A Dual Characterization of Observability for Stochastic Systems ⋆

Jin Won Kim * **Prashant G. Mehta** *

* *Coordinated Science Laboratory and Department of Mechanical Science and Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801 USA (e-mail: jkim684@illinois.edu; mehtapg@illinois.edu)*

Abstract: This paper is concerned with the definition and characterization of the observability for a continuous-time hidden Markov model where the state evolves as a continuous-time Markov process on a compact state space and the observation process is modeled as nonlinear function of the state corrupted by a Gaussian measurement noise. The main technical tool is based on the recently discovered duality relationship between minimum variance estimation and stochastic optimal control: The observability is defined as a dual of the controllability for a certain backward stochastic differential equation. Based on the dual formulation, a test for observability is presented and related to literature. The proposed duality-based framework allows one to easily relate and compare the linear and the nonlinear systems. A side-by-side summary of this relationship is given in a tabular form (Table 1).

Keywords: Observability, Stochastic systems, Duality, Backward stochastic differential equations
MSC 2010: 93B07, 60G35, 93B28

1. INTRODUCTION

This paper is concerned with the definition of observability for a partially observed pair of continuous-time stochastic processes (X,Z) where the state X is a Markov process and the observation Z is a nonlinear function of the state corrupted by the Gaussian measurement noise. The precise mathematical model appears in the main body of the paper.

In deterministic linear time-invariant (LTI) settings, observability (more generally detectability) and its dual relationship to the controllability are foundational concepts in linear systems theory; (Kalman, 1960). It is an important property that a model must satisfy to construct asymptotically stable observers (Kailath et al., 2000). For a partially observed stochastic LTI system, the detectability property of its deterministic counterpart is necessary and also sufficient (under mild additional conditions) to deduce results on asymptotic stability of the optimal (Kalman) filter; (Ocone and Pardoux, 1996).

Generalization of these concepts to nonlinear deterministic and stochastic systems has been an area of historical and current research interest, e.g. (Hermann and Krener, 1977; Moraal and Grizzle, 1995; Liu and Bitmead, 2011). In settings more general than this paper, the fundamental definition of observability is due to van Handel (2009a,b). The definition is used to establish results on asymptotic stability of the nonlinear filter (Chigansky et al., 2009; van Handel, 2010). Certain extensions of van Handel's observability definition appear in recent papers by McDonald and Yüksel (2019).

In this paper, we utilize the recently discovered duality relationship between minimum variance estimation and stochastic optimal control (see Kim et al. (2019)) to define observability as a dual to the controllability. The latter property is somewhat 'natural' because it bears close resemblance to the definition of controllability in deterministic LTI settings. The definition of observability is obtained by using duality. In finite state-space settings, certain Kalman-type rank conditions are derived to verify the observability property. These conditions are shown to be identical to the ones reported by van Handel (2009a) but derived here using alternate means.

The original contributions of our paper are as follows: We presented the observability in terms of the controllability property of the *dual*. This type of duality is different from prior works on duality such as Fleming and Mitter (1982); Mitter and Newton (2003); Goodwin et al. (2005); Todorov (2008). We relate our definition of observability with literature (van Handel, 2009a). Note the background and motivation for our work is different from Van Handel's approach which is entirely probabilistic in nature. The upshot of our work is that we can establish parallels between linear and nonlinear cases (see Table 1). This is expected to be useful in several ways, e.g., to obtain approximation algorithms and for stability analysis of the filter.

The remainder of this paper is organized as follows: The background on the classical deterministic LTI model appears in Sec. 2. The nonlinear model is introduced in Sec. 3 and its stochastic observability defined and discussed in Sec. 4. The finite state case is illustrated in Sec. 5. The conclusions appear in Sec. 6. All the proofs are contained in the Appendix.

2. BACKGROUND: DUALITY IN LINEAR SYSTEMS

In linear algebra, it is an elementary fact that the range space of a matrix is orthogonal to the null space of its transpose. In functional analysis, the closed range theorem provides the necessary generalization of this elementary fact in infinite-dimensional settings. The theorem (Hutson et al., 2005, Theorem 6.5.10) states that

$$\overline{\mathsf{R}(\mathcal{L})} = \mathsf{N}(\mathcal{L}^{\dagger})^{\perp}$$

⋆ Financial support from the NSF grant 1761622 and the ARO grant W911NF1810334 is gratefully acknowledged.

Table 1. Comparison of the controllability–observability duality for linear and nonlinear systems

	Linear-deterministic case (Sec. 2)	**Nonlinear-stochastic case** (Sec. 4)
Signal space	$\mathcal{U} = L^2([0,T];\mathbb{R}^m)$ $\langle u,v\rangle_{\mathcal{U}} = \int_0^T u_t^\top v_t \,\mathrm{d}t$	$\mathcal{U} = L^2_{\mathcal{Z}}([0,T];\mathbb{R}^m)$ $\langle U,V\rangle_{\mathcal{U}} = \tilde{\mathsf{E}}\Big(\int_0^T U_t^\top V_t \,\mathrm{d}t\Big)$
Function space	$\mathcal{Y} = \mathbb{R}^d$ $\langle x,y\rangle_{\mathcal{Y}} = x^\top y$	$\mathcal{Y} = C(\mathbb{S})$, $\mathcal{Y}^\dagger = \mathcal{M}(\mathbb{S})$ $\langle \mu,y\rangle_{\mathcal{Y}} = \mu(y)$
Linear operator	$\mathcal{L}:\mathcal{U}\to\mathcal{Y}$ $u\mapsto y_0$ by ODE (2)	$\mathcal{L}:\mathcal{U}\times\mathbb{R}\to\mathcal{Y}$ $(U,c)\mapsto Y_0$ by BSDE (9)
Adjoint operator	$\mathcal{L}^\dagger:\mathcal{Y}\to\mathcal{U}$ $x_0\mapsto z_t$ by ODE (3)	$\mathcal{L}^\dagger:\mathcal{Y}^\dagger\to\mathcal{U}\times\mathbb{R}$, $\tilde{\pi}_0\mapsto(\tilde{\pi}_t(h),\tilde{\pi}_0(1))$ by Zakai equation (11)
Observability	$\mathsf{R}(\mathcal{L}) = \mathcal{Y} \iff \mathsf{N}(\mathcal{L}^\dagger) = \{0\}$ $\iff$ $He^{At}x_0^{(1)} \equiv He^{At}x_0^{(2)} \Rightarrow x_0^{(1)} = x_0^{(2)}$ (4)	$\overline{\mathsf{R}(\mathcal{L})} = \mathcal{Y} \iff \mathsf{N}(\mathcal{L}^\dagger) = \{0\}$ $\iff$ $\mathsf{E}^\mu(h(X_t)\vert\mathcal{Z}_t) = \mathsf{E}^\nu(h(X_t)\vert\mathcal{Z}_t) \Rightarrow \mu = \nu$ (O3)

where $\overline{\mathsf{R}(\mathcal{L})}$ is closure of the range space of a bounded linear operator $\mathcal{L}$ and $\mathsf{N}(\mathcal{L}^\dagger)$ is the null space of its adjoint operator $\mathcal{L}^\dagger$. This dual relationship is of fundamental importance to understand the duality between controllability and observability. The overall procedure is as follows:

(1) Define the appropriate function spaces and the associated linear operators; and
(2) Express controllability and observability properties in terms of range space and null space of these operators.

We briefly review this well known procedure first in the classical settings.

Function spaces: Denote $\mathcal{U} := L^2([0,T];\mathbb{R}^m)$ to be the Hilbert space of $\mathbb{R}^m$-valued (input or output) square-integrable signals on the time interval $[0,T]$. The space is equipped with the inner product $\langle u,v\rangle_{\mathcal{U}} = \int_0^T u_t^\top v_t \,\mathrm{d}t$ for $u,v\in\mathcal{U}$. Denote $\mathcal{Y} := \mathbb{R}^d$ to be the Euclidean space equipped with the standard inner product $\langle y_0,x_0\rangle_{\mathcal{Y}} := y_0^\top x_0$ for $y_0,x_0\in\mathcal{Y}$.

Operators: For given matrices $A\in\mathbb{R}^{d\times d}$ and $H\in\mathbb{R}^{m\times d}$ define a linear operator $\mathcal{L}:\mathcal{U}\to\mathcal{Y}$ as follows:

$$\mathcal{L}u = \int_0^T e^{A^\top t}H^\top u_t \,\mathrm{d}t =: y_0$$

The definition of the adjoint operator $\mathcal{L}^\dagger:\mathcal{Y}\to\mathcal{U}$ follows from the following calculation:

$$\langle \mathcal{L}u, x_0\rangle_{\mathcal{Y}} = y_0^\top x_0 = \int_0^T u_t^\top He^{At}x_0 \,\mathrm{d}t = \langle u, \mathcal{L}^\dagger x_0\rangle_{\mathcal{U}} \tag{1}$$

Therefore,

$$\big(\mathcal{L}^\dagger x_0\big)(t) = He^{At}x_0 =: z_t \quad \text{for } t\in[0,T]$$

Controllability and observability: The operator $\mathcal{L}$ defines the map from a given input signal $u = \{u_t : 0\le t\le T\}$ to the initial condition y_0 for the linear system [1]:

$$-\dot{y}_t = A^\top y_t + H^\top u_t, \quad y_T = 0 \tag{2}$$

The range space $\mathsf{R}(\mathcal{L})$ is referred to as the *controllable subspace*. The system (2) is said to be *controllable* if $\mathsf{R}(\mathcal{L}) = \mathcal{Y}$.

[1] The system (2) is an example of a backward ordinary differential equation (ODE) because the terminal condition at time $t = T$ is set (to zero in this case).

The adjoint operator $\mathcal{L}^\dagger$ defines the map from a given initial condition x_0 to the observation signal $z = \{z_t : 0\le t\le T\}$ for the linear system

$$\dot{x}_t = Ax_t, \quad \text{with init. cond. } x_0 \tag{3a}$$
$$z_t = Hx_t \tag{3b}$$

The system (3) (henceforth referred to as the linear model (A,H)) is said to be *observable* if $\mathsf{N}(\mathcal{L}^\dagger) = \{0\}$.

By the closed-range theorem (or more directly by simply using (1)), $\mathsf{R}(\mathcal{L}) = \mathsf{N}(\mathcal{L}^\dagger)^\perp$. Therefore, the system (3) is observable if and only if the system (2) is controllable. This is useful in the following ways:

(i) Definition of observability: as the property of the dual system being controllable.
(ii) Geometric interpretation of non-observability: If the controllable subspace $\mathsf{R}(\mathcal{L}) \subsetneq \mathbb{R}^d$ then there exists a non-zero vector $\tilde{x}_0\in\mathsf{N}(\mathcal{L}^\dagger)$ such that $y_0^\top\tilde{x}_0 = 0$ for all $y_0\in\mathsf{R}(\mathcal{L})$. The vector $\tilde{x}_0$ has an interpretation of being the "un-observable" direction in the following sense: For any given $x_0\in\mathbb{R}^d$, $He^{At}x_0 \equiv He^{At}(x_0+\tilde{x}_0)$ for all $t\in[0,T]$. This in turn provides an equivalent definition of observability: The model (A,H) is observable if
$$He^{At}x_0^{(1)} \equiv He^{At}x_0^{(2)} \;\forall\, t\in[0,T] \text{ implies } x_0^{(1)} = x_0^{(2)} \tag{4}$$
(iii) Tests for observability: By the use of the Cayley-Hamilton theorem,
$$\mathsf{R}(\mathcal{L}) = \operatorname{span}\big\{H^\top, A^\top H^\top, \ldots, (A^\top)^{d-1}H^\top\big\} \tag{5}$$
This provides a straightforward test to verify observability: The model (A,H) is observable if the span on the right-hand side of (5) is $\mathbb{R}^d$.

The aim of this paper is to repeat the above program—viz., (i) the definition of the function spaces $\mathcal{U}$ and $\mathcal{Y}$; (ii) the definition of the linear operator $\mathcal{L}$ and its adjoint $\mathcal{L}^\dagger$; (iii) the mathematical characterization of the controllable subspace $\mathsf{R}(\mathcal{L})$; and (iv) its use in definition and geometric interpretation of the observability—for a partially observed nonlinear stochastic system. A summary of the paper appears in the form of a comparison between the linear-deterministic and the nonlinear-stochastic system in Table 1.

3. PROBLEM FORMULATION

3.1 Model & notation

The nonlinear model is defined for a pair of continuous-time stochastic processes denoted as (X,Z). The details of the model are as follows:

(i) The state $X = \{X_t : 0 \le t \le T\}$ is a Markov process that evolves in a compact state-space $\mathbb{S}$ [2]. The generator of the Markov process X is denoted as $\mathcal{A}$.

(ii) The observation process $Z = \{Z_t : 0 \le t \le T\}$ is defined according to the following model:

$$Z_t = \int_0^t h(X_s)\,\mathrm{d}s + W_t \tag{6}$$

where $h : \mathbb{S} \to \mathbb{R}^m$ is a given observation function and $\{W_t : t \ge 0\}$ is an m-dimensional Wiener process (w.p.). It is assumed that W is independent of X.

(iii) We refer to the above model as the nonlinear model $(\mathcal{A},h)$.

Notation: We denote $\mathcal{Z}_t := \sigma(\{Z_s : s \le t\})$ to be the σ-algebra generated by the observations up to time t and $\mathcal{Z} := \{\mathcal{Z}_t : 0 \le t \le T\}$ is the entire filtration.

The law of (X,Z) is denoted as P with the associated expectation operator E. To emphasize the model for initial condition X_0, we use P^μ to denote the law of (X,Z) when the initial condition $X_0 \sim \mu$.

For the state-space $\mathbb{S}$, we let $\mathcal{B}(\mathbb{S})$ denote the Borel σ-algebra on $\mathbb{S}$; $\mathcal{M}(\mathbb{S})$ is the vector space of (signed) Radon (bounded and regular) measures on $\mathcal{B}(\mathbb{S})$; and $\mathcal{P}(\mathbb{S}) \subset \mathcal{M}(\mathbb{S})$ is the set of probability measures. $C(\mathbb{S})$ is used to denote the dual space of $\mathcal{M}(\mathbb{S})$, which is identified by continuous functions on $\mathbb{S}$ (Treves, 1967, Ch. 21). Throughout this paper, we will use the notation:

$$\mu(f) := \int_{\mathbb{S}} f(x)\mu(\mathrm{d}x)$$

to denote the integral of a measurable function f with respect to the measure μ. It is the natural duality paring between $\mathcal{M}(\mathbb{S})$ and $C(\mathbb{S})$.

3.2 Preliminaries

The main concern of this paper is to define and characterize observability for the nonlinear model $(\mathcal{A},h)$. In his paper, van Handel (2009a) proposes the following probabilistic definition [3] of observability for stochastic processes (X,Z):

Definition 1. Suppose X is a Markov process defined on a compact set $\mathbb{S}$ and Z is defined according to model (6). Suppose P^μ and P^ν are two laws of the process (X,Z) with initial measure $X_0 \sim \mu$ and $X_0 \sim \nu$, respectively. The model is said to be *P-observable* if

$$\mathsf{P}^\mu\big|_{\mathcal{Z}_T} = \mathsf{P}^\nu\big|_{\mathcal{Z}_T} \quad \Rightarrow \quad \mu = \nu \tag{7}$$

where $\mathsf{P}^\mu\big|_{\mathcal{Z}_T}$ denotes the restriction of the probability measure P^μ to the σ-algebra $\mathcal{Z}_T$.

[2] The results of this paper are expected to carry over to locally compact spaces.

[3] The definition in van Handel (2009a) applies to a more general class of stochastic processes (X,Z) whereby the independent increment of the measurement noise may not be of the additive Gaussian form (as assumed here).

Before presenting the main result, it is useful to review some concepts from the theory of nonlinear filtering (Xiong, 2008, Ch. 5):

Change of measure: Given P, define a new measure $\tilde{\mathsf{P}}$ according to the Radon-Nikodyn derivative

$$\frac{\mathrm{d}\tilde{\mathsf{P}}}{\mathrm{d}\mathsf{P}}(\omega) := \exp\Big(-\int_0^T h^\top(X_t)\,\mathrm{d}Z_t + \tfrac{1}{2}\int_0^T |h(X_t)|^2\,\mathrm{d}t\Big)$$

By the Girsanov theorem, Z is a $\tilde{\mathsf{P}}$ Wiener process. For a given function f, the un-normalized filter is defined by

$$\sigma_t(f) := \tilde{\mathsf{E}}(D_t f(X_t)|\mathcal{Z}_t)$$

where $\tilde{\mathsf{E}}(\cdot)$ denotes the expectation operator with respect to the new measure $\tilde{\mathsf{P}}$ and

$$D_t = \exp\Big(\int_0^t h^\top(X_s)\,\mathrm{d}Z_s - \tfrac{1}{2}\int_0^t |h(X_s)|^2\,\mathrm{d}s\Big)$$

The un-normalized filter $\sigma_t(f)$ solves the Zakai equation of nonlinear filtering. The nonlinear filter is given by

$$\pi_t(f) := \mathsf{E}(f(X_t)|\mathcal{Z}_t) = \frac{\sigma_t(f)}{\sigma_t(1)} \tag{8}$$

where $1(x) \equiv 1\ \forall\ x \in \mathbb{S}$ denotes the unit constant function. As before, we use superscript (e.g., σ_t^μ, π_t^μ) to emphasize dependence on the initial measure (μ) of X_0.

4. MAIN RESULT: STOCHASTIC OBSERVABILITY

4.1 Function spaces

In nonlinear settings, the signal space $\mathcal{U} = L^2_{\mathcal{Z}}([0,T];\mathbb{R}^m)$ is the Hilbert space of $\mathbb{R}^m$-valued stochastic processes on $[0,T]$. The subscript $\mathcal{Z}$ denotes the fact that the signals are (forward) adapted to the filtration $\mathcal{Z}$. The space is equipped with the inner product

$$\langle U,V\rangle_{\mathcal{U}} := \tilde{\mathsf{E}}\Big(\int_0^T U_t^\top V_t\,\mathrm{d}t\Big)$$

The expectation $\tilde{\mathsf{E}}$ is with respect to the measure $\tilde{\mathsf{P}}$. For the proof that $\mathcal{U}$ is a Hilbert space with respect to this inner product (Le Gall, 2016, p. 99).

The space $\mathcal{Y} = C(\mathbb{S})$ and its dual $\mathcal{Y}^\dagger = \mathcal{M}(\mathbb{S})$. For a function $y \in C(\mathbb{S})$ and a measure $\mu \in \mathcal{M}(\mathbb{S})$, the dual pairing is as follows:

$$\langle \mu, f\rangle_{\mathcal{Y}} = \mu(f) = \int_{\mathbb{S}} f(x)\mu(\mathrm{d}x)$$

A side-by-side comparison of the signal space and the function space for the linear and nonlinear cases appears as first two rows in Table 1.

4.2 Controllability

Parallel to the linear case, we define controllable subspace as the range space of a bounded linear operator. For this purpose, we introduce the following backward stochastic differential equation (BSDE):

$$\begin{aligned} -\mathrm{d}Y_t(x) &= \big(\mathcal{A}Y_t(x) + h^\top(x)(U_t + V_t(x))\big)\,\mathrm{d}t - V_t^\top(x)\,\mathrm{d}Z_t \\ Y_T(x) &= c1(x) \quad \forall\, x \in \mathbb{S} \end{aligned} \tag{9}$$

where $c \in \mathbb{R}$ and the input signal $U \in \mathcal{U}$. The solution $(Y,V) := \{(Y_t(x),V_t(x)) : t \in [0,T],\ x \in \mathbb{S}\}$ of the BSDE is (forward) adapted to the filtration $\mathcal{Z}$. For the purposes of this paper, well-posedness (existence, uniqueness and regularity) of the solution $(Y,V) \in L^2_{\mathcal{Z}}([0,T];C(\mathbb{S})) \times L^2_{\mathcal{Z}}([0,T];C(\mathbb{S}))$ is assumed;

cf., (Ma and Yong, 1997). The BSDE is the nonlinear counterpart of the backward ode (2) in the LTI setting. The justification for considering the BSDE (9) appears in Appendix A where our prior work (Kim et al., 2019) on the topic of duality is briefly reviewed.

The bounded linear operator[4] $\mathcal{L}:\mathcal{U}\times\mathbb{R}\to\mathcal{Y}$ is defined through the solution of the BSDE (9) as follows:

$$\mathcal{L}(U,\,c) = Y_0 \tag{10}$$

and its range space $\mathsf{R}(\mathcal{L}) = \{Y_0 \in \mathcal{Y} : U \in \mathcal{U}, c \in \mathbb{R}\}$ is referred to as the *controllable space*. The BSDE (9) is said to be *controllable* if $\mathsf{R}(\mathcal{L})$ is dense in $\mathcal{Y}$.

In finite state-space settings, when the state space $\mathbb{S}$ is of cardinality d, $\mathsf{R}(\mathcal{L})$ is a subspace of $\mathbb{R}^d$. Therefore, in this setting, the system is *controllable* if $\mathsf{R}(\mathcal{L}) = \mathbb{R}^d$.

Duality is used to propose an indirect definition of observability as follows:

Definition 2. The nonlinear model $(\mathcal{A},h)$ is said to be *observable* if

$$\mathsf{R}(\mathcal{L}) \text{ is dense in } \mathcal{Y} \tag{O1}$$

4.3 Observability

We develop a more direct definition of observability by considering the dual operator. In the Prop. 1 (stated below), it is shown that the adjoint to the BSDE (9) is the *Zakai equation*:

$$\tilde{\pi}_t(f) = \tilde{\pi}_0(f) + \int_0^t \tilde{\pi}_s(\mathcal{A}f)\,\mathrm{d}s + \int_0^t \tilde{\pi}_s(h^\top f)\,\mathrm{d}Z_s \quad \forall f \in \mathcal{Y} \tag{11}$$

where the initial condition $\tilde{\pi}_0 \in \mathcal{M}(\mathbb{S})$ is given[5]. For a given function $f \in \mathcal{Y}$, the solution of the Zakai equation (11) is denoted as $\tilde{\pi}(f) := \{\tilde{\pi}_t(f) : 0 \le t \le T\}$. In finite state-space settings, the Zakai equation is simply a linear SDE on $\mathbb{R}^d$ with initial measure $\tilde{\pi}_0 \in \mathbb{R}^d$.

The following proposition is proved in Appendix B.1:

Proposition 1. Consider the linear operator (10). Its adjoint $\mathcal{L}^\dagger : \mathcal{Y}^\dagger \to \mathcal{U}\times\mathbb{R}$ is given by

$$\mathcal{L}^\dagger \tilde{\pi}_0 = (\tilde{\pi}(h),\, \tilde{\pi}_0(1))$$

where $\tilde{\pi}(h) = \{\tilde{\pi}_t(h) : 0 \le t \le T\}$ is the solution of the Zakai equation (11) with $f = h$ and the initial measure $\tilde{\pi}_0 \in \mathcal{Y}^\dagger$.

For the purposes of defining observability, the adjoint's null space $\mathsf{N}(\mathcal{L}^\dagger) = \{\tilde{\pi}_0 \in \mathcal{Y}^\dagger : \tilde{\pi}(h) = 0,\ \tilde{\pi}_0(1) = 0\}$ is of interest. In the finite state-space settings, $\mathsf{N}(\mathcal{L}^\dagger)$ is a subspace of $\mathbb{R}^d$.

The dual of definition (O1) is as follows:

Definition 3. The nonlinear model $(\mathcal{A},h)$ is said to be *observable* if

$$\mathsf{N}(\mathcal{L}^\dagger) = \{0\} \tag{O2}$$

The two definitions (O1) and (O2) are equivalent: By the closed range theorem $\overline{\mathsf{R}(\mathcal{L})} = \mathsf{N}(\mathcal{L}^\dagger)^\perp$. If the controllable subspace $\overline{\mathsf{R}(\mathcal{L})} \subsetneq \mathcal{Y}$ then there exists a non-zero measure $\tilde{\pi}_0 \in \mathsf{N}(\mathcal{L}^\dagger)$ such that $\tilde{\pi}_0(Y_0) = 0$ for all $Y_0 \in \overline{\mathsf{R}(\mathcal{L})}$. The measure $\tilde{\pi}_0$ has an interpretation of being the *un-observable measure* in the following sense: For given $\mu \in \mathcal{P}(\mathbb{S})$ being a "true" distribution of X_0, suppose $\varepsilon \ne 0$ is chosen such that $\nu = \mu + \varepsilon\tilde{\pi}_0 \in \mathcal{P}(\mathbb{S})$. Then owing to the linearity of (11),

$$\sigma_t^\mu(h) = \sigma_t^\nu(h) \quad t\text{-a.e. } \tilde{\mathsf{P}}^\mu\text{-a.s.}$$

As will be justified more fully in the proof of Theorem 1, this leads to the third equivalent definition of observability:

Definition 4. The nonlinear model $(\mathcal{A},h)$ is said to be *observable* if

$$\pi_t^\mu(h) = \pi_t^\nu(h) \quad t\text{-a.e. } \mathsf{P}^\mu\text{-a.s.} \quad \Rightarrow \quad \mu = \nu \tag{O3}$$

It is noted that (O3) is the stochastic analog of (4).

The proof of the following theorem appears in the Appendix B.2.

Theorem 1. (Observability). The three conditions: (O1), (O2), and (O3) are equivalent.

The following theorem provides an explicit characterization of the controllable space. Its proof appears in the Appendix B.3.

Theorem 2. Consider the linear operator (10). Its range space $\mathsf{R}(\mathcal{L})$ is the smallest such subspace $\mathcal{C} \subset \mathcal{Y}$ that satisfies the following two properties:

(i) The constant function $1 \in \mathcal{C}$;

(ii) If $g \in \mathcal{C}$ then $\mathcal{A}g \in \mathcal{C}$ and $g\cdot h \in \mathcal{C}$. ($g\cdot h$ is the Hadamard (element-wise) product of functions g and h)[6].

4.4 Relationship to P-observability

For the particular (additive Gaussian noise) form of the observation model (6), there is a formula, due to (Clark et al., 1999, Theorem 3.1), for the relative entropy between $\mathsf{P}^\mu_{\mathcal{Z}_T}$ and $\mathsf{P}^\nu_{\mathcal{Z}_T}$:

$$\mathsf{D}(\mathsf{P}^\mu_{\mathcal{Z}_T} \| \mathsf{P}^\nu_{\mathcal{Z}_T}) = \mathsf{E}^\mu \int_0^T |\pi_t^\mu(h) - \pi_t^\nu(h)|^2\,\mathrm{d}t$$

Combined with Theorem 1, a straightforward corollary is the following proposition:

Proposition 2. Consider the observation model of the form (6). The model $(\mathcal{A},h)$ is observable (according to one of the equivalent definitions 2, 3, or 4) if and only if it is P-observable (definition 1).

5. FINITE STATE-SPACE CASE

Although X is allowed to be a general Markov process in a compact state-space, a guiding example is when the state space $\mathbb{S}$ is finite, namely, $\mathbb{S} = \{1,2,\ldots,d\}$. In this setting, any real-valued function or finite measure can be expressed by a vector in $\mathbb{R}^d$, where i^{th} element represent the function value at i. The observation function h is also represented by a matrix $H \in \mathbb{R}^{d\times m}$. The generator $\mathcal{A}$ of the Markov process is identified with a row-stochastic rate matrix $A \in \mathbb{R}^{d\times d}$ which acts on functions (elements of $\mathbb{R}^d$) through right-multiplication: $A : f \mapsto Af$.

Using this notation, the BSDE (9) is expressed as follows:

$$-\mathrm{d}Y_t = \left(AY_t + HU_t + \mathrm{diag}^\dagger(HV_t^\top)\right)\mathrm{d}t - V_t\,\mathrm{d}Z_t, \quad Y_T = c1$$

[4] The bounded-ness property is based on the well-posedness of the solution of the BSDE (9).

[5] In nonlinear filtering, the Zakai equation is considered with initial measure $\tilde{\pi}_0 \in \mathcal{P}(\mathbb{S})$. In this paper, the initial measure is allowed to be a signed measure.

[6] For a vector-valued function $h(x) = [h_1(x),\ldots,h_m(x)]$, $g\cdot h \in \mathcal{C}$ means $g\cdot h_i \in \mathcal{C}$ for each $i = 1,\ldots,m$. The Hadamard product is simply the product of functions, i.e., $(g\cdot h_i)(x) = g(x)h_i(x)$ for all $x \in \mathbb{S}$.

where 1 is a vector of ones in $\mathbb{R}^d$ and $\text{diag}^\dagger(HV_t^\top)$ is the vector of the diagonal elements of the matrix $HV^\top$. The solution pair is $(Y,V) \in L^2_{\mathcal{Z}}([0,T];\mathbb{R}^d) \times L^2_{\mathcal{Z}}([0,T];\mathbb{R}^{d\times m})$.

The controllable space $\mathsf{R}(\mathcal{L})$ is a subspace in $\mathbb{R}^d$:

$$\begin{aligned}\mathsf{R}(\mathcal{L}) = \text{span}\big\{&1, H, AH, A^2H, A^3H, \ldots, \\ &H\cdot H, A(H\cdot H), H\cdot(AH), A^2(H\cdot H), \ldots, \\ &H\cdot(H\cdot H), (AH)\cdot(H\cdot H), H\cdot A(H\cdot H), \ldots\big\}\end{aligned} \tag{12}$$

where the dot denotes the element-wise product. The nonlinear model (A,H) is observable if the vectors in the righthand-side of (12) span $\mathbb{R}^d$. This provides a test for verifying observability of the nonlinear model.

We next compare the above test with the observability test (5) for the linear model (A,H). It is clear that if the linear model (A,H) is observable (in the sense of (5)) then the nonlinear model (A,H) is also observable. However, the latter property is in general much weaker than the observability in linear systems theory. The following proposition is shows such a case. The proof is omitted.

Proposition 3. (A sufficient condition). Consider the nonlinear model (A,H) for the finite state-space. Then (A,H) is observable if $h(x) = H^\top x$ is an injective map from $\mathbb{S}$ into $\mathbb{R}^m$. If $A = 0$ then the injective property of the function h is also necessary for observability.

Remark 1. For the finite state nonlinear model (A,H), the test for observability first appeared in (van Handel, 2009a, Lemma 9). The test was obtained by explicitly calculating the probability of each observation and applying (7). For a general class of linear BSDE-s, the controllable subspace is identically defined by (Peng, 1994, Lemma 3.2). However, its use in the study of observability appears to be new.

6. CONCLUSION AND FUTURE DIRECTIONS

In this paper, the duality introduced in our previous work (Kim et al., 2019) is used for the purposes of defining and characterizing observability of nonlinear stochastic systems. The main idea is to define observability as a dual of the controllability of a certain BSDE (9). Based on the dual formulation, a test for observability is presented and related to literature. The proposed duality-based framework allows one to relate and compare the linear and the nonlinear systems. A side-by-side summary of this relationship is given in a tabular form (Table 1).

The methodology of this and our earlier duality paper is currently being used to investigate nonlinear filter stability; and to develop new control-based algorithms for approximating the nonlinear filter. These are the subjects of continuing research and preprints are in Kim et al. (2021); Kim and Mehta (2021).

REFERENCES

Chigansky, P., Liptser, R., and Van Handel, R. (2009). Intrinsic methods in filter stability. *Handbook of Nonlinear Filtering*.

Clark, J.M.C., Ocone, D.L., and Coumarbatch, C. (1999). Relative entropy and error bounds for filtering of markov processes. *Mathematics of Control, Signals and Systems*, 12(4), 346–360.

Fleming, W. and Mitter, S. (1982). Optimal control and nonlinear filtering for nondegenerate diffusion processes. *Stochastics*, 8, 63–77.

Goodwin, G.C., de Doná, J.A., Seron, M.M., and Zhuo, X.W. (2005). Lagrangian duality between constrained estimation and control. *Automatica*, 41(6), 935–944.

Hermann, R. and Krener, A. (1977). Nonlinear controllability and observability. *IEEE Transactions on automatic control*, 22(5), 728–740.

Hutson, V., Pym, J., and Cloud, M. (2005). *Applications of functional analysis and operator theory*, volume 200. Elsevier.

Kailath, T., Sayed, A.H., and Hassibi, B. (2000). *Linear estimation*. Prentice Hall.

Kalman, R.E. (1960). On the general theory of control systems. *IFAC Proceedings Volumes*, 1(1), 491–502.

Kim, J.W. and Mehta, P.G. (2021). A dual characterization of the stability of the Wonham filter. *submitted for publication*.

Kim, J.W., Mehta, P.G., and Meyn, S.P. (2019). What is the Lagrangian for nonlinear filtering? In *2019 IEEE Conference on Decision and Control (CDC)*, 1607–1614.

Kim, J.W., Mehta, P.G., and Meyn, S.P. (2021). The conditional Poincaré inequality for filter stability. *submitted for publication*.

Krylov, N.V. (2011). On the Itô–Wentzell formula for distribution-valued processes and related topics. *Probability theory and related fields*, 150(1-2), 295–319.

Le Gall, J.F. (2016). *Brownian Motion, Martingales, and Stochastic Calculus*, volume 274. Springer.

Liu, A.R. and Bitmead, R.R. (2011). Stochastic observability in network state estimation and control. *Automatica*, 47(1), 65–78.

Ma, J. and Yong, J. (1997). Adapted solution of a degenerate backward spde, with applications. *Stochastic processes and their applications*, 70(1), 59–84.

McDonald, C. and Yüksel, S. (2019). Observability and filter stability for partially observed markov processes. In *2019 IEEE Conference on Decision and Control (CDC)*, 1623–1628.

Mitter, S.K. and Newton, N.J. (2003). A variational approach to nonlinear estimation. *SIAM journal on control and optimization*, 42(5), 1813–1833.

Moraal, P. and Grizzle, J. (1995). Observer design for nonlinear systems with discrete-time measurements. *IEEE Transactions on automatic control*, 40(3), 395–404.

Ocone, D. and Pardoux, E. (1996). Asymptotic stability of the optimal filter with respect to its initial condition. *SIAM Journal on Control and Optimization*, 34(1), 226–243.

Peng, S. (1994). Backward stochastic differential equation and exact controllability of stochastic control systems. *Progress in natural science*, 4(3), 274–284.

Todorov, E. (2008). General duality between optimal control and estimation. In *2008 47th IEEE Conference on Decision and Control*, 4286–4292.

Treves, F. (1967). *Topological Vector Spaces, Distributions and Kernels: Pure and Applied Mathematics*, volume 25. Elsevier.

van Handel, R. (2009a). Observability and nonlinear filtering. *Probability theory and related fields*, 145(1-2), 35–74.

van Handel, R. (2009b). Uniform observability of hidden Markov models and filter stability for unstable signals. *The Annals of Applied Probability*, 19(3), 1172–1199.

van Handel, R. (2010). Nonlinear filtering and systems theory. In *Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems*.

Xiong, J. (2008). *An Introduction to Stochastic Filtering Theory*, volume 18. Oxford University Press on Demand.

Appendix A. DUALITY BETWEEN ESTIMATION AND CONTROL

This section includes a brief review of the duality between nonlinear filtering and stochastic optimal control introduced in our recent paper (Kim et al., 2019).

Dual optimal control problem:

$$\underset{U\in\mathcal{U}}{\text{Min}} \ \mathsf{J}(U) = \mathsf{E}\Big(\tfrac{1}{2}|Y_0(X_0)-\pi_0(Y_0)|^2 + \int_0^T \ell(Y_t,V_t,U_t;X_t)\,\mathrm{d}t\Big)$$
$$\text{Subj.} \ -\mathrm{d}Y_t(x) = \big(\mathcal{A}Y_t(x) + h^\top(x)(U_t+V_t(x))\big)\mathrm{d}t - V_t^\top(x)\,\mathrm{d}Z_t$$
$$Y_T(x) = f(x)\ \forall x\in\mathbb{S} \tag{A.1}$$

where the set of admissible control $\mathcal{U} := L^2_{\mathcal{Z}}([0,T],\mathbb{R}^m)$ and the cost function

$$l(y,v,u;x) = \tfrac{1}{2}Q(y)(x) + \tfrac{1}{2}(u+v(x))^\top R(u+v(x))$$

where $Q(y)(x)$ is the carré du champ operator asspciated with the state process. It is noted that the constraint is a backward stochastic differential equation (BSDE) with solution $(Y,V) := \{(Y_t,V_t) : t\in[0,T]\} \in L^2_{\mathcal{Z}}([0,T];C(\mathbb{S}))\times L^2_{\mathcal{Z}}([0,T];C^m(\mathbb{S}))$. The terminal condition $f\in C(\mathbb{S})$ is prescribed.

Consider the following linear structure of the estimator:

$$S_T = \pi_0(Y_0) - \int_0^T U_t^\top\,\mathrm{d}Z_t$$

where $U\in\mathcal{U}$ is an admissible control and Y_0 is obtained by (A.1). The precise duality relationship is as follows:

Proposition 4. (Prop. 1 in (Kim et al., 2019)). Consider the observation model (6), together with the dual optimal control problem (A.1). Then for any choice of admissible control $U\in\mathcal{U}$:

$$\mathsf{J}(U) = \tfrac{1}{2}\mathsf{E}(|S_T - f(X_T)|^2)$$

The significance of the duality relationship is as follows: The problem of obtaining the minimum variance estimate S_T of $f(X_T)$ (minimizer of the right-hand side of the equality) is converted into the problem of finding the optimal control U (minimizer of the left-hand side of the identity). Additional details including the use of the dual optimal control problem to derive the nonlinear filter can be found in (Kim et al., 2019).

Appendix B. PROOFS OF THEOREMS

B.1 Proof of Proposition 1

By linearity, $\mathcal{L}(U;c) = \mathcal{L}(U;0) + c1$ for $U\in\mathcal{U}$ and $c\in\mathbb{R}$. Therefore, for $\tilde\pi_0\in\mathcal{Y}^\dagger$,

$$\langle\tilde\pi_0,\mathcal{L}(U;c)\rangle_{\mathcal{Y}} = \langle\tilde\pi_0,\mathcal{L}(U;0)\rangle_{\mathcal{Y}} + c\,\tilde\pi_0(1)$$

Thus, the main calculation is to transform $\langle\tilde\pi_0,\mathcal{L}(U;0)\rangle_{\mathcal{Y}}$. For this purpose, consider (9) with $c=0$ and express $\langle\tilde\pi_0,\mathcal{L}(U;0)\rangle_{\mathcal{Y}} = \tilde\pi_0(Y_0)$. Using the Itô-Wentzell formula for measures (Krylov, 2011, Theorem 1.1),

$$\begin{aligned}\mathrm{d}\big(\tilde\pi_t(Y_t)\big) &= \big(\tilde\pi_t(\mathcal{A}Y_t)\,\mathrm{d}t + \tilde\pi_t(h^\top Y_t)\,\mathrm{d}Z_t\big) + \tilde\pi_t(h^\top V_t)\,\mathrm{d}t\\ &\quad + \big(\tilde\pi_t(-\mathcal{A}Y_t - h^\top U_t - h^\top V_t)\,\mathrm{d}t + \tilde\pi_t(V_t)\,\mathrm{d}Z_t\big)\\ &= -U_t^\top\tilde\pi_t(h)\,\mathrm{d}t + \tilde\pi_t(h^\top Y_t + V_t^\top)\,\mathrm{d}Z_t\end{aligned}$$

Integrating both sides,

$$\tilde\pi_T(Y_T) - \tilde\pi_0(Y_0) = -\int_0^T U_t^\top\tilde\pi_t(h)\,\mathrm{d}t + \int_0^T \tilde\pi_t(h^\top Y_t + V_t^\top)\,\mathrm{d}Z_t$$

Under the probability measure $\tilde{\mathsf{P}}$, Z is a Wiener process. Hence,

$$\tilde\pi_0(Y_0) = \tilde{\mathsf{E}}\Big(\int_0^T U_t^\top\tilde\pi_t(h)\,\mathrm{d}t\Big) = \langle\tilde\pi(h),U\rangle_{\mathcal{U}}$$

Therefore,

$$\langle\tilde\pi_0,\mathcal{L}(U;c)\rangle_{\mathcal{Y}} = \langle\tilde\pi(h),U\rangle_{\mathcal{U}} + c\,\tilde\pi_0(1)$$

B.2 Proof of Theorem 1

(O1) and (O2) are equivalent by the closed range theorem. The proof of (O2) $\iff$ (O3) is presented next.

Necessity: We first show (O3) $\Rightarrow$ (O2). For a given $\tilde\pi_0\in\mathsf{N}(\mathcal{L}^\dagger)$, then for any $\mu,\nu\in\mathcal{P}(\mathbb{S})$ such that $\varepsilon\tilde\pi_0 = \mu-\nu$ for some constant $\varepsilon\neq 0$, we have:

$$\varepsilon\tilde\pi_t(h) = \sigma_t^\mu(h) - \sigma_t^\nu(h)$$

Since $\tilde\pi_0\in\mathsf{N}(\mathcal{L}^\dagger)$ implies $\tilde\pi_t(h)\equiv 0$ $\tilde{\mathsf{P}}$-a.s.,

$$\sigma_t^\mu(h) = \sigma_t^\nu(h) \quad t\text{-a.e. } \tilde{\mathsf{P}}\text{-a.s.} \tag{B.1}$$

Using the Zakai Eq. (11) with $f=1$ (the constant function),

$$\sigma_t^\mu(1) = 1 + \int_0^t \sigma_s^\mu(h^\top)\,\mathrm{d}Z_s \tag{B.2}$$

Therefore (B.1) implies that the normalization constant $\sigma_t^\mu(1) = \sigma_t^\nu(1)$ for all $t\in[0,T]$. Thus, using (8),

$$\pi_t^\mu(h) = \pi_t^\nu(h) \quad t\text{-a.e. } \tilde{\mathsf{P}}\text{-a.s.}$$

Finally, this is also P^μ-a.s. event since $\mathsf{P}^\mu \ll \tilde{\mathsf{P}}$.

Sufficiency: Assume (O3) is not true: There exists $\mu\neq\nu\in\mathcal{P}(\mathbb{S})$ such that $\pi_t^\mu\equiv\pi_t^\nu$. We want to show that $\mu-\nu\in\mathsf{N}(\mathcal{L}^\dagger)$. Equations (B.2) and (8) are combined into:

$$\sigma_t^\mu(1) = 1 + \int_0^t \sigma_s^\mu(1)\pi_s^\mu(h)\,\mathrm{d}Z_s$$

This implies $\sigma_t^\mu(1)\equiv\sigma_t^\nu(1)$, and therefore $\sigma_t^\mu(h)\equiv\sigma_t^\nu(h)$ (P^μ-a.s.) by (8). Again it is $\tilde{\mathsf{P}}$-a.s. event by equivalency. It is obvious that $\mu(1)-\nu(1)=0$, so $\mu-\nu\in\mathsf{N}(\mathcal{L}^\dagger)$.

B.3 Proof of Theorem 2

For notational ease, we assume $m=1$. The objective is to show $\mathcal{C} = \mathsf{R}(\mathcal{L})$. The proof below is adapted from Peng (1994).

The definition of $\mathsf{N}(\mathcal{L}^\dagger)$ is:

$$\tilde\pi_0\in\mathsf{N}(\mathcal{L}^\dagger) \Leftrightarrow \tilde\pi_0(1)=0 \text{ and } \tilde\pi_t(h)\equiv 0 \quad \forall\, t\in[0,T]$$

Since $\mathsf{N}(\mathcal{L}^\dagger)$ is the annihilator of $\mathsf{R}(\mathcal{L})$, we have $1,h\in\mathsf{R}(\mathcal{L})$. Consider next the Zakai equation (11) with the initial condition $\tilde\pi_0\in\mathsf{N}(\mathcal{L}^\dagger)$ and $f=h$:

$$\tilde\pi_t(h) = \tilde\pi_0(h) + \int_0^t \tilde\pi_s(\mathcal{A}h)\,\mathrm{d}s + \int_0^t \tilde\pi_s(h^2)\,\mathrm{d}Z_s$$

Since t is arbitrary, the left-hand side is identically zero for all $t\in[0,T]$ if and only if

$$\tilde\pi_0(h)=0,\quad \tilde\pi_t(\mathcal{A}h)\equiv 0,\quad \tilde\pi_t(h^2)\equiv 0 \quad \forall\, t\in[0,T]$$

and in particular, this implies $\mathcal{A}h, h^2\in\mathsf{R}(\mathcal{L})$.

The subspace $\mathcal{C}$ is obtained by continuing to repeat the steps ad infinitum: If at the conclusion of the k^{th} step, we find a function $g\in\mathcal{C}$ such that $\tilde\pi_t(g)\equiv 0$ for all $t\in[0,T]$. Then through the use of the Zakai equation,

$$\tilde\pi_0(g)=0,\quad \tilde\pi_t(\mathcal{A}g)\equiv 0,\quad \tilde\pi_t(hg)\equiv 0 \quad \forall\, t\in[0,T]$$

so $\mathcal{A}g, hg\in\mathcal{C}$. By construction, because $\tilde\pi_0\in\mathsf{N}(\mathcal{L}^\dagger)$, $\mathcal{C} = \mathsf{R}(\mathcal{L})$.