



A Probabilistic Mechanism for Quark Confinement

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Abstract: The confinement of quarks is one of the enduring mysteries of modern physics. There is a longstanding physics heuristic that confinement is a consequence of ‘unbroken center symmetry’. This article gives mathematical confirmation of this heuristic, by rigorously defining of center symmetry in lattice gauge theories and proving that a theory is confining when center symmetry is unbroken. Furthermore, a sufficient condition for unbroken center symmetry is given: It is shown that if the center of the gauge group is nontrivial, and correlations decay exponentially under arbitrary boundary conditions, then center symmetry does not break.

1. Introduction

Quantum gauge theories, also known as quantum Yang–Mills theories, are components of the Standard Model of quantum mechanics. In spite of many decades of research, physically relevant quantum gauge theories have not yet been constructed in a rigorous mathematical sense. The most popular approach to solving this problem is via the program of constructive field theory [20]. In this approach, one starts with a statistical mechanical model on the lattice; the next step is to pass to a continuum limit of this model; the third step is to show that the continuum limit satisfies certain ‘axioms’; if these axioms are satisfied, then there is a standard machinery which allows the construction of a quantum field theory. Taking this program to its completion is one of the Clay millennium problems [9, 26]. There are various well-known difficulties in adapting the constructive field theory approach to gauge theories, which is probably why the question has remained open for so long; for an alternative (but also unfinished) approach, see [32].

The statistical mechanical models considered in the first step of the above program are known as *lattice gauge theories* (defined in Sect. 2). Lattice gauge theories were

introduced by Wegner [35] to study phase transitions without a local order parameter, and later reintroduced by Wilson [36] to study quark confinement. A lattice gauge theory may be coupled with, for example, a Higgs field, or it may be a *pure* lattice gauge theory. We will only deal with pure lattice gauge theories (also called lattice Yang–Mills theories) in this manuscript. A pure lattice gauge theory is characterized by its gauge group (usually a compact matrix Lie group), the dimension of spacetime, and a parameter known as the coupling strength. These theories on their own, even without passing to the continuum limit or constructing the quantum theory, can yield substantial physically relevant information [22]. Two of the most important open questions in this area have lattice gauge theoretic formulations. The first is the question of Yang–Mills mass gap. In lattice gauge theories, mass gap means exponential decay of correlations. Mass gap is not hard to establish at sufficiently large values of the coupling strength (for example, using the methods of [12]). However, it is widely believed [9, 26] (with some dissent [31]) that certain lattice gauge theories have mass gap at *all* values of the coupling strength. Perhaps the most important example is four-dimensional $SU(3)$ lattice gauge theory. If one can show that this theory has a mass gap at all values of the coupling strength, that would explain why particles known as *glueballs* in the theory of strong interactions have mass [29]. All such questions remain open.

The second big open question is the problem of *quark confinement*. Quarks are the constituents of various elementary particles, such as protons and neutrons. It is an enduring mystery why quarks are never observed freely in nature. The problem of quark confinement has received enormous attention in the physics literature, but the current consensus seems to be that a satisfactory theoretical explanation does not exist [22].

Wilson [36] argued that quark confinement is equivalent to showing that the relevant lattice gauge theory satisfies what's now known as *Wilson's area law* (defined in Sect. 2). Soon after Wilson's work, Osterwalder and Seiler [30] proved that the area law is always satisfied at sufficiently large coupling strength. However, to prove quark confinement, one needs to show that the area law holds at *all* values of the coupling strength—and in particular, for very small values. (Actually, what is really needed is that the area law holds at coupling strengths arbitrarily close to a critical value; in many theories of interest, the critical value is believed to be zero.) This need not always be true; for example, Guth [24] and Fröhlich and Spencer [18] showed that four-dimensional $U(1)$ lattice gauge theory is not confining at weak coupling.

Proving that the area law holds at weak coupling remains a largely open problem above dimension two (where it is relatively easy—see [30, Section 6], and also [2]). A rare instance where the area law has been shown to hold at weak coupling is three-dimensional $U(1)$ lattice gauge theory [21]. But the most important case of four-dimensional $SU(3)$ theory remains out of reach.

Some of the other notable advances in the mathematical study of confinement include the work of Fröhlich [17], who showed that confinement holds in $SU(n)$ theory if it holds in the corresponding \mathbb{Z}_n theory; the work of Durhuus and Fröhlich [13], who showed that confinement in a d -dimensional pure lattice gauge theory holds if there is exponential decay of correlations in a $(d - 1)$ -dimensional nonlinear σ model; the work of Borgs and Seiler [5], who studied long range order for lattice gauge theories on cylinders; and the work of Brydges and Federbush [6] on the related problem of Debye screening. A toy model exhibiting a sharp transition from the confining to the deconfining regime was studied by Fröhlich and Russo [1]. For some recent progress towards understanding confinement in large n lattice gauge theories, see [8, 11].

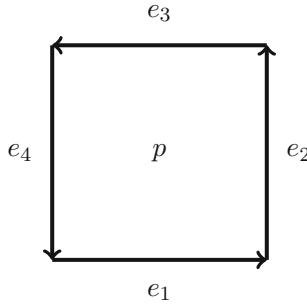


Fig. 1. A plaquette p bounded by four directed edges e_1, e_2, e_3, e_4

In physics, it has been believed since the work of 't Hooft [34] that a lattice gauge theory is confining when a certain kind of symmetry, known as *center symmetry*, is not spontaneously broken. This paper gives mathematical confirmation of this heuristic, by first giving a rigorous definition of center symmetry in lattice gauge theories, and then proving that a theory is confining if center symmetry is unbroken. Furthermore, a sufficient condition for unbroken center symmetry is provided: It is shown that if the center of the gauge group is nontrivial, and correlations decay exponentially under arbitrary boundary conditions, then center symmetry does not break, and therefore the theory is confining.

2. Definitions and Results

This section contains the main results of this paper. We begin with the definitions of lattice gauge theory and Wilson loop variables.

2.1. Lattice gauge theories. Let $n \geq 1$ and $d \geq 2$ be two integers. Let G be a closed connected subgroup of $U(n)$. Let E be the set of directed nearest-neighbor edges of \mathbb{Z}^d , where the direction is from the smaller vertex to the bigger one in the lexicographic ordering. We will call such edges *positively oriented*. For a positively oriented edge $e \in E$, let e^{-1} denote the same edge but directed in the opposite direction. Such edges will be called *negatively oriented*. Let Ω be the set of all functions from E into G . That is, an element $\omega \in \Omega$ assigns a matrix $\omega_e \in G$ to each edge $e \in E$. If $\omega \in \Omega$ and e is a negatively oriented edge, we define $\omega_e := \omega_{e^{-1}}^{-1}$.

A *plaquette* in \mathbb{Z}^d is a set of four directed edges that form the boundary of a square. Let P be the set of all plaquettes. Given some $p \in P$ and $\omega \in \Omega$, we define ω_p as follows. Write p as a sequence of directed edges e_1, e_2, e_3, e_4 , each one followed by the next (see Figure 1). Then let $\omega_p := \omega_{e_1} \omega_{e_2} \omega_{e_3} \omega_{e_4}$. Although there are ambiguities in this definition about the choice of e_1 and the direction of traversal, that is not problematic because we will only use the quantity $\text{Re}(\text{Tr}(\omega_p))$, which is not affected by these ambiguities. (Here ω_e 's are $n \times n$ matrices, and Tr denotes the trace of a matrix.)

Endow the product space $\Omega = G^E$ with the product σ -algebra and let λ denote the normalized product Haar measure on Ω . Pure lattice gauge theory on \mathbb{Z}^d with gauge group G and coupling parameter β (equal to the inverse of the squared coupling strength) is formally defined as the probability measure $d\mu(\omega) = Z^{-1} e^{-\beta H(\omega)} d\lambda(\omega)$ on Ω , where H is the formal Hamiltonian

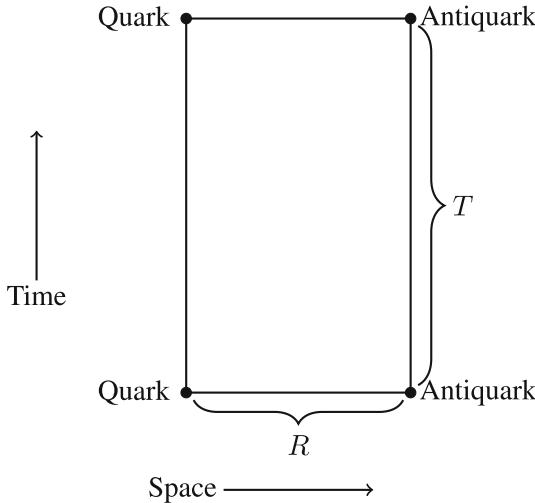


Fig. 2. Rectangular Wilson loop representing a static quark-antiquark pair separated by distance R over a time interval of length T

$$H(\omega) := - \sum_{p \in P} \text{Re}(\text{Tr}(\omega_p)) \quad (2.1)$$

and Z is the normalizing constant. In the language of rigorous mathematical physics, what this actually defines is a *specification*: Although the definition of the probability measure μ as stated above does not make sense since the series defining H may not be convergent, the conditional distribution of any finite set of ω_e 's given all other ω_e 's, under such a hypothetical probability measure μ , is perfectly well-defined. Any actual probability measure μ on Ω which has these specified conditional distributions is called a *Gibbs measure* for this specification. It is not obvious that Gibbs measures exist. In the case of lattice gauge theories with compact gauge groups, the existence of at least one Gibbs measure follows from standard results, such as [19, Theorem 4.22].

Throughout the rest of this paper, we will assume that G , β and d are fixed. In particular, whenever we refer to the lattice gauge theory defined above, we will think of β as a fixed number and part of the definition.

2.2. Wilson loops and area law. Let π be a finite-dimensional irreducible unitary representation of the group G , and let χ_π be the character of π . Along with G , β and d , the representation π will remain fixed throughout this manuscript.

A *loop* ℓ in \mathbb{Z}^d is a sequence of directed edges e_1, e_2, \dots, e_k such that the end vertex of e_i is the same as the beginning vertex of e_{i+1} for $i = 1, \dots, k-1$, and the end vertex of e_k is the beginning vertex of e_1 . Given a loop ℓ and a configuration $\omega \in \Omega(\mathbb{Z}^d)$, the *Wilson loop variable* $W_\ell(\omega)$ is defined as

$$\begin{aligned} W_\ell(\omega) &:= \chi_\pi(\omega_{e_1} \omega_{e_2} \cdots \omega_{e_k}) \\ &= \text{Tr}(\pi(\omega_{e_1}) \pi(\omega_{e_2}) \cdots \pi(\omega_{e_k})). \end{aligned} \quad (2.2)$$

Take any Gibbs measure μ of our lattice gauge theory. The expected value $\langle W_\ell \rangle$ of a Wilson loop variable under μ is known as a *Wilson loop expectation*. A loop is called

rectangular if it forms the boundary of a rectangle. The area enclosed by a rectangular loop is the product of the length and the breadth of the rectangle. The Gibbs measure is said to satisfy *Wilson's area law* for the representation π if

$$|\langle W_\ell \rangle| \leq C_1 e^{-C_2 \text{area}(\ell)} \quad (2.3)$$

for any rectangular loop ℓ , where C_1 and C_2 are positive constants that do not depend on ℓ , and $\text{area}(\ell)$ is the area enclosed by ℓ .

The reason why the area law is thought to imply confinement of quarks is as follows. Let $V(R)$ be the potential energy of a static quark-antiquark pair separated by distance R . Then quantum field theoretic calculations [36] indicate that for a rectangular loop ℓ with side-lengths R and T , $\langle W_\ell \rangle$ should behave like $e^{-V(R)T}$ when R is fixed and T is taken to infinity. Here the sides of length R represent the lines joining the quark-antiquark pair at times 0 and T , and the sides of length T represent the trajectories of the quark and the antiquark in the time direction (Figure 2). So if the area law holds, then $V(R)$ grows linearly in the distance R between the quark and the antiquark. By the conservation of energy, this implies that the pair will not be able to separate beyond a certain distance.

Notice that for quarks to be confined, it is not really necessary to have $V(R)$ growing linearly with R . It suffices to have $V(R) \rightarrow \infty$ as $R \rightarrow \infty$. Thus, for a given Gibbs measure to be confining, it is sufficient to have that for any R ,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log |\langle W_{\ell_{R,T}} \rangle| \leq -V(R)$$

for some V with $V(R) \rightarrow \infty$ as $R \rightarrow \infty$, where $\ell_{R,T}$ is any rectangular loop with side lengths R and T .

2.3. Unbroken center symmetry implies confinement. Physics literature tells us that confinement happens when a certain symmetry of a lattice gauge theory, known as *center symmetry*, is not broken. To my knowledge, there is no rigorous definition of center symmetry in the mathematical physics literature, nor a discussion of what it means for this symmetry to be broken. I will now propose a definition, inspired by the physical definition of center symmetry via Polyakov loops [22].

Instead of the lattice \mathbb{Z}^d , fix a positive integer N and consider the *slab* $S := \{0, 1, \dots, N\} \times \mathbb{Z}^{d-1}$. Let $E(S)$ denote the set of positively oriented edges of S and $P(S)$ denote the set of plaquettes in S .

The slab S has two boundaries, namely, the top boundary $\{N\} \times \mathbb{Z}^{d-1}$, and the bottom boundary $\{0\} \times \mathbb{Z}^{d-1}$. Let $\partial E(S)$ denote the set of all boundary edges. Let us call an element of $G^{\partial E(S)}$ a *boundary condition*. Given a boundary condition δ , let $\Omega(S, \delta)$ denote the set of all $\omega \in G^{E(S)}$ that agree with δ on the boundary. The formal Hamiltonian

$$H_{S,\delta}(\omega) := - \sum_{p \in P(S)} \text{Re}(\text{Tr}(\omega_p))$$

defines a specification on $\Omega(S, \delta)$ with coupling parameter β in the usual way. Let us refer to this specification as our lattice gauge theory on S with boundary condition δ , and denote it by α .

Now consider the following transformation on $\Omega(S, \delta)$. Take any element g_0 in the center of the group G . Take any $\omega \in \Omega(S, \delta)$. For each edge e from the boundary

$\{0\} \times \mathbb{Z}^{d-1}$ to the layer $\{1\} \times \mathbb{Z}^{d-1}$, replace ω_e by $g_0 \omega_e$. Call the resulting configuration $\tau(\omega)$. We will refer to τ as a *center transform*.

Since g_0 is an element of the center of G , it is easy to see that the map τ leaves ω_p invariant for any plaquette p , and hence (formally) leaves the Hamiltonian $H_{S,\delta}$ invariant. From this, it follows that τ is a *symmetry* of the specification α defined above (see [19, Chapter 5] for the definition of a symmetry of a specification). We will call this a *center symmetry* of our lattice gauge theory in the slab S . Following the definition of unbroken symmetry in mathematical physics [19, Definition 5.21], we will say that the specification α has unbroken center symmetry if every Gibbs measure of α is invariant under the action of each center transform.

Definition 2.1. We will say that the lattice gauge theory on \mathbb{Z}^d defined in Sect. 2.1 has unbroken center symmetry if for some $N \geq 1$, the theory restricted to the slab $S = \{0, 1, \dots, N\} \times \mathbb{Z}^{d-1}$, under any boundary condition, has unbroken center symmetry as defined above.

Incidentally, the restriction of Gibbs measures to slabs has similarities with ‘finite temperature’ states in lattice gauge theories, where the possible breaking of the analogous center symmetry has been analyzed in [5].

Having defined center symmetry and what it means for it to be unbroken, we now arrive at the first main result of this paper.

Theorem 2.2. Suppose that the lattice gauge theory defined in Sect. 2.1 has unbroken center symmetry in the sense of Definition 2.1. Let π be a finite-dimensional irreducible unitary representation of G that acts nontrivially on the center of G . Let W_ℓ denote the Wilson loop variable for a loop ℓ , defined using the representation π as in equation (2.2). Then there is a function $V : \mathbb{Z}_+ \rightarrow \mathbb{R}$, satisfying $V(R) \rightarrow \infty$ as $R \rightarrow \infty$, such that for any rectangular loop ℓ with side-lengths $R \leq T$, and for any Gibbs measure of our lattice gauge theory on \mathbb{Z}^d , we have $|\langle W_\ell \rangle| \leq e^{-V(R)T}$.

In the above, the condition ‘ π acts nontrivially on the center of G ’ means that there is at least one element g_0 in the center such that $\pi(g_0)$ is not the identity operator. This requires, in particular, that the center of G is nontrivial. Indeed, it is believed that if the center of the gauge group is trivial, such as in $SO(3)$ theory, then quarks may not confine at weak coupling [23].

2.4. A sufficient condition for unbroken center symmetry. We will say that two edges of \mathbb{Z}^d are neighbors if they both belong to some common plaquette. A measurable map $f : \Omega \rightarrow \mathbb{R}$ will be called a *local function* supported on an edge $e \in E$ if $f(\omega)$ depends only on the values of ω_u for u that are neighbors of e . Given two local functions f and g , let $\text{dist}(f, g)$ denote the Euclidean distance between the midpoints of their supporting edges.

We will say that a subset of \mathbb{Z}^d is a cube if it is a translate of $\{0, \dots, N\}^d$ for some N . A boundary condition on a cube B is an assignment of elements of G to the boundary edges of B . Given a cube B and a boundary condition δ , let $\Omega_{B,\delta}$ denote the set of all assignments of elements of G to edges of B which agree with δ on the boundary. The lattice gauge theory of Sect. 2.1 defines a probability measure on $\Omega_{B,\delta}$ in the natural way. For a measurable map $f : \Omega_{B,\delta} \rightarrow \mathbb{R}$, let $\langle f \rangle_{B,\delta}$ denote the expectation of f under this measure (provided that the expectation exists).

Definition 2.3. Consider the lattice gauge theory defined in Sect. 2.1. We will say that this theory satisfies exponential decay of correlations under arbitrary boundary conditions if there are positive constants K_1 and K_2 depending only on G , β and d , such that for any cube B , for any boundary condition δ on B , and for any local functions f and g supported on edges in B and taking values in $[-1, 1]$, we have $|\langle fg \rangle_{B, \delta} - \langle f \rangle_{B, \delta} \langle g \rangle_{B, \delta}| \leq K_1 e^{-K_2 \text{dist}(f, g)}$.

The above condition is quite strong, but there are situations where it can be proven to hold. For example, it can be shown to hold for essentially any lattice gauge theory when β is small enough, using the methods of [12]. There is also a belief that some form of exponential decay of correlations should hold at large β in four-dimensional non-Abelian theories [9, 26]—this is the Yang–Mills mass gap conjecture—but it is not clear whether this belief includes the strong version stated above.

We now arrive at the second main result of this paper, which says that exponential decay of correlations under arbitrary boundary conditions implies unbroken center symmetry.

Theorem 2.4. *Consider the lattice gauge theory defined in Sect. 2.1. Suppose that it satisfies exponential decay of correlations under arbitrary boundary conditions, according to Definition 2.3. Then it has unbroken center symmetry. Moreover, if Wilson loop variables are defined using a finite-dimensional irreducible unitary representation π that acts nontrivially on the center of G , then any Gibbs measure for the theory satisfies Wilson’s area law (2.3) for rectangular loops.*

Incidentally, there are lattice gauge theories that are believed to be ‘gapped’—that is, exhibiting exponential decay of correlations of local observables—and possessing center symmetry, but not confining at large β . Examples include lattice gauge theories with finite gauge groups [7, 10, 15, 32], and theories coupled with Higgs fields in the adjoint representation [16]. If this is true, then by Theorem 2.2, it must be that center symmetry breaks spontaneously in these models. The reason why this does not contradict Theorem 2.4 is that the ‘exponential decay of correlations’ in these theories refer to exponential decay of two-point truncated correlations under certain kinds of boundary conditions [7, 10]; this is different than what’s commonly understood as ‘decay of correlations’ in mathematical physics, which means absence of long-range order, or equivalently, the uniqueness of the Gibbs measure [19]. Indeed, some of the above theories have been rigorously shown to be possessing multiple Gibbs measures [4, 27] and therefore actually have long-range order at weak coupling.

This completes the statements of results. The rest of the paper is devoted to the proofs of Theorems 2.2 and 2.4.

3. Proof of Theorem 2.2

Let $S = \{0, \dots, N\} \times \mathbb{Z}^{d-1}$ be a slab where our lattice gauge theory has unbroken center symmetry under any boundary condition. Let m be the dimension of the representation π . Let g_0 be an element in the center of G such that $\pi(g_0)$ is not the identity operator. Since $\pi(g_0)$ commutes with $\pi(g)$ for every $g \in G$, Schur’s lemma [25, Corollary 4.30] implies that $\pi(g_0) = cI$ for some $c \neq 1$, where I is the $m \times m$ identity matrix.

We will say that an edge e is a *vertical edge* if its endpoints differ in the first coordinate, imagining the first coordinate as the vertical direction. A sequence of vertical edges that connects the two boundaries of S will be called a *vertical chain of edges*. Let e_1, \dots, e_N

be a vertical chain. A *vertical chain variable* f associated with this chain is a product of one element from each of the matrices $\pi(\omega_{e_1}), \dots, \pi(\omega_{e_N})$. Since π is an m -dimensional representation, there are m^{2N} vertical chain variables associated with a given vertical chain.

For each $R \geq 1$, let $S_R := \{0, \dots, N\} \times \{-R, \dots, R\}^{d-1}$. A boundary condition on S_R is an assignment of elements of G to the boundary edges of S_R . Given a boundary condition δ , let $\Omega_{R,\delta}$ denote set of all assignments of elements of G to edges of S_R which agree with δ on the boundary. The lattice gauge theory of Sect. 2.1 defines a probability measure on $\Omega_{R,\delta}$ in the natural way. For a measurable map $f : \Omega_{R,\delta} \rightarrow \mathbb{R}$, let $\langle f \rangle_{R,\delta}$ denote the expectation of f under this measure (provided that the expectation exists).

Lemma 3.1. *Let ξ be the unique vertical chain containing the origin. There is a function $V : \mathbb{Z}_+ \rightarrow \mathbb{R}$ satisfying $V(R) \rightarrow \infty$ as $R \rightarrow \infty$, such that for any vertical chain variable f associated with ξ , any R , and any boundary condition δ on S_R , we have $|\langle f \rangle_{R,\delta}| \leq e^{-V(R)}$.*

Proof. A different way to state the claim is that

$$\lim_{R \rightarrow \infty} \sup_{f,\delta} |\langle f \rangle_{R,\delta}| = 0,$$

where the supremum is taken over all boundary conditions δ on S_R and all vertical chain variables f associated with ξ . Suppose that the claim is not true. Then the above version shows that there is a real number $\varepsilon > 0$, a sequence of integers $R_k \rightarrow \infty$, a sequence of vertical chain variables $\{f_k\}_{k \geq 1}$ associated with ξ , and a boundary condition δ_k on S_{R_k} for each k , such that $|\langle f_k \rangle_{R_k,\delta_k}| \geq \varepsilon$ for all k . Let $\delta_{k,e}$ denote the group element assigned by δ_k to an edge e on the boundary of S_{R_k} . Passing to a subsequence if necessary, we may assume the following:

- For each edge e in the boundary of the infinite slab S , $\delta_{k,e}$ tends to a limit δ_e as $k \rightarrow \infty$. This defines a boundary condition δ on S .
- For each k , f_k is the same variable f . We can assume this since there are only a finite number of vertical chain variables associated with ξ .

Now take any $R \geq 1$, and any k so large that $R_k \geq R$. Since $|\langle f \rangle_{R_k,\delta_k}| \geq \varepsilon$, and $\langle f \rangle_{R_k,\delta_k}$ is the weighted average of $\langle f \rangle_{R,\eta}$ over all boundary conditions η on S_R that agree with δ_k on the top and bottom faces, it follows that there is a boundary condition η_k for which $|\langle f \rangle_{R,\eta_k}| \geq \varepsilon$. Passing to a subsequence if necessary, we may assume that η_k converges to a limit η as $k \rightarrow \infty$. Since η_k agrees with δ_k on the top and bottom faces of S_R , and $\delta_{k,e} \rightarrow \delta_e$ for each edge e on the boundary of S , it follows that η must agree with δ on the top and bottom faces of S_R .

Thus, we have shown that for any R , there is some boundary condition η_R on S_R that agrees with δ on the top and bottom faces, such that $|\langle f \rangle_{R,\eta_R}| \geq \varepsilon$.

Recall the set $\Omega(S, \delta)$ consisting of all assignments of group elements to edges in S that agree with δ on the boundary of S . Since G is compact, $\Omega(S, \delta)$ is a compact metric space under the product topology. For each R , define a $\Omega(S, \delta)$ -valued random configuration ω as follows:

- For each edge e outside S_R , let ω_e be the identity element of G .
- On the boundary of S_R , let ω be equal to η_R .
- Inside S_R , generate ω from the lattice gauge theory on S_R with boundary condition η_R .

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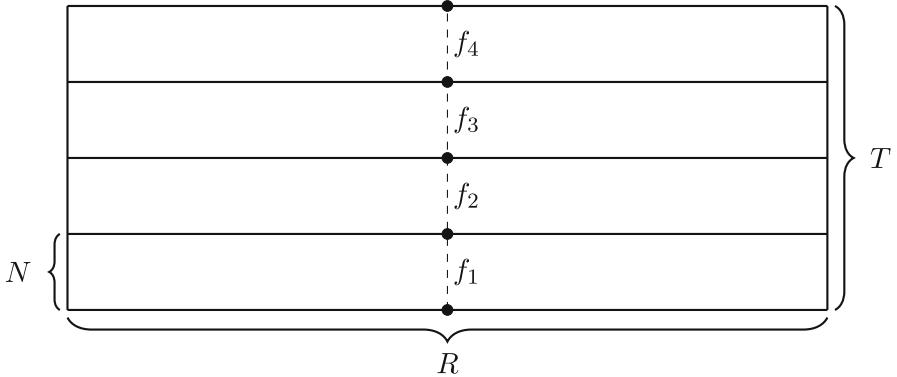


Fig. 3. Bootstrapping by subdividing into smaller slabs

By the compactness of $\Omega(S, \delta)$ and Prokhorov's theorem [3], this sequence of random configurations converges in law through a subsequence. The limit law is easily verified to be a Gibbs measure for our lattice gauge theory on S with boundary condition δ . Since f is a bounded continuous function on $\Omega(S, \delta)$, it follows that the expectation of f under this Gibbs measure has absolute value at least ε .

However, under the center transform described in Sect. 2.3 using our chosen element g_0 , f transforms to cf . So if the center symmetry assumption holds, then we must have $\langle f \rangle = c\langle f \rangle$. Since $c \neq 1$, this implies that $\langle f \rangle = 0$. This contradicts the conclusion of the previous paragraph. \square

We will now bootstrap the bound from Lemma 3.1 to get a bound on expected values of vertical chain variables in finite slabs of arbitrary thickness.

Lemma 3.2. *Take any $T \geq N$ and $R \geq 1$. Let δ be any boundary condition on the slab $\{0, \dots, T\} \times \{-R, \dots, R\}^{d-1}$, and consider the lattice gauge theory on this slab with this boundary condition. Then for any vertical chain variable f defined on the vertical chain ξ containing the origin, we have $|\langle f \rangle| \leq e^{-V(R)[T/N]}$, where V is the function from Lemma 3.1 and $[T/N]$ is the integer part of T/N .*

Proof. Write $T = qN + k$, where $q = [T/N]$ and $k < N$ are the quotient and the remainder when T is divided by N . Divide the slab into $q+1$ slabs S_1, S_2, \dots, S_{q+1} , where S_i has thickness N for $i = 1, \dots, q$, and S_{q+1} has thickness k . Let the top face of S_i , which is the same as the bottom face of S_{i+1} , be called F_i . Conditioning on ω_e for all $e \in \bigcup_{i=1}^{q+1} F_i$, the lattice gauge theory on the slab splits up into independent lattice gauge theories in the slabs S_1, \dots, S_{q+1} . The vertical chain observable f also splits up as a product $f_1 f_2 \cdots f_{q+1}$, where f_i is the part of f coming from S_i . Under the above conditioning, the random variables f_1, \dots, f_{q+1} are independent. (See Figure 3.)

Now let $\langle f_i \rangle'$ denote the conditional expectation of f_i under the above conditioning. For each $1 \leq i \leq q$, Lemma 3.1 gives $|\langle f_i \rangle'| \leq e^{-V(R)}$. Also, since π is a unitary

representation, it follows that $|\langle f_{q+1} \rangle'| \leq 1$. Thus,

$$\begin{aligned} |\langle f \rangle| &= |\langle f_1 f_2 \cdots f_{q+1} \rangle| \\ &= |\langle \langle f_1 f_2 \cdots f_{q+1} \rangle' \rangle| \\ &= |\langle \langle f_1 \rangle' \langle f_2 \rangle' \cdots \langle f_{q+1} \rangle' \rangle| \\ &\leq |\langle f_1 \rangle'| |\langle f_2 \rangle'| \cdots |\langle f_{q+1} \rangle'| \leq e^{-V(R)q}. \end{aligned}$$

This completes the proof of the lemma. \square

We are now ready to complete the proof of Theorem 2.2.

Proof of Theorem 2.2. Let e_1, \dots, e_k be the edges of ℓ , so that $k = 2(R + T)$. Let $f(\omega)$ be a product of one element from each of the matrices $\pi(\omega_{e_1}), \dots, \pi(\omega_{e_k})$. Let us call any such f a *component variable* associated with the loop ℓ . Note that the Wilson loop variable W_ℓ is a sum of m^k component variables.

Let S be a translate of the slab $\{0, \dots, T\} \times \{-R, \dots, R\}^{d-1}$ such that one of the vertical sides of ℓ passes through the center of S and the other three sides belong to the boundary of S . Take any component variable f associated with ℓ . Then f can be written as the product $f_1 f_2$, where f_1 is the product of terms from the vertical side that passes through the center of S , and f_2 is the product of terms from the other three sides. Let $\langle \cdot \rangle'$ denote conditional expectation given ω_e for all e that are either on the boundary of S or outside S . Then

$$\langle f \rangle = \langle f_1 f_2 \rangle = \langle \langle f_1 f_2 \rangle' \rangle = \langle \langle f_1 \rangle' f_2 \rangle, \quad (3.1)$$

where the second identity holds by the tower property of conditional expectation and the third identity holds because f_2 depends only on ω_e for edges e that are on the boundary of S . Now note that f_1 is a vertical chain variable associated with a vertical chain passing through the center of S . Thus, by Lemma 3.2,

$$|\langle f_1 \rangle'| \leq e^{-V(R)[T/N]}. \quad (3.2)$$

Since π is a unitary representation, we have $|f_2| \leq 1$. Thus, by (3.1) and (3.2), we get the bound $|\langle f \rangle| \leq e^{-V(R)[T/N]}$. Since W_ℓ is a sum of $m^{2(R+T)}$ component variables, this gives

$$|\langle W_\ell \rangle| \leq m^{2(R+T)} e^{-V(R)[T/N]}.$$

Now recall that $R \leq T$, and $[x] \geq x/2$ for any $x \geq 1$. Thus,

$$|\langle W_\ell \rangle| \leq m^{4T} e^{-V(R)T/2N}.$$

In other words, $|\langle W_\ell \rangle| \leq e^{-V_1(R)T}$, where $V_1(R) = -4 \log m + V(R)/2N$. Since $V_1(R) \rightarrow \infty$ as $R \rightarrow \infty$, this completes the proof. \square

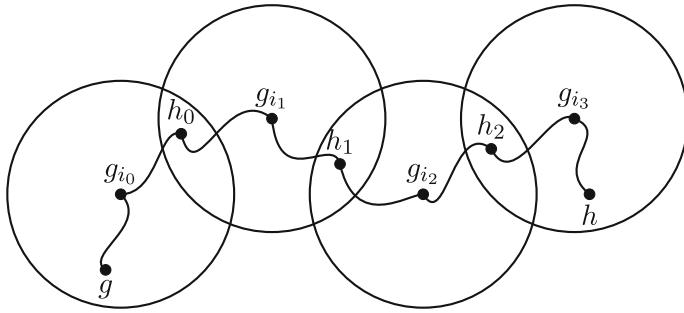


Fig. 4. A piecewise smooth path from g to h , with $m = 3$

4. Paths in the Gauge Group

Let us denote the Euclidean norm of a vector $x \in \mathbb{C}^n$ by $\|x\|$, and the Euclidean inner product of two vectors x and y by $x \cdot y$. Let $M_n(\mathbb{C})$ denote the space of all $n \times n$ complex matrices. We will identify $M_n(\mathbb{C})$ with \mathbb{R}^{2n^2} as a real manifold and view G as a subset of $M_n(\mathbb{C})$. If $f : M_n(\mathbb{C}) \rightarrow \mathbb{R}$ is a smooth function, ∇f will denote the gradient of f , viewing f as a function from \mathbb{R}^{2n^2} into \mathbb{R} . Lastly, for a matrix $A \in M_n(\mathbb{C})$, let $\|A\|$ denote the Euclidean (or Hilbert–Schmidt) norm of A —that is, the square-root of the sum of squared absolute values of the entries of A .

Lemma 4.1. *There is a finite constant K , depending only on G , such that the following is true. For any $g, h \in G$, there is a continuous path $\phi : [0, 1] \rightarrow G$ such that $\phi(0) = g$, $\phi(1) = h$, ϕ is piecewise smooth with a finite number of pieces, and $\|\phi'(t)\| \leq K$ everywhere inside the smooth pieces.*

Proof. Fix some $\varepsilon > 0$, to be chosen later. Let

$$U_\varepsilon := \{g \in G : \|g - I\| < \varepsilon\}.$$

For any $g \in G$, let $U_\varepsilon(g) := \{gh : h \in U_\varepsilon\}$. Each $U_\varepsilon(g)$ is a relatively open subset of G , and these open sets cover G . So by the compactness of G , there exist finitely many $g_1, \dots, g_k \in G$ such that

$$G = \bigcup_{i=1}^k U_\varepsilon(g_i).$$

Let us put a graph structure on $\{U_\varepsilon(g_i)\}_{1 \leq i \leq k}$ by putting an edge between $U_\varepsilon(g_i)$ and $U_\varepsilon(g_j)$ when $U_\varepsilon(g_i) \cap U_\varepsilon(g_j) \neq \emptyset$. If this graph has more than one connected component, then G can be written as a union of two disjoint nonempty relatively open sets, which is impossible since G is connected. Thus, the graph constructed above must be connected.

Take any $g, h \in G$. Find i and j such that $g \in U_\varepsilon(g_i)$ and $h \in U_\varepsilon(g_j)$. By the above paragraph, there is a sequence $i = i_0, i_1, \dots, i_m = j$ such that $U_\varepsilon(g_{i_a}) \cap U_\varepsilon(g_{i_{a+1}}) \neq \emptyset$ for each a . Choose some $h_a \in U_\varepsilon(g_{i_a}) \cap U_\varepsilon(g_{i_{a+1}})$ for $a = 0, 1, \dots, m - 1$. (See Figure 4.)

By the closed subgroup theorem of Lie theory, there exist $\varepsilon, \varepsilon' > 0$ such that for any $g \in U_\varepsilon$, there is some $X \in M_n(\mathbb{C})$ with $\|X\| < \varepsilon'$, such that $e^X = g$ and $e^{tX} \in G$ for all $t \in \mathbb{R}$. (For example, see [25, Corollary 3.44].) Let this ε be our chosen ε .

Take any $0 \leq a \leq m - 1$. Since $h_a \in U_\varepsilon(g_{i_a})$, there is some $u_a \in U_\varepsilon$ such that $h_a = g_{i_a}u_a$. By the previous paragraph, $u_a = e^{tX_a}$ for some $X_a \in M_n(\mathbb{C})$ with $\|X_a\| < \varepsilon'$, such that $e^{tX_a} \in G$ for all $t \in \mathbb{R}$. Define a path ϕ_a from g_{i_a} to h_a as $\phi_a(t) := g_{i_a}e^{tX_a}$, for $0 \leq t \leq 1$. This is a smooth path in G connecting g_{i_a} to h_a , and for each t ,

$$\|\phi'_a(t)\| = \|g_{i_a}X_ae^{tX_a}\| = \|X_a\| \leq \varepsilon',$$

where the second equality holds because g_{i_a} and e^{tX_a} are unitary matrices.

We can similarly construct paths connecting h_a to $g_{i_{a+1}}$ for each a , and paths connecting g to g_i and g_j to h . These paths will all satisfy the above bound on the norm of the derivative. Therefore if we join these $2m + 2$ paths, we get a piecewise smooth path $\phi : [0, 2m + 2] \rightarrow G$ connecting g to h , such that $\|\phi'(t)\| \leq \varepsilon'$ for all t inside the smooth pieces. Rescaling the parameter, we get a piecewise smooth $\phi : [0, 1] \rightarrow G$ connecting g to h such that $\|\phi'(t)\| \leq (2m + 2)\varepsilon'$ for all t inside the smooth pieces. Since $m \leq k$, and k and ε' depend only on G , this completes the proof. \square

As a consequence of the above lemma, we obtain the following corollary, which says that if $f : M_n(\mathbb{C}) \rightarrow \mathbb{R}$ is a smooth function, then the diameter of the set $f(G)$ is bounded by a constant multiple of the maximum value of $\|\nabla f\|$ on G .

Corollary 4.2. *Let K be as in Lemma 4.1. Let $f : M_n(\mathbb{C}) \rightarrow \mathbb{R}$ be a smooth function. Then for any $g, h \in G$, $|f(g) - f(h)| \leq K\|\nabla f\|_G$, where $\|\nabla f\|_G := \sup\{\|\nabla f(g)\| : g \in G\}$.*

Proof. Take any $g, h \in G$. Let ϕ be as in Lemma 4.1. There are numbers $0 = t_0 < t_1 < t_2 < \dots < t_k = 1$ for some k , such that ϕ is smooth in each interval (t_i, t_{i+1}) . By the continuity of ϕ ,

$$\begin{aligned} f(h) - f(g) &= \sum_{i=0}^{k-1} (f(\phi(t_{i+1})) - f(\phi(t_i))) \\ &= \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \nabla f(\phi(t)) \cdot \phi'(t) dt. \end{aligned}$$

Since

$$|\nabla f(\phi(t)) \cdot \phi'(t)| \leq \|\nabla f(\phi(t))\| \|\phi'(t)\| \leq K\|\nabla f\|_G,$$

this completes the proof. \square

5. Total Variation Distance

Let \mathcal{X} be a Polish space equipped with its Borel σ -algebra $\mathcal{B}(\mathcal{X})$. Let μ and ν be two probability measures on \mathcal{X} . Recall that the *total variation distance* between μ and ν is defined as

$$TV(\mu, \nu) := \sup_{A \in \mathcal{B}(\mathcal{X})} |\mu(A) - \nu(A)|.$$

The total variation distance has the following alternate representations (for proofs and discussion, see [28, Chapter 4]). First, we have

$$TV(\mu, \nu) = \frac{1}{2} \sup \left\{ \left| \int_{\mathcal{X}} f(x) d\mu(x) - \int_{\mathcal{X}} f(x) d\nu(x) \right| : \right. \\ \left. f \text{ is a measurable map from } \mathcal{X} \text{ into } [-1, 1] \right\}. \quad (5.1)$$

Next, suppose that μ and ν are both absolutely continuous with respect to a σ -finite measure α . Let f and g be the Radon–Nikodym derivatives of μ and ν with respect to α . The total variation distance between μ and ν can be alternately expressed as

$$TV(\mu, \nu) = \frac{1}{2} \int_{\mathcal{X}} |f(x) - g(x)| d\alpha(x). \quad (5.2)$$

Finally, recall also the *coupling characterization* of total variation distance: If γ is a probability measure on the product space $\mathcal{X} \times \mathcal{X}$, the *marginal probabilities* of γ are the probability measures γ_1 and γ_2 on \mathcal{X} defined as

$$\gamma_1(A) := \gamma(A \times \mathcal{X}), \quad \gamma_2(A) := \gamma(\mathcal{X} \times A).$$

A *coupling* of μ and ν is a probability measure γ on $\mathcal{X} \times \mathcal{X}$ whose marginal probabilities are μ and ν . The coupling characterization of total variation distance says that

$$TV(\mu, \nu) = \inf \{ \gamma(\{(x, y) : x \neq y\}) : \gamma \text{ is a coupling of } \mu \text{ and } \nu \}. \quad (5.3)$$

There is a standard formula for a coupling that attains the infimum in the above formula. It is as follows. Let α be a σ -finite measure such that both μ and ν are absolutely continuous with respect to α . (For example, we can always take $\alpha = \mu + \nu$.) Let f and g be the Radon–Nikodym derivatives of μ and ν with respect to α . Define

$$h(x) := \min\{f(x), g(x)\}, \\ f_1(x) := f(x) - h(x), \quad g_1(x) := g(x) - h(x). \quad (5.4)$$

Note that $\int (f_1 - g_1) d\alpha = \int (f - g) d\alpha = 0$, and $f_1 + g_1 = |f - g|$. Thus, by (5.2),

$$\int f_1(x) d\alpha(x) = \int g_1(x) d\alpha(x) = TV(\mu, \nu). \quad (5.5)$$

For any $S \in \mathcal{B}(\mathcal{X} \times \mathcal{X})$, the set $\tilde{S} := \{x \in \mathcal{X} : (x, x) \in S\}$ is measurable since the map $x \mapsto (x, x)$ is continuous and hence measurable. So we can legitimately define, for any $S \in \mathcal{B}(\mathcal{X} \times \mathcal{X})$,

$$\gamma(S) := \int_{\tilde{S}} h(x) d\alpha(x) + \frac{1}{TV(\mu, \nu)} \int_S f_1(x) g_1(y) d\alpha^{\otimes 2}(x, y), \quad (5.6)$$

where $\alpha^{\otimes 2}$ stands for the product measure $\alpha \times \alpha$ on $\mathcal{X} \times \mathcal{X}$, and the second term is interpreted as zero if $TV(\mu, \nu) = 0$. It is not hard to check that γ is a probability measure on $\mathcal{X} \times \mathcal{X}$, and that it is a coupling of μ and ν which attains the infimum in (5.3). Also, it is not hard to check that γ does not depend on the choice of α .

The reason why we took the trouble of writing down the explicit form of γ is that we will need it in the following lemma. It says that if (μ, ν) and (μ', ν') are pairs of probability measures such that μ is close to μ' and ν is close to ν' in total variation distance, then the optimal coupling of μ and ν is close to the optimal coupling μ' and ν' in total variation distance. I could not find this result in the literature, so a complete proof is given below.

Lemma 5.1. *Let μ, ν, μ' and ν' be probability measures on a Polish space \mathcal{X} . Let γ be an optimal coupling of μ and ν , and let γ' be an optimal coupling of μ' and ν' , both defined according to the formula (5.6). Then*

$$TV(\gamma, \gamma') \leq 10\sqrt{\max\{TV(\mu, \mu'), TV(\nu, \nu')\}}.$$

Proof. Let α be a σ -finite measure on \mathcal{X} such that μ, ν, μ' and ν' are all absolutely continuous with respect to α . Let f and g be the probability density functions of μ and ν with respect to α , and define f_1, g_1 and h as in (5.4). Let f', g', h', f'_1 and g'_1 be the analogous functions for μ' and ν' . Define

$$a := \max\{TV(\mu, \nu), TV(\mu', \nu')\}, \quad b := \max\{TV(\mu, \mu'), TV(\nu, \nu')\}.$$

Suppose that $TV(\mu, \nu)$ and $TV(\mu', \nu')$ are both nonzero. Then for $S \in \mathcal{B}(\mathcal{X} \times \mathcal{X})$,

$$\begin{aligned} & |\gamma(S) - \gamma'(S)| \\ & \leq \int_{\tilde{S}} |h(x) - h'(x)| d\alpha(x) \\ & \quad + \frac{1}{TV(\mu, \nu)} \int_S |f_1(x)g_1(y) - f'_1(x)g'_1(y)| d\alpha^{\otimes 2}(x, y) \\ & \quad + \left| \frac{1}{TV(\mu, \nu)} - \frac{1}{TV(\mu', \nu')} \right| \int_S f'_1(x)g'_1(y) d\alpha^{\otimes 2}(x, y). \end{aligned} \quad (5.7)$$

Let us now estimate the three terms on the right. By the inequality

$$|\min\{u, v\} - \min\{u', v'\}| \leq |u - u'| + |v - v'|,$$

we get

$$|h(x) - h'(x)| \leq |f(x) - f'(x)| + |g(x) - g'(x)|. \quad (5.8)$$

By (5.2), this gives

$$\int_{\tilde{S}} |h(x) - h'(x)| d\alpha(x) \leq 4b. \quad (5.9)$$

Again using (5.8) and (5.2), note that

$$\int_{\mathcal{X}} |f_1(x) - f'_1(x)| d\alpha(x) \leq 6b, \quad \int_{\mathcal{X}} |g_1(x) - g'_1(x)| d\alpha(x) \leq 6b.$$

Thus,

$$\begin{aligned} & \int_S |f_1(x)g_1(y) - f'_1(x)g'_1(y)| d\alpha^{\otimes 2}(x, y) \\ & \leq \int_{\mathcal{X} \times \mathcal{X}} (|f_1(x) - f'_1(x)|g_1(y) + f'_1(x)|g_1(y) - g'_1(y)|) d\alpha^{\otimes 2}(x, y) \\ & \leq 12b. \end{aligned} \quad (5.10)$$

Next, note that by the triangle inequality for total variation distance [28, Remark 4.4],

$$|TV(\mu, \nu) - TV(\mu', \nu')| \leq TV(\mu, \mu') + TV(\nu, \nu') \leq 2b.$$

This gives

$$\begin{aligned} & \left| \frac{1}{TV(\mu, \nu)} - \frac{1}{TV(\mu', \nu')} \right| \int_S f'_1(x) g'_1(y) d\alpha^{\otimes 2}(x, y) \\ & \leq \frac{2b}{TV(\mu, \nu)TV(\mu', \nu')} \int_{\mathcal{X} \times \mathcal{X}} f'_1(x) g'_1(y) d\alpha^{\otimes 2}(x, y) \\ & = \frac{2b}{TV(\mu, \nu)TV(\mu', \nu')} TV(\mu', \nu')^2 \leq \frac{2b}{TV(\mu, \nu)}, \end{aligned} \quad (5.11)$$

where the identity in the last line holds by (5.5). Using the bounds (5.9), (5.10) and (5.11) in (5.7), we get

$$TV(\gamma, \gamma') \leq 4b + \frac{14b}{TV(\mu, \nu)}.$$

But by the symmetry of the problem, the same bound should hold if we replace $TV(\mu, \nu)$ by $TV(\mu', \nu')$ on the right. Thus,

$$TV(\gamma, \gamma') \leq 4b + 14b \min \left\{ \frac{1}{TV(\mu, \nu)}, \frac{1}{TV(\mu', \nu')} \right\} = 4b + \frac{14b}{a}. \quad (5.12)$$

The above bound was derived under the assumption that $TV(\mu, \nu)$ and $TV(\mu', \nu')$ are nonzero. But it is not hard to check that all the steps go through even if one or both of them are zero. Next, note that for any S ,

$$\begin{aligned} \left| \gamma(S) - \int_{\widetilde{S}} h(x) d\alpha(x) \right| & \leq \frac{1}{TV(\mu, \nu)} \int_{\mathcal{X} \times \mathcal{X}} f_1(x) g_1(y) d\alpha^{\otimes 2}(x, y) \\ & = \frac{1}{TV(\mu, \nu)} TV(\mu, \nu)^2 = TV(\mu, \nu). \end{aligned}$$

Note that the bound holds even if $TV(\mu, \nu) = 0$. Similarly,

$$\left| \gamma'(S) - \int_{\widetilde{S}} h'(x) d\alpha(x) \right| \leq TV(\mu', \nu').$$

Thus, by (5.9),

$$\begin{aligned} |\gamma(S) - \gamma'(S)| & \leq \int_{\widetilde{S}} |h(x) - h'(x)| d\alpha(x) + TV(\mu, \nu) + TV(\mu', \nu') \\ & \leq 4b + 2a. \end{aligned}$$

Since this holds for any S , we get

$$TV(\gamma, \gamma') \leq 4b + 2a. \quad (5.13)$$

The proof is now completed by combining (5.12) and (5.13), as follows. If $a < \sqrt{7b}$, we use (5.13) to get

$$TV(\gamma, \gamma') \leq 4b + 2\sqrt{7b} \leq 10\sqrt{b}.$$

On the other hand, if $a \geq \sqrt{7b}$, we use (5.12) to get

$$TV(\gamma, \gamma') \leq 4b + \frac{14b}{\sqrt{7b}} \leq 10\sqrt{b}.$$

This completes the proof of the lemma. □

6. Conditional Probabilities

Let \mathcal{X} and \mathcal{Y} be two Polish spaces, equipped with their respective Borel σ -algebras. A function $\phi : \mathcal{X} \times \mathcal{B}(\mathcal{Y}) \rightarrow [0, 1]$ is called a *conditional probability* from \mathcal{X} to \mathcal{Y} if the following two conditions are satisfied:

- (1) For each $x \in \mathcal{X}$, $\phi(x, \cdot)$ is a probability measure on \mathcal{Y} .
- (2) For each $B \in \mathcal{B}(\mathcal{Y})$, the map $x \mapsto \phi(x, B)$ is measurable.

If α is a σ -finite measure on \mathcal{Y} such that each $\phi(x, \cdot)$ is absolutely continuous with respect to α with Radon–Nikodym derivative $f(x, \cdot)$, and f satisfies the condition that $(x, y) \mapsto f(x, y)$ is measurable, then we say that $f(x, \cdot)$ are *conditional probability densities* from \mathcal{X} to \mathcal{Y} . We will need the following lemma about conditional probabilities. Although elementary, I could not find the exact statement in the literature.

Lemma 6.1. *Let \mathcal{X} , \mathcal{Y} and ϕ be as above. Then, given any probability measure μ on \mathcal{X} , there is a unique probability measure γ on $\mathcal{X} \times \mathcal{Y}$ satisfying*

$$\gamma(A \times B) = \int_A \phi(x, B) d\mu(x) \quad (6.1)$$

for any $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$. Next, suppose that we have two probability measures μ and μ' and two conditional probabilities ϕ and ϕ' having conditional probability densities f and f' with respect to some probability measure α . Suppose that f and f' are uniformly bounded by a constant a . Let γ and γ' be as in (6.1). Then

$$TV(\gamma, \gamma') \leq aTV(\mu, \mu') + \sup_{x \in \mathcal{X}, y \in \mathcal{Y}} |f(x, y) - f'(x, y)|.$$

Proof. The existence and uniqueness of γ can be established by standard measure-theoretic methods, via Carathéodory's extension theorem [14, Theorem A.1.1.]. For the second part, we proceed as follows. Let $\beta := \mu \times \alpha$. For each $S \in \mathcal{B}(\mathcal{X} \times \mathcal{Y})$, let

$$\tilde{\gamma}(S) := \int_S f(x, y) d\beta(x, y).$$

It is easy to verify that $\tilde{\gamma}$ is a probability measure on $\mathcal{X} \times \mathcal{Y}$, and that it satisfies equation (6.1). So by the uniqueness of γ , we get that $\tilde{\gamma} = \gamma$. Therefore by Fubini's theorem, we have

$$\gamma(S) = \tilde{\gamma}(S) = \int_{\mathcal{X}} \int_{\mathcal{Y}} 1_S(x, y) f(x, y) d\alpha(y) d\mu(x), \quad (6.2)$$

where 1_S denotes the indicator function for the set S . A similar formula holds for γ' . Thus, if we let

$$g(x) := \int_{\mathcal{Y}} 1_S(x, y) f(x, y) d\alpha(y),$$

then g is measurable, and

$$\begin{aligned} |\gamma(S) - \gamma'(S)| &\leq \left| \int_{\mathcal{X}} g(x) d\mu(x) - \int_{\mathcal{X}} g(x) d\mu'(x) \right| \\ &\quad + \int_{\mathcal{X}} \int_{\mathcal{Y}} |f(x, y) - f'(x, y)| d\alpha(y) d\mu'(x). \end{aligned} \quad (6.3)$$

By assumption, there is a constant a such that $f(x, y) \leq a$ for all x and y . Since α is a probability measure, this implies that $0 \leq g(x) \leq a$ for all x . Thus, by (5.1),

$$\left| \int_{\mathcal{X}} g(x) d\mu(x) - \int_{\mathcal{X}} g(x) d\mu'(x) \right| \leq a TV(\mu, \mu'). \quad (6.4)$$

Combining (6.3) and (6.4), we get the required bound. \square

7. Lattice Gauge Theory in a cube

For each $N \geq 1$, let B_N be the cube $\{-N, \dots, N\}^d$. The goal of this section is to investigate some properties of our lattice gauge theory restricted to B_N . Let E_N be the set of positively oriented edges of B_N . Let Ω_N be the set of all functions from E_N into G . Let ∂E_N denote the set of positively oriented boundary edges of B_N . Let $\partial\Omega_N$ denote the set of all functions from ∂E_N into G . Let $E_N^\circ := E_N \setminus \partial E_N$ be the set of positively oriented *interior edges* of B_N . Let Ω_N° be the set of all functions from E_N° into G . Thinking of Ω_N as the Cartesian product of Ω_N° and $\partial\Omega_N$, we will write an element of Ω_N as a pair (ω, δ) , where $\omega \in \Omega_N^\circ$ and $\delta \in \partial\Omega_N$.

Let $H_N : \Omega_N \rightarrow \mathbb{R}$ denote the Hamiltonian of our lattice gauge theory restricted to the cube B_N , viewing it as a function of a configuration $\omega \in \Omega_N^\circ$ and a boundary condition $\delta \in \partial\Omega_N$. Given a measurable function $f : \Omega_N \rightarrow \mathbb{R}$ and a boundary condition $\delta \in \partial\Omega_N$, let

$$\langle f \rangle := \frac{\int_{\Omega_N^\circ} f(\omega, \delta) e^{-\beta H_N(\omega, \delta)} d\lambda_N(\omega)}{\int_{\Omega_N^\circ} e^{-\beta H_N(\omega, \delta)} d\lambda_N(\omega)}, \quad (7.1)$$

where λ_N is the normalized product Haar measure on Ω_N° , provided that the numerator is well-defined. Note that this is a function of δ .

Take any edge $e \in \partial E_N$, and consider the Hamiltonian H_N as a function of only δ_e , fixing the matrices assigned to all other edges. Let $\nabla_e H_N$ denote the gradient of this function (identifying $M_n(\mathbb{C})$ with \mathbb{R}^{2n^2} , as before). Then it is easy to see that each component of $\nabla_e H_N$ is a local function supported on e . We will abbreviate this by simply saying that $\nabla_e H_N$ is a local function supported on e . Moreover, observe that $\|\nabla_e H\|$ can be bounded by a constant that depends only on G and d .

Let $f : \Omega_N \rightarrow \mathbb{R}$ be a bounded measurable function. Let $e \in \partial E_N$ be an edge such that f has no dependence on δ_e . Then using the dominated convergence theorem for Lebesgue integrals, it is not hard to see that the gradient can be moved inside the integrals in the following calculation, giving

$$\begin{aligned} \nabla_e \langle f \rangle &= \frac{\int_{\Omega_N^\circ} f(\omega, \delta) \nabla_e (e^{-\beta H_N(\omega, \delta)}) d\lambda_N(\omega)}{\int_{\Omega_N^\circ} e^{-\beta H_N(\omega, \delta)} d\lambda_N(\omega)} \\ &\quad - \frac{\int_{\Omega_N^\circ} f(\omega, \delta) e^{-\beta H_N(\omega, \delta)} d\lambda_N(\omega) \int_{\Omega_N^\circ} \nabla_e (e^{-\beta H_N(\omega, \delta)}) d\lambda_N(\omega)}{(\int_{\Omega_N^\circ} e^{-\beta H_N(\omega, \delta)} d\lambda_N(\omega))^2} \\ &= -\beta \langle f \nabla_e H_N \rangle + \beta \langle f \rangle \langle \nabla_e H_N \rangle, \end{aligned} \quad (7.2)$$

where the last identity holds because

$$\nabla_e (e^{-\beta H_N(\omega, \delta)}) = -\beta e^{-\beta H_N(\omega, \delta)} \nabla_e H_N(\omega, \delta).$$

Note that $f \nabla_e H_N$ and $\nabla_e H_N$ are matrix-valued functions, so the expectations written above are to be interpreted as matrices of expected values.

In the statement of the following lemma and in all that follows, we will use C, C_1, C_2, \dots to denote positive constants that may depend on G, β, d and π , and nothing else. The values of these constants may change from line to line.

Lemma 7.1. *Let $f : \Omega_N \rightarrow [-1, 1]$ be a local function supported on an edge $e \in \partial E_N$. Take any $r \geq 1$. Then there is a function $g : \partial \Omega_N \rightarrow \mathbb{R}$ that depends only on $\{\delta_u : u \in \partial E_N, \text{dist}(e, u) \leq r\}$, such that for any boundary condition δ , $|\langle f \rangle - g(\delta)| \leq C_1 N^{d-1} e^{-C_2 r}$.*

Proof. Take any $u \in \partial E_N$ such that $\text{dist}(e, u) > r$. Then u is not adjacent to e , meaning that they do not belong to a common plaque. Since $r \geq 1$ and f is a local function supported on e , this shows that f has no dependence of δ_u . So, by the formula (7.2) and the fact that f and $\nabla_u H_N$ are local functions whose magnitudes are bounded by constants that depend only on G and d , and whose supporting edges are separated by a distance greater than r , the correlation decay assumption implies that

$$\|\nabla_u \langle f \rangle\| = |\beta| \|\langle f \nabla_u H_N \rangle - \langle f \rangle \langle \nabla_u H_N \rangle\| \leq C_1 e^{-C_2 r}.$$

Note that this bound holds irrespective of the boundary condition. Therefore, Corollary 4.2 implies that if we replace δ_u by I , the value of $\langle f \rangle$ changes by at most $C_1 e^{-C_2 r}$. We can perform this operation successively to replace δ_u by I for each $u \in \partial E_N$ such that $\text{dist}(e, u) > r$. The total change in $\langle f \rangle$ will be bounded by $C_1 N^{d-1} e^{-C_2 r}$. The new value of $\langle f \rangle$ is a function of $\{\delta_u : u \in \partial E_N, \text{dist}(e, u) \leq r\}$. Let this function be denoted by g . Clearly, this g has the desired property. \square

Take any $e \in \partial E_N$ and any $1 \leq r \leq N/4$. Let $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ be the endpoints of e , in lexicographic order. For each i , let

$$a_i := \begin{cases} x_i - 2r & \text{if } -N + 2r \leq x_i \leq N - 2r, \\ -N & \text{if } x_i < -N + 2r, \\ N - 4r & \text{if } x_i > N - 2r. \end{cases}$$

Let $b_i := a_i + 4r$. Let

$$B(e, r) := ([a_1, b_1] \times \dots \times [a_d, b_d]) \cap \mathbb{Z}^d.$$

Clearly, $B(e, r)$ is a cube. Let $E(e, r)$ denote the set of positively oriented edges of $B(e, r)$ and let $\partial E(e, r)$ denote the positively oriented boundary edges of $B(e, r)$. We will refer to $B(e, r)$ as the r -neighborhood of the edge e (see Figure 5). The following lemma gathers some basic facts about the structure of $B(e, r)$.

Lemma 7.2. *The set $B(e, r)$ defined above is a cube of side-length $4r$, and is a subset of B_N . The edge e is a boundary edge of $B(e, r)$. Any edge of B_N that is adjacent to e is also an edge of $B(e, r)$. Lastly, any $u \in \partial E(e, r) \setminus \partial E_N$ must satisfy $\text{dist}(e, u) > r$.*

Proof. For simplicity, let us write B and E instead of $B(e, r)$ and $E(e, r)$. By construction, a_i and b_i are in $[-N, N]$ for each i . Therefore, $B \subseteq B_N$. Also by construction, $b_i - a_i = 4r$ for each i . Thus, B is a cube of side-length $4r$.

By construction, $x_i \in [a_i, b_i]$ for each i . Thus, $x \in B$. Since x and y are neighbors and y is lexicographically bigger than x , there is exactly one coordinate i such that $y_i = x_i + 1$, and $y_j = x_j$ for all $j \neq i$. Since $y \in B_N$, it must be that $x_i < N$, and

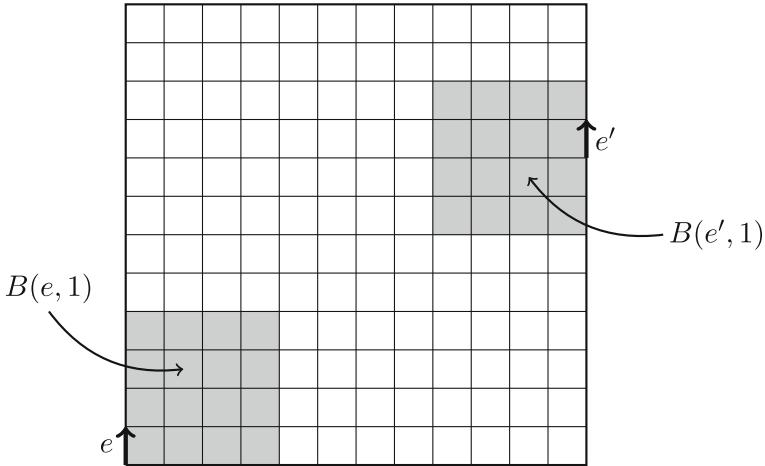


Fig. 5. The 1-neighborhoods of two boundary edges e and e'

hence $y_i \in [a_i, b_i]$. For $j \neq i$, $y_j = x_j \in [a_j, b_j]$. So we conclude that $y \in B$. Since x and y are both in B , we get that $e \in E$. But B is a cube contained in B_N and $e \in \partial E_N$. Therefore e must be a boundary edge of B . Thus, $e \in \partial E$.

Take any $u \in E_N \setminus E$. Let $w = (w_1, \dots, w_d)$ and $z = (z_1, \dots, z_d)$ be the endpoints of u , in lexicographic order. Since $u \notin E$, at least one of w and z is not in B . Suppose that $w \notin B$. Then there is a coordinate j such that $w_j \notin [a_j, b_j]$. That is, either $w_j < a_j$ or $w_j > b_j$. Since $w \in B_N$, the definitions of a_j and b_j now show that $|w_j - x_j| > 2r$. Since $|x_j - y_j| \leq 1$ and $|w_j - z_j| \leq 1$, and $r \geq 1$, this gives

$$\text{dist}(e, u)^2 \geq \left(\frac{w_j + z_j}{2} - \frac{x_j + y_j}{2} \right)^2 > (2r - 1)^2 \geq r^2.$$

A similar argument shows that if $z \notin B$, then $\text{dist}(e, u) > r$. Thus, we conclude that if $u \in E_N \setminus E$, then $\text{dist}(e, u) > r \geq 1$. In particular, u is not adjacent to e .

Finally, take any $u \in \partial E \setminus \partial E_N$. Let $w = (w_1, \dots, w_d)$ and $z = (z_1, \dots, z_d)$ be the endpoints of u , in lexicographic order. Since $u \in \partial E$, there must exist j such that either $w_j = z_j = a_j$ or $w_j = z_j = b_j$. Moreover, since $u \notin \partial E_N$, there must exist j with this property and also satisfying $-N < w_j < N$. Take any such j . Then it is easy to see that $|w_j - x_j| \geq 2r$. Since $|y_j - x_j| \leq 1$ and $r \geq 1$,

$$\begin{aligned} \text{dist}(e, u)^2 &\geq \left(\frac{w_j + z_j}{2} - \frac{x_j + y_j}{2} \right)^2 \\ &= \left(w_j - \frac{x_j + y_j}{2} \right)^2 \geq \left(2r - \frac{1}{2} \right)^2 > r^2. \end{aligned}$$

This completes the proof of the lemma. \square

Take any $e \in \partial E_N$ and any $1 \leq r \leq N/4$. Let $B(e, r)$, $E(e, r)$ and $\partial E(e, r)$ be defined as above. The following lemma gives an upper bound on the correlation between a local function supported on e and a function that depends only on edges outside the r -neighborhood of e .

Lemma 7.3. *Let e and r be as above. Let $f : \Omega_N^\circ \rightarrow [-1, 1]$ be a measurable function that depends only on $\{\omega_u : u \in E_N^\circ \setminus E(e, r)\}$. Let $h : \Omega_N \rightarrow [-1, 1]$ be a local function supported on e . Then under any boundary condition on B_N ,*

$$|\langle fh \rangle - \langle f \rangle \langle h \rangle| \leq C_1 r^{d-1} e^{-C_2 r}.$$

Proof. As before, let us write B and E instead of $B(e, r)$ and $E(e, r)$. Fix some boundary condition δ and let μ be the probability measure defined by our lattice gauge theory in B_N with this boundary condition. Let μ' denote the conditional probability measure given $\{\omega_u : u \in E_N^\circ \setminus E^\circ\}$, where E° is the set of positively oriented interior edges of B . Since a lattice gauge theory is a Markov random field, it is clear that μ' is again a lattice gauge theory on the cube B , with boundary condition δ' on ∂E , where

$$\delta'_u = \begin{cases} \omega_u & \text{if } u \in \partial E \setminus \partial E_N, \\ \delta_u & \text{if } u \in \partial E \cap \partial E_N. \end{cases}$$

Let $\langle h \rangle'$ denote the expected value of h with respect to μ' . By Lemma 7.2, any edge of B_N that is adjacent to e is also an edge of B . Therefore by Lemma 7.1 applied to the cube B , there is a function $g(\delta')$ of the boundary condition δ' , which depends only on $\{\delta'_u : u \in \partial E, \text{dist}(e, u) \leq r\}$, such that

$$|\langle h \rangle' - g(\delta')| \leq C_1 r^{d-1} e^{-C_2 r}.$$

But for any $u \in \partial E \setminus \partial E_N$, Lemma 7.2 tells us that $\text{dist}(e, u) > r$. Thus, $g(\delta')$ depends only on a subset of $\{\delta'_u : u \in \partial E \cap \partial E_N\}$. In particular, $g(\delta')$ is simply a function of the original boundary condition δ and has no dependence on $\{\omega_u : u \in \partial E \setminus \partial E_N\}$. So we can write $g(\delta)$ instead of $g(\delta')$.

Now note that by the tower property of conditional expectation and the fact that f has no dependence on $\{\omega_u : u \in E_N^\circ \cap E\}$, we get $\langle fh \rangle = \langle f \langle h \rangle' \rangle$. Since f maps into $[-1, 1]$, this gives

$$\begin{aligned} |\langle fh \rangle - \langle f \rangle g(\delta)| &= |\langle f(\langle h \rangle' - g(\delta)) \rangle| \\ &\leq |\langle h \rangle' - g(\delta)| \leq C_1 r^{d-1} e^{-C_2 r}. \end{aligned}$$

Similarly, since $\langle h \rangle = \langle \langle h \rangle' \rangle$,

$$\begin{aligned} |\langle f \rangle \langle h \rangle - \langle f \rangle g(\delta)| &\leq |\langle \langle h \rangle' - g(\delta) \rangle| \\ &\leq |\langle h \rangle' - g(\delta)| \leq C_1 r^{d-1} e^{-C_2 r}. \end{aligned}$$

The claim now follows by combining the above inequalities. \square

Combining the above lemma with Corollary 4.2 and equation (7.2), we obtain the following upper bound. It says that the expected value of a function that depends only on edges outside the r -neighborhood of e cannot change by much if δ_e is replaced by a new value.

Corollary 7.4. *Let f and e be as in Lemma 7.3. Then*

$$\|\nabla_e \langle f \rangle\| \leq C_1 r^{d-1} e^{-C_2 r}.$$

Consequently, if δ_e is replaced by any other value δ'_e , the value of $\langle f \rangle$ changes by at most $C_1 r^{d-1} e^{-C_2 r}$.

Proof. Recall that by (7.2), $\nabla_e \langle f \rangle = -\beta \langle f \nabla_e H_N \rangle + \beta \langle f \rangle \langle \nabla_e H_N \rangle$. Also, recall from the discussion in Sect. 7 that $\nabla_e H_N$ is a local function supported on e , whose norm is bounded by a constant that depends only on G and d . These two facts, combined with Lemma 7.3, prove the first claim. For the second, apply the first claim and Corollary 4.2. \square

We will now extend Corollary 7.4 to collections of boundary edges. Let A be any nonempty subset of ∂E_N . Define $E(A, r)$ to be the union of $E(e, r)$ over all $e \in A$.

Lemma 7.5. *Let $E(A, r)$ be as above. Take any measurable function $f : \Omega_N^\circ \rightarrow [-1, 1]$ that depends only on $\{\omega_u : u \in E_N^\circ \setminus E(A, r)\}$. Then for any two boundary conditions δ and δ' that agree on the complement of A , the values of $\langle f \rangle$ differ by at most $C_1 |A| r^{d-1} e^{-C_2 r}$.*

Proof. By Corollary 7.4, $\langle f \rangle$ changes by at most $C_1 r^{d-1} e^{-C_2 r}$ when the value of δ_e is changed for a single $e \in A$. Therefore the cumulative change when δ_e is changed for all $e \in A$ is at most $C_1 |A| r^{d-1} e^{-C_2 r}$. \square

Take any nonempty set $A \subseteq \partial E_N$. Let $\Omega_{A,r}$ be the set of all functions from $E_N^\circ \setminus E(A, r)$ into G , and let $\Omega_{A,r}^*$ be the set of all functions from $E_N^\circ \cap E(A, r)$ into G . Note that Ω_N° can be viewed as the Cartesian product of $\Omega_{A,r}$ and $\Omega_{A,r}^*$. The following corollary of Lemma 7.5 tells us that if μ is the probability measure defined by our lattice gauge theory in B_N , then the marginal probability of μ on $\Omega_{A,r}$ has a very small dependence on $\{\delta_e : e \in A\}$. This is quantified by a bound in total variation distance.

Corollary 7.6. *Let δ and δ' be two boundary conditions on B_N , defining two probability measures μ and μ' according to our lattice gauge theory. Let A be the set of all $e \in \partial E_N$ such that $\delta_e \neq \delta'_e$. Viewing Ω_N° as $\Omega_{A,r} \times \Omega_{A,r}^*$ (defined above), let μ_r and μ'_r be the marginal probabilities of μ and μ' on $\Omega_{A,r}$. Then*

$$TV(\mu_r, \mu'_r) \leq C_1 |A| r^{d-1} e^{-C_2 r}.$$

Proof. Let $f : \Omega_{A,r} \rightarrow [-1, 1]$ be a measurable function. Then f has a natural extension to a function on Ω_N° , which we also denote by f . By Lemma 7.5, $\langle f \rangle$ changes by at most $C_1 |A| r^{d-1} e^{-C_2 r}$ if the boundary condition δ is replaced by δ' . The result now follows by the formula (5.1) for total variation distance. \square

8. Coupling in a Cube

Take any $N \geq 1$ and $1 \leq r \leq N/4$. In this section we give a general prescription for coupling two lattice gauge theories with two different boundary conditions on B_N . This is the first step towards defining a similar coupling on a slab, which is the main component of the proof of Theorem 2.4.

Let us endow the space of all pairs of boundary conditions on B_N with the usual Euclidean metric (viewing it as a subset of a Euclidean space of sufficiently large dimension). Also, let us endow the space of all probability measures on $\Omega_N^\circ \times \Omega_N^\circ$ with the total variation metric. These topologies generate Borel σ -algebras on the two spaces. We will say that a map from one space to the other is measurable if it is measurable with respect to these σ -algebras.

Lemma 8.1. *Let δ and δ' be two boundary conditions on B_N and let μ and μ' be the probability measures defined by our lattice gauge theory with these boundary conditions. Let A be the set of all $e \in \partial E_N$ such that $\delta_e \neq \delta'_e$. Let $E(A, r)$ be the union of $E(e, r)$ over all $e \in A$, where $E(e, r)$ is the r -neighborhood defined in the previous section. Then there is a coupling γ of μ and μ' such that*

$$\begin{aligned} \gamma(\{(\omega, \omega') : \omega_u \neq \omega'_u \text{ for some } u \in E_N^\circ \setminus E(A, r)\}) \\ \leq C_1 |A| r^{d-1} e^{-C_2 r}. \end{aligned} \quad (8.1)$$

Moreover, the coupling can be defined in such a way that the map $(\delta, \delta') \mapsto \gamma$ is measurable in the sense defined above.

Proof. As in Corollary 7.6, let us view Ω_N° as $\Omega_{A,r} \times \Omega_{A,r}^*$. Let μ_r and μ'_r be the marginal probabilities of μ and μ' on $\Omega_{A,r}$. By Corollary 7.6 and the coupling characterization of total variation distance, there is a coupling γ_r of μ_r and μ'_r satisfying

$$\gamma_r(\{(x, x') \in \Omega_{A,r} \times \Omega_{A,r} : x \neq x'\}) \leq C_1 |A| r^{d-1} e^{-C_2 r}. \quad (8.2)$$

Now, given any boundary condition δ , our lattice gauge theory defines a conditional probability ϕ_δ from $\Omega_{A,r}$ to $\Omega_{A,r}^*$. For $x \in \Omega_{A,r}$ and $y \in \Omega_{A,r}^*$, let $f_\delta(x, y)$ denote the conditional probability density (with respect to normalized product Haar measure) of ϕ_δ at the point (x, y) . Then the function $g_{\delta, \delta'}(x, x', y, y') := f_\delta(x, y) f_{\delta'}(x', y')$ is a conditional probability density from $\Omega_{A,r} \times \Omega_{A,r}$ to $\Omega_{A,r}^* \times \Omega_{A,r}^*$, corresponding to the conditional probability $\phi_{\delta, \delta'}((x, x'), \cdot) := \phi_\delta(x, \cdot) \times \phi_{\delta'}(x', \cdot)$. Using the first part of Lemma 6.1 with these conditional probability densities, we can now extend γ_r to a probability measure γ on $\Omega_N^\circ \times \Omega_N^\circ$. By the definition of γ and the property (8.2) of γ_r , we see that γ satisfies (8.1). Let us now verify that γ is a coupling of μ and μ' . Take any $S \in \mathcal{B}(\Omega_N^\circ)$ of the form $S_1 \times S_2$, where $S_1 \in \mathcal{B}(\Omega_{A,r})$ and $S_2 \in \mathcal{B}(\Omega_{A,r}^*)$. Then

$$\begin{aligned} \gamma(S_1 \times \Omega_{A,r} \times S_2 \times \Omega_{A,r}^*) &= \int_{S_1 \times \Omega_{A,r}} \phi_\delta(x, S_2) \phi_{\delta'}(x', \Omega_{A,r}^*) d\gamma_r(x, x') \\ &= \int_{S_1 \times \Omega_{A,r}} \phi_\delta(x, S_2) d\gamma_r(x, x') \\ &= \int_{S_1} \phi_\delta(x, S_2) d\mu_r(x) = \mu(S_1 \times S_2). \end{aligned}$$

Similarly, $\gamma(\Omega_{A,r} \times S_1 \times \Omega_{A,r}^* \times S_2) = \mu'(S_1 \times S_2)$. From this, it is easy to see using the uniqueness part of Carathéodory's extension theorem that γ is indeed a coupling of μ and μ' .

Finally, to prove measurability of $(\delta, \delta') \mapsto \gamma$, we argue as follows. For any $A \subseteq \partial E_N$, let $\Delta(A)$ be the set of all pairs of boundary conditions (δ, δ') such that $A = \{e : \delta_e \neq \delta'_e\}$. The space Δ of all pairs of boundary conditions is the disjoint union of these sets, and each $\Delta(A)$ is a measurable subset of Δ . So it suffices to prove that $(\delta, \delta') \mapsto \gamma$ is measurable on each $\Delta(A)$. We will, in fact, show that this map is continuous on every $\Delta(A)$.

So take any $A \subseteq \partial E_N$. First, suppose that A is nonempty. It is not hard to verify directly from the definition of lattice gauge theory that $\delta \mapsto \mu$ is a continuous map, and therefore so is $(\delta, \delta') \mapsto (\mu, \mu')$. The definition of total variation distance makes it clear that $\mu \mapsto \mu_r$ is continuous, and therefore so is the map $(\mu, \mu') \mapsto (\mu_r, \mu'_r)$. By

the inequality from Lemma 5.1, $(\mu_r, \mu'_r) \mapsto \gamma_r$ is continuous. Combining, we get that $(\delta, \delta') \mapsto \gamma_r$ is a continuous map on $\Delta(A)$.

Now, it is not hard to see that $(\delta, x, y) \mapsto f_\delta(x, y)$ is a continuous map, and therefore uniformly continuous since its domain is compact. Consequently, the map $(\delta, x, y, \delta', x', y') \mapsto g_{\delta, \delta'}(x, x', y, y')$ is uniformly continuous. Thus, by the second part of Lemma 6.1, $(\delta, \delta', \gamma_r) \mapsto \gamma$ is continuous. But as observed above, γ_r is itself a continuous function of (δ, δ') . Thus, $(\delta, \delta') \mapsto \gamma$ is continuous.

If A is empty, the proof is simpler. In this case, $\gamma_r = \gamma$, so we already have the continuity of $(\delta, \delta') \mapsto \gamma$ from the first step above. \square

9. Local Update Map on a Slab

Given any positive integers M and N , define the slab

$$S_{M,N} := \{-N, \dots, N\} \times \{-M, \dots, M\}^{d-1}.$$

The faces of $S_{M,N}$ corresponding to $x_1 = N$ and $x_1 = -N$ will be called the ‘temporal’ faces (because the first coordinate denotes time), and the remaining part of the boundary will be called the ‘spatial boundary’ of $S_{M,N}$.

Let $E_{M,N}$ be the set of positively oriented edges of $S_{M,N}$, $E_{M,N}^\circ$ be the set of positively oriented interior edges, and $\partial E_{M,N}$ to be the set of positively oriented boundary edges. The spatial boundary will be denoted by $\partial' E_{M,N}$. The set of all maps from $E_{M,N}^\circ$ into G will be denoted by $\Omega_{M,N}^\circ$, and the set of all maps from $\partial E_{M,N}$ into G by $\partial \Omega_{M,N}$.

Take any M and N , and let δ and δ' be two boundary conditions on $S_{M,N}$ that agree on the temporal faces. Let μ and μ' be the probability measures defined by these two boundary conditions, according to our lattice gauge theory. Take any $1 \leq r \leq N/4$. We will now define the *local update map*, which maps any coupling of μ and μ' to a ‘better’ coupling of μ and μ' . This is the second step towards the construction of a coupling in a slab.

Take any translate B of the cube B_N that is contained in $S_{M,N}$, such that B has no intersection with the spatial boundary of $S_{M,N}$. (Assume that $M > N$, so that such a B exists.) Then B must be of the form

$$B = ([-N, N] \times [a_2, b_2] \times \dots \times [a_d, b_d]) \cap \mathbb{Z}^d,$$

where a_i and b_i are integers such that $-M < a_i < b_i < M$ and $b_i - a_i = 2N$ for each i . Let \mathcal{B} be the set of all such B . (See Figure 6.)

Let E° be the set of interior edges of B . Let Ω_B be the set of all functions from $E_{M,N}^\circ \setminus E^\circ$ into G , and let Ω_B^* be the set of all functions from E° into G . Let γ be a coupling of μ and μ' . Viewing $\Omega_{M,N}^\circ \times \Omega_{M,N}^\circ$ as $\Omega_B \times \Omega_B \times \Omega_B^* \times \Omega_B^*$, let γ_B be the marginal probability of γ on $\Omega_B \times \Omega_B$.

Let μ_B and μ'_B be the marginal probabilities of μ and μ' on Ω_B , viewing Ω_N° as $\Omega_B \times \Omega_B^*$. We claim that μ_B and μ'_B are also the marginal probabilities of the probability measure γ_B . To see this, take any $S \in \mathcal{B}(\Omega_B)$. Then

$$\begin{aligned} \gamma_B(S \times \Omega_B) &= \gamma(S \times \Omega_B \times \Omega_B^* \times \Omega_B^*) \\ &= \mu(S \times \Omega_B^*) \quad (\text{since } \gamma \text{ is a coupling of } \mu \text{ and } \mu') \\ &= \mu_B(S), \end{aligned}$$

and similarly, $\gamma_B(\Omega_B \times S) = \mu'_B(S)$.

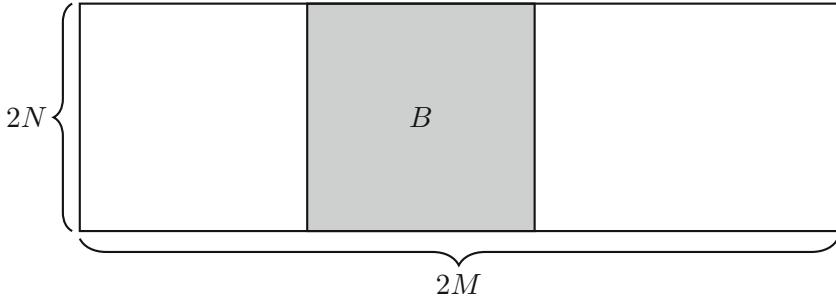


Fig. 6. A cube of width $2N$ sitting inside the slab $S_{M,N}$

Now, any pair $(x, x') \in \Omega_B \times \Omega_B$ defines a pair of boundary conditions on B . By Lemma 8.1, this pair of boundary conditions lets us define a probability measure $\phi((x, x'), \cdot)$ on $\Omega_B^* \times \Omega_B^*$, which is a coupling of the two lattice gauge theories μ_x and $\mu_{x'}$ defined by the two boundary conditions. By the measurability assertion of Lemma 8.1, ϕ is a conditional probability from $\Omega_B \times \Omega_B$ to $\Omega_B^* \times \Omega_B^*$. Using γ_B and ϕ , we can now invoke Lemma 6.1 to define a probability measure $\tilde{\gamma}$ on $\Omega_{M,N}^* \times \Omega_{M,N}^*$. We claim that $\tilde{\gamma}$ is a coupling of μ and μ' . To see this, take any $S_1 \in \mathcal{B}(\Omega_B)$ and $S_2 \in \mathcal{B}(\Omega_B^*)$. Then

$$\begin{aligned}\tilde{\gamma}(S_1 \times \Omega_B \times S_2 \times \Omega_B^*) &= \int_{S_1 \times \Omega_B} \phi((x, x'), S_2 \times \Omega_B^*) d\gamma_B(x, x') \\ &= \int_{S_1 \times \Omega_B} \mu_x(S_2) d\gamma_B(x, x') \\ &= \int_{S_1} \mu_x(S_2) d\mu_B(x) = \mu(S_1 \times S_2),\end{aligned}$$

where the second-to-last identity holds because μ_B is a marginal probability of γ_B , as deduced above. Similarly,

$$\tilde{\gamma}(\Omega_B \times S_1 \times \Omega_B^* \times S_2) = \mu'(S_1 \times S_2).$$

Thus, $\tilde{\gamma}$ is indeed a coupling of μ and μ' . Let us denote $\tilde{\gamma}$ by $\tau_B(\gamma)$, viewing it as a function of γ . The map τ_B will be called the *local update map* corresponding to the cube B . Note that the definition of τ_B depends not only on B , but also on M, N, δ, δ' and r , but we will consider those as fixed.

For any coupling γ of μ and μ' , and any edge $e \in E_{M,N}^o$, let

$$\rho(\gamma, e) := \gamma(\{(\omega, \omega') : \omega_e \neq \omega'_e\}). \quad (9.1)$$

For any edge $e \in E_{M,N}^o$, let $U(e)$ be the set of all edges $u \in E_{M,N}^o$ such that e and u are both in some common cube of width $4r$, and let $V(e)$ be the set of all $u \in E_{M,N}^o$ such that both e and u are in some common cube of width $2N$. The following lemma estimates how ρ changes under a local update map.

Lemma 9.1. *Let all notation be as above. Take any $B \in \mathfrak{B}$, any $e \in E_{M,N}^o$ and any coupling γ of μ and μ' . Let E^o be the set of interior edges of B , and let $\partial' E$ be the*

spatial boundary of B . If $e \notin E^\circ$, then $\rho(\tau_B(\gamma), e) = \rho(\gamma, e)$. On the other hand, if $e \in E^\circ$, then

$$\rho(\tau_B(\gamma), e) \leq C_1 N^{d-1} e^{-C_2 r} \sum_{u \in \partial' E \cap V(e)} \rho(\gamma, u) + \sum_{u \in \partial' E \cap U(e)} \rho(\gamma, u).$$

Proof. For simplicity of notation, let $\tilde{\gamma} := \tau_B(\gamma)$. Take any $(\omega, \omega') \in \Omega_{M,N}^\circ \times \Omega_{M,N}^\circ$. Write (ω, ω') as an element (x, x', y, y') of $\Omega_B \times \Omega_B \times \Omega_B^* \times \Omega_B^*$. If $e \notin E^\circ$, then $\omega_e \neq \omega'_e$ means that $x_e \neq x'_e$. So in this case, by the definitions of γ and $\tilde{\gamma}$,

$$\begin{aligned} \rho(\tilde{\gamma}, e) &= \tilde{\gamma}(\{(\omega, \omega') : \omega_e \neq \omega'_e\}) \\ &= \gamma_B(\{(x, x') : x_e \neq x'_e\}) = \rho(\gamma, e). \end{aligned}$$

Next, suppose that $e \in E^\circ$. Then $\omega_e \neq \omega'_e$ means that $y_e \neq y'_e$. Thus,

$$\begin{aligned} \rho(\tilde{\gamma}, e) &= \tilde{\gamma}(\{(\omega, \omega') : \omega_e \neq \omega'_e\}) \\ &= \int_{\Omega_B \times \Omega_B} \phi((x, x'), \{(y, y') : y_e \neq y'_e\}) d\gamma_B(x, x'). \end{aligned} \quad (9.2)$$

Take any (x, x') . Let

$$p := \phi((x, x'), \{(y, y') : y_e \neq y'_e\}).$$

Let A be the set of $u \in \partial' E$ such that $x_u \neq x'_u$. Since δ and δ' agree on the temporal faces, and B has no intersection with the spatial boundary of $S_{M,N}$, we see that A consists of all boundary edges of B where the boundary conditions defined by x and x' disagree. So by the construction of ϕ (using Lemma 8.1), we see that if $U(e)$ has no intersection with A , then

$$p \leq C_1 |A| r^{d-1} e^{-C_2 r} \leq C_1 N^{d-1} e^{-C_2 r} |A|.$$

If $U(e)$ intersects A , we simply use $p \leq 1$. Combining, we get that for any (x, x') and e ,

$$p \leq C_1 N^{d-1} e^{-C_2 r} |A| + |U(e) \cap A|. \quad (9.3)$$

But note that by the definition of γ_B ,

$$\begin{aligned} \int_{\Omega_B \times \Omega_B} |A| d\gamma_B(x, x') &= \int_{\Omega_{M,N}^\circ \times \Omega_{M,N}^\circ} |\{u \in \partial' E : \omega_u \neq \omega'_u\}| d\gamma(\omega, \omega') \\ &= \sum_{u \in \partial' E} \gamma(\{(\omega, \omega') : \omega_u \neq \omega'_u\}) \\ &= \sum_{u \in \partial' E} \rho(\gamma, u). \end{aligned}$$

We can replace $\partial' E$ by $\partial' E \cap V(e)$ in the above sum since the two sets are equal for any $e \in E^\circ$. Thus,

$$\int_{\Omega_B \times \Omega_B} |A| d\gamma_B(x, x') = \sum_{u \in \partial' E \cap V(e)} \rho(\gamma, u). \quad (9.4)$$

Similarly,

$$\int_{\Omega_B \times \Omega_B} |U(e) \cap A| d\gamma_B(x, x') = \sum_{u \in \partial' E \cap U(e)} \rho(\gamma, u). \quad (9.5)$$

Using the information obtained from (9.3), (9.4) and (9.5) in (9.2), we get the required upper bound. \square

10. Global Update Map on a Slab

Let us continue to use the notations introduced in the previous section. In particular, let us fix M, N, δ, δ' and r as before. We define the *global update map* on the space of all couplings of μ and μ' as

$$\tau(\gamma) := \frac{1}{|\mathfrak{B}|} \sum_{B \in \mathfrak{B}} \tau_B(\gamma).$$

If \mathfrak{B} is empty (which happens if $M \leq N$), just let $\tau(\gamma) := \gamma$. For any edge e , let $\mathfrak{B}(e)$ be the set of all $B \in \mathfrak{B}$ such that e is an interior edge of B . The following lemma quantifies how the global update map ‘improves’ any given coupling. This is the third step towards the construction of the coupling in a slab.

Lemma 10.1. *Suppose that $M > N$, so that \mathfrak{B} is nonempty. Then for any coupling γ of μ and μ' , and any $e \in E_{M,N}^\circ$,*

$$\begin{aligned} \rho(\tau(\gamma), e) &\leq \left(1 - \frac{|\mathfrak{B}(e)|}{|\mathfrak{B}|}\right) \rho(\gamma, e) + \frac{C_1 N^{2d-3} e^{-C_2 r}}{|\mathfrak{B}|} \sum_{u \in V(e)} \rho(\gamma, u) \\ &\quad + \frac{C_3 N^{d-2}}{|\mathfrak{B}|} \sum_{u \in U(e)} \rho(\gamma, u). \end{aligned}$$

Proof. Note that for any edge e , there can be at most CN^{d-2} cubes $B \in \mathfrak{B}$ such that $e \in \partial' E$ (where, as before, $\partial' E$ denotes the spatial boundary of B). Therefore by Lemma 9.1,

$$\begin{aligned} \sum_{B \in \mathfrak{B}} \rho(\tau_B(\gamma), e) &\leq (|\mathfrak{B}| - |\mathfrak{B}(e)|) \rho(\gamma, e) + C_1 N^{2d-3} e^{-C_2 r} \sum_{u \in V(e)} \rho(\gamma, u) \\ &\quad + C_3 N^{d-2} \sum_{u \in U(e)} \rho(\gamma, u). \end{aligned}$$

Since the function ρ is linear in γ ,

$$\rho(\tau(\gamma), e) = \frac{1}{|\mathfrak{B}|} \sum_{B \in \mathfrak{B}} \rho(\tau_B(\gamma), e).$$

Combining the above identity with the inequality from the previous display, we get the required bound. \square

11. Coupling in a Slab

Let M, N, δ, δ' and r be fixed as in the previous two sections. We will now construct a coupling of μ and μ' using an infinite number of iterations of the global update map τ . Start with the coupling $\gamma_0 := \mu \times \mu'$. For each n , let $\gamma_n := \tau^n(\gamma)$. Since $\{\gamma_n\}_{n \geq 0}$ is a sequence of probability measures on the compact Polish space $\Omega_{M,N}^\circ \times \Omega_{M,N}^\circ$, it has a subsequence $\{\gamma_{n_k}\}_{k \geq 1}$ converging weakly to a limit γ . It is not hard to see that γ is also a coupling of μ and μ' . The following lemma gives the main property of this coupling that will be useful for us.

Lemma 11.1. *There is some N_0 depending only on G, β and d , such that if $N \geq N_0$, and r is chosen to be $\lfloor (\log N)^2 \rfloor$, then for any M and any $e \in E_{M,N}^\circ$,*

$$\rho(\gamma, e) \leq C_1 \exp\left(-\frac{C_2 \text{dist}(e, \partial' E_{M,N})}{N}\right), \quad (11.1)$$

where $\text{dist}(e, \partial' E_{M,N})$ is the minimum value of $\text{dist}(e, u)$ over all $u \in \partial' E_{M,N}$.

Proof. Note that the distance of any edge e to the spatial boundary of $S_{M,N}$ is bounded above by a constant times M , where the constant depends only on d . This shows that if $M \leq N$, the proof is trivial, because the right side is bounded below by a positive constant depending only on d , whereas the left side is bounded above by 1. So let us assume that $M > N$, which renders \mathfrak{B} nonempty. For each e , let

$$p(e) := \limsup_{n \rightarrow \infty} \rho(\gamma_n, e).$$

Since $\{(\omega, \omega') : \omega_e \neq \omega'_e\}$ is a relatively open subset of $\Omega_{M,N}^\circ \times \Omega_{M,N}^\circ$, the portmanteau lemma for weak convergence [14, Theorem 3.9.1] gives us that

$$\begin{aligned} \rho(\gamma, e) &\leq \liminf_{k \rightarrow \infty} \rho(\gamma_{n_k}, e) \\ &\leq \limsup_{k \rightarrow \infty} \rho(\gamma_{n_k}, e) \leq \limsup_{n \rightarrow \infty} \rho(\gamma_n, e) = p(e). \end{aligned} \quad (11.2)$$

Thus, it suffices to get an upper bound for $p(e)$. Now, by Lemma 10.1,

$$\begin{aligned} p(e) &\leq \left(1 - \frac{|\mathfrak{B}(e)|}{|\mathfrak{B}|}\right)p(e) + \frac{C_1 N^{2d-3} e^{-C_2 r}}{|\mathfrak{B}|} \sum_{u \in V(e)} p(u) \\ &\quad + \frac{C_3 N^{d-2}}{|\mathfrak{B}|} \sum_{u \in U(e)} p(u) \end{aligned}$$

for every e . If $\mathfrak{B}(e)$ is nonempty, this can be rearranged as

$$p(e) \leq \frac{C_1 N^{2d-3} e^{-C_2 r}}{|\mathfrak{B}(e)|} \sum_{u \in V(e)} p(u) + \frac{C_3 N^{d-2}}{|\mathfrak{B}(e)|} \sum_{u \in U(e)} p(u). \quad (11.3)$$

Let E' be the set of all $e \in E_{M,N}^\circ$ such that $\text{dist}(e, \partial' E_{M,N}) > 3N$. Clearly, it suffices to prove the required bound for $e \in E'$, because if $\text{dist}(e, \partial' E_{M,N}) \leq 3N$, then the right side of (11.1) is bounded below by a positive constant depending only on d . So

let us take any such e . It is easy to see that $|\mathfrak{B}(e)| > CN^{d-1}$. Also, $|V(e)| \leq CN^d$ and $|U(e)| \leq Cr^d$. Therefore by (11.3), we get

$$p(e) \leq C_1 N^{2d-2} e^{-C_2 r} \max_{u \in V(e)} p(u) + \frac{C_3 r^d}{N} \max_{u \in U(e)} p(u).$$

Now, if N is large enough (depending on G , β and d) and $r = \lfloor (\log N)^2 \rfloor$, then the coefficients in front both the maxima on the right are less than $1/4$. Since $U(e) \subseteq V(e)$, this implies that there is some $e_1 \in V(e)$ such that

$$p(e) \leq \frac{p(e_1)}{2}.$$

If e_1 is also in E' , then by the above argument applied to e_1 instead of e , we get an edge e_2 such that $p(e_1) \leq p(e_2)/2$, and so on. Since $e_{i+1} \in V(e_i)$ for each i (with $e_0 = e$), it follows that each e_{i+1} is within distance CN from e_i . Therefore, the minimum i such that $e_i \notin E'$ must be at least $CN^{-1} \text{dist}(e, \partial' E_{M,N})$. The final $p(e_i)$ can simply be bounded by 1. This completes the proof of the lemma. \square

12. Proof of Theorem 2.4

Let N_0 be as in Lemma 11.1, and take any $N \geq N_0$ and any $M \geq 1$. Take any two boundary conditions δ and δ' for $S_{M,N}$ that agree on the temporal faces. Let μ and μ' be the probability measures defined by our lattice gauge theory on $S_{M,N}$ with these boundary conditions.

By Lemma 11.1, it is possible to construct a coupling of μ and μ' such that if (ω, ω') is a pair of coupled configurations, then for any edge e in the slab, the chance of $\omega_e \neq \omega'_e$ is exponentially small in the distance of e from the spatial boundary. From this, it follows that if f is a bounded measurable function of $\{\omega_e\}_{e \in A}$ for some fixed set A , then $|\int f d\mu - \int f d\mu'|$ falls off exponentially in the distance of A from the spatial boundary of $S_{M,N}$. In particular, if we let $M \rightarrow \infty$, then it is impossible to produce a sequence of boundary conditions that differ only on the spatial boundaries, such that $\int f d\mu - \int f d\mu'$ does not tend to zero. And from this, it follows that for any boundary condition on the infinite slab $\{-N, \dots, N\} \times \mathbb{Z}^{d-1}$, there is a unique Gibbs measure for our lattice gauge theory on the slab. Uniqueness of the Gibbs measure trivially implies that center symmetry cannot be spontaneously broken. This proves the first assertion of Theorem 2.4.

The above argument also implies that if f is a vertical chain variable associated with the vertical chain through the center of $S_{M,N}$, its expected value in $S_{M,N}$ must tend to its expected value under the unique Gibbs measure on the infinite slab exponentially fast in M as $M \rightarrow \infty$. But its expected value under the Gibbs measure must be zero due to center symmetry and the fact that f transforms to cf under the center transform defined in Sect. 3. This shows that the function V from Lemma 3.1 must have at least linear growth. Thus, from the proof of Theorem 2.2, we now see that in the setting of Theorem 2.4, we have

$$|\langle W_\ell \rangle| \leq e^{(C_1 - C_2 R)T} \tag{12.1}$$

for any rectangular loop with side-lengths $R \leq T$. This almost proves the second assertion of Theorem 2.4, except that we have to remove the $C_1 T$ term from the exponent.

The first step towards eliminating this term is the following lemma. Recall that π is a finite-dimensional irreducible unitary representation of G which acts nontrivially on the center of G .

Lemma 12.1. *Let λ_0 denote the normalized Haar measure on G . Let ρ be a probability density with respect to λ_0 . Suppose that there are positive constants a and b such that $a \leq \rho(g) \leq b$ for all $g \in G$. Then there is some $\varepsilon \in (0, 1)$ depending only on G , π , a , and b , such that for any function $f : G \rightarrow \mathbb{C}$ which is a composition of π followed by a linear map,*

$$\left| \int_G f(g) \rho(g) d\lambda_0(g) \right|^2 \leq (1 - \varepsilon)^2 \int_G |f(g)|^2 \rho(g) d\lambda_0(g).$$

Proof. Let $\lambda_0^{\otimes 2}$ denote the normalized product Haar measure on $G \times G$. Then

$$\begin{aligned} & \frac{1}{2} \int_{G \times G} |f(g) - f(g')|^2 \rho(g) \rho(g') d\lambda_0^{\otimes 2}(g, g') \\ &= \frac{1}{2} \int_{G \times G} (|f(g)|^2 - f(g) \overline{f(g')} - \overline{f(g)} f(g') + |f(g')|^2) \rho(g) \rho(g') d\lambda_0^{\otimes 2}(g, g') \\ &= \int_G |f(g)|^2 \rho(g) d\lambda_0(g) - \left| \int_G f(g) \rho(g) d\lambda_0(g) \right|^2. \end{aligned} \quad (12.2)$$

Let g_0 be an element of the center of G such that $\pi(g_0)$ is not the identity operator. As in the proof of Theorem 2.2, we note that by Schur's lemma, $\pi(g_0)$ must be cI for some $c \neq 1$. Since f is the composition of π followed by a linear map, the invariance of the Haar measure gives us

$$\begin{aligned} & \int_{G \times G} |f(g) - f(g')|^2 \rho(g) \rho(g') d\lambda_0^{\otimes 2}(g, g') \\ &= \int_{G \times G} |f(g) - f(g_0 g')|^2 \rho(g) \rho(g_0 g') d\lambda_0^{\otimes 2}(g, g') \\ &= \int_{G \times G} |f(g) - cf(g')|^2 \rho(g) \rho(g_0 g') d\lambda_0^{\otimes 2}(g, g'). \end{aligned}$$

Since $a \leq \rho(g) \leq b$ for all g , we have

$$\rho(g_0 g') \geq a \geq \frac{a}{b} \rho(g').$$

Applying this to the previous display gives

$$\begin{aligned} & \int_{G \times G} |f(g) - f(g')|^2 \rho(g) \rho(g') d\lambda_0^{\otimes 2}(g, g') \\ &\geq \frac{a}{b} \int_{G \times G} |f(g) - cf(g')|^2 \rho(g) \rho(g') d\lambda_0^{\otimes 2}(g, g'). \end{aligned}$$

But trivially, since $a/b \leq 1$,

$$\begin{aligned} & \int_{G \times G} |f(g) - f(g')|^2 \rho(g) \rho(g') d\lambda_0^{\otimes 2}(g, g') \\ &\geq \frac{a}{b} \int_{G \times G} |f(g) - f(g')|^2 \rho(g) \rho(g') d\lambda_0^{\otimes 2}(g, g'). \end{aligned}$$

Combining the last two displays, we get

$$\begin{aligned} & \int_{G \times G} |f(g) - f(g')|^2 \rho(g) \rho(g') d\lambda_0^{\otimes 2}(g, g') \\ & \geq \frac{a}{2b} \int_{G \times G} (|f(g) - f(g')|^2 + |f(g) - cf(g')|^2) \rho(g) \rho(g') d\lambda_0^{\otimes 2}(g, g'). \end{aligned}$$

But by the inequality $|w - z|^2 \leq 2|w|^2 + 2|z|^2$, we have

$$|f(g) - f(g')|^2 + |f(g) - cf(g')|^2 \geq \frac{|1 - c|^2}{2} |f(g')|^2.$$

Therefore,

$$\begin{aligned} & \int_{G \times G} |f(g) - f(g')|^2 \rho(g) \rho(g') d\lambda_0^{\otimes 2}(g, g') \\ & \geq \frac{a|1 - c|^2}{4b} \int_{G \times G} |f(g')|^2 \rho(g) \rho(g') d\lambda_0^{\otimes 2}(g, g') \\ & = \frac{a|1 - c|^2}{4b} \int_G |f(g)|^2 \rho(g) d\lambda_0(g). \end{aligned}$$

The proof is now easily completed by combining the above inequality with the identity (12.2). \square

In what follows, we will need some basic facts about matrix norms. First, note that for any $A, B \in M_n(\mathbb{C})$, the Cauchy–Schwarz inequality implies that

$$|\mathrm{Tr}(AB)| \leq \|A\| \|B\|. \quad (12.3)$$

Let $\|A\|_{op}$ denote the L^2 operator norm of a matrix $A \in M_n(\mathbb{C})$, defined as

$$\|A\|_{op} := \sup\{\|Ax\| : x \in \mathbb{C}^n, \|x\| = 1\}.$$

It is easy to see that the operator norm satisfies

$$\|AB\|_{op} \leq \|A\|_{op} \|B\|_{op}. \quad (12.4)$$

Take any $A, B \in M_n(\mathbb{C})$. Let b_1, \dots, b_n be the columns of B . Then

$$\|AB\|^2 = \sum_{i=1}^n \|Ab_i\|^2 \leq \sum_{i=1}^n \|A\|_{op}^2 \|b_i\|^2 = \|A\|_{op}^2 \|B\|^2.$$

As a consequence of this inequality and the inequalities (12.3) and (12.4), we get that for any sequence $A_1, \dots, A_k \in M_n(\mathbb{C})$, where $k \geq 3$,

$$\begin{aligned} |\mathrm{Tr}(A_1 A_2 \cdots A_k)| & \leq \|A_1 \cdots A_{k-1}\| \|A_k\| \\ & \leq \|A_1\|_{op} \|A_2\|_{op} \cdots \|A_{k-2}\|_{op} \|A_{k-1}\| \|A_k\|. \end{aligned} \quad (12.5)$$

We will now use Lemma 12.1 to show that for any edge e , the conditional expectation of $\pi(\omega_e)$ given $\{\omega_u : u \neq e\}$ is a matrix whose operator norm is strictly less than 1. Moreover, the gap is uniformly bounded below by a constant.

Lemma 12.2. Consider any Gibbs measure for our lattice gauge theory on \mathbb{Z}^d . For any edge e , let $\langle \pi(\omega_e) \rangle'$ denote the matrix of conditional expectations of the entries of $\pi(\omega_e)$ given $\{\omega_u : u \neq e\}$. There is a constant $\varepsilon \in (0, 1)$, depending only on G, β, π and d , such that $\|\langle \pi(\omega_e) \rangle'\|_{op} \leq 1 - \varepsilon$.

Proof. It is not hard to see that the conditional probability density (with respect to Haar measure) of ω_e given $\{\omega_u : u \neq e\}$ is bounded above and below by two positive constants a and b , which depend only on G, β and d . Take any $x \in \mathbb{C}^m$ such that $\|x\| = 1$. Then each component of the vector $\pi(\omega_e)x$ is a linear function of $\pi(\omega_e)$. Therefore by Lemma 12.1,

$$\|\langle \pi(\omega_e) \rangle'\|^2 \leq (1 - \varepsilon)^2 \langle \|\pi(\omega_e)x\|^2 \rangle',$$

where $\varepsilon \in (0, 1)$ depends only on G, β, π and d . Since $\pi(\omega_e)$ is a unitary matrix, $\|\pi(\omega_e)x\| = \|x\| = 1$. Thus, $\|\langle \pi(\omega_e) \rangle'x\| = \|\langle \pi(\omega_e) \rangle'\| \leq 1 - \varepsilon$. Taking supremum over x , we get the desired result. \square

We will now use the previous lemma to prove the perimeter law for Wilson loop expectations. A proof of the perimeter law already appears in an old paper of [33]; however, that result is conditional on a certain uniqueness assumption about the Gibbs measure. The result given below is unconditional, showing that the perimeter law upper bound holds in complete generality.

Lemma 12.3. For any rectangular loop ℓ with side-lengths R and T , $|\langle W_\ell \rangle| \leq C_1 e^{-C_2(R+T)}$ for any Gibbs measure of our lattice gauge theory on \mathbb{Z}^d .

Proof. Without loss of generality, suppose that $R \leq T$. Let e_1, e_2, \dots, e_k be the edges of ℓ , where the first T edges belong to a side of length T . Let $\langle \cdot \rangle'$ denote the conditional expectation given $\{\omega_e : e \notin \{e_1, \dots, e_T\}\}$. Under this conditioning, $\omega_{e_1}, \dots, \omega_{e_T}$ are independent random matrices because no two of these edges share a common plaquette. Moreover, for any $1 \leq i \leq T$, the conditional distribution of ω_{e_i} given $\{\omega_e : e \notin \{e_1, \dots, e_T\}\}$ is the same as the conditional distribution of ω_{e_i} given $\{\omega_e : e \neq e_i\}$. Applying the conditional independence to each summand in the formula for the trace in the following display, we get

$$\begin{aligned} \langle W_\ell \rangle' &= \langle \text{Tr}(\pi(\omega_{e_1}) \cdots \pi(\omega_{e_k})) \rangle' \\ &= \text{Tr}(\langle \pi(\omega_{e_1}) \rangle' \langle \pi(\omega_{e_2}) \rangle' \cdots \langle \pi(\omega_{e_T}) \rangle' \pi(\omega_{e_{T+1}}) \cdots \pi(\omega_{e_k})). \end{aligned}$$

Thus, by Lemma 12.2, the matrix inequality (12.5) (observing that $k - 3 \geq r$), and the fact that $\|\pi(\omega_e)\|_{op} = 1$ for all e (since π is a unitary representation), we get

$$\begin{aligned} |\langle W_\ell \rangle'| &= |\text{Tr}(\langle \pi(\omega_{e_1}) \rangle' \langle \pi(\omega_{e_2}) \rangle' \cdots \langle \pi(\omega_{e_T}) \rangle' \pi(\omega_{e_{T+1}}) \cdots \pi(\omega_{e_k}))| \\ &\leq \|\langle \pi(\omega_{e_1}) \rangle'\|_{op} \cdots \|\langle \pi(\omega_{e_T}) \rangle'\|_{op} \|\pi(\omega_{e_{T+1}}) \cdots \pi(\omega_{e_k})\| \\ &\quad \cdots \|\pi(\omega_{e_{k-2}})\|_{op} \|\pi(\omega_{e_{k-1}})\| \|\pi(\omega_{e_k})\| \\ &\leq C(1 - \varepsilon)^T \leq C(1 - \varepsilon)^{(R+T)/2}, \end{aligned}$$

where $C > 0$ and $\varepsilon \in (0, 1)$ depend only on G, β, π and d . Thus, the same bound holds for $|\langle W_\ell \rangle|$. \square

Finally, we are ready to complete the proof of the second assertion of Theorem 2.4. Let ℓ be a rectangular loop with side-lengths $R \leq T$. Consider any Gibbs measure for our lattice gauge theory on \mathbb{Z}^d . By inequality (12.1),

$$|\langle W_\ell \rangle| \leq e^{C_1 T - C_2 R T}. \quad (12.6)$$

On the other hand, by Lemma 12.3,

$$|\langle W_\ell \rangle| \leq C_3 e^{-C_4(R+T)} \leq C_3 e^{-C_4 T}. \quad (12.7)$$

Let $a := C_4/(C_1 + C_4)$. Since $a \in [0, 1]$, we may combine (12.6) and (12.7) as follows:

$$\begin{aligned} |\langle W_\ell \rangle| &\leq (e^{C_1 T - C_2 R T})^a (C_3 e^{-C_4 T})^{1-a} \\ &= C_3^{1-a} e^{-C_2 a R T}, \end{aligned}$$

where the last equality holds because $C_1 a - C_4(1 - a) = 0$. This completes the proof of Theorem 2.4.

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References

1. Aizenman, M., Chayes, J.T., Chayes, L., Fröhlich, J., Russo, L.: On a sharp transition from area law to perimeter law in a system of random surfaces. *Commun. Math. Phys.* **92**(1), 19–69 (1983)
2. Balian, R., Drouffe, J.M., Itzykson, C.: Gauge fields on a lattice. I. General outlook. *Phys. Rev. D* **10**(10), 3376–3395 (1974)
3. Billingsley, P.: *Convergence of Probability Measures*, 2nd edn. Wiley, New York (1999)
4. Borgs, C.: Translation symmetry breaking in four-dimensional lattice gauge theories. *Commun. Math. Phys.* **96**(2), 251–284 (1984)
5. Borgs, C., Seiler, E.: Lattice Yang-Mills theory at nonzero temperature and the confinement problem. *Commun. Math. Phys.* **91**(3), 329–380 (1983)
6. Brydges, D.C., Federbush, P.: Debye screening. *Commun. Math. Phys.* **73**(3), 197–246 (1980)
7. Cao, S.: Wilson loop expectations in lattice gauge theories with finite gauge groups. *Commun. Math. Phys.* **380**, 1439–1505 (2020)
8. Chatterjee, S.: Rigorous solution of strongly coupled $SO(N)$ lattice gauge theory in the large N limit. *Commun. Math. Phys.* **366**, 203–268 (2019)
9. Chatterjee, S.: Yang-Mills for probabilists. In: *Probability and Analysis in Interacting Physical Systems*, pp. 1–16. Springer, Cham (2019)
10. Chatterjee, S.: Wilson loops in Ising lattice gauge theory. *Commun. Math. Phys.* **377**, 307–340 (2020)
11. Chatterjee, S., Jafarov, J.: The $1/N$ expansion for $SO(N)$ lattice gauge theory at strong coupling. arXiv preprint [arXiv:1604.04777](https://arxiv.org/abs/1604.04777) (2016)
12. Dobrushin, R.L., Shlosman, S.B.: Constructive criterion for the uniqueness of Gibbs field. In: *Statistical Physics and Dynamical Systems*, pp. 347–370. Birkhäuser Boston, Boston, MA (1985)
13. Durhuus, B., Fröhlich, J.: A connection between ν -dimensional Yang-Mills theory and $(\nu - 1)$ -dimensional, non-linear σ -models. *Commun. Math. Phys.* **75**(2), 103–151 (1980)
14. Durrett, R.: *Probability: Theory and Examples*, 4th edn. Cambridge University Press, Cambridge (2010)
15. Forström, M. P., Lenells, J., Viklund, F.: Wilson loops in finite Abelian lattice gauge theories. arXiv preprint [arXiv:2001.07453](https://arxiv.org/abs/2001.07453) (2020)

16. Fradkin, E., Shenker, S.H.: Phase diagrams of lattice gauge theories with Higgs fields. *Phys. Rev. D* **19**(12), 3682–3697 (1979)
17. Fröhlich, J.: Confinement in \mathbb{Z}_n lattice gauge theories implies confinement in $SU(n)$ lattice Higgs theories. *Phys. Lett. B* **83**(2), 195–198 (1979)
18. Fröhlich, J., Spencer, T.: Massless phases and symmetry restoration in abelian gauge theories and spin systems. *Commun. Math. Phys.* **83**(3), 411–454 (1982)
19. Georgii, H.-O.: *Gibbs Measures and Phase Transitions*, 2nd edn. Walter de Gruyter & Co., Berlin (2011)
20. Glimm, J., Jaffe, A.: *Quantum Physics. A Functional Integral Point of View*, 2nd edn. Springer, New York (1987)
21. Göpfert, M., Mack, G.: Proof of confinement of static quarks in 3-dimensional $U(1)$ lattice gauge theory for all values of the coupling constant. *Commun. Math. Phys.* **82**(4), 545–606 (1982)
22. Greensite, J.: *An Introduction to the Confinement Problem*. Springer, Heidelberg (2011)
23. Greensite, J., Lautrup, B.: First-order phase transition in four-dimensional $SO(3)$ lattice gauge theory. *Phys. Rev. Lett.* **47**(1), 9–11 (1981)
24. Guth, A.H.: Existence proof of a nonconfining phase in four-dimensional $U(1)$ lattice gauge theory. *Phys. Rev. D* **21**(8), 2291–2307 (1980)
25. Hall, B.C.: *Lie Groups, Lie Algebras, and Representations. An Elementary Introduction*, 2nd edn. Springer, Cham (2015)
26. Jaffe, A., Witten, E.: Quantum Yang–Mills theory. In: *The Millennium Prize Problems*, pp. 129–152, Clay Math. Inst., Cambridge, MA (2006)
27. Kotecký, R., Shlosman, S.B.: First-order phase transitions in large entropy lattice models. *Commun. Math. Phys.* **83**(4), 493–515 (1982)
28. Levin, D.A., Peres, Y., Wilmer, E.L.: *Markov Chains and Mixing Times*. American Mathematical Society, Providence, RI
29. Morningstar, C.J., Peardon, M.: Glueball spectrum from an anisotropic lattice study. *Phys. Rev. D* **60**(3), 034509 (1999)
30. Osterwalder, K., Seiler, E.: Gauge field theories on a lattice. *Ann. Phys.* **110**(2), 440–471 (1978)
31. Patrascioiu, A., Seiler, E.: Is the 2D $O(3)$ nonlinear σ model asymptotically free? *Phys. Lett. B* **430**(3–4), 314–319 (1998)
32. Seiler, E.: *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*. Springer, Berlin (1982)
33. Simon, B., Yaffe, L.G.: Rigorous perimeter law upper bound on Wilson loops. *Phys. Lett. B* **115**(2), 145–147 (1982)
34. 't Hooft, G.: On the phase transition towards permanent quark confinement. *Nuclear Phys. B* **138**(1), 1–25 (1978)
35. Wegner, F.J.: Duality in generalized Ising models and phase transitions without local order parameters. *J. Math. Phys.* **12**(10), 2259–2272 (1971)
36. Wilson, K.G.: Confinement of quarks. *Phys. Rev. D* **10**(8), 2445–2459 (1974)

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