

PARTIAL REGULARITY OF SUITABLE WEAK SOLUTIONS OF THE NAVIER–STOKES–PLANCK–NERNST–POISSON EQUATION*

HUAJUN GONG[†], CHANGYOU WANG[‡], AND XIAOTAO ZHANG[§]

Abstract. In this paper, inspired by the seminal work by Caffarelli, Kohn, and Nirenberg [*Comm. Pure Appl. Math.*, 35 (1982), pp. 771–831] on the incompressible Navier–Stokes equation, we establish the existence of a suitable weak solution to the Navier–Stokes–Planck–Nernst–Poisson equation in dimension three, which is smooth away from a closed set whose 1-dimensional parabolic Hausdorff measure is zero.

Key words. Navier–Stokes–Planck–Nernst–Poisson equation, suitable weak solution, partial regularity

AMS subject classifications. 35Q35, 35K60

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1. Introduction. Let $\Omega \subset \mathbb{R}^3$ be a bounded, smooth domain and $0 < T < \infty$. We consider the following Navier–Stokes–Nernst–Planck–Poisson equation:

$$(1.1) \quad \begin{cases} \partial_t u + (u \cdot \nabla)u - \Delta u + \nabla P = -(n^+ - n^-)\nabla \Psi, \\ \operatorname{div} u = 0, \\ \partial_t n^+ + (u \cdot \nabla)n^+ - \Delta n^+ = \operatorname{div} (n^+ \nabla \Psi) \\ \partial_t n^- + (u \cdot \nabla)n^- - \Delta n^- = -\operatorname{div} (n^- \nabla \Psi), \\ -\Delta \Psi = n^+ - n^-, \end{cases} \quad \text{in } \Omega \times (0, T),$$

where $u : \Omega \times (0, T) \rightarrow \mathbb{R}^3$ denotes the velocity field of the fluid, $P : \Omega \times (0, T) \rightarrow \mathbb{R}$ denotes the pressure function, $n^+, n^- : \Omega \times (0, T) \rightarrow \mathbb{R}$ are the number densities of positively and negatively charged constituents, and Ψ is the quasi-electrostatic potential function. The initial and boundary values of (1.1) are given by

$$(1.2) \quad (u, n^+, n^-) = (u_0, n_0^+, n_0^-) \quad \text{in } \Omega \times \{0\},$$

$$(1.3) \quad u = 0, \quad \frac{\partial n^+}{\partial \nu} = \frac{\partial n^-}{\partial \nu} = \frac{\partial \Psi}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, T),$$

where ν denotes the outward unit normal vector of $\partial\Omega$.

The system (1.1) models an isothermal, incompressible, viscous Newtonian fluid of uniform and homogeneous composition of a high number of positively and negatively

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[†]Shenzhen Key Laboratory of Advance Machine Learning and Applications, College of Mathematics and Statistics, Shenzhen University, Shenzhen 518060, Guangdong, China (huajun84@szu.edu.cn).

[‡]Department of Mathematics, Purdue University, West Lafayette, IN 47907 USA (wang2482@purdue.edu).

[§]South China Research Center for Applied Mathematics and Interdisciplinary Studies, South China Normal University, Zhong Shan Avenue West 55, Guangzhou 510631, China (xtzhang@m.scnu.edu.cn).

charged particles ranging from colloidal to nano size. It was proposed by Rubinstein [20] to model electro-kinetic fluids, which describes the interaction between the macroscopic fluid motion and the microscopic charge transportation. See Castellanos [2] for more discussions on the physical background on (1.1). In the system (1.1), we assume a dilute fluid and therefore the electromagnetic forces are neglected. There have been considerable interests in the mathematical analysis of the system (1.1). For example, Jerome [9] has proved the existence of local strong solutions by employing Kato's semigroup framework. Deng, Zhao, and Cui [4] have established the existence and well-posedness of mild solutions in the Triebel-Lizorkin and Besov spaces of negative indices. We refer to Zhao, Zhang, and Liu [31] for some time decay results of (1.1). The existence of global weak solutions of (1.1), (1.2), and (1.3), has been established by Schmuck [21] under the Neumann boundary condition (for bounded initial data n_0^+ and n_0^-), and Jerome and Sacco [10] under the mixed Dirichlet boundary condition. Fan, Li, and Nakamura [5] have proved some regularity criteria of weak solutions to (1.1) on $\Omega = \mathbb{R}^3$ in the spirit of Serrin. More recently, there are some interesting works by Wang, Liu, and Tan [29, 30] on generalized Navier-Stokes-Planck-Nernst-Poisson equations through an energetic variational approach.

When the underlying fluid is at rest, i.e., $u = 0$, the system (1.1) reduces to the Planck-Nernst-Poisson (PNP) equation, which is the drift-diffusion model for semiconductor devices, first proposed by Roosbroeck [19] in 1950, that has been widely accepted and applied in the semiconductor industry and in device simulations. See Gajewski [7], Mock [18], Seidman and Troianiello [25], and Fang and Ito [6] for results on the existence of global weak solutions to the PNP equation.

It remains an interesting question to investigate regularity properties of weak solutions in dimension three. Motivated by the celebrated work by Scheffer [22], Caffarelli, Kohn, and Nirenberg [1], and Lin [13] on the Navier-Stokes equation, we introduce the notion of a suitable weak solution of (1.1)–(1.3) and establish both the existence and partial regularity for such a weak solution. See also [8, 14, 3] for related works on other complex fluids.

A constitutive equation of the Navier-Stokes-Nernst-Planck-Poisson system (1.1) is the Navier-Stokes equation: for $0 < T \leq \infty$,

$$(1.4) \quad \begin{cases} \partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla P = f, \\ \nabla \cdot u = 0 \end{cases} \quad \text{in } Q_T = \Omega \times (0, T),$$

with the initial and boundary conditions

$$(1.5) \quad u(\cdot, 0) = u_0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \times [0, T].$$

The existence of global weak solutions of (1.4) and (1.5) ($T = \infty$) was established by Leray [11]; see also Teman [27]. While it is an outstanding open question whether (1.4) and (1.5) has a global smooth solution when $\Omega = \mathbb{R}^3$, there have been many studies concerning partial regularity of suitable weak solutions of (1.4) initiated by Scheffer [22] and then by Caffarelli, Kohn, and Nirenberg [1], where it was proven that the singular set has 1-dimensional Hausdorff measure zero. Such a theorem was later simplified by Lin [13]; see also Vasseur [28]. There has also been a lot of work on the regularity criteria of (1.4), going back to Serrin [23] where it has been proven that $u \in C^\infty(Q_T)$, provided $u \in L_t^q L_x^p(Q_T)$ and $p \geq 3$ and $2 \leq q < \infty$ satisfy

$$(1.6) \quad \frac{2}{q} + \frac{3}{p} = 1.$$

The end point case $p = 3$ and $q = \infty$ for (1.6) was resolved by [24].

The goal of this paper is to extend the partial regularity theory on the Navier–Stokes equation by Caffarelli, Kohn, and Nirenberg [1] to the system (1.1). We first recall the definition of suitable weak solutions to the system (1.1). For $T > 0$, denote $Q_T = \Omega \times (0, T)$ and

$$\mathcal{D} := \left\{ X \mid X \in C_0^\infty(Q_T, \mathbb{R}^3), \operatorname{div} X = 0 \right\}.$$

DEFINITION 1.1. We say that (u, n^+, n^-, Ψ) is a weak solution of (1.1) in Q_T if

$$\begin{aligned} u &\in L^\infty([0, T], L^2(\Omega, \mathbb{R}^3)) \cap L^2([0, T], H^1(\Omega, \mathbb{R}^3)), \\ \Psi &\in L^\infty([0, T], H^1(\Omega)) \cap L^2([0, T], H^2(\Omega)), \quad n^+, n^- \in L^2(Q_T, \mathbb{R}_+), \end{aligned}$$

and the system (1.1) holds in the sense of distributions: for any $\varphi \in \mathcal{D}$,

$$\int_{Q_T} (\langle u, \partial_t \varphi \rangle - \langle \nabla u, \nabla \varphi \rangle + u \otimes u : \nabla \varphi) \, dxdt = \int_{Q_T} \langle (n^+ - n^-) \nabla \Psi, \varphi \rangle \, dxdt,$$

and, for any $\phi \in C_0^\infty(Q_T)$,

$$\begin{aligned} \int_{Q_T} u \cdot \nabla \phi \, dxdt &= 0, \\ \int_{Q_T} (\langle n^+, \partial_t \phi \rangle + \langle n^+, \Delta \phi \rangle + \langle n^+, u \cdot \nabla \phi \rangle) \, dxdt &= \int_{Q_T} \langle n^+ \nabla \Psi, \nabla \phi \rangle \, dxdt, \\ \int_{Q_T} (\langle n^-, \partial_t \phi \rangle + \langle n^-, \Delta \phi \rangle + \langle n^-, u \cdot \nabla \phi \rangle) \, dxdt &= - \int_{Q_T} \langle n^- \nabla \Psi, \nabla \phi \rangle \, dxdt, \end{aligned}$$

and

$$\int_{\Omega} \langle \nabla \Psi, \nabla \phi \rangle \, dx = \int_{\Omega} (n^+ - n^-) \phi \, dx \quad \forall 0 < t < T,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathbb{R}^3$.

A weak solution (u, P, n^+, n^-, Ψ) is called a suitable weak solution of (1.1) if, in addition, it enjoys the local energy inequality property defined as follows.

DEFINITION 1.2. A weak solution (u, P, n^+, n^-, Ψ) of (1.1) is called a suitable weak solution of (1.1) in Q_T , if the following conditions are true:

- (a) $P \in L^{\frac{3}{2}}(Q_T)$.
- (b) $n^+, n^- \in L^2(Q_T)$.
- (c) There exist positive constants $0 < E_1, E_2 < \infty$ such that

$$\begin{cases} \int_{\Omega} (|u|^2 + |\nabla \Psi|^2)(x, t) \, dx \leq E_1 \quad \forall t \in (0, T), \\ \int_{Q_T} (|\nabla u|^2 + |\nabla^2 \Psi|^2) \, dxdt \leq E_2. \end{cases}$$

- (d) (u, P, n^+, n^-, Ψ) satisfy (1.1) in the sense of distributions on Q_T .
- (e) For any $\phi \in C^\infty(Q_T)$, $\phi \geq 0$, the generalized energy inequality (1.7) holds:

$$\begin{aligned} (1.7) \quad 2 \int_{Q_T} |\nabla u|^2 \phi \, dxdt &\leq \int_{Q_T} |u|^2 (\partial_t \phi + \Delta \phi) \, dxdt + \int_{Q_T} (|u|^2 + 2P) u \cdot \nabla \phi \, dxdt \\ &\quad - 2 \int_{Q_T} \left(\nabla \Psi \otimes \nabla \Psi - \frac{1}{2} |\nabla \Psi|^2 I_3 \right) : \nabla (u \phi) \, dxdt. \end{aligned}$$

Now we state our main theorem.

THEOREM 1.1. *For any $0 < T \leq \infty$, $u_0 \in L^2(\Omega, \mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$, and $0 \leq n_0^+, n_0^- \in L^2(\Omega)$ with $\int_{\Omega} n_0^+ dx = \int_{\Omega} n_0^- dx$, there exists a suitable weak solution (u, P, n^+, n^-, Ψ) of (1.1)–(1.3) in Q_T such that*

- (i) $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T)$, $P \in L^{\frac{5}{3}}(Q_T)$, $0 \leq n^+, n^- \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T)$, $\Psi \in L_t^\infty H_x^2 \cap L_t^2 H_x^3(Q_T)$, and

$$(1.8) \quad \begin{aligned} & \| (u, n^+, n^-) \|_{L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T)} + \| P \|_{L^{\frac{5}{3}}(Q_T)} + \| \Psi \|_{L_t^\infty H_x^2 \cap L_t^2 H_x^3(Q_T)} \\ & \leq C(\|u_0\|_{L^2(\Omega)}, \|n_0^+\|_{L^2(\Omega)}, \|n_0^-\|_{L^2(\Omega)}); \end{aligned}$$

- (ii) (u, n^+, n^-, Ψ) satisfies the following global energy inequality: for any $0 < t \leq T$,

$$(1.9) \quad \begin{aligned} & \int_{\Omega} (|u|^2 + |\nabla \Psi|^2)(x, t) dx + 2 \int_{Q_t} (|\nabla u|^2 + |n^+ - n^-|^2 + (n^+ + n^-)|\nabla \Psi|^2) dx ds \\ & \leq \int_{\Omega} (|u_0|^2 + |\nabla \Psi_0|^2)(x) dx, \end{aligned}$$

where $\Psi_0 \in H^2(\Omega)$ solves

$$-\Delta \Psi_0 = n_0^+ - n_0^- \text{ in } \Omega, \quad \frac{\partial \Psi_0}{\partial \nu} = 0 \text{ on } \partial \Omega;$$

- (iii) there exists a closed set $\Sigma \subset Q_T$ with $\mathcal{P}^1(\Sigma) = 0$, such that $(u, n^+, n^-, \Psi) \in C^\infty(Q_T \setminus \Sigma)$.

Here $\mathcal{P}^k$, $0 \leq k \leq 4$, denotes the k -dimensional Hausdorff measure on $\mathbb{R}^4$ with respect to the parabolic metric:

$$(1.10) \quad \delta((x, t), (y, s)) = \max \{ |x - y|, \sqrt{|t - s|} \} \quad \forall (x, t), (y, s) \in \mathbb{R}^4.$$

We would like to point out the major steps in the proof of Theorem 1.1:

1. The existence of suitable weak solutions to (1.1) is established by first studying approximate systems of (1.1) through modifying a “retarded” mollification of its drifting coefficients, $\Theta_\epsilon(u)$, originally due to [1] on the Navier-Stokes equation. Here we need to modify it so that its normal component vanishes on the boundary of Ω in order to guarantee the equations for $n^\pm$ enjoy both the positivity and maximum principle property. For the existence of suitable weak solutions to an approximate version of (1.1), we employ a contraction map theorem on the function spaces $L_t^4 L_x^2(Q_T)$ for $n^\pm$ which was first employed by Schmuck [21]. Then we prove that such a sequence of suitable weak solutions to the approximate equation enjoys some uniform estimates and hence converges to a suitable weak solution to (1.1).
2. The partial regularity of a suitable weak solution constructed in step (1) above is proven by first employing the fact $\Psi \in L_t^\infty H_x^2(Q_T)$ and performing a blowing up argument to establish an ϵ_0 -decay property for (u, P) in the renormalized $L^3 \times L^{\frac{3}{2}}$ -norms, and then applying the Reisz potential estimate of u in parabolic Morrey spaces to obtain L^q -boundedness of u for any $1 < q < \infty$, which eventually yields the ϵ_0 -smoothness of (u, n^+, n^-, Ψ) via the bootstrap argument.

3. To obtain the size estimate of the singular set, we improve the ϵ_0 -regularity from (2) through the process, similar to that of the Navier–Stokes equation by [1], that establishes the so-called A-B-C-D lemmas.

The paper is organized as follows. In section 2, we will establish the existence of the suitable weak solutions of (1.1)–(1.3). In section 3, we will prove an ϵ_0 -regularity for suitable weak solutions to (1.1). In section 4, we will improve the ϵ_0 -regularity from section 3 and provide a proof of Theorem 1.1.

2. Existence of suitable weak solutions. In order to obtain the existence of suitable weak solutions of (1.1), we first consider the following system: given $w \in C^\infty(\bar{\Omega} \times [0, T], \mathbb{R}^3)$ with $\operatorname{div} w = 0$ in Q_T and $w \cdot \nu = 0$ on $\partial\Omega \times [0, T]$, let (u, P, n^+, n^-, Ψ) solve

$$(2.1) \quad \begin{cases} \partial_t u + (w \cdot \nabla)u - \Delta u + \nabla P = -(n^+ - n^-)\nabla \Psi, \\ \operatorname{div} u = 0, \\ \partial_t n^+ + (w \cdot \nabla)n^+ - \Delta n^+ = \operatorname{div}([n^+]_+ \nabla \Psi), \\ \partial_t n^- + (w \cdot \nabla)n^- - \Delta n^- = -\operatorname{div}([n^-]_+ \nabla \Psi), \\ -\Delta \Psi = n^+ - n^-, \end{cases}$$

subject to the initial and boundary conditions

$$(2.2) \quad (u, n^+, n^-)|_{t=0} = (u_0, n_0^+, n_0^-) \text{ in } \Omega,$$

$$(2.3) \quad u = 0, \quad \frac{\partial n^+}{\partial \nu} = \frac{\partial n^-}{\partial \nu} = \frac{\partial \Psi}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0, T).$$

Here $[y]_+ = \max\{y, 0\}$ denotes the positive part of $y \in \mathbb{R}$.

We shall use the following function spaces:

$$\mathcal{V} = C_0^\infty(\Omega, \mathbb{R}^3) \cap \{u : \operatorname{div} u = 0\};$$

$$\mathbf{H} = \text{Closure of } \mathcal{V} \text{ in } L^2(\Omega);$$

$$\mathbf{V} = \text{Closure of } \mathcal{V} \text{ in } H^1(\Omega).$$

Concerning (2.1), (2.2), and (2.3), we have the following existence theorem.

THEOREM 2.1. *For a bounded and smooth domain $\Omega \subset \mathbb{R}^3$, $u_0 \in \mathbf{H}$, and two nonnegative $n_0^+, n_0^- \in L^2(\Omega)$ satisfying*

$$\int_{\Omega} n_0^+(x) dx = \int_{\Omega} n_0^-(x) dx,$$

if $w \in C^\infty(\bar{\Omega} \times [0, T], \mathbb{R}^3)$ with $\operatorname{div} w = 0$ in Q_T and $w \cdot \nu = 0$ on $\partial\Omega \times [0, T]$, then there is a unique weak solution (u, P, n^+, n^-, Ψ) of (2.1), (2.2), and (2.3) such that $n^+, n^- \geq 0$ in $\Omega \times [0, T]$, and

$$(2.4) \quad \begin{cases} u \in C([0, T], \mathbf{H}) \cap L^2([0, T], \mathbf{V}), \\ \Psi \in L^\infty([0, T], H^2(\Omega)) \cap L^2([0, T], H^3(\Omega)), \\ n^+, n^- \in L^\infty([0, T], L^2(\Omega)) \cap L^2([0, T], H^1(\Omega)). \end{cases}$$

The existence of weak solutions (u, P, n^+, n^-, Ψ) to (2.1), (2.2), and (2.3) will be established by the contraction mapping theorem. The uniqueness of such weak solutions (u, P, n^+, n^-, Ψ) can be employed to show the nonnegativity of n^+, n^- as follows.

LEMMA 2.1. *Under the assumptions of Theorem 2.1, the weak solution (u, P, n^+, n^-, Ψ) of (2.1), (2.2), and (2.3), satisfying (2.4), must satisfy $n^+, n^- \geq 0$ in Q_T .*

Proof. This proof is similar to that of [21, Lemma 1]. In order to prove that n^+, n^- are nonnegative, let $(\tilde{u}, \tilde{P}, \tilde{n}^+, \tilde{n}^-, \tilde{\Psi})$, that satisfy (2.4), be a weak solution of the system:

$$(2.5) \quad \begin{cases} \partial_t \tilde{u} + (w \cdot \nabla) \tilde{u} - \Delta \tilde{u} + \nabla \tilde{P} = -(\tilde{n}^+ - \tilde{n}^-) \nabla \tilde{\Psi}, \\ \operatorname{div} \tilde{u} = 0, \\ \partial_t \tilde{n}^+ + (w \cdot \nabla) \tilde{n}^+ - \Delta \tilde{n}^+ = \operatorname{div} ([\tilde{n}^+]_+ \nabla \tilde{\Psi}) \\ \partial_t \tilde{n}^- + (w \cdot \nabla) \tilde{n}^- - \Delta \tilde{n}^- = -\operatorname{div} ([\tilde{n}^-]_+ \nabla \tilde{\Psi}), \\ -\Delta \tilde{\Psi} = \tilde{n}^+ - \tilde{n}^-, \end{cases} \quad \text{in } Q_T,$$

subject to the initial and boundary conditions

$$(2.6) \quad (\tilde{u}, \tilde{n}^+, \tilde{n}^-)|_{t=0} = (u_0, n_0^+, n_0^-) \quad \text{in } \Omega,$$

$$(2.7) \quad \tilde{u} = 0, \quad \frac{\partial \tilde{n}^+}{\partial \nu} = \frac{\partial \tilde{n}^-}{\partial \nu} = \frac{\partial \tilde{\Psi}}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times (0, T).$$

The existence of such a weak solution $(\tilde{u}, \tilde{P}, \tilde{n}^+, \tilde{n}^-, \tilde{\Psi})$ will be constructed by Theorem 2.1 below.

Note that $\tilde{n}^+ = [\tilde{n}^+]_+ - [-\tilde{n}^+]_+$. Multiplying (2.29)₃ by $[-\tilde{n}^+]_+$ and integrating over Ω , we have that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |[-\tilde{n}^+]_+|^2 dx + \int_{\Omega} |\nabla [-\tilde{n}^+]_+|^2 dx = \int_{\Omega} [\tilde{n}^+]_+ \nabla [-\tilde{n}^+]_+ \cdot \nabla \tilde{\Psi} dx = 0.$$

This implies that

$$\int_{\Omega} |[-\tilde{n}^+]_+|^2 dx \leq \int_{\Omega} |[-n_0]_+|^2 dx = 0,$$

since n_0^+ is nonnegative. Thus we conclude that $\tilde{n}^+ \geq 0$ in Q_T . Similarly, we can show that $\tilde{n}^- \geq 0$ in Q_T . Therefore, we see that $(\tilde{u}, \tilde{P}, \tilde{n}^+, \tilde{n}^-, \tilde{\Psi})$ is also a weak solution of (2.1), (2.2), and (2.3). From Theorem 2.1, the uniqueness holds for weak solutions to (2.1), (2.2), and (2.3) that satisfy (2.4). Thus

$$(\tilde{u}, \tilde{P}, \tilde{n}^+, \tilde{n}^-, \tilde{\Psi}) \equiv (u, P, n^+, n^-, \Psi) \quad \text{in } Q_T.$$

In particular, $n^+ \equiv \tilde{n}^+ \geq 0$ and $n^- \equiv \tilde{n}^- \geq 0$ in Q_T . $\square$

PROPOSITION 2.1. *Under the same assumptions as Theorem 2.1, if, in addition, $n_0^+, n_0^- \in L^p(\Omega)$ for some $p \geq 2$, then the weak solution (u, P, n^+, n^-, Ψ) of (2.1), (2.2), and (2.3), satisfying (2.4), would enjoy*

$$(2.8) \quad n^+, n^- \in L^\infty([0, T], L^p(\Omega)), \quad \Psi \in L^\infty([0, T], W^{2,p}(\Omega))$$

and

$$(2.9) \quad \begin{aligned} & \int_{\Omega} (|n^+|^p + |n^-|^p)(x, t) dx + p(p-1) \int_{Q_t} [(n^+)^{p-2} |\nabla n^+|^2 + (n^-)^{p-2} |\nabla n^-|^2] dx dt \\ & \leq \int_{\Omega} (|n_0^+|^p + |n_0^-|^p)(x) dx, \quad 0 \leq t < T. \end{aligned}$$

Proof. Multiplying (2.1)₃ by $|n^+|^{p-2}n^+$ and (2.1)₄ by $|n^-|^{p-2}n^-$, integrating the resulting equations over Ω , and applying (2.1)₅, we obtain that

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega} (|n^+|^p + |n^-|^p) dx \\ & + (p-1) \int_{\Omega} (|\nabla n^+|^2 |n^+|^{p-2} + |\nabla n^-|^2 |n^-|^{p-2}) dx \\ & = -\frac{p-1}{p} \int_{\Omega} \nabla \Psi \cdot \nabla (|n^+|^p - |n^-|^p) dx \\ & = -\frac{p-1}{p} \int_{\Omega} (|n^+|^p - |n^-|^p)(n^+ - n^-) dx \leq 0, \end{aligned}$$

where we have used in the last step the fact that n^+, n^- are nonnegative, and

$$(|n^+|^p - |n^-|^p)(n^+ - n^-) = [(n^+)^p - (n^-)^p][n^+ - n^-] \geq 0.$$

Therefore, we obtain that

$$\frac{d}{dt} \int_{\Omega} (|n^+|^p + |n^-|^p) dx + (p-1) \int_{Q_t} [(n^+)^{p-2} |\nabla n^+|^2 + (n^-)^{p-2} |\nabla n^-|^2] dx dt \leq 0.$$

This implies (2.9) and completes the proof. $\square$

Proof of Theorem 2.1. Step 1: Existence. We will modify the approach by Schmuck [21] as follows. For $T > 0$, set the function space

$$Y_T \equiv \{\mathbf{y} = (n^+, n^-) : n^{\pm} \in L^4([0, T], L^2(\Omega))\},$$

which is equipped with the norm

$$\|(n^+, n^-)\|_{Y_T} = \|(n^+, n^-)\|_{L^4([0, T], L^2(\Omega))}.$$

Now we define a map $F : Y_T \mapsto Y_T$ as follows. For any $\bar{\mathbf{y}} = (\bar{n}^+, \bar{n}^-) \in Y_T$, define $F(\bar{\mathbf{y}}) = \mathbf{y} = (n^+, n^-)$, where $\mathbf{y}$ is a solution of the system

$$\begin{aligned} (2.10) \quad & -\Delta \bar{\Psi} = \bar{n}^+ - \bar{n}^- \text{ in } \Omega, \quad \frac{\partial \bar{\Psi}}{\partial \nu} = 0 \text{ on } \partial\Omega, \\ (2.11) \quad & \begin{cases} \partial_t n^+ + (w \cdot \nabla) n^+ - \Delta n^+ = \operatorname{div} ([n^+]_+ \nabla \bar{\Psi}) & \text{in } Q_T, \\ \partial_t n^- + (w \cdot \nabla) n^- - \Delta n^- = -\operatorname{div} ([n^-]_- \nabla \bar{\Psi}) & \text{in } Q_T, \\ (n^+, n^-) = (n_0^+, n_0^-) & \text{on } \Omega \times \{t = 0\}, \\ \frac{\partial n^+}{\partial \nu} = \frac{\partial n^-}{\partial \nu} = 0 & \text{on } \partial\Omega \times [0, T]. \end{cases} \end{aligned}$$

Note that for any $f, g \in L^1(\Omega)$, it holds that

$$|[f]_+| \leq |f|, \quad |[f]_+ - [g]_+| \leq |f - g| \text{ a.e. } \Omega.$$

Since $\bar{n}^+ - \bar{n}^- \in L^4([0, T], L^2(\Omega))$, it follows from the $W^{2,2}$ -theory of the Laplace equation that $\bar{\Psi} \in L^4([0, T], W^{2,2}(\Omega))$, and

$$(2.12) \quad \|\bar{\Psi}\|_{L^4([0, T], W^{2,2}(\Omega))} \leq C \|\bar{n}^+ - \bar{n}^-\|_{L^4([0, T], L^2(\Omega))} \leq C \|\bar{\mathbf{y}}\|_{Y_T}.$$

By the theory of linear parabolic systems [12], there exists a unique solution (n^+, n^-) of (2.11) in $L^\infty([0, T], L^2(\Omega)) \cap L^2([0, T], H^1(\Omega))$ for any $T > 0$. Moreover, by multiplying (2.11)₁ by n^+ and (2.11)₂ by n^- , integrating the resulting equation over Ω , and adding these two equations, we obtain that

$$\begin{aligned}
 (2.13) \quad & \frac{1}{2} \frac{d}{dt} \int_{\Omega} (|n^+|^2 + |n^-|^2) dx + \int_{\Omega} (|\nabla n^+|^2 + |\nabla n^-|^2) dx \\
 &= - \int_{\Omega} \langle \nabla \bar{\Psi}, [n^+]_+ \nabla n^+ - [n^-]_- \nabla n^- \rangle dx \\
 &\leq C \|\nabla \bar{\Psi}\|_{L^6(\Omega)} \| |n^+| + |n^-| \|_{L^3(\Omega)} \| |\nabla n^+| + |\nabla n^-| \|_{L^2(\Omega)} \\
 &\leq C \|\bar{\Psi}\|_{W^{2,2}(\Omega)} (\|n^+\|_{L^2(\Omega)} + \|n^-\|_{L^2(\Omega)})^{\frac{1}{2}} \\
 &\quad \cdot [(\|n^+\|_{L^2(\Omega)} + \|n^-\|_{L^2(\Omega)}) + (\|\nabla n^+\|_{L^2(\Omega)} + \|\nabla n^-\|_{L^2(\Omega)})]^{\frac{1}{2}} \\
 &\quad \cdot [\|\nabla n^+\|_{L^2(\Omega)} + \|\nabla n^-\|_{L^2(\Omega)}] \\
 &\leq C [\|\bar{n}^+\|_{L^2(\Omega)} + \|\bar{n}^-\|_{L^2(\Omega)}] (\|n^+\|_{L^2(\Omega)} + \|n^-\|_{L^2(\Omega)})^{\frac{1}{2}} \\
 &\quad \cdot [(\|n^+\|_{L^2(\Omega)} + \|n^-\|_{L^2(\Omega)}) + (\|\nabla n^+\|_{L^2(\Omega)} + \|\nabla n^-\|_{L^2(\Omega)})]^{\frac{1}{2}} \\
 &\quad \cdot [\|\nabla n^+\|_{L^2(\Omega)} + \|\nabla n^-\|_{L^2(\Omega)}] \\
 &\leq \frac{1}{2} (\|\nabla n^+\|_{L^2(\Omega)}^2 + \|\nabla n^-\|_{L^2(\Omega)}^2) \\
 &\quad + C [1 + (\|\bar{n}^+\|_{L^2(\Omega)}^4 + \|\bar{n}^-\|_{L^2(\Omega)}^4)] \cdot [\|n^+\|_{L^2(\Omega)}^2 + \|n^-\|_{L^2(\Omega)}^2].
 \end{aligned}$$

This implies that

$$\begin{aligned}
 (2.14) \quad & \frac{d}{dt} \int_{\Omega} (|n^+|^2 + |n^-|^2) dx + \int_{\Omega} (|\nabla n^+|^2 + |\nabla n^-|^2) dx \\
 &\leq C [1 + (\|\bar{n}^+\|_{L^2(\Omega)}^4 + \|\bar{n}^-\|_{L^2(\Omega)}^4)] \cdot [\|n^+\|_{L^2(\Omega)}^2 + \|n^-\|_{L^2(\Omega)}^2].
 \end{aligned}$$

Applying Gronwall's inequality, we obtain that

$$\begin{aligned}
 (2.15) \quad & \sup_{0 \leq t \leq T} \int_{\Omega} (|n^+|^2 + |n^-|^2) dx + \int_{Q_T} (|\nabla n^+|^2 + |\nabla n^-|^2) dx dt \\
 &\leq e^{Ct} \exp \left\{ C \int_0^t (\|\bar{n}^+\|_{L^2(\Omega)}^4 + \|\bar{n}^-\|_{L^2(\Omega)}^4) d\tau \right\} \int_{\Omega} (|n_0^+|^2 + |n_0^-|^2)(x) dx.
 \end{aligned}$$

For $R > 0$, if $\bar{\mathbf{y}} = (\bar{n}^+, \bar{n}^-) \in Y_T$ belongs to

$$B_R^Y = \{\bar{\mathbf{y}} : \|\bar{\mathbf{y}}\|_{Y_T} \leq R\},$$

then (2.15) yields that

$$\|F(\bar{\mathbf{y}})\|_{Y_T}^4 = \int_0^T \left(\int_{\Omega} (|n^+|^2 + |n^-|^2)(x, t) dx \right)^2 dt \leq C_0 \exp(CT + CR^4) T \leq \left(\frac{R}{2} \right)^4,$$

provided that $T = T_1 \in (0, 1]$ is chosen sufficiently small. Hence there exists a sufficiently small $T = T_1 \in (0, 1]$ such that $F(\bar{\mathbf{y}}) \in B_{\frac{Y}{2}}^Y \subset B_R^Y$.

Next we want to show that $F : B_R^Y \mapsto B_R^Y$ is a contractive map. For $i = 1, 2$, let $\bar{\mathbf{y}}_i = (\bar{n}_i^+, \bar{n}_i^-) \in B_R^Y$ and $\mathbf{y}_i = (n_i^+, n_i^-) = F(\bar{\mathbf{y}}_i) \in B_R^Y$ be the solutions of (2.10) and (2.17). Then $n_1^+ - n_2^+$ and $n_1^- - n_2^-$ solve

$$(2.16) \quad -\Delta(\bar{\Psi}_1 - \bar{\Psi}_2) = (\bar{n}_1^+ - \bar{n}_1^-) - (\bar{n}_2^+ - \bar{n}_2^-) \text{ in } \Omega, \quad \frac{\partial(\bar{\Psi}_1 - \bar{\Psi}_2)}{\partial \nu} = 0 \text{ on } \partial\Omega,$$

(2.17)

$$\begin{cases} \partial_t(n_1^+ - n_2^+) + (w \cdot \nabla)(n_1^+ - n_2^+) - \Delta(n_1^+ - n_2^+) \\ = \operatorname{div}([n_1^+]_+ \nabla(\bar{\Psi}_1 - \bar{\Psi}_2)) + \operatorname{div}([n_2^+]_+ \nabla \bar{\Psi}_2) & \text{in } Q_T, \\ \partial_t(n_1^- - n_2^-) + (w \cdot \nabla)(n_1^- - n_2^-) - \Delta(n_1^- - n_2^-) \\ = -\operatorname{div}([n_1^-]_+ \nabla(\bar{\Psi}_1 - \bar{\Psi}_2)) - \operatorname{div}([n_2^-]_+ \nabla \bar{\Psi}_2) & \text{in } Q_T, \\ (n_1^+ - n_2^+, n_1^- - n_2^-) = (0, 0) & \text{on } \Omega \times \{t = 0\}, \\ \frac{\partial(n_1^+ - n_2^+)}{\partial \nu} = \frac{\partial(n_1^- - n_2^-)}{\partial \nu} = 0 & \text{on } \partial\Omega \times [0, T]. \end{cases}$$

Multiplying (2.17)₁ by $(n_1^+ - n_2^+)$, (2.17)₂ by $(n_1^- - n_2^-)$, integrating the resulting equations over Ω , and adding them together, we obtain that

(2.18)

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} (|n_1^+ - n_2^+|^2 + |n_1^- - n_2^-|^2) dx \\ & + \int_{\Omega} (|\nabla(n_1^+ - n_2^+)|^2 + |\nabla(n_1^- - n_2^-)|^2) dx \\ & = - \int_{\Omega} [n_1^+]_+ \langle \nabla(\bar{\Psi}_1 - \bar{\Psi}_2), \nabla(n_1^+ - n_2^+) \rangle + ([n_1^+]_+ - [n_2^+]_+) \langle \nabla \bar{\Psi}_2, \nabla(n_1^+ - n_2^+) \rangle dx \\ & + \int_{\Omega} [n_1^-]_+ \langle \nabla(\bar{\Psi}_1 - \bar{\Psi}_2), \nabla(n_1^- - n_2^-) \rangle + ([n_1^-]_+ - [n_2^-]_+) \langle \nabla \bar{\Psi}_2, \nabla(n_1^- - n_2^-) \rangle dx \\ & \leq C \|\nabla \bar{\Psi}_2\|_{L^6(\Omega)} (\|[n_1^+]_+ - [n_2^+]_+\|_{L^3(\Omega)} \|\nabla(n_1^+ - n_2^+)\|_{L^2(\Omega)} \\ & \quad + \|[n_1^-]_+ - [n_2^-]_+\|_{L^3(\Omega)} \|\nabla(n_1^- - n_2^-)\|_{L^2(\Omega)}) \\ & + C \|\nabla(\bar{\Psi}_1 - \bar{\Psi}_2)\|_{L^6(\Omega)} (\|[n_1^+]_+\|_{L^3(\Omega)} \|\nabla(n_1^+ - n_2^+)\|_{L^2(\Omega)} \\ & \quad + \|[n_1^-]_+\|_{L^3(\Omega)} \|\nabla(n_1^- - n_2^-)\|_{L^2(\Omega)}) \\ & \leq C \|\bar{n}_2^+ - \bar{n}_2^-\|_{L^2(\Omega)} \left\{ \|n_1^+ - n_2^+\|_{L^2(\Omega)} \|\nabla(n_1^+ - n_2^+)\|_{L^2(\Omega)} \right. \\ & \quad + \|n_1^+ - n_2^+\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla(n_1^+ - n_2^+)\|_{L^2(\Omega)}^{\frac{3}{2}} \\ & \quad + \|n_1^- - n_2^-\|_{L^2(\Omega)} \|\nabla(n_1^- - n_2^-)\|_{L^2(\Omega)} \\ & \quad \left. + \|n_1^- - n_2^-\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla(n_1^- - n_2^-)\|_{L^2(\Omega)}^{\frac{3}{2}} \right\} \\ & + C \left[\|\bar{n}_1^+ - \bar{n}_2^+\|_{L^2(\Omega)} + \|\bar{n}_1^- - \bar{n}_2^-\|_{L^2(\Omega)} \right] \\ & \quad \left\{ \|n_1^+\|_{L^2(\Omega)} \|\nabla(n_1^+ - n_2^+)\|_{L^2(\Omega)} \right. \\ & \quad + \|n_1^+\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla n_1^+\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla(n_1^+ - n_2^+)\|_{L^2(\Omega)} \\ & \quad + \|n_1^-\|_{L^2(\Omega)} \|\nabla(n_1^- - n_2^-)\|_{L^2(\Omega)} \\ & \quad \left. + \|n_1^-\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla n_1^-\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla(n_1^- - n_2^-)\|_{L^2(\Omega)} \right\} \\ & \leq \frac{1}{2} \left(\|\nabla(n_1^+ - n_2^+)\|_{L^2(\Omega)}^2 + \|\nabla(n_1^- - n_2^-)\|_{L^2(\Omega)}^2 \right) \\ & + C \left(1 + \|\bar{n}_1^+\|_{L^2(\Omega)}^4 + \|\bar{n}_2^-\|_{L^2(\Omega)}^4 \right) \left(\|n_1^+ - n_2^+\|_{L^2(\Omega)}^2 + \|n_1^- - n_2^-\|_{L^2(\Omega)}^2 \right) \\ & + C \left\{ \|n_1^+\|_{L^2(\Omega)} \|n_1^+\|_{H^1(\Omega)} + \|n_1^-\|_{L^2(\Omega)} \|n_1^-\|_{H^1(\Omega)} \right\} \\ & \quad \cdot \left[\|\bar{n}_1^+ - \bar{n}_2^+\|_{L^2(\Omega)}^2 + \|\bar{n}_1^- - \bar{n}_2^-\|_{L^2(\Omega)}^2 \right], \end{aligned}$$

where we have used the following inequalities: for any $f, g \in H^1(\Omega)$,

$$\begin{cases} \|f_+ \|_{L^3(\Omega)} \leq \|f\|_{L^3(\Omega)} \leq C \|f\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla f\|_{L^2(\Omega)}^{\frac{1}{2}}, \\ \|f_+ - g_+ \|_{L^3(\Omega)} \leq \|f - g\|_{L^3(\Omega)} \leq C \|f - g\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla(f - g)\|_{L^2(\Omega)}^{\frac{1}{2}}. \end{cases}$$

Therefore we conclude that

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} (|n_1^+ - n_2^+|^2 + |n_1^- - n_2^-|^2) dx + \int_{\Omega} (|\nabla(n_1^+ - n_2^+)|^2 + |\nabla(n_1^- - n_2^-)|^2) dx \\ & \leq C \left(1 + \|\bar{n}_1^-\|_{L^2(\Omega)}^4 + \|\bar{n}_2^-\|_{L^2(\Omega)}^4 \right) \left(\|n_1^+ - n_2^+\|_{L^2(\Omega)}^2 + \|n_1^- - n_2^-\|_{L^2(\Omega)}^2 \right) \\ & \quad + C \left\{ \|n_1^+\|_{L^2(\Omega)} \|n_1^+\|_{H^1(\Omega)} + \|n_1^-\|_{L^2(\Omega)} \|n_1^-\|_{H^1(\Omega)} \right\} \\ (2.19) \quad & \cdot \left[\|\bar{n}_1^+ - \bar{n}_2^+\|_{L^2(\Omega)}^2 + \|\bar{n}_1^- - \bar{n}_2^-\|_{L^2(\Omega)}^2 \right]. \end{aligned}$$

Applying Gronwall's inequality, we obtain that

$$\begin{aligned} & \sup_{0 \leq t \leq T} \int_{\Omega} (|n_1^+ - n_2^+|^2 + |n_1^- - n_2^-|^2) dx + \int_Q (|\nabla(n_1^+ - n_2^+)|^2 + |\nabla(n_1^- - n_2^-)|^2) dx dt \\ (2.20) \quad & \leq \alpha(T) \beta^{\frac{1}{2}}(T) \left\{ \int_0^T [\|\bar{n}_1^+ - \bar{n}_2^+\|_{L^2(\Omega)}^4 + \|\bar{n}_1^- - \bar{n}_2^-\|_{L^2(\Omega)}^4] dt \right\}^{\frac{1}{2}}, \end{aligned}$$

where

$$\alpha(T) = \exp \left(C \int_0^T (1 + \|\bar{n}_1^-\|_{L^2(\Omega)}^4 + \|\bar{n}_2^-\|_{L^2(\Omega)}^4) dt \right)$$

and

$$\beta(T) = \left(\|n_1^+\|_{L^\infty([0,T], L^2(\Omega))}^2 + \|n_1^-\|_{L^\infty([0,T], L^2(\Omega))}^2 \right) \int_0^T (\|n_1^+\|_{H^1(\Omega)}^2 + \|n_1^-\|_{H^1(\Omega)}^2) dt.$$

It follows from $(\bar{n}_1^+, \bar{n}_1^-) \in B_R^Y$ and (2.15) that for $0 < T \leq T_1$,

$$\max \{ \alpha(T), \beta(T) \} \leq C(R).$$

Hence (2.20) yields that for $0 < T \leq T_1$,

$$\begin{aligned} \|(n_1^+, n_1^-) - (n_2^+, n_2^-)\|_{Y_T}^4 & \leq \int_0^T \left\{ \int_{\Omega} (|n_1^+ - n_2^+|^2 + |n_1^- - n_2^-|^2) dx \right\}^2 dt \\ & \leq T \left\{ \sup_{0 \leq t \leq T} \int_{\Omega} (|n_1^+ - n_2^+|^2 + |n_1^- - n_2^-|^2) dx \right\}^2 \\ & \leq T \alpha^2(T) \beta(T) \int_0^T [\|\bar{n}_1^+ - \bar{n}_2^+\|_{L^2(\Omega)}^4 + \|\bar{n}_1^- - \bar{n}_2^-\|_{L^2(\Omega)}^4] dt \\ & \leq C(R) T \|(\bar{n}_1^+, \bar{n}_1^-) - (\bar{n}_2^+, \bar{n}_2^-)\|_{Y_T}^4 \\ (2.21) \quad & \leq 2^{-4} \|(\bar{n}_1^+, \bar{n}_1^-) - (\bar{n}_2^+, \bar{n}_2^-)\|_{Y_T}^4, \end{aligned}$$

provided $T = T_2 \leq \min\{T_1, \frac{1}{16}C(R)\}$.

This implies that $F : B_R^Y \mapsto B_R^Y$ is a contractive map with a contraction constant $\frac{1}{2}$, provided T_2 and R are chosen sufficiently small. Therefore, there exists a unique fixed point $\mathbf{y} = (n^+, n^-) \in B_R^Y$ of F , i.e., $\mathbf{y} = F(\mathbf{y})$. In particular, (n^+, n^-, Ψ) is a solution of

$$(2.22) \quad -\Delta \Psi = n^+ - n^- \text{ in } \Omega, \quad \frac{\partial \Psi}{\partial \nu} = 0 \text{ on } \partial \Omega,$$

$$(2.23) \quad \begin{cases} \partial_t n^+ + (w \cdot \nabla) n^+ - \Delta n^+ = \operatorname{div} ([n^+]_+ \nabla \Psi) & \text{in } Q_T, \\ \partial_t n^- + (w \cdot \nabla) n^- - \Delta n^- = -\operatorname{div} ([n^-]_+ \nabla \Psi) & \text{in } Q_T, \\ (n^+, n^-) = (n_0^+, n_0^-) & \text{on } \Omega \times \{t = 0\}, \\ \frac{\partial n^+}{\partial \nu} = \frac{\partial n^-}{\partial \nu} = 0 & \text{on } \partial \Omega \times [0, T_2] \end{cases}$$

on the interval $[0, T_2]$ such that $n^\pm \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_{T_2})$, $\Psi \in L_t^\infty H_x^2 \cap L_t^2 H_x^3(Q_{T_2})$, and

$$(2.24) \quad \|(n^+, n^-)\|_{L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_{T_2})} + \|\Psi\|_{L_t^\infty H_x^2 \cap L_t^2 H_x^3(Q_{T_2})} \leq C(\|(n_0^+, n_0^-)\|_{L^2(\Omega)}, T_2).$$

For such a solution (n^+, n^-, Ψ) to (2.22) and (2.23), let $u \in L^\infty([0, T_2], \mathbf{H}) \cap L^2([0, T_2], \mathbf{V})$ be a weak solution to the system

$$(2.25) \quad \begin{cases} \partial_t u + (w \cdot \nabla) u - \Delta u + \nabla P = -(n^+ - n^-) \nabla \Psi & \text{in } Q_T, \\ \operatorname{div} u = 0 & \text{in } Q_T, \\ u = u_0 & \text{in } \Omega \times \{0\}, \\ u = 0 & \text{on } \partial \Omega \times [0, T_2]. \end{cases}$$

Since $(n^+ - n^-) \nabla \Psi \in L^\infty([0, T_2], L^{\frac{3}{2}}(\Omega))$, it follows from the regularity theory of the linear Stokes system that $\partial_t u, \nabla^2 u \in L^{\frac{3}{2}}(Q_{T_2})$ and $\nabla P \in L^{\frac{3}{2}}(Q_{T_2})$ and

$$(2.26) \quad \begin{aligned} & \|u\|_{L^\infty([0, T_2], L^2(\Omega))} + \|u\|_{L^2([0, T_2], H^1(\Omega))} + \|(\partial_t u, \nabla^2 u)\|_{L^{\frac{3}{2}}(Q_{T_2})} + \|\nabla P\|_{L^{\frac{3}{2}}(Q_{T_2})} \\ & \leq C(\|u_0\|_{L^2(\Omega)}, \|(n_0^+, n_0^-)\|_{L^2(\Omega)}, T_2). \end{aligned}$$

From the estimates (2.24) and (2.26), we can extend (u, P, n^+, n^-, Ψ) beyond T_2 to get a global weak solution of (2.1)–(2.3) on the interval $[0, T]$ such that both (2.24) and (2.26) hold with T_2 replaced by T . Finally, we know that by Lemma 2.1, (u, P, n^+, n^-, Ψ) is also a weak solution of the system (2.1) in Q_T .

Since the solution (u, P, n^+, n^-, Ψ) to (2.1) constructed in Step 1 satisfies the estimates (2.24) and (2.26) (with $T_2 = T$), it is not hard to verify that the L^p -theory of linear parabolic equations [12] implies that $\partial_t n^+, \partial_t n^- \in L^{\frac{5}{4}}(Q_T)$. From

$$(2.27) \quad -\Delta(\partial_t \Psi) = \partial_t n^+ - \partial_t n^- \text{ in } \Omega, \quad \frac{\partial}{\partial \nu}(\partial_t \Psi) = 0 \text{ on } \partial \Omega,$$

we can conclude by the L^p -theory of linear elliptic equations that $\nabla^2 \partial_t \Psi \in L^{\frac{5}{4}}(Q_T)$.

Multiplying (2.27) by Ψ , (2.1)₁ by u , integrating over Ω , and applying integration by parts, and adding these two resulting equations together, we can obtain that

$$(2.28) \quad \begin{aligned} & \int_{\Omega} (|u|^2 + |\nabla \Psi|^2)(x, t) \, dx + 2 \int_{Q_t} (|\nabla u|^2 + |n^+ - n^-|^2 + (n^+ + n^-)|\nabla \Psi|^2) \, dx ds \\ &= \int_{\Omega} (|u_0|^2 + |\nabla \Psi_0|^2)(x) \, dx + 2 \int_{Q_t} (n^+ - n^-)(w - u) \cdot \nabla \Psi \, dx ds \end{aligned}$$

holds for all $0 < t \leq T$.

Step 2: Uniqueness. We want to prove that there exists at most one weak solution of (2.1)–(2.3) satisfying the estimates (2.24) and (2.26). Let $(u_1, P_1, \Psi_1, n_1^+, n_1^-)$ and $(u_2, P_2, \Psi_2, n_2^+, n_2^-)$ be two weak solutions of (2.1), (2.2), and (2.3), satisfying (2.24) and (2.26). Set

$$U = u_1 - u_2, \quad P = P_1 - P_2, \quad \Psi = \Psi_1 - \Psi_2, \quad N^+ = n_1^+ - n_2^+, \quad N^- = n_1^- - n_2^-.$$

Then

$$(2.29) \quad \begin{cases} \partial_t U + (w \cdot \nabla)U - \Delta U + \nabla P = -(N^+ - N^-)\nabla \Psi_1 - (n_2^+ - n_2^-)\nabla \Psi, \\ \operatorname{div} U = 0, \\ \partial_t N^+ + (w \cdot \nabla)N^+ - \Delta N^+ = \operatorname{div} (N^+ \nabla \Psi_1) + \operatorname{div} (n_2^+ \nabla \Psi), \\ \partial_t N^- + (w \cdot \nabla)N^- - \Delta N^- = -\operatorname{div} (N^- \nabla \Psi_1) - \operatorname{div} (n_2^- \nabla \Psi), \\ -\Delta \Psi = N^+ - N^-, \end{cases}$$

subject to the initial and boundary conditions

$$(2.30) \quad \begin{cases} (U, N^+, N^-)|_{t=0} = (0, 0, 0) \text{ on } \Omega, \\ U = 0, \frac{\partial N^+}{\partial \nu} = \frac{\partial N^-}{\partial \nu} = \frac{\partial \Psi}{\partial \nu} = 0 \text{ on } \partial \Omega \times (0, T). \end{cases}$$

Multiplying (2.29)₁ by U , (2.29)₃ by N^+ , (2.29)₄ by N^- , and (2.29)₅ by Ψ , integrating the resulting equations over Ω , and adding all these equations together, we obtain that

$$(2.31) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} (|U|^2 + |\nabla \Psi|^2 + |N^+|^2 + |N^-|^2) \, dx \\ &+ \int_{\Omega} (|\nabla U|^2 + |N^+ - N^-|^2 + |\nabla N^+|^2 + |\nabla N^-|^2 + (n_2^+ + n_2^-)|\nabla \Psi|^2) \, dx \\ &= - \int_{\Omega} \left[(N^+ - N^-)U \cdot \nabla \Psi_1 + (n_2^+ - n_2^-)U \cdot \nabla \Psi - (N^+ - N^-)w \cdot \nabla \Psi \right. \\ &\quad \left. + (N^+ - N^-)\nabla \Psi_1 \cdot \nabla \Psi + N^+ \nabla \Psi_1 \cdot \nabla N^+ + n_2^+ \nabla \Psi \cdot \nabla N^+ - n_1^- \nabla \Psi \cdot \nabla N^- \right] \, dx \\ &\leq \frac{1}{2} \|N^+ - N^-\|_{L^2(\Omega)}^2 + C \|U\|_{L^3(\Omega)}^2 \|\nabla \Psi_1\|_{L^6(\Omega)}^2 + C \|w\|_{L^\infty(Q)}^2 \|\nabla \Psi\|_{L^2(\Omega)}^2 \\ &\quad + \|(n_2^+, n_2^-)\|_{L^6(\Omega)}^2 \|U\|_{L^2(\Omega)}^2 + C(1 + \|\nabla \Psi_1\|_{L^6(\Omega)}^2) \|\nabla \Psi\|_{L^3(\Omega)}^2 \\ &\quad + C \|\nabla \Psi_1\|_{L^6(\Omega)}^2 \|N^+\|_{L^3(\Omega)}^2 + \|(n_1^-, n_2^+)\|_{L^6(\Omega)}^2 \|\nabla \Psi\|_{L^3(\Omega)}^2 \\ &\quad + \frac{1}{2} (\|\nabla N^+\|_{L^2(\Omega)}^2 + \|\nabla N^-\|_{L^2(\Omega)}^2). \end{aligned}$$

By the interpolation inequality, Sobolev's embedding theorem, and (2.24), we have

$$\begin{aligned}
\|U\|_{L^3(\Omega)}^2 &\leq C\|U\|_{L^2(\Omega)}\|\nabla U\|_{L^2(\Omega)}, \\
\|\nabla\Psi_1(t)\|_{L^6(\Omega)} &\leq C\|\Psi_1(t)\|_{H^2(\Omega)} \leq C, \text{ a.e. } t \in [0, T], \\
\|(n_2^+, n_2^-)\|_{L^6(\Omega)} + \|n_1^-\|_{L^6(\Omega)} &\leq C \sum_{i=1}^2 \|(n_i^+, n_i^-)\|_{H^1(\Omega)}, \\
\|N^+\|_{L^3(\Omega)}^2 &\leq C\|N^+\|_{L^2(\Omega)}^2 + C\|N^+\|_{L^2(\Omega)}\|\nabla N^+\|_{L^2(\Omega)}, \\
\|\nabla\Psi\|_{L^3(\Omega)}^2 &\leq \|\nabla\Psi\|_{L^2(\Omega)}\|\nabla\Psi\|_{L^6(\Omega)} \\
&\leq C\|\nabla\Psi\|_{L^2(\Omega)}\|\Psi\|_{H^2(\Omega)} \\
&\leq C\|\nabla\Psi\|_{L^2(\Omega)}\|N^+ - N^-\|_{L^2(\Omega)}.
\end{aligned}$$

Putting these estimates into (2.31) and applying Young's inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt} \int_{\Omega} (|U|^2 + |\nabla\Psi|^2 + |N^+|^2 + |N^-|^2) dx \\
&\quad + \int_{\Omega} (|\nabla U|^2 + |N^+ - N^-|^2 + |\nabla N^+|^2 + |\nabla N^-|^2 + (n_2^+ + n_2^-)|\nabla\Psi|^2) dx \\
(2.32) \quad &\leq C \left(1 + \|w\|_{L^\infty(Q)}^2 + \|(n_1^-, n_2^+)\|_{H^1(\Omega)}^2 \right) \\
&\quad \cdot \left(\|U\|_{L^2(\Omega)}^2 + \|N^+\|_{L^2(\Omega)}^2 + \|\nabla\Psi\|_{L^2(\Omega)}^2 \right) \\
&\quad + \frac{1}{2} \left(\|\nabla N^+\|_{L^2(\Omega)}^2 + \|\nabla U\|_{L^2(\Omega)}^2 + \|N^+ - N^-\|_{L^2(\Omega)}^2 \right).
\end{aligned}$$

This, combined with

$$\gamma(T) = \exp \left(C \int_0^T (1 + \|w\|_{L^\infty(Q)}^2 + \|(n_1^-, n_2^+)\|_{H^1(\Omega)}^2) dt \right) < \infty,$$

implies that for any $0 < t < T$,

$$\begin{aligned}
&\int_{\Omega} (|U|^2 + |\nabla\Psi|^2 + |N^+|^2 + |N^-|^2)(x, t) dx \\
&\leq \gamma(T) \int_{\Omega} (|U|^2 + |\nabla\Psi|^2 + |N^+|^2 + |N^-|^2)(x, 0) dx = 0.
\end{aligned}$$

Hence the proof is complete. $\square$

Next we want to provide a global $L^{\frac{5}{3}}$ -estimate of the pressure function P of the weak solution (u, P, n^+, n^-, Ψ) of the system (2.1). More precisely, we have the following.

THEOREM 2.2. *Assume $n_0^+, n_0^- \in L^2(\Omega)$ are nonnegative, $u_0 \in \mathbf{H}$, and $w \in C^\infty(Q_T)$ satisfies $\operatorname{div} w = 0$ in Q_T and $w \cdot \nu = 0$ on $\partial\Omega \times [0, T]$. Let (u, P, n^+, n^-, Ψ) with $\int_{\Omega} P dx = 0$, be the weak solution of the system (2.1) in Q_T that satisfies (2.4). Then $P \in L^{\frac{5}{3}}(Q_T)$ and*

$$(2.33) \quad \|P\|_{L^{\frac{5}{3}}(Q_T)} \leq C \left(1 + \|w\|_{L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T)} + \|n_0^+\|_{L^2(\Omega)} + \|n_0^-\|_{L^2(\Omega)} + \|u_0\|_{L^2(\Omega)} \right).$$

Furthermore, for every nonnegative $\phi \in C_0^\infty(Q_T)$, it holds that

$$(2.34) \quad \begin{aligned} 2 \int_{Q_T} |\nabla u|^2 \phi \, dx dt &= \int_{Q_T} |u|^2 (\partial_t \phi + \Delta \phi) \, dx dt + \int_{Q_T} (|u|^2 w + 2Pu) \cdot \nabla \phi \, dx dt \\ &\quad - 2 \int_{Q_T} \left(\nabla \Psi \otimes \nabla \Psi - \frac{1}{2} |\nabla \Psi|^2 I_3 \right) : \nabla(u\phi) \, dx dt. \end{aligned}$$

Proof. Equation (2.1)_{1,2} can be written as the Stokes system:

$$\begin{cases} \partial_t u - \Delta u + \nabla P = f, \\ \operatorname{div} u = 0 \end{cases} \quad \text{in } Q_T,$$

where $f = -(w \cdot \nabla)u + \Delta \Psi \nabla \Psi$. By Hölder's inequality, we have

$$(2.35) \quad \begin{aligned} \|f\|_{L^{\frac{5}{3}}([0,T], L^{\frac{15}{14}}(\Omega))} &\leq C \left[\|w\|_{L_t^{10} L_x^{\frac{30}{13}}(Q_T)} \|\nabla u\|_{L^2(Q_T)} + \|n^+ - n^-\|_{L^\infty([0,T], L^2(\Omega))} \|\nabla \Psi\|_{L^\infty([0,T], L^6(\Omega))} \right] \\ &\leq C \left(1 + \|w\|_{L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T)} + \|n_0^+\|_{L^2(\Omega)} + \|n_0^-\|_{L^2(\Omega)} + \|u_0\|_{L^2(\Omega)} \right). \end{aligned}$$

Here we have used the Sobolev-interpolation inequality,

$$\|w\|_{L_t^{10} L_x^{\frac{30}{13}}(Q_T)} \leq C \|w\|_{L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T)}.$$

In particular, $f \in L^{\frac{5}{3}}([0,T], L^{\frac{15}{14}}(\Omega))$. Applying the theorem by Sohr and von Wahl [26] and (2.35), we obtain that $\nabla P \in L^{\frac{5}{3}}([0,T], L^{\frac{15}{14}}(\Omega))$ and

$$\begin{aligned} \|\nabla P\|_{L^{\frac{5}{3}}([0,T], L^{\frac{15}{14}}(\Omega))} &\leq C \|f\|_{L^{\frac{5}{3}}([0,T], L^{\frac{15}{14}}(\Omega))} \\ &\leq C \left(1 + \|w\|_{L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T)} + \|n_0^+\|_{L^2(\Omega)} + \|n_0^-\|_{L^2(\Omega)} + \|u_0\|_{L^2(\Omega)} \right). \end{aligned}$$

This, combined with Sobolev's inequality, implies that $P \in L^{\frac{5}{3}}(Q_T)$ satisfies (2.33).

Mollify u, P, f in $\mathbb{R}^4$ as follows. Choose $m_0 \in \mathbb{N}^+$ so that $\frac{1}{m_0} < \delta(\operatorname{supp} \phi, \partial(\Omega \times [0, T]))$, where $\delta(\cdot, \cdot)$ denotes the parabolic distance function given by (1.10). Let $\eta \in C_0^\infty(B_1 \times (-1, 0))$ be a standard mollifier function, and $\eta_m(x, t) = m^5 \eta(mx, m^2 t)$ for $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$. For $m \geq m_0$, define sequences of smooth functions $u_m = \eta_m * u$, $P_m = \eta_m * P$, $f_m = \eta_m * f$. Then for $m \geq m_0$, we have that

$$(2.36) \quad \partial_t u_m - \Delta u_m + \nabla P_m = f_m, \quad \operatorname{div} u_m = 0,$$

holds in a small neighborhood of $\operatorname{supp} \phi$. Moreover,

$$\begin{aligned} u_m &\rightarrow u \quad \text{in } L_{\operatorname{loc}}^3(Q_T), \quad \nabla u_m \rightarrow \nabla u \quad \text{in } L_{\operatorname{loc}}^2(Q_T), \\ P_m &\rightarrow P \quad \text{in } L_{\operatorname{loc}}^{\frac{5}{3}}(Q_T), \quad f_m \rightarrow f \quad \text{in } (L_t^2 L_x^{\frac{3}{2}})_{\operatorname{loc}}(Q_T). \end{aligned}$$

Multiplying (2.36) by $2u_m \phi$ and integrating by parts, we obtain that

$$\begin{aligned} 2 \int_{Q_T} |\nabla u_m|^2 \phi \, dx dt &= \int_{Q_T} |u_m|^2 (\partial_t \phi + \Delta \phi) \, dx dt + 2 \int_{Q_T} P_m u_m \cdot \nabla \phi \, dx dt \\ &\quad + 2 \int_{Q_T} f_m \cdot u_m \phi \, dx dt. \end{aligned}$$

Sending $m \rightarrow \infty$, we have

$$\begin{aligned} 2 \int_{Q_T} |\nabla u|^2 \phi \, dx dt &= \int_{Q_T} |u|^2 (\partial_t \phi + \Delta \phi) \, dx dt + 2 \int_{Q_T} Pu \cdot \nabla \phi \, dx dt \\ &\quad + 2 \int_{Q_T} (-w \cdot \nabla u + \Delta \Psi \nabla \Psi) \cdot u \phi \, dx dt. \end{aligned}$$

Note that since $\operatorname{div} w = 0$, we have that

$$-2 \int_{Q_T} w \cdot \nabla u \cdot u \phi \, dx dt = \int_{Q_T} |u|^2 w \cdot \nabla \phi \, dx dt$$

and

$$2 \int_{Q_T} \Delta \Psi \nabla \Psi \cdot u \phi \, dx dt = -2 \int_{Q_T} \left(\nabla \Psi \otimes \nabla \Psi - \frac{1}{2} |\nabla \Psi|^2 I_3 \right) : \nabla (u \phi) \, dx dt.$$

Thus (2.34) holds. This completes the proof. $\square$

Now recall Aubin–Lions’ compactness lemma; see [27, section III].

LEMMA 2.2. *Let X_0, X_1, X_2 be three Banach spaces with X_0 and X_2 self-reflexive, that satisfy $X_0 \subset X_1 \subset X_2$. Suppose that the embedding of X_0 into X_1 is compact and the embedding of X_1 into X_2 is continuous. For $p, q \in (0, \infty)$, assume that*

$$\{u_k\}_{k \in N} \subset L^p([0, T], X_0)$$

is a bounded sequence such that each u_k has a weak derivative $\partial_t u_k$ and the sequence

$$\{\partial_t u_k\}_{k \in N} \in L^q([0, T], X_2)$$

is also bounded. Then there is a subsequence of u_k converging strongly in $L^p([0, T], X_1)$.

Now we utilize Theorem 2.1 to obtain a suitable weak solution to the system (1.1). To do this, we adapt the retarded mollification technique by Caffarelli, Kohn, and Nirenberg [1] on the Navier–Stokes equation.

Let $\zeta \in C_0^\infty(\mathbb{R}^4)$ be nonnegative and satisfy

$$\int_{\mathbb{R}^4} \zeta \, dx dt = 1 \text{ and } \operatorname{supp} \zeta \subset \{(x, t) \in \mathbb{R}^4 : |x|^2 < t, 1 < t < 2\}.$$

For $f \in L^1(Q_T)$, let $\bar{f} = \mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R}^3$ be

$$\bar{f} = \begin{cases} f(x, t) & \text{if } (x, t) \in \Omega_T, \\ 0 & \text{otherwise.} \end{cases}$$

For $\epsilon > 0$, define the retarded mollifier of f by

$$(2.37) \quad \Theta_\epsilon(f)(x, t) = \epsilon^{-4} \int_{\mathbb{R}^4} \zeta\left(\frac{y}{\epsilon}, \frac{\tau}{\epsilon}\right) \bar{f}(x - y, t - \tau) dy d\tau.$$

Then it is well known (see [1, Lemma A.8]) that

$$\begin{cases} \operatorname{div} (\Theta_\epsilon(f)) = 0 \text{ if } \operatorname{div} f = 0, \\ \sup_{0 \leq t \leq T} \int_{\Omega} |\Theta_\epsilon(f)|^2(x, t) \, dx + \int_{Q_T} |\nabla (\Theta_\epsilon(f))|^2 \, dx dt \\ \leq \sup_{0 \leq t \leq T} \int_{\Omega} |f|^2(x, t) \, dx + \int_{Q_T} |\nabla f|^2 \, dx dt, \end{cases}$$

and if $f \in L^p(Q_T)$ for $1 \leq p < \infty$, then $\Theta_\epsilon(f) \rightarrow f$ in $L^p(Q_T)$ as $\epsilon \rightarrow 0$. Since $\Theta_\epsilon(f) \cdot \nu$ may not be 0 on $\partial\Omega \times [0, T]$, we want to modify it as follows. For $\delta > 0$, let $\Omega_\delta = \{y \in \mathbb{R}^3 : \text{dist}(y, \Omega) \leq \delta\}$ be the δ -neighborhood of Ω , and $\Phi_\delta : \Omega \rightarrow \Omega_\delta$ be a smooth diffeomorphism such that

$$\|\Phi_\delta - Id\|_{C^1(\Omega)} \leq C\delta,$$

where $Id(x) = x$, $x \in \Omega$, is the identity map. From the definition, we see that $\Theta_\epsilon(f) = 0$ in $(\mathbb{R}^3 \setminus \Omega_{2\epsilon}) \times [0, T]$. Hence $\tilde{\Theta}_\epsilon(f)(x, t) = \Theta_\epsilon(f(\Phi_{2\epsilon}(x), t))$, $(x, t) \in Q_T$, satisfies that $\tilde{\Theta}_\epsilon(f) = 0$ on $\partial\Omega \times [0, T]$. If $\text{div}(f) = 0$ in Q_T , then

$$\text{div } \tilde{\Theta}_\epsilon(f)(x, t) = \text{tr}[\nabla \Theta_\epsilon(f)(\Phi_{2\epsilon}(x), t)(\nabla \Phi_{2\epsilon}(x) - I_3)], \quad (x, t) \in Q_T.$$

Therefore we have that

$$\begin{aligned} & \sup_{0 \leq t \leq T} \int_{\Omega} |\tilde{\Theta}_\epsilon(f)|^2(x, t) dx + \int_{Q_T} |\nabla(\tilde{\Theta}_\epsilon(f))|^2 dx dt \\ & \leq C \left(\sup_{0 \leq t \leq T} \int_{\Omega} |f|^2(x, t) dx + \int_{Q_T} |\nabla f|^2 dx dt \right) \end{aligned}$$

and

$$\int_{Q_T} |\text{div}(\tilde{\Theta}_\epsilon(f))|^2 dx dt \leq C\epsilon^2 \int_{Q_T} |\nabla f|^2 dx dt.$$

For $0 < t < T$, let $g_\epsilon(t) \in C^\infty(\bar{\Omega})$ satisfy $\int_{\Omega} g_\epsilon(x, t) dx = 0$, and solve

$$-\Delta g_\epsilon(x, t) = \text{div}(\tilde{\Theta}_\epsilon(f))(x, t) \quad \text{in } \Omega, \quad \frac{\partial g_\epsilon}{\partial \nu}(x, t) = 0 \quad \text{on } \partial\Omega.$$

By the standard elliptic theory, we have that for any $0 < t < T$,

$$\begin{cases} \int_{\Omega} |\nabla g_\epsilon|^2(x, t) dx \leq C \int_{\Omega} |\tilde{\Theta}_\epsilon(f)|^2(x, t) dx \leq C \int_{\Omega} |f|^2(x, t) dx, \\ \int_{\Omega} |\nabla^2 g_\epsilon|^2(x, t) dx \leq C \int_{\Omega} |\text{div} \tilde{\Theta}_\epsilon(f)|^2(x, t) dx \leq C\epsilon^2 \int_{\Omega} |\nabla f|^2(x, t) dx. \end{cases}$$

Now we define $\hat{\Theta}_\epsilon(f) \in C^\infty(\bar{\Omega} \times (0, T), \mathbb{R}^3)$ by letting

$$\hat{\Theta}_\epsilon(f)(x, t) = \tilde{\Theta}_\epsilon(f)(x, t) + \nabla g_\epsilon(x, t), \quad (x, t) \in \bar{\Omega} \times [0, T].$$

Then it is easy to check that for $f \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T)$ with $\text{div}(f) = 0$ in Q_T ,

$$\begin{aligned} & \text{div}(\hat{\Theta}_\epsilon(f)) = 0 \quad \text{in } Q_T, \quad \hat{\Theta}_\epsilon(f) \cdot \nu = 0 \quad \text{on } \partial\Omega \times [0, T], \\ & \sup_{0 \leq t \leq T} \int_{\Omega} |\hat{\Theta}_\epsilon(f)|^2(x, t) dx + \int_{Q_T} |\nabla(\hat{\Theta}_\epsilon(f))|^2 dx dt \\ & \leq C \left(\sup_{0 \leq t \leq T} \int_{\Omega} |f|^2(x, t) dx + \int_{Q_T} |\nabla f|^2 dx dt \right), \end{aligned}$$

and

$$\hat{\Theta}_\epsilon(f) \rightarrow f \quad \text{in } L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T), \quad \text{as } \epsilon \rightarrow 0.$$

For any large positive integer M , set $\epsilon = \frac{T}{M}$ and let $(u_M, P_M, n_M^+, n_M^-, \Psi_M)$ solve the following system:

$$(2.38) \quad \begin{cases} \partial_t u_M + (\hat{\Theta}_\epsilon(u_M) \cdot \nabla) u_M - \Delta u_M + \nabla P_M = -(n_M^+ - n_M^-) \nabla \Psi_M, \\ \operatorname{div} u_M = 0, \\ \partial_t n_M^+ + (\hat{\Theta}_\epsilon(u_M) \cdot \nabla) n_M^+ - \Delta n_M^+ = \operatorname{div} (n_M^+ \nabla \Psi_M), \\ \partial_t n_M^- + (\hat{\Theta}_\epsilon(u_M) \cdot \nabla) n_M^- - \Delta n_M^- = -\operatorname{div} (n_M^- \nabla \Psi_M), \\ -\Delta \Psi_M = n_M^+ - n_M^-, \end{cases} \quad \text{in } Q_T,$$

subject to the initial and boundary conditions (2.2) and (2.3).

Since $\hat{\Theta}_\epsilon(u_M) = 0$ in Q_ϵ , the system (2.38) decomposes into the PNP equation and the inhomogeneous Stokes equation, both of which can be solved in the standard ways. While in the interval $[\epsilon, 2\epsilon]$, $\hat{\Theta}_\epsilon(u_M)$ are smooth and their values depend only on the values of u_M and Ψ_M at interval $[0, \epsilon]$. Hence by Theorem 2.1, we can solve the solution $(u_M, P_M, n_M^+, n_M^-, \Psi_M)$ of (2.38) on the interval $\Omega \times [\epsilon, 2\epsilon]$ with the initial condition $(u_M, n_M^+, n_M^-)(\cdot, \epsilon)$ and the boundary condition (2.3). Keeping this process in each interval $(m\epsilon, (m+1)\epsilon)$, $0 \leq m \leq M-1$, we obtain a global solution $(u_M, P_M, n_M^+, n_M^-, \Psi_M)$ to (2.38), (2.2), and (2.3).

It follows from Lemma 2.1, Proposition 2.1 (for $p = 2$), (2.24) and (2.26) of the proof of Theorem 2.1, and (2.33) of Theorem 2.2 that $\{u_M\}$ is bounded in $L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T)$, $\{n_M^\pm\}$ are nonnegative and bounded in $L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T)$, Ψ_M is bounded in $L_t^\infty H_x^2 \cap L_t^2 H_x^3(Q_T)$, and $\{P_M\}$ is bounded in $L^{\frac{5}{3}}(Q_T)$.

By (2.38)₁, (2.38)₃, (2.38)₄, we have that

$$\begin{cases} \partial_t u_M = -\operatorname{div} (u_M \otimes \hat{\Theta}_\epsilon(u_M) - \nabla u_M + P_M I_3) - (n_M^+ - n_M^-) \nabla \Psi_M, \\ \partial_t n_M^+ = -\operatorname{div} (n_M^+ \hat{\Theta}_\epsilon(u_M) - \nabla n_M^+ + n_M^+ \nabla \Psi_M), \\ \partial_t n_M^- = -\operatorname{div} (n_M^- \hat{\Theta}_\epsilon(u_M) - \nabla n_M^- + n_M^- \nabla \Psi_M). \end{cases}$$

Hence $\{\partial_t u_M\}_{M \in \mathbb{N}}$, $\{\partial_t n_M^+\}_{M \in \mathbb{N}}$, $\{\partial_t n_M^-\}_{M \in \mathbb{N}}$ are bounded in the space

$$L^{\frac{5}{3}}([0, T], W^{-1, \frac{5}{2}}(\Omega)).$$

We can apply Lemma 2.2 with

$$\begin{cases} X_0 := H^1(\Omega), \\ X_1 := L^2(\Omega), \\ X_2 := W^{-1, \frac{5}{2}}, \end{cases}$$

to conclude that there exist $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T)$, $n^\pm \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T)$, $\Psi \in L_t^\infty H_x^2 \cap L_t^2 H_x^3(Q_T)$, and $P \in L^{\frac{5}{3}}(Q_T)$ such that as $M \rightarrow \infty$, after passing to a subsequence,

$$(2.39) \quad u_m \rightharpoonup u \text{ in } L_t^2 H_x^1(Q_T), \quad u_M \rightarrow u \text{ in } L^q(Q_T) \quad \forall 1 < q < \frac{10}{3},$$

$$(2.40) \quad \begin{cases} (n_M^+, n_M^-) \rightharpoonup (n^+, n^-) \text{ in } L_t^2 H_x^1(Q_T), \\ (n_M^+, n_M^-) \rightarrow (n^+, n^-) \text{ in } L^l(\Omega_T) \quad \forall 1 < l < \frac{10}{3}, \end{cases}$$

$$(2.41) \quad \nabla \Psi_M \rightarrow \nabla \Psi \quad \text{in } L^4(Q_T),$$

and

$$(2.42) \quad P_M \rightharpoonup P \quad \text{in } L^{\frac{5}{3}}(Q_T).$$

With (2.39), (2.40), (2.41), and (2.42), we can easily verify that (u, P, n^+, n^-, Ψ) is a weak solution of (1.1), (2.2), and (2.3).

Since $(u_M, n_M^+, n_M^-, \Psi_M)$ satisfies the global energy equality (2.28), with (u, n^+, n^-, Ψ) and w replaced by $(u_M, n_M^+, n_M^-, \Psi_M)$ and $\widehat{\Theta}_\epsilon(u_M)$, respectively, and since

$$n_M^+ \rightarrow n^+, \quad n_M^- \rightarrow n^-, \quad \widehat{\Theta}_\epsilon(u_M) \rightarrow u, \quad \nabla \Psi_M \rightarrow \nabla \Psi \quad \text{in } L^3(Q_T),$$

it is not hard to verify that as $\epsilon \rightarrow 0$,

$$2 \int_{Q_t} (n_M^+ - n_M^-)(\widehat{\Theta}_\epsilon(u_M) - u) \cdot \nabla \Psi_M \, dx ds \rightarrow 0 \quad \forall 0 < t \leq T,$$

so that for any $0 < t < T$,

$$\begin{aligned} & \int_{\Omega} (|u|^2 + |\nabla \Psi|^2) \, dx + 2 \int_{Q_t} (|\nabla u|^2 + |n^+ - n^-|^2 + (n^+ + n^-)|\nabla \Psi|^2) \, dx ds \\ & \leq \liminf_{\epsilon \rightarrow 0} \left\{ \int_{\Omega} (|u_M|^2 + |\nabla \Psi_M|^2) \, dx \right. \\ & \quad \left. + 2 \int_{Q_t} (|\nabla u_M|^2 + |n_M^+ - n_M^-|^2 + (n_M^+ + n_M^-)|\nabla \Psi_M|^2) \, dx ds \right\} \\ & = \liminf_{\epsilon \rightarrow 0} \left(\int_{\Omega} (|u_0|^2 + |\nabla \Psi_0|^2) \, dx + 2 \int_{Q_t} (n_M^+ - n_M^-)(\widehat{\Theta}_\epsilon(u_M) - u_M) \cdot \nabla \Psi_M \, dx ds \right) \\ & = \int_{\Omega} (|u_0|^2 + |\nabla \Psi_0|^2) \, dx; \end{aligned}$$

this yields that (u, n^+, n^-, Ψ) satisfies the global energy inequality (1.9).

Finally, we need to verify that (u, P, n^+, n^-, Ψ) satisfies the local energy inequality (1.7). For this, consider a test function $\phi \in C^\infty(\overline{Q_T})$ with $\phi \geq 0$ and $\text{supp } \phi \Subset Q_T$. By Theorem 2.2, we have

$$\begin{aligned} (2.43) \quad 2 \int_{Q_T} |\nabla u_M|^2 \phi \, dx dt &= \int_{Q_T} |u_M|^2 (\partial_t \phi + \Delta \phi) \, dx dt \\ &+ \int_{Q_T} (|u_M|^2 \widehat{\Theta}_\epsilon(u_M) + 2P_M u_M) \cdot \nabla \phi \, dx dt \\ &- 2 \int_{Q_T} \left(\nabla \Psi_M \otimes \nabla \Psi_M - \frac{1}{2} |\nabla \Psi_M|^2 I_3 \right) : \nabla (u_M \phi) \, dx dt. \end{aligned}$$

As $M \rightarrow \infty$, by the lower semicontinuity we have that

$$2 \int_{Q_T} |\nabla u|^2 \phi \, dx dt \leq \liminf_{M \rightarrow \infty} \int_{Q_T} |\nabla u_M|^2 \phi \, dx dt,$$

while by (2.39)–(2.42) and $\widehat{\Theta}_\epsilon(u_M) \rightarrow u$ in $L^3(Q_T)$ as $\epsilon \rightarrow 0$, we have

$$\begin{aligned} & \int_{Q_T} |u_M|^2 (\partial_t \phi + \Delta \phi) \, dx dt + \int_{Q_T} (|u_M|^2 \widehat{\Theta}_\epsilon(u_M) + 2P_M u_M) \cdot \nabla \phi \, dx dt \\ & - 2 \int_{Q_T} \left(\nabla \Psi_M \otimes \nabla \Psi_M - \frac{1}{2} |\nabla \Psi_M|^2 I_3 \right) : \nabla (u_M \phi) \, dx dt \end{aligned}$$

$$\begin{aligned} & \rightarrow \int_{Q_T} |u|^2 (\partial_t \phi + \Delta \phi) dxdt + \int_{Q_T} (|u|^2 u + 2Pu) \cdot \nabla \phi dxdt \\ & - 2 \int_{Q_T} \left(\nabla \Psi \otimes \nabla \Psi - \frac{1}{2} |\nabla \Psi|^2 I_3 \right) : \nabla (u\phi) dxdt. \end{aligned}$$

Hence (1.7) follows.

3. The ϵ -regularity, part I. In this section, we will prove the partial regularity of suitable weak solutions to (1.1). The crucial steps are the two levels of ϵ -regularities.

For $(x, t) \in Q_T$ and $r > 0$, set

$$B_r(x) = \{y \in \mathbb{R}^3 : |y - x| < r\}, \quad Q_r(x, t) = \{(y, \tau) \mid |y - x| < r, t - r^2 < \tau < t\},$$

and denote $B_r(0)$ and $Q_r(0, 0)$ by B_r and Q_r .

LEMMA 3.1. *There exist $\epsilon_0 > 0$ and $\theta_0 \in (0, \frac{1}{2})$ such that if (u, P, n^+, n^-, Ψ) is a suitable weak solution of the system (1.1) in Q_T , which satisfies, for an $(x_0, t_0) \in Q_T$ and $0 < r_0 < \min\{\text{dist}(x_0, \partial\Omega), \sqrt{t_0}\}$,*

$$\begin{aligned} (3.1) \quad & r_0^{-2} \int_{Q_{r_0}(x_0, t_0)} |u|^3 dxdt + \left(r_0^{-1} \int_{Q_{r_0}(x_0, t_0)} |\nabla \Psi|^4 dxdt \right)^{\frac{3}{4}} \\ & + \left(r_0^{-2} \int_{Q_{r_0}(x_0, t_0)} |P|^{\frac{3}{2}} dxdt \right)^2 < \epsilon_0^3, \end{aligned}$$

then

$$\begin{aligned} (3.2) \quad & (\theta_0 r_0)^{-2} \int_{Q_{\theta_0 r_0}(x_0, t_0)} |u|^3 dxdt + \left((\theta_0 r_0)^{-2} \int_{Q_{\theta_0 r_0}(x_0, t_0)} |P|^{\frac{3}{2}} dxdt \right)^2 \\ & \leq \frac{1}{2} \left[r_0^{-2} \int_{Q_{r_0}(x_0, t_0)} |u|^3 dxdt + \left(r_0^{-1} \int_{Q_{r_0}(x_0, t_0)} |\nabla \Psi|^4 dxdt \right)^{\frac{3}{4}} \right. \\ & \quad \left. + \left(r_0^{-2} \int_{Q_{r_0}(x_0, t_0)} |P|^{\frac{3}{2}} dxdt \right)^2 \right]. \end{aligned}$$

Proof. For $z_0 = (x_0, t_0) \in Q_T$ and $r_0 > 0$, define the rescaling functions

$$(\tilde{u}, \tilde{P}, \tilde{n}^+, \tilde{n}^-, \tilde{\Psi})(x, t) = (r_0 u, r_0^2 P, n^+, n^-, \Psi)(x_0 + r_0 x, t_0 + r_0^2 t).$$

Then $(\tilde{u}, \tilde{P}, \tilde{n}^+, \tilde{n}^-, \tilde{\Psi})$ solves the following system:

$$(3.3) \quad \begin{cases} \partial_t \tilde{u} + (\tilde{u} \cdot \nabla) \tilde{u} - \Delta \tilde{u} + \nabla \tilde{P} = -r_0^2 (\tilde{n}^+ - \tilde{n}^-) \nabla \tilde{\Psi}, \\ \text{div } \tilde{u} = 0, \\ \partial_t \tilde{n}^+ + (\tilde{u} \cdot \nabla) \tilde{n}^+ - \Delta \tilde{n}^+ = \text{div } (\tilde{n}^+ \nabla \tilde{\Psi}), \\ \partial_t \tilde{n}^- + (\tilde{u} \cdot \nabla) \tilde{n}^- - \Delta \tilde{n}^- = -\text{div } (\tilde{n}^- \nabla \tilde{\Psi}), \\ -\Delta \tilde{\Psi} = r_0^2 (\tilde{n}^+ - \tilde{n}^-). \end{cases}$$

From (3.3)₅, we can see that

$$-r_0^2 (\tilde{n}^+ - \tilde{n}^-) \nabla \tilde{\Psi} = \Delta \tilde{\Psi} \cdot \nabla \tilde{\Psi} = \text{div} \left(\nabla \tilde{\Psi} \otimes \nabla \tilde{\Psi} - \frac{1}{2} |\nabla \tilde{\Psi}|^2 I_3 \right).$$

Thus (3.3)₁ can be rewritten as

$$(3.4) \quad \partial_t \tilde{u} + (\tilde{u} \cdot \nabla) \tilde{u} - \Delta \tilde{u} + \nabla \tilde{P} = \operatorname{div} \left(\nabla \tilde{\Psi} \otimes \nabla \tilde{\Psi} - \frac{1}{2} |\nabla \tilde{\Psi}|^2 I_3 \right).$$

Because of the invariance of the first four equations of (3.3) under translations and scalings, we will assume $z_0 = (0, 0)$ and $r_0 = 1$. We prove (3.2) by contradiction. Suppose the conclusion were false. Then for any $\theta \in (0, \frac{1}{2})$, there would exist a sequence of suitable weak solutions $(u_i, P_i, n_i^+, n_i^-, \Psi_i)$ of (1.1) in Q_1 such that

$$(3.5) \quad \int_{Q_1} |u_i|^3 dxdt + \left(\int_{Q_1} |\nabla \Psi_i|^4 dxdt \right)^{\frac{3}{4}} + \left(\int_{Q_1} |P_i|^{\frac{3}{2}} dxdt \right)^2 = \epsilon_i^3 \rightarrow 0$$

and

$$(3.6) \quad \begin{aligned} & \theta^{-2} \int_{Q_\theta} |u_i|^3 dxdt + \left(\theta^{-2} \int_{Q_\theta} |P_i|^{\frac{3}{2}} dxdt \right)^2 \\ & > \frac{1}{2} \left[\int_{Q_1} |u_i|^3 dxdt + \left(\int_{Q_1} |\nabla \Psi_i|^4 dxdt \right)^{\frac{3}{4}} + \left(\int_{Q_1} |P_i|^{\frac{3}{2}} dxdt \right)^2 \right]. \end{aligned}$$

Now we define the blowing up sequences $v_i = \frac{u_i}{\epsilon_i}$, $R_i = \frac{P_i}{\epsilon_i}$, $\Phi_i = \frac{\Psi_i}{\epsilon_i}$ on Q_1 . Then (v_i, R_i) solves the system

$$(3.7) \quad \begin{cases} \partial_t v_i + \epsilon_i v_i \cdot \nabla v_i - \Delta v_i + \nabla R_i = \epsilon_i \operatorname{div} \left(\nabla \Phi_i \otimes \nabla \Phi_i - \frac{1}{2} |\nabla \Phi_i|^2 I_3 \right), \\ \operatorname{div} v_i = 0, \end{cases}$$

and satisfies

$$(3.8) \quad \int_{Q_1} |v_i|^3 dxdt + \left(\int_{Q_1} |\nabla \Phi_i|^4 dxdt \right)^{\frac{3}{4}} + \left(\int_{Q_1} |R_i|^{\frac{3}{2}} dxdt \right)^2 = 1,$$

$$(3.9) \quad \theta^{-2} \int_{Q_\theta} |v_i|^3 dxdt + \left(\theta^{-2} \int_{Q_\theta} |R_i|^{\frac{3}{2}} dxdt \right)^2 > \frac{1}{2}.$$

Moreover, since (u_i, P_i, Ψ_i) satisfies the local energy inequality (1.7), we can see that (v_i, R_i, Φ_i) satisfies a rescaled version of (1.7): for any $0 \leq \phi \in C_0^\infty(Q_1)$,

$$(3.10) \quad \begin{aligned} & 2 \int_{Q_1} |\nabla v_i|^2 \phi dxdt \\ & \leq \int_{Q_1} |v_i|^2 (\phi_t + \Delta \phi) dxdt + \int_{Q_1} (\epsilon_i |v_i|^2 + 2R_i) v_i \cdot \nabla \phi dxdt \\ & \quad - 2 \int_{Q_1} \epsilon_i \left(\nabla \Phi_i \otimes \nabla \Phi_i - \frac{1}{2} |\nabla \Phi_i|^2 I_3 \right) : \nabla (v_i \phi) dxdt \\ & \leq \int_{Q_1} |v_i|^2 (\phi_t + \Delta \phi) dxdt + \int_{Q_1} (\epsilon_i |v_i|^2 + 2R_i) v_i \cdot \nabla \phi dxdt \\ & \quad + C \epsilon_i \int_{Q_1} |\nabla \Phi_i|^2 (\phi + |v_i| |\nabla \phi|) dxdt + \int_{Q_1} |\nabla v_i|^2 \phi dxdt. \end{aligned}$$

By choosing suitable test functions ϕ , (3.10) and (3.8) imply that $v_i \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_{\frac{1}{2}})$ and there exists $C > 0$ such that

$$(3.11) \quad \sup_{i \geq 1} \|v_i\|_{L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_{\frac{1}{2}})} \leq C.$$

Moreover, we see from (3.7) that

$$(3.12) \quad \|\partial_t v_i\|_{L^{\frac{3}{2}}([-1,0], W^{-1, \frac{3}{2}}(B_1))} \leq C.$$

Indeed, for $\phi \in L^3([-1,0], W_0^{1,3}(B_1))$, we have

$$\begin{aligned} & \left| \int_{Q_1} \partial_t v_i \phi \, dxdt \right| \\ &= \left| \int_{Q_1} [(\epsilon_i v_i \otimes v_i - \nabla v_i) : \nabla \phi + R_i \operatorname{div} \phi - \epsilon_i \left(\nabla \Phi_i \otimes \nabla \Phi_i - \frac{1}{2} |\nabla \Phi_i|^2 I_3 \right) : \nabla \phi] \, dxdt \right| \\ &\leq C(\|v_i\|_{L^3(Q_1)}^2 + \|\nabla \Phi_i\|_{L^3(Q_1)}^2 + \|R_i\|_{L^{\frac{3}{2}}(Q_1)}) \|\nabla \phi\|_{L^3(Q_1)} \\ &\leq C\|\phi\|_{L^3([-1,0], W_0^{1,3}(B_1))}. \end{aligned}$$

From (3.11) and (3.12), we can apply Lemma 2.2 to conclude that after passing to a subsequence, there exist $v \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_{\frac{1}{2}})$, $R \in L^{\frac{5}{3}}(Q_{\frac{1}{2}})$, and $\Phi \in L_t^4 W_x^{1,4}(Q_{\frac{1}{2}})$ such that

$$(3.13) \quad v_i \rightharpoonup v \text{ in } L_t^2 H_x^1(Q_{\frac{1}{2}}), \quad v_i \rightarrow v \text{ in } L^3(Q_{\frac{1}{2}}),$$

and

$$(3.14) \quad R_i \rightharpoonup R \text{ in } L^{\frac{5}{3}}(Q_{\frac{1}{2}}), \quad \Phi_i \rightharpoonup \Phi \text{ in } L_t^4 W_x^{1,4}(Q_{\frac{1}{2}}).$$

Passing to the limit in (3.7), we see that (v, R) solves the Stokes equation:

$$(3.15) \quad \partial_t v - \Delta v + \nabla R = 0, \quad \operatorname{div} v = 0 \quad \text{in } Q_{\frac{1}{2}}.$$

Therefore by the standard theory on the Stokes equation, we conclude that $v \in C^\infty(Q_{\frac{1}{2}})$, and for any $\theta \in (0, \frac{1}{2})$,

$$(3.16) \quad \theta^{-2} \int_{Q_\theta} |v|^3 \, dxdt \leq C\theta^3 \int_{Q_{\frac{1}{2}}} |v|^3 \, dxdt \leq C\theta^3.$$

This and (3.13) imply that for i sufficiently large,

$$(3.17) \quad \theta^{-2} \int_{Q_\theta} |v_i|^3 \, dxdt \leq C\theta^3 + o(1).$$

Here $o(1)$ denotes a quantity such that $\lim_i o(1) = 0$.

As for the pressure function R_i , taking the divergence of (3.7)₁ yields that R_i solves the Poisson equation:

$$(3.18) \quad \Delta R_i = \epsilon_i \operatorname{div}^2 \left(\nabla \Phi_i \otimes \nabla \Phi_i - \frac{1}{2} |\nabla \Phi_i|^2 I_3 - v_i \otimes v_i \right) \quad \text{in } B_{\frac{1}{2}}.$$

By the Calderon–Zygmund theory, we can show that

$$\begin{aligned} (3.19) \quad \theta^{-2} \int_{Q_\theta} |R_i|^{\frac{3}{2}} \, dxdt &\leq C\theta^{-2} \epsilon_i^{\frac{3}{2}} \int_{Q_1} (|v_i|^3 + |\nabla \Phi_i|^3) \, dxdt + C\theta^3 \int_{Q_1} |R_i|^{\frac{3}{2}} \, dxdt \\ &\leq C\theta^{-2} \epsilon_i^{\frac{3}{2}} + C\theta^3. \end{aligned}$$

Adding (3.17) and (3.19) together, we obtain that

$$(3.20) \quad \theta^{-2} \int_{Q_\theta} |v_i|^3 dxdt + \left(\theta^{-2} \int_{Q_\theta} |R_i|^{\frac{3}{2}} dxdt \right)^2 \leq C\theta^3 + C\theta^{-2}\epsilon_i^{\frac{3}{2}} + o(1) \leq \frac{1}{4},$$

provided we choose a sufficiently small $\theta \in (0, \frac{1}{2})$ and a sufficiently large i . It is clear that (3.20) contradicts (3.9). The proof is complete. $\square$

Keep iterating Lemma 3.1; we then obtain the following decay property.

COROLLARY 3.1. *There exist $\epsilon_0 > 0$ and $\theta_0 \in (0, \frac{1}{2})$ such that if (u, P, n^+, n^-, Ψ) is a suitable weak solution of the system (1.1) in Q_T , which satisfies, for a $z_0 = (x_0, t_0) \in Q_T$, $0 < r_0 < \min\{\text{dist}(x_0, \partial\Omega), \sqrt{t_0}\}$, and $0 < \alpha < 4$*

$$(3.21) \quad \max \left\{ r_0^{-2} \int_{Q_{r_0}(z_0)} |u|^3 dxdt + \left(r_0^{-2} \int_{Q_{r_0}(z_0)} |P|^{\frac{3}{2}} dxdt \right)^2, \right. \\ \left. \sup_{0 < r \leq r_0} \left(r^{-(1+\alpha)} \int_{Q_r(z_0)} |\nabla \Psi|^4 dxdt \right)^{\frac{3}{4}} \right\} < \frac{1}{2} \epsilon_0^3,$$

then for any positive integer $k \in \mathbb{N}^+$,

$$(3.22) \quad (\theta_0^k r_0)^{-2} \int_{Q_{\theta_0^k r_0}(z_0)} |u|^3 dxdt + \left((\theta_0^k r_0)^{-2} \int_{Q_{\theta_0^k r_0}(z_0)} |P|^{\frac{3}{2}} dxdt \right)^2 \leq C\epsilon_0^3 \left(\frac{1}{2} \right)^k.$$

Proof. It is readily seen that (3.22) follows from Lemma 3.1 for $k = 1$. Note that (3.21) and (3.22) for $k = 1$ yield that

$$(\theta_0 r_0)^{-2} \int_{Q_{\theta_0 r_0}(z_0)} |u|^3 dxdt + \left((\theta_0 r_0)^{-1} \int_{Q_{\theta_0 r_0}(z_0)} |\nabla \Psi|^4 dxdt \right)^{\frac{3}{4}} \\ + \left((\theta_0 r_0)^{-2} \int_{Q_{\theta_0 r_0}(z_0)} |P|^{\frac{3}{2}} dxdt \right)^2 < \epsilon_0^3.$$

Hence applying Lemma 3.1, we obtain that

$$(\theta_0^2 r_0)^{-2} \int_{Q_{\theta_0^2 r_0}(z_0)} |u|^3 dxdt + \left((\theta_0^2 r_0)^{-2} \int_{Q_{\theta_0^2 r_0}(z_0)} |P|^{\frac{3}{2}} dxdt \right)^2 \\ \leq \frac{1}{2} \left[(\theta_0 r_0)^{-2} \int_{Q_{\theta_0 r_0}(z_0)} |u|^3 dxdt + \left((\theta_0 r_0)^{-1} \int_{Q_{\theta_0 r_0}(z_0)} |\nabla \Psi|^4 dxdt \right)^{\frac{3}{4}} \right. \\ \left. + \left((\theta_0 r_0)^{-2} \int_{Q_{\theta_0 r_0}(z_0)} |P|^{\frac{3}{2}} dxdt \right)^2 \right] \\ \leq \frac{1}{2} \left[(\theta_0 r_0)^{-2} \int_{Q_{\theta_0 r_0}(z_0)} |u|^3 dxdt + \left((\theta_0 r_0)^{-2} \int_{Q_{\theta_0 r_0}(z_0)} |P|^{\frac{3}{2}} dxdt \right)^2 \right]$$

$$\begin{aligned}
& + \left((\theta_0 r_0)^{-1} \int_{Q_{\theta_0 r_0}(z_0)} |\nabla \Psi|^4 dx dt \right)^{\frac{3}{4}} \Big] \\
& \leq \frac{1}{2} \left[\frac{1}{2} \left(r_0^{-2} \int_{Q_{r_0}(z_0)} |u|^3 dx dt + \left(r_0^{-1} \int_{Q_{r_0}(z_0)} |\nabla \Psi|^4 dx dt \right)^{\frac{3}{4}} \right. \right. \\
& \quad \left. \left. + \left(r_0^{-2} \int_{Q_{r_0}(z_0)} |P|^{\frac{3}{2}} dx dt \right)^2 \right) + \left((\theta_0 r_0)^{-1} \int_{Q_{\theta_0 r_0}(z_0)} |\nabla \Psi|^4 dx dt \right)^{\frac{3}{4}} \right] \\
& \leq \left(\frac{1}{2} \right)^2 \left[r_0^{-2} \int_{Q_{r_0}(z_0)} |u|^3 dx dt + \left(r_0^{-2} \int_{Q_{r_0}(z_0)} |P|^{\frac{3}{2}} dx dt \right)^2 \right] \\
& \quad + \left(\frac{1}{2} \right)^2 \left(r_0^{-1} \int_{Q_{r_0}(z_0)} |\nabla \Psi|^4 dx dt \right)^{\frac{3}{4}} + \frac{1}{2} \left((\theta_0 r_0)^{-1} \int_{Q_{\theta_0 r_0}(z_0)} |\nabla \Psi|^4 dx dt \right)^{\frac{3}{4}} \\
& \leq \left(\frac{1}{2} \right)^2 \left[r_0^{-2} \int_{Q_{r_0}(z_0)} |u|^3 dx dt + \left(r_0^{-2} \int_{Q_{r_0}(z_0)} |P|^{\frac{3}{2}} dx dt \right)^2 \right] \\
& \quad + \left(\frac{1}{2} \right)^2 \epsilon_0^3 r_0^\alpha \left[\theta_0^\alpha + \frac{1}{2} \right].
\end{aligned}$$

Hence we have that for $k \geq 1$,

$$\begin{aligned}
& (\theta_0^k r_0)^{-2} \int_{Q_{\theta_0^k r_0}(z_0)} |u|^3 dx dt + \left((\theta_0^k r_0)^{-1} \int_{Q_{\theta_0^k r_0}(z_0)} |\nabla \Psi|^4 dx dt \right)^2 \\
& \leq \left(\frac{1}{2} \right)^k \left[r_0^{-2} \int_{Q_{r_0}(z_0)} |u|^3 dx dt + \left(r_0^{-2} \int_{Q_{r_0}(z_0)} |P|^{\frac{3}{2}} dx dt \right)^2 \right] \\
& \quad + \left(\frac{1}{2} \right)^2 \epsilon_0^3 r_0^\alpha \left[\theta_0^{\alpha(k-1)} + \frac{1}{2} \theta_0^{\alpha(k-2)} + \cdots + \left(\frac{1}{2} \right)^{k-2} \theta_0^\alpha \right] \\
& \leq \left(\frac{1}{2} \right)^k \left[r_0^{-2} \int_{Q_{r_0}(z_0)} |u|^3 dx dt + \left(r_0^{-2} \int_{Q_{r_0}(z_0)} |P|^{\frac{3}{2}} dx dt \right)^2 \right] + 2^{-(k-1)} (\theta_0 r_0)^\alpha \epsilon_0^3 \\
& \leq C \epsilon_0^3 2^{-k}.
\end{aligned}$$

This yields (3.22) and completes the proof. $\square$

With (3.22), we can now prove the following ϵ_0 -regularity property.

THEOREM 3.1. *There exists $\epsilon_0 > 0$ such that for any $0 < T \leq \infty$, $u_0 \in \mathbf{H}$, and $0 \leq n_0^\pm \in L^2(\Omega)$ with $\int_\Omega n_0^+ dx = \int_\Omega n_0^- dx$, if (u, P, n^+, n^-, Ψ) is the suitable weak solution obtained by Theorem 3.1(i), which satisfies*

$$(3.23) \quad r_0^{-2} \int_{Q_{r_0}(z_0)} |u|^3 dx dt + \left(r_0^{-1} \int_{Q_{r_0}(z_0)} |\nabla \Psi|^4 dx dt \right)^{\frac{3}{4}} + \left(r_0^{-2} \int_{Q_{r_0}(z_0)} |P|^{\frac{3}{2}} dx dt \right)^2 \leq \epsilon_0^3,$$

for $z_0 = (x_0, t_0) \in \Omega \times (0, \infty)$ and $0 < r_0 < \min \{ \text{dist}(x_0, \partial\Omega), \sqrt{t_0} \}$, then $(u, n^+, n^-, \Psi) \in C^\infty(Q_{\frac{r_0}{2}}(z_0))$.

Proof. It follows from (1.8) and Sobolev's embedding theorem that $\nabla\Psi \in L_t^\infty L_x^6(Q_T)$ and

$$(3.24) \quad \|\nabla\Psi\|_{L_t^\infty L_x^6(Q_T)} \leq C\|\Psi\|_{L_t^\infty H_x^2(Q_T)} \leq C(\|u_0\|_{L^2(\Omega)}, \|(n_0^+, n_0^-)\|_{L^2(\Omega)}).$$

This implies that

$$(3.25) \quad \int_{Q_r(z)} |\nabla\Psi|^4 dxdt \leq Cr^3 \|\nabla\Psi\|_{L_t^\infty L_x^6(Q_T)}^4 \leq Cr^3 \forall Q_r(z) \subset Q_T.$$

It follows from (3.25) and (3.23) that for any $\alpha \in (0, 2)$, the condition (3.21) holds on $Q_{\frac{r_0}{2}}(z_1)$ for any $z_1 \in Q_{\frac{r_0}{2}}(z_0)$, provided we may choose a smaller $r_0 > 0$, depending on ϵ_0 . Thus by Corollary (3.1), we conclude that there exists $\theta_0 \in (0, \frac{1}{2})$ such that

$$(3.26) \quad (\theta_0^k r_0)^{-2} \int_{Q_{\theta_0^k r_0}(z_1)} |u|^3 dxdt + \left((\theta_0^k r_0)^{-2} \int_{Q_{\theta_0^k r_0}(z_1)} |P|^{\frac{3}{2}} dxdt \right)^2 \leq C\epsilon_0^3 \left(\frac{1}{2}\right)^k$$

for any $z_1 \in Q_{\frac{r_0}{2}}(z_0)$. Therefore there exists $\tau_0 \in (0, 1)$ such that

$$(3.27) \quad s^{-2} \int_{Q_s(z_1)} |u|^3 dxdt + \left(s^{-2} \int_{Q_s(z_1)} |P|^{\frac{3}{2}} dxdt \right)^2 \leq Cs^{3\tau_0}$$

for all $z_1 \in Q_{\frac{r_0}{2}}(z_0)$ and $0 < s < \frac{r_0}{2}$. From (3.27), we can repeat the same argument of Lemma 3.1 and Corollary 3.1 to improve the exponent τ_0 such that (3.27) remains to be true for all $\tau_0 \in (0, 1)$.

Now we plan to apply the Riesz potential estimates between parabolic Morrey spaces to show that $u \in L^q(Q_{\frac{r_0}{2}}(z_0))$ for any $1 < q < \infty$, analogous to that by Huang and Wang [15], Hineman and Wang [16], and Huang, Lin, and Wang [17].

For any open set $U \subset \mathbb{R}^3 \times \mathbb{R}$, $1 \leq p < \infty$, and $0 \leq \lambda \leq 5$, define the Morrey space $M^{p,\lambda}(U)$ by

$$M^{p,\lambda}(U) := \left\{ f \in L_{\text{loc}}^p(U) : \|f\|_{M^{p,\lambda}(U)}^p = \sup_{z \in U, r > 0} r^{\lambda-5} \int_{Q_r(z)} |f|^p dxdt < \infty \right\}.$$

It follows from (3.25) and (3.27) that for any $\alpha \in (0, 1)$, it holds that

$$(u, \nabla\Psi) \in M^{3,3(1-\alpha)}(Q_{\frac{r_0}{2}}(z_0)), \quad P \in M^{\frac{3}{2},3(1-\alpha)}(Q_{\frac{r_0}{2}}(z_0)).$$

We now proceed with the estimation of u . Let $\eta \in C_0^\infty(\mathbb{R}^4)$ be a cut-off function of $Q_{\frac{r_0}{2}}(z_0)$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $Q_{\frac{r_0}{4}}(z_0)$, and $|\partial_t \eta| + |\nabla^2 \eta| \leq Cr_0^{-2}$. Let $v : \mathbb{R}^3 \times (0, \infty) \mapsto \mathbb{R}^3$ solve the Stokes equation:

$$(3.28) \quad \begin{cases} \partial_t v - \Delta v + \nabla P = -\text{div} \left[\eta^2 (u \otimes u + (\nabla\Psi \otimes \nabla\Psi - \frac{1}{2} |\nabla\Psi|^2 I_3)) \right] & \text{in } \mathbb{R}_+^4, \\ \text{div } v = 0 & \text{in } \mathbb{R}_+^4, \\ v(\cdot, 0) = 0 & \text{in } \mathbb{R}^3. \end{cases}$$

By using the Oseen kernel (see Leray [11]), an estimate of v can be given by

$$(3.29) \quad |v(x, t)| \leq C\mathcal{I}_1(|X|)(x, t) \quad \forall (x, t) \in \mathbb{R}^3 \times (0, \infty),$$

where

$$X = \eta^2 \left[u \otimes u + \left(\nabla \Psi \otimes \nabla \Psi - \frac{1}{2} |\nabla \Psi|^2 I_3 \right) \right],$$

and $\mathcal{I}_1$ is the Riesz potential of order 1 on $\mathbb{R}^4$ defined by

$$\mathcal{I}_1(g)(x, t) = \int_{\mathbb{R}^4} \frac{|g(y, s)|}{\delta^4((x, t), (y, s))} dy ds \quad \forall g \in L^1(\mathbb{R}^4).$$

We can verify that $X \in M^{\frac{3}{2}, 3(1-\alpha)}(\mathbb{R}^4)$ and

$$\begin{aligned} \|X\|_{M^{\frac{3}{2}, 3(1-\alpha)}(\mathbb{R}^4)} &\leq C \left[\|u\|_{M^{3, 3(1-\alpha)}(Q_{\frac{r_0}{2}}(z_0))}^2 + \|\nabla \Psi\|_{M^{3, 3(1-\alpha)}(Q_{\frac{r_0}{2}}(z_0))}^2 \right] \\ &\leq C(1 + \epsilon_0). \end{aligned}$$

Hence we conclude that $v \in M^{\frac{3(1-\alpha)}{1-2\alpha}, 3(1-\alpha)}(\mathbb{R}^4)$ and

$$(3.30) \quad \|v\|_{M^{\frac{3(1-\alpha)}{1-2\alpha}, 3(1-\alpha)}(\mathbb{R}^4)} \leq C \|X\|_{M^{\frac{3}{2}, 3(1-\alpha)}(\mathbb{R}^4)} \leq C(1 + \epsilon_0).$$

By taking $\alpha \uparrow \frac{1}{2}$, we conclude that for any $1 < q < \infty$, $v \in L^q(Q_{\frac{r_0}{2}}(z_0))$ and

$$(3.31) \quad \|v\|_{L^q(Q_{\frac{r_0}{2}}(z_0))} \leq C(q, r_0, \epsilon_0).$$

Note that $u - v$ solves the linear homogeneous Stokes equation:

$$\partial_t(u - v) - \Delta(u - v) + \nabla P = 0, \quad \operatorname{div}(u - v) = 0 \quad \text{in } Q_{\frac{r_0}{2}}(z_0).$$

Hence $u - v \in L^\infty(Q_{\frac{r_0}{4}}(z_0))$, and for any $1 < q < \infty$, $u \in L^q(Q_{\frac{r_0}{4}}(z_0))$ and

$$(3.32) \quad \|u\|_{L^q(Q_{\frac{r_0}{4}}(z_0))} \leq C(q, r_0, \epsilon_0).$$

From $\Psi \in L_t^\infty H_x^2 \cap L_t^2 H_x^3(Q_T)$ and the Sobolev inequality, we have that $\Delta \Psi \in L^{\frac{10}{3}}(Q_T)$, $\nabla \Psi \in L^q(Q_T)$ for $q > 5$, and

$$\|\Delta \Psi\|_{L^{\frac{10}{3}}(Q_T)} + \|\nabla \Psi\|_{L^q(Q_T)} \leq C \|\Psi\|_{L_t^\infty H_x^2 \cap L_t^2 H_x^3(Q_T)} \leq C.$$

Since n^+ solves

$$\partial_t n^+ - \Delta n^+ = (\Delta \Psi) n^+ - (u - \nabla \Psi) \cdot \nabla n^+ \quad \text{in } Q_{\frac{r_0}{4}}(z_0),$$

where $(u - \nabla \Psi) \in L^q(Q_T)$ and $\Delta \Psi \in L^{\frac{q}{2}}(Q_T)$ for some $q > 5$, we can apply the standard theory of linear parabolic equation [12] to conclude that there exists $\beta \in (0, 1)$ such that $n^+ \in C^\beta(Q_{\frac{r_0}{4}}(z_0))$ and

$$(3.33) \quad \|n^+\|_{C^\beta(Q_{\frac{r_0}{4}}(z_0))} \leq C(r_0, \epsilon_0).$$

Similarly, $n^- \in C^\beta(Q_{\frac{r_0}{4}}(z_0))$ and

$$(3.34) \quad \|n^-\|_{C^\beta(Q_{\frac{r_0}{4}}(z_0))} \leq C(r_0, \epsilon_0).$$

Substituting the estimates (3.33) and (3.34) into (1.1)₅ for Ψ , we conclude that $\nabla^2 \Psi \in L^\infty([t_0 - \frac{r_0^2}{64}, t_0], C^\alpha(B_{\frac{r_0}{8}}(x_0)))$ and

$$(3.35) \quad \|\nabla^2 \Psi\|_{L^\infty([t_0 - \frac{r_0^2}{64}, t_0], C^\alpha(B_{\frac{r_0}{8}}(x_0)))} \leq C(r_0, \epsilon_0).$$

Substituting (3.33), (3.34), and (3.35) into (1.1)_{1,2}, we conclude that $u \in C^\beta(Q_{\frac{r_0}{16}}(z_0))$ and

$$(3.36) \quad \|u\|_{C^\beta(Q_{\frac{r_0}{16}}(z_0))} \leq C(r_0, \epsilon_0).$$

By a bootstrap argument, we eventually show that $(u, n^+, n^-, \Psi) \in C^\infty(Q_{\frac{r_0}{32}}(z_0))$. $\square$

Remark 3.1. Similar to [22] and [1], Theorem 3.1 yields that (u, n^+, n^-, Ψ) is smooth away from a closed set Σ with $\mathcal{P}^{\frac{5}{3}}(\Sigma) = 0$.

4. The ϵ -regularity, part II. In this section, we will improve the size estimate of the singular set Σ for suitable weak solutions (u, P, n^+, n^-, Ψ) obtained by Theorem 1.1. The argument is based on the A-B-C-D lemmas, originally due to [1]. Namely, we want to establish the following theorem.

THEOREM 4.1. *Under the same assumptions as in Theorem 1.1, there exists $\epsilon_1 > 0$ such that if (u, P, n^+, n^-, Ψ) is the suitable weak solution of (1.1) given by Theorem 1.1, and satisfies*

$$(4.1) \quad \limsup_{r \rightarrow 0} \frac{1}{r} \int_{Q_r(z_0)} |\nabla u|^2 dx dt < \epsilon_1^2$$

for $z_0 = (x_0, t_0) \in Q_T$, then (u, n^+, n^-, Ψ) is smooth near z_0 .

For simplicity, we will assume $z_0 = (0, 0) \in Q_T$. In order to prove Theorem 4.1, we first recall the following interpolation inequality; see [1].

LEMMA 4.1. *For $u \in H^1(\mathbb{R}^3)$,*

$$\int_{B_r} |u|^q dx \leq C \left(\int_{B_r} |\nabla u|^2 dx \right)^{\frac{q}{2}-a} \left(\int_{B_r} |u|^2 dx \right)^a + Cr^{3(1-\frac{q}{2})} \left(\int_{B_r} |u|^2 dx \right)^{\frac{q}{2}}$$

for any $B_r \subset \mathbb{R}^3$, $2 \leq q \leq 6$, and $a = \frac{3}{2}(1 - \frac{q}{6})$.

Assume $z_0 = (0, 0)$. Set

$$A(r) = \sup_{-r^2 \leq t \leq 0} r^{-1} \int_{B_r \times \{t\}} |u|^2 dx,$$

$$B(r) = r^{-1} \int_{Q_r} |\nabla u|^2 dx dt,$$

$$C(r) = r^{-2} \int_{Q_r} |u|^3 dx dt,$$

$$D(r) = r^{-2} \int_{Q_r} |P|^{\frac{3}{2}} dx dt.$$

By Lemma 4.1, we see that for any $0 < r \leq \rho$, it holds that

$$(4.2) \quad C(r) \leq C_0 \left[\left(\frac{r}{\rho} \right)^3 A^{\frac{3}{2}}(\rho) + \left(\frac{\rho}{r} \right)^3 A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho) \right].$$

Now we need to estimate the pressure function.

LEMMA 4.2. *Let (u, P, n^+, n^-, Ψ) be a suitable weak solution of (1.1) in Q_1 given by Theorem 1.1. Then for any $0 < r \leq \frac{\rho}{2}$, we have*

$$(4.3) \quad D(r) \leq C \left[\frac{r}{\rho} D(\rho) + \left(\frac{\rho}{r} \right)^2 A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho) + \left(\frac{\rho}{r} \right)^2 \rho^{\frac{3}{2}} \right].$$

Proof. Taking the divergence of $(1.1)_1$, we obtain

$$(4.4) \quad -\Delta P = \operatorname{div}^2 \left[(u - (u)_\rho) \otimes (u - (u)_\rho) + \left(\nabla \Psi \otimes \nabla \Psi - \frac{1}{2} |\nabla \Psi|^2 I_3 \right) \right] \quad \text{in } B_\rho.$$

Here $(u)_\rho$ denotes the average of u over B_ρ .

Let $\eta \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function of $B_{\frac{\rho}{2}}$ such that

$$(4.5) \quad \begin{cases} \eta = 1 & \text{in } B_{\frac{\rho}{2}}, \\ \eta = 0 & \text{outside } B_\rho, \\ 0 \leq \eta \leq 1 & |\nabla \eta| \leq 8\rho^{-1}. \end{cases}$$

Define an auxiliary function

$$\begin{aligned} P_1(x, t) = - \int_{\mathbb{R}^3} \nabla_y^2 G(x - y) : \eta^2(y) & \left[(u - (u)_\rho) \otimes (u - (u)_\rho) \right. \\ & \left. + \left(\nabla \Psi \otimes \nabla \Psi - \frac{1}{2} |\nabla \Psi|^2 I_3 \right) \right] (y, t) dy. \end{aligned}$$

Then we have

$$-\Delta P_1 = \operatorname{div}^2 \left[(u - (u)_\rho) \otimes (u - (u)_\rho) + \left(\nabla \Psi \otimes \nabla \Psi - \frac{1}{2} |\nabla \Psi|^2 I_3 \right) \right] \quad \text{in } B_{\frac{\rho}{2}}$$

and

$$-\Delta(P - P_1) = 0 \quad \text{in } B_{\frac{\rho}{2}}.$$

For P_1 , we apply the Calderon–Zygmund theory to deduce

$$\begin{aligned} \int_{\mathbb{R}^3} |P_1|^{\frac{3}{2}} dx & \leq C \int_{\mathbb{R}^3} \eta^3 (|u - (u)_\rho|^3 + |\nabla \Psi|^3) dx \\ (4.6) \quad & \leq C \int_{B_\rho} (|u - (u)_\rho|^3 + |\nabla \Psi|^3) dx. \end{aligned}$$

Since $P - P_1$ is harmonic in $B_{\frac{\rho}{2}}$, we get that for $0 < r < \frac{\rho}{2}$,

$$\begin{aligned} \frac{1}{r^2} \int_{B_r} |P - P_1|^{\frac{3}{2}} dx & \leq C \left(\frac{r}{\rho} \right) \frac{1}{\rho^2} \int_{B_{\frac{\rho}{2}}} |P - P_1|^{\frac{3}{2}} dx \\ & \leq C \left(\frac{r}{\rho} \right) \left[\frac{1}{\rho^2} \int_{B_{\frac{\rho}{2}}} |P|^{\frac{3}{2}} dx + \frac{1}{\rho^2} \int_{B_{\frac{\rho}{2}}} |P_1|^{\frac{3}{2}} dx \right]. \end{aligned}$$

Integrating it over $[-r^2, 0]$, we can show that

$$\begin{aligned} & \frac{1}{r^2} \int_{Q_r} |P|^{\frac{3}{2}} dx dt \\ & \leq C \left(\frac{r}{\rho} \right) \frac{1}{\rho^2} \int_{Q_\rho} |P|^{\frac{3}{2}} dx dt + C \left(\frac{\rho}{r} \right)^2 \frac{1}{\rho^2} \int_{Q_\rho} (|u - (u)_\rho|^3 + |\nabla \Psi|^3) dx dt \\ & \leq C \left(\frac{r}{\rho} \right) \frac{1}{\rho^2} \int_{Q_\rho} |P|^{\frac{3}{2}} dx dt + C \left(\frac{\rho}{r} \right)^2 \frac{1}{\rho^2} \int_{Q_\rho} |u - (u)_\rho|^3 dx dt + C \left(\frac{\rho}{r} \right)^2 \rho^{\frac{3}{2}}, \end{aligned}$$

where we have used the inequality (3.25) in the last step.

This, combined with the interpolation inequality

$$\begin{aligned} & \frac{1}{\rho^2} \int_{Q_\rho} |u - (u)_\rho|^3 dx dt \\ & \leq C \sup_{-\rho^2 \leq t \leq 0} \left(\frac{1}{\rho} \int_{B_\rho} |u|^2 dx \right)^{\frac{3}{4}} \cdot \left(\frac{1}{\rho} \int_{Q_\rho} |\nabla u|^2 dx dt \right)^{\frac{3}{4}}, \end{aligned}$$

implies that

$$D(r) \leq C \left[\left(\frac{r}{\rho} \right) D(\rho) + \left(\frac{\rho}{r} \right)^2 A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho) + \left(\frac{\rho}{r} \right)^2 \rho^{\frac{3}{2}} \right].$$

This completes the proof. $\square$

Proof of Theorem 4.1. Here we closely follow the presentation by [3]. For $0 < \theta < \frac{1}{2}$ and $0 < \rho < 1$, let $0 \leq \phi \in C_0^\infty(Q_{\theta\rho})$ be such that

$$\phi = 1 \quad \text{in } Q_{\frac{\theta\rho}{2}}, \quad |\nabla \phi| \leq \frac{4}{\theta\rho}, \quad |\nabla^2 \phi| + |\partial_t \phi| \leq \frac{16}{(\theta\rho)^2}.$$

Applying the local energy inequality (1.7) and $\operatorname{div} u = 0$, we obtain

$$\begin{aligned} & \sup_{-(\theta\rho)^2 \leq t \leq 0} \int_{\Omega} |u|^2 \phi^2 dx + 2 \int_{\Omega \times [-(\theta\rho)^2, 0]} |\nabla u|^2 \phi^2 dx dt \\ (4.7) \quad & \leq \int_{\Omega \times [-(\theta\rho)^2, 0]} |u|^2 (|\partial_t \phi| + |\nabla \phi|^2 + |\nabla^2 \phi|) dx dt \\ & \quad + \int_{\Omega \times [-(\theta\rho)^2, 0]} (|u|^2 - (|u|^2)_\rho + 2|P|) |u| |\nabla \phi| dx dt \\ & \quad + 2 \int_{\Omega \times [-(\theta\rho)^2, 0]} \left| \nabla \Psi \otimes \nabla \Psi - \frac{1}{2} |\nabla \Psi|^2 I_3 \right| (|\nabla u| \phi + |u| |\nabla \phi|) dx dt, \end{aligned}$$

where

$$(|u|^2)_\rho = \oint_{B_\rho} |u|^2 dx$$

is the average of $|u|^2$ over B_ρ . By Sobolev's inequality, we have

$$\left(\int_{B_\rho} |u|^2 - (|u|^2)_\rho^{\frac{3}{2}} dx \right)^{\frac{2}{3}} \leq C \int_{B_\rho} |u| |\nabla u| dx.$$

By Hölder's inequality, we can bound

$$\begin{aligned}
& \int_{\Omega \times [-(\theta\rho)^2, 0]} \left| \nabla \Psi \otimes \nabla \Psi - \frac{1}{2} |\nabla \Psi|^2 I_3 (|\nabla u| \phi + |u| |\nabla \phi|) \right| dx dt \\
& \leq c \int_{Q_{\theta\rho}} |\nabla \Psi|^2 |\nabla u| dx dt + c(\theta\rho)^{-1} \int_{Q_{\theta\rho}} |\nabla \Psi|^2 |u| dx dt \\
& \leq c(\theta\rho)^{\frac{1}{2}} B^{\frac{1}{2}}(\theta\rho) \left(\int_{Q_{\theta\rho}} |\nabla \Psi|^4 dx dt \right)^{\frac{1}{2}} \\
& \quad + c(\theta\rho)^{-1} \left(\int_{Q_{\theta\rho}} |\nabla \Psi|^3 dx dt \right)^{\frac{2}{3}} \left(\int_{Q_{\theta\rho}} |u|^3 dx dt \right)^{\frac{1}{3}} \\
& \leq c(\theta\rho)^2 \left(B^{\frac{1}{2}}(\theta\rho) + C^{\frac{1}{3}}(\theta\rho) \right),
\end{aligned}$$

where we have used (3.25) in the last step and

$$\int_{Q_{\theta\rho}} |\nabla \Psi|^4 dx dt \leq c(\theta\rho)^{\frac{3}{2}}.$$

Substituting these two estimates into (4.7), we obtain

$$\begin{aligned}
(4.8) \quad A\left(\frac{1}{2}\theta\rho\right) + B\left(\frac{1}{2}\theta\rho\right) & \leq c \left[C^{\frac{2}{3}}(\theta\rho) + A^{\frac{1}{2}}(\theta\rho) B^{\frac{1}{2}}(\theta\rho) C^{\frac{1}{3}}(\theta\rho) \right. \\
& \quad \left. + C^{\frac{1}{3}}(\theta\rho) D^{\frac{2}{3}}(\theta\rho) + (\theta\rho)^2 B^{\frac{1}{2}}(\theta\rho) + (\theta\rho)^2 C^{\frac{1}{3}}(\theta\rho) \right] \\
& \leq c \left[C^{\frac{2}{3}}(\theta\rho) + A(\theta\rho) B(\theta\rho) + (\theta\rho)^4 + (\theta\rho)^2 B^{\frac{1}{2}}(\theta\rho) + D^{\frac{4}{3}}(\theta\rho) \right].
\end{aligned}$$

Thus we obtain

$$A^{\frac{3}{2}}\left(\frac{1}{2}\theta\rho\right) \leq c \left[C(\theta\rho) + A^{\frac{3}{2}}(\theta\rho) B^{\frac{3}{2}}(\theta\rho) + D^2(\theta\rho) + (\theta\rho)^6 + (\theta\rho)^3 B^{\frac{3}{4}}(\theta\rho) \right].$$

While we also have

$$\begin{aligned}
D^2(\theta\rho) & \leq c\theta^2 [D^2(\rho) + \theta^{-6} A^{\frac{3}{2}}(\rho) B^{\frac{3}{2}}(\rho) + \theta^{-6} \rho^3], \\
C(\theta\rho) & \leq c \left[\theta^3 A^{\frac{3}{2}}(\rho) + \theta^{-3} A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho) \right],
\end{aligned}$$

and

$$A^{\frac{3}{2}}(\theta\rho) B^{\frac{3}{2}}(\theta\rho) \leq \theta^{-3} A^{\frac{3}{2}}(\rho) B^{\frac{3}{2}}(\rho).$$

Putting all these estimates together, we arrive at

$$\begin{aligned}
& A^{\frac{3}{2}}\left(\frac{1}{2}\theta\rho\right) + D^2\left(\frac{1}{2}\theta\rho\right) \\
& \leq c \left[\theta^2 (D^2(\rho) + A^{\frac{3}{2}}(\rho)) + \theta^{-8} A^{\frac{3}{2}}(\rho) B^{\frac{3}{2}}(\rho) + \theta^2 + \theta^{-4} \rho^3 + \theta^6 \rho^6 + \theta^{\frac{9}{4}} \rho^3 B^{\frac{3}{4}}(\rho) \right] \\
& \leq c(\theta^2 + \theta^{-8} B^{\frac{3}{2}}(\rho)) (A^{\frac{3}{2}}(\rho) + D^2(\rho)) + c(\theta^2 + \theta^{-4} \rho^3 + \theta^6 \rho^6 + \theta^{\frac{9}{4}} \rho^3 B^{\frac{3}{4}}(\rho)).
\end{aligned}$$

For $\epsilon_1 > 0$ given by Theorem 3.1, let $\theta_0 \in (0, \frac{1}{2})$ be such that

$$c\theta_0^2 = \min \left\{ \frac{1}{4}, \frac{1}{8}\epsilon_1^2 \right\}.$$

Since

$$\limsup_{r \rightarrow 0} \frac{1}{r} \int_{Q_r} |\nabla u|^2 dx dt < \epsilon_1^2,$$

we can choose $\rho_0 > 0$ such that

$$c\theta_0^{-2} B^{\frac{3}{2}}(\rho) \leq \frac{1}{4} \quad \forall 0 < \rho < \rho_0$$

and

$$c(\theta_0^2 + \theta_0^{-4}\rho^3 + \theta_0^6\rho^6 + \theta_0^{\frac{9}{4}}\rho^3 B^{\frac{3}{4}}(\rho)) \leq \frac{1}{2}\epsilon_1^2 \quad \forall 0 < \rho < \rho_0.$$

Therefore we obtain that there exist $\theta_0 \in (0, \frac{1}{2})$ and $\rho_0 > 0$ such that

$$A^{\frac{3}{2}}\left(\frac{1}{2}\theta_0\rho\right) + D^2\left(\frac{1}{2}\theta_0\rho\right) \leq \frac{1}{2}\left(A^{\frac{3}{2}}(\rho) + D^2(\rho)^2\right) + \frac{1}{2}\epsilon_1^2 \quad \forall 0 < \rho < \rho_0.$$

Iterating this inequality yields that

$$(4.9) \quad A^{\frac{3}{2}}\left(\left(\frac{1}{2}\theta_0\right)^k \rho\right) + D^2\left(\left(\frac{1}{2}\theta_0\right)^k \rho\right) \leq \frac{1}{2^k}\left(A^{\frac{3}{2}}(\rho) + D^2(\rho)\right) + \epsilon_1^2$$

holds for all $0 < \rho < \rho_0$ and $k \geq 1$.

Employing (4.9) and (4.2), we obtain that

$$(4.10) \quad \begin{aligned} C\left(\left(\frac{1}{2}\theta_0\right)^k \rho\right) &\leq c\left[\left(\frac{1}{2}\theta_0\right)^3 A^{\frac{3}{2}}\left(\left(\frac{1}{2}\theta_0\right)^{k-1} \rho\right) \right. \\ &\quad \left. + \left(\frac{1}{2}\theta_0\right)^{-3} A^{\frac{3}{4}}\left(\left(\frac{1}{2}\theta_0\right)^{k-1} \rho\right) B^{\frac{3}{4}}\left(\left(\frac{1}{2}\theta_0\right)^{k-1} \rho\right) \right] \\ &\leq c\left[\left(\frac{1}{2}\theta_0\right)^3 + \left(\frac{1}{2}\theta_0\right)^{-3} \epsilon_1^{\frac{3}{2}}\right] \left[\frac{1}{2^{k-1}}\left(A^{\frac{3}{2}}(\rho) + D^2(\rho)\right) + \epsilon_1^2\right] \end{aligned}$$

holds for all $0 < \rho < \rho_0$ and $k \geq 1$.

Putting (4.9) and (4.10) together, we obtain that

$$(4.11) \quad \begin{aligned} \limsup_{k \rightarrow \infty} \left[C\left(\left(\frac{1}{2}\theta_0\right)^k \rho\right) + D^2\left(\left(\frac{1}{2}\theta_0\right)^k \rho\right) \right] \\ \leq c\left[1 + \left(\frac{1}{2}\theta_0\right)^3 + \left(\frac{1}{2}\theta_0\right)^{-3} \epsilon_1^{\frac{3}{2}}\right] \epsilon_1^2 \leq \frac{1}{2}\epsilon_0^3 \end{aligned}$$

holds for all $\rho \in (0, \rho_0)$, provided $\epsilon_1 = \epsilon_1(\theta_0, \epsilon_0) > 0$ is chosen sufficiently small. It follows from Theorem 3.1 that (u, n^+, n^-, Ψ) is smooth near $z_0 = (0, 0)$. This completes the proof. $\square$

Completion of Proof of Theorem 1.1. Define the singular set of (u, n^+, n^-, Ψ) by

$$\Sigma = \left\{ (x, t) \in Q_T \left| \limsup_{r \rightarrow 0} r^{-1} \int_{Q_r(x, t)} |\nabla u|^2 dx dt > \epsilon_1^2 \right. \right\}.$$

From Theorem 4.1, we know that Σ is closed and $(u, n^+, n^-, \Psi) \in C^\infty(Q_T \setminus \Sigma)$.

Let U be a small neighborhood of Σ and let $\delta > 0$. For each $(x, t) \in \Sigma$, choose $0 < r < \delta$ such that

$$r^{-1} \int_{Q_r(x, t)} |\nabla u|^2 dx dt > \epsilon_1^2 \text{ and } Q_r(x, t) \subset U.$$

By Vitali's five time covering lemma, there exists a disjoint subfamily $\{Q_{r_i}(x_i, t_i)\}$ such that

$$\Sigma \subset \bigcup_i Q_{5r_i}(x_i, t_i).$$

Hence

$$\mathcal{P}_{5\delta}^1(\Sigma) \leq \sum_i 5r_i \leq 5\epsilon_1^{-2} \sum_i \int_{Q_{r_i}(x_i, t_i)} |\nabla u|^2 dx dt \leq 5\epsilon_1^{-2} \int_U |\nabla u|^2 dx dt.$$

Sending $\delta \rightarrow 0$, this implies that $\mathcal{P}^1(\Sigma) = 0$. The proof is now complete. $\square$

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REFERENCES

- [1] L. CAFFARELLI, R. KOHN, AND L. NIRENBERG, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math., 35 (1982), pp. 771–831.
- [2] A. CASTELLANOS, *Electrohydrodynamics*, Springer, Vienna, 1998.
- [3] H. R. DU, X. P. HU, AND C. Y. WANG, *Suitable weak solutions for the co-rotational Beris-Edwards system in dimension three*, Arch. Ration. Mech. Anal., 238 (2020), pp. 749–803.
- [4] C. DENG, J. ZHAO, AND S. CUI, *Well-posedness for the Navier-Stokes-Nernst-Planck-Poisson system in Triebel-Lizorkin space and Besov space with negative indices*, J. Math. Anal. Appl., 377 (2011), pp. 392–405.
- [5] J. FAN, F. LI, AND G. NAKAMURA, *Regularity criteria for a mathematical model for the deformation of electrolyte droplets*, Appl. Math. Lett. 26 (203), pp. 494–499.
- [6] W. FANG AND K. ITO, *On the time-dependent drift-diffusion model for semiconductors*, J. Differential Equations, 117 (1995), pp. 245–280.
- [7] H. GAJEWSKI, *On existence, uniqueness and asymptotic behavior of solutions of the basic equations for carrier transport in semiconductors*, Z. Angew. Math. Mech., 65 (1985), pp. 101–108.
- [8] R. HYND, *Partial regularity of weak solutions of the viscoelastic Navier-Stokes equations with damping*, SIAM J. Math. Anal., 45 (2013), pp. 495–517.
- [9] J. JEROME, *Analytical approaches to charge transport in a moving medium*, Transp. Theory Statist. Phys., 31 (2002), pp. 333–366.
- [10] J. JEROME AND R. SACCO, *Global weak solutions for an incompressible charged fluid with multi-scale couplings: Initial-boundary-value problem*, Nonlinear Anal., 71 (2009), pp. e2487–e2497.
- [11] J. LERAY, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta. Math., 63 (1934), pp. 183–248.
- [12] O. A. LADYZHENSKAYA, V. A. SOLONNIKOV, AND N. N. URAL'CEVA, *Linear and Quasi-linear Equations of Parabolic Type*, American Mathematical Society, Providence, RI, 1968.

- [13] F. LIN, *A new proof of the Caffarelli-Kohn-Nirenberg theorem*, Comm. Pure Appl. Math., 51 (1998), pp. 241–257.
- [14] F. LIN AND C. LIU, *Partial regularity of the dynamic system modeling the flow of liquid crystals*, Discrete Contin. Dyn. Syst., 2 (1996), pp. 1–22.
- [15] T. HUANG AND C. WANG, *Notes on the regularity of harmonic map systems*, Proc. Amer. Math. Soc., 138 (2010), pp. 2015–2023.
- [16] J. HINEMAN AND C. WANG, *Well-posedness of nematic liquid crystal flow in $L^3_{\text{uloc}}(\mathbb{R}^3)$* , Arch. Ration. Mech. Anal., 210 (2013), pp. 177–218.
- [17] J. HUANG, F. LIN, AND C. WANG, *Regularity and existence of global solutions to the Ericksen-Leslie system in $\mathbb{R}^3$* , Comm. Math. Phys., 331 (2014), pp. 805–850.
- [18] M. S. MOCK, *An initial value problem from semiconductor device theory*, SIAM J. Math. Anal., 5 (1974), pp. 597–612.
- [19] V. ROOSBROECK, *Theory of the flow of electrons and holes in germanium and other semiconductors*, Bell System Tech. J., 29 (1950), pp. 560–607.
- [20] I. RUBINSTEIN, *Electro-Diffusion of Ions*, SIAM Stud. Appl. Math. 11, SIAM, Philadelphia, 1990.
- [21] M. SCHMUCK, *Analysis of the Navier-Stokes-Nernst-Planck-Poisson system*, Math. Models Methods Appl. Sci., 19 (2009), pp. 993–1015.
- [22] V. SCHEFFER, *Partial regularity of solutions to the Navier-Stokes equations*, Pacific J. Math., 66 (1976), pp. 535–552.
- [23] J. SERRIN, *On the interior regularity of weak solutions of the Navier-Stokes equations*, Arch. Ration. Mech. Anal., 9 (1962), pp. 187–195.
- [24] L. ISKAURIAZA, G. SEREGIN, AND V. SVERAK, *$L^{3,\infty}$ -solutions of Navier-Stokes equations and backward uniqueness*, Uspekhi Mat. Nauk 58 (2003), 2 (350), pp. 3–44 (in Russian); Russian Math. Surveys 58 (2003), pp. 211–250 (in English).
- [25] T. SEIDMAN AND G. TROIANIELLO, *Time-dependent solutions of a nonlinear system arising in semiconductor theory*, Nonlinear Anal., 9 (1985), pp. 1137–1157.
- [26] H. SOHR AND W. VON WAHL, *On the regularity of the pressure of weak solutions of Navier-Stokes equations*, Arch. Math. (Basel), 46 (1986), pp. 428–439.
- [27] R. TEMAM, *Navier-Stokes Equations: Theory and Numerical Analysis*, American Mathematical Society, Providence, RI, 2001.
- [28] A. F. VASSEUR, *A new proof of partial regularity of solutions to Navier-Stokes equations*, NoDEA Nonlinear Differential Equations Appl., 14 (2007), pp. 753–785.
- [29] Y. WANG, C. LIU, AND Z. TAN, *A generalized Poisson-Nernst-Planck-Navier-Stokes model on the fluid with the crowded charged particles: Derivation and its well-posedness*, SIAM J. Math. Anal., 48 (2016), pp. 3191–3235.
- [30] Y. WANG, C. LIU, AND Z. TAN, *Well-posedness on a new hydrodynamic model of the fluid with the dilute charged particles*, J. Differential Equations, 262 (2017), pp. 68–115.
- [31] J. ZHAO, T. ZHANG, AND Q. LIU, *Global well-posedness for the dissipative system modeling electro-hydrodynamics with large vertical velocity component in critical Besov spaces*, Discrete Contin. Dyn. Syst., 35 (2015), pp. 555–582.