the modulation parameters, $\Theta(t), X(t), \rho(t)$ are well defined for all $t \in D_{\delta_0}$. Let δ_1 to be specified later such that $0 < \delta_1 < \delta_0$. Assume that $t \in D_{\delta_1}$. Let $g_{-X} = \rho Q_{-X}\Psi + h$, then from Proposition 3.3 with Lemma 4.9 and (3.8), we have

$$u(t,x) = e^{i\theta(t)+it}Q(x - X(t))\Psi(x) + g(t, x - X(t))e^{i\theta(t)+it} \quad \text{and}$$
$$\|g\|_{H_0^1(\Omega)} \le C\delta(t). \tag{4.52}$$

We claim that for large R,

$$\forall\theta_0 \in \mathbb{R},\ \forall x_0 \in \mathbb{R}^3, \quad A_R\left(e^{i\theta_0}Q(\cdot + x_0)\right) = 0 \tag{4.53}$$

Indeed, fix $R > 0$ large enough so that $\varphi(x/R) = |x|^2$ if x is in a neighborhood of the obstacle Θ. Consider the solution $U(t,x) = e^{i(t+\theta_0)}Q(x + x_0)$ of $(\text{NLS})_{\mathbb{R}^3}$. We note that for this solution,

$$\forall t \in \mathbb{R}, \quad \int_{\mathbb{R}^3} R^2\varphi\left(\frac{x}{R}\right)|U(t,x)|^2 dx = \int_{\mathbb{R}^3} R^2\varphi\left(\frac{x}{R}\right)|Q(x)|^2 dx$$

(which is independent of t), and

$$8\|\nabla U(t)\|_{L^2}^2 - 6\|U(t)\|_{L^4}^4 = 0.$$

By the same explicit computation as the one leading to (4.50), but on the whole space $\mathbb{R}^3$, we obtain

$$0 = \frac{d^2}{dt^2}\int_{\mathbb{R}^3} R^2\varphi\left(\frac{x}{R}\right)|U(t,x)|^2 = A_R(U(t)),$$

which proves (4.53). Note that we have used that by our assumption on R, all the integrands in the definition (4.49) of A_R are zero in a neighborhood of the obstacle Θ. Using the change of variable $y = x - X(t)$ in (4.49), we get

$$\begin{aligned}|A_R(u(t))| &= \left|A_R(u(t)) - A_R(e^{i\theta(t)+it}Q(x - X(t))\right| \\ &\le C \int_{|y+X(t)|\ge R} \Big(|\nabla Q(y)||\nabla g(y)| + |\nabla g(y)|^2 + |Q(y)||g(y)| + |Q(y)||g(y)|^3 \\ &\qquad + |g(y)|^2 + |g(y)|^4\Big) dy\end{aligned}$$

$$\leq C \int_{|y+X(t)|\geq R} \left(\frac{e^{-|y|}}{|y|}\left(|\nabla g(y)| + |g(y)| + |g(y)|^3\right) + |\nabla g(y)|^2 + |g(y)|^2 + |g(y)|^4\right) dy.$$

By (4.52), we have $\|g\|_{H_0^1(\Omega)} \leq C\delta(t)$, which yields

$$\begin{aligned} R \geq R_0 + |X(t)| \Longrightarrow |A_R(u(t))| &\leq C\left[e^{-R_0}(\delta(t)+\delta(t)^3) + \delta(t)^2 + \delta(t)^4\right] \\ &\leq C\left[e^{-R_0} + e^{-R_0}\delta(t)^2 + \delta(t) + \delta(t)^3\right]\delta(t) \\ &\leq \varepsilon\delta(t), \end{aligned}$$

provided $R_0 > 0$ is such that $Ce^{-R_0} \leq \frac{\varepsilon}{2}$ and δ_1 is such that $Ce^{-R_0}\delta_1^2 + \delta_1 + \delta_1^3 \leq \frac{\varepsilon}{2}$. Since $0 < \delta_1 < \delta_0$ and $x(t) = X(t)$ on D_{δ_0}, we obtain (4.51) for $\delta(t) < \delta_1$.
Now consider the second case, i.e., $\delta(t) \geq \delta_1$. By (4.49), we have

$$|A_R(u(t))| \leq C \int_{|x-x(t)|\geq R-|x(t)|} |\nabla u(t)|^2 + |u(t)|^4 + |u(t)|^2 dx.$$

By the compactness of K, there exists $R_1 > 0$ such that

$$R \geq |x(t)| + R_1 \text{ and } \delta(t) \geq \delta_1 \Longrightarrow |A_R(u(t))| \leq \varepsilon\delta_1 \leq \varepsilon\delta(t), \tag{4.54}$$

which concludes the proof of (4.51) and completes Step 1.

- **Step 2:** Conclusion of the proof.
By (4.50) and (4.51), we get that there exists $R_2 > 0$ such that,

$$R \geq R_2(1 + |x(t)|) \Longrightarrow |\mathcal{Y}_R''(t)| \geq 2\delta(t).$$

Let $R = R_2(1 + \sup_{\sigma\leq t\leq\tau} |x(t)|)$. Then

$$2\int_\sigma^\tau \delta(t)dt \leq \int_\sigma^\tau \mathcal{Y}_R''(t)\, dt \leq \mathcal{Y}_R'(\tau) - \mathcal{Y}_R'(\sigma). \tag{4.55}$$

If $\delta(t) < \delta_0$, then by Step 1, changing the variable $y = x - X(t)$ and since $\Psi(x) = 1$ for large $|x|$, we obtain

$$\begin{aligned} \mathcal{Y}_R'(t) = 2R\operatorname{Im} \int \bar{g}(y)\, \nabla\varphi\left(\frac{y+X(t)}{R}\right) \cdot \nabla\left(Q(y)\Psi(y+X(t)\right) \\ + 2R\operatorname{Im}\int Q(y)\Psi(y+X(t))\nabla\varphi\left(\frac{y+X(t)}{R}\right)\cdot\nabla g(y)\, dy \\ + 2R\operatorname{Im}\int \bar{g}(y)\nabla\varphi\left(\frac{y+X(t)}{R}\right)\cdot\nabla g(y)dy, \end{aligned}$$

which yields

$$|\mathcal{Y}_R'(t)| \leq CR(\delta(t) + \delta(t)^2) \leq CR\delta(t).$$

This inequality is also valid for $\delta(t) \geq \delta_0$, by straightforward estimates. Using (4.55), we obtain

$$\begin{aligned}\int_\sigma^\tau \delta(t)dt &\leq C\,R(\delta(\sigma) + \delta(\tau)) \\ &\leq C\,R_2 \left(1 + \sup_{\sigma \leq t \leq \tau} |x(\tau)|\right) (\delta(\sigma) + \delta(\tau)).\end{aligned}$$

This concludes the proof of Lemma 4.10. □

Lemma 4.11. *There exists a constant $C > 0$ such that*

$$\forall \sigma, \tau > 0 \quad \textit{with} \quad \sigma + 1 \leq \tau, \quad |x(\tau) - x(\sigma)| \leq C \int_\sigma^\tau \delta(t)dt. \tag{4.56}$$

Proof. Let $\delta_0 > 0$ be as in Section 3. Let us first show that there exists $\delta_1 > 0$ such that,

$$\forall \tau \geq 0 \qquad \inf_{t \in [\tau, \tau+2]} \delta(t) \geq \delta_1 \quad \text{or} \quad \sup_{t \in [\tau, \tau+2]} \delta(t) < \delta_0. \tag{4.57}$$

If not, there exist $t_n, t_n' \geq 0$ such that

$$\delta(t_n) \xrightarrow[n \to +\infty]{} 0, \quad \delta(t_n') \geq \delta_0, \quad |t_n - t_n'| \leq 2, \tag{4.58}$$

extracting a subsequence if necessary, we may assume

$$\lim_{n \to +\infty} t_n - t_n' = \tau \in [-2, 2]. \tag{4.59}$$

Note that if t_n' goes to $+\infty$, then $|x(t_n')|$ converges (after extraction) to a limit $X_0 \in \mathbb{R}^3$. If not $|x(t_n')| \longrightarrow +\infty$ and by Lemma 4.8, $\delta(t_n') \longrightarrow 0$, which contradicts (4.58).

By the compactness of K, we have

$$\underline{u}(t_n', \cdot + x(t_n')) \xrightarrow[n \longrightarrow +\infty]{} w_0 \in H^1(\mathbb{R}^3).$$

Denote $v_0(x) = w_0(x - X_0)$. We have

$$\underline{u}(t_n', \cdot + x(t_n')) \xrightarrow[n \longrightarrow +\infty]{} v_0(\cdot + X_0) \in H^1(\mathbb{R}^3). \tag{4.60}$$

Thus,

$$\underline{u}(t'_n) \xrightarrow[n\longrightarrow+\infty]{} v_0 \in H^1(\mathbb{R}^3).$$

In particular, $v_0 = 0$ on Ω^c and we obtain,

$$u(t'_n) \xrightarrow[n\longrightarrow+\infty]{} v_0 \in H^1_0(\Omega). \tag{4.61}$$

Since $\delta(t'_n) = \int |\nabla Q|^2 - \int |\nabla \underline{u}(t'_n, \cdot + x(t'_n))|^2 \geq \delta_0 > 0$, we have

$$\|\nabla v_0\|_{L^2(\Omega)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)}\,.$$

Let $v(t)$ be a solution of (NLS_Ω) with initial data v_0 at $t = 0$ and maximal time of existence I. Then by continuity of the flow of the NLS_Ω equation, we have for all $t \in I$,

$$\|\nabla v(t)\|_{L^2(\Omega)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)}\,. \tag{4.62}$$

As a consequence, $I = \mathbb{R}$ and by continuity of the flow of the NLS_Ω equation, (4.59) and (4.61), we have

$$u(t_n) \xrightarrow[n\longrightarrow+\infty]{} v(\tau) \in H^1_0(\Omega).$$

Since $\delta(t_n) \to 0$, $\|\nabla v(\tau)\|_{L^2(\Omega)} = \|\nabla Q\|_{L^2(\mathbb{R}^3)}$, which contradicts (4.62).

Now, we prove (4.56) with an additional condition that $\tau < \sigma + 2$. By (4.57), we may assume that

$$\inf_{t\in[\sigma,\tau]} \delta(t) \geq \delta_1 \quad \text{ or } \quad \sup_{t\in[\sigma,\tau]} \delta(t) < \delta_0.$$

In the first case, we have $\int_\sigma^\tau \delta(t) \geq \delta_1$ and by a straightforward consequence of the compactness of K and the continuity of the flow of (NLS_Ω) equation, we have

$$\exists C > 0,\ \forall t, s \geq 0, \quad |t - s| \leq 2 \Longrightarrow |X(t) - X(s)| \leq \frac{C}{\delta_1} \int_\sigma^\tau \delta(t)dt.$$

In the second case, by Corollary 4.6 we have, $\forall t \in D_{\delta_0}$, $x(t) = X(t)$, and from Lemmas 3.4 and 4.9, we have

$$|X'(t)| \leq C\delta(t). \tag{4.63}$$

Thus, (4.56) follows from the time integration of (4.63) for $\tau < \sigma + 2$.

To conclude the proof of Lemma 4.11, we divide $[\sigma, \tau]$ into intervals of length at least 1 and at most 2 and combine together the previous inequalities to get (4.56). □

Proof of the Proposition 4.7. We argue by contradiction. Assume that there exists $\tau_n \longrightarrow +\infty$ such that $|x(\tau_n)| \longrightarrow +\infty$ and $|x(\tau_n)| = \sup_{t\in[0,\tau_n]} |x(t)|$. By Lemma 4.8, $\delta(\tau_n) \xrightarrow[n\to+\infty]{} 0$.

Let N_0 be such that $C\delta(\tau_n) \leq \frac{1}{100}$ for all $n \geq N_0$. By Lemmas 4.10 and 4.11 we have

$$\begin{aligned}|x(\tau_n) - x(\tau_{N_0})| &\leq C \int_{\tau_{N_0}}^{\tau_n} \delta(t)dt \\ &\leq C(1 + |x(\tau_n)|)(\delta(\tau_{N_0}) + \delta(\tau_n)),\end{aligned}$$

hence,

$$|x(\tau_n)| \leq C|x(\tau_{N_0})|,$$

which gives a contradiction. This concludes the proof of Proposition 4.7. □

4.3. Convergence in mean

Lemma 4.12. *Consider a solution $u(t)$ of* (NLS$_\Omega$) *satisfying assumptions of Proposition 4.7. Then*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T \delta(t)dt = 0. \tag{4.64}$$

Corollary 4.13. *Under the assumptions of Proposition 4.7, there exists a sequence of times t_n such that $t_n \to +\infty$ and*

$$\lim_{n\longrightarrow+\infty} \delta(t_n) = 0.$$

Proof of Lemma 4.12. Consider the localized variance defined in (4.47) and recall that from the proof of Lemma 4.10, we have

$$\mathcal{Y}_R''(t) = 4\delta(t) + A_R(u(t)) + 2\int_{\partial\Omega} |\nabla u|^2 |x\cdot\vec{n}|d\sigma(x), \tag{4.65}$$

where $\vec{n}$ is outward normal vector and A_R is defined in (4.49).

If $|y| \leq 1, (\Delta^2\varphi)(y) = 0, \partial^2_{x_j}\varphi(y) = 2$ and $\Delta\varphi(y) = 6$. Thus,

$$|A_R(u(t))| \leq C \int_{|x|\geq R} |\nabla u|^2 + |u|^4 + \frac{1}{R^2}|u|^2. \tag{4.66}$$

Let $x(t)$ be as in Corollary 4.6 and K be defined by (4.2). Let $\varepsilon > 0$. By the compactness of K and Proposition 4.7, there exists $R_0(\varepsilon) > 0$ such that

$$\forall t \geq 0, \quad \int_{|x-X(t)|\geq R_0(\varepsilon)} |\nabla u|^2 + |u|^2 + |u|^4 \leq \varepsilon. \tag{4.67}$$

Furthermore, $x(t)$ is bounded, and thus, $\frac{x(t)}{t} \xrightarrow[t\to+\infty]{} 0$. There exists $t_0(\varepsilon)$ such that

$$\forall t \geq t_0(\varepsilon), \ |x(t)| \leq \varepsilon t.$$

Let

$$T \geq t_0(\varepsilon), \ R = \varepsilon T + R_0(\varepsilon) + 1 \quad \text{for } \ t \in [t_0(\varepsilon), T].$$

Next, we use the fact that $|x(t)| \leq \varepsilon T$ and $R_0(\varepsilon) + \varepsilon T \leq R$, to get

$$\begin{aligned} \int_{|x|\geq R} |\nabla u|^2 + |u|^4 + \frac{1}{R^2}|u|^2 \leq & \int_{|x-x(t)|+|x(t)|\geq R} |\nabla u|^2 + |u|^4 + \frac{1}{R^2}|u|^2 \\ & \leq \int_{|x-x(t)|\geq R_0(\varepsilon)} |\nabla u|^2 + |u|^4 + \frac{1}{R^2}|u|^2 \leq \varepsilon. \end{aligned} \tag{4.68}$$

By (4.48), we have

$$\int_{t_0(\varepsilon)}^{T} \mathcal{Y}_R''(t)dt \leq |\mathcal{Y}_R'(T)| + |\mathcal{Y}_R'(t_0(\varepsilon))| \leq C\,R.$$

From (4.65), (4.66) and (4.68) we have

$$\int_{t_0(\varepsilon)}^{T} \delta(t)dt \leq C(R + T\varepsilon) \leq CR_0(\varepsilon) + \varepsilon T + 1,$$

where $C > 0$, independent of T and ε.

This yields

$$\frac{1}{T}\int_0^T \delta(t)dt \leq \frac{1}{T}\int_0^{t_0(\varepsilon)} \delta(t)\,dt + \frac{C}{T}(R_0(\varepsilon) + 1) + C\varepsilon.$$

Taking first limsup as $T \to +\infty$, and letting ε tend to 0, we obtain (4.64). $\square$

Proposition 4.14. *Let u be a solution of* (NLS_Ω) *such that*

$$M[u] = M_{\mathbb{R}^3}[Q],\ E[u] = E_{\mathbb{R}^3}[Q],\ \|\nabla u_0\|_{L^2(\Omega)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)} \tag{4.69}$$

and $K = \{u(t); t \geq 0\}$ has a compact closure in $H_0^1(\Omega)$. Then $u \equiv 0$.

Proof. If not, there exists a solution $u \neq 0$ such that the assumptions of this Proposition are satisfied. From Lemma 4.12, there exists t_n such that $t_n \longrightarrow +\infty$ and $\delta(t_n)$ tends to 0. By the compactness of the closure of K, $u(t_n)$ converges in $H_0^1(\Omega)$ to some $v_0 \in H_0^1(\Omega)$ and the fact that $\delta(t_n)$ tends to 0 implies that $E[v_0] = E_{\mathbb{R}^3}[Q], M[v_0] = M_{\mathbb{R}^3}[Q]$ and $\|\nabla v_0\|_{L^2(\Omega)} = \|\nabla Q\|_{L^2(\mathbb{R}^3)}$. Thus, $v_0 = e^{i\theta_0}Q(x - x_0) \notin H_0^1(\Omega)$, for some parameters $\theta_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^3$, which contradicts the fact that $v_0 \in H_0^1(\Omega)$. □

Appendix A. Proof of the existence of initial data covered by Theorem 1

In this appendix, we prove the existence of initial data $u_0 \in H_0^1(\Omega)$ that satisfy

$$M_\Omega[u_0]E_\Omega[u_0] = M_{\mathbb{R}^3}[Q]E_{\mathbb{R}^3}[Q] \tag{A.1}$$

$$\|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} < \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}. \tag{A.2}$$

Let $\lambda > 0$, $\varphi \in H_0^1(\Omega)\backslash\{0\}$ and let $u_\lambda(t)$ be a solution of the NLS_Ω equation with initial data $u_\lambda(t_0) := u_{0,\lambda} = \lambda\varphi \in H_0^1(\Omega)$. Let us assume, without loss of generality, $M_\Omega[\varphi] = M_{\mathbb{R}^3}[Q]$.

We have

$$E_\Omega[u_\lambda]M_\Omega[u_\lambda] = M_{\mathbb{R}^3}[Q]\mathcal{F}(\lambda),\ \text{where}\ \mathcal{F}(\lambda) := \frac{\lambda^4}{2}\int_\Omega |\nabla\varphi|^2 - \frac{\lambda^6}{4}\int_\Omega |\varphi|^4.$$

One can see that $\mathcal{F}'(\lambda) = 0$ for $\lambda_0 := \left(\frac{4\int|\nabla\varphi|^2}{3\int|\varphi|^4}\right)^{\frac{1}{2}}$, $\mathcal{F}'(\lambda) > 0$ if $\lambda < \lambda_0$ and $\mathcal{F}'(\lambda) < 0$ if $\lambda > \lambda_0$.

Let us recall that we can extend the function $\varphi \in H_0^1(\Omega)$ by 0 on the obstacle and it can be identified to an element of $H^1(\mathbb{R}^3)$, which we have denoted by $\underline{\varphi}$. Thus, we can apply the Gagliardo-Nirenberg inequality (2.2) to $\underline{\varphi}$.

Using (2.2) with the sharp constant $C_{GN} = \frac{4}{3\|Q\|_{L^2(\mathbb{R}^3)}\|\nabla Q\|_{L^2(\mathbb{R}^3)}}$ and the fact that $M_\Omega[\varphi] := M_{\mathbb{R}^3}[\underline{\varphi}] = M_{\mathbb{R}^3}[Q]$, we have

$$\|\underline{\varphi}\|_{L^4(\mathbb{R}^3)}^4 \leq \frac{4}{3}\frac{\|\nabla\underline{\varphi}\|_{L^2(\mathbb{R}^3)}^3}{\|\nabla Q\|_{L^2(\mathbb{R}^3)}}, \tag{A.3}$$

which yields

$$\mathcal{F}(\lambda_0) = \frac{8}{27}\frac{\left(\int \left|\nabla \underline{\varphi}\right|^2\right)^3}{\left(\int |\underline{\varphi}|^4\right)^2} > \frac{1}{6}\int_{\mathbb{R}^3} |\nabla Q|^2 = E_{\mathbb{R}^3}[Q].$$

Thus, there exists a unique $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 < \lambda_0 < \lambda_2$ and $E_{\mathbb{R}^3}[Q] = \mathcal{F}(\lambda_1) = \mathcal{F}(\lambda_2)$, i.e., $E_\Omega[u_{0,\lambda_{1,2}}]M_\Omega[u_{0,\lambda_{1,2}}] = E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q]$. It remains to prove that u_{0,λ_1} satisfies (A.2) and u_{0,λ_2} satisfies $\|u_{0,\lambda_2}\|_{L^2(\Omega)} \|\nabla u_{0,\lambda_2}\|_{L^2(\Omega)} > \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}$.

Using (A.3) and the fact that $\lambda_0^4 \int \left|\nabla \underline{\varphi}\right|^2 = \left(\frac{4\int|\nabla\underline{\varphi}|^2}{3\int|\underline{\varphi}|^4}\right)^2 \int \left|\nabla \underline{\varphi}\right|^2$, we have

$$\int_{\mathbb{R}^3} |\nabla Q|^2 < \lambda_0^4 \int_{\mathbb{R}^3} \left|\nabla \underline{\varphi}\right|^2.$$

Thus, there exists $\lambda_3 > 0$ such that $\lambda_3 < \lambda_0$, and $\lambda_3^4 \int_{\mathbb{R}^3} \left|\nabla \underline{\varphi}\right|^2 = \int_{\mathbb{R}^3} |\nabla Q|^2$. Next, we show that $\lambda_1 < \lambda_3$ or equivalently that $\mathcal{F}(\lambda_1) < \mathcal{F}(\lambda_3)$. Using (A.3), we obtain

$$\mathcal{F}(\lambda_3) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla Q|^2 - \frac{1}{4}\frac{\left(\int_{\mathbb{R}^3} |\nabla Q|^2\right)^{\frac{3}{2}}}{\left(\int_{\mathbb{R}^3} \left|\nabla \underline{\varphi}\right|^2\right)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \left|\underline{\varphi}\right|^4 > E_{\mathbb{R}^3}[Q] = \mathcal{F}(\lambda_1).$$

Since $\lambda_1 < \lambda_3$, we have

$$\lambda_1^4 \int_{\mathbb{R}^3} \left|\nabla \underline{\varphi}\right|^2 = \lambda_1^4 \int_{\Omega} |\nabla \varphi|^2 < \int_{\mathbb{R}^3} |\nabla Q|^2,$$

which implies that u_{0,λ_1} satisfies (A.2) using that $M_\Omega[\varphi] = M_{\mathbb{R}^3}[Q]$. Similarly, we obtain

$$\int_{\mathbb{R}^3} |\nabla Q|^2 < \lambda_2^4 \int_{\mathbb{R}^3} \left|\nabla \underline{\varphi}\right|^2 = \lambda_2^4 \int_{\Omega} |\nabla \varphi|^2.$$

Hence,

$$\|u_{0,\lambda_2}\|_{L^2(\Omega)} \|\nabla u_{0,\lambda_2}\|_{L^2(\Omega)} > \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}.$$

Then, there exists a unique $\lambda_1 > 0$, such that u_{0,λ_1} satisfy (A.1) and (A.2).

Appendix B. Existence of a continuous translation parameter

In this appendix, we prove:

Lemma B.1. *Let $u(t)$ be a solution of (NLS_Ω) defined for $t \geq 0$. Assume that for all sequence of times $t_n \geq 0$, there exists a sequence $x_n \in \mathbb{R}^3$ such that $\left(\underline{u}(t_n, x + x_n)\right)_n$ has a subsequence that converges in $H^1(\mathbb{R}^3)$. Then there exists a continuous function $x(t)$ such that*

$$K = \{\underline{u}(x + x(t), t),\ t \in [0, +\infty)\} \tag{B.1}$$

has a compact closure in $H^1(\mathbb{R}^3)$.

Proof. We can of course assume that u is not identically 0. We let χ be a nonincreasing radial cutoff function such that $\chi(x) = 1$ if $|x| \leq 1/4$ and $\chi(x) = 0$ if $|x| \geq 1/2$. We let, for $t \geq 0$, $R > 0$,

$$A(t, R) = \sup_{y \in \mathbb{R}^3} \int \chi\left(\frac{x - y}{R}\right) |\underline{u}(t, x)|^2 dx.$$

At fixed t, $R \mapsto A(t, R)$ is a nondecreasing continuous function such that $\lim_{R\to 0} A(t, R) = 0$ and $\lim_{R\to+\infty} A(t, R) = \|u_0\|_{L^2}^2$. We choose $R(t) > 0$ such that

$$A(t, R(t)) = \frac{7}{8}\|u_0\|_{L^2}^2.$$

- Step 1. In this step, we prove that $R(t)$ is uniformly bounded for $t \geq 0$. We argue by contradiction, assuming that there exists a sequence $(t_n)_n$ such

$$\lim_{n\to\infty} R(t_n) = \infty. \tag{B.2}$$

  By the assumptions of the lemma, there exists a sequence $x_n \in \mathbb{R}^3$, and $\varphi \in H^1(\mathbb{R}^3)$ such that (after extraction)

$$\lim_{n\to\infty} \|\underline{u}(t, \cdot + x_n) - \varphi\|_{H^1} = 0.$$

  Since $\|\varphi\|_{L^2}^2 = \|u_0\|_{L^2}^2$, there exists $\rho > 0$ such that $\|\varphi\|_{L^2(B(0,\rho))}^2 \geq \frac{8}{9}\|u_0\|_{L^2}^2$. This implies that $\liminf_{n\to\infty} \|\underline{u}(t_n)\|_{L^2(B(x_n,\rho)}^2 \geq \frac{8}{9}\|u_0\|_{L^2}^2$, and thus, for large n, that $\rho \geq R(t_n)$, a contradiction.
- Step 2. By Step 1, taking $R = \sup_{t\geq 0} R(t) < \infty$, we have

$$\forall t \geq 0, \quad \sup_{y\in\mathbb{R}^3} \int \chi\left(\frac{x - y}{R}\right) |\underline{u}(t, x)|^2 dx \geq \frac{7}{8}\|u_0\|_{L^2}^2.$$

For $t \geq 0$, we fix $y(t)$ such that

$$\int \chi\left(\frac{x-y(t)}{R}\right)|\underline{u}(t,x)|^2dx \geq \frac{4}{5}\|u_0\|_{L^2}^2. \tag{B.3}$$

We claim that there exists $\delta > 0$ such that

$$\forall t, s \geq 0, \quad |t-s| \leq \delta \Longrightarrow \int \chi\left(\frac{x-y(t)}{R}\right)|\underline{u}(s,x)|^2dx \geq \frac{3}{4}\|u_0\|_{L^2}^2 \tag{B.4}$$

$$\forall t, s \geq 0, \quad |t-s| \leq \delta \Longrightarrow |y(t)-y(s)| \leq R. \tag{B.5}$$

Indeed

$$\frac{d}{ds}\int \chi\left(\frac{x-y(t)}{R}\right)|\underline{u}(s,x)|^2dx = -2\Im\frac{1}{R}\int \nabla\chi\left(\frac{x-y(t)}{R}\right)\cdot\nabla u(s,x)\overline{u}(s,x)dx$$

and (B.4) follows the fact that $\underline{u}$ is bounded in $H^1(\mathbb{R}^3)$ by the assumptions of the lemma. By (B.4), and the definition of $y(s)$,

$$\int \chi\left(\frac{x-y(t)}{R}\right)|\underline{u}(s,x)|^2dx+\int \chi\left(\frac{x-y(s)}{R}\right)|\underline{u}(s,x)|^2dx \geq \frac{4}{3}\|u_0\|_{L^2}^2 = \frac{4}{3}\|\underline{u}(t)\|_{L^2}^2,$$

and (B.5) follows from the fact that $x \mapsto \chi((x-y(t))/R)$ and $x \mapsto \chi((x-y(s))/R)$ have disjoint support if $|y(t)-y(s)| > R$.

- Step 3. We define $x(t)$ as the function such that for all integer $n \geq 0$, $x(n\delta) = y(n\delta)$ and x is affine on $\big(n\delta, (n+1)\delta\big)$. We claim that K defined by (B.1) has compact closure in $H^1(\mathbb{R}^3)$. Indeed, using (B.3) and the assumptions of the lemma, it is easy to see that

$$\widetilde{K} = \{\underline{u}(x+y(t),t),\ t \in [0,+\infty)\}$$

has compact closure in $H^1(\mathbb{R}^3)$. Noting that (B.5) and the definition of $x(t)$ implies that $|x(t)-y(t)| \leq 2R$ for all $t \geq 0$, we see that K has compact closure, concluding the proof. □

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