# FINITE-TIME STABILITY OF POLYHEDRAL SWEEPING PROCESSES WITH APPLICATION TO ELASTOPLASTIC SYSTEMS *

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**Abstract.** We use the ideas of Adly-Attouch-Cabot [Adv. Mech. Math., 12, Springer, 2006] on finite-time stabilization of dry friction oscillators to establish a theorem on finite-time stabilization of differential inclusions with a moving polyhedral constraint (known as polyhedral sweeping processes) of the form $C+c(t)$. We then employ the ideas of Moreau [New variational techniques in mathematical physics, CIME, 1973] to apply our theorem to a system of elastoplastic springs with a displacement-controlled loading. We show that verifying the condition of the theorem ultimately leads to the following two problems: (i) identifying the active vertex "A" or the active face "A" of the polyhedron that the vector $c'(t)$ points at; (ii) computing the distance from $c'(t)$ to the normal cone to the polyhedron at "A". We provide a computational guide for solving problems (i)-(ii) in the case of an arbitrary elastoplastic system and apply it to a particular example. Due to the simplicity of the particular example, we can solve (i)-(ii) by the methods of linear algebra and basic combinatorics.

**Key words.** Polyhedral constraint, Normal cone, Vertex enumeration, Sweeping process, Finite-time stability, Lyapunov function

**AMS subject classifications.** 47H11; 70H45; 26B30; 34A60

## 1. Introduction.

Finite-time stability of an attractor is typical for differential equations with nonsmooth right-hand-sides. This fact is used in control theory since long ago, see e.g. Orlov [26]. Finite-time stability is often proved by showing that a Lyapunov function $V$ satisfies the estimate (Bernuau et al. [5], Bhat-Bernstein [6], Oza et al. [27], Sanchez et al. [30], etc.)

$$\frac{d}{dt}[V(x(t))] + 2\varepsilon\sqrt{V(x(t))} \le 0 \quad \text{a.e. on } [0,\infty) \tag{1.1}$$

for some $\varepsilon > 0$, where $x$ is a solution. Specifically, if (1.1) holds for a function $x(t)$, then $V(x(t_1)) = 0$ at some $t_1 \ge 0$, where (see Lemma B.1)

$$t_1 \le \frac{1}{\varepsilon}V(x(0)).$$

Motivated by applications in frictional mechanics, Adly et al. [2] extended the Lyapunov function approach to finite-time stability analysis of differential inclusions. Let $\nabla f(x)$ be the gradient of a function $f : \mathbb{R} \to \mathbb{R}^m$, $\partial\Phi(x)$ be the subdifferential of a convex function $\Phi : \mathbb{R} \to \mathbb{R}^n$, and $B_\varepsilon(0)$ be the ball of $\mathbb{R}^n$ of radius $\varepsilon$ centered at 0. By focusing on differential inclusions of the form

$$-x''(t) - \nabla f(x(t)) \in \partial\Phi(x'(t)), \tag{1.2}$$

the paper [2] discovered (see the proof of [2, Theorem 24.8]) that the property

$$-\nabla f(x(t)) + B_\varepsilon(0) \subset \partial\Phi(0) \quad \text{a.e. on } [0,\infty) \tag{1.3}$$

implies (1.1) for a suitable Lyapunov function $V$ that measures the distance from $x'(t)$ to 0 and for any solution $x$ of (1.2).

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More recently, a significant interest in the study of finite-time stability of differential inclusions has developed due to new applications in elastoplasticity (see e.g. Gudoshnikov et al. [18]). We remind the reader (see Section 6 for full details) that according to the pioneering work by Moreau [25] (revisited in Gudoshnikov-Makarenkov [16]), the stresses in a network of $m$ elastoplastic springs with time-varying displacement-controlled loadings are governed by

$$(1.4)\qquad -y' \in N^A_{C(t)}(y), \quad y \in \mathcal{V}, \quad \begin{array}{l}\mathcal{V} \text{ is a } d-\text{dimensional subspace of } \mathbb{R}^m \\ \text{with the scalar product } (x,y)_A = \langle x, Ay\rangle,\end{array}$$

where $A$ is a positive diagonal $m \times m$-matrix, $N^A_{C(t)}(y)$ is a normal cone to the set $C(t) = C + c(t)$ at a point $y$, $C \subset \mathbb{R}^m$ is a polyhedron, and $c(t) \in \mathbb{R}^m$ is a vector. The solutions $y(t)$ of differential inclusion (1.4) never escape from $C(t)$ (i.e. $y(t)$ is swept by $C(t)$) for which reason (1.4) is called the *sweeping process.* Spring $j$ undergoes *plastic deformation* when the inequality $c_j^- < \langle n_j, Ay\rangle < c_j^+$ is violated, where $c_j^-, c_j^+ \in \mathbb{R}$, $n_j \in \mathbb{R}^m$ are mechanical parameters of the spring (yield stresses) and vectors $n_j \in \mathbb{R}^m$ come from geometry of the network (see Section 6 for details). Therefore, knowing the evolution of $y(t)$ allows to make conclusions about the regions of plastic deformation (that lead to *low-cycle fatigue* or *incremental failure*, see Yu [33, §4.6]).

Krejci [22] proved that if $c(t)$ is $T$-periodic then the set $Y$ of $T$-periodic solutions of (1.4) is always asymptotically stable. Examining finite-time stability of $Y$ is related to the interaction of $Y$ with the boundary of the moving constraint $C + c(t)$ (plastic deformation). In particular, the analysis of finite-time stability helps to understand whether the plastic deformation will repeatedly progress or cease which is related to the phenomena of alternating plasticity, ratchetting, and shakedown in the theory of elastoplasticity, see e.g. Yu [33], Boissier et al [7]. In the case where $Y$ consists of just one solution (and the state space is 2-dimensional), the finite-time stability is established in Gudoshnikov et al. [18] (with an application to an elastoplastic system). However, as shown in Gudoshnikov-Makarenkov [17], the case where the periodic attractor consists of a family of solutions is often structurally stable for sweeping processes of elastoplastic systems. The assumption that $Y$ consists of one solution was dropped by Colombo et al. [10] who worked in higher-dimensional states spaces and allowed the size of the moving constraint to change but focused on moving constraints of parallelepipedal shape (naturally arising in applications to soft locomotors with dry friction). The mathematical contribution of the present paper is that we simultaneously address the case where $Y$ is a family of solutions and the moving constraint is a translation of an arbitrary polyhedron.

Predicting the behavior of solutions of sweeping process (1.4) within a guaranteed time is of importance for materials science. Current methods of computing the asymptotic response of networks of elastoplastic springs (see e.g. Boudy et al. [8], Zouain-SantAnna [34]) run numeric routines until the difference between the responses corresponding to two successive cycles of loading get smaller than a prescribed tolerance (without an estimate as for how soon such a desired accuracy will be reached).

The present paper adapts condition (1.3) in order to predict the behavior of solutions of (1.4) within a guaranteed finite time. We do not prove the finite-time stability of $Y$, but still prove that all solutions of (1.4) will reach a certain computable set $\mathcal{Y}$ in finite time. Specifically, let $F(t)$ be a face of $C(t)$. Then, as we clarify in Section 2, $F(t) = F + c(t)$. We prove that if

$$(1.5)\qquad -c'(t) + B^A_\varepsilon(0) \cap N^A_F(y) \subset N^A_C(y), \quad y \in F, \text{ a.e. } t \in [0, \tau_d],$$

where $B_\varepsilon^A(0)$ is a ball in the norm induced by the scalar product $(x,y)_A = \langle x, Ay\rangle$ (cf. (1.4)), then, for any solution $y(t)$ of (1.4), the function

$$x(t) = y(t) - c(t) \tag{1.6}$$

satisfies estimate (1.1) on $[0, \tau_d]$ for a suitable Lyapunov function $V$ that measures the distance from $x(t)$ to $F$. Since, by (2.2), the distances from $x(t)$ to $F$ equals the distance from $y(t)$ to $F(t)$, then the relation $\tau_d \geq V(x(0))/\varepsilon$ ensures that the solution $y(t)$ sticks to the face $F(t)$ by the time $V(x(0))/\varepsilon$. In particular, when $\tau_d \geq \tau$ for $\tau = \max\limits_{v\in C} V(v)/\varepsilon$, our result implies that the dynamics of sweeping process (1.4) is fully determined by the trajectories $\mathcal{Y}$ with the initial conditions $y(\tau) \in F(\tau)$. When $c(t)$ is $T$-periodic with $T \geq \tau$ and $F(\tau)$ is a singleton (i.e. a vertex), $T$-periodicity of $\mathcal{Y}$ follows from Krejci [22]. When $c(t)$ is $T$-periodic, but $F(\tau)$ is not a singleton, our result doesn't imply $T$-periodicity of the trajectories of $\mathcal{Y}$, but still implies $T$-periodic occurrence of plastic deformations in applications of (1.4) to elastoplastic systems.

The paper is organized as follows. Sections 2-5 present the theory of finite-time stability of sweeping process (1.4), where we first establish an abstract theorem (Theorem 3.1) and then propose various criteria to verify condition (1.5) depending on the dimension of $F(t)$ (Corollary 4.1, Corollary 5.2, Corollary 5.4). Section 6 summarizes the approach by Moreau [25] to the use of sweeping process (1.4) for modeling networks of elastoplastic springs. Section 7 combines the results of Sections 3-6 to formulate a simple algebraic condition (7.1) identifying possible options for plastic deformations in a network of elastoplastic springs with a displacement-controlled loading, which is the main contribution of the present paper. The workings of formula (7.1) are further explained in Propositions 7.3, 7.8, and 7.10 where both the case of gradually stretching and cyclic loadings are considered. Section 8 illustrates the efficiency of the methodology of Section 7 through a benchmark example (taken from Rachinskiy [28]). We show that all formulas of Section 7 can be computed in closed form (using Wolfram Mathematica) and explicit conditions for one or another scenario of the distribution of plastic deformations can be obtained (Propositions 8.1, 8.2, 8.3). The Mathematica notebook is uploaded as supplementary material and stored at Zenodo [15]. Appendix A contains a few proofs omitted in the main text. Appendex B collects more substantial auxiliary results that are known (the proofs are included where we were unable to find the exact required formulations in the literature).

**2. Formulation of the moving constraint.** We will work with $C(t)$ given by

$$C(t) = C + c(t), \quad C = \bigcap_{j=1}^{m} \mathcal{V}_j, \qquad \begin{aligned} \mathcal{V}_j &= L(-1,j) \cap L(+1,j), \\ L(\alpha,j) &= \left\{ y \in \mathcal{V} \colon \langle \alpha n_j, Ay\rangle \leq \alpha c_j^\alpha \right\}. \end{aligned} \tag{2.1}$$

The particular face $F(t)$ of interest will be defined through the set of indexes $I_i \subset \{-1,1\} \times \overline{1,m}$, $i \in \overline{0,M}$, as follows:

$$\begin{aligned} &F(t) = F + c(t), \quad F = \left( \bigcap_{(\alpha,j)\in I_0} \overline{\overline{L}}(\alpha,j) \right) \cap \left( \bigcap_{i=1}^{M} \bigcap_{(\alpha,j)\in I_i} L(\alpha,j) \right), \\ &\overline{\overline{L}}(\alpha,j) = \left\{ y \in \mathcal{V} \colon \langle n_j, Ay\rangle = c_j^\alpha \right\}, \end{aligned} \tag{2.2}$$

where the ingredients in (2.2) satisfy the following assumptions (and where $\mathrm{ri}(F)$ stays for the relative interior of $F$):

$$\text{(2.3)}\quad \begin{array}{l}\text{each } I_0\cup I_i \text{ defines}\\ \text{a vertex of } F:\end{array}\quad \begin{array}{l}\text{for each } i\in\overline{1,M},\ \{y: y\in \overline{\overline{L}}(\alpha,j),\ (\alpha,j)\in I_0\cup I_i\}\\ \text{is a non-empty singleton } \{y_{*,i}\},\end{array}$$

$$\text{(2.4)}\quad \begin{array}{l}F \text{ has no other vertices, all}\\ \text{constraints are accounted for}:\end{array}\quad \begin{array}{l}\text{each } y\in F \text{ defines } J_y\subset \overline{1,M}\cup\emptyset, \text{ such}\\ \text{that } \{(\alpha,j)\colon y\in\overline{\overline{L}}(\alpha,j)\} = I_0\cup\bigcap_{i\in J_y} I_i,\end{array}$$

$$\text{(2.5)}\quad \text{vertices do not coincide}:\quad J_{y_{*,i}}=\{i\},\ i\in\overline{1,M},$$

$$\text{(2.6)}\quad \text{vertices do not reduce } \mathrm{ri}(F):\quad \{(\alpha,j)\colon y\in\overline{\overline{L}}(\alpha,j)\}=I_0,\ y\in\mathrm{ri}(F),$$

$$\text{(2.7)}\quad F \text{ is feasible}:\quad F\subset C,$$

$$\text{(2.8)}\quad \overline{\overline{L}}(\alpha,j),\ (\alpha,j)\in I_0\cup I_i, \text{ are independent}:\qquad |I_0|+|I_i|=d,\ i\in\overline{1,M}.$$

The case where $F$ is a vertex of $C$ is accounted for by $M=0$, see formulas (4.1)-(4.3) below for the corresponding reduced form of (2.2)-(2.8).

Assumptions (2.3)-(2.8) control the construction of the sets $I_i$ (to make the verification of assumption (1.5) of Theorem 3.1 possible algorithmically). They do not restrict the generality of the results. Corollary 4 about the finite-time convergence to a vertex uses (2.3) and (2.7) with $M=0$, which, by definition, simplify to (4.2) ($F$ is a singleton) and (4.3) ($F\subset C$). The full power of assumptions (2.3)-(2.8) is explored in Corollaries 5.2 and 5.4 about the finite-time convergence to a face. Assumption (2.4) is a way (convenient for proofs and verification) to state that $F$ has no vertexes other than $y_{*,i}$, $i\in\overline{1,M}$, while assumption (2.3) relates $y_{*,i}$ to normals $\alpha n_j$ (termed *vertex enumeration* in discrete geometry), thus allowing to express $N_C^A(y_{*,i})$ through the normals $\alpha n_j$, see formula (5.6). The combination (2.3)-(2.4) is used in Corollary 5.2 to reduce the verification of (1.5) on the entire $F$ to the verification at the vertexes of $F$. Assumption (2.5) doesn't restrict the generality because coinciding vertexes can always be dropped. We use (2.5) to have certainty concerning the normal vectors in formulas for $N_C^A(y_{*,i})$ and $N_F^A(y_{*,i})$, see (5.6). Assumption (2.6) simply ensures that the indexes $I_0$ are chosen correctly, i.e. the dimension of the space formed by the normal vectors with indexes from $I_0$ coincides with the dimension of $\mathrm{lin}(F)$. Therefore, the same normal vectors define $N_F^A(y)$ at $y\in\mathrm{ri}(F)$ and at $y\in F$, which is used in Corollary 5.4 to reduce the verification of (1.5) from the entire $F$ to $\mathrm{ri}(F)$. If (2.3) holds, then (2.8) can be achieved by removing appropriate indexes from $I_i$, $i\in\overline{0,M}$. In other words, assumption (2.8) doesn't restrict generality. It is used e.g. to ensure that the spaces $\mathrm{lin}\{n_j:(\alpha,j)\in I_0\}$ and $\mathrm{lin}\{n_j:(\alpha,j)\in I_i\}$ in the definition of the map $\mathcal{L}_i$ of Lemma 5.3 are linearly independent. Finally, assumption (2.7) is just a part of the definition of the face $F$ of $C$.

**3. A sufficient condition for finite-time stability.** We remind the reader that the normal cone $N_C^A(y)$ to the set $C$ at a point $y\in C$ in a scalar product space $\mathcal{V}$ with the scalar product

$$\text{(3.1)}\qquad (x,y)_A=\langle x,Ay\rangle\,,\quad \text{where } A \text{ is a diagonal positive } m\times m\text{-matrix},$$

is defined as (see Bauschke-Combettes [4, §6.4])

$$N_C^A(y)=\left\{\begin{array}{ll}\{x\in\mathcal{V}:\langle x,A(\xi-y)\rangle\leqslant 0 \text{ for any } \xi\in C\} & \text{if } y\in C,\\ \emptyset, & \text{if } y\notin C.\end{array}\right.$$

In what follows (see Bauschke-Combettes [4, §3.2]),

$$\|x\|^A = \sqrt{\langle x, Ax\rangle}, \qquad \mathrm{proj}^A(v,F) = \underset{v'\in F}{\mathrm{argmin}}\|v-v'\|^A.$$

Finally, recall that the solution of an initial-value problem for a sweeping processes with Lipschitz continuous moving set exists, is unique, and features Lipschitz continuity (thus differentiability almost everywhere) and continuous dependence on initial conditions (see e.g. Kunze and Monteiro Marques [23, Theorems 1-3]).

Theorem 3.1. *Let $\mathcal{V}$ be a $d$-dimensional linear subspace of $\mathbb{R}^m$ with scalar product (3.1), $c:[0,\infty)\to\mathcal{V}$ be Lipschitz continuous, and $F\subset C\subset\mathcal{V}$ be closed convex sets. Assume that there exists an $\varepsilon>0$ such that condition (1.5) holds on an interval $[0,\tau_d]$ with*

$$\tau_d \ge \tau, \quad \tau = \frac{1}{\varepsilon}\cdot \max_{\ v_1\in C,\ v_2\in F}\ \|v_1-v_2\|^A.$$

*Then, every solution $y$ of (1.4) with $C(t)=C+c(t)$ and any initial condition $y(0)\in C(0)$, satisfies $y(t)\in F+c(t)$, $t\in[\tau,\tau_d]$, and $y'(t)-c'(t)=0$ a.e. on $[\tau,\tau_d]$.*

What we will effectively prove is that the function

$$V(v) = \left\langle v-\mathrm{proj}^A(v,F), A\left(v-\mathrm{proj}^A(v,F)\right)\right\rangle \tag{3.2}$$

is a Lyapunov function for the sweeping process

$$-x'(t)-c'(t)\in N_C^A(x(t)), \tag{3.3}$$

which is related to (1.4) through the change of variables (1.6). Since (see Proposition B.9)

$$\mathrm{proj}^A(v,F)+c = \mathrm{proj}^A(v+c,F+c),$$

we have

$$V(x(t)) = \left(\|x(t)-\mathrm{proj}^A(x(t),F)\|^A\right)^2 = \left(\|y(t)-\mathrm{proj}^A(y(t),F+c(t))\|^A\right)^2$$

for the function $x(t)$ given by (1.6). Therefore, as expected, $V(x(t_1))=0$ will imply $y(t_1)\in F+c(t_1)$.

In what follows, $D_\xi f(u)$ is the bilateral directional derivative (Giorgi et al. [14, §2.6], Correa-Thibault [12]) of $f:\mathcal{V}\to\mathcal{V}_1$ at the point $u\in\mathcal{V}$ in the direction $\xi\in\mathcal{V}$, i.e.

$$D_\xi f(u) = \lim_{\tau\to 0}(f(v+\tau\xi)-f(v))/\tau.$$

Here $\mathcal{V}_1$ are finite-dimensional scalar product spaces.

If the bilateral directional derivative $D_\xi \mathrm{proj}^A(\cdot,F)(v)$ of $v\mapsto \mathrm{proj}^A(v,F)$ at the point $v\in\mathcal{V}$ in the direction $\xi\in\mathcal{V}$ exists, then the existence of $D_\xi V(v)$ and the formula

$$D_\xi V(v) = 2\left\langle \xi - D_\xi\mathrm{proj}^A(\cdot,F)(v), A(v-\mathrm{proj}^A(v,F))\right\rangle \tag{3.4}$$

follow by observing that

$$\left(\|v-\mathrm{proj}^A(v,F)\|^A\right)^2 = \left\langle v-\mathrm{proj}^A(v,F), A(v-\mathrm{proj}^A(v,F))\right\rangle,$$

see Lemma B.2. What we significantly use in the proof of Theorem 3.1 is that any directional derivative of $v\mapsto\mathrm{proj}^A(v,F)$ is orthogonal to $v-\mathrm{proj}^A(v,F)$, as it is a projection on the respective critical cone (see Lemma B.4 in Appendix B), so that formula (3.4) reduces to $D_\xi V(v) = 2\left\langle \xi, A(v-\mathrm{proj}^A(v,F))\right\rangle$, meaning that $D_\xi V(v)$ is actually linear in $\xi$.

**Proof of Theorem 3.1.** Let $y(t)$ be an arbitrary solution of (1.4). For the function $x(t)$ given by (1.6) consider

$$v(t) = V(x(t)).$$

Note, that $x(t)$ is differentiable almost everywhere on $[0,\infty)$ because $c(t)$ is Lipschitz continuous. Since $v\mapsto\mathrm{proj}^A(v,F)$ is Lipschitz continuous (see e.g. Bauschke-Combettes [4, Proposition 4.16]), the function $t\mapsto\mathrm{proj}^A(x(t),F)$ is differentiable almost everywhere on $[0,\infty)$. Let us fix some $t \geq 0$ such that both $\mathrm{proj}^A(x(t),F)$ and $x(t)$ are differentiable at $t$. Then $D_{x'(t)}\mathrm{proj}^A(\cdot,F)(x(t))$ exists (see Lemma B.3) and by Lemma B.4 we conclude

$$D_{x'(t)}V(x(t)) = 2\left\langle x'(t), A(x(t) - \mathrm{proj}^A(x(t),F))\right\rangle.$$

Without loss of generality we can assume that $t \geq 0$ is chosen so that $V(x(t))$ is differentiable at $t$. Then (see Lemma B.3),

$$v'(t) = D_{x'(t)}V(x(t)) = 2\left\langle x'(t), A\left(x(t) - \mathrm{proj}^A(x(t),F)\right)\right\rangle. \tag{3.5}$$

By definition of normal cone, (3.3) implies

$$\langle -x'(t) - c'(t), A(\xi - x(t))\rangle \leq 0, \quad \text{for any } \xi \in C.$$

Therefore, taking $\xi = \mathrm{proj}^A(x(t),F)$ we conclude from (3.5) that

$$v'(t) \leq 2\left\langle -c'(t), A\left(x(t) - \mathrm{proj}^A(x(t),F)\right)\right\rangle. \tag{3.6}$$

Now we use assumption (1.5), which is equivalent to

$$-c'(t) + \varepsilon\zeta/\|\zeta\|^A \in N_C^A(v), \quad \text{for any } \zeta \in N_F^A(v),\ v \in F,$$

or, using the definition of the normal cone,

$$\left\langle -c'(t) + \varepsilon\zeta/\|\zeta\|^A, A(\xi - v)\right\rangle \leq 0, \quad \text{for any } \zeta \in N_F^A(v),\ v \in F,\ \xi \in C.$$

Therefore, letting $\xi = x(t)$, $v = \mathrm{proj}^A(x(t),F)$, and $\zeta = x(t) - \mathrm{proj}^A(x(t),F)$, we get

$$\left\langle -c'(t) + \varepsilon\frac{x(t) - \mathrm{proj}^A(x(t),F)}{\|x(t) - \mathrm{proj}^A(x(t),F)\|^A}, A\left(x(t) - \mathrm{proj}^A(x(t),F)\right)\right\rangle \leq 0,$$

which allows to further rewrite inequality (3.6) as

$$v'(t) \leq -2\varepsilon\left\langle \frac{x(t) - \mathrm{proj}^A(x(t),F)}{\|x(t) - \mathrm{proj}^A(x(t),F)\|^A}, A(x(t) - \mathrm{proj}^A(x(t),F))\right\rangle = -2\varepsilon\sqrt{v(t)}.$$

Therefore, the Lyapunov function (3.2) satisfies (1.1) and so $v(t) = 0$, $t \in [\tau, \tau_d]$.

To prove that $y(t) - c(t)$ is constant on $[\tau, \tau_d]$ we introduce $z(t) = y(\tau) - c(\tau) + c(t)$, $t \in [\tau, \tau_d]$. Then $-z'(t) = -c'(t)$ a.e. on $[\tau, \tau_d]$. Since we already proved that $y(t) \in F + c(t)$ on $[\tau, \tau_d]$, we have $y(t) - c(t) \in F$ on $[\tau, \tau_d]$, and so, by (1.5),

$$-c'(t) \in N_C^A(y(t) - c(t)) = N_{C(t)}^A(y(t)) \quad \text{a.e. on } [\tau, \tau_d].$$

Therefore, $z(t)$ is a solution of (1.4) on $[\tau, \tau_d]$ and since $z(\tau) = y(\tau)$ by construction, we get $z(t) = y(t)$ on $[\tau, \tau_d]$ by the uniqueness of solutions of (1.4), see e.g. Kunze and Monteiro Marques [23, Theorem 3]. The proof of the theorem is complete. □

**4. Finite-time convergence to a vertex.** In this section we consider the case of $\mathrm{ri}(F) = \emptyset$ or, equivalently, $M = 0$. When $M = 0$, formula (2.2) reduces to

$$F = \bigcap_{(\alpha,j)\in I_0} \overline{\overline{L}}(\alpha, j). \tag{4.1}$$

In this case, of all the conditions (2.3), (2.4), (2.5), (2.6), (2.8), and (2.7), only conditions (2.3) and (2.7) are needed. These two conditions take the following form:

(4.2) Condition (2.3) : $\{y : y \in \overline{\overline{L}}(\alpha, j),\ (\alpha, j) \in I_0\}$ is a singleton $\{y_{*,0}\} \neq \emptyset$,

(4.3) Condition (2.7) : $y_{*,0} \in C$.

Corollary 4.1. *Let $\mathcal{V}$ be a $d$-dimensional linear subspace of $\mathbb{R}^m$ with scalar product (3.1), and $c : [0,\infty) \to \mathcal{V}$ be Lipschitz continuous. Assume that $C$ is given by (2.1) and $F$ is given by (4.1) with conditions (4.2) and (4.3) satisfied. Assume that there exists an $\varepsilon > 0$ such that*

$$-c'(t)+B_\varepsilon^A(0) \subset N_C^A(y_{*,0}) \text{ a.e. on } [0,\tau_d], \quad \tau_d \ \geq \tau, \ \ \tau = \frac{1}{\varepsilon}\cdot \max_{v\in C} \|v-y_{*,0}\|^A. \tag{4.4}$$

*Then, every solution $y$ of sweeping process (1.4) with the initial condition $y(0) \in C(0)$ satisfies $y(t) = y_{*,0} + c(t)$, $t \in [\tau, \tau_d]$. Furthermore, let $y_*(t)$, $t \geq \tau$, be the solution of (1.4) with the initial condition $y_*(\tau) = y_{*,0} + c(\tau)$. If $c(t)$ is $T$-periodic with $T \geq \tau$, then $y_*$ is a globally one-period stable $T$-periodic solution of (1.4) satisfying $y_*(t) = y_{*,0} + c(t)$, $t \in [\tau + jT, \tau_d + jT]$, $j \in \mathbb{N}$.*

**Proof of Corollary 4.1.** Since

$$N^A_{\{y_{*,0}\}}(y_{*,0}) = \mathcal{V},$$

the inclusion (1.5) takes the form of that of (4.4). Therefore, by Theorem 3.1, $y(\tau_d) = y_{*,0} + c(\tau_d)$. Since $c(t)$ is Lipschitz continuous, sweeping process (1.4) features uniqueness of solutions and so $y(t) = y_*(t)$, $t \geq \tau_d$. Since $c(t)$ is $T$-periodic, sweeping process (1.4) admits a $T$-periodic solution (by Brouwer fixed point theorem). Therefore, $y_*$ is a unique $T$-periodic solution of (1.4) (on $[\tau_d, \infty)$). Therefore, $y_*$ is the attractor of (1.4) by Massera-Krejci theorem (see Krejci [22, Theorem 3.14] or Gudoshnikov-Makarenkov [16, Theorem 4.6]). The proof is complete. □

More properties of $N_F^A(y)$ and $N_C^A(y)$ are required to obtain an applicable corollary of Theorem 3.1 in the case where $\mathrm{ri}(F) \neq \emptyset$.

**5. Finite-time convergence to a face.** Assume now that $\mathrm{ri}(F) \neq \emptyset$. To compute $N_C^A(y)$ and $N_F^A(y)$ for $y \in F$, we want to use the following corollary of (Rockafellar-Wets [29, Theorem 6.46]). Recall that $\mathrm{cone}\{\xi_1, ..., \xi_K\}$ stays for the cone formed by vectors $\xi_1, ..., \xi_K$.

Lemma 5.1. *Let $\mathcal{V}$ be a $d$-dimensional linear subspace of $\mathbb{R}^m$ with the scalar product (3.1). Consider*

$$\widetilde{C} = \bigcap_{k=1}^{K} \{y \in \mathcal{V} : \langle \widetilde{n}_k, Ay\rangle \leq c_k\}, \tag{5.1}$$

*where $\widetilde{n}_k \in \mathcal{V}$, $c_k \in \mathbb{R}$, $K \in \mathbb{N}$. If $\widetilde{I}(y) = \left\{k \in \overline{1,K} : \langle \widetilde{n}_k, Ay\rangle = c_k\right\}$, then*

$$N^A_{\widetilde{C}}(y) = \mathrm{cone}\left\{\widetilde{n}_k : k \in \widetilde{I}(y)\right\}.$$

Both, the statement of [29, Theorem 6.46] and a proof of Lemma 5.1 are given in appendix B.

In what follows, we call $\widetilde{n}_k$, $k \in \widetilde{I}(y)$, the active normal vectors of the set $\widetilde{C}$ at $y$.

To match the format of formula (5.1), we rewrite $L(\alpha, j)$ and $\overline{\overline{L}}(\alpha, j)$ as

$$L(\alpha, j) = \left\{y \in \mathcal{V} : \langle \alpha n_j, Ay\rangle \leq \alpha c_j^\alpha\right\}, \tag{5.2}$$

$$\overline{\overline{L}}(\alpha, j) = \left\{y \in \mathcal{V} : \langle -n_j, Ay\rangle \leq -c_j^\alpha\right\} \cap \left\{y \in \mathcal{V} : \langle n_j, Ay\rangle \leq c_j^\alpha\right\}. \tag{5.3}$$

Using the representations (5.2)-(5.3), we can formulate the active normals of $C$ and $F$ at $y$ as follows:

(5.4) active normals of $C$ at $y \in F : \{\alpha n_j : y \in \overline{\overline{L}}(\alpha, j)\} \cup \{\alpha n_j : (\alpha, j) \in I_0\}$,

(5.5) active normals of $F$ at $y \in F : \{\alpha n_j : y \in \overline{\overline{L}}(\alpha, j)\} \cup \{-n_j, n_j : (\alpha, j) \in I_0\}$.

Formula (5.4) uses condition (2.7) to make sure that the term $\{\alpha n_j : (\alpha, j) \in I_0\}$ is a part of $\{\alpha n_j : y \in \overline{\overline{L}}(\alpha, j)\}$. We keep this "redundant" term to ease the comparison between the formulas (5.4) and (5.5). Formula (5.5) uses assumption (2.4) to claim that all vectors of $\{\alpha n_j : y \in \overline{\overline{L}}(\alpha, j)\}$ are normal vectors of $F$.

Using assumption (2.5) we can conclude from (5.4)-(5.5) that the sets of active normals are given by

active normals of $C$ at $y_{*,i} : \{\alpha n_j : (\alpha, j) \in I_i\} \cup \{\alpha n_j : (\alpha, j) \in I_0\}, \quad i \in \overline{1, M}$,
active normals of $F$ at $y_{*,i} : \{\alpha n_j : (\alpha, j) \in I_i\} \cup \{-n_j, n_j : (\alpha, j) \in I_0\}, \quad i \in \overline{1, M}$.

Therefore, by (2.3) and Lemma 5.1,

$$
\begin{aligned}
N_C^A(y_{*,i}) &= \operatorname{cone}\{\alpha n_j : (\alpha, j) \in I_0,\ \alpha n_j : (\alpha, j) \in I_i\}, \qquad i \in \overline{1, M}, \\
N_F^A(y_{*,i}) &= \operatorname{cone}\{-n_j, n_j : (\alpha, j) \in I_0,\ \alpha n_j : (\alpha, j) \in I_i\}, \qquad i \in \overline{1, M}.
\end{aligned}
\tag{5.6}
$$

Using assumption (2.4) we can specify the lists of active normals at $y \in F$ as follows:

active normals of $C$ at $y \in F : \{\alpha n_j : (\alpha, j) \in \bigcap_{i \in J_y} I_i\} \cup \{\alpha n_j : (\alpha, j) \in I_0\}$,
active normals of $F$ at $y \in F : \{\alpha n_j : (\alpha, j) \in \bigcap_{i \in J_y} I_i\} \cup \{-n_j, n_j : (\alpha, j) \in I_0\}$.

Therefore, by (5.6),

$$
N_C^A(y) = \bigcap_{i \in J_y} N_C^A(y_{*,i}), \quad N_F^A(y) = \bigcap_{i \in J_y} N_F^A(y_{*,i}), \quad y \in F,
\tag{5.7}
$$

where $J_y$ are given by assumption (2.4).

Provided that condition (2.3) holds, the boundary of the $d$-dimensional cone $N_C^A(y_{*,i})$ is the following union of $(d-1)$-dimensional cones:

$$
\partial N_C^A(y_{*,i}) = \bigcup_{(\alpha_*, j_*) \in I_0 \cup I_i} \operatorname{cone}\{\alpha n_j : (\alpha, j) \in (I_0 \cup I_i) \backslash \{(\alpha_*, j_*)\}\},
\tag{5.8}
$$

see appendix A for a proof. We note that symbol $\partial$ refers to the boundary or subdifferential according to whether it is used with a set or with a function.

Now we use assumption (2.6) for the first time. This assumption allows us to conclude from (5.4)-(5.5) that

active normals of $C$ at $y \in \operatorname{ri}(F) : \ \{\alpha n_j : (\alpha, j) \in I_0\}$,
active normals of $F$ at $y \in \operatorname{ri}(F) : \ \{-n_j, n_j : (\alpha, j) \in I_0\}$.

Therefore, by (2.6) and Lemma 5.1,

$$
\begin{aligned}
N_C^A(y) &= \operatorname{cone}\{\alpha n_j : (\alpha, j) \in I_0\}, \qquad y \in \operatorname{ri}(F), \\
N_F^A(y) &= \operatorname{cone}\{-n_j, n_j : (\alpha, j) \in I_0\}, \qquad y \in \operatorname{ri}(F).
\end{aligned}
\tag{5.9}
$$

Corollary 5.2. *Let $\mathcal{V}$ be a d-dimensional linear subspace of $\mathbb{R}^m$ with scalar product (3.1), and $c : [0,\infty) \to \mathcal{V}$ be Lipschitz continuous. Assume that $F(t)$ is given by (2.2) with $F$ satisfying properties (2.3)-(2.5) and (2.7). Let $y_{*,i}$ be the vertices of $F$ given by (2.3). If there exists an $\varepsilon > 0$ such that*

$$-c'(t) + B^A_\varepsilon(0) \cap N^A_F(y_{*,i}) \subset N^A_C(y_{*,i}) \quad \text{a.e. on } [0,\tau_d],\ i \in \overline{1,M}, \tag{5.10}$$
$$\text{and } \tau_d \geq \tau, \quad \tau = \frac{1}{\varepsilon} \max_{v\in C,\ i\in\overline{1,M}} \|v - y_{*,i}\|^A,$$

*then every solution $y$ of (1.4) with the initial condition $y(0) \in C(0)$ satisfies $y(t) \in F(t)$, $t \in [\tau,\tau_d]$. In particular, if $c(t)$ is $T$-periodic with $T \geq \tau$ and if $\mathcal{Y}$ denotes the set of all solutions $y$ of (1.4) with the initial conditions $y(\tau) \in F(\tau)$, then the set $\mathcal{Y}$ is globally one-period stable for which and every $y_* \in \mathcal{Y}$ verifies $y_*(t) \in F(t)$ and $y'_*(t) - c'(t) = 0$, a.e. $t \in [\tau + jT, \tau_d + jT]$, $j \in \overline{1,\infty}$.*

**Proof.** We need to prove that (5.10) implies (1.5). Formulas (5.7) and (5.10) allow to conclude that, for any $y \in F$,

$$-c'(t) + B^A_\varepsilon(0) \cap N^A_F(y) \subset -c'(t) + B^A_\varepsilon(0) \cap N^A_F(y_{*,i}) \subset N^A_C(y_{*,i}), \quad i \in J_y.$$

Therefore,

$$-c'(t) + B^A_\varepsilon(0) \cap N^A_F(y) \subset \bigcap_{i\in J_y} N^A_C(y_{*,i}), \quad y \in F,$$

which implies (1.5) due to formula (5.7). □

For each $i \in \overline{1,M}$, we denote by $\mathcal{L}_i : \mathcal{V} \to \mathcal{V}$ the linear map which projects each element $\xi \in \mathcal{V}$ onto the subspace $\mathrm{lin}\{n_j : (\alpha,j) \in I_0\}$ along the subspace $\mathrm{lin}\{n_j : (\alpha,j) \in I_i\}$. The map $\mathcal{L}_i$ is well defined if condition (2.8) holds.

Lemma 5.3. *Let $\mathcal{V}$ be a d-dimensional linear subspace of $\mathbb{R}^m$ with scalar product (3.1). Let the polyhedron $C$ and its face $F$ be given by formulas (2.1) and (2.2). Assume that $|I_0| < d$ and*

$$-c_1 + B^A_{\varepsilon_0}(0) \cap N^A_F(y) \subset N^A_C(y), \quad \text{for any } y \in \mathrm{ri}(F), \tag{5.11}$$

*and for some $c_1 \in \mathcal{V}$. If conditions (2.3)-(2.8) hold, then*

$$-c_1 + B^A_{\varepsilon_i}(0) \cap N^A_F(y_{*,i}) \subset N^A_C(y_{*,i}), \quad \text{for any } i \in \overline{1,M},$$

*where*

$$\varepsilon_i = \varepsilon_0 / \|\mathcal{L}_i\|^A. \tag{5.12}$$

**Proof.** Fix $i \in \overline{1,M}$. By conditions of the lemma, the vectors $\{n_j : (\alpha,j) \in I_0 \cup I_i\}$ form a basis of $\mathcal{V}$. Therefore, using formulas (5.6) and (5.9), any $\xi \in N^A_F(y_{*,i})$ is uniquely decomposable as

$$\xi = \xi_1 + \xi_2, \quad \xi_1 \in N^A_F(y), \quad \xi_2 \in \mathrm{span}\{n_j : (\alpha,j) \in I_i\},$$

where $\xi_1 = \mathcal{L}_i\xi$ and $y$ is an arbitrary element of $\mathrm{ri}(F)$. Fix an arbitrary $\xi \in N^A_F(y_{*,i})$ with $\|\xi\|^A < \varepsilon_i$. Using condition (5.12), we can estimate the norm of $\xi_1$ as follows

$$\|\xi_1\|^A \leq \|\mathcal{L}_i\|^A \cdot \|\xi\|^A = \|\mathcal{L}_i\|^A \cdot \varepsilon_i = \varepsilon_0.$$

By (5.11),

$$-c_1 + \xi_1 \in N^A_C(y), \quad y \in \mathrm{ri}(F)$$

and so

$$-c_1 + \xi \in N^A_C(y) + \xi_2, \quad y \in \mathrm{ri}(F).$$

But $N^A_C(y) + \xi_2 \subset N^A_C(y_{*,i})$ by (5.6) and (5.9), which completes the proof. □

Corollary 5.4. *Let $\mathcal{V}$ be a d-dimensional linear subspace of $\mathbb{R}^m$ with scalar product (3.1). Let the polyhedron $C(t)$ and its face $F(t)$ be given by formulas (2.1) and (2.2). Assume that $\mathrm{ri}(F) \neq \emptyset$ and $c'(t) = c_1$ for all $t \geq 0$. Assume that $F$ satisfies conditions (2.3)-(2.8). If*

$$-c_1 \in \mathrm{ri}(N_C^A(y)), \quad y \in \mathrm{ri}(F),$$

*then there exists an $\varepsilon > 0$ such that (1.5) holds on any interval $[0, \tau_d]$ and, in particular, the solution $y$ of sweeping process (1.4) with any initial condition $y(0) \in C(0)$ satisfies $y(t) \in F(t)$ and $y'(t) - c_1 = 0$, for almost all sufficiently large $t > 0$.*

The conclusion of Corollary 5.4 follows by combining Lemma 5.3 and Corollary 5.2. The assumption on $c_1$ of Corollary 5.4 implies that the respective assumption of Lemma 5.3 holds for some $\varepsilon > 0$.

The statement of the following remark is a part of the proof of [16, Proposition 3.14].

*Remark* 5.5. Both $\max\limits_{v\in C,\ i\in\overline{1,M}} \|v - y_{*,i}\|^A$ from Corollary 5.2 and $\max\limits_{v\in C} \|v - y_{*,0}\|^A$ from Corollary 4.1 can be estimated using the following inequality:

$$\max_{u,v\in C} \|u - v\|^A \leq \|A^{-1}c^+ - A^{-1}c^-\|^A. \tag{5.13}$$

For completeness, we included a proof of formula (5.13) in Appendix A.

**6. Finite-time stability of elastoplastic systems with uniaxial displacement-controlled loading.** We remind the reader that according to Moreau [25] a network of $m$ elastoplastic springs on $n$ nodes with 1 displacement-controlled loading is fully defined by an $m \times n$ kinematic matrix $D$ associated with the topology of the network, $m \times m$ matrix of stiffnesses (Hooke's coefficients) $A = \mathrm{diag}(a_1, ..., a_m)$, an $m$-dimensional hyperrectangle $S = \prod_{j=1}^m [c_j^-, c_j^+]$ of the achievable stresses of springs (beyond which plastic deformation begins), a vector $R \in \mathbb{R}^m$ of the location of the displacement-controlled loading, and a scalar function $l(t)$ that defines the magnitude of the displacement-controlled loading. Such an elastoplastic system will be referred to as $(D, A, S, R, l(t))$. When all springs are connected (form a connected graph), we have (see Bapat [3, Lemma 2.2])

$$\mathrm{rank}\, D = n - 1. \tag{6.1}$$

We furthermore assume that

$$m \geq n \quad \text{and} \quad \mathrm{rank}(D^T R) = 1.$$

To formulate the Moreau sweeping process corresponding to the elastoplastic system $(D, A, S, R, l(t))$, we follow the 3 steps described in Gudoshnikov-Makarenkov [17, §5]:

1. Find an $n \times (n-2)-$matrix $M$ of $\mathrm{rank}(DM) = n-2$ that solves $R^T DM = 0$ and use $M$ to introduce $\mathcal{U}_{basis} = DM$.
2. Find a matrix $\mathcal{V}_{basis}$ of $m - n + 2$ linearly independent column vectors of $\mathbb{R}^m$ that solves $(\mathcal{U}_{basis})^T A \mathcal{V}_{basis} = 0$.
3. Find an $m \times (m - n + 1)-$matrix $D^\perp$ that solves $(D^\perp)^T D = 0_{(m-n+1)\times n}$ and

$$\mathrm{rank}(D^\perp) = m - n + 1. \tag{6.2}$$

With the new matrices introduced, the parameters of the sweeping process (1.4) corresponding to the elastoplastic system $(D, A, S, R, l(t))$ are given by

$$n_j = \mathcal{V}_{basis} W^{-1} \begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} e_j, \qquad W = \begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} \mathcal{V}_{basis}, \qquad \mathcal{V} = \mathcal{V}_{basis}\mathbb{R}^d,$$
$$c(t) = -\mathcal{V}_{basis} W^{-1} \begin{pmatrix} 1 \\ 0_{d-1} \end{pmatrix} l(t), \quad e_j = (\underbrace{0,...,0}_{j-1}, 1, 0, ..., 0)^T, \quad d = m - n + 2. \tag{6.3}$$

The existence of $W^{-1}$ is demonstrated in Gudoshnikov-Makarenkov [17, 16] for particular examples. Since this section intends to offer a general recipe, Lemma B.5 in the appendix features a proof of the invertibility of $W$ in the general case. The solution $y(t)$ of sweeping process (1.4) is linked to the vector of stresses $s(t) = (s_1(t), ..., s_m(t))^T$ of the springs of elastoplastic system $(D, A, S, R, l(t))$ by the formula $y(t) = A^{-1}s(t) + c(t)$.

Therefore, the conclusion $y(\tau_d) \in F(\tau_d)$ of Theorem 3.1 is equivalent to the inclusion $s(\tau_d) \in AF$, or, upon combining with Lemma B.10, to

$$s(\tau_d) \in \operatorname{conv}\{Ay_{*,1}, ..., Ay_{*,M}\}. \tag{6.4}$$

## 7. A step-by-step guide for analytic computations.

**Step 1. Find appropriate indexes $I_0$** *(springs that will reach plastic deformation).* Spot an $I_0 \subset \{-1, 1\} \times \overline{1, m}$ such that

$$\begin{pmatrix} 1 \\ 0_{m-n+1} \end{pmatrix} l'(t) \in \operatorname{cone}\left( \begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} \{\alpha e_j : (\alpha, j) \in I_0\} \right). \tag{7.1}$$

Definition 7.1. *We say that a family of indexes $I_0$ is irreducible, if $I_0$ cannot be represented in the form*

$$I_0 = \widetilde{I}_0 \cup \{(\alpha_*, j_*)\}, \tag{7.2}$$

*where $\widetilde{I}_0$ satisfies*

$$\begin{pmatrix} 1 \\ 0_{m-n+1} \end{pmatrix} l'(t) \in \operatorname{cone}\left( \begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} \left\{\alpha e_j : (\alpha, j) \in \widetilde{I}_0\right\} \right). \tag{7.3}$$

By Corollary B.7 (see below), $I_0$ with $|I_0| = d$ always exists. However, some $I_0$ with $|I_0| = d$ may appear to be reducible, in which case an irreducible subset of $I_0$ needs to be considered. Proposition 7.11 below explains why our results do not apply when $I_0$ is reducible. Intuitively, a vertex cannot be finite-time stable, if finite-time stability holds for the entire face that the vertex belongs to.

*Remark* 7.2. Relation (7.1) implies (see Appendix A for a proof) that

$$-c'(t) \in \operatorname{cone}\{\alpha n_j : (\alpha, j) \in I_0\}. \tag{7.4}$$

**Step 2. Fix appropriate indexes $I_i$** *(springs that may reach plastic deformation and that affect the convergence of springs $I_0$ to plastic deformation).* **Skip this step, if $|I_0| = d$.** We will consider the simplest possible way to design $I_i$ which ensures that $F \neq \emptyset$ and assumptions (2.2)-(2.7) are satisfied. This simplest way utilizes the minimal possible number $d - |I_0|$ of springs. The conditions to be imposed on the remaining $m-d$ springs will ensure that those $m-d$ springs do not affect the convergence of the stress vector to $F(t)$ and, in particular, do not undergo plastic deformation when close to $F(t)$.

Find some $I_1$ such that

$$|I_0| + |I_1| = d \tag{7.5}$$

and

$$\text{rank}\left(\left(\begin{array}{c} R^T \\ (D^\perp)^T \end{array}\right)(\{\alpha e_j : (\alpha, j) \in I_0 \cup I_1\})\right) = d. \tag{7.6}$$

Based on $I_1$ we can obtain more vertexes $I_i$ by changing the elements of $I_1$ from $(\alpha, n)$ to $(-\alpha, n)$. Let $I_i$, where $i \in \overline{1, M}$,

$$M = 2^{d-|I_0|}, \tag{7.7}$$

be all different families of indexes obtained through this process, i.e.

$$I_{i_1} \neq I_{i_2}, \quad i_1 \neq i_2, \quad i_1, i_2 \in \overline{1, M}. \tag{7.8}$$

Use $I_0$ and $I_i$, $i \in \overline{1, M}$, to define $F$ by formula (2.2).

**Step 3. Compute the vertexes of $F$ and impose conditions ensuring feasibility of $F$.** Depending on whether $|I_0| = d$ or $|I_0| < d$, compute $y_{*,0}$ or $y_{*,i}$, $i \in \overline{1, M}$, respectively, using the formula (see Appendix A for a proof)

$$y_{*,i} = \mathcal{V}_{basis}\left((\{e_j, (\alpha, j) \in I_0 \cup I_i\})^T A\mathcal{V}_{basis}\right)^{-1}\left(\left\{c_j^\alpha, (\alpha, j) \in I_0 \cup I_i\right\}\right)^T. \tag{7.9}$$

The feasibility condition (2.7) holds if

$$\begin{array}{lll} |I_0| < d: & c_j^- < \langle e_j, Ay_{*,i}\rangle < c_j^+, & i \in \overline{1, M},\ (\alpha, j) \notin I_0 \cup I_1 \cup ... \cup I_M, \\ |I_0| = d: & c_j^- < \langle e_j, Ay_{*,0}\rangle < c_j^+, & (\alpha, j) \notin I_0. \end{array} \tag{7.10}$$

Assumption (2.5) concerning non-coincidence of the vertices holds if

$$\begin{array}{ll} |I_0| < d: & c_j^- < c_j^+, \ \text{ for all } (\alpha, j) \in I_i,\ i \in \overline{1, M}, \\ |I_0| = d: & \text{does not apply.} \end{array} \tag{7.11}$$

We will say that relation (7.1) holds in a strict sense if, in addition to (7.1), the following property is satisfied:

$$\left(\begin{array}{c} 1 \\ 0_{m-n+1} \end{array}\right) l'(t) \notin \text{rb}\left(\text{cone}\left(\left(\begin{array}{c} R^T \\ (D^\perp)^T \end{array}\right)\{\alpha e_j : (\alpha, j) \in I_0\}\right)\right). \tag{7.12}$$

With the moving constraint $C(t)$ introduced in Section 6 and with the face $F$ introduced in Steps 1-3, the Corollaries 4.1 and 5.4 lead to the following qualitative description of the asymptotic behavior of the elastoplastic system $(D, A, C, R, l(t))$ and the associated sweeping process (1.4). In the sequel, by loosely saying "springs with indexes $I_0$" we refer to all springs $j$ for which either $(-1, j) \in I_0$ or $(1, j) \in I_0$.

Proposition 7.3. **(Conclusion of Steps 1-3).** *If $l'(t) = const$, relation (7.1) holds in a strict sense, and if properties (7.10) and (7.11) hold, then there exists an $\varepsilon > 0$ such that condition (1.5) is satisfied on any $[0, \tau_d]$. Accordingly, there exists $t_0 > 0$ such that, beginning $t = t_0$ and regardless of the initial distribution of stresses,* (i) *the springs of the elastoplastic system $(D, A, C, R, l(t))$ with the indexes $I_0$ all stay in the plastic mode;* (ii) *the stress vector $s(t)$ of $(D, A, C, R, l(t))$ holds a constant value satisfying (6.4).*

When $|I_0| = d$, the statement of Proposition 7.3 follows from Corollary 4.1 almost directly. Assumption (4.2) holds because $I_0$ is irreducible. Assumption (4.3) is satisfied by (7.10). Conditions (7.1) and (7.12) ensure the existence of $\varepsilon > 0$ for which (4.4) holds for any $t \geq 0$.

Considering $|I_0| < d$ and deriving the statement of Proposition 7.3 from Corollary 5.4 requires establishing validity of assumptions (2.3), (2.4), (2.5), (2.6), (2.8), and (2.7). Property (2.3) follows from (7.6). Property (2.5) follows from (7.8) and (7.11). Property (2.7) follows from (7.10). Property (2.8) coincides with (7.5). Verifying conditions (2.4) and (2.6) is less straightforward. This is done in the two propositions below.

Proposition 7.4. *Assume $M \geq 1$. Let $F$ be the face defined in Step 2. If (7.10) holds, then (2.4) holds as well. In other words, (7.10) implies that*

$$\{(\alpha, j) : y \in \overline{\overline{L}}(\alpha, j)\} = I_0 \cup \bigcap_{i \in I} I_i \quad \textit{for some } I \subset \overline{1, M} \textit{ or for } I = \emptyset.$$

**Proof.** Let $y \in F$ and $I_* = \{(\alpha, j) : y \in \overline{\overline{L}}(\alpha, j)\}$. By condition (7.10), $I_* = I_0 \cup I_{**}$, where $I_{**} \subset I_1 \cup \ldots \cup I_M$. By construction,

$$I_i = \{(\pm, n_{j_1}), \ldots, (\pm, n_{j_{d-|I_0|}})\}, \tag{7.13}$$

where different $i$ correspond to different choices of $''+''$ and $''-''$ in each symbol $''\pm''$. Therefore, $I_{**}$ can either be an empty set or a set of the form

$$I_{**} = \{(\alpha_1^{**}, n_{j_1^{**}}), \ldots, (\alpha_{d-|I_0|}^{**}, n_{j_{d-|I_0|}^{**}})\}, \tag{7.14}$$

where $\{j_1^{**}, \ldots, j_{d-|I_0|}^{**}\} \subset \{j_1, \ldots, j_{d-|I_0|}\}$. If $I_{**} = \emptyset$, then the proof is complete. So, from now on we assume that $I_{**} \neq \emptyset$.

From expressions (7.13) and (7.14), we see that $I_{**} \subset I_i$ for at least one index $i_0 \in \overline{1, M}$. Define $I$ as $I = \{i \in \overline{1, M} : I_{**} \subset I_i\}$.
Therefore

$$I_{**} \subset \bigcap_{i \in I} I_i. \tag{7.15}$$

Since the elements of $I_i \backslash I_{**}$ are obtained from the elements of $I_{i_0} \backslash I_{**}$ by taking all possible replacements of $(\alpha, n)$ by $(-\alpha, n)$, we have $\bigcap_{i \in I} (I_i \backslash I_{**}) = \emptyset$.
Therefore, $\bigcap_{i \in I} I_i \subset I_{**}$, and so (7.15) turns into equality. □

Lemma 7.5. *Assume that the face $F$ is given by (2.2). Assume that conditions (2.3) and (2.8) hold. If*

$$\textit{there exists } \bar{y} \in F \textit{ such that } \langle a e_j, A\bar{y}\rangle < \alpha c_j^{\alpha},\ (\alpha, j) \in I_i,\ i \in \overline{1, M}, \tag{7.16}$$

*then condition (2.6) is satisfied.*

**Proof.** *Part 1.* If (7.16) holds then there exists a full-dimensional ball $B_\delta(\bar{y})$ in $V$ such that $B_\delta(\bar{y}) \subset L(\alpha, j), \quad (\alpha, j) \in I_i,\ i \in \overline{1, M}$.

Therefore,

$$\operatorname{aff}(F) \supset \operatorname{aff}\left(\bigcap_{(\alpha,j)\in I_0} \overline{\overline{L}}(\alpha, j) \cap B_\delta(\bar{y})\right) = \operatorname{aff}\left(\bigcap_{(\alpha,j)\in I_0} \overline{\overline{L}}(\alpha, j)\right),$$

where $\operatorname{aff}(A)$ is the affine hull of the set $A$ (see [24]) Directly from the definition of $F$,

$$\operatorname{aff}(F) \subset \operatorname{aff}\left(\bigcap_{(\alpha,j)\in I_0} \overline{\overline{L}}(\alpha, j)\right).$$

So we conclude that

$$\operatorname{aff}(F) = \operatorname{aff}\left(\bigcap_{(\alpha,j)\in I_0} \overline{\overline{L}}(\alpha, j)\right). \tag{7.17}$$

*Part 2.* Consider $y \in F$ and assume that $y \in \overline{\overline{L}}(\alpha_*, j_*)$ for some $(\alpha_*, j_*) \in I_{i_*}$ and some $i_* \in \overline{1, M}$. By properties (2.3) and (2.8), the subspace (7.17) intersects the subspace $\overline{\overline{L}}(\alpha_*, j_*)$ transversally. Therefore, if we consider a ball of space (7.17) centered at $y$, then part of this ball will lie outside of $L(\alpha_*, j_*)$, hence $y \in \operatorname{rb}(F)$. Therefore, if $y \in \operatorname{ri}(F)$, then $y \notin \overline{\overline{L}}(\alpha, j)$, $(\alpha, j) \in I_i$, $i \in \overline{1, M}$, which completes the proof. □

Proposition 7.6. *Assume $M \geq 1$. Let $F$ be the face defined in Step 2. If (7.11) holds, then (7.16) holds as well.*

**Proof.** We will construct the required $\bar{y}$ as the solution of the following system of $d$ algebraic equations:

$$\begin{cases} \langle e_j, A\bar{y}\rangle = c_j^\alpha, & (\alpha, j) \in I_0, \\ \langle e_j, A\bar{y}\rangle = \dfrac{c_j^- + c_j^+}{2}, & (\alpha, j) \in I_i,\ i \in \overline{1, M}. \end{cases}$$

As in the proof of formula (7.9), this system admits a unique solution $\bar{y}$ because $F$ satisfies assumptions (2.3) and (2.8). Condition (7.11) yields $c_j^- < \dfrac{c_j^- + c_j^+}{2} < c_j^+$. Therefore, by construction,

$$\bar{y} \in \bigcap_{(\alpha,j)\in I_0} \overline{\overline{L}}(\alpha, j), \quad c_j^- < \langle e_j, A\bar{y}\rangle < c_j^+, \quad (\alpha, j) \in I_i,\ i \in \overline{1, M},$$

which implies (7.16). □

One has to proceed to Steps 4 and 5, if an estimate for $\tau_d$ is of interest.

**Step 4. Compute $\varepsilon_0$.** Our next argument will be based on application of Corollary 4.1 (when $|I_0| = d$) and Corollary 5.2 in combination with Lemma 5.3. This step is devoted to finding $\varepsilon$ for which the respective assumptions (4.4) and (5.11) hold. Assumptions (4.4) and (5.11) require computing the distance from $-c'(t)$ to the boundary of the cone $N_C^A(y)$ at the point $F$ when $F$ is a singleton and at the points of $\operatorname{ri}(F)$ when $\operatorname{ri}(F) \neq \emptyset$. In either case, the required boundary is $\partial\,\operatorname{cone}\{\alpha n_j : (\alpha, j) \in I_0\}$.

Using formula (5.8), we compute

$$\begin{aligned} \bar{\varepsilon}_0(t) &= \operatorname{dist}^A\left(-c'(t), \partial\operatorname{cone}\{\alpha n_j : (\alpha, j) \in I_0\}\right) = \\ &= \min_{(\alpha_*, j_*)\in I_0} \operatorname{dist}^A\left(-c'(t), \operatorname{cone}\{\alpha n_j : (\alpha, j) \in I_0 \backslash \{(\alpha_*, j_*)\}\}\right). \end{aligned} \tag{7.18}$$

The following lemma can be used to compute the distances from $-c'(t)$ to the required cones (see Appendix B for a proof of the lemma).

Lemma 7.7. *Assume that* $\{n_{i_1}, ..., n_{i_k}\}$ *is a linearly independent subset of vectors* $\{n_1, ..., n_m\}$. *Introduce* $\mathcal{N} = (n_{i_1} \ ... \ n_{i_k})$. *Then the matrix* $\mathcal{N}^T A \mathcal{N}$ *is invertible and, for any* $c' \in \mathbb{R}^m$,

$$\mathrm{dist}^A\left(-c', \mathrm{span}\left\{n_{i_1}, ..., n_{i_k}\right\}\right) = \left\|-c' - \mathrm{proj}^A\left(-c', \mathrm{span}\left\{n_{i_1}, ..., n_{i_k}\right\}\right)\right\|^A,$$

$$\mathrm{proj}^A\left(-c', \mathrm{span}\left\{n_{i_1}, ..., n_{i_k}\right\}\right) = -\mathcal{N}\left[\mathcal{N}^T A \mathcal{N}\right]^{-1} \mathcal{N}^T A c'. \tag{7.19}$$

Based on Lemma 7.7 we can rewrite formula (7.18) as

$$\bar{\varepsilon}_0(t) = \min_{(\alpha_*, j_*) \in I_0} \left\|-c'(t) - \mathrm{proj}^A\left(-c'(t), \mathrm{span}\left\{\alpha n_j : (\alpha, j) \in I_0 \backslash \{(\alpha_*, j_*)\}\right\}\right)\right\|^A,$$

$$\begin{aligned} &\mathrm{proj}^A\left(-c'(t), \mathrm{span}\left\{\alpha n_j : (\alpha, j) \in I_0 \backslash \{(\alpha_*, j_*)\}\right\}\right) = \\ &= -(\{n_j, \ (\alpha, j) \in I_0 \backslash \{(\alpha_*, j_*)\}\})\circ \\ &\circ \left[(\{n_j, \ (\alpha, j) \in I_0 \backslash \{(\alpha_*, j_*)\}\})^T A (\{n_j, \ (\alpha, j) \in I_0 \backslash \{(\alpha_*, j_*)\}\})\right]^{-1} \circ \\ &\circ (\{n_j, \ (\alpha, j) \in I_0 \backslash \{(\alpha_*, j_*)\}\})^T A c'(t). \end{aligned} \tag{7.20}$$

Choose $\varepsilon_0 > 0$ such that $\varepsilon_0 \le \bar{\varepsilon}_0(t)$ for all $t \in [0, \tau_d]$. Corollary 4.1, Remark 5.5 and formula (6.4) lead to the following conclusion.

Proposition 7.8. **(Conclusion of Steps 1-4).** *Assume that* $|I_0| = d$, *i.e.* $F = \{y_{*,0}\}$. *If conditions (7.1) and (7.10) hold on* $[0, \tau_d]$, *then (1.5) holds on the same time interval. If, in addition,*

$$\tau_d \ge \tau, \quad \tau = \frac{1}{\varepsilon_0} \cdot \|A^{-1}c^+ - A^{-1}c^-\|^A,$$

*then, for any initial distribution of stresses* $s(0)$ *in the elastoplastic system* $(D, A, C, R, l(t))$, *(i) the springs with the indexes* $I_0$ *undergo plastic deformations on the time interval* $[\tau, \tau_d]$, *(ii)* $s(t) = Ay_{*,0}$, $t \in [\tau, \tau_d]$. *If* $l(t)$ *is* $T$*-periodic with* $T \ge \tau_d$, *then (i) the springs with the indexes* $I_0$ *undergo plastic deformations on* $[\tau + jT, \tau_d + jT]$, $j \in \overline{0, \infty}$, *(ii)* $s(t) = Ay_{*,0}$ *on the same time intervals. In particular, if* $l(t)$ *is* $T$*-periodic, then* $s(t)$ *exhibits a unique* $T$*-periodic behavior after the time* $\tau$.

One more step is required to produce an estimate for $\tau_d$ when $|I_0| < d$.

**Step 5. Compute $\boldsymbol{\sigma_i}$.** Having found $\varepsilon_0$ for which (5.11) holds, we can now use Lemma 5.3 to compute $\varepsilon$ for which assumption (5.10) of Corollary 5.2 is satisfied. Specifically, formula (5.12) of Lemma 5.3 implies that the required $\varepsilon$ is given by

$$\varepsilon = \varepsilon_0 \min_{i \in \overline{1,M}} (1/\|\mathcal{L}_i\|^A) = \varepsilon_0 \max_{i \in \overline{1,M}} \|\mathcal{L}_i\|^A.$$

Next lemma provides a computational formula for the linear map $\mathcal{L}_i$. In what follows, $[A]_k$ stays for the matrix formed by the first $k$ lines of the matrix $A$.

Lemma 7.9. *In the settings of Lemma 5.3, assume that the parameters of sweeping process (1.4) are given by (6.3). Then the linear map* $\mathcal{L}_i : \mathcal{V} \to \mathcal{V}$ *can be expressed through the following* $m \times m$ *matrix:*

$$\begin{aligned} &\mathcal{L}_i = (\{n_j, \ (\alpha, j) \in I_0\})\circ \\ &\circ \left[\left(\begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} (\{e_j, \ (\alpha, j) \in I_0\}, \{e_j, \ (\alpha, j) \in I_i\})\right)^{-1}\right]_{|I_0|} \begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix}. \end{aligned} \tag{7.21}$$

**Proof.** Since $\{n_j : (\alpha, j) \in I_0 \cup I_i\}$ is a basis of $\mathcal{V}$, we can decompose $\xi \in \mathcal{V}$ as

$$\xi = (\{n_j, \ (\alpha, j) \in I_0\}, \{n_j, \ (\alpha, j) \in I_i\}) \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$$

for some $\zeta_1 \in \mathbb{R}^{|I_0|}$ and $\zeta_2 \in \mathbb{R}^{|I_i|}$. On the other hand, $\xi = \mathcal{V}_{basis}v$ for some $v \in \mathbb{R}^d$. Combining this formula and formula (6.3) for normals $n_j$, we get

$$\mathcal{V}_{basis}v = \mathcal{V}_{basis}W^{-1}\begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix}(\{e_j,\ (\alpha,j)\in I_0\},\{e_j,\ (\alpha,j)\in I_i\})\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}$$

or, equivalently,

$$\left[\begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix}(\{e_j,\ (\alpha,j)\in I_0\},\{e_j,\ (\alpha,j)\in I_i\})\right]^{-1}Wv = \begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}.$$

Therefore,

$$\left[\left(\begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix}(\{e_j,\ (\alpha,j)\in I_0\},\{e_j,\ (\alpha,j)\in I_i\})\right)^{-1}\right]_{|I_0|}\circ\begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix}\xi = \zeta_1,$$

which implies (7.21). □

In order to compute $\|\mathcal{L}_i\|^A$, we first observe that

$$\|\mathcal{L}_i\xi\|^A = \sqrt{\langle\mathcal{L}_i\xi, A\mathcal{L}_i\xi\rangle} = \sqrt{\left\langle\sqrt{A}\mathcal{L}_i\xi, \sqrt{A}\mathcal{L}_i\xi\right\rangle} = \|\sqrt{A}\mathcal{L}_i\xi\| =$$
$$= \|\sqrt{A}\mathcal{L}_i\sqrt{A^{-1}}\sqrt{A}\xi\| \le \|\sqrt{A}\mathcal{L}_i\sqrt{A^{-1}}\|\cdot\|\sqrt{A}\xi\| = \|\sqrt{A}\mathcal{L}_i\sqrt{A^{-1}}\|\cdot\|\xi\|^A.$$

Therefore, $\|\mathcal{L}_i\|^A \le \|\sqrt{A}\mathcal{L}_i\sqrt{A^{-1}}\|$. But, based on e.g. Friedberg et al. [13, §6.10, Corollary 1], $\|\sqrt{A}\mathcal{L}_i\sqrt{A^{-1}}\| = \sqrt{\sigma_i}$, where

(7.22) $\quad\sigma_i$ is the largest eigenvalue of the matrix $\left(\sqrt{A}\mathcal{L}_i\sqrt{A^{-1}}\right)^T\sqrt{A}\mathcal{L}_i\sqrt{A^{-1}}$.

Therefore, $\varepsilon_i$ can be computed as $\varepsilon_i = \varepsilon_0 \,/\, \max\limits_{i\in\overline{1,M}}\sqrt{\sigma_i}$.

Corollary 5.2, Remark 5.5, and formula (6.4) can now be summarized as follows.

Proposition 7.10. **(Conclusion of Steps 1-5).** *Assume that $|I_0| < d$. If (7.1) and (7.10) hold on $[0,\tau_d]$ then (1.5) holds on the same time interval. If, in addition,*

$$\tau_d \ge \tau, \quad \tau = \frac{\max\{\sqrt{\sigma_1},\ldots,\sqrt{\sigma_M}\}}{\varepsilon_0}\cdot\|A^{-1}c^+ - A^{-1}c^-\|^A,$$

*then all conclusions about the time intervals of plastic deformations stay the same as in Proposition 7.8. As for the stress vector $s(t)$, it holds a constant value $s(t)\in$ $\mathrm{conv}\{Ay_{*,1},...,Ay_{*,M}\}$ during each of the above-mentioned plastic deformations.*

We remind the reader that inclusion (7.1) is called strict, if (7.12) holds.

Proposition 7.11. *If $I_0$ is reducible, then inclusion (7.1) is never strict and, in particular, $\bar{\varepsilon}_0(t)$ given by formula (7.18) is necessarily zero.*

**Proof.** By definition, $I_0$ is representable as (7.2). Therefore, as in the proof of formula (5.8), we can conclude that $\mathrm{cone}\{\alpha n_j : (\alpha,j)\in\widetilde{I_0}\} \subset \mathrm{rb}\,(\mathrm{cone}\,\{\alpha n_j : (\alpha,j)\in I_0\})$. Hence, by (7.3), $-c'(t)\in\mathrm{rb}\,(\mathrm{cone}\,\{\alpha n_j : (\alpha,j)\in I_0\})$. Therefore, inclusion (7.1) is not strict (we use Remark 7.2 again) and $\bar{\varepsilon}_0(t)$ given by (7.18) vanishes. □

**8. Application to a system of elastoplastic springs.** The focus of the present section is on the elastoplastic model shown in Fig. 1 (earlier introduced in Rachinskii [28]), which allows to fully illustrate the practical implementation of Theorem 3.1. According to Gudoshnikov-Makarenkov [17, §2], the elastoplastic system of Fig. 1 leads to the following expressions for $D$ and $R$:

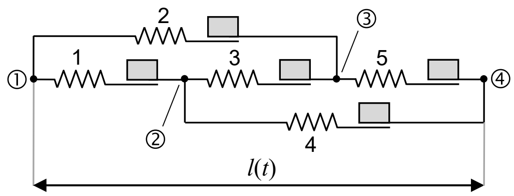


FIG. 1. *A system of 5 elastoplastic springs on 4 nodes that we investigate to illustrate our method. A displacement-controlled loading $l(t)$ is applied as the arrows show.*

$$D = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

We now follow Section 6 to formulate a sweeping process (1.4) corresponding to the elastoplastic system $(D, A, C, R, l(t))$. First of all, based on (6.3), we compute the dimension of sweeping process (1.4) as $d = m - n + q + 1 = 5 - 4 + 1 + 1 = 3$.

According to [17, §5, Step 1], we then look for an $4\times 2$ matrix $\mathcal{M}$ such that $R^T D\mathcal{M} = 0$ and such that the matrix $D\mathcal{M}$ is full rank. Such a matrix $\mathcal{M}$ can be taken as

$$\mathcal{M} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix} \qquad \text{with} \qquad D\mathcal{M} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & -2 \\ -1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The next step is determining $\mathcal{V}_{basis}$ which consists of $d = 3$ linearly independent columns of $\mathbb{R}^m = \mathbb{R}^5$ and solves $(DM)^T A\mathcal{V}_{basis} = 0$. Such a $\mathcal{V}_{basis}$ can be taken as

$$\mathcal{V}_{basis} = \begin{pmatrix} 0 & 1/a_1 & 1/a_1 \\ 0 & 1/a_2 & -1/a_2 \\ 1/a_3 & 0 & 1/a_3 \\ -1/a_4 & 1/a_4 & 0 \\ 1/a_5 & 1/a_5 & 0 \end{pmatrix} \qquad \text{with} \qquad A\mathcal{V}_{basis} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Finally, a $5 \times 2$ full rank matrix $D^\perp$ satisfying $(D^\perp)^T D = 0$ can be taken as

$$D^\perp = \begin{pmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 1 \\ -1 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{leading to} \qquad \begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 & 0 \end{pmatrix}.$$

In what follows, we consider two types of loading:

$$l(t) = l_0 + l_1 \cdot t, \quad t \geq 0, \tag{8.1}$$

$$l(t) = \begin{cases} l_0 + l_1 \cdot t, & t \in [0, T/2], \quad \text{extended to } [0, \infty) \\ l_0 + l_1 \cdot (T/2) - l_1 \cdot (t - T/2), & t \in [T/2, T], \quad \text{by } T\text{-periodicity}, \end{cases} \tag{8.2}$$

where $l_0, l_1, T > 0$ are fixed constants.

**Step 1.** To shorten the presentation, we address only two possible $I_0$ (of different cardinality) along with the corresponding inclusion (7.1):

$$\widehat{I}_0 = \{(+,1),(+,2)\}: \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} l_1 \in \operatorname{cone}\left(\left\{\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}\right),$$

$$\widetilde{I}_0 = \{(+,1),(-,3),(+,5)\}: \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} l_1 \in \operatorname{cone}\left(\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}\right).$$

**Step 2.** Since $|\widehat{I}_0| < d$, we need some $I_1$ satisfying (7.6). Let us pick $\widehat{I}_1 = \{(-,3)\}$. Since, by (7.7), $M = 2$, we have to use $I_1$ to make one additional vertex $I_2$, which is determined by $\widehat{I}_1$ uniquely as $\widehat{I}_2 = \{(+,3)\}$.

**Step 3.** For the above $\widehat{I}_0$, $\widehat{I}_1$, and $\widehat{I}_2$, we use formula (7.9) in order to compute

$$A\widehat{y}_{*,i} = \left(c_1^+, c_2^+, c_3^{2i-3}, c_1^+ - c_3^{2i-3}, c_2^+ + c_3^{2i-3}\right)^T, \quad i \in \{1,2\}, \tag{8.3}$$

as well as to formulate the respective feasibility condition (7.10) which consists of 4 two-sided inequalities

$$\text{for } \widehat{y}_{*,1}: \begin{array}{l} c_4^- \le c_1^+ - c_3^- \le c_4^+, \\ c_5^- \le c_2^+ + c_3^- \le c_5^+, \end{array} \qquad \text{for } \widehat{y}_{*,2}: \begin{array}{l} c_4^- \le c_1^+ - c_3^+ \le c_4^+, \\ c_5^- \le c_2^+ + c_3^+ \le c_5^+. \end{array} \tag{8.4}$$

For $\widetilde{I}_0$, formulas (7.9) and (7.10) give the following single vertex along with the respective feasibility condition:

$$A\widetilde{y}_{*,0} = \left(c_1^+, -c_3^- + c_5^+, c_3^-, -c_3^- + c_1^+, c_5^+\right), \qquad \begin{array}{l} c_2^- \le -c_3^- + c_5^+ \le c_2^+, \\ c_4^- \le -c_3^- + c_1^+ \le c_4^+. \end{array} \tag{8.5}$$

Sine each of the inclusions in Step 1 holds in a strict sense (i.e. the vector $(1,0,0)^T$ never belongs to the boundary of the respective cone), we can now use Proposition 7.3 to obtain the following statement about the evolution of the model shown in Fig. 1.

PROPOSITION 8.1. *(i) If elastic limits $c_i^-, c_i^+$ of the elastoplastic springs of the model of Fig. 1 satisfy the feasibility condition (8.4) with displacement-controlled loading (8.1), then there exists an $\varepsilon > 0$ such that springs with the indexes 1 and 2 undergo plastic deformation for all sufficiently large $t > 0$. During this plastic deformation, the stress vector of all 5 springs holds a constant value from the line segment $[A\widehat{y}_{*,1}, A\widehat{y}_{*,2}]$. (ii) If elastic limits $c_i^-, c_i^+$ satisfy the feasibility condition (8.5), then there exists an $\varepsilon > 0$ such springs 1, 3, and 5 undergo plastic deformation for all sufficiently large $t > 0$. During this plastic deformation, the stress vector of all 5 springs holds the constant value $A\widetilde{y}_{*,0}$.*

**Step 4. $|I_0| = 2$.** In this case, for any $(\alpha_*, j_*) \in I_0$, the set $I_0 \backslash \{(\alpha_*, j_*)\}$ consists of just one element $\{(\alpha, j)\}$ and formula (7.20) takes the form $\varepsilon_0(t) = \min\limits_{(\alpha,j)\in I_0} S_j$, with

$$S_j = \left\|-c'(t) - \operatorname{proj}^A(-c'(t), \operatorname{span}\{n_j\})\right\|^A, \ \operatorname{proj}^A(-c'(t), \operatorname{span}\{n_j\}) = -n_j \frac{n_j^T A c'(t)}{n_j^T A n_j}.$$

Therefore, for $\widehat{I}_0 = \{(+,1),(+,2)\}$, we get $\varepsilon_0 = \min\{\widehat{S}_1, \widehat{S}_2\}$, and a computation in Mathematica gives

$$\widehat{S}_1 = h(2,4), \quad \widehat{S}_2 = h(1,5), \quad h(i,j) = l_1 \sqrt{\frac{a_i\left(a_4 a_5 + a_3\left(a_4 + a_5\right)\right)}{a_i\left(a_3 + a_j\right) + a_4 a_5 + a_3\left(a_4 + a_5\right)}}.$$

$|\boldsymbol{I_0}| = \mathbf{3.}$ In this case, for any $(\alpha_*, j_*) \in I_0$, the set $I_0 \backslash \{(\alpha_*, j_*)\}$ consists of two elements $\{(\alpha_1, j_1), (\alpha_2, j_2)\}$ and formula (7.20) can be rewritten as

$$\varepsilon_0(t) = \min_{(\alpha_1,j_1),(\alpha_2,j_2)\in I_0} S_{j_1j_2}, \; S_{j_1j_2} = \|c'(t) - \mathrm{proj}(c'(t), \mathrm{span}\{n_{j_1}, n_{j_2}\})\|^A,$$

$$\mathrm{proj}^A(c'(t), \mathrm{span}\{n_{j_1}, n_{j_2}\}) = (n_{j_1} \; n_{j_2}) \begin{pmatrix} n_{j_1}^T A n_{j_1} & n_{j_1}^T A n_{j_2} \\ n_{j_2}^T A n_{j_1} & n_{j_2}^T A n_{j_2} \end{pmatrix}^{-1} \begin{pmatrix} n_{j_1}^T \\ n_{j_2}^T \end{pmatrix} A c'(t).$$

Therefore, for $\widetilde{I}_0 = \{(+,1), (-,3), (+,5)\}$, we have $\varepsilon_0 = \min\{\widetilde{S}_{13}, \widetilde{S}_{15}, \widetilde{S}_{35}\}$, and computation in Mathematica gives

$$\widetilde{S}_{13} = l_1\sqrt{\frac{a_2a_5}{a_2+a_5}}, \quad \widetilde{S}_{35} = l_1\sqrt{\frac{a_1a_4}{a_1+a_4}}, \quad \widetilde{S}_{15} = l_1\sqrt{\frac{a_2a_3a_4}{a_2a_3+a_2a_4+a_3a_4}}.$$

Proposition 7.8 (applied with $\tau_d = T/2$) leads to the following result about the evolution of the model of Fig. 1 .

PROPOSITION 8.2. *Assume that elastic limits of the elastoplastic springs of the model of Fig. 1 with loading (8.1) satisfy feasibility condition (8.5). Put*

$$\tau = \left(1/\min\left\{\widetilde{S}_{13}, \widetilde{S}_{15}, \widetilde{S}_{35}\right\}\right) \cdot \|A^{-1}c^+ - A^{-1}c^-\|^A.$$

*Then, for any initial distribution of stresses, springs 1, 3, and 5 undergo plastic deformation for $t \geq \tau$. For cyclic loading (8.2) with $T/2 > \tau$ (and the same feasibility condition), springs 1, 3, and 5 undergo plastic deformation on time intervals $[\tau + jT, T/2 + jT]$, $j \in \overline{0,\infty}$. During the above-mentioned plastic deformations, the stress vector of the 5 springs holds the constant value $A\widetilde{y}_{*,0}$ given by (8.5).*

**Step 5. Computing $\boldsymbol{\sigma_i}$.** For each of the vertexes $\widehat{I}_1$ and $\widehat{I}_2$ we setup the matrixes $\widehat{\mathcal{L}}_1$ and $\widehat{\mathcal{L}}_2$ according to formula (7.21) and use Mathematica to compute the corrections $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ as defined by formula (7.22). This gives

$$\widehat{\sigma}_1 = \widehat{\sigma}_2 = \max\left\{1, \frac{(a_1+a_4)(a_2+a_5)(a_4a_5+a_3(a_4+a_5))}{a_4a_5(a_3a_4+a_2(a_3+a_4)+a_3a_5+a_4a_5+a_1(a_2+a_3+a_5))}\right\}.$$

PROPOSITION 8.3. *Assume that elastic limits of the elastoplastic springs of the model of Fig. 1 with loading (8.1) satisfy feasibility condition (8.4). Put*

$$\tau = \left(\sqrt{\max\{\widehat{\sigma}_1, \widehat{\sigma}_2\}}/\min\left\{\widehat{S}_1, \widehat{S}_2\right\}\right) \cdot \|A^{-1}c^+ - A^{-1}c^-\|^A.$$

*Then, for any initial values of stresses, springs 1 and 2 undergo plastic deformation for $t \geq \tau$. For the cyclic loading (8.2) with $T/2 > \tau$ (and the same feasibility condition), springs 1 and 2 undergo plastic deformation on time intervals $[\tau + jT, T/2 + jT]$, $j \in \overline{0,\infty}$. During the above-mentioned plastic deformations, the stress-vector $s(t)$ of the 5 springs holds a constant value (that depends on $s(0)$) from the line segment $[A\widehat{y}_{*,1}, A\widehat{y}_{*,2}]$ given by (8.3).*

*Remark* 8.4. **(Implications for shakedown theory of continuous media)** In terms of the shakedown theory for continuous media (Kachanov [20, Ch. 9], Yu [33, §4.6]) the conclusions of Propositions 7.8, 7.10, 8.2, 8.3 imply that an elastoplastic structure that periodically crosses the extreme values will not shakedown (i.e. will not cease to fully elastic behavior over time), but will keep deforming plastically upon each cycle of loading. The corresponding result is known as Koiter's shakedown theorem in elastoplasticity literature (as opposed to Melan's shakedown theorem which gives conditions for shakedown to occur). The repeating plastic deformation guaranteed by the present paper usually appears in the form of alternating plasticity or ratcheting in the literature (see e.g. Yu [33, p. 62], Boissier et al [7, Fig. 1]) but our framework is not capable to identify the type of the underlying plastic deformation.

**9. Conclusions.** In this paper we adapted and applied the ideas of Adly et al. [2] about finite-time stability of frictional systems to finite-time stability of sweeping processes with polyhedral moving constraints. Our condition (1.5) takes the form (1.3) (copied from [2]) in the particular case when $F$ is a singleton, in which sense we extended the condition of [2] to cover the case of finite-time convergence to a given face instead of a single point. Based on these results, we proposed a step-by-step guide to analyze finite-time reachability of plastic deformation in networks of elastoplastic springs. The analysis was applied to an example of 5 elastoplastic springs on 4 nodes.

Our step-by-step guide of Section 7 addresses a particular (most straightforward) way of creating the list of scenarios of how the terminal distribution of plastic deformations (7.1) can be reached. Specifically, as seen from Step 2 of Section 7, we complemented the polyhedron given by the planes with indexes from (7.1) by suitable pairs of parallel facets to form a bounded polyhedron of dimension $d$. Extending the list of scenarios by using arbitrary available facets (not necessarily pairs of parallel facets) is a technical task in the field of discrete geometry that we omitted in the present paper. Generalizing (7.1) to the case of multiple displacement-controlled loadings and allowing for stress-controlled loadings (where the shape of the moving constraint changes with time [16]) is a subject of future research. Investigating finite-time stability of sweeping process (1.4) in the case of moving constraint of changing shape will also be required to account for hardening and softening of the elastoplastic system (of Section 6), see Chaboche [9]. Another limitation of the paper is that, when loading is periodic and the periodic attractor is a family of functions, we prove one-period stability of the entire face that contains the periodic attractor, not one-period stability of the periodic attractor itself (unlike Colombo et al. [10]). The non-periodic properties of the finite-time attractor in case of a non-periodically moving constraint is also an open question (the ideas of Kamenskii et al. [21] can help in the quasi-periodic case).

Although the focus of the present paper is on applications in elastoplasticity, the finite-time stability results of Section 3 can be applied to sweeping processes of other applied sciences, e.g. electrical circuits (see Acary et al. [1]). Extending the results to perturbed sweeping processes would enlarge the domain of applications further (to allow for more complex electric circuits, swarms of robots, traffic control problems, see Acary et al. [1], Colombo et al. [11], Hedjar-Bounkhel [19]). The results on disturbance rejection for Lyapunov functions of type (1.1) (see e.g. Orlov [26], Santiesteban et al. [31]) might be useful in this regard (the perturbation term of perturbed sweeping process could be viewed as the disturbance).

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## Appendix

### Appendix A. Skipped proofs.

**Proof of implication (2.3) $\Longrightarrow$ (5.8).** By definition (5.6), if $\xi \in N_C^A(y_{*,i})$, then there exist non-negative numbers $\lambda_1, ..., \lambda_d$ such that

$$\xi = (\{\alpha n_j : (\alpha, j) \in (I_0 \cup I_i)\}\})(\lambda_1, ..., \lambda_d)^T.$$

But by (2.3), $\{\alpha n_j : (\alpha, j) \in (I_0 \cup I_i)\}$ is a basis of $V$. Therefore, the correspondence between $\xi \in N_C^A(y_{*,i})$ and non-negative $\lambda_1, ..., \lambda_d$ is one-to-one. Therefore, any $\xi \in N_C^A(y_{*,i})$ for which the corresponding $\lambda_1, ..., \lambda_d$ contains $\lambda_i = 0$ is from $\partial N_C^A(y_{*,i})$, which is exactly the statement of formula (5.8). □

**Proof of formula (5.13).** By (2.1), $\frac{1}{a_j} c_j^- \le y_j \le \frac{1}{a_j} c_j^+$, for all $y \in C$. Therefore,

$$\max_{u,v\in C} \left(\|u - v\|^A\right)^2 = \sum_{j=1}^m a_j (u_j - v_j)^2 \le \sum_{j=1}^m \frac{1}{a_j} (c_j^+ - c_j^-)^2 = \left(\|A^{-1}c^+ - A^{-1}c^-\|^A\right)^2 . \square$$

**Proof of the equivalence (7.1) $\Longleftrightarrow$ (7.4).** Statement (7.1) implies the existence of $(\lambda_1, ..., \lambda_{|I_0|})$ such that

$$\mathcal{V}_{basis} W^{-1} \begin{pmatrix} 1 \\ 0_{m-n+1} \end{pmatrix} l'(t) = \mathcal{V}_{basis} W^{-1} \left\{ \begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} (\{\alpha e_j : (\alpha, j) \in I_0\}) \right\} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{|I_0|} \end{pmatrix}.$$

Due to (6.3), the latter formula just coincides with

$$-c'(t) = \{\alpha n_j : (\alpha, j) \in I_0\} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{|I_0|} \end{pmatrix},$$

which yields (7.4). □

**Proof of the implication (2.2)-(2.7) $\Longrightarrow$ (7.9).** Based on formula (2.3), finding $y_{*,i}$ means solving a system of $d$ algebraic equations

$$\langle e_j, A y_{*,i} \rangle = c_j^\alpha, \quad (\alpha, j) \in I_0 \cup I_i,$$

or, equivalently,

$$\left(\{e_j, (\alpha, j) \in I_0 \cup I_i\}\right)^T A \mathcal{V}_{basis} v_{*,i} = \left(\{c_j^\alpha, (\alpha, j) \in I_0 \cup I_i\}\right)^T,$$

where $y_{*,i} = \mathcal{V}_{basis} v_{*,i}$. □

## Appendix B. Technical lemmas.

LEMMA B.1. *If a non-negative continuously differentiable function $v(t)$ satisfies the differential inequality $v'(t) \le -2\varepsilon\sqrt{v(t)}$, then $v(t_1) = 0$ for some $t_1 \le \frac{1}{\varepsilon} v(0)$.*

**Proof.** The proof follows by observing that the solution of the differential equation $\bar{v}'(t) = -2\varepsilon\sqrt{\bar{v}(t)}$ with $\bar{v}(0) \ge 0$ is given by $\bar{v}(t) = \left(-\varepsilon t + \sqrt{\bar{v}(0)}\right)^2$ on $[0, \bar{t}_1]$, where $\bar{t}_1 = (1/\varepsilon)\sqrt{\bar{v}(0)}$. □

LEMMA B.2. *Consider $f, g : \mathcal{V} \to \mathcal{V}_1$, where $\mathcal{V}$, $\mathcal{V}_1$ are scalar product spaces. If both $D_\xi f(v)$ and $D_\xi g(v)$ exist then $D_\xi \langle f(\cdot), g(\cdot) \rangle (v)$ exists and*

$$D_\xi \langle f(\cdot), g(\cdot) \rangle (v) = \langle D_\xi f(v), g(v) \rangle + \langle f(v), D_\xi g(v) \rangle .$$

**Proof.** We have

$$D_\xi \langle f(\cdot), g(\cdot) \rangle (v) = \lim_{\tau \to 0} \frac{\langle f(v + \tau\xi), g(v + \tau\xi) \rangle - \langle f(v), g(v) \rangle}{\tau} =$$
$$= \left\langle \lim_{\tau \to 0} \frac{f(v + \tau\xi) - f(v)}{\tau}, g(v) \right\rangle + \left\langle f(v), \lim_{\tau \to 0} \frac{g(v + \tau\xi) - g(v)}{\tau} \right\rangle +$$
$$+ \lim_{\tau \to 0} \left\langle f(v + \tau\xi) - f(v), \frac{g(v + \tau\xi) - g(v)}{\tau} \right\rangle = \langle D_\xi f(v), g(v) \rangle + \langle f(v), D_\xi g(v) \rangle ,$$

where we used that

$$\left|\left\langle f(v+\tau\xi)-f(v),\frac{g(v+\tau\xi)-g(v)}{\tau}\right\rangle\right|^2\le\tau\cdot\left\|\frac{f(v+\tau\xi)-f(v)}{\tau}\right\|\cdot\left\|\frac{g(v+\tau\xi)-g(v)}{\tau}\right\|$$

by Cauchy-Schwartz inequality. □

LEMMA B.3. *Consider $f:\mathcal{V}\to\mathcal{V}_1$ and $u:\mathbb{R}\to\mathcal{V}$, where $\mathcal{V}$, $\mathcal{V}_1$ are scalar product spaces. If both $u'(t_0)$ and the derivative $(f\circ u)'(t_0)$ of $f\circ u$ exist at a point $t_0$ and if $f$ is Lipschitz continuous in the neighborhood of $u_0=u(t_0)$, then $D_{u'(t_0)}f(u_0)$ exists and $D_{u'(t_0)}f(u_0)=(f\circ u)'(t_0)$.*

**Proof.** We have

$$D_{u'(t_0)}f(u_0)=\lim_{\tau\to0}\frac{f(u_0+\tau u'(t_0))-f(u_0)}{\tau}=$$
$$=\lim_{\tau\to0}\left(\frac{f(u(t_0)+\tau u'(t_0))-f(u(t_0+\tau))}{\tau}+\frac{f(u(t_0+\tau))-f(u_0)}{\tau}\right)=(f\circ u)'(t_0),$$

where we used Lipschitz continuity of $f$ to conclude that the first fraction in the limit converges to 0 as $\tau\to0$. □

LEMMA B.4. *Let $\mathcal{V}$ be a scalar product space, $C\subset\mathcal{V}$ be a nonempty convex polyhedral set, $v\in\mathcal{V}$. Then $\mathrm{proj}(v,C)$ is directionally differentiable at $v$, and, for any $\xi\in\mathcal{V}$,*

$$D_\xi\,\mathrm{proj}(v,C)=\mathrm{proj}(\xi,\mathcal{C}_v),$$

*where $\mathcal{C}_v:=\{h\in T_C(\mathrm{proj}(v,C)):\langle v-\mathrm{proj}(v,C),h\rangle=0\}$ is the so-called critical cone and $T_C(\mathrm{proj}(x,C))$ is the tangent cone to $C$ at $\mathrm{proj}(v,C)$. In particular, if $v\in C$, then $\mathcal{C}_v=T_C(v)$.*

Lemma B.4 is a particular (polyhedral) case of [32, Theorem 3.1].

LEMMA B.5. *For $m\ge n$, consider a $m\times n$-matrix $D$ and $m\times(m-n+1)$-matrix $D^\perp$, such that $(D^\perp)^TD=0_{n\times(m-n+1)}$. If (6.1) and (6.2) hold, then*

$$D^\perp\mathbb{R}^{m-n+1}=(D\mathbb{R}^n)^\perp. \tag{B.1}$$

**Proof.** By the definition of $D^\perp$,

$$D^\perp\mathbb{R}^{m-n+1}\subset\mathrm{Ker}D^T. \tag{B.2}$$

Furthermore, we have

$$(D\mathbb{R}^n)^\perp=\mathrm{Ker}\,D^T, \tag{B.3}$$

see e.g. Friedberg et al. [13, Exercise 17, p. 367]. To prove the backwards implication in (B.1), we use (B.3) and assumption (6.1) to conclude that $\mathrm{Ker}\,D^T=\dim\left((D\mathbb{R}^n)^\perp\right)=m-n+1$. On the other hand, assumption (6.2) implies that $\dim\left(D^\perp\mathbb{R}^{m-n+1}\right)=m-n+1$ too. Therefore, the dimensions of the spaces in the two sides of (B.2) coincide and the inclusion (B.2) is actually an equality. □

COROLLARY B.6. *Assume that $m\ge n$. Let $R$ be an $m\times q$-matrix. Let $D^\perp$ be as defined in Lemma B.5. Consider $\mathcal{U}=\left\{x\in D\mathbb{R}^n:R^Tx=0\right\}$. If conditions (6.1) and (6.2) hold, then $x\in\mathcal{U}$ if and only if $\begin{pmatrix}R^T\\(D^\perp)^T\end{pmatrix}x=0$.*

**Proof.** The proof follows by observing that $(D^\perp)^T x = 0$ if and only if

$$x \in \mathrm{Ker}\left((D^\perp)^T\right) = \left(D^\perp \mathbb{R}^{m-n+1}\right)^\perp = \left((D\mathbb{R}^n)^\perp\right)^\perp = D\mathbb{R}^n,$$

where the first equality is the property that we already used in the proof of Lemma B.5 (see formula (B.3)) and the second equality is the conclusion of Lemma B.5. □

COROLLARY B.7. *In the settings of Corollary B.6, assume that* $\mathrm{rank}(D^T R) = q$, *in addition to (6.1) and (6.2). Put* $d = m - n + q + 1$. *Let* $\mathcal{V}_{basis}$ *be a matrix of* $d$ *linearly independent vectors of* $\mathbb{R}^m$ *which are orthogonal to vectors of* $\mathcal{U}$ *in some scalar product. Then,*

*(i) the* $d \times d$*-matrix* $\begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} \mathcal{V}_{basis}$ *is invertible,*

*(ii)* $\mathrm{rank}\begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} = m - n + q + 1.$

**Proof.** (i) If $\begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} \mathcal{V}_{basis} v = 0$ for some $v \in \mathbb{R}^d$, then $\mathcal{V}_{basis} v$ must be an element of $\mathcal{U}$ by Corollary B.6. On the other hand, vector $\mathcal{V}_{basis} v$ is orthogonal to the vectors of $\mathcal{U}$, which implies $\mathcal{V}_{basis} v = 0$ which can only happen if $v = 0$.

(ii) By the rank-nullity theorem (see e.g. Friedberg et al. [13, Theorem 2.3]) and by Corollary B.6 we have $\mathrm{rank}\begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix} = m - \dim\left(\ker\begin{pmatrix} R^T \\ (D^\perp)^T \end{pmatrix}\right) = m - \dim(\mathcal{U})$.

In this formula, $\dim(\mathcal{U}) = n - q - 1$ by Gudoshnikov-Makarenkov [16, Lemma 3.8]. □

LEMMA B.8. (Rockafellar-Wets [29, Theorem 6.46]) *Consider a polyhedron*

$$C = \bigcap_{k=1}^{K} \left\{v \in \mathbb{R}^d : \langle n_k, v\rangle \le c_k\right\},$$

*where* $n_k \in \mathbb{R}^d$, $c_k \in \mathbb{R}$, $K \in \mathbb{N}$. *If* $I(v) = \left\{k \in \overline{1,K} : \langle n_k, v\rangle = c_k\right\}$, *then*

$$N_C(y) = \mathrm{cone}\left\{n_k : k \in I(v)\right\}.$$

**Proof of Lemma 5.1.** Fix $y \in \mathcal{V}$. The definition of $N^A_{\widetilde{C}}(y)$ reads as

$$\left\langle N^A_{\widetilde{C}}(y), A(\widetilde{c} - y)\right\rangle \le 0, \quad \widetilde{c} \in \widetilde{C}. \tag{B.4}$$

Let $d$ be the dimension of $\mathcal{V}$ and let $\mathcal{V}_{basis}$ be a $m \times d$-matrix of some linearly independent vectors of $\mathcal{V}$. Then we can represent $\widetilde{C}$ as

$$\widetilde{C} = \mathcal{V}_{basis} C, \quad \text{where } C = \bigcap_{k=1}^{K} \left\{v \in \mathbb{R}^d : \langle n_k, v\rangle \le c_k\right\}, \ n_k = (A\mathcal{V}_{basis})^T \widetilde{n}_k.$$

Defining $v \in \mathbb{R}^d$ in such a way that $y = \mathcal{V}_{basis} v$, statement (B.4) can be rewritten as

$$\left\langle N^A_{\widetilde{C}}(\mathcal{V}_{basis} v), A(\widetilde{c} - \mathcal{V}_{basis} v)\right\rangle \le 0, \quad \widetilde{c} \in \mathcal{V}_{basis} C,$$

or

$$\left\langle (A\mathcal{V}_{basis})^T N^A_{\widetilde{C}}(\mathcal{V}_{basis} v), c - v\right\rangle \le 0, \quad c \in C.$$

But the definition of $N_C(v)$ reads as $\langle N_C(v), c - v\rangle \le 0$, $c \in C$.

Therefore, $(A\mathcal{V}_{basis})^T N^A_{\widetilde{C}}(\mathcal{V}_{basis} v) = N_C(v)$ or, incorporating the conclusion of Lemma B.8, $(A\mathcal{V}_{basis})^T N^A_{\widetilde{C}}(\mathcal{V}_{basis} v) = \mathrm{cone}\left\{(A\mathcal{V}_{basis})^T \widetilde{n}_k : k \in I(v)\right\}$,

from where the required statement follows. □

PROPOSITION B.9. *For any convex set* $F \subset \mathbb{R}^m$,

$$\mathrm{proj}^A(v, F) + c = \mathrm{proj}^A(v + c, F + c), \quad v, c \in F.$$

**Proof.** Indeed, let $v''_* = \mathrm{proj}^A(v+c, F+c)$. Then $v''_*$ satisfies one of the following three properties

$\min_{v''\in F+c} \|v+c-v''\|^A = \|v+c-v''_*\|^A$,
$\|v+c-v''\|^A > \|v+c-v''_*\|^A \quad$ for all $v'' \in F+c,\ v'' \neq v''_*$,
$\|v-v'\|^A > \|v+c-v''_*\|^A \quad$ for all $v' \in F,\ c+v' \neq v''_*$.

Introducing $v'_* = v''_* - c$, $\|v-v'\|^A > \|v-v'_*\|^A \quad$ for all $v' \in F,\ v' \neq v'_*$. Therefore, $\min_{v'\in F} \|v-v'\|^A = \|v-v'_*\|^A$, i.e. $v'_* = \mathrm{proj}(v,F)$. □

**Proof of Lemma 7.7.** Invertibility of the $k\times k$-matrix $\mathcal{N}^T A\mathcal{N}$ follows from the fact that $\mathrm{rank}(\sqrt{A}\mathcal{N}) = k$ and so $\mathrm{rank}(\mathcal{N}^T A\mathcal{N}) = \mathrm{rank}((\sqrt{A}\mathcal{N})^T\sqrt{A}\mathcal{N}) = k$, see e.g. Friedberg et al. [13, §6.3, Lemma 2]. To prove formula (7.19), we observe that

$$\mathrm{dist}^A\left(-c', \mathrm{span}\left\{n_{i_1},...,n_{i_k}\right\}\right) = \left\|-c' - \mathrm{proj}^A\left(-c', \mathrm{span}\left\{n_{i_1},...,n_{i_k}\right\}\right)\right\|^A.$$

By the definition of projection (see e.g. Bauschke-Combettes [4, §3.2]),

$$\mathrm{proj}^A\left(-c', \mathrm{span}\left\{n_{i_1},...,n_{i_k}\right\}\right) = \lambda_1 n_{i_1} + \ldots + \lambda_k n_{i_k},$$

where $\lambda_1,\ldots,\lambda_k \in \mathbb{R}$ minimize the quantity

$$\left\langle -c' - \lambda_1 n_{i_1} - \ldots - \lambda_k, A(-c' - \lambda_1 n_{i_1} - \ldots - \lambda_k)\right\rangle.$$

Therefore,

$$\begin{array}{rcl} \left\langle -c' - \lambda_1 n_{i_1} - \ldots - \lambda_k, A n_{i_1}\right\rangle & = & 0, \\ & \vdots & \\ \left\langle -c' - \lambda_1 n_{i_1} - \ldots - \lambda_k, A n_{i_k}\right\rangle & = & 0, \end{array}$$

for the unknown $\lambda_1,...,\lambda_k$, or, equivalently, $-\mathcal{N}^T Ac' - \mathcal{N}^T A\mathcal{N}(\lambda_1\ldots\lambda_k)^T = 0$.
Formula (7.19) follows by solving this equation for $(\lambda_1\ldots\lambda_k)^T$ and by plugging the result into $\mathrm{proj}^A\left(-c', \mathrm{span}\left\{n_{i_1},...,n_{i_k}\right\}\right) = \mathcal{N}(\lambda_1\ldots\lambda_k)^T$. □

LEMMA B.10. *If conditions (2.3), (2.4), and (2.8) hold, then all vertices of $F$ are contained in the set $\{y_{*,1},...,y_{*,M}\}$.*

**Proof.** Assume that $F$ has a vertex $\tilde{y}_* \notin \{y_{*,1},...,y_{*,M}\}$. We have

$$\{\tilde{y}_*\} = \{y : y \in \overline{\overline{L}}(\alpha,j),\ (\alpha,j) \in I_0 \cup \{j_1,...,j_{d-|I_0|}\}\},$$

where $|I_0 \cup \{j_1,...,j_{d-|I_0|}\}| = d$. By (2.4), $\{j_1,...,j_{d-|I_0|}\} = I_0 \cup \bigcap_{i\in J_{\tilde{y}_*}} I_i$. But $|I_i| = d - |I_0|$ by (2.8). Therefore, there exists $\tilde{i} \in J_{\tilde{y}_*}$ such that $\{j_1,...,j_{d-|I_0|}\} = I_{\tilde{i}}$, i.e. $\tilde{y}_* = y_{*,\tilde{i}}$. The proof of the lemma is complete. □

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