Extremal hinged lattices do not obey the theory of elasticity

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Abstract

Hinged lattices that attain extremal values of Poisson's ratio do not in general obey the theory of elasticity. Hinged structures that attain a Poisson's ratio of -1 are easy to stretch or deform volumetrically but they resist bending, in contrast to the predictions of elasticity theory. Their behavior corresponds to that of a Cosserat solid with divergent characteristic length. Hinged structures that attain a maximum Poisson's ratio are rigid with respect to hydrostatic tension, are easy to shear or to stretch but are unstable with respect to hydrostatic compression.

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1 Introduction and rationale

Hinged lattice structures have been the subject of recent study in the context of extremal elastic behavior. If one knows energy based bounds on continuum elastic properties, is it possible to attain the bounds? Composites, including cellular solids, lattices, and hinged lattices allow one to demonstrate attainability of bounds. Lattices are periodic two phase composites in which one phase is void space. As for extremal properties, the range of Poisson's ratio ν allowed for isotropic materials by energy considerations in the theory of elasticity [1] is $-1 < \nu < 0.5$. The upper limit is easily approached in soft rubbers as the density of cross-links in the polymer is reduced. A structure with ideal sliders and hinges [2] was predicted to exhibit a Poisson's ratio of -1. Foams with re-entrant folded in cells were made and were observed to have a Poisson's ratio of -0.7 or lower [3]. Lattices of hinged squares can attain a Poisson's ratio of -1. In the analysis of hinged lattice structures, it is assumed that the hinges are ideal, with no friction and no resistance to rotation, and that the lattice elements are perfectly rigid. These assumptions allow considerable simplification of analysis in comparison with lattices with flexible ribs or walls.

Hinged lattices are not continuous media; the theory of elasticity is a continuum theory. However common materials such as aluminum and glass are not continuous either; they are comprised of atoms. Heterogeneity in a physical material gives rise to deviations from predictions of classical elasticity. In the case of atomic structure, the heterogeneity gives rise to dispersion of waves at frequencies above 10^{11} Hz and has minimal effect in static experiments or in acoustics or ultrasound.

Defects in crystal structures can give rise to viscoelasticity. As for the hinged lattices, if they are ideal, one can scale down the size of the structural elements until they are invisible to the eye. The solid then appears to be continuous. Because it is so idealized, it cannot be physically made.

The theory of elasticity allows prediction of stress and deformation under various circumstances beyond those used in initial analysis or experiment. There is, however, more than one theory of elasticity. The classical theory allows points to translate freely. Stress is defined as force per area in the limit of small area. There are two independent elastic constants. A more restrictive version with one elastic constant had been considered based on a model of interatomic interactions; it required Poisson's ratio of 1/4 for all materials and was found to be inconsistent with experiment. If one allows points in the continuum to rotate as well as translate, the solid has the additional freedom of a Cosserat solid, which admits moment per area or couple stress as well as the usual stress, force per area. Some materials with microstructure have been observed to behave as Cosserat solids. Use of such a generalized continuum theory for lattice materials is intended to attain allow greater predictive power than is possible via classical elasticity.

In the present research, it is shown by geometrical construction that hinged lattices with zero bulk modulus or zero shear modulus do not obey the classical theory of elasticity.

2 Hinged structures

2.1 Overview

Conceptual models of hinged structures as elastic solids can be traced to Kelvin [4] who envisaged a pivoted lattice in the context of a mechanical model of the luminiferous aether. More recently, hinged structures comprised of periodic arrays of polygons or polyhedra have been studied from a mathematical perspective [5] [6] [7]. Two dimensional arrays containing rotating hexamers [8] [9] were predicted to exhibit a negative Poisson's ratio.

A Poisson's ratio of -1 was predicted via inverse homogenization analysis to occur in two dimensional arrays of rotating rigid squares connected at the corners by ideal hinges [10]. Arrays of rotating squares were considered [11] as a model for the negative Poisson's ratio observed in some crystals. A two dimensional model (Figure 1) containing an array of hinged rigid rotating squares [12] [13] was predicted to exhibit a Poisson's ratio of -1. Young's modulus is zero under tension or compression. The solid easily allows area change so the two dimensional bulk modulus is zero. The solid is rigid with respect to shear.

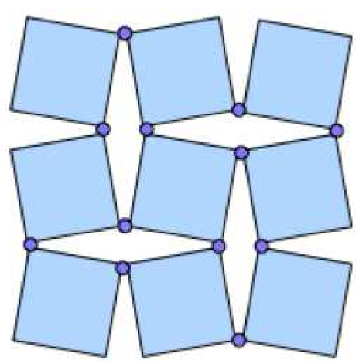

Figure 1: Hinged structure containing rotating squares adapted from [12]. Hinges are shown as small circles. Poisson's ratio is -1 provided the hinges are ideal.

Periodic arrays of hinged rotating triangles [14], rhombi [15], prisms [16] and cubes [17], also give rise to negative Poisson's ratio. Structures or materials with only one easy mode of deformation have been called unimode; the easy mode is volumetric deformation which entails a negative Poisson's ratio equal to the lower limit as well as a zero bulk modulus. Examples of such hinged structures have been analyzed [18]; some of these can attain arbitrarily large strain.

Extremal materials and structures have been classified according to the eigenvalues of the elasticity matrix [19] [20]. The material is unimode if there is one small eigenvalue. An isotropic solid with a Poisson's ratio approaching the lower limit -1 is described as unimode provided the deformation is affine. There is only one easy mode of deformation, a volume change. A material that is easy to deform in shear is called pentamode because there are five eigenvalues associated with shear.

Observe that for isotropic materials in three dimensions, the elastic constants Young's modulus E, shear modulus G and bulk modulus B are related by

$$E = \frac{9GB}{3B+G}. \tag{1}$$

If either the shear modulus or the bulk modulus vanishes, then Young's modulus vanishes as well. Such materials are easy to stretch.

2.2 Lattices with zero bulk modulus: bending

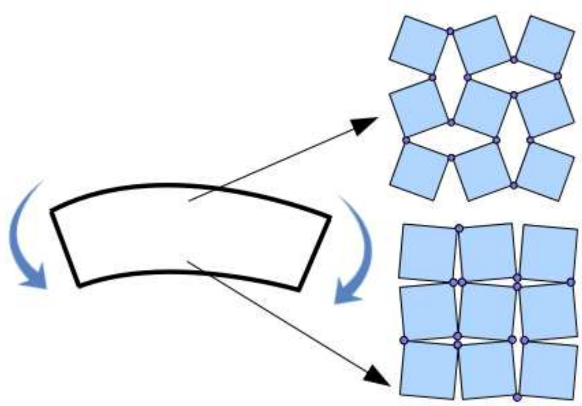

Figure 2: An attempt to bend a hinged structure containing rotating squares results in misfit of angular displacement of the squares. The solid is therefore rigid with respect to bending.

However if one attempts to bend an extremal hinged structure of squares, one observes that bending, in contrast to extension, is not an easy mode of deformation as illustrated in Figure 2. The squares rotate in one direction for tension and in the other direction for compression. If one attempts bending, the solid is rigid. The rotation of squares in the tensile region does not match the rotation in the compressive region. So the only allowable rotation if one attempts bending is zero. If the solid obeyed classical elasticity, the same zero Young's modulus observed in tension or compression should apply to bending as well.

A three-dimensional version of this structure is shown in Figure 3. In each layer, the cubes are hinged at their edges; adjacent layers are connected via pivots. The structure is shown fully extended in the xy plane. An attempt to bend the structure in the z direction by moments in the x or y directions results in no deformation because the hinged cubes cannot tilt with respect to each other out of plane. The solid is therefore rigid with respect to bending out of plane. The structure is also rigid with respect to torsion because the cubes cannot tilt out of plane.

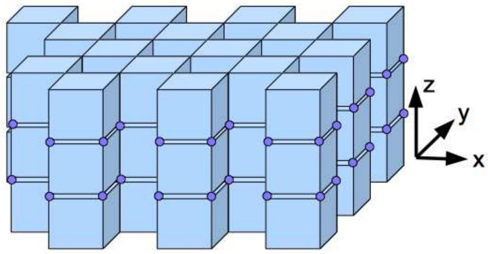

Figure 3: Three dimensional lattice of cubes hinged at their edges.

An anisotropic cube structure with cubes connected by pivots at corners [17] was predicted to exhibit zero Young's moduli in tension along principal directions, anisotropic negative Poisson's ratio and to be rigid with respect to bending and torsion. The role of anisotropy in bending is discussed in §3.1. The rigidity in both tension and bending depends on Young's modulus E in elasticity theory. There is a strain gradient in bending but classical elasticity is insensitive to strain gradients even if the solid is anisotropic.

2.3 Lattices with zero shear modulus: stability

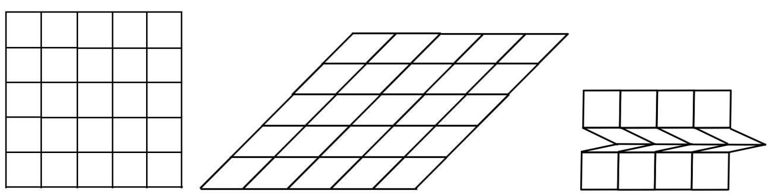

Figure 4: Square lattice of hinged ribs (left) is easy to shear globally (center) but it is also subject to local heterogeneous deformation (right).

A square lattice of ribs connected by hinges is easy to shear. It is also easy to stretch in directions 45° with respect to principal directions but is rigid to tension in principal directions so it is highly anisotropic. The lattice is rigid with respect to changes in area under hydrostatic tension. Such a lattice can exhibit local instability associated with heterogeneous deformation as illustrated in Figure 4. This is an example of non-affine deformation.

In three dimensions, a simple cubic hinged lattice has similar characteristics: compliance in shear, high anisotropy, and rigidity with respect to volume changes. Simple cubic hinged lattices also allow heterogeneous deformation and collapse.

The predictions of extremal behavior for unimode structures with zero bulk modulus is in the context of affine deformation [18]. Non-affine deformation can occur in structures that have additional modes of deformation that allow heterogeneous deformation and collapse [21]. Collapse modes are accessible in view of the instability in compression as discussed in §3.2.

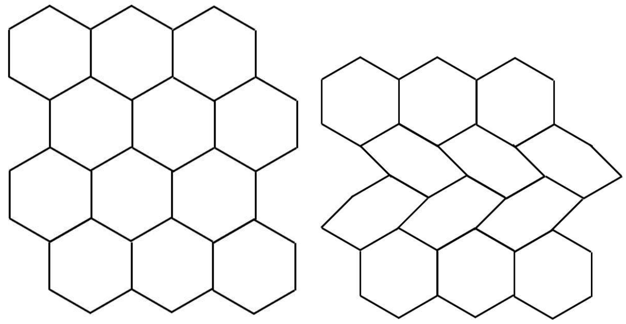

Figure 5: Hexagonal lattice of hinged ribs is easy to stretch or shear globally but it is also subject to local heterogeneous deformation.

A lattice of hinged ribs in an array of regular hexagons is also easy to shear or to stretch in any direction in its plane. It is described as bimode [19]. It has a Poisson's ratio +1 in plane for small strains. The lattice is rigid with respect to changes in area. As with the square lattice, the hexagonal lattice is subject to local heterogeneous deformation as illustrated in Figure 5. These deformation modes entail a decrease in volume.

In three dimensions, soft rubber has a high bulk modulus and a shear modulus that can be made as low as desired by reducing the concentration of cross links between polymer molecules. Poisson's ratio tends to 0.5 in the limit. If one prefers a hinged lattice, a structure in which four ribs meet at each pivoted point in a diamond structure is an example of a pentamode material [19] that is easy to shear but is stiff with respect to hydrostatic stress. Poisson's ratio is 0.5 for small strain. It is called pentamode because there are five eigenvalues associated with shear. The structure is shown in Figure 6. Polycrystalline diamond [22], by contrast, has a much lower Poisson's ratio of about 0.07; the interatomic bonds resist rotation and have compliance for stretching.

These lattices with zero shear modulus are rigid with respect to hydrostatic tension but are unstable in hydrostatic compression. There is no linear region about zero. This nonlinearity differs from that in the tilting square lattices which freely deform until a hard nonlinearity occurs because the squares come in contact or the lattice is fully extended. All the lattices will exhibit similar limits as well.

2.4 Lattices with zero shear modulus: bending and torsion

A hexagonal lattice constructed in three dimensions with plates hinged at their edges will have the same behavior as the two dimensional hexagonal lattice for in-plane deformation. The lattice is compliant in tension and in shear. The lattice is, however, rigid with respect to bending out of plane because the hinged plates cannot accommodate the out of plane tilt required for such bending. This rigidity, in contrast to the compliance for in-plane tension, is not consistent with the theory of elasticity, even with anisotropy.

As for the three dimensional hinged lattice with diamond structure (Figure 6), ball and socket pivots allow rotation of the ribs in all directions. If the joints are instead hinged to allow angles angles ξ, η, ζ to freely vary but not angle ψ, then the lattice becomes rigid with respect to torsion because rotation of ribs in the z direction about their own axis is constrained. In contrast to a classical continuum in which torsion is fully accommodated by a distribution of shear of differential elements, torsion of a lattice entails twisting or rotation of rib elements. Because the ribs are assumed to be rigid, rotation of the ribs is required. Rigidity in torsion combined with compliance

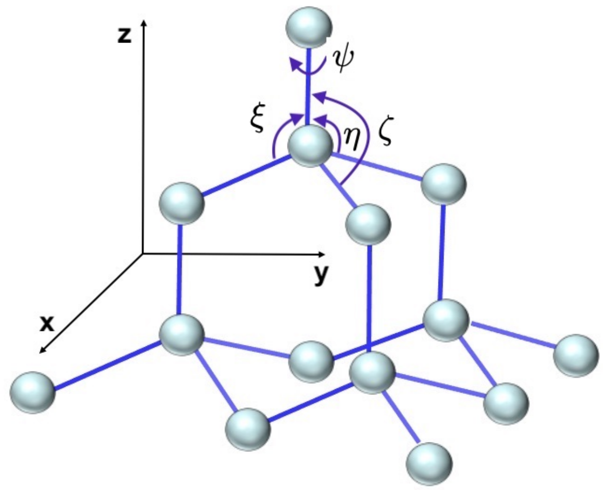

Figure 6: Diamond structure lattice of pivoted ribs is easy to stretch or shear globally and is rigid with respect to hydrostatic stress.

in shear is not consistent with the theory of elasticity. This rigidity, as with the rigidity with respect to hydrostatic compression, may not be stable in view of the rotational freedom remaining in the lattice.

2.5 Lattices with finite nonzero properties

For the above hinged lattices, effective elastic moduli are either zero or divergent. To obtain lattices with finite nonzero properties, one may envisage compliant rotational springs at the hinges or pivots to provide a restoring action so that the corresponding effective moduli are slightly greater than zero. Moreover one may envisage rib or plate elements that are not perfectly rigid so that corresponding moduli are finite. The comparative simplicity of analysis provided by the simplifying assumptions would then be lost, but the solid would be more realistic. One may also consider flexible lattices with structures similar to those of the hinged lattices, as follows.

2.6 Flexible lattices: comparison

Flexible lattices differ from hinged lattices but have some similarities. Flexible lattices in two dimensions, called honeycombs [23], allow extremal behavior provided the cell walls are slender. A honeycomb of regular hexagons has a Poisson's ratio for small strains approaching +1 in plane, the maximum allowable in two dimensions. Deformation is via bending of ribs or walls in contrast to the pure rotation that occurs in hinged lattices. Heterogeneous deformation can occur under compression but there is a threshold stress for this to occur because the physical mechanism is buckling of the cell ribs or walls [23]. By contrast, a hinged hexagonal lattice is unstable even for a minuscule compressive stress, provided the hinges are perfect as was assumed. It is indeed rigid for tensile hydrostatic stress but for two dimensional hydrostatic compression it is subject to collapse as discussed in §3.2.

For a lattice of slender ribs in cells with tetrakaidecahedron (truncated octahedron) shape [24] analysis predicts the bulk modulus to greatly exceed the shear modulus and Poisson's ratio to approach 0.5. To approach the limit, the ribs must become progressively more slender.

3 Analysis

3.1 Anisotropy and bending

Anisotropy cannot account for the different behavior in bending and tension. In anisotropic elastic solids [25], the deformation in bending is governed by the elastic constant for the long direction of the bar; this is the same elastic constant that governs compliance in tension. Specifically, tension of a bar of cross section area A in its long direction z by load P results in a displacement

$$u_z = \frac{P}{A}(S_{33}z + S_{34}y + S_{35}x) \tag{2}$$

with S_{33} as the compliance in the z direction. The stretch depends on that compliance which is the inverse of the corresponding Young's modulus. The other terms represent stretch-shear coupling from off diagonal components of the compliance.

The displacement in pure bending by a moment M in the x direction of a bar of length L and area moment of inertia I is

$$u_y = \frac{M}{2I}(-S_{33}(L - z^2) - S_{13}x^2 + S_{23}y^2 + S_{35}x(L - z)) \tag{3}$$

in which the bar long direction is z and the compliance in the z direction is S_{33}. If off-diagonal compliance elements such as S_{13} are nonzero, components of displacement in other directions occur.

The structural rigidity in bending and tension are governed by the same compliance element S_{33} therefore anisotropy cannot account for the different behavior in bending and tension.

3.2 Stability

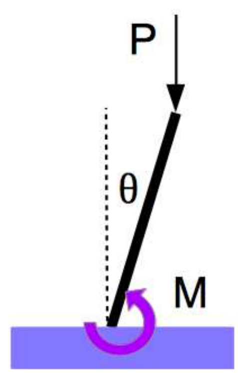

Figure 7: Stability of a column, pivoted at the base, under compression.

Columns with a hinge joint under compression (Figure 7) are known to be unstable; they can be stabilized by a spring which provides a restoring moment at the base as indicated by the curved arrow. Ribs in the lattices will behave in this way if they are provided with sufficient rotational freedom.

Hinged bars with rotational freedom are unstable with respect to compression as is illustrated in a classic buckling example, a hinged bar of length L is pressed by a vertical force P; a rotational spring provides a restoring moment $M = k\theta$ with θ as the angle with respect to the vertical (Figure 7). For small angles the total moment is $-k\theta - PL\theta$. The system is stable provided $k > PL$. If there is no stabilizing spring as for the hinged lattices with zero shear modulus, then instability occurs for any compressive force P no matter how small. These lattices that exhibit zero shear moduli are rigid for hydrostatic tension but they do not have a linear region about zero; they are unstable in compression. Indeed, for the lattices shown in Figure 4 and in Figure 5, accessible modes of heterogeneous deformation entail a volume decrease, hence a pathway to collapse.

The three dimensional hinged lattice with diamond structure (Figure 6) can decrease in volume as well: if it is stretched in one direction, for example the z direction, the ribs become aligned. As angles ξ, η, ζ increase, the volume decreases to small values. The unimode lattices, by contrast, are neutrally stable because a rotation of one structural element determines the rotation of all of the elements.

3.3 Cosserat elasticity

Cosserat elasticity predicts different response in bending vs. tension because, in contrast to classical elasticity, there is sensitivity to strain gradient. The constitutive equations for linear isotropic Cosserat [26] elasticity are:

$$\sigma_{ij} = 2G\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij} + \kappa e_{ijk}(r_k - \phi_k) \tag{4}$$

$$m_{ij} = \alpha\phi_{k,k}\delta_{ij} + \beta\phi_{i,j} + \gamma\phi_{j,i}. \tag{5}$$

Cosserat solids that admit local inertia are called micropolar [27]. The points in a Cosserat solid have rotational freedom ϕ_k, called micro-rotation. This local rotation in general differs from $r_k = \frac{1}{2}e_{klm}u_{m,l}$ which is the "macro" rotation based on the antisymmetric part of gradient of displacement u_i. e_{jkm} is the permutation symbol. The Cauchy stress σ_{ij} (force per unit area) in Cosserat elasticity can be asymmetric, in contrast to classical elasticity in which the stress is symmetric. Cosserat theory includes a couple stress m_{ij} (a torque per unit area) which balances the distributed moment from the asymmetric stress.

Cosserat elasticity admits six elastic constants for isotropic materials. These are λ, G, α, β, γ, κ. Constants λ and G have the same meaning as in classical elasticity. G is the shear modulus in the absence of gradients. Constants α, β, γ provide sensitivity to rotation gradient. The Cosserat constant κ quantifies the coupling between local rotation ϕ_k and the rotation r_k associated with displacement gradients. The characteristic length for torsion is defined as $\ell_t = \sqrt{\frac{\beta+\gamma}{2G}}$. The characteristic length for bending is defined as $\ell_b = \sqrt{\frac{\gamma}{4G}}$. The dimensionless coupling number is defined as $N = \sqrt{\frac{\kappa}{2G+\kappa}}$.

The bending size effect analysis for a square cross section bar [28] is complicated but it simplifies considerably if $\beta/\gamma = -\nu$. The rigidity ratio is defined as $\Omega = \frac{MR}{EI}$, with M as the applied moment, R as the radius of curvature, and E as Young's modulus. The rigidity ratio with w as the bar full width and depth, is then

$$\Omega = 1 + 24(\frac{\ell_b}{w})^2(1-\nu). \tag{6}$$

in which ν is Poisson's ratio. This relation is similar to the relation [29] for the round bar torsion size effect for $N = 1$. This equation is also equivalent to the exact solution obtained for bending of a plate to a cylindrical shape [29]. Bending of a round rod [30] results in similar size effects if $\beta/\gamma = -\nu$.

The bending analysis reveals the rigidity in bending exceeds the value expected based on Young's modulus in tension or compression; moreover, the excess stiffness increases as the bar thickness is reduced. As the characteristic length ℓ_b becomes large compared with the bar width w, the effective modulus in bending greatly exceeds Young's modulus in tension or compression. Such behavior occurs in the lattices with Poisson's ratio -1.

If the material is anisotropic, the characteristic lengths as well as the Young's and shear moduli may be considered as technical constants; the usual isotropic interrelations between elastic constants do not apply.

Also, Cosserat solids exhibit lower stress concentration factors [31] around holes and notches than classical elastic solids. This can be beneficial in reducing the vulnerability of materials to fracture in the presence of stress concentrators.

In summary, Cosserat elasticity predicts a higher bending rigidity than is anticipated from the Young's modulus in tension or compression. The unimode lattices exhibit an extreme example of this phenomenon.

4 Discussion

The affine unimode hinged solids may be regarded as strongly Cosserat elastic in the limit of diverging characteristic length. The solids are rigid with respect to bending or twisting but compliant with respect to tension. If a finite characteristic length is desired in these lattices, the hinges may be provided with a weak rotational spring that provides a restoring torque and the structural elements made stiff but not perfectly rigid. Imperfect hinges may also be realized as flexible links in lattices made by additive manufacturing. The flexible links are not hinges and they tend to approximate hinges poorly. Even so, some features of the hinged lattices may be approximated in all-flexible physical lattices. For example an ideal hinged cube structure [17] had zero stiffness in tension and negative Poisson's ratio but was rigid with respect to bending; a physical lattice made with flexible links rather than hinges [35] had a nonzero modulus in compression and was Cosserat elastic with a characteristic length on the order of the cell size rather than infinitely large. A lattice was designed and made with flexible hinge-like substructure in the ribs was observed to exhibit strong Cosserat size effects [36] of a factor of 30 compared with classical predictions. This lattice exhibited a positive Poisson's ratio of about 0.05 so extremal Poisson's ratio is not required for strong Cosserat effects to occur.

Saint-Venant's principle is of interest in extremal solids. For unimode materials and structures, there is only one easy mode of deformation, a volume change in three dimensions or an area change in two dimensions. If such a solid with Poisson's ratio -1 is subjected to a localized input such as a pair of forces in opposite directions, the effects will propagate through the full solid rather than decay with distance. Such a solid therefore does not obey Saint-Venant's principle. It is known that Saint-Venant's principle does not apply to solids at the limits of Poisson's ratio [32]. These effects are predicted in a classically elastic continuum and are not dependent on a heterogeneous hinged structure.

Not all lattices with a negative Poisson's ratio are unimode. For example, a honeycomb with inverted hexagonal cells has a negative Poisson's ratio, whether the ribs are flexible [33] [34] or hinged. The hinged version is considered to be bimode [19]. The hexagonal structure allows more freedom than that present in the lattice of rotating hinged squares.

Hinged rib lattices need not be either extreme Cosserat or unstable. Triangulated lattices such as the octet under the ideal assumptions are rigid under all deformation modes. If finite lattice rigidity is desired, the ribs may be given a finite modulus rather than assumed to be rigid. The Cosserat characteristic length will then be expected to be less than the cell size.

The size scale of a hinged lattice may be made arbitrarily small without altering the conclusions provided the ideal assumptions are fully satisfied. If the lattice is not ideal, the size scale has physical significance; the Cosserat characteristic lengths will then be finite and of magnitude on the order of the cell size as is known in flexible lattices [37].

5 Conclusion

Hinged lattices that attain extremal values of Poisson's ratio do not in general obey the theory of elasticity. Lattices with only one easy mode of deformation, a volume change or an area change, offer no resistance to stretching but are rigid with respect to bending. They may be regarded as extreme Cosserat solids. Lattices that are easy to shear but rigid with respect to hydrostatic tension are unstable in hydrostatic compression and are subject to collapse.

6 Acknowledgment

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