An Optimal Transport Formulation of the Ensemble Kalman Filter

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***Abstract*—Controlled interacting particle systems such as the ensemble Kalman filter (EnKF) and the feedback particle filter (FPF) are numerical algorithms to approximate the solution of the nonlinear filtering problem in continuous time. The distinguishing feature of these algorithms is that the Bayesian update step is implemented using a feedback control law. It has been noted in the literature that the control law is not unique. This is the main problem addressed in this paper. To obtain a unique control law, the filtering problem is formulated here as an optimal transportation problem. An explicit formula for the (mean-field type) optimal control law is derived in the linear Gaussian setting. Comparisons are made with the control laws for different types of EnKF algorithms described in the literature. Via empirical approximation of the mean-field control law, a finite-N controlled interacting particle algorithm is obtained. For this algorithm, the equations for empirical mean and covariance are derived and shown to be identical to the Kalman filter. This allows strong conclusions on convergence and error properties based on the classical filter stability theory for the Kalman filter. It is shown that, under certain technical conditions, the mean squared error (m.s.e.) converges to zero *even* with a finite number of particles. A detailed propagation of chaos analysis is carried out for the finite-N algorithm. The analysis is used to prove weak convergence of the empirical distribution as $N \to \infty$. For a certain simplified filtering problem, analytical comparison of the m.s.e. with the importance sampling-based algorithms is described. The analysis helps explain the favorable scaling properties of the control-based algorithms reported in several numerical studies in recent literature.**

I. Introduction

The subject of this paper concerns Monte-Carlo methods for simulating a nonlinear filter (conditional distribution) in continuous-time settings. The mathematical abstraction of any filtering problem involves two processes: a hidden Markov process $\{X_t\}_{t\geq 0}$ and the observation process $\{Z_t\}_{t\geq 0}$. The numerical problem is to compute the *posterior distribution* $\mathsf{P}(X_t \in \cdot | \mathscr{Z}_t)$ where $\mathscr{Z}_t := \sigma\{Z_s; 0 \leq s \leq t\}$ is the filtration generated by the observations. A standard solution approach is the particle filter which relies on importance sampling to implement the effect of conditioning [3], [4]. In numerical implementations, this often leads to the particle degeneracy issue (or weight collapse) whereby only a few particles have large weights. To combat this issue, various types of resampling schemes have been proposed in the literature [5], [6].

In the past decade, an alternate class of algorithms has attracted growing attention. These algorithms can be regarded as a controlled interacting particle system where the central idea is to implement the effect of conditioning using feedback control. Mathematically, this involves construction of controlled stochastic process, denoted by $\{\bar{X}_t\}_{t\geq 0}$. In continuous-time settings, the model for the $\bar{X}_t$ is a stochastic differential equation (sde):

$$\mathrm{d}\bar{X}_t = u_t(\bar{X}_t)\mathrm{d}t + \mathsf{K}_t(\bar{X}_t)\mathrm{d}Z_t + [\text{additional terms}], \quad \bar{X}_0 \overset{\mathrm{d}}{=} X_0 \tag{1}$$

where the [additional terms] are pre-specified (these terms may be zero). The control problem is to design mean-field type control law $\{u_t(\cdot)\}_{t\geq 0}$ and $\{\mathsf{K}_t(\cdot)\}_{t\geq 0}$ such that the conditional distribution of $\bar{X}_t$ (given $\mathscr{Z}_t$) is equal to the posterior distribution of X_t. If this property holds, the filter is said to be *exact*. In a numerical implementation, the mean-field terms in the control law are approximated empirically by simulating N copies of (1). The resulting system is a controlled interacting particle system with a finite number of N interacting particles. The particles have uniform importance weights by construction. Therefore, the particle degeneracy issue does not arise. Resampling is no longer necessary and steps such as rules for reproduction, death or birth of particles are altogether avoided.

The focus of this paper is on (i) formal methods for design of control laws ($u_t(\cdot)$ and $\mathsf{K}_t(\cdot)$) for (1); (ii) algorithms for empirical approximation of the control laws using N particles; and (iii) error analysis of the finite-N interacting particle models as $N \to \infty$. The main problem highlighted and addressed in this paper is the issue of uniqueness: one can interpret the controlled system (1) as transporting the initial distribution at time $t = 0$ (prior) to the conditional distribution at time t (posterior). Clearly, there are infinitely many maps that transport one distribution into another. This suggests that there are infinitely many choices of control laws that all lead to exact filters. This is not surprising: The exactness condition specifies only the marginal distribution of the stochastic process $\{\bar{X}_t\}_{t\geq 0}$ at times $t \geq 0$, which is not enough to uniquely identify a stochastic process, e.g., the joint distributions at two time instants are not specified.

Although these issues are relevant more generally, the scope of this paper is limited to the linear Gaussian problem. A motivation comes from the widespread use of the ensemble Kalman filter (EnKF) algorithm in applications. It is noted that the mean-field limit of the EnKF algorithm is exact only in linear Gaussian settings. The issue of non-uniqueness is manifested in the different types of EnKF algorithms reported in literature. Some of these EnKF types are discussed as part of the literature survey (in Sec. I-B) and in the main body of the paper (in Sec. II-B).

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A. Contributions of this paper

The following contributions are made in this paper:

1) **Non-uniqueness issue:** For the linear Gaussian problem, an error process is introduced to help explain the non-uniqueness issue in the selection of the control law in (1). The error process helps clarify the relationship between the different types of control laws leading to the different types of EnKF algorithms that have appeared over the years in the literature.
2) **Optimal transport FPF:** To select a unique control law, an optimization problem is proposed in the form of a time-stepping procedure. The optimality concept is motivated by the optimal transportation theory [7], [8]. The solution of the time-stepping procedure yields a unique optimal control law. The resulting filter is referred to as the optimal transport FPF. The procedure is suitably adapted to handle the case, important in Monte-Carlo implementations with finitely many N particles, where the covariance is singular. In this case, the optimal (deterministic) transport maps are replaced by optimal (stochastic) couplings. The general form of the optimal FPF includes stochastic terms which are zero when the covariance is non-singular.
3) **Error analysis:** A detailed error analysis is carried out for the deterministic form of the optimal FPF for the finite but large N limit. For the purposes of error analysis, it is assumed that the linear system is controllable and observable, and the initial empirical covariance matrix is non-singular. The main results are as follows:
   (i) Empirical mean and covariance of particles is shown to converge almost surely to exact mean and covariance as $t \to \infty$ even for finite N (Prop. 2-(i));
   (ii) Mean-squared error is shown to be bounded by $\frac{Ce^{-\lambda t}}{\sqrt{N}}$ where the constant C has polynomial dependence on the problem dimension (Prop. 2-(ii));
   (iii) A propagation of chaos analysis is carried out to show that empirical distribution of the particles converges in a weak sense to the exact filter posterior distribution (Cor. 1).
4) **Comparison to importance sampling:** For a certain simplified filtering problem, a comparison of the m.s.e. between the importance sampling and control-based filters is described. The main result is to show that using an important sampling approach, the number of particles N must grow exponentially with the dimension d. In contrast, with a control-based approach, N scales at most as order d^2 in order to maintain the same error (Prop. 4). The conclusions are also verified numerically (Fig. 1).

This paper extends and completes the preliminary results reported in our prior conference papers [1], [2]. The optimal transport formulation of the FPF and the time-stepping procedure was originally introduced in [1]. However, its extension to the singular covariance matrix case, in Sec. III-D, is original and has not appeared before. Preliminary error analysis of the deterministic form of optimal FPF appeared in our prior work [2]. The current paper extends the results in [2] in two key aspects: (i) The error bounds for the convergence of the empirical mean and covariance, in Prop. 2-(ii), reveal the scaling with the problem dimension (see Remark 3-(ii)), whereas previous results did not; (ii) The propagation of chaos analysis is carried out for the vector case (Cor. 1), whereas previous result was only valid for the scalar case. These improvements became possible with a proof approach that is entirely different than the one used in [2]. Finally, the analytical comparison with the importance sampling particle filter, in Sec. V, is new.

B. Literature survey

Two examples of the controlled interacting particle systems are the classical ensemble Kalman filter (EnKF) [9]–[12] and the more recently developed feedback particle filter (FPF) [13], [14]. The EnKF algorithm is the workhorse in applications (such as weather prediction) where the state dimension d is very high; cf., [12], [15]. The high-dimension of the state-space provides a significant computational challenge *even* in linear Gaussian settings. For such problems, an EnKF implementation may require less computational resources (memory and FLOPS) than a Kalman filter [15], [16]. This is because the particle-based algorithm avoids the need to store and propagate the error covariance matrix (whose size scales as d^2).

An expository review of the continuous-time filters including the progression from the Kalman filter (1960s) to the ensemble Kalman filter (1990s) to the feedback particle filter (2010s) appears in [17]. In continuous-time settings, the first interacting particle representation of the nonlinear filter appears in the work of Crisan and Xiong [18]. Also in continuous-time settings, Reich and collaborators have derived deterministic forms of the EnKF [12], [19]. In discrete-time settings, Daum and collaborators have pioneered the development of closely related particle flow algorithms [20], [21].

The technical approach of this paper has its roots in the optimal transportation theory. These methods have been widely applied for uncertainty propagation and Bayesian inference: The ensemble transform particle filter is based upon computing an optimal coupling by solving a linear optimization problem [22]; Polynomial approximations of the Rosenblatt transport maps for Bayesian inference appears in [23], [24]; Solution of such problems using the Gibbs flow is the subject of [25]. The time stepping procedure of this paper is inspired by the J-K-O construction in [26]. Its extension to the filtering problem appears in [27]–[30].

Closely related to error analysis of this paper is the recent literature on stability and convergence of the EnKF algorithm. For the discrete-time EnKF algorithm, these results appear in [31]–[35]. The analysis for continuous-time EnKF is more recent. For continuous-time EnKF with perturbed observation, under additional assumptions (stable and fully observable), it has been shown that the empirical distribution of the ensemble converges to the mean-field distribution uniformly for all time with the rate $O(\frac{1}{\sqrt{N}})$ [36]. This result has been extended to the nonlinear setting for the case with Langevin type dynamics with a strongly convex potential and full linear

observation [37]. The stability assumption is recently relaxed in [38]. Under certain conditions, convergence and long term stability results appear in [39].

In independent numerical evaluations and comparisons, it has been observed that EnKF and FPF exhibit smaller simulation variance and better scaling properties – with respect to the problem dimension – when compared to the traditional methods [40]–[44]. The error analysis (in Sec. IV) together with the analytical bounds on comparison with the importance sampling (in Sec. V) provide the first such rigorous justification for the performance improvement reported in literature. The analysis of this paper is likely to spur wider adoption of the control-based algorithms for the purposes of sampling and simulation.

C. Paper outline

Sec. II includes the preliminaries along with a discussion of the non-uniqueness issue. Its resolution is provided in Sec. III where the optimal FPF is derived. The error analysis appears in Sec. IV and comparison with importance sampling particle filter is given in Sec. V. The proofs appear in the Appendix.

Notation: For a vector m, $\|m\|_2$ denotes the Euclidean norm. For a square matrix Σ, $\|\Sigma\|_F$ denotes the Frobenius norm, $\|\Sigma\|_2$ is the spectral norm, $\mathrm{Tr}(\Sigma)$ is the matrix-trace, $\mathrm{Ker}(\Sigma)$ denotes the null-space, $\mathrm{Range}(\Sigma)$ denotes the range space, and $\mathrm{Spec}(\Sigma)$ denotes the spectrum. For a symmetric matrix Σ, $\lambda_{\max}(\Sigma)$ and $\lambda_{\min}(\Sigma)$ denote the maximum and minimum eigenvalues of Σ respectively. The partial order of positive definite matrices is denoted by $\succ$ such that $A \succ B$ means $A-B$ is positive definite. $\mathcal{N}(m,\Sigma)$ is a Gaussian probability distribution with mean m and covariance Σ.

II. The Non-Uniqueness Issue

A. Preliminaries

The linear Gaussian filtering problem is described by the linear stochastic differential equations (sde-s):

$$\mathrm{d}X_t = AX_t\mathrm{d}t + \sigma_B \mathrm{d}B_t, \tag{2a}$$

$$\mathrm{d}Z_t = HX_t\mathrm{d}t + \mathrm{d}W_t, \tag{2b}$$

where $X_t \in \mathbb{R}^d$ and $Z_t \in \mathbb{R}^m$ are the state and observation at time t, $\{B_t\}_{t\geq 0}, \{W_t\}_{t\geq 0}$ are mutually independent standard Wiener processes taking values in $\mathbb{R}^q$ and $\mathbb{R}^m$, respectively, and A, H, σ_B are matrices of appropriate dimension. The initial condition X_0 is assumed to have a Gaussian distribution $\mathcal{N}(m_0,\Sigma_0)$. The filtering problem is to compute the posterior distribution,

$$\pi_t(\cdot) := \mathsf{P}(X_t \in \cdot \,|\, \mathscr{Z}_t), \tag{3}$$

where $\mathscr{Z}_t := \sigma(Z_s;\, 0 \leq s \leq t)$.

Kalman-Bucy filter: In this linear Gaussian case, the posterior distribution π_t is Gaussian $\mathcal{N}(m_t,\Sigma_t)$, whose mean m_t and variance Σ_t evolve according to the Kalman-Bucy filter [45]:

$$\mathrm{d}m_t = Am_t\mathrm{d}t + \mathsf{K}_t(\mathrm{d}Z_t - Hm_t\mathrm{d}t), \tag{4a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Sigma_t = \mathrm{Ricc}(\Sigma_t) := A\Sigma_t + \Sigma_t A^\top + \Sigma_B - \Sigma H^\top H \Sigma_t, \tag{4b}$$

Variable	Notation	Equation
State of the hidden process	X_t	Eq. (2a)
State of the i^{th} particle in finite-N sys.	X_t^i	Eq. (8), (21)
State of the mean-field model	$\bar{X}_t$	Eq. (5), (19)
Kalman filter mean and covariance	m_t, Σ_t	Eq. (4a)-(4b)
Empirical mean and covariance	$m_t^{(N)}, \Sigma_t^{(N)}$	Eq. (9)
Mean-field mean and covariance	$\bar{m}_t, \bar{\Sigma}_t$	Eq. (5)-(19)
Conditional distribution of X_t	π_t	Eq. (3)
Conditional distribution of $\bar{X}_t$	$\bar{\pi}_t$	Eq. (6)
Empirical distribution of particles $\{X_t^i\}$	$\pi_t^{(N)}$	Eq. (10)

TABLE I: Nomenclature.

where $\mathsf{K}_t := \Sigma_t H^\top$ is the Kalman gain and $\Sigma_B := \sigma_B \sigma_B^\top$.

Feedback particle filter: The stochastic linear FPF [14, Eq. (26)] (and also the square-root form of the EnKBF [12, Eq (3.3)]) is described by the Mckean-Vlasov sde:

$$\mathrm{d}\bar{X}_t = A\bar{X}_t\mathrm{d}t + \sigma_B \mathrm{d}\bar{B}_t + \underbrace{\bar{\mathsf{K}}_t\big(\mathrm{d}Z_t - \frac{H\bar{X}_t + H\bar{m}_t}{2}\mathrm{d}t\big)}_{\text{FPF control law}}, \tag{5}$$

where $\bar{\mathsf{K}}_t := \bar{\Sigma}_t H^\top$ is the Kalman gain, $\bar{B}_t$ is a standard Wiener process, $\bar{m}_t := \mathsf{E}[\bar{X}_t \,|\, \mathscr{Z}_t]$, $\bar{\Sigma}_t := \mathsf{E}[(\bar{X}_t - \bar{m}_t)(\bar{X}_t - \bar{m}_t)^\top \,|\, \mathscr{Z}_t]$ are the mean-field terms, and $\bar{X}_0 \sim \mathcal{N}(m_0,\Sigma_0)$. We use

$$\bar{\pi}_t(\cdot) := \mathsf{P}(\bar{X}_t \in \cdot \,|\, \mathscr{Z}_t) \tag{6}$$

to denote the conditional distribution of mean-field process $\bar{X}_t$.

The FPF control law is exact. The exactness result appears in the following theorem which is a special case of the [14, Thm. 1] that describes the exactness result for the general nonlinear non-Gaussian case. A proof is included in Appendix A. The proof is useful for studying the non-uniqueness issue described in Sec. II-B.

Theorem 1: **(Exactness of linear FPF)** Consider the linear Gaussian filtering problem (2a)-(2b) and the linear FPF (5). If $\pi_0 = \bar{\pi}_0$ then

$$\pi_t = \bar{\pi}_t, \quad \forall t \geq 0.$$

The notation nomenclature is tabulated in Table I.

B. The non-uniqueness issue

In the proof of Thm. 1 (given in Appendix A), it is shown that (i) the conditional mean process $\{\bar{m}_t\}_{t\geq 0}$ evolves according to (4a); and (ii) the conditional variance process $\{\bar{\Sigma}_t\}_{t\geq 0}$ evolves according to (4b).

Define an error process $\xi_t := \bar{X}_t - \bar{m}_t$ for $t \geq 0$. The equation for ξ_t is obtained by subtracting the equation for the mean, (38) in Appendix A, from (5):

$$\mathrm{d}\xi_t = (A - \frac{1}{2}\bar{\Sigma}_t H^T H)\xi_t + \sigma_B \mathrm{d}\bar{B}_t$$

This is a linear system and therefore, the variance of ξ_t, easily seen to be given by $\bar{\Sigma}_t$, evolves according to the Lyapunov equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\bar{\Sigma}_t &= (A - \frac{1}{2}\bar{\Sigma}_t H^T H)\bar{\Sigma}_t + \bar{\Sigma}_t(A - \frac{1}{2}\bar{\Sigma}_t H^T H)^\top + \Sigma_B \\ &= \mathrm{Ricc}(\bar{\Sigma}_t)\end{aligned}$$

which is identical to (4b).

These arguments suggest the following general procedure to construct an exact $\bar{X}_t$ process: Express $\bar{X}_t$ as a sum of two terms:

$$\bar{X}_t = \bar{m}_t + \xi_t,$$

where $\bar{m}_t$ evolves according to the Kalman-Bucy equation (4a) and the evolution of ξ_t is defined by the sde:

$$\mathrm{d}\xi_t = G_t \xi_t \mathrm{d}t + \sigma_t \mathrm{d}\bar{B}_t + \sigma_t' \mathrm{d}\bar{W}_t$$

where $\{\bar{W}\}_{t\geq 0}$ and $\{\bar{B}\}_{t\geq 0}$ are independent Brownian motions, and G_t, σ_t, and σ_t' are solutions to the matrix equation

$$G_t \bar{\Sigma}_t + \bar{\Sigma}_t G_t^T + \sigma_t \sigma_t^\top + \sigma_t' (\sigma_t')^\top = \mathrm{Ricc}(\bar{\Sigma}_t) \tag{7}$$

By construction, the equation for the variance is given by the Riccati equation (4b).

In general, there are infinitely many solutions for (7). Below, we describe three solutions that lead to three established form of EnKF and linear FPF:

1) EnKF with perturbed observation [19, Eq. (27)]:

$$G_t = A - \bar{\Sigma}_t H^\top H, \quad \sigma_t = \bar{\Sigma}_t H^\top, \quad \sigma_t' = \sigma_B$$

2) Stochastic linear FPF [14, Eq. (26)] or square-root form of the EnKF [12, Eq (3.3)] :

$$G_t = A - \frac{1}{2}\bar{\Sigma}_t H^\top H, \quad \sigma_t = \sigma_B, \quad \sigma_t' = 0$$

3) Deterministic linear FPF [2]:

$$G_t = A - \frac{1}{2}\bar{\Sigma}_t H^\top H + \frac{1}{2}\bar{\Sigma}_t^{-1}\Sigma_B, \quad \sigma_t = 0, \quad \sigma_t' = 0$$

Given a particular solution G_t, one can construct a family of solutions $G_t + \bar{\Sigma}_t^{-1}\Omega_t$, where Ω_t is *any* skew-symmetric matrix. For the linear Gaussian problem, the non-uniqueness issue has been discussed in literature: At least two forms of EnKF, the perturbed observation form [19] and the square-root form [12], are well known. A homotopy of exact deterministic and stochastic filters is given in [46]. An explanation for the non-uniqueness in terms of the Gauge transformation appears in [47].

C. Finite-N implementation

In a numerical implementation, one simulates N stochastic processes (particles) $\{X_t^i : 1 \leq i \leq N\}_{t\geq 0}$, where X_t^i is the state of the i^{th}-particle at time t. The evolution of X_t^i is obtained upon empirically approximating the mean-field terms. The finite-N filter for the linear FPF (5) is an interacting particle system:

$$\mathrm{d}X_t^i = AX_t^i \mathrm{d}t + \sigma_B \mathrm{d}B_t^i + \mathsf{K}_t^{(N)}(\mathrm{d}Z_t - \frac{HX_t^i + Hm_t^{(N)}}{2}\mathrm{d}t) \tag{8}$$

where $\mathsf{K}_t^{(N)} := \Sigma_t^{(N)} H^\top$; $\{B_t^i\}_{i=1}^N$ are independent copies of B_t; $X_0^i \overset{\text{i.i.d}}{\sim} \mathscr{N}(m_0, \Sigma_0)$ for $i = 1, 2, \ldots, N$; and the empirical approximations of the two mean-field terms are as follows:

$$\begin{aligned} m_t^{(N)} &:= \frac{1}{N}\sum_{j=1}^N X_t^i \\ \Sigma_t^{(N)} &:= \frac{1}{N-1}\sum_{j=1}^N (X_t^i - m_t^{(N)})(X_t^i - m_t^{(N)})^\top \end{aligned} \tag{9}$$

We use the notation

$$\pi_t^{(N)} := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i} \tag{10}$$

to denote the empirical distribution of the particles. Here, δ_x denotes the Dirac delta distribution at x.

III. Optimal Transport FPF

The problem is to uniquely identify an exact stochastic process $\bar{X}_t$. The solution is based on an optimality concept motivated by the optimal transportation theory [7], [8]. The background on this theory is briefly reviewed next.

A. Background on optimal transportation

Let μ_X and μ_Y be two probability measures on $\mathbb{R}^d$ with finite second moments. The (Monge) optimal transportation problem with quadratic cost is to minimize

$$\min_T \ \mathsf{E}[(T(X) - X)^2] \tag{11}$$

over all measurable maps $T : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$X \sim \mu_X, \ T(X) \sim \mu_Y \tag{12}$$

If it exists, the minimizer T is called the *optimal transport map* between μ_X and μ_Y. The optimal cost is referred to as L^2-Wasserstein distance between μ_X and μ_Y [8].

Theorem 2: (Optimal map between Gaussians [48, Prop. 7]) Consider the optimization problem (11)-(12). Suppose μ_X and μ_Y are Gaussian distributions, $\mathscr{N}(m_X, \Sigma_X)$ and $\mathscr{N}(m_Y, \Sigma_Y)$, with $\Sigma_X, \Sigma_Y \succ 0$. Then the optimal transport map between μ_X and μ_Y is given by

$$T(x) = m_Y + F(x - m_X) \tag{13}$$

where $F = \Sigma_Y^{\frac{1}{2}}(\Sigma_Y^{\frac{1}{2}}\Sigma_X\Sigma_Y^{\frac{1}{2}})^{-\frac{1}{2}}\Sigma_Y^{\frac{1}{2}}$.

B. The time-stepping optimization procedure

To uniquely identify an exact stochastic process $\bar{X}_t$, the following time stepping optimization procedure is proposed:

1) Divide the time interval $[0, T]$ into $n \in \mathbb{N}$ equal time steps with the time instants $t_0 = 0 < t_1 < \ldots < t_n = T$.
2) Initialize a discrete time random process $\{\bar{X}_{t_k}\}_{k=1}^n$ according to the initial distribution (prior) of X_0,

$$\bar{X}_{t_0} \sim \pi_0$$

3) For each time step $[t_k, t_{k+1}]$, evolve the process $\bar{X}_{t_k}$ according to

$$\bar{X}_{t_{k+1}} = T_k(\bar{X}_{t_k}), \quad \text{for} \quad k = 0, \ldots, n-1 \tag{14}$$

where the map T_k is the optimal transport map between the probability measures π_{t_k} and $\pi_{t_{k+1}}$.
4) Take the limit as $n \to \infty$ to obtain the continuous-time process $\bar{X}_t$ and the sde:

$$\mathrm{d}\bar{X}_t = u_t(\bar{X}_t)\mathrm{d}t + \mathsf{K}_t(\bar{X}_t)\mathrm{d}Z_t \tag{15}$$

The procedure leads to the control laws u_t and K_t that depend upon π_t. Since π_t is unknown, one simply replaces

it with $\bar{\pi}_t$ – the two are meant to be identical by construction. The resulting sde (15) is referred to as the optimal transport FPF or simply the optimal FPF. A definition is needed to state the main result.

Definition 1: For a given positive-definite matrix $Q \succ 0$, define $\sqrt{\text{Ricc}}(Q)$ as the (unique such) symmetric solution to the matrix equation:

$$\sqrt{\text{Ricc}}(Q)Q + Q\sqrt{\text{Ricc}}(Q) = \text{Ricc}(Q) \tag{16}$$

Remark 1: The symmetric solution to the matrix equation (16) is explicitly given by:

$$\sqrt{\text{Ricc}}(Q) = \int_0^\infty e^{-sQ}\text{Ricc}(Q)e^{-sQ}\mathrm{d}s$$

The solution can also be expressed as:

$$\sqrt{\text{Ricc}}(Q) = A - \frac{1}{2}QH^\top H + \frac{1}{2}\Sigma_B Q^{-1} + \Omega Q^{-1} \tag{17}$$

where Ω is the (unique such) skew-symmetric matrix that solves the matrix equation

$$\begin{aligned}\Omega Q^{-1} + Q^{-1}\Omega = (A^\top - A) + \tfrac{1}{2}(QH^\top H - H^\top HQ) \\ + \tfrac{1}{2}(\Sigma_B Q^{-1} - Q^{-1}\Sigma_B)\end{aligned} \tag{18}$$

The main result is as follows. Its proof appears in the Appendix B.

Proposition 1: Consider the linear Gaussian filtering problem (2a)-(2b). Assume the initial covariance $\Sigma_0 \succ 0$. Then the optimal transport FPF is given by

$$\mathrm{d}\bar{X}_t = A\bar{m}_t\mathrm{d}t + \bar{\mathsf{K}}_t(\mathrm{d}Z_t - H\bar{m}_t\mathrm{d}t) + G_t(\bar{X}_t - \bar{m}_t)\mathrm{d}t \tag{19}$$

where $\bar{\mathsf{K}}_t := \bar{\Sigma}_t H^T$, $\bar{m}_t = \mathsf{E}[\bar{X}_t|\mathscr{Z}_t]$, $\bar{\Sigma}_t = \mathsf{E}[(\bar{X}_t - \bar{m}_t)(\bar{X}_t - \bar{m}_t)^\top|\mathscr{Z}_t]$, $\bar{X}_0 \sim N(m_0, \Sigma_0)$, and $G_t = \sqrt{\text{Ricc}}(\bar{\Sigma}_t)$

The filter is exact: That is, the conditional distribution of $\bar{X}_t$ is Gaussian $N(\bar{m}_t, \bar{\Sigma}_t)$ with $\bar{m}_t = m_t$ and $\bar{\Sigma}_t = \Sigma_t$.

Using the form of the solution (17) for $G_t = \sqrt{\text{Ricc}}(\bar{\Sigma}_t)$, the optimal transport sde (19) is expressed as

$$\begin{aligned}\mathrm{d}\bar{X}_t =& A\bar{X}_t\mathrm{d}t + \frac{1}{2}\Sigma_B\bar{\Sigma}_t^{-1}(\bar{X}_t - \bar{m}_t)\mathrm{d}t \\ &+ \frac{1}{2}\bar{\mathsf{K}}_t(\mathrm{d}Z_t - \frac{H\bar{X}_t + H\bar{m}_t}{2}\mathrm{d}t) + \Omega_t\bar{\Sigma}_t^{-1}(\bar{X}_t - \bar{m}_t)\mathrm{d}t\end{aligned} \tag{20}$$

Compared to the original (linear Gaussian) FPF (5), the optimal transport FPF (20) has two differences:

1) The stochastic term $\sigma_B\mathrm{d}\bar{B}_t$ is replaced with the deterministic term $\frac{1}{2}\Sigma_B\bar{\Sigma}_t^{-1}(\bar{X}_t - \bar{m}_t)\mathrm{d}t$. Given a Gaussian prior, the two terms yield the same posterior. However, in a finite-N implementation, the difference becomes significant. The stochastic term serves to introduce an additional error of order $O(\frac{1}{\sqrt{N}})$.
2) The sde (20) has an extra term involving the skew-symmetric matrix Ω_t. The extra term does not effect the posterior distribution. This term is viewed as a correction term that serves to make the dynamics symmetric and hence optimal in the optimal transportation sense. It is noted that for the scalar ($d = 1$) case, the skew-symmetric term is zero. Therefore, in the scalar case, the update formula in the original FPF (5) is optimal. In the vector case, it is optimal iff $\Omega_t \equiv 0$.

C. Finite-N implementation in non-singular covariance case

The finite-N implementation of the optimal transport sde (19) is given by the following system of N equations:

$$\begin{aligned}\mathrm{d}X_t^i =& Am_t^{(N)}\mathrm{d}t + \mathsf{K}_t^{(N)}(\mathrm{d}Z_t - Hm_t^{(N)}\mathrm{d}t) \\ &+ \sqrt{\text{Ricc}}(\Sigma_t^{(N)})(X_t^i - m_t^{(N)})\mathrm{d}t\end{aligned} \tag{21}$$

for $i = 1, \ldots, N$, where $\mathsf{K}_t^{(N)} := \Sigma_t^{(N)}H^\top$; $X_0^i \overset{\text{i.i.d}}{\sim} \mathscr{N}(m_0, \Sigma_0)$; and empirical approximations of mean and variance are defined in (9).

The matrix $\sqrt{\text{Ricc}}(\Sigma_t^{(N)})$ is not well-defined if $\Sigma_t^{(N)}$ is a singular matrix. This is a problem because in a finite-N implementation, it is possible that $\Sigma_t^{(N)} \not\succ 0$ even though $\Sigma_t \succ 0$. In particular, when $N < d$, the empirical covariance matrix is of rank at most N and hence singular. Note that this problem *only* arises for the optimal and deterministic forms of the FPF. In particular, the stochastic FPF (8) does not suffer from this issue. It can be simulated for any choice of N. In Sec. III-D, we extend the optimal transportation formulation to handle also the singular forms of the covariance matrix.

D. The singular covariance case

The derivation of the optimal FPF (19) crucially relies on the assumption that $\bar{\Sigma}_0 \succ 0$ which in turn implies that, in the time-stepping procedure, $\bar{\Sigma}_{t_k} \succ 0$ for $k = 0, 1, \ldots, n-1$. In the proof of Prop. 1, the assumption is used to derive the optimal transport map T_k. In general, when the covariance of Gaussian random variables $\bar{X}_{t_k}$ or $\bar{X}_{t_{k+1}}$ is singular, the optimal transport map T_k may not exist.

In the singular case, a relaxed form of the optimal transportation problem, first introduced by Kantorovich, is used to search for optimal (stochastic) couplings instead of (deterministic) transport maps [8]. The following example helps illuminate the issue:

Example 1: Consider Gaussian random variable X and Y with distributions, $\mathscr{N}(m_X, \Sigma_X)$ and $\mathscr{N}(m_Y, \Sigma_Y)$, respectively. Suppose

$$m_X = m_Y = \begin{bmatrix}0\\0\end{bmatrix}, \quad \Sigma_X = \begin{bmatrix}1 & 0\\0 & \varepsilon\end{bmatrix}, \quad \Sigma_Y = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

where $\varepsilon \geq 0$ is small. If $\varepsilon > 0$, the optimal transportation map exists, and is given by

$$Y = \begin{bmatrix}1 & 0\\0 & \frac{1}{\sqrt{\varepsilon}}\end{bmatrix} X$$

If $\varepsilon = 0$ then there is no transport map that satisfies the constraints of the optimal transportation problem.

The Kantorovich relaxation of the optimal transportation problem (11) is the following optimization problem:

$$\min_{\mu} \ \mathsf{E}_{(X,Y)\sim\mu}[|X - Y|^2] \tag{22}$$

where μ is a joint distribution on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals fixed to μ_X and μ_Y.

Although a deterministic map does not exist for the $\varepsilon = 0$ problem, a (stochastic) coupling that solves the Kantorovich problem (22) exists and is given by

$$Y = X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} B$$

where $B \sim \mathcal{N}(0,1)$ is independent of X.

In Appendix C, the Kantorovich relaxation is used to motivate an optimization problem whose solution yields the following extension of the optimal FPF:

$$\mathrm{d}\bar{X}_t = A\bar{m}_t + \bar{\mathsf{K}}_t(\mathrm{d}Z_t - H\bar{m}_t\mathrm{d}t) + G_t(\bar{X}_t - \bar{m}_t)\mathrm{d}t + \sigma_t \mathrm{d}\bar{B}_t \quad (23)$$

with $\sigma_t := P_K \sigma_B$ where P_K is the projection matrix into the kernel of the matrix $\bar{\Sigma}_t$, and G_t is any symmetric solution of the matrix equation

$$G_t\bar{\Sigma}_t + \bar{\Sigma}_t G_t = \mathrm{Ricc}(\bar{\Sigma}_t) - \sigma_t(\sigma_t)^\top \quad (24)$$

Remark 2: When $\bar{\Sigma}_t$ is singular, the solution to the matrix equation (24) is not unique. It is shown in Appendix C that the solution is of the following general form:

$$\begin{aligned} G_t =& A - \frac{1}{2}\bar{\Sigma}_t HH^\top + \frac{1}{2}P_R\Sigma_B\bar{\Sigma}_t^+ + P_K\Sigma_B\bar{\Sigma}_t^+ + \bar{\Sigma}_t^+\Sigma_B P_K \\ &+ P_R\Omega^{(1)}\bar{\Sigma}_t^+ + P_K(\Omega^{(0)} - A)P_K \end{aligned} \quad (25)$$

where $\bar{\Sigma}_t^+$ is the pseudo inverse[1] of $\bar{\Sigma}_t$, P_R is projection matrix onto the range of $\bar{\Sigma}_t$, and $\Omega^{(1)} \in \mathbb{R}^{d\times d}$ is a skew-symmetric matrix that solves a certain matrix equation (46) and $\Omega^{(0)}$ is an arbitrary symmetric matrix.

Using the formula (25), the optimal FPF (23) is expressed as follows:

$$\begin{aligned} \mathrm{d}\bar{X}_t =& A\bar{X}_t\mathrm{d}t + \frac{1}{2}(\sigma_B + \sigma_t)\sigma_B^\top\bar{\Sigma}_t^+(\bar{X}_t - \bar{m}_t)\mathrm{d}t + \sigma_t\mathrm{d}\bar{B}_t \\ &+ \bar{\mathsf{K}}_t(\mathrm{d}Z_t - \frac{H\bar{X}_t + H\bar{m}_t}{2}\mathrm{d}t) + [\text{additional terms}] \end{aligned} \quad (26)$$

The formula (26) allows one to clearly see the relationship between the deterministic and stochastic forms of the optimal FPF. In particular, when the covariance matrix is non-singular, $\bar{\Sigma}_t^+ = \bar{\Sigma}_t^{-1}$, and $\sigma_t = P_K\sigma_B = 0$. This results in the deterministic form of the optimal transport FPF (20). When the covariance matrix is singular, then the effect of the linear term $\frac{1}{2}\Sigma_B\bar{\Sigma}_t^{-1}(\bar{X}_t - \bar{m}_t)\mathrm{d}t$ in (20) is now simulated with the two terms $\frac{1}{2}(\sigma_B + \sigma_t)\sigma_B^\top\bar{\Sigma}_t^+(\bar{X}_t - \bar{m}_t)\mathrm{d}t + \sigma_t\mathrm{d}\bar{B}_t$ in (26). This is conceptually similar to the Example 1, where the deterministic optimal transport map is replaced with a stochastic coupling. The [additional terms] in (26) do not affect the distribution.

E. Finite-N implementation in the singular covariance case

The finite-N implementation of the optimal transport sde (23) is given by the following system of N equations:

$$\begin{aligned} \mathrm{d}X_t^i =& Am_t^{(N)}\mathrm{d}t + \mathsf{K}_t^{(N)}(\mathrm{d}Z_t - Hm_t^{(N)}\mathrm{d}t) + G_t^{(N)}(X_t^i - m_t^{(N)})\mathrm{d}t + \\ &\sigma_t^{(N)}\mathrm{d}B_t^i, \quad X_0^i \sim \mathcal{N}(m_0,\Sigma_0) \end{aligned} \quad (27)$$

[1]The pseudo inverse of matrix Q is the unique matrix Q^+ that satisfies $Q^+QQ^+ = Q^+$, $QQ^+Q = Q$, Q^+Q is symmetric, and QQ^+ is symmetric [49].

where $\mathsf{K}_t^{(N)} := \Sigma_t^{(N)}H^\top$; $\{B_t^i\}_{i=1}^N$ are independent copies of B_t, $\sigma_t^{(N)} = \sigma_B - \Sigma_t^{(N)}(\Sigma_t^{(N)})^+\sigma_B$, $G_t^{(N)}$ is a symmetric matrix solution to the matrix equation

$$G_t^{(N)}\Sigma_t^{(N)} + \Sigma_t^{(N)}G_t^{(N)} = \mathrm{Ricc}(\Sigma_t^{(N)}) - \sigma_t^{(N)}\sigma_t^{(N)^\top}$$

and $m_t^{(N)}$ and $\Sigma_t^{(N)}$ are empirical mean and covariance defined in (9). Note that the stochastic term is zero when $\sigma_B \in \mathrm{Range}(\Sigma_t^{(N)})$, which is true, e.g., when $\sigma_B \in \mathrm{span}\{X_t^1,\dots,X_t^N\}$.

IV. Error analysis

This section is concerned with error bounds in the large but finite N regime. Given that N is large, we restrict ourselves to the non-singular case. For the purposes of the error analysis, the following assumptions are made:

Assumption (I): The system (A,H) is detectable and (A,σ_B) is stabilizable.

Assumption (II): Assume $N > d$ and the initial empirical covariance matrix $\Sigma_0^{(N)} \succ 0$ almost surely.

The main result for the finite-N deterministic FPF (21) is as follows with the proof given in Appendix D.

Proposition 2: Consider the Kalman filter (4a)-(4b) initialized with the prior $\mathcal{N}(m_0,\Sigma_0)$ and the finite-N deterministic form of the optimal FPF (21) initialized with $X_0^i \overset{\text{i.i.d}}{\sim} \mathcal{N}(m_0,\Sigma_0)$ for $i = 1,2,\dots,N$. Under Assumption (I) and (II), the following characterizes the convergence and error properties of the empirical mean and covariance $(m_t^{(N)},\Sigma_t^{(N)})$ obtained from the finite-N filter relative to the mean and covariance (m_t,Σ_t) obtained from the Kalman filter:

(i) Convergence: For any finite $N > 1$, as $t \to \infty$:

$$\lim_{t\to\infty} e^{\lambda t}\|m_t^{(N)} - m_t\|_2 = 0 \quad \text{a.s}$$
$$\lim_{t\to\infty} e^{2\lambda t}\|\Sigma_t^{(N)} - \Sigma_t\|_F = 0 \quad \text{a.s}$$

for all $\lambda \in (0,\lambda_0)$ where λ_0 is a fixed positive constant (see (48) in Appendix D).

(ii) Mean-squared error: For any $t > 0$, as $N \to \infty$:

$$\mathsf{E}[\|m_t^{(N)} - m_t\|_2^2] \le (\text{const.})e^{-2\lambda t}\frac{\mathrm{Tr}(\Sigma_0) + \mathrm{Tr}(\Sigma_0)^2}{N} \quad (28a)$$
$$\mathsf{E}[\|\Sigma_t^{(N)} - \Sigma_t\|_F^2] \le (\text{const.})e^{-4\lambda t}\frac{\mathrm{Tr}(\Sigma_0)^2}{N} \quad (28b)$$

for all $\lambda \in (0,\lambda_0)$. The constant depends on λ, and spectral norms $\|\Sigma_0\|$, $\|\Sigma_\infty\|_2$ and $\|H\|_2$, where Σ_∞ is the solution to the algebraic Riccati equation (see Lemma 2).

Remark 3: Two remarks are in order:

1) Asymptotically (as $t \to \infty$) the empirical mean and variance of the finite-N filter becomes exact. This is because of the stability of the Kalman filter whereby the filter forgets the initial condition. In fact, for any (not necessarily i.i.d Gaussian) $\{X_0^i\}_{i=1}^N$, that satisfy the assumption $\Sigma_0^{(N)} \succ 0$, the result holds.
2) (Scaling with dimension) If the parameters of the linear Gaussian filtering problem (2a)-(2b) scale with the dimension in a way that the spectral norms $\|\Sigma_0\|_2$, $\|\Sigma_\infty\|_2$,

$\|H\|_2$, and λ_0 do not change, then the constant in the error bounds (28a)-(28b) do not change. The only term that scales with the dimension is $\mathrm{Tr}(\Sigma_0)$. For example, with $\Sigma_0 = \sigma_0^2 I_{d\times d}$, $\mathrm{Tr}(\Sigma_0) = d\sigma_0^2$. Therefore, the bound on the error typically scales as d^2 in problem dimension.

A. Propagation of chaos

In this section, we study the convergence of the empirical distribution of the particles $\pi_t^{(N)}$ for the finite-N system (21) to the exact posterior distribution π_t. Derivation of error estimates involve construction of N independent copies of the exact process (19). Consistent with the convention to denote mean-field variables with a bar, the stochastic processes are denoted as $\{\bar{X}_t^i : 1 \le i \le N\}$ where $\bar{X}_t^i$ denotes the state of the i^{th} particle at time t. The particle evolves according to the mean-field equation (19) as

$$\mathrm{d}\bar{X}_t^i = A\bar{m}_t\mathrm{d}t + \bar{\mathsf{K}}_t(\mathrm{d}Z_t - H\bar{m}_t\mathrm{d}t) + \sqrt{\mathrm{Ricc}}(\bar{\Sigma}_t)(\bar{X}_t^i - \bar{m}_t) \quad (29)$$

where $\bar{\mathsf{K}}_t = \bar{\Sigma}_t H^\top$ is the Kalman gain and the initial condition $\bar{X}_0^i = X_0^i$, the right-hand side being the initial condition of the i^{th} particle in the finite-N FPF (21). The mean-field process $\bar{X}_t^i$ is thus coupled to X_t^i through the initial condition.

In order to carry out the error analysis, the following event is defined for an arbitrary choice of a fixed matrix $\Lambda_0 \succ 0$:

$$S_{\Lambda_0} := \{\Sigma_0^{(N)} \succ \Lambda_0\} \quad (30)$$

The following proposition characterizes the error between X_t^i and $\bar{X}_t^i$, when the event S_{Λ_0} is true (the estimate is key to the propagation of chaos analysis). The proof appears in the Appendix E.

Proposition 3: Consider the stochastic processes X_t^i and $\bar{X}_t^i$ whose evolution is defined according to the optimal transport FPF (21) and its mean-field model (29), respectively. The initial condition $X_0^i \overset{\text{i.i.d}}{\sim} \mathcal{N}(m_0, \Sigma_0)$ for $i = 1, 2, \ldots, N$ Then, under Assumptions (I) and (II):

$$\mathsf{E}[\|X_t^i - \bar{X}_t^i\|_2^2 \mathbb{1}_{S_{\Lambda_0}}]^{1/2} \le \frac{(\text{const.})}{\sqrt{N}} \quad (31)$$

The estimate (31) is used to prove the following important result that the empirical distribution of the particles in the linear FPF converges weakly to the true posterior distribution. Its proof appears in the Appendix E.

Corollary 1: Consider the linear filtering problem (2a)-(2b) and the finite-N deterministic FPF (21). The initial condition $X_0^i \overset{\text{i.i.d}}{\sim} \mathcal{N}(m_0, \Sigma_0)$ for $i = 1, 2, \ldots, N$. Under Assumptions (I) and (II), for any bounded and Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\mathsf{E}\left[\left|\frac{1}{N}\sum_{i=1}^N f(X_t^i) - \mathsf{E}[f(X_t)|\mathscr{Z}_t]\right|^2\right]^{1/2} \le \frac{(\text{const.})}{\sqrt{N}}$$

V. Comparison to Importance Sampling

For the purposes of the comparison of the optimal FPF with the importance sampling-based particle filter, we restrict to the static filtering example with a fully observed observation model:

$$\begin{aligned} \mathrm{d}X_t &= 0 \qquad X_0 \sim \mathcal{N}(0, \sigma_0^2 I_d) \\ \mathrm{d}Z_t &= X_t\mathrm{d}t + \sigma_w \mathrm{d}W_t \end{aligned} \quad (32)$$

for $t \in [0,1]$, where $\sigma_W, \sigma_0 > 0$. The posterior distribution at time $t = 1$, denoted as π_{EXACT}, is a Gaussian $\mathcal{N}(m_1, \Sigma_1)$ with $m_1 = \frac{\sigma_0^2}{\sigma_0^2+\sigma_w^2} Z_1$ and $\Sigma_1 = \frac{\sigma_0^2\sigma_w^2}{\sigma_0^2+\sigma_w^2} I_d$.

Let $\{X_0^i\}_{i=1}^N$ be N i.i.d samples of X_0. The importance sampling-based particle filter yields an empirical approximation of the posterior distribution π_{EXACT} as follows:

$$\pi_{\text{PF}}^{(N)} = \sum_{i=1}^N w_i \delta_{X_0^i}, \quad w_i = \frac{e^{-\frac{\|Z_1 - X_0^i\|_2^2}{2\sigma_w^2}}}{\sum_{i=1}^N e^{-\frac{\|Z_1 - X_0^i\|_2^2}{2\sigma_w^2}}} \quad (33)$$

In contrast, given the initial samples $\{X_0^i\}_{i=1}^N$, the FPF approximates the posterior by implementing a feedback control law as follows:

$$\pi_{\text{FPF}}^{(N)} = \frac{1}{N}\sum_{i=1}^N \delta_{X_1^i}, \quad \mathrm{d}X_t^i = \frac{1}{\sigma_w^2}\Sigma_t^{(N)}(\mathrm{d}Z_t - \frac{X_t^i + m_t^{(N)}}{2}\mathrm{d}t) \quad (34)$$

where the empirical mean $m_t^{(N)}$ and covariance $\Sigma_t^{(N)}$ are approximated as (9).

The m.s.e in estimating the conditional expectation of a given function f is defined as follows:

$$\text{m.s.e}_*(f) = \mathsf{E}[\|\pi_*^{(N)}(f) - \pi_{\text{EXACT}}(f)\|_2^2]$$

where the subscript $*$ is either the PF or the FPF.

For $f(x) = \frac{1}{\sqrt{d}} 1^\top x$, a numerically computed plot of the level-sets of m.s.e, as a function of N and d, is depicted in Figure 1-(a)-(b). The expectation is approximated by averaging over $M = 1000$ independent simulations. It is observed that, in order to have the same error, the importance sampling-based approach requires the number of samples N to grow exponentially with the dimension d, whereas the growth using the FPF for this numerical example is $O(d^{\frac{1}{2}})$. This conclusion is consistent with other numerical studies reported in the literature [43].

For the purposes of the analysis, a modified form of the particle filter is considered whereby the denominator is replaced by its exact form:

$$\pi_{\overline{\text{PF}}}^{(N)} = \frac{1}{N}\sum_{i=1}^N \bar{w}_i \delta_{X_0^i}, \quad \bar{w}_i = \frac{e^{-\frac{\|Z_1 - X_0^i\|_2^2}{2\sigma_w^2}}}{\mathsf{E}[e^{-\frac{\|Z_1 - X_0^i\|_2^2}{2\sigma_w^2}} | \mathscr{Z}_1]} \quad (35)$$

The proof of the following result on the scaling of the m.s.e. with the state dimension d appears in the Appendix F:

Proposition 4: Consider the filtering problem (32) with state dimension d. Suppose $\sigma_0 = \sigma_w = \sigma > 0$ and $f(x) = a^\top x$ where $a \in \mathbb{R}^d$ with $\|a\|_2 = 1$. Then:

(i) The m.s.e. for the modified form of the importance sampling estimator (35) is given by

$$\text{m.s.e}_{\overline{\text{PF}}}(f) = \frac{\sigma^2}{N}\left(3(2^d) - \frac{1}{2}\right) \ge \frac{\sigma^2}{N} 2^{d+1} \quad (36)$$

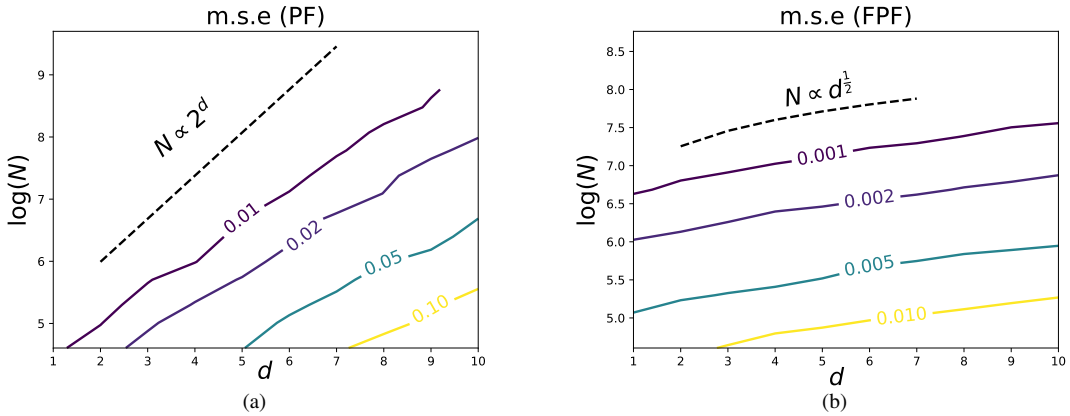

Fig. 1: Level sets of the m.s.e. for the filtering problem (32): (a) importance sampling algorithm (33) and (b) the FPF (34) algorithm.

(ii) The m.s.e for the FPF estimator (34) is bounded as

$$\text{m.s.e}_{\text{FPF}}(f) \leq \frac{\sigma^2}{N}(3d^2+2d) \tag{37}$$

Remark 4: In the limit as $d \to \infty$, the performance of the importance sampling-based particle filters has been studied in the literature [50]–[53]. The main result in these papers is concerned with the particle degeneracy (or the weight collapse) issue: it is shown that if $\frac{\log N \log d}{d} \to 0$ then the largest weight $\max_{1\leq i\leq N} w^i \to 1$ in probability. Consequently, in order to prevent the weight collapse, the number of particles should grow exponentially with the dimension. This phenomenon is referred to as the curse of dimensionality for the particle filters.

Remark 5: The scaling with dimension depicted in Figure 1 (b) suggests that the $O(d^2)$ bound for the m.s.e in the linear FPF is loose. This is the case because, in deriving the bound, the inequality $\|\cdot\|_2 \leq \|\cdot\|_F$ was used. The inequality is loose particularly so as the dimension grows. Also, it is observed that the m.s.e for the particle filter grows slightly slower than the lower-bound 2^d. This is because the lower-bound is obtained for the modified particle filter (35), while the m.s.e is numerically evaluated for the standard particle filter (33). The correlation between the numerator and denominator in (33) reduces the m.s.e error.

VI. Conclusions & Directions for Future Work

In this paper, a principled approach is presented for design of the EnKF algorithm. The approach is based upon a reformulation of the filtering problem into an optimal transportation problem. Its solution is referred to as the optimal transport FPF. Empirical approximation of the mean-field terms in the control law yield a finite-N form of the algorithm as a controlled interacting particle system. Detailed error analysis is presented for the finite-N algorithm including a comparison with the importance sampling-based approach. Taken together with numerical comparisons in recent literature, the analysis of this paper is likely to spur research and application of controlled interacting particle algorithms for filtering and data assimilation. There are many directions for future work:

1) Apart from the optimal transportation formulation stressed in this paper, one may consider alternative approaches for control design. One possible direction is based on the Schrödinger bridge problem [54], [55]. It is an interesting question whether such an approach can lead to stochastic forms of FPF, in contrast to the deterministic form obtained using the optimal transport formulation.
2) An important research direction is to extend the stability and error analysis to the class of finite-N stochastic EnKF and FPF algorithms. The results in this paper serve as baseline, in terms of assumptions and order scalings, for the analysis of the stochastic algorithm.
3) It will be of interest to construct optimization formulations that directly yield a finite-N algorithm without the need for empirical approximation as an intermediate step. Such constructions may lead to better error properties by design.
4) Finally, it is extremely important to understand the curse of dimensionality (CoD) for general types of controlled interacting particle systems. The result in Prop. 4 is very exciting because it suggests that CoD is avoided in the linear Gaussian case. Whether such a property also holds for the non-Gaussian case remains an open question.

Appendix A
Proof of Thm. 1

Proof: It is first shown that the conditional mean and variance of $\bar{X}_t$ evolve according to Kalman filtering equations. Express the sde (5) in its integral form,

$$\bar{X}_t = \bar{X}_0 + \int_0^t A\bar{X}_s \mathrm{d}s + \int_0^t \sigma_B \mathrm{d}\bar{B}_s + \int_0^t \bar{\mathsf{K}}_s(\mathrm{d}Z_s - \frac{H\bar{X}_s + H\bar{m}_s}{2}\mathrm{d}s)$$

Upon taking the conditional expectation of both sides

$$\begin{aligned}\bar{m}_t &= \mathsf{E}[\bar{X}_0|\mathscr{Z}_t] + \mathsf{E}[\int_0^t A\bar{X}_s \mathrm{d}s|\mathscr{Z}_t] + \mathsf{E}[\int_0^t \sigma_b \mathrm{d}\bar{B}_s|\mathscr{Z}_t] \\ &+ \mathsf{E}[\int_0^t \bar{\mathsf{K}}_s(\mathrm{d}Z_s - \frac{H\bar{X}_s + H\bar{m}_s}{2}\mathrm{d}s)|\mathscr{Z}_t] \\ &= \mathsf{E}[\bar{X}_0|\mathscr{Z}_0] + \int_0^t \mathsf{E}[\bar{\mathsf{K}}_s|\mathscr{Z}_s]\mathrm{d}Z_s \\ &+ \int_0^t \mathsf{E}[A\bar{X}_s - \bar{\mathsf{K}}_s\frac{H\bar{X}_s + H\bar{m}_s}{2}|\mathscr{Z}_s]\mathrm{d}s \\ &= \bar{m}_0 + \int_0^t A\bar{m}_s \mathrm{d}s + \int_0^t \bar{\mathsf{K}}_s(\mathrm{d}Z_s - H\bar{m}_s \mathrm{d}s)\end{aligned}$$

where we used the fact that $\bar{X}_t$ is adapted to the filteration $\mathscr{Z}_t$ to obtain the second identity (see [56, Lemma 5.4]). As a result, the sde for the conditional mean is

$$\mathrm{d}\bar{m}_t = A\bar{m}_t\mathrm{d}t + \bar{\mathsf{K}}_t(\mathrm{d}Z_t - H\bar{m}_t\mathrm{d}t) \tag{38}$$

Define the error ξ_t according to $\xi_t := \bar{X}_t - \bar{m}_t$. The equation for ξ_t is obtained by simply subtracting (38) from (5):

$$\mathrm{d}\xi_t = (A - \frac{\bar{\Sigma}_t H^T H}{2})\xi_t + \sigma_B \mathrm{d}\bar{B}_t$$

By application of the Itô's rule

$$\begin{aligned}\mathrm{d}(\xi_t\xi_t^\top) =& (A - \frac{1}{2}\bar{\Sigma}_t H^\top H)\xi_t\xi_t^\top \mathrm{d}t + \sigma_B \mathrm{d}\bar{B}_t\xi_t^\top \\ &+ \xi_t\xi_t^\top(A^\top - \frac{1}{2}H^\top H\bar{\Sigma}_t)\mathrm{d}t + \xi_t(\sigma_B\mathrm{d}\bar{B}_t)^\top + \Sigma_B\mathrm{d}t\end{aligned}$$

which, following the same procedure as for the conditional mean, leads to the sde for the conditional covariance $\bar{\Sigma}_t = \mathsf{E}[\xi_t\xi_t^\top|\mathscr{Z}_t]$. It is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\Sigma}_t = A\bar{\Sigma}_t + \bar{\Sigma}_t A^T + \Sigma_B - \bar{\Sigma}_t H^T H\bar{\Sigma}_t$$

identical to the Ricatti equation (4b). Hence, because $\bar{\Sigma}_0 = \Sigma_0$, $\bar{\Sigma}_t = \Sigma_t$ for all $t \geq 0$. This also implies $\bar{\mathsf{K}}_t = \mathsf{K}_t$, which in turn implies that the sde for conditional mean (38) is identical to the Kalman filter equation (4a). Therefore, again because $\bar{m}_0 = m_0$, $\bar{m}_t = m_t$ for all $t \geq 0$.

Given $\bar{\Sigma}_t = \Sigma_t$ and $\bar{m}_t = m_t$, the mean-field terms in the McKean-Vlasov sde (5) can be treated as exogenous processes. Therefore, the McKean-Vlasov sde (5) simplifies to a Ornstein-Uhlenbeck sde. Because the distribution of the initial condition $\bar{X}_0$ is Gaussian, the distribution of $\bar{X}_t$ is also Gaussian given by $\mathscr{N}(m_t, \Sigma_t)$ which is equal to the posterior distribution given by Kalman filter and concludes the proof. ∎

Appendix B
Proof of Prop. 1

The key step in the proof is the following Lemma:

Lemma 1: Consider the ode (4b). Let Σ_t be its solution for $t \in [0,T]$. Then

$$\Sigma_{t+\Delta t}^{\frac{1}{2}}(\Sigma_{t+\Delta t}^{\frac{1}{2}}\Sigma_t\Sigma_{t+\Delta t}^{\frac{1}{2}})^{-\frac{1}{2}}\Sigma_{t+\Delta t}^{\frac{1}{2}} = I + G_t\Delta t + O(\Delta t^2) \tag{39}$$

where G_t is the solution to the matrix equation

$$G_t\Sigma_t + \Sigma_t G_t = A\Sigma_t + \Sigma_t A^T + \Sigma_B - \Sigma_t H^T H\Sigma_t \tag{40}$$

and the $O(\Delta t^2)$ in (39) is uniformly bounded for all $t \in [0,T]$.

Proof: From the theory of dynamic Riccati equations, the solution is bounded over any finite time horizon [57]. Moreover, because $\Sigma_0 \succ 0$, $\Sigma_t \succ 0$. Fix $t \in [0,T]$, and define

$$F(s) := \Sigma_{t+s}^{\frac{1}{2}}(\Sigma_{t+s}^{\frac{1}{2}}\Sigma_t\Sigma_{t+s}^{\frac{1}{2}})^{-\frac{1}{2}}\Sigma_{t+s}^{\frac{1}{2}}$$

Equation (39) is obtained by considering the Taylor series of $F(s)$ at $s = 0$

$$F(\Delta t) = I + \dot{F}(0)\Delta t + \frac{1}{2}\ddot{F}(\tau)\Delta t^2$$

and showing that $\dot{F}(0) = G_t$; here $\tau \in [0, \Delta t]$. The second order term is uniformly bounded for all $t \in [0,T]$ because Σ_t is positive definite and bounded. In order to verify $\dot{F}(0) = G_t$, express

$$F(s)\Sigma_t F(s) = \Sigma_{t+s}$$

Evaluating the derivative with respect to s at $s = 0$

$$\dot{F}(0)\Sigma_t + \Sigma_t\dot{F}(0) = A\Sigma_t + \Sigma_t A^T + \Sigma_B - \Sigma_t H^\top H\Sigma_t$$

By the uniqueness property of the solution to the Lyapunov equation (40), $\dot{F}(0) = G_t$. ∎

Proof: (Prop. 1) The proof of exactness is similar to the proof of Thm. 1 and is omitted. In order to obtain the optimal transport sde, the time stepping procedure is used. The key step in the procedure is to obtain the optimal transport map T_k. The optimal map is between two Gaussians, $\mathscr{N}(m_{t_k}, \Sigma_{t_k})$ and $\mathscr{N}(m_{t_{k+1}}, \Sigma_{t_{k+1}})$. By Thm. 2, the optimal map is,

$$\bar{X}_{t_{k+1}} = m_{t_{k+1}} + F_k(\bar{X}_{t_k} - m_{t_k})$$

where $F_k = \Sigma_{t_{k+1}}^{\frac{1}{2}}(\Sigma_{t_{k+1}}^{\frac{1}{2}}\Sigma_{t_k}\Sigma_{t_{k+1}}^{\frac{1}{2}})^{-\frac{1}{2}}\Sigma_{t_{k+1}}^{\frac{1}{2}}$. Using Lemma 1,

$$\bar{X}_{t_{k+1}} = m_{t_{k+1}} + (\bar{X}_{t_k} - m_{t_k}) + G_k(\bar{X}_{t_k} - m_{t_k})\Delta t + O(\Delta t^2)$$

To obtain the sde, take a sum over $k = 0, 1, \ldots, n-1$,

$$\bar{X}_{t_n} = \bar{X}_{t_0} + m_{t_n} - m_{t_0} + \sum_{k=0}^{n-1}\left[G_k(\bar{X}_{t_k} - m_{t_k})\Delta t + O(\Delta t^2)\right]$$

In the limit as $\Delta t \to 0$,

$$\bar{X}_{t_n} = \bar{X}_{t_0} + m_{t_n} - m_{t_0} + \int_0^t G_s(\bar{X}_s - m_s)\mathrm{d}s$$

where the uniform boundedness of the second order term is used. The associated sde is,

$$\mathrm{d}\bar{X}_t = \mathrm{d}m_t + G_t(\bar{X}_t - m_t)\mathrm{d}t$$

where $\mathrm{d}m_t$ is given by (4a). Finally one obtains (19) by replacing m_t and Σ_t with $\bar{m}_t$ and $\bar{\Sigma}_t$ respectively. ∎

APPENDIX C
DERIVATION OF OPTIMAL FPF IN SINGULAR COVARIANCE CASE

Consider the following general form of the controlled process:

$$\mathrm{d}\bar{X}_t = G_t(\bar{X}_t - \bar{m}_t)\mathrm{d}t + \mathrm{d}v_t + \sigma_t \mathrm{d}\bar{B}_t \tag{41}$$

The problem is to choose G_t, v_t and σ_t such that the stochastic map $\bar{X}_t \to \bar{X}_{t+\Delta t}$ is optimal in the limit as $\Delta t \to 0$. The optimality criterion is the Kantorovich form (22) of the optimal transportation problem. The particular choice (41) of the sde for $\{\bar{X}_t\}_{t\geq 0}$ is motivated by the optimal transport sde (19) derived in Prop. 1. We expect to recover the deterministic form of the filter ($\sigma_t = 0$) for the special case when the covariance is non-singular.

The stochastic map $\bar{X}_t \to \bar{X}_{t+\Delta t}$ is given by

$$\begin{aligned}\bar{X}_{t+\Delta t} &= \bar{X}_t + \int_t^{t+\Delta t} G_s(\bar{X}_s - \bar{m}_s)\mathrm{d}s + (v_{t+\Delta t} - v_t) + \int_t^{t+\Delta t} \sigma_s \mathrm{d}B_s \\ &= \bar{X}_t + \Delta t G_t(\bar{X}_t - \bar{m}_t) + v_{t+\Delta t} - v_t + \sqrt{\Delta t}\sigma_t \zeta + o(\Delta t)\end{aligned}$$

where $\zeta \sim \mathcal{N}(0,1)$. The stochastic map is optimal if (i) the marginals $\bar{X}_t \sim \mathcal{N}(m_t, \Sigma_t)$ and $\bar{X}_{t+\Delta t} \sim \mathcal{N}(m_{t+\Delta t}, \Sigma_{t+\Delta t})$, and (ii) the transport cost $\mathsf{E}[|\bar{X}_{t+\Delta t} - \bar{X}_t|^2]$ is minimized.

Now, given $\bar{X}_t \sim \mathcal{N}(m_t, \Sigma_t)$, the marginal constraint is satisfied by the following choice:

$$\begin{aligned}&m_t + v_{t+\Delta t} - v_t = m_{t+\Delta t} \\ &(I + \Delta t G_t)\Sigma_t(I + \Delta t G_t) + \Delta t \sigma_t \sigma_t^\top + o(\Delta t) = \Sigma_{t+\Delta t}\end{aligned}$$

The first constraint simply means that the increment of v_t must be chosen according to

$$\mathrm{d}v_t = \mathrm{d}m_t = Am_t\mathrm{d}t + \mathsf{K}_t(\mathrm{d}Z_t - Hm_t\mathrm{d}t)$$

Dividing the second constraint by Δt and taking the limit as $\Delta t \to 0$ gives

$$G_t\Sigma_t + \Sigma_t G_t^\top + \sigma_t\sigma_t^\top = \mathrm{Ricc}(\Sigma_t) \tag{42}$$

which means that, in the limit as $\Delta t \to 0$, G_t and σ_t must satisfy the constraint (42). Clearly, there are infinitely many possible choices for G_t and σ_t which accounts for the non-uniqueness of the control law as discussed in Sec. II.

A unique choice is obtained by minimizing the optimal transportation cost

$$\begin{aligned}\mathsf{E}[|\bar{X}_{t+\Delta t} - \bar{X}_t|^2] = |m_{t+\Delta t} - m_t|^2 + \Delta t \underbrace{\mathrm{Tr}(\sigma_t\sigma_t^\top)}_{f_1(\sigma_t)} \\ + (\Delta t)^2 \underbrace{\mathrm{Tr}(G_t\Sigma_t G_t^\top)}_{f_2(G_t)} + o(\Delta t^2)\end{aligned}$$

Taking the limit as $\Delta t \to 0$ suggests the following sequence of problems: (i) Choose σ_t to first minimize $f_1(\sigma_t)$; and (ii) Choose G_t to next minimize $f_2(G_t)$. These formal considerations lead to the following optimization problem:

Optimization problem: Define $f_1(\sigma) := \mathrm{Tr}(\sigma\sigma^\top)$, $f_2(G) := \mathrm{Tr}(G\Sigma G^\top)$ together with the sets

$$\begin{aligned}&\mathscr{D}_\Sigma := \{(\sigma, G) \in \mathbb{R}^{d\times d_B} \times \mathbb{R}^{d\times d};\ G\Sigma + \Sigma G^\top + \sigma\sigma^\top = \mathrm{Ricc}(\Sigma)\} \\ &\mathscr{D}_\Sigma|_{\sigma^*} := \{G \in \mathbb{R}^{d\times d};\ G\Sigma + \Sigma G^\top + \sigma^*\sigma^{*\top} = \mathrm{Ricc}(\Sigma)\}\end{aligned}$$

The pair $(\sigma^*, G^*) \in \mathscr{D}_\Sigma$ are said to be *optimal* if

$$\begin{aligned}\mathrm{Tr}(\sigma^*(\sigma^*)^\top) &= \min_{(\sigma,G)\in\mathscr{D}_\Sigma} f_1(\sigma), \\ \mathrm{Tr}(G^*\Sigma G^*) &= \min_{G\in\mathscr{D}_\Sigma|_{\sigma^\star}} f_2(G)\end{aligned} \tag{43}$$

Let P_R and P_K be the orthogonal projection matrices onto the range and kernel space of Σ.

Proposition 5: Consider the optimization problem (43). Its optimal solution (σ^*, G^*) is as follows: $\sigma^* = P_K\sigma_B$ and G^* is the (unique such) symmetric solution of the matrix equation

$$G^*\Sigma + \Sigma G^* = \mathrm{Ricc}(\Sigma) - \sigma^*(\sigma^*)^\top \tag{44}$$

solve the optimization problem (43).

These choices yield the formula (23) for the optimal FPF described in Sec. III-D. It remains to prove the Proposition.

Proof: (of Prop. 5) For any $(\sigma, G) \in \mathscr{D}_\Sigma$, multiply both sides of constraint (42) from left and right by P_K to obtain

$$P_K\sigma\sigma^\top P_K = P_K\sigma_B\sigma_B P_K$$

where $P_K\Sigma = \Sigma P_K = 0$ is used. Therefore,

$$\begin{aligned}f_1(\sigma) = \mathrm{Tr}(\sigma\sigma^\top) &= \mathrm{Tr}(P_K\sigma\sigma^\top P_K) + \mathrm{Tr}(P_R\sigma\sigma^\top P_R) \\ &= \mathrm{Tr}(P_K\sigma_B\sigma_B^\top P_K) + \mathrm{Tr}(P_R\sigma\sigma^\top P_R) \\ &= \mathrm{Tr}(\sigma^*(\sigma^*)^\top) + \mathrm{Tr}(P_R\sigma\sigma^\top P_R)\end{aligned}$$

The second term is non-negative and zero iff $\sigma = \sigma^*$. Therefore, σ^* minimizes $f_1(\sigma)$.

It remains to show that G^* minimizes $f_2(G)$ over all $G \in \mathscr{D}_\Sigma|_{\sigma^*}$. We begin by showing that any symmetric solution of the Lyapunov equation (44) exists and is well-defined. The formula for the solution is given by

$$G^* = \int_0^\infty e^{-t\Sigma}(\mathrm{Ricc}(\Sigma) - P_K\sigma_B\sigma_B P_K)e^{-t\Sigma}\mathrm{d}t + P_K\Omega^{(0)}P_K$$

where $\Omega^{(0)}$ is any symmetric matrix. The integral is well-defined because

$$\begin{aligned}G^* - P_K\Omega^{(0)}P_K &= (P_K + P_R)(G^* - P_K\Omega^{(0)}P_K)(P_K + P_R) \\ &= \int_0^\infty (P_K + P_R)e^{-t\Sigma}(\mathrm{Ricc}(\Sigma) - P_K\sigma_B\sigma_B P_K)e^{-t\Sigma}(P_K + P_R)\mathrm{d}t \\ &= \int_0^\infty e^{-t\Sigma}P_R(\mathrm{Ricc}(\Sigma) - P_K\sigma_B\sigma_B P_K)e^{-t\Sigma}\mathrm{d}t \\ &\quad + \int_0^\infty e^{-t\Sigma}P_K(\mathrm{Ricc}(\Sigma) - P_K\sigma_B\sigma_B P_K)P_R e^{-t\Sigma}\mathrm{d}t\end{aligned}$$

where $P_K(\mathrm{Ricc}(\Sigma) - P_K\sigma_B\sigma_B P_K)P_K = 0$ is used. The integral is bounded because $\|e^{-t\Sigma}P_R\|_2 = \|P_R e^{-t\Sigma}\|_2 = e^{-t\mu}$ where $\mu > 0$ is the smallest non-zero eigenvalue of Σ.

It remains to show that G^* thus defined is optimal. Express an arbitrary $G \in \mathscr{D}_\Sigma|_{\sigma^*}$ as $G = G^* + V$. Since $G, G^* \in \mathscr{D}_\Sigma|_{\sigma^*}$, it is an easy calculation to see that

$$V\Sigma + \Sigma V^\top = 0$$

Now,

$$\begin{aligned}f_2(G) = \mathrm{Tr}(G\Sigma G^\top) &= \mathrm{Tr}((G^* + V)\Sigma(G^* + V)^\top) \\ &= \mathrm{Tr}(G^*\Sigma G^*) + \mathrm{Tr}(G^*\Sigma V^\top) + \mathrm{Tr}(V\Sigma G^*) + \mathrm{Tr}(V\Sigma V^\top) \\ &= \mathrm{Tr}(G^*\Sigma G^*) + \mathrm{Tr}(G^*(V\Sigma + \Sigma V^\top)) + \mathrm{Tr}(V\Sigma V^\top) \\ &= \mathrm{Tr}(G^*\Sigma G^*) + \mathrm{Tr}(V\Sigma V^\top)\end{aligned}$$

Therefore, the choice $G = G^*$ minimizes $f_2(G)$. ∎

Justification for formula (25): The goal is to show that any symmetric solution to the matrix equation (24) is of the form (25). Without loss of generality, express the solution as

$$G_t = A - \frac{1}{2}\bar{\Sigma}_t HH^\top + \frac{1}{2}P_R\Sigma_B\bar{\Sigma}_t^+ + P_K\Sigma_B\bar{\Sigma}_t^+ + \bar{\Sigma}_t^+\Sigma_B P_K + \tilde{G}_t$$

where $\tilde{G}_t$ is the new variable. Because G_t is symmetric, $\tilde{G}_t$ should satisfy

$$\begin{aligned}\tilde{G}_t - \tilde{G}_t^\top =& A^\top - A + \frac{1}{2}(\bar{\Sigma}_t HH^\top - H^\top H\bar{\Sigma}_t) \\ &- \frac{1}{2}(P_R\Sigma_B\bar{\Sigma}_t^+ - \bar{\Sigma}_t^+\Sigma_B P_R)\end{aligned} \tag{45}$$

Inserting this form of the solution for G_t to the matrix equation (24) yields:

$$\begin{aligned}&A\bar{\Sigma}_t + \bar{\Sigma}_t A^\top - \bar{\Sigma}_t HH^\top\bar{\Sigma}_t + \frac{1}{2}P_R\Sigma_B\bar{\Sigma}_t^+\bar{\Sigma}_t + \frac{1}{2}\bar{\Sigma}_t\bar{\Sigma}_t^+\Sigma_B P_R \\ &+ P_K\Sigma_B\bar{\Sigma}_t^+\bar{\Sigma}_t + \bar{\Sigma}_t\bar{\Sigma}_t^+\Sigma_B P_K + \tilde{G}_t\bar{\Sigma}_t + \bar{\Sigma}_t\tilde{G}_t^\top = \mathrm{Ricc}(\bar{\Sigma}_t) - \sigma_t\sigma_t^\top\end{aligned}$$

Using $\bar{\Sigma}_t^+\bar{\Sigma}_t = \bar{\Sigma}_t\bar{\Sigma}_t^+ = P_R$, $\sigma_t = P_K\sigma_B$ and $P_K + P_R = I$ concludes:

$$\tilde{G}_t\bar{\Sigma}_t + \bar{\Sigma}_t\tilde{G}_t^\top = 0$$

All the solutions to this linear matrix equation can be expressed as:

$$\tilde{G}_t = P_K\Omega^{(0)}P_K + P_R\Omega^{(1)}\bar{\Sigma}_t^+$$

where $\Omega^{(0)} \in \mathbb{R}^{d\times d}$ is arbitrary and $\Omega^{(1)} \in \mathbb{R}^{d\times d}$ is skew-symmetric. Next, conditions for $\Omega^{(0)}$ and $\Omega^{(1)}$ are obtained so that the constraint (45) is true. For $\Omega^{(0)}$, multiply (45) by P_K from left and right to obtain

$$P_K\Omega^{(0)}P_K - P_K\Omega^{(0)}P_K = P_K(A^\top - A)P_K$$

This condition is satisfied when $\Omega^{(0)} + A$ is symmetric. For $\Omega^{(1)}$, multiply (45) by P_R from left and right to obtain

$$\begin{aligned}&P_R\Omega^{(1)}\bar{\Sigma}_t^+ + \bar{\Sigma}_t^+\Omega_t^{(1)}P_R = P_R(A^\top - A)P_R \\ &+ \frac{1}{2}P_R(\bar{\Sigma}_t HH^\top - H^\top H\bar{\Sigma}_t)P_R\frac{1}{2}P_R(P_R\Sigma_B\bar{\Sigma}_t^+ - \bar{\Sigma}_t^+\Sigma_B, P_R)P_R\end{aligned} \tag{46}$$

This is the condition that is satisfied by $\Omega^{(1)}$.

Appendix D
Proof of the Prop. 2

The proof of the Prop. 2 relies on the stability theory of the Kalman filter. The following results are quoted without proof:

Lemma 2: Consider the Kalman filter (4a)-(4b) with initial condition (m_0, Σ_0). Then, under Assumption (I):

(i) ([58, Thm. 4.11]) There exists a solution $\Sigma_\infty \succ 0$ to the algebraic Riccati equation (ARE)

$$A\Sigma_\infty + \Sigma_\infty A^\top + \sigma_B\sigma_B^\top - \Sigma_\infty H^\top H\Sigma_\infty = 0 \tag{47}$$

such that $A - \Sigma_\infty H^\top H$ is Hurwitz. Let

$$0 < \lambda_0 = \min\{-\mathrm{Real}\,\lambda : \lambda \in \mathrm{Spec}(A - \Sigma_\infty H^\top H)\} \tag{48}$$

(ii) ([57, Thm. 1.1]) If the initial covariance matrix $\Sigma_0 \succ 0$, then there exists matrices $\Lambda_{\min}, \Lambda_{\max} \succ 0$ such that the solution Σ_t satisfies

$$\Lambda_{\min} \preceq \Sigma_t \preceq \Lambda_{\max}$$

(iii) ([59, Lem. 2.2]) The error covariance $\Sigma_t \to \Sigma_\infty$ exponentially fast for *any* initial condition Σ_0 (not necessarily the prior): for all $\lambda \in (0, \lambda_0)$, there exists a constant c_λ such that

$$\lim_{t\to\infty} \|\Sigma_t - \Sigma_\infty\|_F \le c_\lambda e^{-2\lambda t} \to 0$$

(iv) [59, Thm. 2.3] Starting from two initial conditions (m_0, Σ_0) and $(\tilde{m}_0, \tilde{\Sigma}_0)$, the means converge in the following senses:

$$\begin{aligned}\lim_{t\to\infty} \mathsf{E}[\|m_t - \tilde{m}_t\|_2^2] &\le \text{(const.)}\, e^{-2\lambda t} \to 0 \\ \lim_{t\to\infty} \|m_t - \tilde{m}_t\|_2 e^{\lambda t} &= 0 \quad \text{a.s.}\end{aligned}$$

for all $\lambda \in (0, \lambda_0)$.

In the remainder of this paper, the notation Σ_∞ is used to denote the positive definite solution of the ARE (47) and λ_0 is used to denote the spectral bound as defined in (48).

Proof of Prop. 2: Consider the finite-N filter (21) for the deterministic FPF. The empirical mean and covariance are defined in Eq. (9). The error is defined as

$$\xi_t^i := X_t^i - m_t^{(N)} \quad \text{for } i = 1, 2, \ldots, N$$

The evolution equations for the mean, covariance, and the error are as follows:

$$\mathrm{d}m_t^{(N)} = Am_t^{(N)}\mathrm{d}t + \mathsf{K}_t^{(N)}(\mathrm{d}Z_t - Hm_t^{(N)}\mathrm{d}t) \tag{49a}$$

$$\frac{\mathrm{d}\Sigma_t^{(N)}}{\mathrm{d}t} = \mathrm{Ricc}(\Sigma_t^{(N)}) \tag{49b}$$

$$\frac{\mathrm{d}\xi_t^i}{\mathrm{d}t} = \sqrt{\mathrm{Ricc}}(\Sigma_t^{(N)})\xi_t^i \tag{49c}$$

Eq. (49a) is obtained by summing up Eq. (21) for the i^{th} particle from $i = 1, \ldots, N$. Equation (49b) is obtained by application of Itö rule

$$\mathrm{d}(\xi_t^i\xi_t^{i\top}) = \sqrt{\mathrm{Ricc}}(\Sigma_t^{(N)})\xi_t^i\xi_t^{i\top}\mathrm{d}t + \xi_t^i\xi_t^{i\top}\sqrt{\mathrm{Ricc}}(\Sigma_t^{(N)})\mathrm{d}t$$

and summing over $i = 1, \ldots, N$ and dividing by $(N-1)$. Equation (49c) is obtained by subtracting (21) for X_t^i from (49a) for $m_t^{(N)}$.

Because the equations for the empirical mean (49a) and the empirical covariance (49b) are identical to the Kalman filter (4a)-(4b), the a.s. convergence of mean and variance follows from Lemma 2 on the filter stability theory. It remains to derive the mean-squared estimates. This is done in the following steps:

1) Denote $F_\infty := A - \Sigma_\infty H^\top H$. In the step 1, an estimate for the spectral norm of the transition matrix e^{tF_∞} is obtained. From Lemma 2, the eigenvalues of F_∞ have negative real parts smaller than $-\lambda_0$. Consider the Jordan decomposition $J = P^{-1}F_\infty P$ to bound

$$\|e^{tF_\infty}\|_2 \le \|P\|_2\|P^{-1}\|_2\left(\max_{0\le k\le n}\frac{t^k}{k!}\right)e^{-\lambda_0 t}, \quad \forall t > 0$$

where n the largest multiplicity of the eigenvalues of F_∞. As a result, for all $\lambda < \lambda_0$, there exists a constant $c'_\lambda := \|P\|_2\|P^{-1}\|_2 \sup_{t\geq 0} e^{-(\lambda_0-\lambda)t}\left(\max_{0\leq k\leq n} \frac{t^k}{k!}\right)$ such that

$$\|e^{tF_\infty}\|_2 \leq c'_\lambda e^{-\lambda t}$$

2) Denote $F_t := A - \Sigma_t H^\top H$ and consider the linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}x_t = F_t x_t = F_\infty x_t + (\Sigma_\infty - \Sigma_t)H^\top H x_t \tag{50}$$

Therefore

$$x_t = e^{tF_\infty}x_s + \int_s^t e^{(t-\tau)F_\infty}(\Sigma_\infty - \Sigma_\tau)H^\top H x_\tau \mathrm{d}\tau$$

Upon taking the norm and using the triangle inequality

$$\begin{aligned}\|x_t\|_2 \leq & c_\lambda e^{-t\lambda}\|x_s\|_2 \\ & + \int_s^t c_\lambda e^{-(t-\tau)\lambda}\|\Sigma_\tau - \Sigma_\infty\|_2\|H^\top H\|_2\|x_\tau\|_2 \mathrm{d}\tau\end{aligned}$$

The Gronwall inequality is then used to conclude that

$$\|x_t\|_2 \leq c'_\lambda e^{-\lambda(t-s)}\|x_s\|_2 e^{c'_\lambda\|H^\top H\|_2\int_s^t\|\Sigma_\tau-\Sigma_\infty\|_2\mathrm{d}\tau}$$

This shows that the transition matrix $\Phi_{t,s}$ for the linear system (50) is bounded as follows:

$$\|\Phi_{t,s}\|_2 \leq c'_\lambda e^{-\lambda(t-s)}e^{c'_\lambda\|H^\top H\|_2\int_s^t\|\Sigma_\tau-\Sigma_\infty\|_2\mathrm{d}\tau}$$

Now, because of the exponential convergence $\|\Sigma_t - \Sigma_\infty\|_2 \leq c_\lambda e^{-2\lambda t}$ from Lemma 2,

$$\|\Phi_{t,s}\|_2 \leq c'_\lambda e^{-\lambda(t-s)}e^{c'_\lambda\|H^\top H\|_2\frac{c_\lambda\|\Sigma_0-\Sigma_\infty\|_2}{2\lambda}}$$

and therefore $\|\Phi_{t,s}\|_2 \leq c_\lambda e^{-\lambda(t-s)}$ for some constant c_λ.

3) Consider the empirical counterpart of the linear system (50) defined using $F_t^{(N)} := A - \Sigma_t^{(N)}H^\top H$. Denote the associated transition matrix as $\Phi_{t,s}^{(N)}$. Then, because $\Sigma_t^{(N)}$ also evolves according to the Riccati equation and converges exponentially to Σ_∞, by repeating the steps above, we also obtain $\|\Phi_{t,s}^{(N)}\|_2 \leq c_\lambda e^{-\lambda(t-s)}$.

4) We are now ready to derive an estimate for the error $\Sigma_t^{(N)} - \Sigma_t$. From the Riccati equation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\Sigma_t^{(N)} - \Sigma_t) = & (A - \Sigma_t HH^\top)(\Sigma_t^{(N)} - \Sigma_t) \\ & + (\Sigma_t^{(N)} - \Sigma_t)(A - \Sigma_t^{(N)}HH^\top)^\top\end{aligned}$$

whose solution is given by

$$\Sigma_t^{(N)} - \Sigma_t = \Phi_{t,0}(\Sigma_0^{(N)} - \Sigma_0)(\Phi_{t,0}^{(N)})^\top$$

Therefore,

$$\begin{aligned}\|\Sigma_t^{(N)} - \Sigma_t\|_F &\leq \|\Phi_t^{\Sigma_t}\|_2\|\Phi_t^{\Sigma_t^{(N)}}\|_2\|\Sigma_0^{(N)} - \Sigma_0\|_F \\ &\leq c_\lambda^2 e^{-2\lambda t}\|\Sigma_0^{(N)} - \Sigma_0\|_F\end{aligned}$$

Upon squaring and taking the expectation of both sides

$$\begin{aligned}\mathsf{E}[\|\Sigma_t^{(N)} - \Sigma_t\|_F^2] &\leq c_\lambda^4 e^{-4\lambda t}\mathsf{E}[\|\Sigma_0^{(N)} - \Sigma_0\|_F^2] \\ &= c_\lambda^4 e^{-4\lambda t}\frac{1}{N}\mathsf{E}[\mathrm{Tr}((\xi_0^i\xi_0^{i\top} - \Sigma_0)^2)] \\ &\leq c_\lambda^4 e^{-4\lambda t}\frac{\mathsf{E}[\|\xi_0^i\|_2^4]}{N} = c_\lambda^4 e^{-4\lambda t}\frac{3\mathrm{tr}(\Sigma_0)^2}{N}\end{aligned}$$

5) Finally, a bound for the mean-squared error in estimating the mean is derived. Subtracting (4a) for the conditional mean from (49a) for the empirical mean yields:

$$\begin{aligned}\mathrm{d}m_t^{(N)} - \mathrm{d}m_t = & (A - \Sigma_t^{(N)}H^\top H)(m_t^{(N)} - m_t)\mathrm{d}t \\ & + (\Sigma_t^{(N)} - \Sigma_t)H^\top H \mathrm{d}I_t\end{aligned}$$

where $\mathrm{d}I_t = \mathrm{d}Z_t - Hm_t\mathrm{d}t$ is the increment of the innovation process. Its solution is given by

$$\begin{aligned}m_t^{(N)} - m_t = & \Phi_t^{(N)}(m_0^{(N)} - m_0) \\ & + \int_0^t \Phi_{t,s}^{(N)}(\Sigma_s^{(N)} - \Sigma_s)H^\top H \mathrm{d}I_s\end{aligned}$$

The norm of the first term is bounded by:

$$\begin{aligned}\mathsf{E}[\|\Phi_t^{(N)}(m_0^{(N)} - m_0)\|_2^2] &\leq c_\lambda^2 e^{-2\lambda t}\mathsf{E}[\|m_0^{(N)} - m_0\|_2^2] \\ &\leq c_\lambda^2 e^{-2\lambda t}\frac{\mathrm{Tr}(\Sigma_0)}{N}\end{aligned}$$

The norm of the second term is bounded by:

$$\begin{aligned}&\mathsf{E}\left[\left\|\int_0^t \Phi_{t,s}^{(N)}(\Sigma_s^{(N)} - \Sigma_s)H^\top \mathrm{d}I_s\right\|_2^2\right] = \\ &\int_0^t \mathsf{E}\left[\mathrm{Tr}\left(\Phi_{t,s}^{(N)}(\Sigma_s^{(N)} - \Sigma_s)H^\top H(\Sigma_s^{(N)} - \Sigma_s)\Phi_{t,s}^{(N)\top}\right)\right]\mathrm{d}s \\ &\leq \int_0^t \mathsf{E}[\|\Phi_{t,s}^{(N)}(\Sigma_s^{(N)} - \Sigma_s)\|_F^2]\|H\|_2^2\mathrm{d}s \\ &\leq \|H\|_2^2\int_0^t c_\lambda^2 e^{-2\lambda(t-s)}c_\lambda^4 e^{-4\lambda s}\mathsf{E}[\|\Sigma_0^{(N)} - \Sigma_0\|_F^2]\mathrm{d}s \\ &\leq c_\lambda^6\|H\|_2^2\frac{3\mathrm{tr}(\Sigma_0)^2}{N}\frac{e^{-2\lambda t}}{2\lambda}\end{aligned}$$

where we used the fact that the innovation process is a Brownian motion [56, Lemma 5.6] and Itô isometry in the first step. Adding the two bounds,

$$\begin{aligned}\mathsf{E}[\|m_t - m_t^{(N)}\|_2^2] \leq & e^{-2\lambda t}\frac{2c_\lambda^2\mathrm{Tr}(\Sigma_0)}{N} \\ & + e^{-2\lambda t}\frac{6c_\lambda^6\|H\|_2^2\mathrm{tr}(\Sigma_0)^2}{2\lambda N}\end{aligned}$$

Appendix E
Proofs of the Prop. 3 and Cor. 1

Proof: In the proof S is used to denote S_{Λ_0}. Use the decomposition

$$X_t^i = m_t^{(N)} + \xi_t^i, \quad \bar{X}_t^i = \bar{m}_t + \bar{\xi}_t^i$$

to bound the error as

$$\begin{aligned}\mathsf{E}[\|X_t^i - \bar{X}_t^i\|_2^2\mathbb{1}_S]^{1/2} &\leq \mathsf{E}[\|m_t^{(N)} - \bar{m}_t\|_2^2]^{1/2} + \mathsf{E}[\|\xi_t^i - \bar{\xi}_t^i\|_2^2\mathbb{1}_S]^{1/2} \\ &\leq \frac{(\text{const.})}{\sqrt{N}} + \mathsf{E}[\|\xi_t^i - \bar{\xi}_t^i\|_2^2\mathbb{1}_S]^{1/2}\end{aligned}$$

where we have used the exactness property $\bar{m}_t = m_t$ and the bound (28a) derived in Prop. 2-(ii). The hard part of the proof is to establish the following bound:

$$\mathsf{E}[\|\xi_t^i - \bar{\xi}_t^i\|_2^2\mathbb{1}_S]^{\frac{1}{2}} \leq \frac{(\text{const.})}{\sqrt{N}} \tag{51}$$

This is done next.

The two processes evolve as follows:

$$\mathrm{d}\xi_t^i = \sqrt{\mathrm{Ricc}}(\Sigma_t^{(N)})\xi_t^i \mathrm{d}t, \quad \xi_0^i = X_0^i - m_0^{(N)}$$
$$\mathrm{d}\bar{\xi}_t^i = \sqrt{\mathrm{Ricc}}(\bar{\Sigma}_t)\bar{\xi}_t^i \mathrm{d}t, \quad \bar{\xi}_0^i = X_0^i - \bar{m}_0$$

To express the solution, define the state transition matrix according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_{t,s}^{(Q_t)} = \sqrt{\mathrm{Ricc}}(Q_t)\Psi_{t,s}^{(Q_t)}, \quad \Psi_{s,s}^{(Q_t)} = I$$

Using this definition,

$$\begin{aligned}\xi_t^i - \bar{\xi}_t^i =& \Psi_{t,0}^{(\Sigma_t^{(N)})}(\xi_0^i - \bar{\xi}_0^i) \\ &+ \int_0^t \Psi_{t,s}^{(\Sigma_s^{(N)})}(\sqrt{\mathrm{Ricc}}(\bar{\Sigma}_s) - \sqrt{\mathrm{Ricc}}(\Sigma_s^{(N)}))\bar{\xi}_s^i \mathrm{d}s\end{aligned} \tag{52}$$

We claim: There exists $c_1, c_2 > 0$ and a matrix $\Lambda_{\min} \succ 0$ such that for all $t \geq s \geq 0$:

$$\mathsf{P}(\{\Sigma_t^{(N)} \succ \Lambda_{\min}\} \cap S) = 1 \tag{53a}$$
$$\mathsf{P}(\{\|\Psi_{t,s}^{(\Sigma_t^{(N)})}\|_2 \leq c_1\} \cap S) = 1 \tag{53b}$$
$$\mathsf{E}[\|\sqrt{\mathrm{Ricc}}(\bar{\Sigma}_t) - \sqrt{\mathrm{Ricc}}(\Sigma_t^{(N)})\|_2^2 \mathbb{1}_S]^{\frac{1}{2}} \leq \frac{c_2}{\sqrt{N}} e^{-2\lambda t} \tag{53c}$$

Assuming the claim is true, the bound (51) is obtained as follows:

$$\begin{aligned}\mathsf{E}[\|\xi_t^i - \bar{\xi}_t^i\|_2^2 \mathbb{1}_S]^{\frac{1}{2}} &\leq c_1 \mathsf{E}[\|\xi_0^i - \bar{\xi}_0^i\|_2^2 \mathbb{1}_S]^{\frac{1}{2}} \\ &\quad + c_1 \frac{c_2}{\sqrt{N}} \int_0^t e^{-2\lambda t} \mathsf{E}[\|\bar{\xi}_s^i\|^2]^{\frac{1}{2}} \mathrm{d}s \\ &\leq c_1 \mathsf{E}[\|m_0^{(N)} - \bar{m}_0\|_2^2]^{\frac{1}{2}} + c_1 \frac{c_2}{\sqrt{N}} \int_0^t e^{-2\lambda t} \mathrm{tr}(\bar{\Sigma}_s)^{\frac{1}{2}} \mathrm{d}s \\ &\leq c_1 \frac{\mathrm{Tr}(\Sigma_0)^{\frac{1}{2}}}{\sqrt{N}} + c_1 \frac{c_2}{\sqrt{N}} \frac{\mathrm{tr}(\Lambda_{\max})^{\frac{1}{2}}}{2\lambda}\end{aligned}$$

where we used the identity $\xi_0^i - \bar{\xi}_0^i = \bar{m}_0 - m_0^{(N)}$ and $\bar{\Sigma}_t = \Sigma_t \prec \Lambda_{\max}$ from Lemma 2-(ii).

It remains to prove the claim. The three bounds (53b)-(53a) are obtained in the following three steps:

1) (Bound (53a)): On the event S, $\Sigma_0^{(N)} \succ \Lambda_0$. Let Λ_t be the solution of the Riccati equation initialized at Λ_0. By Lemma 2-(ii), $\exists\, \Lambda_{\min} \succ 0$ such that $\Lambda_t \succ \Lambda_{\min}$. Because $\Sigma_0^{(N)} \succ \Lambda_0$, by the monotonicity property of the solution to Riccati equation (see [57, prop. 4.1]), $\Sigma_t^{(N)} \succ \Lambda_t$. Therefore, $\Sigma_t^{(N)} \succ \Lambda_{\min}$.
2) (Bound (53b)): We begin by deriving a bound for $\|\Psi_{t,s}^{(\Sigma_\infty)}\|_2$. Consider the linear system:
$$\frac{\mathrm{d}}{\mathrm{d}t}x_t = \sqrt{\mathrm{Ricc}}(\Sigma_\infty)^\top x_t$$
and a function $V(x) := x^\top \Sigma_\infty x$. Then,
$$\frac{\mathrm{d}}{\mathrm{d}t}V(x_t) = x_t^\top \sqrt{\mathrm{Ricc}}(\Sigma_\infty)\Sigma_\infty x_t + x_t^\top \Sigma_\infty \sqrt{\mathrm{Ricc}}(\Sigma_\infty)^\top x_t = 0$$
where we used the definition (16). As a result, for all $t \geq s \geq 0$,
$$\begin{aligned}x_t^\top \Sigma_\infty x_t = x_s^\top \Sigma_\infty x_s \quad &\Rightarrow \quad \|x_t\|_2^2 \leq \frac{\lambda_{\max}(\Sigma_\infty)}{\lambda_{\min}(\Sigma_\infty)}\|x_s\|_2^2 \\ &\Rightarrow \quad \|(\Psi_{t,s}^{(\Sigma_\infty)})^\top x_s\|_2 \leq c_3\|x_s\|_2 \\ &\Rightarrow \quad \|(\Psi_{t,s}^{(\Sigma_\infty)})\|_2 \leq c_3\end{aligned}$$
where the constant $c_3 := \sqrt{\frac{\lambda_{\max}(\Sigma_\infty)}{\lambda_{\min}(\Sigma_\infty)}}$.

To obtain a bound for $\|\Psi_{t,s}^{(\Sigma_t^{(N)})}\|_2$, consider the linear system
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}x_t &= \sqrt{\mathrm{Ricc}}(\Sigma_t^{(N)})x_t \\ &= \sqrt{\mathrm{Ricc}}(\Sigma_\infty)x_t + (\sqrt{\mathrm{Ricc}}(\Sigma_t^{(N)}) - \sqrt{\mathrm{Ricc}}(\Sigma_\infty))x_t\end{aligned}$$
whose solution is given by
$$x_t = \Psi_{t,0}^{(\Sigma_\infty)}x_0 + \int_0^t \Psi_{t,s}^{(\Sigma_\infty)}(\sqrt{\mathrm{Ricc}}(\Sigma_s^{(N)}) - \sqrt{\mathrm{Ricc}}(\Sigma_\infty))x_s \mathrm{d}s$$
Therefore, using the bound $\|\Psi_{t,s}^{(\Sigma_\infty)}\|_2 \leq c_3$,
$$\|x_t\|_2 \leq c_3\|x_0\|_2 + c_3 \int_0^t \|\sqrt{\mathrm{Ricc}}(\Sigma_s^{(N)}) - \sqrt{\mathrm{Ricc}}(\Sigma_\infty)\|_2 \|x_s\|_2 \mathrm{d}s$$
Now,
$$\begin{aligned}&\|\sqrt{\mathrm{Ricc}}(\Sigma_t^{(N)}) - \sqrt{\mathrm{Ricc}}(\Sigma_\infty)\|_2 \\ &= \|\frac{1}{2}\Sigma_B((\Sigma_t^{(N)})^{-1} - \Sigma_\infty^{-1}) - \frac{1}{2}(\Sigma_t^{(N)} - \Sigma_\infty)H^\top H\|_2 \\ &\leq \frac{1}{2}(\|\Sigma_B\|_2\|\Sigma_\infty^{-1}\|_2\|(\Sigma_t^{(N)})^{-1}\|_2 + \|H^\top H\|_2)\|\Sigma_t^{(N)} - \Sigma_\infty\|_2 \\ &\leq \underbrace{(\|\Sigma_B\|_2\|\Sigma_\infty^{-1}\|_2\|\Lambda_{\min}^{-1}\|_2 + \|H^\top H\|_2)c_\lambda}_{c_4} \; e^{-2\lambda t}\end{aligned}$$
where we used Lemma 2-(iii) (because $\Sigma_t^{(N)}$ is a solution of the Riccati equation (49b)) and $\|(\Sigma_t^{(N)})^{-1}\|_2 \leq \|\Lambda_{\min}^{-1}\|_2$ from (53a). Therefore,
$$\|x_t\|_2 \leq c_3\|x_0\|_2 + c_3c_4 \int_0^t e^{-2\lambda s}\|x_s\|_2 \mathrm{d}s$$
By an application of the Grönwall inequality
$$\|x_t\|_2 \leq c_3 e^{c_3c_4\int_0^t e^{-2\lambda s}\mathrm{d}s}\|x_0\|_2 \leq \underbrace{c_3 e^{c_3c_4\frac{1}{2\lambda}}}_{=:c_1}\|x_0\|_2$$
which implies $\|\Psi_{t,s}^{(\Sigma_t^{(N)})}\| \leq c_1$ for all $t \geq s \geq 0$. This is the bound (53b).
3) (Bound (53c)): We have
$$\begin{aligned}&\|\sqrt{\mathrm{Ricc}}(\Sigma_t^{(N)}) - \sqrt{\mathrm{Ricc}}(\bar{\Sigma}_t)\|_2 = \|\sqrt{\mathrm{Ricc}}(\Sigma_t^{(N)}) - \sqrt{\mathrm{Ricc}}(\Sigma_t)\|_2 \\ &\leq \frac{1}{2}(\|\Sigma_B\|_2\|\Sigma_t^{-1}\|_2\|\Sigma_t^{(N)^{-1}}\|_2 + \|H^\top H\|_2)\|\Sigma_t^{(N)} - \Sigma_t\|_2 \\ &\leq \frac{1}{2}(\|\Sigma_B\|_2\|\Lambda_{\min}^{-1}\|_2^2 + \|H^\top H\|_2)\|\Sigma_t^{(N)} - \Sigma_t\|_F\end{aligned}$$
where we used $\bar{\Sigma}_t = \Sigma_t$ (by the exactness property), $\|\Sigma_t^{(N)} - \Sigma_t\|_2 \leq \||\Sigma_t^{(N)} - \Sigma_t\|_F$, and $\|(\Sigma_t^{(N)})^{-1}\|_2 \leq \|\Lambda_{\min}^{-1}\|_2$ from (53a). Taking the squared expectation and using (28b) in Prop. 2-(ii) gives (53c).

■

Proof of the Corollary 1: Consider the event $S = S_{\frac{1}{2}\Sigma_0}$. We derive the following bounds:

$$\mathsf{E}\left[\left|\frac{1}{N}\sum_{i=1}^N f(X_t^i) - \mathsf{E}[f(X_t)|\mathscr{Z}_t]\right|^2 \mathbb{1}_S\right]^{1/2} \leq \frac{(\text{const.})}{\sqrt{N}} \tag{54}$$
$$\mathsf{E}\left[\left|\frac{1}{N}\sum_{i=1}^N f(X_t^i) - \mathsf{E}[f(X_t)|\mathscr{Z}_t]\right|^2 \mathbb{1}_{S^c}\right]^{1/2} \leq \frac{(\text{const.})}{\sqrt{N}} \tag{55}$$

1) (Bound (54)) Using the triangle inequality,

$$\mathsf{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N} f(X_t^i) - \mathsf{E}[f(X_t)|\mathscr{Z}_t]\right|^2 \mathbb{1}_S\right]^{1/2} \le \mathsf{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N} f(X_t^i) - \frac{1}{N}\sum_{i=1}^{N} f(\bar{X}_t^i)\right|^2 \mathbb{1}_S\right]^{1/2} + \mathsf{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N} f(\bar{X}_t^i) - \mathsf{E}[f(X_t)|\mathscr{Z}_t]\right|^2\right]^{1/2}$$

Now, because $\bar{X}_t^i$ are i.i.d with distribution equal to the conditional distribution, the second term on the right-hand side

$$\mathsf{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N} f(\bar{X}_t^i) - \mathsf{E}[f(X_t)|\mathscr{Z}_t]\right|^2\right]^{1/2} = \frac{\mathrm{Var}(f(X_t)|\mathscr{Z}_t)}{\sqrt{N}}$$

The first term on the right-hand side is bounded as follows:

$$\mathsf{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N} f(X_t^i) - \frac{1}{N}\sum_{i=1}^{N} f(\bar{X}_t^i)\right|^2 \mathbb{1}_S\right]^{1/2} \le \frac{1}{N}\sum_{i=1}^{N} \mathsf{E}\left[\left|f(X_t^i) - f(\bar{X}_t^i)\right|^2 \mathbb{1}_S\right]^{1/2} \le \frac{(\text{const.})}{N}\sum_{i=1}^{N} \mathsf{E}\left[\left\|X_t^i - \bar{X}_t^i\right\|_2^2 \mathbb{1}_S\right]^{1/2} \le \frac{(\text{const.})}{\sqrt{N}}$$

where we used triangle inequality in the first step, the Lipschitz property of f in the second step, and the estimate (31) from Prop. 3 in the final step.

2) (Bound (55)) The function f is assumed bounded. So,

$$\mathsf{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N} f(X_t^i) - \mathsf{E}[f(X_t)|\mathscr{Z}_t]\right|^2 \mathbb{1}_{S^c}\right]^{1/2} \le (\text{const})\mathsf{P}(S^c)^{\frac{1}{2}}$$

The probability of the event S^c is bounded as follows:

$$\mathsf{P}(S^c) = \mathsf{P}(\Sigma_0 - \Sigma_0^{(N)} \succ \frac{1}{2}\Sigma_0) \le \mathsf{P}(\|\Sigma_0^{(N)} - \Sigma_0\|_F^2 \ge \frac{1}{4}\|\Sigma_0\|_F^2) \le \frac{\mathsf{E}[\|\Sigma_0^{(N)} - \Sigma_0\|_F^2]}{\frac{1}{4}\|\Sigma_0\|_F^2} \le \frac{12\mathrm{Tr}(\Sigma_0^2)}{N\|\Sigma_0\|_F^2} = \frac{12}{N}$$

■

Appendix F
Proof of the Prop. 4

Proof: **Part (i)** Express the m.s.e as:

$$\text{m.s.e}_{\overline{\text{PF}}}(f) = \mathsf{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N} \bar{w}_i f(X_0^i) - \mathsf{E}[\bar{w}_i f(X_0^i)|\mathscr{Z}_1]\right|^2\right]$$

where we used $\pi(f) = \mathsf{E}[\bar{w}_i f(X_0^i)|\mathscr{Z}_1]$. The expectation simplifies to:

$$\text{m.s.e}_{\overline{\text{PF}}} = \frac{1}{N}\left(\underbrace{\mathsf{E}[|\bar{w}_i f(X_0^i)|^2]}_{\text{1st term}} - \underbrace{\mathsf{E}[|\mathsf{E}[\bar{w}_i f(X_0^i)|\mathscr{Z}_1]|^2]}_{\text{2nd term}}\right)$$

The two terms are simplified separately:

1) (2nd term) Note that $\mathsf{E}[\bar{w}_i f(X_0^i)|\mathscr{Z}_1] = \mathsf{E}[a^\top X_0|\mathscr{Z}_1] = \frac{\sigma_x^2}{\sigma_x^2+\sigma_w^2} a^\top Z_1 = \frac{1}{2}a^\top Z_1$ where we used $\sigma_x^2 = \sigma_w^2 = \sigma^2$ in the last step. Therefore, the (2nd term) is evaluated as

$$\mathsf{E}[|\frac{1}{2}a^\top Z_1|^2] = \frac{1}{4}a^\top \mathsf{E}[Z_1 Z_1^\top]a = \frac{\sigma^2}{2}$$

where we used $\mathsf{E}[Z_1 Z_1^\top] = \mathsf{E}[X_0 X_0^\top] + \sigma_w^2 I_d = 2\sigma^2 I_d$.

2) (1st term) We have

$$\mathsf{E}[|\bar{w}_i f(X_0^i)|^2] = \mathsf{E}\left[\frac{|f(X_0^i)|^2 e^{-\frac{\|Z_1 - X_0^i\|_2^2}{\sigma_w^2}}}{\left|\mathsf{E}[e^{-\frac{\|Z_1 - X_0^i\|_2^2}{2\sigma_w^2}}|\mathscr{Z}_1]\right|^2}\right]$$

The denominator

$$\left|\mathsf{E}[e^{-\frac{\|Z_1 - X_0^i\|_2^2}{2\sigma_w^2}}|\mathscr{Z}_1]\right|^2 = \left|\frac{(2\pi\sigma_w^2)^{d/2}}{(2\pi(\sigma_w^2+\sigma_0^2))^{d/2}} e^{-\frac{\|Z_1\|_2^2}{2(\sigma_w^2+\sigma_x^2)}}\right|^2 = \frac{1}{2^d} e^{-\frac{\|Z_1\|_2^2}{2\sigma^2}}$$

The conditional expectation of the numerator

$$\mathsf{E}[|f(X_0^i)|^2 e^{-\frac{\|Z_1 - X_0^i\|_2^2}{\sigma_w^2}}|\mathscr{Z}_1] = \frac{(\pi\sigma_w^2)^{d/2}}{(\pi(\sigma_w^2+2\sigma_0^2))^{d/2}} e^{-\frac{\|Z_1\|^2}{2\sigma_0^2+\sigma_w^2}} \mathsf{E}_{\zeta\sim\mathscr{N}(\frac{2}{3}Z_1, \frac{\sigma^2}{3}I_d)}[|f(\zeta)|^2] = \frac{1}{3^{d/2}} e^{-\frac{\|Z_1\|^2}{3\sigma^2}} (\frac{4}{9}|a^\top Z_1|^2 + \frac{\sigma^2}{3})$$

Using the tower property of the conditional expectation

$$\mathsf{E}[|\bar{w}_i f(X_0^i)|^2 = \frac{2^d}{3^{d/2}} \mathsf{E}[e^{\frac{\|Z_1\|^2}{6\sigma^2}} (\frac{4}{9}|a^\top Z_1|^2 + \frac{\sigma^2}{3})] = \frac{2^d}{3^{d/2}} \frac{(12\pi\sigma^2)^{d/2}}{(4\pi\sigma^2)^{d/2}} \mathsf{E}_{\zeta\sim\mathscr{N}(0,6\sigma^2 I_d)}[\frac{4}{9}|a^\top\zeta|^2 + \frac{\sigma^2}{3}] = 2^d(3\sigma^2)$$

The two terms are combined to obtain the formula (36).

Part (ii) The FPF estimator is $\pi_{\text{FPF}}(f) = a^\top m_1^{(N)}$ where

$$\mathrm{d}m_t^{(N)} = \mathsf{K}_t^{(N)}(\mathrm{d}Z_t - m_t^{(N)}\mathrm{d}t)$$

where $\mathsf{K}_t^{(N)} = \frac{1}{\sigma_w^2}\Sigma_t^{(N)}$. The exact mean evolves according to:

$$\mathrm{d}m_t = \mathsf{K}_t(\mathrm{d}Z_t - m_t\mathrm{d}t)$$

where $\mathsf{K}_t = \frac{1}{\sigma_w^2}\Sigma_t$. Therefore, the difference $m_t^{(N)} - m_t$ solves the sde:

$$\mathrm{d}m_t^{(N)} - \mathrm{d}m_t = -\mathsf{K}_t^{(N)}(m_t^{(N)} - m_t)\mathrm{d}t + (\mathsf{K}_t^{(N)} - \mathsf{K}_t)\mathrm{d}I_t$$

where $\mathrm{d}I_t = \mathrm{d}Z_t - m_t\mathrm{d}t$ is the increment of the innovation process. Let $\Phi_{t,s}$ be the state transition matrix for the linear system $\frac{\mathrm{d}}{\mathrm{d}t}x_t = -\mathsf{K}_t^{(N)}x_t$. In terms of this matrix

$$m_1^{(N)} - m_1 = \Phi_{1,0}(m_0^{(N)} - m_0) + \int_0^1 \Phi_{1,t}(\mathsf{K}_t^{(N)} - \mathsf{K}_t)\mathrm{d}I_t$$

Taking an inner product of both sides with a yields

$$\begin{aligned} a^\top m_1^{(N)} - a^\top m_1 =& a^\top\Phi_{1,0}(m_0^{(N)} - m_0) \\ &+ \int_0^1 a^\top\Phi_{1,t}(\mathsf{K}_t^{(N)} - \mathsf{K}_t)\mathrm{d}I_t \end{aligned}$$

Therefore,

$$\begin{aligned} &\mathsf{E}[|a^\top m_1^{(N)} - a^\top m_1|^2] \le \\ &2\mathsf{E}[|a^\top\Phi_{1,0}(m_0^{(N)} - m_0)|^2] + 2\mathsf{E}[(\int_0^1 a^\top\Phi_{1,t}(\mathsf{K}_t^{(N)} - \mathsf{K}_t)\mathrm{d}I_t)^2] \end{aligned}$$

The formula (37) follows by showing the following bounds for the two terms:

$$\mathsf{E}[|a^\top\Phi_{1,0}(m_0^{(N)} - m_0)|^2] \le \frac{2d\sigma^2}{N} \tag{56a}$$

$$\mathsf{E}[(\int_0^1 a^\top\Phi_{1,t}(\mathsf{K}_t^{(N)} - \mathsf{K}_t)\mathrm{d}I_t)^2] \le \frac{3d^2\sigma^2}{N} \tag{56b}$$

1) (Bound (56a)) The spectral norm $\|\Phi_{t,s}\|_2 \le 1$ because $\mathsf{K}_t^{(N)} = \frac{1}{\sigma^2}\Sigma_t^{(N)} \succeq 0$. Therefore, $|a^\top\Phi_{1,0}(m_0^{(N)} - m_0)| \le \|a\|_2\|\Phi_{1,0}\|_2\|(m_0^{(N)} - m_0)\|_2 \le \|(m_0^{(N)} - m_0)\|_2$ and

$$\mathsf{E}[|a^\top\Phi_{1,0}(m_0^{(N)} - m_0)|^2] \le \mathsf{E}[\|m_0^{(N)} - m_0\|_2^2] = \frac{\sigma_0^2 d}{N}$$

2) (Bound (56b)) By the Itô isometry, because the innovation process is a Brownian motion [56, Lemma 5.6],

$$\begin{aligned} &\mathsf{E}[(\int_0^1 a^\top\Phi_{1,t}(\mathsf{K}_t^{(N)} - \mathsf{K}_t)\mathrm{d}I_t)^2] \\ &= \sigma_w^2\mathsf{E}[\int_0^1 a^\top\Phi_{1,t}(\mathsf{K}_t^{(N)} - \mathsf{K}_t)^2\Phi_{1,t}^\top a\mathrm{d}t] \\ &\le \frac{1}{\sigma_w^2}\int_0^1 \mathsf{E}[\|\Sigma_t^{(N)} - \Sigma_t)\|_2^2]\mathrm{d}t \end{aligned}$$

where we used $\|\Phi_{1,t}\|_2 \le 1$ and $\|a\|_2 = 1$ to derive the inequality.

Next, we bound the spectral norm $\|\Sigma_t^{(N)} - \Sigma\|_2$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Sigma_t^{(N)} = -\frac{1}{\sigma_w^2}(\Sigma_t^{(N)})^2, \quad \frac{\mathrm{d}}{\mathrm{d}t}\Sigma_t = -\frac{1}{\sigma_w^2}\Sigma_t^2$$

and thus

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\Sigma_t^{(N)} - \Sigma_t) =& -\frac{1}{2\sigma_w^2}(\Sigma_t^{(N)} + \Sigma_t)(\Sigma_t^{(N)} - \Sigma_t) \\ &- \frac{1}{2\sigma_w^2}(\Sigma_t^{(N)} - \Sigma_t)(\Sigma_t^{(N)} + \Sigma_t) \end{aligned}$$

Its solution is obtained as

$$\Sigma_t^{(N)} - \Sigma_t = \Phi_t(\Sigma_0^{(N)} - \Sigma_0)\Phi_t^\top$$

where Φ_t solves $\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t = -\frac{1}{2\sigma^2}(\Sigma_t + \Sigma_t^{(N)})\Phi_t$ with $\Phi_0 = I$. Because $\Sigma_t + \Sigma_t^{(N)} \succeq 0$, the spectral norm $\|\Phi_t\|_2 \le 1$. Therefore,

$$\|\Sigma_t^{(N)} - \Sigma_t\|_2 \le \|\Sigma_0^{(N)} - \Sigma_0\|_2$$

and

$$\begin{aligned} \frac{1}{\sigma_w^2}\int_0^1 \mathsf{E}\|\Sigma_t^{(N)} - \Sigma_t)\|_2^2\mathrm{d}t &\le \frac{1}{\sigma_w^2}\|\Sigma_0^{(N)} - \Sigma_0\|_2 \\ &\le \frac{1}{\sigma_w^2}\|\Sigma_0^{(N)} - \Sigma_0\|_F \\ &\le \frac{1}{\sigma_w^2}\frac{1}{N}\mathsf{E}[\|X_0^i\|^4] \\ &\le \frac{1}{\sigma_w^2}\frac{3\sigma_0^4 d^2}{N} \end{aligned}$$

■

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