Quantum Scrambling of Observable Algebras

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In this paper we describe an algebraic/geometrical approach to quantum scrambling. Generalized quantum subsystems are described by an hermitian-closed unital subalgebra $\mathcal{A}$ of operators evolving through a unitary channel. Qualitatively, quantum scrambling is defined by how the associated physical degrees of freedom get mixed up with others by the dynamics. Quantitatively, this is accomplished by introducing a measure, the geometric algebra anti-correlator (GAAC), of the self-orthogonalization of the commutant of $\mathcal{A}$ induced by the dynamics. This approach extends and unifies averaged bipartite OTOC, operator entanglement, coherence generating power and Loschmidt echo. Each of these concepts is indeed recovered by a special choice of $\mathcal{A}$. We compute typical values of GAAC for random unitaries, we prove upper bounds and characterize their saturation. For generic energy spectrum we find explicit expressions for the infinite-time average of the GAAC which encode the relation between $\mathcal{A}$ and the full system of Hamiltonian eigenstates. Finally, a notion of $\mathcal{A}$-chaoticity is suggested.

Introduction.— Quantum dynamics can quickly spread information, which was initially encoded in some physical degrees of freedom, into a larger set of degrees of freedom, in this way quantum information gets *delocalized* and highly non-local correlations can be built. This so-called *quantum scrambling*, has over the last few years attracted a growing amount of attention in the context of quantum chaos and also quantum computing. The Out of Time Order Correlation functions (OTOCs) are among the most popular tools to analyze scrambling from a quantitative point view [1–7].

The goal of this paper is to lay down a novel formalism for quantum scrambling. Roughly speaking, we will characterize scrambling by how much a *whole* set of distinguished degrees of freedom gets far from itself by unitary evolution. The underlying philosophy of this paper is an extension of the observable-algebra approach to quantum subsystems originally advocated in [8, 9] (see also recent developments in [10, 11]). As such the strategy can be applied to situations in which there is no an *a priori* locality structure which gives a natural way of defining subsystems e.g., see [12].

We will show that specific instances of our construction allow one to recover apparently different concepts including operator entanglement [13, 14], averaged bipartite OTOCs [15, 16], coherence generating power [17–19] and Loschmidt echo [20, 21]. This conceptual unification provides one of the main motivations for this work. Another one is to design candidate tools for unveiling novel facets of quantum chaos.

For the sake of clarity, the main technical results of this paper are organized in "Propositions" whose proofs are in [1].

Preliminaries.— In this section we introduce the main formal ingredients utilized in this paper and set the notation. Let $\mathcal{H} = \text{span}\{|m\rangle\}_{m=1}^d$ be a d-dimensional Hilbert space and $L(\mathcal{H})$ its the full operator algebra (see [2] for further notation). In the following by the notation $\mathbf{C}\{X\}$ we will denotes the vector space spanned by the X's.

The key formal ingredients of this investigation are *hermitian-closed unital subalgebras* $\mathcal{A} \subset L(\mathcal{H})$ and their commutants $\mathcal{A}' := \{X \in L(\mathcal{H}) \,/\, [X, Y] = 0, \forall Y \in \mathcal{A}\}$. The intersection $\mathcal{A} \cap \mathcal{A}' =: \mathcal{Z}(\mathcal{A})$ is the *center* of the algebra $\mathcal{A}$. The fundamental structure theorem of these objects states that the Hilbert space breaks into a direct sum of $d_Z := \dim \mathcal{Z}(\mathcal{A})$ orthogonal blocks and each of them has a tensor product bi-partite structure: $\mathcal{H} = \oplus_J \mathcal{H}_J$, $\mathcal{H}_J, \cong \mathbf{C}^{n_J} \otimes \mathbf{C}^{d_J}$. Moreover,

$$\mathcal{A} \cong \oplus_J \mathbb{1}_{n_J} \otimes L(\mathbf{C}^{d_J}), \qquad \mathcal{A}' \cong \oplus_J L(\mathbf{C}^{n_J}) \otimes \mathbb{1}_{d_J}. \tag{1}$$

Whence, $d = \sum_J n_J d_J$, $\dim \mathcal{A} = \sum_J d_J^2 =: d(\mathcal{A})$ and $\dim \mathcal{A}' = \sum_J n_J^2 =: d(\mathcal{A}')$. Also, $\mathcal{Z}(\mathcal{A}) = \mathbf{C}\{\Pi_J :=$

[1] See supplemental material

[2] $L(\mathcal{H})$ has a Hilbert space structure via the Hilbert-Schmidt scalar product: $\langle X, Y\rangle := \text{Tr}\left(X^\dagger Y\right)$ and norm $\|X\|_2^2 := \langle X, X\rangle$. This equips the space of superoperators i.e., $L(L(\mathcal{H}))$ with the scalar product $\langle \mathcal{T}, \mathcal{F}\rangle := \text{Tr}_{HS}\left(\mathcal{T}^\dagger \mathcal{F}\right) = \sum_{l,m} \langle m|\mathcal{T}^\dagger \mathcal{F}(|m\rangle\langle l|)|l\rangle$, and the norm $\|\mathcal{T}\|_{HS}^2 = \langle \mathcal{T}, \mathcal{T}\rangle = \sum_{l,m} \|\mathcal{T}(|m\rangle\langle l|)\|_2^2$. If $\mathcal{T}(X) = \sum_i A_i X A_i^\dagger$, then $\|\mathcal{T}\|_{HS}^2 = \sum_{i,j} |\text{Tr}(A_i^\dagger A_j)|^2$

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$\mathbb{1}_{n_J}\otimes\mathbb{1}_{d_J}\}$, namely the center of $\mathcal{A}$ is spanned by the projections over the $\mathcal{H}_J$ blocks.

Associated to any algebra $\mathcal{A}$ we have an orthogonal (super) projection CP-map: $\mathbb{P}_{\mathcal{A}}^\dagger = \mathbb{P}_{\mathcal{A}}$, $\mathbb{P}_{\mathcal{A}}^2 = \mathbb{P}_{\mathcal{A}}$ and $\mathrm{Im}\,\mathbb{P}_{\mathcal{A}} = \mathcal{A}$. Such maps can be written in the Kraus form $\mathbb{P}_{\mathcal{A}}(X) = \sum_{\alpha=1}^{d(\mathcal{A}')} e_\alpha X e_\alpha^\dagger$, where the e_α are a suitable *orthogonal* basis of $\mathcal{A}'$. Notice that $\mathrm{Tr}_{HS}\mathbb{P}_{\mathcal{A}} = \sum_{\alpha=1}^{d(\mathcal{A}')} |\mathrm{Tr}\, e_\alpha|^2 = d(\mathcal{A})$. In terms of the decomposition (1) one has $\mathbb{P}_{\mathcal{A}}(X) = \sum_J \frac{\mathbb{1}_{n_J}}{n_J}\otimes \mathrm{tr}_{n_J}(X)$, and $\mathbb{P}_{\mathcal{A}'}(X) = \sum_J \mathrm{tr}_{d_J}(X)\otimes\frac{\mathbb{1}_{d_J}}{d_J}$.

These structural results provide the mathematical underpinnings of the theory of decoherence-free subspaces [22, 23], noiseless subsystems [24, 25] and in general to all quantum-information stabilizing techniques [25]. From the physical point of view two special cases are worth emphasizing:

Factors: $\mathcal{Z}(\mathcal{A}) = \mathbf{C}\mathbb{1}$, in this case $\mathcal{H}\cong\mathbf{C}^{n_1}\otimes\mathbf{C}^{d_1}$ namely the algebra $\mathcal{A}$ endows $\mathcal{H}$ with a bipartition into *virtual* subsystems [8, 9]. The case in which $\mathcal{H} = \mathcal{H}_A\otimes\mathcal{H}_B$ with $\mathcal{A} = L(\mathcal{H}_A)\otimes\mathbb{1}_B$ and $\mathcal{A}' = \mathbb{1}_A\otimes L(\mathcal{H}_B)$ clearly falls in this category.

Super-Selection: $\mathcal{A}'\subset\mathcal{A}$ this is when the commutant is an Abelian algebra. This implies $n_J = 1$, $(\forall J)$ and therefore the Hilbert space breaks into d_J-dimensional *super-selection* sectors i.e., $\mathcal{H}\cong\oplus_{J=1}^{d(\mathcal{A}')}\mathcal{H}_J$, $\mathcal{H}_J\cong\mathbf{C}^{d_J}$ and $\mathcal{A}\cong\oplus_{J=1}^{d(\mathcal{A}')}L(\mathbf{C}^{d_J})$. If $\mathcal{A}$ is a maximal abelian subalgebra one has $\mathcal{A} = \mathcal{A}'$ and $n_J = d_J = 1$, $(\forall J)$. This is the case that is relevant to the study of quantum coherence [26] and its dynamical generation [17, 27].

When the d_Z-dimensional (integer-valued) vectors $\mathbf{d} := (d_J)_J$, and $\mathbf{n} := (n_J)_J$ are proportional to each other i.e., $\mathbf{d} = \lambda\mathbf{n}$ one has that $d^2 = d(\mathcal{A})d(\mathcal{A}')$. If this is the case we shall say that the pair $(\mathcal{A},\mathcal{A}')$ is *collinear*. Note that both factors and maximal abelian subalgebras are of this type.

General results.— We are now in the position to define the central mathematical object of this paper: the *geometric algebra anti-correlator* (GAAC) by

$$G_{\mathcal{A}}(U) := 1 - \frac{\langle\mathbb{P}_{\mathcal{A}'},\mathbb{P}_{\mathcal{U}(\mathcal{A}')}\rangle}{\|\mathbb{P}_{\mathcal{A}'}\|_{HS}^2}. \tag{2}$$

The geometrical meaning of GAAC should be evident from Eq. (2): the larger $G_{\mathcal{A}}(U)$ the smaller is the intersection between $\mathcal{A}'$ and its unitarily evolved image $\mathcal{U}(\mathcal{A}') := \{\mathcal{U}(X)\,/\,X\in\mathcal{A}'\}$ [3]

Remark.– In the RHS of Eq. (2) we use the $\mathcal{A}'$ (and not $\mathcal{A}$) as the dynamics $\mathcal{U}$ is in the Heisenberg picture. Symmetries mapped out of $\mathcal{A}'$ by $\mathcal{U}$ is equivalent to states mapped out of $\mathcal{A}$ by $\mathcal{U}^\dagger$. This choice is somewhat arbitrary (See Prop. 1).

Algebraically, (2) measures how much the symmetries of the generalized quantum subsystem associated to $\mathcal{A}$ are dynamically broken by the channel $\mathcal{U}$. Let us now start by further unveiling the geometrical nature of GAACs. First notice that, using the algebra superprojections, one can define a distance between two algebras $\mathcal{A}$ and $\mathcal{B}$: $D(\mathcal{A},\mathcal{B}) := \|\mathbb{P}_{\mathcal{A}} - \mathbb{P}_{\mathcal{B}}\|_{HS}$. This metric structure allows one to draw a quite simple geometrical picture of algebra scrambling.

Proposition 1. *i) The GAAC is the (squared and normalized) distance between the algebra $\mathcal{A}'$ and its image $\mathcal{U}(\mathcal{A}')$.*

$$G_{\mathcal{A}}(U) = \frac{1}{2}\frac{D^2(\mathcal{A}',\mathcal{U}(\mathcal{A}'))}{d(\mathcal{A}')} \tag{3}$$

ii) $G_{\mathcal{A}}(U) = 0 \Leftrightarrow \mathcal{U}(\mathcal{A}') = \mathcal{A}' \Leftrightarrow \mathcal{U}(\mathcal{A}) = \mathcal{A}$. In words: the GAAC Eq. (2) vanishes if and only if both algebras $\mathcal{A}$ and $\mathcal{A}'$ are invariant under $\mathcal{U}$ i.e., there is no algebra scrambling.

The definition of GAAC given by Eq. (2) has the drawback of relying of superoperator projections and therefore may seem somewhat abstract and removed from practical calculations. Hence, before moving on to physical examples and applications of our formalism, we would like to re-express the GAAC at the more familiar operator level.

Proposition 2. *i) One can find an orthogonal basis of $\mathcal{A}$ $\{e_\alpha\}_{\alpha=1}^{d(\mathcal{A})}$ and an orthonormal basis of $\mathcal{A}'$ $\{f_\gamma\}_{\gamma=1}^{d(\mathcal{A}')}$ such that*

$$1 - G_{\mathcal{A}}(U) = \frac{\langle\Omega_{\mathcal{A}},\mathcal{U}^{\otimes 2}(\Omega_{\mathcal{A}})\rangle}{\|\Omega_{\mathcal{A}}\|_2^2} = \frac{\langle\tilde\Omega_{\mathcal{A}},\mathcal{U}^{\otimes 2}(\tilde\Omega_{\mathcal{A}})\rangle}{\|\tilde\Omega_{\mathcal{A}}\|_2^2} \tag{4}$$

where $\Omega_{\mathcal{A}} := \sum_{\alpha=1}^{d(\mathcal{A})} e_\alpha\otimes e_\alpha^\dagger$, and $\tilde\Omega_{\mathcal{A}} = \sum_{\gamma=1}^{d(\mathcal{A}')} f_\gamma\otimes f_\gamma^\dagger$. Also, $\tilde\Omega_{\mathcal{A}} = S\Omega_{\mathcal{A}}$, where S is the swap on $\mathcal{H}^{\otimes 2}$, and $\|\Omega_{\mathcal{A}}\|_2^2 = \|\tilde\Omega_{\mathcal{A}}\|_2^2 = d(\mathcal{A}')$.

ii) If $(\mathcal{A},\mathcal{A}')$ is collinear then $G_{\mathcal{A}}(U) = G_{\mathcal{A}'}(U), (\forall U)$.

In the above proposition, all the (Hilbert-Schmidt) scalar products and norms are ordinary operators ones. Moreover, the Ω's operator can be expressed in the same way if the bases e_α's and f_α's are replaced by unitarily equivalent ones. The connection between Eqs (2) and (4) is given by

$$\mathbb{P}_{\mathcal{A}'}(X) = \mathrm{Tr}_1\left(S\Omega_{\mathcal{A}}(X\otimes\mathbb{1})\right) = \mathrm{Tr}_1\left(\tilde\Omega_{\mathcal{A}}(X\otimes\mathbb{1})\right). \tag{5}$$

Interestingly, the no-scrambling condition $G_{\mathcal{A}}(U) = 0$ using Prop. 2 can be expressed by the operator fixed-point equations $\mathcal{U}^{\otimes 2}(\Omega_{\mathcal{A}}) = \Omega_{\mathcal{A}}$. The (unsurprising) price to pay is that now the Hilbert space is doubled.

[3] This can be seen from the fact that given two projectors P, and Q of rank d one has: $\dim(V_P\cap V_Q)\le \mathrm{tr}(PQ)\le d$. Where $V_{P/Q}$ are the images of P/Q. The lower (upper) bound is achieved when P and Q commute (coincide).

Another advantage of the formulation (4) is that it makes clear that the GAAC can be computed in terms of 2-point correlation functions. In fact, from Eq. (4) one finds (see appendix)

$$1 - G_{\mathcal{A}}(U) = \frac{1}{d(\mathcal{A}')} \sum_{\alpha,\beta=1}^{d(\mathcal{A})} |\langle e_\alpha, \mathcal{U}(e_\beta)\rangle|^2, \quad (6)$$

(a similar expression hold for the f_α's). This expression suggests how one could measure the GAAC by resorting to process tomography for $\mathcal{U}$. Notice also that operational protocols to measure the GAAC were already discussed, for the cases **1)** and **2)** here below, in [16] and [17] respectively.

Physical Cases.— To concretely illustrate the formalism let us now consider several physically motivated examples in which the GAAC can be fully computed analytically.

The first two examples show how the GAAC formalism allows one to understand two ostensibly unrelated physical problems, operator entanglement [13] and coherence generating power (CGP) [17, 28], from a single vantage point. The first (second) concept is obtained when $\mathcal{A}$ is a factor (maximal abelian). This means that one can also think of the GAAC either as an extension of operator entanglement to algebras that are not factors, or as an extension of coherence generating power to algebras that are not maximal abelian subalgebras.

The third and fourth examples are "dual" to each other and show that, in general, $G_{\mathcal{A}}(U) \neq G_{\mathcal{A}'}(U)$. Finally, the fifth illustrates in which sense even the concept of *Loschmidt echo*, a valuable tool in the study of quantum chaos [29–32], is comprised by the GAAC. This last connection is perhaps unsurprising as the Loschmidt echo is indeed a measure of auto-correlation of a dynamicaly evolving state which is precisely what $1 - G_{\mathcal{A}}(U)$ does at the more general algebra level.

The special results **1**–**5** reported here below can be obtained by Eqs. (4) and (6) by rather straightforward manipulations.

1) Now we consider a bipartite quantum system with $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\mathcal{A} = L(\mathcal{H}_A) \otimes \mathbb{1}_B$ and, therefore, $\mathcal{A}' = \mathbb{1}_A \otimes L(\mathcal{H}_B)$. In this case one finds that $\mathbb{P}_{\mathcal{A}'}(X) = \frac{\mathbb{1}}{d_A} \otimes \mathrm{Tr}_A(X)$, $\Omega_{\mathcal{A}} = \frac{S_{AA'}}{d_A}$, where $S_{AA'}$ is the swap between the A factors in $\mathcal{H}^{\otimes 2}$ and $d_X = \dim \mathcal{H}_X$ $(X = A, B)$. One gets

$$G_{\mathcal{A}}(U) = 1 - \frac{1}{d^2} \langle S_{AA'}, \mathcal{U}^{\otimes 2}(S_{AA'})\rangle, \quad (7)$$

where $d = d_B d_A = \dim \mathcal{H}$. The same relation is true with $S_{AA'} \to S_{BB'} = S S_{AA'} = d_A \tilde{\Omega}_{\mathcal{A}}$.

Eq. (7) *coincides exactly* with the averaged OTOC discussed in [16] i.e., $d^{-1}\mathbb{E}_{X\in\mathcal{A},Y\in\mathcal{A}'}\left[\|[X, \mathcal{U}(Y)]\|_2^2\right]$ (here $\mathbb{E}$ denotes the Haar average over the unitary groups of $\mathcal{A}$ and $\mathcal{A}'$.). Remarkably, this quantity was shown to be equal to the *operator entanglement* [13, 33] of the unitary U.

The latter concept has found important applications to a variety of quantum information-theoretic problems [14, 34–37]. More recently, it has been shown that operator entanglement requires *exponentially scaled* computational resources to simulate [38].

Remark.– The bi-partite OTOC Eq. (7), because of the averages over the two full sub-algebras, does not satisfy Lieb-Robinson type of bounds with associated effective 'light-cone" structures. Indeed the regions A and B are complementary and therefore contiguous (zero distance). The same is, in general true, for the GAAC which does not even require a locality (tensor product) structure to begin with.

2) Let $\mathcal{A}_B$ the algebra of operators which are diagonal with respect to an orthonormal basis $B := \{|i\rangle\}_{i=1}^d$ i.e., $\mathcal{A}_B = \mathbf{C}\,\{\Pi_i := |i\rangle\langle i|\}_{i=1}^d$. This is a d-dimensional maximal abelian subalgebra of $L(\mathcal{H})$ such that $\mathcal{A} = \mathcal{A}'$. In this case $\mathbb{P}_{\mathcal{A}'}(X) = \sum_{i=1}^d \Pi_i X \Pi_i$, $\Omega_{\mathcal{A}} = \sum_{i=1}^d \Pi_i^{\otimes 2}$, and

$$G_{\mathcal{A}_B}(U) = 1 - \frac{1}{d} \sum_{i,j=1}^{d} |\langle i|U|j\rangle|^4, \quad (8)$$

This expression coincides with the *coherence generating power* (CGP) of U introduced in [17, 39]. CGP is there defined as the average coherence (measured by the the sum of the square of off-diagonal elements, with respect B) generated by U starting from any of the pure incoherent states Π i.e., $G_{\mathcal{A}_B}(U) = \frac{1}{d}\sum_{i=1}^d \|\mathbb{Q}_B \mathcal{U}(\Pi_i)\|_2^2$, where $\mathbb{Q} = 1 - \mathbb{P}_{\mathcal{A}_B}$ projects onto the orthogonal complement of $\mathcal{A}_B$. [17, 28]. The fact that the CGP is related to the distance between maximal abelian subalgebras was already established in [27]. CGP has been applied to the detection of the localization transitions in many-body systems [39], detection of quantum chaos in closed and open systems [40].

3) $\mathcal{H} = \mathbf{C}^d \otimes \mathbf{C}^d$, $\mathcal{A} = L(\mathcal{H})_s$, $\mathcal{A}' = \mathbf{C}\{\mathbb{1}, S\} \cong \mathbf{C}\mathbf{Z}_2$. Here, $L(\mathcal{H})_s$ denotes the algebra of symmetric operators i.e., commuting with the swap S. One can readily check that $\tilde{\Omega}_{\mathcal{A}} = \frac{1}{2}\sum_{J=\pm 1}\left(\frac{\mathbb{1}+JS}{\sqrt{d(d+J)}}\right)^{\otimes 2}$, and

$$G_{L(\mathcal{H})_s}(U) = \frac{1}{2}\left(1 - \left|\frac{1 - \langle S, \mathcal{U}(S)\rangle}{d^2 - 1}\right|^2\right) \quad (9)$$

Here $J = \pm 1$ is labeling the symmetric/anti-symmetric representation of the permutation group generated by S.

4) $\mathcal{H} = \mathbf{C}^d \otimes \mathbf{C}^d$, $\mathcal{A} = \mathbf{C}\mathbf{Z}_2$ and $\mathcal{A}' = L(\mathcal{H})_s$. Here, $\Omega_{\mathcal{A}} = \frac{1}{2}\left(\mathbb{1}^{\otimes 2} + S^{\otimes 2}\right)$, $d(\mathcal{A}') = \mathrm{Tr}\,\Omega_{\mathcal{A}} = d^2(d^2+1)/2$, whence

$$G_{\mathbf{CZ}_2}(U) = \frac{1}{2}\frac{d^4 - |\langle S, \mathcal{U}(S)\rangle|^2}{d^2(d^2+1)} \tag{10}$$

Consistently with Prop. (1) both functions vanish iff $\langle S, \mathcal{U}(S)\rangle = d^2 \Leftrightarrow \mathcal{U}(S) = S$. That is to say that non-scrambling U are such that $[U, S] = 0$ i.e., $U \in L(\mathcal{H})_s$.

5) Let $|\psi\rangle \in \mathcal{H}$ and $\Pi = |\psi\rangle\langle\psi|$. We define $\mathcal{A}_{LE} = \mathbf{C}\{\mathbb{1}, \Pi\}$ i.e., the unital *-closed algebra generated by the projection Π. The commutant $\mathcal{A}'_{LE}$ is the algebra of operators leaving the subspace $\mathbf{C}|\psi\rangle$ and its orthogonal complement invariant. One has, $\Omega_{\mathcal{A}_{LE}} = \Pi^{\otimes 2} + (\mathbb{1} - \Pi)^{\otimes 2}$, $d(\mathcal{A}'_{LE}) = (d-1)^2 + 1$.

$$G_{\mathcal{A}_{LE}}(U) = \frac{2(1-\mathcal{L}^2)[d - 2(1-\mathcal{L}^2)]}{(d-1)^2+1}, \tag{11}$$

where $\mathcal{L} := |\langle\psi|U|\psi\rangle|$ is the Loschmidt echo. Notice, $G_{\mathcal{A}_{LE}}(U) = \frac{2}{d}(1-\mathcal{L}^2) + O(1/d^2)$ and that $2(1-\mathcal{L}^2) = \|\Pi - \mathcal{U}(\Pi)\|_2^2$, i.e., the distance between the algebras $\mathcal{A}'_{LE}$ and its image $\mathcal{U}(\mathcal{A}'_{LE})$, as captured by the GAAC [see Eq. (3)], in high dimension is directly related to the Hilbert-Schmidt distance between the states Π and $\mathcal{U}(\Pi)$. From Eq. (11) one can see that the GAAC is a monotonic decreasing function of $\mathcal{L}$ for $d > 4$ and that $\mathcal{L} = 1 \Rightarrow G_{\mathcal{A}_{LE}}(U) = 0$. For $d = 2$ one is back to **2)**. The case $\mathcal{L} = 0$ corresponds to $U\Pi U^\dagger = \mathbb{1} - \Pi$.

Upper bounds and Expectations.— What are the bounds to algebra scrambling as measured by the GAAC? Now we would like to answer this question and to see whether and how those bounds might be saturated.

Proposition 3. *i)*

$$G_{\mathcal{A}}(U) \le \min\{1 - \frac{1}{d(\mathcal{A})}, 1 - \frac{1}{d(\mathcal{A}')}\} =: G_{UB}(\mathcal{A}) \tag{12}$$

ii) if $d(\mathcal{A}') \le d(\mathcal{A})$ then the bound above is achieved iff $\mathbb{P}_{\mathcal{A}'}\mathcal{U}\mathbb{P}_{\mathcal{A}'} = \mathcal{T}$ where $\mathcal{T}\colon X \mapsto \mathrm{Tr}(X)\frac{\mathbb{1}}{d}$. iii) If $\mathcal{A}'$ is Abelian the bound $1 - \frac{1}{d(\mathcal{A}')}$ is always achieved. iv) In the collinear case ii) and iii) above hold true with $\mathcal{A} \leftrightarrow \mathcal{A}'$.

The saturation condition $\mathbb{P}_{\mathcal{A}'}\mathcal{U}\mathbb{P}_{\mathcal{A}'} = \mathcal{T}$ is quite transparent and intuitive: maximal scrambling is achieved when, from the point of view of the commutant, the dynamics generated by $\mathcal{U}$ is just full depolarization. Physical degrees of freedom supported in $\mathcal{A}'$ are, quite properly, fully *scrambled*.

Let us now briefly discuss Prop. (3) for the physical cases **1**–**5)**. In the bipartite example **1)**, if $d_A = d_B$, then (12) is achieved for $U = S$ (swap) [16]. In the maximal abelian case **2)** the bound $1 - d^{-1}$ is saturated by those U's such that $|\langle i|U|j\rangle| = d^{-1/2}$, $(\forall i, j)$ [17]. In case **3)** the bound $\frac{1}{2}$ is achieved for $\langle S, \mathcal{U}(S)\rangle = 1$, which amounts to the condition ii). On the other hand, in case **4)** from Eq. (10) we see that $\langle S, \mathcal{U}(S)\rangle = 0 \Rightarrow \max_U G_{\mathbf{CZ}_2}(U) = \frac{1}{2}(1 + 1/d^2)^{-1} < \frac{1}{2}$ i.e., bound (12) is *not* always achieved. **5)** The bound $1/2$ is achieved for $d = 2$ only ($\mathcal{A}'$ is abelian). For $d > 4$ the maximun is for $\mathcal{L} = 0$ and it is $O(1/d)$.

The next general question that we would like to address is: what is the typical value of the GAAC for generic unitaries? To answer this question we perform an average of (4) over random, Haar distributed, unitaries.

Proposition 4.

$$i) \qquad \overline{G_{\mathcal{A}}(U)}^U = \frac{(d^2 - d(\mathcal{A}'))(d(\mathcal{A}') - 1)}{d(\mathcal{A}')(d^2-1)} \tag{13}$$

ii) $\mathbf{Prob}_U\left[|G_{\mathcal{A}}(U) - \overline{G_{\mathcal{A}}(U)}^U| \ge \epsilon\right] \le \exp[-\frac{d\epsilon^2}{4K^2}]$.

iii) In the collinear case $G_{UB}(\mathcal{A}) - \overline{G_{\mathcal{A}}(U)}^U = O(1/d)$ *and* $\mathbf{Prob}_U\left[G_{UB}(\mathcal{A}) - G_{\mathcal{A}}(U) \ge d^{-1/3}\right] \le \exp[-\frac{d^{1/3}}{16K^2}]$.

In ii) and iii) one can choose $K \ge 40$.

As a sanity check, note that Eq. (13) implies that the GAAC of any U vanishes when $d(\mathcal{A}') = d^2$ i.e., $\mathcal{A} = \mathbf{C}\mathbb{1}$, or $d(\mathcal{A}') = 1$ i.e., $\mathcal{A} = L(\mathcal{H})$. In fact in these cases $\mathcal{A}$ is obviously invariant under the action of *any* U. Point ii) is a direct application of the Levy Lemma on measure concentration: in high dimension (13) is the *typical* value of the GAAC. Finally, point iii) shows that, in the collinear case, the average value of the GAAC converges to the bound (12) when d grows and that the GAAC is, with overwhelming probability, close to $G_{UB}(\mathcal{A})$.

Time dynamics: infinite averages and fluctuations.— In this final section we will consider a one-parameter group of unitary channels $\{U_t := e^{-iHt}\}_t$ generated by an Hamiltonian H. The idea is that the behavior of the infinite-time average $\overline{G_{\mathcal{A}}(U_t)}^t := \lim_{T\to\infty} T^{-1}\int_0^T dt\, G_{\mathcal{A}}(U_t)$ contains information about the "chaoticity" of the dynamics as seen from the physical degrees of freedom in the algebra. These calculations greatly extends the corresponding results, for the bipartite averaged OTOC, reported in [16].

Proposition 5. $\overline{G_{\mathcal{A}}(U_t)}^t \le \overline{G_{\mathcal{A}}(U_t)}^{NRC} \le \overline{G_{\mathcal{A}}(U)}^U$ *where*

$$\overline{G_{\mathcal{A}}(U_t)}^{NRC} = 1 - \frac{1}{d(\mathcal{A}')}\sum_{\alpha=0,1}\left[\|R^{(\alpha)}\|_2^2 - \frac{1}{2}\|R_D^{(\alpha)}\|_2^2\right], \tag{14}$$

$R_{lk}^{(0)} := \|\mathbb{P}_{\mathcal{A}'}(|\Psi_l\rangle\langle\Psi_k|)\|_2^2$, $R_{lk}^{(1)} := \langle\mathbb{P}_{\mathcal{A}'}(\Pi_l), \mathbb{P}_{\mathcal{A}'}(\Pi_k)\rangle$, *and* $(R_D^{(\alpha)})_{lk} := \delta_{lk}R_{lk}^{(\alpha)}$, $(l, k = 1, \ldots, d)$. *Moreover, the first inequality above* *becomes an equality if H fulfills the so-called Non Resonance Condition (NRC).*

Remark.– The NRC condition amounts to to say $E_l + E_k = E_n + E_m$ iff $l = n$, $k = m$ or $l = m$, $k = n$ In words: the Hamiltonian spectrum and its gaps are non-degenerate. This fact holds true for *generic* (non-interacting) Hamiltonians.

The result above, which holds for any observable algebra $\mathcal{A}$, has the very same structure of the corresponding one proved for the averaged bipartite OTOC (see Prop. 4 in [16]). The matrices $R^{(\alpha)}$, $(\alpha = 0, 1)$ encode the connection between the algebra and the *full* system of eigenstates of H.

A further simplification occurs, as usual, for the collinear situation $\mathbf{d} = \lambda \mathbf{n}$: $\lambda R^{(0)}_{lk} = \langle \mathbb{P}_{\mathcal{A}}(\Pi_l), \mathbb{P}_{\mathcal{A}}(\Pi_k)\rangle =: R^{(1)}_{lk}(\mathcal{A}')$. In this case Eq. (14) can be written in way in which $\mathcal{A}$ and $\mathcal{A}'$ appear symmetrically and the following upper bound holds:

$$\overline{G_{\mathcal{A}}(U_t)}^{NRC} \leq 1 - \frac{1}{d(\mathcal{A}')} - \frac{1}{d(\mathcal{A})} + \frac{1}{d\, d(\mathcal{A}')}. \quad (15)$$

This bound is saturated iff $\mathbb{P}_{\mathcal{A}'}(\Pi_l) = \mathbb{P}_{\mathcal{A}}(\Pi_l) = \frac{\mathbb{1}}{d}$, $(\forall l)$. Namely, Hamiltonians whose eigenstates are fully scrambled by the two algebra projections correspond to maximal infinite-time averaged GAAC. For these Hamiltonians infinite-time averages of arbitrary observables are, from the point of view of $\mathcal{A}$ and $\mathcal{A}'$, *completely* randomized [4]. Conceptually, this seems a natural way of characterizing chaoticity relative to the distinguished algebra of observables.

For example: In the bipartite case **1)** with $d_A = d_B$ the bound (15) is achieved if the (non-degenerate) Hamiltonian has a fully-entangled eigenstates [16]. In the maximal abelian algebra case **2)** the bound saturation corresponds to Hamiltonians with eigenstates that have maximum coherence with respect to the basis associated with $\mathcal{A}$ [17]. In both these two important physical situations, the RHS of Eq. (15) is equal to $(1 - \frac{1}{d})^2$ and $\overline{G_{\mathcal{A}}(U)}^U - \overline{G_{\mathcal{A}}(U_t)}^{NRC} = O(\frac{1}{d^2})$; whereby, assuming that NRC holds, using iii) in Prop. (4) and the Markov inequality, one can bound temporal fluctuations:

$$\mathbf{Prob}_t \left[G_{UB}(\mathcal{A}) - G_{\mathcal{A}}(U_t) \geq \epsilon\right] \leq O(\frac{1}{d\, \epsilon}), \quad (16)$$

one sees e.g., by choosing $\epsilon = d^{-1/3}$, that Hamiltonians achieving bound (15) have, in high dimension, highly suppressed temporal fluctuations below the value (12).

In [16] this concentration phenomenon has been numerically observed for the bi-partite case in chaotic many-body systems and *not* in integrable systems. For the same type of physical systems, suppression of temporal variance of CGP has been noticed in [40]. These findings were used to suggest that both the bi-partite averaged OTOC and CGP can be used as diagnostic tools to detect some aspects of quantum chaotic behavior. The results above show how this picture may extend to the general algebraic setting developed in this paper.

In fact,we would like to *define* $\mathcal{A}$-chaotic the dynamics generated by U_t's such that the (relative) difference between its infinite-time average and the Haar-average of the GAAC is approaching zero sufficiently fast as the system dimension grows. More formally,

$$1 - \overline{G_{\mathcal{A}}(U_t)}^t / \overline{G_{\mathcal{A}}(U)}^U =: \epsilon = O(d^{-\gamma}) \quad (\gamma \geq 1). \quad (17)$$

In particular, in the collinear case, this condition would allow one to prove the “equilibration" result for the GAAC (16). The intuition behind this definition is quite simple: if Eq. (17) holds the long time behavior of the GAAC gets, as the system dimension grows, quickly indistinguishable from the one of a typical Haar random unitary i.e., a “fully chaotic" one.

Before concluding, we would like to illustrate $\mathcal{A}$-chaos with the simple Loschmidt case **5)**. Here one has $\epsilon = \overline{\mathcal{L}_t^2}^t + O(1/d)$ where $\mathcal{L}_t = |\langle\psi|U_t|\psi\rangle|$. The infinite-time average is given by the purity of the Hamiltonian dephased state $\overline{\mathcal{L}_t^2}^t = \|\overline{\mathcal{U}_t(|\psi\rangle\langle\psi|)}^t\|_2^2$ [41] Whence the "chaoticity" condition is achieved if this purity is $O(1/d)$ which in turn implies that the dephased state is $O(1/d)$ away from the maximally mixed state. This condition is known to be a sufficient one to bound time-fluctuations of the expectation value of observables with initial state $|\psi\rangle$ [42]. Namely, $\mathcal{A}_{LE}$-chaos amounts to temporal-equilibration [41].

Conclusions.— In this paper we have proposed a novel approach to quantum scrambling based on algebras of observables. A quantitative measure of scrambling is introduced in terms of anti-correlation between the whole commutant algebra and its (unitarily) evolved image. This quantity, which we named the Geometric Algebra Anti Correlator (GAAC), has also a clear geometrical meaning as it describes the distance between the two algebras or, equivalently, the degree of self-orthogonalization induced by the dynamics.

We explicitly computed the GAAC for several physically motivated cases and characterized its behavior in terms of typical values, upper bounds and temporal fluctuations. We have shown that the GAAC formalism provides an unified mathematical and conceptual framework for concepts like operator entanglement, averaged bipartite OTOC, coherence generating power and Loschmidt echo.

[4] Indeed, for any observable $\mathbb{P}_{\mathcal{A}}(\overline{A(t)}^t) = \mathbb{P}_{\mathcal{A}}(\sum_l A_l \Pi_l) = \sum_l A_l \mathbb{P}_{\mathcal{A}}(\Pi_l) = \frac{\mathbb{1}}{d}\sum_l A_l = \frac{\mathrm{Tr}(A)\mathbb{1}}{d} = \mathcal{T}(A)$ where $A_l := \mathrm{Tr}(A\Pi_l)$. Same holds for $\mathcal{A}'$.

Finally, we suggested an approach to quantum chaos in terms of the behavior of infinite-time average of the GAAC for large system dimension. To assess the effectiveness of such an approach is one of the challenges of future investigations.

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A Supplemental Material

A.1 Proof of Prop 1

i) It is a direct computation: $D^2(\mathcal{A}', \mathcal{U}(\mathcal{A}')) = \|\mathbb{P}_{\mathcal{A}'} - \mathbb{P}_{\mathcal{U}(\mathcal{A}')}\|_{HS}^2 = \|\mathbb{P}_{\mathcal{A}'}\|_{HS}^2 + \|\mathbb{P}_{\mathcal{U}(\mathcal{A}')}\|_{HS}^2 - 2\,\langle \mathbb{P}_{\mathcal{A}'}, \mathbb{P}_{\mathcal{U}(\mathcal{A}')}\rangle$. Now, $\|\mathbb{P}_{\mathcal{A}'}\|_{HS}^2 = \|\mathcal{U}\mathbb{P}_{\mathcal{A}'}\mathcal{U}^\dagger\|_{HS}^2 = \|\mathbb{P}_{\mathcal{U}(\mathcal{A}')}\|_{HS}^2 = d(\mathcal{A}')$. Whence, by dividing D^2 by $2\,d(\mathcal{A}') = 2\,\|\mathbb{P}_{\mathcal{A}'}\|_{HS}^2$, Eq. (3) follows.

ii) Since D is a metric from Eq. (3) one has $G_{\mathcal{A}}(U) = 0 \Leftrightarrow \mathcal{U}(\mathcal{A}') = \mathcal{A}' \Leftrightarrow \mathcal{U}(\mathcal{A}) = \mathcal{A}$. Last equivalence is obtained by taking the commutant of both sides, using $\mathcal{A}'' = \mathcal{A}$ (double commutant theorem) and that $\mathcal{U}(\mathcal{A})' = \mathcal{U}(\mathcal{A}')$ (true for unitary auto-morphisms).

A.2 Proof of Prop 2

i) Let us write the Algebra projections in the Kraus form $\mathbb{P}_{\mathcal{A}'}(X) = \sum_\alpha e_\alpha X e_\alpha^\dagger$ and $\mathbb{P}_{\mathcal{U}(\mathcal{A}')}(X) = \sum_\alpha (U e_\alpha U^\dagger) X (U e_\alpha U^\dagger)^\dagger$. Here, because of the structure theorem (1), one can choose the *orthogonal* basis of $\mathcal{A}$ given by

$$e_\alpha = \frac{1}{\sqrt{d_J}} \mathbb{1}_{n_J} \otimes |l\rangle\langle m|, \in \mathcal{A} \qquad \alpha := (J, l, m)\, (l, m = 1, \ldots, d_J). \tag{18}$$

(note that $|\{\alpha\}| = \sum_J d_J^2 = d(\mathcal{A})$ and that the set $\{e_\alpha\}_\alpha$ is closed under hermitian conjugation). Hence $\langle \mathbb{P}_{\mathcal{A}'}, \mathbb{P}_{\mathcal{U}(\mathcal{A}')}\rangle = \mathrm{Tr}_{HS}\left(\mathbb{P}_{\mathcal{A}'}\mathbb{P}_{\mathcal{U}(\mathcal{A}')}\right) = \sum_{\alpha,\beta} |\mathrm{Tr}\left(e_\alpha U e_\beta U^\dagger\right)|^2 = \sum_{\alpha,\beta} |\langle e_\alpha, \mathcal{U}(e_\beta)\rangle|^2$. Here we've used that $\mathcal{T} = \sum_i T_i X T_i^\dagger \Rightarrow \mathrm{Tr}_{HS}\,\mathcal{T} = \sum_i |\mathrm{tr}\, T_i|^2$. Because of the definition (2) this proves Eq. (6).

On the other hand, if $\Omega_{\mathcal{A}} = \sum_\alpha e_\alpha \otimes e_\alpha^\dagger$ one has $\langle \Omega_{\mathcal{A}}, \mathcal{U}^{\otimes 2}(\Omega_{\mathcal{A}})\rangle = \sum_{\alpha,\beta} \langle e_\alpha \otimes e_\alpha^\dagger, U e_\beta U^\dagger \otimes U e_\beta^\dagger U^\dagger\rangle = \sum_{\alpha,\beta} \langle e_\alpha, U e_\beta U^\dagger\rangle \langle e_\alpha^\dagger, U e_\beta^\dagger U^\dagger\rangle = \sum_{\alpha,\beta} |\langle e_\alpha, \mathcal{U}(e_\beta)\rangle|^2$. Moreover, from (18) one has

$$\Omega_{\mathcal{A}} = \sum_J \frac{\mathbb{1}_{n_J}^{\otimes 2}}{d_J} \otimes \sum_{l,m=1}^{d_J} |lm\rangle\langle ml| =: \sum_J \frac{\mathbb{1}_{n_J}^{\otimes 2} \otimes S_{d_J}}{d_J}. \tag{19}$$

Therefore, $\|\Omega_{\mathcal{A}}\|_2^2 = \mathrm{Tr}\left(\Omega_{\mathcal{A}}^2\right) = \mathrm{Tr}\left(\sum_J \frac{(\mathbb{1}_{n_J} \otimes \mathbb{1}_{d_J})^{\otimes 2}}{d_J^2}\right) = \sum_J (n_J d_J)^2 / d_J^2 = \sum_J n_J^2 = d(\mathcal{A}')$. This completes the proof of the first equality in Eq. (4). Now, if S is the swap operator $S\Omega_{\mathcal{A}}S = \sum_\alpha e_\alpha^\dagger \otimes e_\alpha = \sum_\alpha e_\alpha \otimes e_\alpha^\dagger = \Omega_{\mathcal{A}}$, i.e.,$[S, \Omega_{\mathcal{A}}] = 0$. Since $[U^{\otimes 2}, S] = 0$ and $\|\Omega_{\mathcal{A}}\|_2^2 = \|S\Omega_{\mathcal{A}}\|_2^2 = \|\tilde{\Omega}_{\mathcal{A}}\|_2^2$ the second equality in Eq. (4) follows. Also, since $S(\mathbb{1}_{n_J}^{\otimes 2} \otimes S_{d_J}) = S_{n_J} \otimes \mathbb{1}_{d_J}^{\otimes 2}$, ($S_{n_J}$ is a swap operator defined over the $\mathbf{C}^{n_J}$ factors) one finds $\tilde{\Omega}_{\mathcal{A}} =: \sum_J \frac{S_{n_J} \otimes \mathbb{1}_{d_J}^{\otimes 2}}{d_J} = \sum_\gamma f_\gamma \otimes f_\gamma^\dagger$, where $\gamma := (J, p, q)$, $p, q = 1, \ldots, n_J$ and $f_\gamma = \frac{1}{\sqrt{d_J}} |p\rangle\langle q| \otimes \mathbb{1}_{d_J} \in \mathcal{A}'$. This is an *orthonormal* basis of $\mathcal{A}'$.

By direct computation $\mathrm{Tr}_1\left(S\Omega_{\mathcal{A}}(X \otimes \mathbb{1}\right) = \sum_J \mathrm{tr}_{d_J}(X) \otimes \frac{\mathbb{1}_{d_J}}{d_J} = \mathbb{P}_{\mathcal{A}'}(X)$ which proves Eq. (5).

ii) In the collinear case $\tilde{\Omega}_{\mathcal{A}} = \frac{1}{\lambda}\Omega_{\mathcal{A}'} = \frac{1}{\lambda}\sum_J \frac{S_{n_J} \otimes \mathbb{1}_{d_J}^{\otimes 2}}{n_J}$ where $d_J = \lambda n_J$, $(\forall J)$. Inserting this in Eq. (4) and using $\lambda^2 \|\Omega_{\mathcal{A}}\|_2^2 = \lambda^2 \sum_J n_J^2 = \sum_J d_J^2 = d(\mathcal{A}) = \|\Omega_{\mathcal{A}'}\|_2^2$ one sees that in this collinear case $G_{\mathcal{A}}(U) = G_{\mathcal{A}'}(U)$.

A.3 Proof of Prop 3

i) First, notice that for any two orthogonal projections P and Q one has that $\mathrm{Tr}\,(PQ) \geq \dim\,(\mathrm{Im}P \cap \mathrm{Im}Q)$. Since both $\mathcal{A}'$ and $\mathcal{U}(\mathcal{A}')$ contain the identity $\mathbb{1}$ one has $\langle \mathbb{P}_{\mathcal{A}'}, \mathbb{P}_{\mathcal{U}(\mathcal{A}')}\rangle \geq 1$, from which the bound $G_{\mathcal{A}}(U) \leq 1 - 1/d(\mathcal{A}')$, immediately follows.

To prove the bound $G_{\mathcal{A}}(U) \leq 1 - 1/d(\mathcal{A})$, we begin by observing that $\Omega_{\mathcal{A}} = \sum_\alpha \tilde{e}_\alpha \otimes \tilde{e}_\alpha^\dagger$ for any basis $\tilde{e}_\alpha = \sum_\beta U_{\beta,\alpha} e_\beta$, where the e_β's are given by (18) and the matrix $U_{\beta,\alpha}$ is unitary. Now $\frac{\mathbb{1}}{\sqrt{d(\mathcal{A})}} = \sum_J \sqrt{\frac{d_J}{d(\mathcal{A})}} \sum_{l=1}^{d_J} e_{(J,l,l)} =: \tilde{e}_1$. Since $\sum_{J,l} |\sqrt{\frac{d_J}{d(\mathcal{A})}}|^2 = \frac{1}{d(\mathcal{A})} \sum_J d_J^2 = 1$, we see that on can always unitarily move to a new basis such that $\tilde{e}_1 = \frac{\mathbb{1}}{\sqrt{d(\mathcal{A})}}$. Whence $\Omega_{\mathcal{A}} = \frac{\mathbb{1}}{d(\mathcal{A})} + \sum_{\alpha > 1} \tilde{e}_\alpha \otimes \tilde{e}_\alpha^\dagger =: \frac{\mathbb{1}}{d(\mathcal{A})} + \Omega'_{\mathcal{A}}$ and $\langle \Omega_{\mathcal{A}}, \mathcal{U}^{\otimes 2}(\Omega_{\mathcal{A}})\rangle = \frac{\mathrm{Tr}\,\Omega_{\mathcal{A}}}{d(\mathcal{A})} + \langle \Omega_{\mathcal{A}}, \mathcal{U}^{\otimes 2}(\Omega'_{\mathcal{A}})\rangle \geq \frac{d(\mathcal{A}')}{d(\mathcal{A})}$ (note $\langle \Omega_{\mathcal{A}}, \mathcal{U}^{\otimes 2}(\Omega'_{\mathcal{A}})\rangle = \sum_\alpha \sum_{\beta > 1} |\langle e_\alpha, \mathcal{U}(\tilde{e}_\beta)\rangle|^2 \geq 0$.). Plugging this inequality in Eq. (4) one finds $G_{\mathcal{A}}(U) \leq 1 - 1/d(\mathcal{A})$.

In summary, $G_{\mathcal{A}}(U) \leq \min\{1 - 1/d(\mathcal{A}),\, 1 - 1/d(\mathcal{A}')\}$.

ii) One has that $\langle \mathbb{P}_{\mathcal{A}'}, \mathbb{P}_{\mathcal{U}(\mathcal{A}')}\rangle = \|\mathbb{P}_{\mathcal{A}'}\mathcal{U}\mathbb{P}_{\mathcal{A}'}\|_{HS}^2$. This last norm is always larger than the (square of the) operator norm of the CP-map $\mathcal{F} := \mathbb{P}_{\mathcal{A}'}\mathcal{U}\mathbb{P}_{\mathcal{A}'}$ which is one. The lower bound is achieved *iff* $\mathcal{F}$ has rank one, but the only rank one unital trace-preserving CP-map is the depolarizing channel $\mathcal{T}$.

iii) If $\mathcal{A}'$ is Abelian one take the orhonormal basis $f_J = \frac{\mathbb{1}_1 \otimes \mathbb{1}_{d_J}}{\sqrt{d_J}} = \frac{\Pi_J}{\sqrt{d_J}}$.It follows, $\langle \tilde{\Omega}_{\mathcal{A}}, \mathcal{U}^{\otimes 2}(\tilde{\Omega}_{\mathcal{A}})\rangle = \sum_{J,K} \frac{1}{d_J d_K} |\langle \Pi_J, \mathcal{U}(\Pi_K)\rangle|^2$. Now if $\Pi_J = \sum_{l=1}^{d_J} |Jl\rangle\langle Jl|$ and $\Pi_K = \sum_{m=1}^{d_K} |Km\rangle\langle Km|$, one has $\langle \Pi_J, \mathcal{U}(\Pi_K)\rangle = \sum_{l=1}^{d_J}\sum_{m=1}^{d_K} |\langle Km|U|Jl\rangle|^2$. Therefore, if U maps the basis $|Jl\rangle$ into mutually unbiased one i.e., $|\langle Km|U|Jl\rangle| = 1/\sqrt{d}$, $(\forall J, K, l, m)$ one finds $\langle \Pi_J, \mathcal{U}(\Pi_K)\rangle = \frac{d_J d_K}{d}$ whence $\sum_{J,K} \frac{1}{d_J d_K} |\langle \Pi_J, \mathcal{U}(\Pi_K)\rangle|^2 = \sum_{JK} \frac{d_J^2 d_K^2}{d_J d_K d^2} = 1$ (here we used that, in the Abelian case, $\sum_J d_J = d$.) By Eq. (4) this last relation implies that, for these U's, the upper bound $1 - 1/d(\mathcal{A}')$ is saturated. Notice that in this case, $\mathbb{P}_{\mathcal{A}'}\mathcal{U}(\frac{\Pi_J}{\sqrt{d_J}}) = \sum_K \frac{\Pi_K}{\sqrt{d_K}} \langle \frac{\Pi_K}{\sqrt{d_K}}, \mathcal{U}(\frac{\Pi_J}{\sqrt{d_J}})\rangle = \frac{1}{d}\sum_K \frac{\Pi_K}{\sqrt{d_K}} \frac{d_J d_K}{\sqrt{d_J}\sqrt{d_K}} = \sqrt{d_J}\frac{1}{d}\sum_K \Pi_K = \sqrt{d_J}\frac{\mathbb{1}}{d} = \mathcal{T}(\frac{\Pi_J}{\sqrt{d_J}}), \forall J$ which implies $\mathbb{P}_{\mathcal{A}'}\mathcal{U}\mathbb{P}_{\mathcal{A}'} = \mathcal{T}$.

iv) Since in the collinear case $G_{\mathcal{A}}(U) = G_{\mathcal{A}'}(U)$ ii) and iii) above holds with $\mathcal{A}$ replacing $\mathcal{A}'$.

A.4 Proof of Prop 4

i) Averaging over the Haar measure gives you a projector: $\overline{\mathcal{U}^{\otimes 2}(X)}^U = \overline{U^{\otimes 2} X U^{\dagger \otimes 2}}^U =: \mathbb{P}_{Haar}(X)$ over the commutant of the algebra generated by $\{U^{\otimes 2} \,/\, U \in U(\mathcal{H})\}$. By Schur-Weyl duality this commutant is generated by $\mathbb{1}$ and the swap S:

$$\mathbb{P}_{Haar}(X) = \frac{1}{2}\sum_{\alpha=\pm 1} \frac{\mathbb{1} + \alpha S}{d(d+\alpha)} \langle \mathbb{1} + \alpha S,\, X\rangle. \tag{20}$$

Therefore, $\overline{\langle \Omega_{\mathcal{A}}, \mathcal{U}^{\otimes 2}(\Omega_{\mathcal{A}})\rangle}^U = \langle \Omega_{\mathcal{A}}, \mathbb{P}_{Haar}(\Omega_{\mathcal{A}})\rangle = \|\mathbb{P}_{Haar}(\Omega_{\mathcal{A}})\|_2^2 = \frac{1}{2}\sum_{\alpha=\pm 1} \frac{|\langle \mathbb{1} + \alpha S, \Omega_{\mathcal{A}}\rangle|^2}{d(d+\alpha)}$. Now $\langle \mathbb{1}, \Omega_{\mathcal{A}}\rangle = \mathrm{Tr}\,\Omega_{\mathcal{A}} = d(\mathcal{A}')$, and $\langle S, \Omega_{\mathcal{A}}\rangle = \mathrm{Tr}\,\tilde{\Omega}_{\mathcal{A}} = \sum_J n_J d_J^2/d_J = \sum_J n_J d_J = d$. Proving Eq. (13) is now straightforward algebra from these equations and (4).

ii) This is an application of the Levy's Lemma for the GAAC: $U \in U(\mathcal{H}) \mapsto G_{\mathcal{A}}(U) := \frac{\langle \Omega_{\mathcal{A}}, \mathcal{U}^{\otimes 2}(\Omega_{\mathcal{A}})\rangle}{\|\Omega_{\mathcal{A}}\|_2^2}$.

Levy's Lemma:

$$|G_{\mathcal{A}}(U) - G_{\mathcal{A}}(V)| \le K\|U - V\|_2 \Rightarrow \mathbf{Prob}_U\left[|G_{\mathcal{A}}(U) - \overline{G_{\mathcal{A}}(U)}^U| \ge \epsilon\right] \le \exp\left(-\frac{d\epsilon^2}{4K^2}\right). \tag{21}$$

Let us show that this is Lipschitz function. $|G_{\mathcal{A}}(U) - G_{\mathcal{A}}(V)| = \|\Omega_{\mathcal{A}}\|_2^{-2} |\langle \Omega_{\mathcal{A}}, (\mathcal{U}^{\otimes 2} - \mathcal{V}^{\otimes 2})(\Omega_{\mathcal{A}})\rangle| \le \|\mathcal{U}^{\otimes 2} - \mathcal{V}^{\otimes 2}\|_{2,2}$ [here $\|\mathcal{T}\|_{2,2} := \sup_{\|X\|_2 = 1} \|\mathcal{T}(X)\|_2$.] If $\mathcal{U} - \mathcal{V} = \Delta$ one has $\|\mathcal{U}^{\otimes 2} - \mathcal{V}^{\otimes 2}\|_{2,2} \le \|\Delta\|_{2,2}(\|\Delta\|_{2,2} + 2)$ [where we used $\|X \otimes Y\|_{2,2} = \|X\|_{2,2}\|Y\|_{2,2}$, and $\|\mathcal{V}\|_{2,2} = 1$.] Moreover, if $U - V = \delta$ then $\Delta(X) = \delta X \delta^\dagger + \delta X V^\dagger + V X \delta^\dagger$. From this one finds $\|\Delta\|_{2,2} \le \sup_{\|X\|_2=1}(\|\delta\|_2\|X\delta^\dagger\|_2 + \|\delta X\|_2 + \|X\delta^\dagger\|_2) \le 4\,\|\delta\|_2$. Notice also, $\|X\delta\|_2 = \|X(U - V)\|_2 \le 2\|X\|_2$, whence $\|\Delta\|_{2,2} \le \sup_{\|X\|_2=1}(2\|X\|_2 + 2\|X\|_2 + 2\|X\delta^\dagger\|_2) \le 8$. Bringing everything together:

$$|G_{\mathcal{A}}(U) - G_{\mathcal{A}}(V)| \le \|\Delta\|_{2,2}(\|\Delta\|_{2,2} + 2) \le 10\,\|\Delta\|_{2,2} \le 40\|\delta\|_2 = 40\,\|U - V\|_2. \tag{22}$$

This shows that one can choose a Lipschitz constant $K \ge 40$ for f.

iii) In this collinear case, since $d^2 = d(\mathcal{A})d(\mathcal{A}')$, the Haar average (13) takes the form $\overline{G_{\mathcal{A}}(U)}^U = (1 - 1/d^2)^{-1}(1 - 1/d(\mathcal{A}'))(1 - 1/d(\mathcal{A}))$. Suppose $d(\mathcal{A}') \le d$, $(d(\mathcal{A}) \ge d)$ then $G_{UB}(\mathcal{A}) = 1 - 1/d(\mathcal{A}')$ One has $G_{UB}(\mathcal{A}) - \overline{G_{\mathcal{A}}(U)}^U = \frac{1 - 1/d(\mathcal{A}')}{1 - 1/d^2}(1/d(\mathcal{A}) - 1/d^2) \le 1/d(\mathcal{A}) - 1/d^2 \le 1/d$. The case $d(\mathcal{A}') > d$, works exactly in the same way (with $\mathcal{A} \leftrightarrow \mathcal{A}'$.) This proves that $G_{UB}(\mathcal{A}) - \overline{G_{\mathcal{A}}(U)}^U = O(1/d)$.

Now, $G_{UB}(\mathcal{A}) - G_{\mathcal{A}}(U) = (G_{UB}(\mathcal{A}) - \overline{G_{\mathcal{A}}(U)}^U) + (\overline{G_{\mathcal{A}}(U)}^U - G_{\mathcal{A}}(U)) \ge \epsilon$ implies, for large d, that $\overline{G_{\mathcal{A}}(U)}^U - G_{\mathcal{A}}(U)) \ge \epsilon - 1/d \ge \epsilon/2$. It follows that $\mathbf{Prob}_U\left[G_{UB}(\mathcal{A}) - G_{\mathcal{A}}(U) \ge \epsilon\right] \le \mathbf{Prob}_U\left[\overline{G_{\mathcal{A}}(U)}^U - G_{\mathcal{A}}(U)) \ge \epsilon/2\right] \le \exp[-\frac{d\epsilon^2}{16K^2}]$.

A.5 Proof of Prop 5

Suppose the unitary evolution has the spectral resolution $U_t = \sum_n \Pi_n e^{-iE_n t}$ (here n ranges over the set of distinct eigenvalues) then one has $\mathcal{U}_t^{\otimes 2}(X) = \sum_{k,h,p,m} \Pi_k \otimes \Pi_h\, X\, \Pi_p \otimes \Pi_m \exp[-it(E_k + E_h - E_p - E_m)]$.