Two–Sided Bounds for PDF's Maximum of a Sum of Weighted Chi-square Variables

Sergey G. Bobkov[1,2], Alexey A. Naumov[2], and Vladimir V. Ulyanov[2,3(✉)]

[1] University of Minnesota, Vincent Hall 228,
206 Church St SE, Minneapolis, MN 55455, USA
[2] Faculty of Computer Science, HSE University, 109028 Moscow, Russian Federation
[3] Faculty of Computational Mathematics and Cybernetics,
Lomonosov Moscow State University, 119991 Moscow, Russian Federation
`vulyanov@cs.msu.ru`

Abstract. Two–sided bounds are constructed for a probability density function of a weighted sum of chi-square variables. Both cases of central and non-central chi-square variables are considered. The upper and lower bounds have the same dependence on the parameters of the sum and differ only in absolute constants. The estimates obtained will be useful, in particular, when comparing two Gaussian random elements in a Hilbert space and in multidimensional central limit theorems, including the infinite-dimensional case.

Keywords: Two–sided bounds · Weighted sum · Chi-square variable · Gaussian element

1 Introduction

In many statistical and probabilistic applications, we have to solve the problem of Gaussian comparison, that is, one has to evaluate how the probability of a ball under a Gaussian measure is affected, if the mean and the covariance operators of this Gaussian measure are slightly changed. In [1] we present particular examples motivating the results when such "large ball probability" problem naturally arises, including bootstrap validation, Bayesian inference and high-dimensional CLT, see also [2]. The tight non-asymptotic bounds for the Kolmogorov distance between the probabilities of two Gaussian elements to hit a ball in a Hilbert space have been derived in [1] and [3]. The key property of these bounds is that they are dimension-free and depend on the nuclear (Schatten-one) norm of the difference between the covariance operators of the elements and on the norm of the mean shift. The obtained bounds significantly improve the bound based on Pinsker's inequality via the Kullback–Leibler divergence. It was also established an anti-concentration bound for a squared norm $||Z - a||^2$, $a \in \mathbf{H}$, of a shifted Gaussian element Z with zero mean in a Hilbert space $\mathbf{H}$. The decisive role in

proving the results was played by the upper estimates for the maximum of the probability density function $g(x, a)$ of $||Z - a||^2$, see Theorem 2.6 in [1]:

$$\sup_{x \geq 0} g(x, a) \leq c \, (\Lambda_1 \Lambda_2)^{-1/4}, \tag{1}$$

where c is an absolute constant and

$$\Lambda_1 = \sum_{k=1}^{\infty} \lambda_k^2, \qquad \Lambda_2 = \sum_{k=2}^{\infty} \lambda_k^2$$

with $\lambda_1 \geq \lambda_2 \geq \ldots$ are the eigenvalues of a covariance operator Σ of Z.

It is well known that $g(x, a)$ can be considered as a density function of a weighted sum of non-central χ^2 distributions. An explicit but cumbersome representation for $g(x, a)$ in finite dimensional space $\mathbf{H}$ is available (see, e.g., Sect. 18 in Johnson, Kotz and Balakrishnan [4]). However, it involves some special characteristics of the related Gaussian measure which makes it hard to use in specific situations. Our result (1) is much more transparent and provides sharp uniform upper bounds. Indeed, in the case $\mathbf{H} = \mathbf{R}^d$, $a = 0$, Σ is the unit matrix, one has that the distribution of $||Z||^2$ is the standard χ^2 with d degrees of freedom and the maximum of its probability density function is proportional to $d^{-1/2}$. This is the same as what we get in (1).

At the same time, it was noted in [1] that obtaining lower estimates for $\sup_x g(x, a)$ remains an open problem. The latter problem was partially solved in [5], Theorem 1. However, it was done under additional conditions and we took into account the multiplicity of the largest eigenvalue.

In the present paper we get two–sided bounds for $\sup_x g(x, 0)$ in the finite-dimensional case $\mathbf{H} = \mathbf{R}^d$, see Theorem 1 below. The bounds are dimension-free, that is they do not depend on d. Thus, for the upper bounds (1), we obtain a new proof, which is of independent interest. And new lower bounds show the optimality of (1), since the upper and lower bounds differ only in absolute constants. Moreover, new two-sided bounds are constructed for $\sup_x g(x, a)$ with $a \neq 0$ in the finite-dimensional case $\mathbf{H} = \mathbf{R}^d$, see Theorem 2 below. Here we consider a typical situation, where λ_1 does not dominate the other coefficients.

2   Main Results

For independent standard normal random variables $Z_k \sim N(0, 1)$, consider the weighted sum

$$W_0 = \lambda_1 Z_1^2 + \cdots + \lambda_n Z_n^2, \qquad \lambda_1 \geq \cdots \geq \lambda_n > 0.$$

It has a continuous probability density function $p(x)$ on the positive half-axis. Define the functional

$$M(W_0) = \sup_x p(x).$$

Theorem 1. *Up to some absolute constants c_0 and c_1, we have*

$$c_0(A_1A_2)^{-1/4} \le M(W_0) \le c_1(A_1A_2)^{-1/4}, \tag{2}$$

where

$$A_1 = \sum_{k=1}^{n} \lambda_k^2, \qquad A_2 = \sum_{k=2}^{n} \lambda_k^2$$

and

$$c_0 = \frac{1}{4e^2\sqrt{2\pi}} > 0.013, \qquad c_1 = \frac{2}{\sqrt{\pi}} < 1.129.$$

Theorem 1 can be extended to more general weighted sums:

$$W_a = \lambda_1(Z_1 - a_1)^2 + \cdots + \lambda_n(Z_n - a_n)^2 \tag{3}$$

with parameters $\lambda_1 \ge \cdots \ge \lambda_n > 0$ and $a = (a_1, \ldots, a_n) \in \mathbf{R}^n$.

It has a continuous probability density function $p(x, a)$ on the positive half-axis $x > 0$. Define the functional

$$M(W_a) = \sup_x p(x, a).$$

Remark. It is known that for any non-centred Gaussian element Y in a Hilbert space, the random variable $||Y||^2$ is distributed as $\sum_{i=1}^{\infty} \lambda_i(Z_i - a_i)^2$ with some real a_i and λ_i such that

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge 0 \quad \text{and} \quad \sum_{i=1}^{\infty} \lambda_i < \infty.$$

Therefore, the upper bounds for $M(W_a)$ immediately imply the upper bounds for the probability density function of $||Y||^2$.

Theorem 2. *If $\lambda_1^2 \le A_1/3$, then one has a two-sided bound*

$$\frac{1}{4\sqrt{3}} \frac{1}{\sqrt{A_1 + B_1}} \le M(W_a) \le \frac{2}{\sqrt{A_1 + B_1}},$$

where

$$A_1 = \sum_{k=1}^{n} \lambda_k^2, \qquad B_1 = \sum_{k=1}^{n} \lambda_k^2 a_k^2.$$

Moreover, the left inequality holds true without any assumption on λ_1^2.

Remark. In Theorem 2 we only consider a typical situation, where λ_1 does not dominate the other coefficients. Moreover, the condition $\lambda_1^2 \le A_1/3$ necessarily implies that $n \ge 3$. If this condition is violated, the behaviour of $M(W_a)$ should be studied separately.

3 Auxiliary Results

For the lower bounds in the theorems, one may apply the following lemma, which goes back to the work by Statulyavichus [6], see also Proposition 2.1 in [7].

Lemma 1. *Let η be a random variable with $M(\eta)$ denoting the maximum of its probability density function. Then one has*

$$M^2(\eta)\,\mathrm{Var}(\eta) \geq \frac{1}{12}. \tag{4}$$

Moreover, the equality in (4) is attained for the uniform distribution on any finite interval.

Remark. There are multidimensional extensions of (4), see e.g. [8,9] and Section III in [10].

Proof. Without loss of generality we may assume that $M(\eta) = 1$.

Put $H(x) = \mathbf{P}(|\eta - \mathbf{E}\eta| \geq x), \quad x \geq 0$.

Then, $H(0) = 1$ and $H'(x) \geq -2$, which gives $H(x) \geq 1 - 2x$, so

$$\mathrm{Var}(\eta) = 2\int_0^\infty x H(x)\,dx \geq 2\int_0^{1/2} x H(x)\,dx$$
$$\geq 2\int_0^{1/2} x(1 - 2x)\,dx = \frac{1}{12}.$$

Lemma is proved.

The following lemma will give the lower bound in Theorem 2.

Lemma 2. *For the random variable W_a defined in (3), the maximum $M(W_a)$ of its probability density function satisfies*

$$M(W_a) \geq \frac{1}{4\sqrt{3}}\,\frac{1}{\sqrt{A_1 + B_1}}, \tag{5}$$

where

$$A_1 = \sum_{k=1}^n \lambda_k^2, \qquad B_1 = \sum_{k=1}^n \lambda_k^2 a_k^2.$$

Proof. Given $Z \sim N(0,1)$ and $b \in \mathbf{R}$, we have

$$\mathbf{E}\,(Z - b)^2 = 1 + b^2, \qquad \mathbf{E}\,(Z - b)^4 = 3 + 6b^2 + b^4,$$

so that $\mathrm{Var}((Z - b)^2) = 2 + 4b^2$. It follows that

$$\mathrm{Var}(W_a) = \sum_{k=1}^n \lambda_k^2\,(2 + 4a_k^2) = 2A_1 + 4B_1 \leq 4(A_1 + B_1).$$

Applying (4) with $\eta = W_a$, we arrive at (5).

Lemma is proved.

The proofs of the upper bounds in the theorems are based on the following lemma.

Lemma 3. *Let*
$$\alpha_1^2 + \cdots + \alpha_n^2 = 1.$$
If $\alpha_k^2 \leq 1/m$ for $m = 1, 2, \ldots$, then the characteristic function $f(t)$ of the random variable
$$W = \alpha_1 Z_1^2 + \cdots + \alpha_n Z_n^2$$
satisfies
$$|f(t)| \leq \frac{1}{(1 + 4t^2/m)^{m/4}}. \tag{6}$$
In particular, in the cases $m = 4$ and $m = 3$, W has a bounded density with $M(W) \leq 1/2$ and $M(W) < 0.723$ respectively.

Proof. Necessarily $n \geq m$. The characteristic function has the form
$$f(t) = \prod_{k=1}^{n} (1 - 2\alpha_k it)^{-1/2},$$
so
$$-\log |f(t)| = \frac{1}{4} \sum_{k=1}^{n} \log(1 + 4\alpha_k^2 t^2).$$

First, let us describe the argument in the simplest case $m = 1$.
For a fixed t, consider the concave function
$$V(b_1, \ldots, b_n) = \sum_{k=1}^{n} \log(1 + 4b_k t^2)$$
on the simplex
$$Q_1 = \Big\{ (b_1, \ldots, b_n) : b_k \geq 0, \ b_1 + \cdots + b_n = 1 \Big\}.$$
It has n extreme points $b^k = (0, \ldots, 0, 1, 0, \ldots, 0)$. Hence
$$\min_{b \in Q_1} V(b) = V(b^k) = \log(1 + 4t^2),$$
that is, $|f(t)| \leq (1 + 4t^2)^{-1/4}$, which corresponds to (6) for $m = 1$.
If $m = 2$, we consider the same function V on the convex set
$$Q_2 = \Big\{ (b_1, \ldots, b_n) : 0 \leq b_k \leq \frac{1}{2}, \ b_1 + \cdots + b_n = 1 \Big\},$$
which is just the intersection of the cube $[0, \frac{1}{2}]^n$ with the hyperplane. It has $n(n-1)/2$ extreme points
$$b^{kj}, \quad 1 \leq k < j \leq n,$$

with coordinates $1/2$ on the j-th and k-th places and with zero elsewhere. Indeed, suppose that a point

$$b = (b_1, \ldots, b_n) \in Q_2$$

has at least two non-zero coordinates $0 < b_k, b_j < 1/2$ for some $k < j$. Let x be the point with coordinates

$$x_l = b_l \quad \text{for} \quad l \neq k, j, \quad x_k = b_k + \varepsilon, \quad \text{and} \quad x_j = b_j - \varepsilon,$$

and similarly, let y be the point such that

$$y_l = b_l \quad \text{for} \quad l \neq k, j, \quad y_k = b_k - \varepsilon, \quad \text{and} \quad y_j = b_j + \varepsilon.$$

If $\varepsilon > 0$ is small enough, then both x and y lie in Q_2, while

$$b = (x + y)/2, \quad x \neq y.$$

Hence such b cannot be an extreme point. Equivalently, any extreme point b of Q_2 is of the form

$$b^{kj}, \quad 1 \leq k < j \leq n.$$

Therefore, we conclude that

$$\min_{b \in Q_2} V(b) = V(b^{kj}) = 2 \log(1 + 2t^2),$$

which is the first desired claim.

In the general case, consider the function V on the convex set

$$Q_m = \left\{ (b_1, \ldots, b_n) : 0 \leq b_k \leq \frac{1}{m}, \; b_1 + \cdots + b_n = 1 \right\}.$$

By a similar argument, any extreme point b of Q_m has zero for all coordinates except for m places where the coordinates are equal to $1/m$. Therefore,

$$\min_{b \in Q_m} V(b) = V\left(\frac{1}{m}, \ldots, \frac{1}{m}, 0, \ldots, 0 \right) = m \log(1 + 4t^2/m),$$

and we are done.

In case $m = 4$, using the inversion formula, we get

$$M(W) \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(t)| \, dt \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 + t^2} \, dt = \frac{1}{2}.$$

Similarly, in the case $m = 3$,

$$M(W) \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(1 + \frac{4}{3} t^2)^{3/4}} \, dt < 0.723.$$

Lemma is proved.

4   Proofs of Main Results

Proof of Theorem 1. In the following we shall write W instead of W_0.

If $n = 1$, then the distribution function and the probability density function of $W = \lambda_1 Z_1^2$ are given by

$$F(x) = 2\,\Phi\left(\sqrt{\frac{x}{\lambda_1}}\right) - 1, \quad p(x) = \frac{1}{\sqrt{2\pi\lambda_1}}\,e^{-x/(2\lambda_1)} \qquad (x > 0),$$

respectively. Therefore, p is unbounded near zero, so that $M(W) = \infty$. This is consistent with (2), in which case $A_1 = \lambda_1^2$ and $A_2 = 0$.

If $n = 2$, the density $p(x)$ is described as the convolution

$$p(x) \;=\; \frac{1}{2\pi\sqrt{\lambda_1\lambda_2}} \int_0^1 \frac{1}{\sqrt{(1-t)t}}\,\exp\left\{-\frac{x}{2}\left[\frac{1-t}{\lambda_1} + \frac{t}{\lambda_2}\right]\right\} dt \qquad (x > 0). \quad (7)$$

Hence, p is decreasing and attains maximum at $x = 0$:

$$M(W) = \frac{1}{2\pi\sqrt{\lambda_1\lambda_2}} \int_0^1 \frac{1}{\sqrt{(1-t)t}}\,dt = \frac{1}{2\sqrt{\lambda_1\lambda_2}}.$$

Since $A_1 = \lambda_1^2 + \lambda_2^2$ and $A_2 = \lambda_2^2$, we conclude, using the assumption $\lambda_1 \geq \lambda_2$, that

$$\frac{1}{2}\,(A_1 A_2)^{-1/4} \leq M(W) \leq \frac{1}{2^{3/4}}\,(A_1 A_2)^{-1/4}.$$

As for the case $n \geq 3$, the density p is vanishing at zero and attains maximum at some point $x > 0$.

The further proof of Theorem 1 is based on the following observations and Lemma 3.

By homogeneity of (2), we may assume that $A_1 = 1$.

If $\lambda_1 \leq 1/2$, then all $\lambda_k^2 \leq 1/4$, so that $M(W) \leq 1/2$, by Lemma 3. Hence, the inequality of the form

$$M(W) \leq \frac{1}{2}\,(A_1 A_2)^{-1/4}$$

holds true.

Now, let $\lambda_1 \geq 1/2$, so that $A_2 \leq 3/4$. Write

$$W = \lambda_1 Z_1^2 + \sqrt{A_2}\,\xi, \qquad \xi = \sum_{k=2}^n \alpha_k Z_k^2, \qquad \alpha_k = \frac{\lambda_k}{\sqrt{A_2}}.$$

By construction, $\alpha_2^2 + \cdots + \alpha_n^2 = 1$.

Case 1: $\lambda_2 \geq \sqrt{A_2}/2$. Since the function $M(W)$ may only decrease when adding an independent random variable to W, we get using (7) that

$$M(W) \leq M(\lambda_1 Z_1^2 + \lambda_2 Z_2^2) = \frac{1}{2\sqrt{\lambda_1\lambda_2}} \leq c\,(A_1 A_2)^{-1/4},$$

where the last inequality holds with $c = 1$. This gives the upper bound in (2) with constant 1.

Case 2: $\lambda_2 \leq \sqrt{A_2}/2$. It implies that $n \geq 5$ and all $\alpha_k^2 \leq 1/4$ for $k > 1$. By Lemma 3 with $m = 4$, the random variable ξ has the probability density function q bounded by $1/2$. The distribution function of W may be written as

$$\mathbf{P}\{W \leq x\} = \int_0^{x/\sqrt{A_2}} \mathbf{P}\left\{|Z_1| \leq \frac{1}{\sqrt{\lambda_1}}(x - y\sqrt{A_2})^{1/2}\right\} q(y)\, dy, \quad x > 0,$$

and its density has the form

$$p(x) = \frac{1}{\sqrt{2\pi\lambda_1}} \int_0^{x/\sqrt{A_2}} \frac{1}{\sqrt{x - y\sqrt{A_2}}}\, e^{-(x - y\sqrt{A_2})/(2\lambda_1)}\, q(y)\, dy.$$

Equivalently,

$$p(x\sqrt{A_2}) = \frac{1}{\sqrt{2\pi\lambda_1}} A_2^{-1/4} \int_0^x \frac{1}{\sqrt{x - y}}\, e^{-(x - y)\sqrt{A_2}/(2\lambda_1)}\, q(y)\, dy. \qquad (8)$$

Since $\lambda_1 \geq 1/2$, we immediately obtain that

$$M(W) \leq A_2^{-1/4} \frac{1}{\sqrt{\pi}} \sup_{x>0} \int_0^x \frac{1}{\sqrt{x - y}}\, q(y) dy.$$

But, using $q \leq 1/2$, we get

$$\int_0^x \frac{1}{\sqrt{x - y}}\, q(y) dy = \int_{0<y<x,\ x-y<1} \frac{1}{\sqrt{x - y}}\, q(y) dy$$
$$+ \int_{0<y<x,\ x-y>1} \frac{1}{\sqrt{x - y}}\, q(y) dy$$
$$\leq \frac{1}{2} \int_0^1 \frac{1}{\sqrt{z}}\, dz + 1 = 2.$$

Thus,

$$M(W) \leq 2A_2^{-1/4} \frac{1}{\sqrt{\pi}}.$$

Combining the obtained upper bounds for $M(W)$ in all cases we get the upper bound in (2).

For the lower bound, one may apply the inequality (4) in Lemma 1. Thus, we obtain that

$$M(W) \geq \frac{1}{2\sqrt{6}}$$

due to the assumption $A_1 = 1$ and the property $\mathrm{Var}(Z_1^2) = 2$.

If $\lambda_1^2 \leq 1/2$, we have $A_2 \geq 1/2$. Hence,

$$M(W) \geq \frac{1}{2\sqrt{6}} \geq c_0\, (A_1 A_2)^{-1/4}, \qquad (9)$$

where the last inequality holds true with

$$c_0 = \frac{1}{2^{5/4}\sqrt{6}} \geq 0.171.$$

In case $\lambda_1^2 \geq \frac{1}{2}$, we have $A_2 \leq 1/2$. Returning to the formula (8), let us choose $x = \mathbf{E}\xi + 2$ and restrict the integration to the interval

$$\Delta : \max(\mathbf{E}\xi - 2, 0) < y < \mathbf{E}\xi + 2.$$

On this interval necessarily

$$x - y \leq 4.$$

Therefore, (8) yields

$$M(W) \geq \frac{A_2^{-1/4}}{2\sqrt{2\pi}\lambda_1} \cdot e^{-2\sqrt{A_2}/\lambda_1} \, \mathbf{P}\{\xi \in \Delta\}.$$

Here,

$$\frac{A_2}{\lambda_1^2} = \frac{1}{\lambda_1^2} - 1 \leq 1,$$

and we get

$$M(W) \geq \frac{A_2^{-1/4}}{2\sqrt{2\pi}} \cdot e^{-2} \, \mathbf{P}\{\xi \in \Delta\}.$$

Now, recall that $\xi \geq 0$ and $\mathrm{Var}(\xi) = 2\,(\alpha_2^2 + \cdots + \alpha_n^2) = 2$. Hence, by Chebyshev's inequality,

$$\mathbf{P}\{|\xi - \mathbf{E}\xi| \geq 2\} \leq \frac{1}{4}\,\mathrm{Var}(\xi) = \frac{1}{2}.$$

That is, $\mathbf{P}\{\xi \in \Delta\} \geq 1/2$, and thus

$$M(W) \geq \frac{(A_1 A_2)^{-1/4}}{4\sqrt{2\pi}}\, e^{-2} \geq 0.013 \cdot (A_1 A_2)^{-1/4}.$$

Theorem 1 is proved.

Proof of Theorem 2. In the following we shall write W instead of W_a.

The lower bound in Theorem 2 immediately follows from (5) in Lemma 2 without any assumption on λ_1^2.

Our next aim is to reverse this bound up to a numerical factor under suitable natural assumptions.

Without loss of generality, let $A_1 = 1$. Our basic condition will be that $\lambda_1^2 \leq 1/3$, similarly to the first part of the proof of Theorem 1. Note that if $\lambda_1^2 \leq 1/3$ then necessarily $n \geq 3$.

As easy to check, for $Z \sim N(0,1)$ and $a \in \mathbf{R}$,

$$\mathbf{E}\,e^{it\,(Z-a)^2} = \frac{1}{\sqrt{1-2it}}\,\exp\left\{a^2\,\frac{it}{1-it}\right\}, \qquad t \in \mathbf{R},$$

so that

$$\left| \mathbf{E} \, e^{it \, (Z-a)^2} \right| = \frac{1}{(1 + 4t^2)^{1/4}} \exp \left\{ - 2a^2 \, \frac{t^2}{1 + 4t^2} \right\}.$$

Hence, the characteristic function $f(t)$ of W satisfies

$$- \log |f(t)| = \frac{1}{4} \sum_{k=1}^{n} \log(1 + 4\lambda_k^2 t^2) + 2 \sum_{k=1}^{n} a_k^2 \, \frac{\lambda_k^2 t^2}{1 + 4\lambda_k^2 t^2}.$$

Since $\lambda_1^2 \leq \frac{1}{3}$, by the monotonicity, all $\lambda_k^2 \leq \frac{1}{3}$ as well. But, as we have already observed, under the conditions

$$0 \leq b_k \leq \frac{1}{3}, \quad b_1 + \cdots + b_k = 1,$$

and for any fixed value $t \in \mathbf{R}$, the function

$$\psi(b_1, \ldots, b_n) = \sum_{k=1}^{n} \log(1 + 4b_k t^2)$$

is minimized for the vector with coordinates

$$b_1 = b_2 = b_3 = \frac{1}{3} \quad \text{and} \quad b_k = 0 \quad \text{for} \quad k > 3.$$

Hence,

$$\psi(b_1, \ldots, b_n) \geq 3 \log(1 + 4t^2/3) \geq 3 \log(1 + t^2).$$

Therefore, one may conclude that

$$|f(t)| \leq \frac{1}{(1 + t^2)^{3/4}} \exp \left\{ - 2 \sum_{k=1}^{n} a_k^2 \, \frac{\lambda_k^2 t^2}{1 + 4\lambda_k^2 t^2} \right\}. \tag{10}$$

It is time to involve the inversion formula which yields the upper bound

$$M(W) \leq \frac{1}{\pi} \int_0^{\infty} |f(t)| \, dt. \tag{11}$$

In the interval

$$0 < t < T = \frac{1}{2\lambda_1},$$

we have $\lambda_k^2 t^2 \leq 1/4$ for all k, and the bound (8) is simplified to

$$|f(t)| \leq \frac{1}{(1 + t^2)^{3/4}} \, e^{-B_1 t^2}.$$

This gives

$$\int_0^T |f(t)| \, dt \leq I(B_1) \equiv \int_0^{\infty} \frac{1}{(1 + t^2)^{3/4}} \, e^{-B_1 t^2} \, dt.$$

If $B_1 \leq 1$,

$$I(B_1) \leq \int_0^\infty \frac{1}{(1+t^2)^{3/4}} \, dt < 3,$$

while for $B_1 \geq 1$,

$$I(B_1) \leq \int_0^\infty e^{-B_1 t^2} \, dt = \frac{\sqrt{\pi}}{2\sqrt{B_1}} < \frac{1}{\sqrt{B_1}}.$$

The two estimates can be united by

$$I(B_1) \leq \frac{3\sqrt{2}}{\sqrt{1+B_1}}.$$

To perform the integration over the half-axis $t \geq T$, a different argument is needed. Put $p_k = a_k^2 \lambda_k^2 / B_1$, so that $p_k \geq 0$ and $p_1 + \cdots + p_k = 1$. By Jensen's inequality applied to the convex function $V(x) = 1/(1+x)$ for $x \geq 0$ with points $x_k = 4\lambda_k^2 t^2$, we have

$$\sum_{k=1}^n a_k^2 \frac{\lambda_k^2 t^2}{1 + 4\lambda_k^2 t^2} = B_1 t^2 \sum_{k=1}^n p_k V(x_k)$$

$$\geq B_1 t^2 \, V(p_1 x_1 + \ldots p_n x_n)$$

$$= \frac{B_1 t^2}{1 + \frac{4t^2}{B_1} \sum_{k=1}^n a_k^2 \lambda_k^4} \geq \frac{B_1 t^2}{1 + \frac{4t^2}{3B_1} \sum_{k=1}^n a_k^2 \lambda_k^2} = \frac{B_1 t^2}{1 + \frac{4}{3} t^2},$$

where we used the property $\lambda_k^2 \leq 1/3$. Moreover, since

$$t^2 \geq \frac{1}{(2\lambda_1)^2} \geq \frac{3}{4},$$

necessarily

$$\frac{t^2}{1 + \frac{4}{3} t^2} \geq \frac{3}{8}.$$

Hence, from (10) we get

$$|f(t)| \leq \frac{1}{(1+t^2)^{3/4}} e^{-3B_1/4}, \quad t \geq T,$$

and

$$\int_T^\infty |f(t)| \, dt \leq e^{-3B_1/4} \int_{\sqrt{3}/2}^\infty \frac{1}{(1+t^2)^{3/4}} \, dt < 1.68 \, e^{-3B_1/4} < \frac{1.85}{\sqrt{1+B_1}}.$$

Combining the two estimates together for different regions of integration with $(3\sqrt{2} + 1.85)/\pi < 1.94$, the bound (11) leads to

$$M(W) < \frac{2}{\sqrt{A_1 + B_1}}.$$

Thus, this inequality, together with Lemma 2, completes the proof of the theorem.

References

1. Götze, F., Naumov, A.A., Spokoiny, V.G., Ulyanov, V.V.: Large ball probabilities, Gaussian comparison and anti-concentration. Bernoulli **25**(4A), 2538–2563 (2019). https://doi.org/10.3150/18-BEJ1062

2. Prokhorov, Y., Ulyanov, V.: Some approximation problems in statistics and probability. In: Limit Theorems in Probability, Statistics and Number Theory. Springer Proceedings in Mathematics and Statistics, vol. 42, pp. 235–249. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-36068-8_11

3. Naumov, A.A., Spokoiny, V.G., Tavyrikov, Y.E., Ulyanov, V.V.: Nonasymptotic estimates for the closeness of Gaussian measures on balls. Doklady Math. **98**(2), 490–493 (2018). https://doi.org/10.1134/S1064562418060248

4. Johnson, N., Kotz, S., Balakrishnan, N.: Continuous Univariate Distributions, vol. 1. Wiley, New York (1994)

5. Christoph, G., Prokhorov, Y.V., Ulyanov, V.V.: On distribution of quadratic forms in Gaussian random variables. Theory Probab. Appl. **40**(2), 250–260 (1996). https://doi.org/10.1137/1140028

6. Statulyavichus, V.A.: Limit theorems for densities and asymptotic expansions for distributions of sums of independent random variables. Theory Probab. Appl. **10**(4), 582–595 (1965). https://doi.org/10.1137/1110074

7. Bobkov, S.G., Chistyakov, G.P.: On concentration functions of random variables. J. Theor. Probab. **28**(3), 976–988 (2013). https://doi.org/10.1007/s10959-013-0504-1

8. Ball, K.: Logarithmically concave functions and sections of convex sets in R^n. Studia Math. **88**(1), 69–84 (1988)

9. Hensley, D.: Slicing convex bodies-bounds for slice area in terms of the body's covariance. Proc. Am. Math. Soc. **79**(4), 619–625 (1980). https://doi.org/10.2307/2042510

10. Bobkov, S., Madiman, M.: The entropy per coordinate of a random vector is highly constrained under convexity conditions. IEEE Trans. Inf. Theory **57**(8), 4940–4954 (2011). https://doi.org/10.1109/TIT.2011.2158475