

On the Lyapunov Foster criterion and Poincaré inequality for Reversible Markov Chains

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Abstract—This paper presents an elementary proof of stochastic stability of a discrete-time reversible Markov chain starting from a Foster-Lyapunov drift condition. Besides its relative simplicity, there are two salient features of the proof: (i) it relies entirely on functional-analytic non-probabilistic arguments; and (ii) it makes explicit the connection between a Foster-Lyapunov function and Poincaré inequality. The proof is used to derive an explicit bound for the spectral gap. An extension to the non-reversible case is also presented.

I. INTRODUCTION

This paper presents an elementary functional-analytic proof of stochastic stability of a discrete-time reversible Markov chain. The main hypothesis is the existence of a Foster Lyapunov function, drift condition (v4) in [11, Ch. 15, Ch. 16]. The main result is to establish Poincaré inequality and relate it to a spectral gap under additional hypothesis. The spectral gap yields geometric convergence as an easy consequence.

The use of Lyapunov drift condition (v4) to establish geometric convergence rate is standard in the theory of Markov chains; cf., [11] and references therein. It is known that the geometric ergodicity is equivalent to a spectral gap for the corresponding Markov operator in a certain normed vector space L_∞^V [10]. The spectral gap in L_∞^V implies a spectral gap in L^2 for reversible Markov chains [13]. Explicit bounds on the convergence rate are obtained in [12], [14]. However, in a general setting, the existing bounds can be difficult to compute.

The techniques and tools used in [11] and the related literature are probabilistic in nature. In contrast, the short proof in this paper is entirely analytical and relies on elementary arguments. The key is to use the Lyapunov Foster condition (v4) to derive a Poincaré inequality. This is then related to existence of the spectral gap from which the convergence result follows. The approach of this paper is inspired by [3], [1] [2, Ch. 4] where Lyapunov function is related to Poincaré inequality for a continuous-time Markov processes. To the best of our knowledge, the extension of this connection, between Poincaré inequality and Lyapunov function, in discrete-time setting is not known. Given the elementary nature of the proof and the explicit bound on spectral gap, the results of this paper are expected to be broadly useful to the practitioners who use

the discrete-time reversible Markov chain for Markov chain Monte-Carlo (MCMC) and simulation purposes.

Analysis of geometric ergodicity based on Lyapunov drift condition appears in [7]. Their main result [7, Thm. 1.3] is based on introducing a family weighted normed spaces $L_\infty^{\beta V}$ and establishing the spectral gap in this space, for a particular weight β . This is different compared to this paper where a direct connection between Lyapunov condition and Poincaré inequality is established, and explicit bounds on the L^2 spectral gap are derived.

The outline of the remainder of this paper is as follows: The preliminaries and problem statement appears in Sec. II. The main result for the reversible Markov chain appears in Sec. III. Some extensions to reversible and to non-reversible cases are discussed in Sec. IV. The main result is illustrated with examples in Sec. V. Some concluding remarks appear in Sec. VI.

II. PRELIMINARIES

A. Model and definitions

Consider a time-homogeneous discrete-time Markov process $\{X_n\}_{n \geq 0}$ taking values in Polish state space $\mathcal{X}$, equipped with the Borel σ -field $\mathcal{B}$. Let P denote the corresponding Markov operator defined such that

$$Pf(x) = \mathbb{E}[f(X_1)|X_0 = x],$$

for all bounded measurable functions $f : \mathcal{X} \rightarrow \mathbb{R}$. Let $p : \mathcal{X} \times \mathcal{B} \rightarrow [0, 1]$ be the probability transition kernel associated with P . In terms of this kernel, the action of P on bounded measurable functions as follows:

$$Pf(\cdot) = \int_{\mathcal{X}} f(y)p(\cdot, dy).$$

The action of P on probability measure μ on $(\mathcal{X}, \mathcal{B})$ is as follows:

$$\mu P(\cdot) = \int_{\mathcal{X}} p(x, \cdot) d\mu(x).$$

A probability measure π is invariant for P if $\pi P = \pi$.

Consider the space of square integrable functions with respect to π denoted as $L^2(\pi)$ equipped with the inner product

$$\langle f, g \rangle_\pi := \int f(x)g(x) d\pi(x),$$

and the norm $\|f\|_{2,\pi}^2 := \langle f, f \rangle_\pi$. It follows from Jensen's inequality that P is a bounded linear operator on $L^2(\pi)$, when π is the invariant measure [2, pp. 10]. The invariant measure π is said to be reversible for the Markov operator P if P is self-adjoint on $L^2(\pi)$, i.e.,

$$\langle f, Pg \rangle_\pi = \langle Pf, g \rangle_\pi, \quad \forall f, g \in L^2(\pi).$$

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In this paper, we consider Markov chains P with a unique reversible invariant measure π , formalized below as an assumption:

Assumption 1: P admits a unique reversible invariant measure π .

The main question is to establish a spectral gap (in $L^2(\pi)$) for P . Since P has an eigenvalue $\lambda = 1$ with eigenfunction $f(x) \equiv 1$, we consider the orthogonal subspace $L_0^2(\pi) = \{f \in L^2(\pi); \int f(x) d\pi(x) = 0\}$. P is said to admit a spectral gap $\beta > 0$ in $L_0^2(\pi)$ if

$$\|P\|_{L_0^2(\pi)} = \sup_{f \in L_0^2(\pi)} \frac{\|Pf\|_{2,\pi}}{\|f\|_{2,\pi}} \leq 1 - \beta. \quad (1)$$

Two immediate consequences of the spectral gap are as follows:

- 1) Geometric convergence of the moments in $L^2(\pi)$

$$\|P^n f - \pi(f)\|_{2,\pi} \leq (1 - \beta)^n \|f - \pi(f)\|_{2,\pi},$$

where $\pi(f) := \int f(x) d\pi(x)$ is the mean of f with respect to the invariant measure π .

- 2) Geometric convergence of the probability distribution in the total-variation distance [4, Thm. 2.1],

$$\|\mu P^n - \pi\|_{TV} \leq (1 - \beta)^n \|h - 1\|_{2,\pi},$$

for any initial distribution $d\mu = h d\pi$.

For reversible Markov chains, the spectral gap is related to the Poincaré inequality as explained in the following section.

B. Spectral gap and Poincaré inequality

Define the Dirichlet forms

$$\mathcal{E}(f, f) := \langle f, (I - P)f \rangle_\pi, \quad \tilde{\mathcal{E}}(f, f) := \langle f, (I + P)f \rangle_\pi.$$

Then P is said to satisfy the Poincaré inequality, if there are positive constants β_+ and β_- such that

$$\|f\|_{2,\pi}^2 \leq \frac{1}{\beta_+} \mathcal{E}(f, f), \quad \forall f \in L_0^2(\pi), \quad (2)$$

$$\|f\|_{2,\pi}^2 \leq \frac{1}{\beta_-} \tilde{\mathcal{E}}(f, f), \quad \forall f \in L_0^2(\pi). \quad (3)$$

Lemma 1: Under Assumption 1,

- (i) If P satisfies the Poincaré inequality (2) with constant $\beta_+ > 0$, then the spectrum of P on $L_0^2(\pi)$ is bounded above by $1 - \beta_+$.
- (ii) If P satisfies the Poincaré inequality (3) with constant $\beta_- > 0$, then the spectrum of P on $L_0^2(\pi)$ is bounded below by $-1 + \beta_-$.
- (iii) If P satisfies the Poincaré inequalities (2)-(3), then it admits spectral gap $\beta = \min(\beta_+, \beta_-)$.

Proof: Omitted. See [15, Sec. 5.2.1, pp. 115]

Remark 1: For a continuous-time reversible Markov process with the infinitesimal generator L and the semigroup $P_t = e^{tL}$, the Dirichlet form is defined as $\mathcal{E}(f, f) := -\langle f, Lf \rangle_\pi$. Therefore, the Poincaré inequality (2) is expressed as $\langle f, Lf \rangle_\pi \leq -\beta_+ \|f\|_{2,\pi}^2$, from which the spectral gap for the semigroup, $\|P_t\|_{L^2(\pi)} = \|e^{tL}\|_{L^2(\pi)} \leq e^{-\beta_+ t} < 1$,

readily follows. In discrete-time settings, the second Poincaré inequality (3) is also required. This is to rule out periodicity, eigenvalue at -1 for the reversible case [15, Ch. 5].

III. MAIN RESULT

The main hypothesis of the paper is the Foster Lyapunov condition (v4):

Assumption 2: Suppose P satisfies

$$PV \leq (1 - \lambda)V + b\mathbf{1}_K, \quad (4)$$

$$P\mathbf{1}_A(x) \geq \alpha\nu(A)\mathbf{1}_K(x), \quad \forall A \in \mathcal{B}, \quad (5)$$

for a positive function $V : \mathbb{R}^d \rightarrow [1, \infty)$, numbers $b < \infty$, $\alpha, \lambda > 0$, a set $K \subseteq \mathcal{X}$, and a probability measure ν .

The condition (4) is known as the drift condition and condition (5) is known as the minorization condition. The main result of this paper is as follows:

Theorem 1: Under Assumptions 1-2, P admits a Poincaré inequality (2) with constant $\beta_+ = \frac{\lambda}{1 + \frac{2b}{\alpha}}$.

For continuous-time processes, the analogous result appears in [2, Thm. 4.6.2 pp. 202]. Unlike the continuous-time case, the conclusion of the Theorem 1 is not sufficient to establish a spectral gap without also establishing (3), except for the case when P is positive semi-definite.

Corollary 1: Under the assumptions of Theorem 1, if P is a positive semi-definite operator, then P admits a spectral gap $\beta = \beta_+ = \frac{\lambda}{1 + \frac{2b}{\alpha}}$.

Later, in Sec. IV-A, additional assumption is introduced to establish spectral gap for Markov operators that are not necessarily positive semi-definite.

A. Proof of Theorem 1

Remark 2: If $K = \mathcal{X}$ then the minorization condition (5) is the Doeblin's condition which directly implies the spectral gap $\|P\|_{L_0^2(\pi)} \leq 1 - \frac{\alpha}{2}$ [15, Sec. 2.2, pp. 28]. However, Doeblin's condition is a very strong assumption for Markov-chains. In the other extreme when $K = \emptyset$ then the drift condition (4) implies the spectral gap $\|P\|_{L_0^2(\pi)} \leq 1 - \lambda$ from the spectral theory of positive operators [6, Thm. 13.1.6, pp. 383]. Owing to the eigenvalue at 1, this case does not apply to Markov operators. However, by suitably adapting the proof from the theory of positive operators to accomodate the minorization condition (5), one obtains an elementary proof of Theorem 1 as presented next.

Proof of the Theorem 1: From the variational characterization of the mean, we have $\|f - \pi(f)\|_{2,\pi} \leq \|f - m\|_{2,\pi}$ for all constants $m \in \mathbb{R}$. Therefore, in order to prove the Poincaré inequality (2), it suffices to show that

$$\|f - m\|_{2,\pi}^2 \leq \frac{1}{C} \langle f, (I - P)f \rangle_\pi, \quad \forall f \in L^2(\pi), \quad (6)$$

for some constant $m = m(f)$ to be chosen later.

Consider the Lyapunov drift condition (4). Multiply both sides by $\frac{(f-m)^2}{V}$ to obtain

$$\begin{aligned} \frac{(f-m)^2}{V}PV &\leq (1-\lambda)(f-m)^2 + b\frac{1}{V}(f-m)^2\mathbb{1}_K \\ &\leq (1-\lambda)(f-m)^2 + b(f-m)^2\mathbb{1}_K, \end{aligned}$$

where the second inequality follows because $V \geq 1$. Rearranging the terms

$$\lambda(f-m)^2 \leq \frac{(f-m)^2}{V}(I-P)V + b(f-m)^2\mathbb{1}_K,$$

and integrating both sides with respect to π ,

$$\lambda\|f-m\|_{2,\pi}^2 \leq \left\langle \frac{(f-m)^2}{V}, (I-P)V \right\rangle_\pi + b\|(f-m)\mathbb{1}_K\|_{2,\pi}^2.$$

It is claimed that

$$\left\langle \frac{(f-m)^2}{V}, (I-P)V \right\rangle_\pi \leq \langle f, (I-P)f \rangle_\pi, \quad (7)$$

$$\|(f-m)\mathbb{1}_K\|_{2,\pi}^2 \leq \frac{2}{\alpha} \langle f, (I-P)f \rangle_\pi, \quad (8)$$

with $m = \frac{1}{\pi(K)} \int_K f d\pi$. If the claims are true then

$$\|f-m\|_{2,\pi}^2 \leq \frac{1+\frac{2b}{\alpha}}{\lambda} \langle f, (I-P)f \rangle_\pi,$$

which proves (6), hence the Poincaré inequality (2) with $\beta_+ = \frac{\lambda}{1+\frac{2b}{\alpha}}$. It remains to prove the two claims:

- 1) Proof of the claim (7): Let $g = f - m$. Then, using $(I-P)\mathbb{1} = 0$, (7) is equivalently expressed as

$$\langle g, Pg \rangle_\pi \leq \left\langle \frac{g^2}{V}, PV \right\rangle_\pi. \quad (9)$$

Note that

$$\begin{aligned} 0 &\leq \int \int V(x)V(y) \left(\frac{g(y)}{V(y)} - \frac{g(x)}{V(x)} \right)^2 p(x, dy) d\pi(x) \\ &= \int \int \frac{g(x)^2}{V(x)} V(y)p(x, dy) d\pi(x) \\ &\quad + \int \int \frac{g(y)^2}{V(y)} V(x)p(x, dy) d\pi(x) \\ &\quad - 2 \int \int g(x)g(y)p(x, dy) d\pi(x) \\ &= \left\langle \frac{g^2}{V}, PV \right\rangle_\pi + \left\langle V, P\frac{g^2}{V} \right\rangle_\pi - 2\langle g, Pg \rangle_\pi. \end{aligned}$$

Using the self-adjoint property of P , it follows that $\left\langle \frac{g^2}{V}, PV \right\rangle_\pi = \left\langle V, P\frac{g^2}{V} \right\rangle_\pi$ which in turn proves (9).

- 2) Proof of the claim (8): Note that

$$\begin{aligned} \langle f, (I-P)f \rangle_\pi &= \frac{1}{2} \int \int (f(x) - f(y))^2 p(x, dy) d\pi(x) \\ &\stackrel{(1)}{\geq} \frac{\alpha}{2} \int \int (f(x) - f(y))^2 d\nu(y) d\pi(x) \\ &\stackrel{(2)}{\geq} \frac{\alpha}{2} \int_K (f(x) - \int f(y) d\nu(y))^2 d\pi(x) \\ &\stackrel{(3)}{\geq} \frac{\alpha}{2} \int_K (f(x) - m)^2 d\pi(x). \end{aligned}$$

The first inequality follows from the use of the minorization condition (5). The second inequality is the Jensen's inequality. The third inequality follows from the variational characterization of the variance of the function f because $m = \frac{1}{\pi(K)} \int_K f d\pi$ is the mean. ■

B. A counter-example

The following counter-example serves to show that it is not possible to obtain a bound for β_- using *only* the Foster Lyapunov condition (v4).

Example 1: Consider the Markov transition matrix

$$P = \begin{bmatrix} \epsilon & 1-\epsilon \\ 1-\epsilon & \epsilon \end{bmatrix},$$

on the state-space $\mathbb{S} = \{1, 2\}$. The invariant measure $\pi = [\frac{1}{2}, \frac{1}{2}]$ is reversible. The eigenvalues of P are $\lambda = 1, -1+2\epsilon$. Therefore, $\beta_+ = 1-2\epsilon$ and $\beta_- = 2\epsilon$. In the following, we show that the conditions (4)-(5) hold, with constants that are independent of ϵ . As a result, it is not possible to derive a bound on β_- simply from the constants that appear in the conditions (4)-(5).

- 1) Let the subset $K = \{1\} \subset \mathbb{S}$. Then the condition (5) holds with $\alpha = 1$ and $\nu = [\nu_1, \nu_2] = [\epsilon, 1-\epsilon]$ because

$$P_{11} = \epsilon = \alpha\nu_1, \quad P_{12} = 1-\epsilon = \alpha\nu_2.$$

- 2) For all $\epsilon \leq \frac{1}{4}$, the condition (4) holds with $V = [1, 3]^\top$, $\lambda = \frac{1}{2}$, and $b = 3$ because

$$\begin{aligned} (i=1) \quad P_{11}V_1 + P_{12}V_2 &= \epsilon + 3(1-\epsilon) \\ &\leq \frac{1}{2} + 3 = (1-\lambda)V_1 + b, \end{aligned}$$

$$(i=2) \quad P_{21}V_1 + P_{22}V_2 = 1-\epsilon + 3\epsilon \leq \frac{3}{2} = (1-\lambda)V_2,$$

The resulting bound for β_+ is $\frac{\frac{1}{2}}{1+\frac{2 \cdot 3}{1}} = \frac{1}{14}$.

IV. EXTENSIONS

By imposing additional conditions, the analysis of this paper is useful to obtain bounds for the spectral gap in the reversible and also the non-reversible cases. Two sets of results are described next.

A. Spectral gap under stronger condition

From Corollary 1, a spectral gap is obtained whenever P is positive semi-definite. Therefore, one way to prove the spectral gap for P is to consider P^2 which is always positive semi-definite for reversible Markov processes. Given the counter-example 1, it is not true that P^2 satisfies condition (v4) (if P does), without imposing some additional condition. One such condition, based on [7, Assumption 2], is as follows:

Assumption 3: In condition (v4) (in Assumption 2), the set

$$K = \{x \in \mathcal{X}; V(x) \leq R\},$$

for some $R > \frac{2b}{\lambda}$.

Proposition 1: Suppose P is a Markov operator that satisfies Assumptions 1, 2, and 3. Then P^2 satisfies

$$P^2V \leq (1 - \lambda')V + b'\mathbb{1}_K, \quad (10)$$

$$P^2\mathbb{1}_A(x) \geq \alpha'\nu(A)\mathbb{1}_K(x), \quad \forall A \in \mathcal{B} \quad (11)$$

where $\lambda' = \lambda(\frac{3}{2} - \lambda)$, $b' = (2 - \lambda)b$, and $\alpha' = \alpha^2\nu(K)$. Consequently,

$$\|P\|_{L_0^2(\pi)} \leq (1 - \beta^+)^{\frac{1}{2}}. \quad (12)$$

with $\beta^+ = \frac{\lambda(\frac{3}{2} - \lambda)}{1 + \frac{2b(2 - \lambda)}{\alpha^2\nu(K)}}$.

Proof: Proof of (10): Because P is a positive operator¹, $Pf \leq Pg$ whenever $f \leq g$. Therefore, applying P to both sides of the inequality (4),

$$\begin{aligned} P^2V &\leq (1 - \lambda)PV + bP\mathbb{1}_K \\ &\leq (1 - \lambda)^2V + (1 - \lambda)b\mathbb{1}_K + bP\mathbb{1}_K \\ &\stackrel{(3)}{\leq} (1 - \lambda)^2V + (1 - \lambda)b\mathbb{1}_K + b(\mathbb{1}_K + \frac{V}{R}) \\ &\stackrel{(4)}{\leq} ((1 - \lambda)^2 + \frac{\lambda}{2})V + (2 - \lambda)b\mathbb{1}_K, \end{aligned}$$

where the third inequality follows from $P\mathbb{1}_K \leq P\mathbb{1} = \mathbb{1} \leq \mathbb{1}_K + \frac{V}{R}$ and the fourth inequality is because $R > \frac{b}{2\lambda}$. This completes the proof of (10).

Proof of (11): Letting $A = K$ in the minorization condition (5),

$$P\mathbb{1}_K(x) \geq \alpha\nu(K)\mathbb{1}_K(x).$$

As a result, applying P to both sides of (5),

$$P^2\mathbb{1}_A(x) \geq \alpha\nu(A)P\mathbb{1}_K(x) \geq \alpha^2\nu(K)\nu(A)\mathbb{1}_K(x),$$

which proves (11).

Spectral gap (12) follows from application of Corollary 1 to positive semi-definite operator P^2 satisfying conditions (10)-(11). ■

B. Spectral gap for a non-reversible chain

Suppose P is a Markov operator with a unique invariant measure π and suppose $P^\dagger$ is its adjoint in $L^2(\pi)$. Then $P^\dagger P$ is a Markov operator with a reversible invariant measure π .

Proposition 2: Suppose both P and its adjoint $P^\dagger$ satisfy condition (v4) (inequality (4)-(5)) with the same Foster Lyapunov function V , set K and constants λ , b and α . Then $P^\dagger P$ satisfies

$$P^\dagger PV \leq (1 - \lambda')V + b'\mathbb{1}_K, \quad (13)$$

$$P^\dagger P\mathbb{1}_A(x) \geq \alpha'\nu(A)\mathbb{1}_K(x), \quad \forall A \in \mathcal{B} \quad (14)$$

where $\lambda' = \lambda(\frac{3}{2} - \lambda)$, $b' = (2 - \lambda)b$, and $\alpha' = \alpha^2\nu(K)$. Consequently,

$$\|P\|_{L_0^2(\pi)} \leq (1 - \beta^+)^{\frac{1}{2}}. \quad (15)$$

with $\beta^+ = \frac{\lambda(\frac{3}{2} - \lambda)}{1 + \frac{2b(2 - \lambda)}{\alpha^2\nu(K)}}$.

¹An operator P is positive if $Pf \geq 0$ whenever $f \geq 0$.

Proof: The proof of (13)-(14) is entirely analogous to the proof of Proposition 1. It follows from applying $P^\dagger$ to the inequalities (4)-(5) upon using the fact that $P^\dagger$ also satisfies the inequalities (4)-(5). The bound (15) then follows from application of Corollary 1 to $P^\dagger P$ which is reversible, positive-definite, and satisfies the conditions (13)-(14). ■

V. EXAMPLES

A. Ornstein-Uhlenbeck process

Consider the discrete-time Markov chain $\{X_n\}_{n \geq 0}$ taking values in $\mathbb{R}$ that evolves according to

$$X_{n+1} = (1 - a)X_n + \sigma B_n,$$

where $a \in (0, 1)$, $\sigma > 0$ and $\{B_n\}_{n \geq 0}$ are independent Gaussian random variables. The associated Markov operator

$$\begin{aligned} Pf(x) &= \mathbb{E}[f((1 - a)x + \sigma B_1)] \\ &= \int_{\mathbb{R}} (2\pi\sigma^2)^{-\frac{1}{2}} \exp(-\frac{(y - (1 - a)x)^2}{2\sigma^2}) f(y) dy, \end{aligned}$$

with a reversible Gaussian invariant measure

$$d\pi(x) = (\frac{2\pi\sigma^2}{a^2})^{-\frac{1}{2}} \exp(-\frac{x^2}{2\frac{\sigma^2}{a^2}}) dx.$$

The Markov operator P is an example of the Ornstein-Uhlenbeck Diffusion semigroup [2, Sec. 2.7.1] with spectrum

$$\lambda_n = (1 - a)^n, \quad n = 0, 1, \dots$$

yielding the spectral gap $\|P\|_{L_0^2(\pi)} = 1 - a$.

Our goal in this example is to apply the results of this paper to obtain a bound for the spectral gap of P and compare it to the exact spectral gap. Consider the Lyapunov function $V(x) = 1 + x^2$. Then,

$$\begin{aligned} PV(x) &= 1 + (1 - a)^2x^2 + \sigma^2 \\ &\leq (1 - a)V + (\sigma^2 + 1)\mathbb{1}_{|x| \leq R}, \end{aligned}$$

with $R^2 = \frac{\sigma^2 + 1}{a(1 - a)}$. The minorization condition (5) also holds:

$$\begin{aligned} P\mathbb{1}_A(x) &= \mathbb{P}\{(1 - a)x + \sigma B_1 \in A\} \\ &\geq \alpha \mathbb{P}\{\frac{\sigma}{\sqrt{2}}B_1 \in A\} \mathbb{1}_{|x| \leq R}, \end{aligned}$$

where $\alpha = \exp(-\frac{\sigma^2 + 1}{\sigma^2} \frac{1 - a}{a})$. Since P is a positive definite operator on $L^2(\pi)$, Corollary 1 applies and one obtains

$$\|P\|_{L_0^2(\pi)} \leq 1 - \frac{a}{1 + (\sigma^2 + 1) \exp(\frac{\sigma^2 + 1}{\sigma^2} \frac{1 - a}{a})}.$$

This is a conservative bound based on the exact spectral gap. The bound may be improved with another choice of Lyapunov function (e.g., $\exp(|x|)$). In general, it is known that the Lyapunov method is only able to provide a conservative bound for the spectral gap; cf. [2, pp. 203].

B. Diffusion map

The diffusion map T_ϵ is a Markov operator defined as

$$T_\epsilon(f)(x) := \frac{\int_{\mathbb{R}^d} g_\epsilon(x-y) e^{-U(y)} f(y) dy}{\int_{\mathbb{R}^d} g_\epsilon(x-y) e^{-U(y)} dy}, \quad (16)$$

where $g_\epsilon(z) = \exp^{-\frac{|z|^2}{4\epsilon}}$ is the Gaussian kernel, $U(x)$ is a potential function of sufficient regularity, and $\epsilon > 0$ is a positive parameter. The diffusion map was introduced and studied in spectral clustering literature as asymptotic limit of graph Laplacian matrix [5], [9]. Explicit bounds on the spectral gap of T_ϵ are important for analysis of diffusion map-based algorithms such as the gain function approximation algorithm in the feedback particle filter. [16].

The spectral gap is obtained via an application of Corollary 1. It is straightforward to check that T_ϵ is a Markov operator with reversible invariant probability density

$$\pi(x) := \gamma e^{-U(x)} \int g_\epsilon(x-y) e^{-U(y)} dy,$$

where γ is the normalization constant. Moreover, T_ϵ is positive-definite because

$$\langle f, T_\epsilon f \rangle_\pi = \int_{\mathbb{R}^d} g_\epsilon(x-y) (e^{-U} f)(x) (e^{-U} f)(y) dx dy \geq 0,$$

for all $f \in L^2(\pi)$. It remains to verify the Lyapunov conditions (4)-(5). This requires additional assumptions on the potential function $U(x)$, an example of which appears in the following Proposition.

Proposition 3: Consider the diffusion map operator (16). Suppose U is bounded from below and twice continuously differentiable with a bounded Hessian $\|\nabla^2 U\|_\infty < \infty$. Also, suppose $\exists \lambda_0, R > 0$ such that

$$\frac{1}{2} |\nabla U(x)|^2 \geq \lambda_0 U(x) + \|\nabla^2 U\|_\infty, \quad \forall |x| \geq R \quad (17)$$

Then, for all $\epsilon \in (0, \frac{1}{4\|\nabla^2 U\|_\infty})$ the Lyapunov conditions (4)-(5) hold and T_ϵ admits a spectral gap

$$\|T_\epsilon\|_{L_0^2(\pi)} \leq 1 - \frac{\epsilon \lambda_0}{1 + \frac{2\epsilon b_0}{\alpha}} \quad (18)$$

where

$$b_0 = \|\nabla^2 U\|_\infty + \max_{\|x\| \leq R} \left\{ \lambda_0 U(x) - \frac{1}{2} |\nabla U(x)|^2 \right\},$$

$$\alpha = \min_{|x| \leq R} \frac{1}{\sqrt{2}} e^{-2\epsilon |\nabla U(x)|^2 - 3\epsilon \|\nabla^2 U\|_\infty - \frac{2}{\sigma^2} (x - \sigma^2 \nabla U(x))^2}.$$

and $\sigma^2 = \frac{2\epsilon}{1+2\|\nabla^2 U\|_\infty \epsilon}$.

Remark 3: The assumption (17) on U is a type of a dissipative condition for a dynamical systems with drift ∇U [8]. It is satisfied by any potential function U that has a quadratic growth as $|x| \rightarrow \infty$. For example, $U(x) = U_0(x) + \frac{1}{2}\delta|x|^2$ satisfies this assumption provided U_0 is Lipschitz and $\delta > 0$.

Proof: Without loss of generality assume $U(x) \geq 1$ for all x . The Lyapunov condition (4) holds with $V = U$ because

$$\begin{aligned} (T_\epsilon U)(x) &\stackrel{(1)}{\leq} \log((T_\epsilon e^U)(x)) = -\log\left(\int g_\epsilon(x-y) e^{-U(y)} dy\right) \\ &\stackrel{(3)}{\leq} -\log\left(\int g_\epsilon(x-y) e^{-U(x) - \langle \nabla U(x), y-x \rangle - \frac{m}{2}|y-x|^2} dy\right) \\ &= U(x) - \frac{\sigma^2}{2} |\nabla U(x)|^2 - \frac{1}{2} \log\left(\frac{\sigma^2}{2\epsilon}\right) \end{aligned}$$

where the Jensen's inequality is used in the first step, and the inequality $U(y) \leq U(x) + \langle \nabla U(x), y-x \rangle + \frac{m}{2}|y-x|^2$ is used in the third step, with $m = \|\nabla^2 U\|_\infty$ and $\sigma^2 = \frac{2\epsilon}{1+2m\epsilon}$. Then, using (17)

$$\begin{aligned} (T_\epsilon U)(x) &\leq U(x) - \sigma^2 \lambda_0 U(x) - \frac{1}{2} \log\left(\frac{\sigma^2}{2\epsilon}\right) - \sigma^2 m \\ &\leq (1 - \epsilon \lambda_0) U(x), \quad \text{if } |x| \geq R \end{aligned}$$

where $\sigma^2 \geq \epsilon$ and $\log(\frac{\sigma^2}{2\epsilon}) \geq -2\epsilon m$ are used in the second step. This proves Lyapunov condition (4), with $\lambda = \epsilon \lambda_0$, $K = \{x \in \mathbb{R}^d; |x| \leq R\}$, and $b = \epsilon b_0$.

The minorization condition (5) holds because

$$\begin{aligned} T_\epsilon \mathbb{1}_A(x) &= \frac{\int_A g_\epsilon(x-y) e^{-U(y)} dy}{\int_{\mathbb{R}^d} g_\epsilon(x-y) e^{-U(y)} dy} \\ &\geq \frac{e^{\frac{\epsilon}{2} |\nabla U(x)|^2 - \frac{m\epsilon}{2}} \mathbb{P}(x - \sigma^2 \nabla U(x) + \sigma B_1 \in A)}{e^{2\epsilon |\nabla U(x)|^2 + 2m\epsilon}} \\ &\geq \alpha \mathbb{P}\left(\frac{\sigma}{\sqrt{2}} B_1 \in A\right), \quad \text{if } |x| \leq R \end{aligned}$$

This proves the Lyapunov conditions (4)-(5), which together with the fact that T_ϵ is reversible and positive-definite, proves the spectral gap (18) by Corollary 1. ■

VI. CONCLUSION

In this paper, a straightforward analytical approach is presented to establish stochastic stability starting from the Lyapunov drift condition (v4) (Theorem 1). A key message of this paper is that the Lyapunov Foster drift condition (v4), in of itself, only implies a bound on the spectral gap from eigenvalue at 1. This is formalized in this paper as a relationship between condition (v4) and the Poincaré inequality for the operator $I - P$ (Theorem 1). From this main theorem, two sets of bounds are obtained here under certain hypotheses (Corollary 1 and Proposition 1). An extension to the non-reversible chain is also described in Proposition 2. Two illustrative examples are presented. The diffusion map example is of independent interest for analysis of gain function approximation in the feedback particle filter.

REFERENCES

- [1] D. Bakry, P. Cattiaux, and A. Guillin. Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré. *Journal of Functional Analysis*, 254(3):727–759, 2008.
- [2] D. Bakry, I. Gentil, and M. Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348. Springer Science & Business Media, 2013.
- [3] Dominique Bakry, Franck Barthe, Patrick Cattiaux, and Arnaud Guillin. A simple proof of the Poincaré inequality for a large class of probability measures including the log-concave case. *Electron. Commun. Probab.*, 13:60–66, 2008.

- [4] P. Cattiaux and A. Guillin. Trends to equilibrium in total variation distance. In *Annales de l'IHP Probabilités et statistiques*, volume 45, pages 117–145, 2009.
- [5] R. R. Coifman and S. Lafon. Diffusion maps. *Applied and computational harmonic analysis*, 21(1):5–30, 2006.
- [6] E. B. Davies. *Linear operators and their spectra*, volume 106. Cambridge University Press, 2007.
- [7] M. Hairer and J. C. Mattingly. Yet another look at Harris’ ergodic theorem for Markov chains. In *Seminar on Stochastic Analysis, Random Fields and Applications VI*, pages 109–117. Springer, 2011.
- [8] J. K. Hale. Asymptotic behavior of dissipative systems (mathematical surveys and monographs vol 25)(providence, ri: American mathematical society). 1988.
- [9] M. Hein, J. Audibert, and U. Von Luxburg. From graphs to manifolds—weak and strong pointwise consistency of graph Laplacians. In *Learning theory*, pages 470–485. Springer, 2005.
- [10] I. Kontoyiannis and S. P. Meyn. Spectral theory and limit theorems for geometrically ergodic Markov processes. *Annals of Applied Probability*, pages 304–362, 2003.
- [11] S. Meyn and R. L. Tweedie. *Markov chains and stochastic stability*, cambridge, 2009.
- [12] S. P. Meyn and R. L. Tweedie. Computable bounds for geometric convergence rates of Markov chains. *The Annals of Applied Probability*, 4(4):981–1011, 1994.
- [13] G. Roberts and J. Rosenthal. Geometric ergodicity and hybrid Markov chains. *Electronic Communications in Probability*, 2:13–25, 1997.
- [14] J. S. Rosenthal. Minorization conditions and convergence rates for Markov chain Monte Carlo. *Journal of the American Statistical Association*, 90(430):558–566, 1995.
- [15] D. W. Stroock. *An introduction to Markov processes*, volume 230. Springer Science & Business Media, 2013.
- [16] A. Taghvaei and P. G. Mehta. Diffusion map-based algorithm for gain function approximation in the feedback particle filter. *arXiv preprint arXiv:1902.07263 (To appear in SIAM Journal of Uncertainty Quantification)*, 2019.