



Dualizing spheres for compact p -adic analytic groups and duality in chromatic homotopy

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Abstract The primary goal of this paper is to study Spanier–Whitehead duality in the $K(n)$ -local category. One of the key players in the $K(n)$ -local category is the Lubin–Tate spectrum E_n , whose homotopy groups classify deformations of a formal group law of height n , in the implicit characteristic p . It is known that E_n is self-dual up to a shift; however, that does not fully take into account the action of the automorphism group $\mathbb{G}_n$ of the formal group in question. In this paper we find that the $\mathbb{G}_n$ -equivariant dual of E_n is in fact E_n twisted by a sphere with a non-trivial (when $n > 1$) action by $\mathbb{G}_n$. This sphere is a dualizing module for the group $\mathbb{G}_n$, and we construct and study such an object $I_{\mathcal{G}}$ for any compact p -adic analytic group $\mathcal{G}$. If we restrict the action of $\mathcal{G}$ on $I_{\mathcal{G}}$ to certain type of small subgroups, we identify $I_{\mathcal{G}}$ with a specific representation sphere coming from the Lie algebra of $\mathcal{G}$. This is done by a

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classification of p -complete sphere spectra with an action by an elementary abelian p -group in terms of characteristic classes, and then a specific comparison of the characteristic classes in question. The setup makes the theory quite accessible for computations, as we demonstrate in the later sections of this paper, determining the $K(n)$ -local Spanier–Whitehead duals of E_n^{hH} for select choices of p and n and finite subgroups H of $\mathbb{G}_n$.

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1 Introduction

This project began with a contemplation of duality in the $K(n)$ -local stable homotopy category, so let us start with explaining why this is an interesting topic. One of the standard approaches to stable homotopy theory emphasizes complex oriented cohomology theories, those with a natural theory of Chern classes. Any such cohomology theory determines a smooth 1-parameter formal group, and the algebraic geometry of formal groups can be used to organize calculations and the search for large scale phenomena. Over an algebraically closed field of characteristic p , formal groups are classified up to isomorphism by a single invariant, the height. Fix a prime p and a height n , and choose a representative F_n for this isomorphism class. It is convenient to assume F_n is defined over some finite field $\mathbb{F}_q$ of characteristic p , and usually $q = p^n$. There is a complex oriented cohomology theory $\mathbf{K}^* := K(n)^* = K(\mathbb{F}_q, F_n)^*$ determined by the pair $(\mathbb{F}_q, F_n)$. Let $\mathbf{K}$ be the representing spectrum; this is a Morava K -theory. One standard approach is to then study the homotopy theory

of $\mathbf{K}$ -local spectra in the sense of Bousfield [15] and, perhaps later, assemble that information into a more global picture.

In many ways, the $\mathbf{K}$ -local category is much better behaved than the full stable homotopy category. To begin with there is a very effective computational tool based on Lubin–Tate (or Morava) E -theory. The universal deformation of F_n determines a Landweber exact complex oriented cohomology theory $\mathbf{E}^* := E_n^* = E(\mathbb{F}_q, F_n)^*$. Let $\mathbf{E}$ be the representing spectrum. If we let $\mathbb{G}$ be the automorphism group of the pair $(\mathbb{F}_q, F_n)$, then $\mathbb{G}$ is a profinite group and acts on $\mathbf{E}$. If X is any spectrum we define

$$\mathbf{E}_*X = \pi_*L_{\mathbf{K}}(\mathbf{E} \wedge X).$$

Then $\mathbf{E}_*X$ comes equipped with an $\mathbf{E}_*$ -module structure and a compatible continuous action of $\mathbb{G}$; we say $\mathbf{E}_*X$ is a *Morava module*. The $\mathbf{K}$ -local Adams–Novikov Spectral Sequence based on $\mathbf{E}$ reads

$$H^s(\mathbb{G}, \mathbf{E}_tX) \implies \pi_{t-s}L_{\mathbf{K}}X.$$

Here we are using continuous group cohomology. If p is large with respect to n , this spectral sequence collapses for some important examples, such as $X = S^0$. Even when it does not collapse the Morava module $\mathbf{E}_*X$ is a very sensitive and informative algebraic invariant of X .

The group $\mathbb{G}$ is not simply a profinite group; it is a compact p -adic analytic group. It is a classical observation that the category of continuous modules over such groups behaves very much like the category of quasi-coherent sheaves on a very nice projective scheme; for example, there is a very good notion of Grothendieck–Serre duality. See [62] for a classical source and especially [64] for a thorough modern treatment. The assignment $X \mapsto \mathbf{E}_*X$ is not an equivalence of categories in general. (But see [59] for more on this point.) Nonetheless, the structure of continuous $\mathbb{G}$ -modules is a very good indicator of what might be true for the $\mathbf{K}$ -local category. In particular, there are rich theories of duality in the $\mathbf{K}$ -local category that can be glimpsed by studying Morava modules. The investigation of these dualities has been an important aspect of research in chromatic homotopy theory.

Here, we will concentrate on Spanier–Whitehead Duality in the $\mathbf{K}$ -local category. Any spectrum X has a $\mathbf{K}$ -local Spanier–Whitehead dual defined by the function spectrum

$$DX = F(X, L_{\mathbf{K}}S^0).$$

The spectrum $L_{\mathbf{K}}S^0$ is the unit for the natural symmetric monoidal structure on the $\mathbf{K}$ -local category, so this is a very basic duality. There is another duality, more closely related to Grothendieck–Serre duality, known as Gross–Hopkins

(or $\mathbf{K}$ -local Brown-Comenetz) duality. Without going into detail, if $I_n(-)$ denotes Gross–Hopkins duality, then $D(X) \wedge I_n \simeq I_n(X)$ for a certain invertible object I_n in the $\mathbf{K}$ -local category, so the two dualities are closely related. For more on this point, see [63] and [45].

Spanier–Whitehead duality in the global (i.e. unlocalized) stable homotopy category is often extremely hard to compute and behaves in strange and surprising ways when X is not finite. However, it is much better behaved in the $\mathbf{K}$ -local category. For example, quite a few years ago, the third author noted that $D\mathbf{E}$ is essentially $\mathbf{E}$; in fact, since $\mathbb{G}$ is a compact p -adic analytic group of rank (or dimension) n^2 and since $\mathbf{E}^*\mathbf{E}$ is isomorphic, as a graded $\mathbb{G}$ -module to the profinite group ring $\mathbf{E}^*[[\mathbb{G}]]$, there is an isomorphism of continuous Morava modules

$$\pi_* D\mathbf{E} \cong \pi_* \Sigma^{-n^2} \mathbf{E}. \quad (1.1)$$

See [63] for details. Note that we could equally write $\pi_* D\mathbf{E} \cong \pi_* F(S^{n^2}, \mathbf{E})$. We could then hope that this isomorphism is induced by a $\mathbb{G}$ -equivariant equivalence $D\mathbf{E} \simeq \Sigma^{-n^2} \mathbf{E}$, but this is false. In fact, it is not true even when restricted to certain key finite subgroups of $\mathbb{G}$. Indeed, Behrens [9] and Bobkova [14] have computed $D(\mathbf{E})^{hH}$ for maximal finite subgroups H of $\mathbb{G}$ at $n = 2$, when $p = 3$ and $p = 2$, respectively, and their results, in particular, show that $D\mathbf{E}$ is not $\mathbb{G}$ -equivariantly equivalent to $\Sigma^{-n^2} \mathbf{E}$. Our first goal was to understand this subtlety.

Ultimately, it turns out that this is not a question about $\mathbf{E}$, but a question about $\mathbb{G}$. In fact, it is fairly formal, if technically formidable, to arrive at the following result.

Theorem 11.16 *There is a p -complete spectrum $I_{\mathbb{G}}$ with an action of $\mathbb{G}$, whose underlying p -complete homotopy type is S^{n^2} such that there is a $\mathbb{G}$ -equivariant equivalence*

$$D\mathbf{E} \simeq F(I_{\mathbb{G}}, \mathbf{E}).$$

By combining Remark 4.23 and Proposition 5.9 we also have that the action of $\mathbb{G}$ on $H_{n^2} I_{\mathbb{G}} \cong \mathbb{Z}_p$ is trivial. These results imply the formula (1.1).

The question then is to understand the equivariant homotopy type of $I_{\mathbb{G}}$. To do so, it is helpful to generalize and replace $\mathbb{G}$ by an arbitrary compact p -adic analytic group $\mathcal{G}$. Any such $\mathcal{G}$ has an exhaustive nested sequence of open subgroups, all normal in $\mathcal{G}$,

$$\cdots \subseteq \Gamma_{i+1} \subseteq \Gamma_i \subseteq \cdots \subseteq \Gamma_1 \subseteq \Gamma_0 = \mathcal{G},$$

with the property that for large enough i , each Γ_i is a particularly nice p -profinite group known as a uniformly powerful group. We review this theory in Sect. 2. We have that $\Gamma_{i+1} \subseteq \Gamma_i$ is of finite index and $\mathcal{G}$ acts on the classifying space $B(\Gamma_i/\Gamma_j)$ by conjugation. We first define the $\mathcal{G}$ -space

$$B\Gamma_i = \operatorname{holim}_j B(\Gamma_i/\Gamma_j),$$

where the limit is over the projections $\Gamma_i/\Gamma_{j+1} \rightarrow \Gamma_i/\Gamma_j$ and then we define a p -complete $\mathcal{G}$ -spectrum

$$I_{\mathcal{G}} = \left(\operatorname{hocolim}_i \Sigma_+^{\infty} B\Gamma_i \right)_p^{\wedge},$$

where Σ_+^{∞} denotes the suspension spectrum after adding a disjoint basepoint. The colimit is over the transfer maps

$$\operatorname{tr}: \Sigma_+^{\infty} B\Gamma_i \rightarrow \Sigma_+^{\infty} B\Gamma_{i+1},$$

which are equivariant with respect to the $\mathcal{G}$ -action; see Remark 5.3. Then $I_{\mathcal{G}}$ has the underlying homotopy type of a p -complete d -sphere, where d is the rank of $\mathcal{G}$. See Proposition 5.6. Even in the cases where $\mathcal{G} = \mathbb{G}$ or $\operatorname{Gl}_n(\mathbb{Z}_p)$ the resulting action of $\mathcal{G}$ on $I_{\mathcal{G}}$ is rich and potentially very non-linear, since the action on the building blocks is rich and highly non-linear.

There is a linear analog of $I_{\mathcal{G}}$ which is much simpler for computations. Because $\mathcal{G}$ is a compact p -adic analytic group it has a p -adic Lie algebra $\mathfrak{g}$; this is a free $\mathbb{Z}_p$ -module of rank d , where d is the rank of $\mathcal{G}$. The conjugation action makes $\mathfrak{g}$ into a $\mathcal{G}$ -module called the adjoint representation. We can define a linear $\mathcal{G}$ -sphere

$$S^{\mathfrak{g}} = \left(\operatorname{hocolim}_i \Sigma_+^{\infty} B(p^i \mathfrak{g}) \right)_p^{\wedge}$$

using this representation, where again the colimit is along transfers. Then $S^{\mathfrak{g}}$ has the underlying homotopy type of a p -complete d -sphere and there is an isomorphism of $\mathcal{G}$ -modules

$$H_d(I_{\mathcal{G}}, \mathbb{Z}_p) \cong \Lambda^d \mathfrak{g} \cong H_d(S^{\mathfrak{g}}, \mathbb{Z}_p),$$

as we recall in Proposition 5.9. Here $\Lambda^d \mathfrak{g}$ denotes the top exterior power of the adjoint representation. This is the classical dualizing module for continuous $\mathcal{G}$ -modules. Again, see [62] or [64] for details. In analogy with the classical algebraic duality statements discussed in these sources, one is led to the following conjecture.

Linearization Hypothesis 1 In an appropriate category of continuous $\mathcal{G}$ -spheres there is a $\mathcal{G}$ -equivariant equivalence $S^{\mathcal{G}} \simeq I_{\mathcal{G}}$.

This conjecture appeared in the work of Clausen, [20, §6.4]. Subsequently, in [21], Clausen announced a proof of this conjecture in full generality, including a discussion of what category is a natural home for the spectrum $I_{\mathcal{G}}$ with its $\mathcal{G}$ -action. Looking further back, duality and dualizing modules in the $\mathbf{K}$ -local category featured in work from the early '90s by the third author and his collaborators. See [26], especially the last section, and [39].

Remark 1.2 The analogue of the Linearization Hypothesis for compact Lie groups was stated and proved by Klein in [47]. In a different (genuinely equivariant) setting, the Wirthmüller isomorphism can also be cast as a duality statement for compact Lie groups as in Fausk-Hu-May [29].

In this paper, we do not prove the Linearization Hypothesis in its full strength, but we show it holds when the action of $\mathcal{G}$ is restricted to certain small subgroups. Here we write $Z(\mathcal{G})$ for the center of $\mathcal{G}$.

Theorem 8.11 *Let $\mathcal{G}$ be compact p -adic analytic group and let H be a closed subgroup of $\mathcal{G}$ such that $H/H \cap Z(\mathcal{G})$ is finite. Suppose the p -Sylow subgroup of $H/H \cap Z(\mathcal{G})$ is an elementary abelian p -group. Then there is an H -equivariant equivalence*

$$I_{\mathcal{G}} \simeq S^{\mathcal{G}}.$$

Although we do not prove the Linearization Hypothesis in its full strength, our methods are very different from Clausen's and are valuable in and of themselves: They allow us to access the result for specific computations, as we demonstrate in the latter sections of this paper. A major input in the proof are techniques from Lannes theory [48], leading up to the following result, which we deduce directly from the work of Castellana [19].

Theorem 6.24 *Let F be an elementary abelian p -group and let X be a p -complete F -sphere of virtual dimension k . Then there is stable vector bundle ξ over BF of virtual dimension k and a p -equivalence of spectra*

$$M\xi \simeq EF_+ \wedge_F X.$$

Furthermore there is an F -equivalence $X \simeq Y$ of p -complete F -spheres if and only if there is an isomorphism of modules over the Steenrod algebra

$$H^*(EF_+ \wedge_F X) \cong H^*(EF_+ \wedge_F Y).$$

Such an isomorphism uniquely determines the F -equivalence up to F -homotopy.

This result tells us that any stable H -sphere, and in particular either $I_{\mathcal{G}}$ or $S^{\mathfrak{g}}$, is linearizable and, as such, is determined by its characteristic classes. We have included the proof here in its entirety, as many of the details are missing in loc.cit. We then dive deeply into the constructions of $I_{\mathcal{G}}$ and $S^{\mathfrak{g}}$ to show that their characteristic classes match.

The natural question that arises at this point is whether our methods for proving Theorem 8.11 can be bootstrapped to the case when $H/H \cap Z(\mathcal{G})$ is a larger finite group. This is tied to the problem of classifying stable spheres with an action of a finite group. This problem, as well as its unstable version, are well studied in homotopical representation theory, though the methods there for using Lannes theory to obtain results about general groups do not apply here. Nevertheless, work of Jesper Grodal and Jeff Smith [35] may point the way toward generalizations.

The essence of linearizability is distilled in the following result, which makes Theorem 8.11 accessible for computations. It gives a hold on the H -equivariant homotopy type of $S^{\mathfrak{g}}$. We prove this by implementing ideas from geometric topology. For simplicity, we limit our attention to finite subgroups of $\mathcal{G}$, but this is not restrictive in practice.

Proposition 9.8 and Proposition 9.14 *Let $\mathcal{G}$ be a compact p -adic analytic group and let $F \subseteq \mathcal{G}$ be a finite subgroup. Suppose there is a finitely generated free abelian group $L \subseteq \mathfrak{g}$ with the properties that*

- (1) L is stable under the adjoint action of F on $\mathfrak{g}$, and
- (2) $L/pL \cong \mathfrak{g}/p\mathfrak{g}$ (or, more generally, $\mathbb{Q}_p \otimes L \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathfrak{g}$).

Let $V = \mathbb{R} \otimes L$ and let S^V be the one-point compactification of V . Then there is an F -equivariant map $S^V \rightarrow S^{\mathfrak{g}}$ which becomes a weak equivalence after completion at p .

This result unlocks the applications of Theorem 8.11 in the case of the Morava stabilizer group $\mathbb{G}$ in chromatic homotopy theory. Specifically, we develop a unified strategy to address the problem of determining $\mathbf{K}$ -local Spanier–Whitehead duals $D(\mathbf{E}^{hH})$ of spectra of the form $\mathbf{E}^{hH}$, for H as in Theorem 8.11. For this application, we need more than the above duality results applied to the profinite group $\mathbb{G}$. Namely, we need a good understanding of invertible equivariant $\mathbf{E}$ -modules. We develop this in Sect. 12, where again the theory of characteristic classes plays a leading role. Further, we revisit the known examples mentioned above but also explore new cases. An important part of this paper develops this new strategy for addressing familiar questions in chromatic homotopy theory.

With that in hand, we apply the observation that if H is finite, then Tate vanishing [36] implies that there is an equivalence

$$D(\mathbf{E}^{hH}) \simeq (DE)^{hH}.$$

This (now) standard but subtle fact allows us to combine our results on the equivariant homotopy type of $D\mathbf{E}$ and the results on equivariant invertible $\mathbf{E}$ -modules to calculate $D(\mathbf{E}^{hH})$ for interesting H .

As a first example, if $H \subset Z(\mathbb{G})$ is a finite subgroup of the center of $\mathbb{G}$, then

$$D(\mathbf{E}^{hH}) \simeq \Sigma^{-n^2} \mathbf{E}^{hH}. \quad (1.3)$$

This follows from Theorem 11.16 and the fact that H acts trivially on $I_{\mathbb{G}}$. The center of $\mathbb{G}$ is not large; indeed, $Z(\mathbb{G}) \cong \mathbb{Z}_p^\times$, the units in the p -adic integers. Hence, this example has the most impact when $p = 2$ and

$$H = C_2 = \{\pm 1\} \subseteq \text{Aut}(F_n/\mathbb{F}_{p^n}) \subseteq \mathbb{G}.$$

The full force of Theorem 11.16, Theorem 8.11, and Proposition 9.8 can also be used to recover the results of Behrens [9] and Bobkova [14]. If n is small with respect to p , then $\mathbb{G}$ can contain p -torsion elements. For example if $n = 2$ and $p = 2$, the maximal 2-torsion subgroup is isomorphic to the quaternionic group Q_8 of order 8. If $n = 2$ and $p = 3$, the maximal 3-torsion subgroup is a cyclic group C_3 of order 3. In the result below, $\mathbb{G} = \text{Aut}(\mathbb{F}_{p^2}, F_C)$ where F_C is the formal group law of certain supersingular elliptic curves.

Theorem 13.12 and Theorem 13.25 *Let $n = 2$ and $p = 2$ or 3 . Let $F \subseteq G \subseteq \mathbb{G}$ where G is a maximal finite subgroup of $\mathbb{G}$ containing the maximal p -torsion subgroup. Then*

$$D(\mathbf{E}^{hF}) \simeq \Sigma^{44} \mathbf{E}^{hF}.$$

The spectrum $\mathbf{E}^{hG}$ is the $\mathbf{K}$ -localization of the spectrum of topological modular forms. At either prime, the Morava module $\mathbf{E}_* \mathbf{E}^{hG}$ is 24-periodic; from this and (1.1) one can prove $\mathbf{E}_* D(\mathbf{E}^{hG}) \simeq \Sigma^{24k-4} \mathbf{E}^{hG}$ for some integer k . The difficulty is to understand why k must be 2.

For $F = G$, Theorem 13.12 ($p = 3$) was originally proved by Behrens in [9] and Theorem 13.25 ($p = 2$) by Bobkova in [14]. Both papers used delicate calculations based on the theory of topological resolutions from [31] and [10]. We have replaced that style of argument with one that depends ultimately on the representation theory of C_3 and Q_8 , respectively.

The case $p = 3$ of Theorem 13.12 can be extended to an arbitrary odd prime p and height $n = p - 1$. In that case, the maximal p -torsion subgroup of $\mathbb{G}$ is C_p , a cyclic group of order p , and its representation theory helps us prove the following result. Here, $\mathbb{G} = \text{Aut}(\mathbb{F}_{p^n}, F_n)$ where F_n is the Honda formal group law.

Theorem 14.14 *Let $p > 2$ and $n = p - 1$. Let $G \subseteq \mathbb{G}$ be a maximal finite subgroup containing the maximal p -torsion subgroup of $\mathbb{G}$. Then*

$$D(\mathbf{E}^{hG}) \simeq \Sigma^{-(p-1)^2(2p+1)} \mathbf{E}^{hG}.$$

Note that at $n = 2$ and $p = 3$, the spectrum $\mathbf{E}^{hG}$ is 72-periodic, so $\Sigma^{-28} \mathbf{E}^{hG} \simeq \Sigma^{44} \mathbf{E}^{hG}$. Compare Theorem 13.12 and Theorem 14.14.

Finally, a direct application of Theorem 14.14 gives us a new result about the $\mathbf{K}$ -local Picard group Pic_n . Namely, we compare our result on Spanier–Whitehead duality with results about Gross–Hopkins duals of the same spectra from [6]. At the very end of this paper, we prove the following result.

Theorem 14.16 *Let $n = p - 1$ and $F = C_p$. Then, there is an element $P_n \in \text{Pic}_n$ such that $\mathbf{E}_* P_n \cong \mathbf{E}_*$ as Morava modules but*

$$P_n \wedge \mathbf{E}^{hF} \simeq \Sigma^{p^2+p} \mathbf{E}^{hF}.$$

In particular, P_n is a non-trivial element of subgroup κ_n of exotic elements in the Picard group Pic_n .

In an analogous way, our main result in the case $n = 2 = p$ from (1.3) is used by Heard–Li–Shi [40] in combination with their computation of $I_n(\mathbf{E}^{hC_2})$ to prove that there are exotic invertible elements in the $\mathbf{K}$ -local category at $p = 2$.

Organization of the paper

The next three sections are a review of results found throughout the literature needed later in the paper. In Sect. 2, we begin by reviewing some theory on compact p -adic analytic groups and introduce what will later be our main example, the Morava stabilizer group $\mathbb{G}$. In Sect. 3, we discuss properties of the mod p cohomology of these groups. Section 4 is a review of Poincaré duality and of Frobenius reciprocity for compact p -adic analytic groups.

Section 5 begins our investigation of new results. There, we introduce the spheres $I_{\mathbb{G}}$ and $S^{\mathfrak{g}}$ and state the Linearization Hypothesis. In Sect. 6, we study spheres with actions of finite subgroups, reviewing the classical theory of Lannes and key facts about characteristic classes. This is where we deduce Theorem 6.24 from the work of Castellana. We apply it to $I_{\mathbb{G}}$ and $S^{\mathfrak{g}}$ in Theorem 6.25. This reduces the proof that $I_{\mathbb{G}} \simeq_H S^{\mathfrak{g}}$ (Theorem 8.11) to a cohomological calculation. For this computation, we need to do a tricky analysis of Lyndon–Serre–Hochschild spectral sequences. We have opted to isolate these technical aspects to Sect. 7, which can easily be skipped on a first read. Section 8 is dedicated to the proof of Theorem 8.11.

In Sect. 9, we get ready for our computation in the examples to follow. This is a study of Frobenius reciprocity for G -manifolds. We do not claim originality for the results of this section, but include them here because we did not find them conveniently gathered in literature. Based on these geometric results, we prove Propositions 9.8 and 9.14 at the end of this section.

The last sections are applications of our results to chromatic homotopy theory. Sections 10 and 11 study Lubin–Tate theory **E**. In Sect. 10 we introduce technical background from chromatic homotopy theory and in Sect. 11 we prove Theorem 11.16. In Sect. 12, we review equivariant techniques established by Hill–Hopkins–Ravenel that can be used to study the Picard groups of categories of R -modules in G -spectra, as well as establish our general strategy for computing the duals $D(\mathbf{E}^{hF})$ for finite subgroups $F \subseteq \mathbb{G}$ in the final parts of the paper. The last two sections are the computational applications of our theory. Section 13 contains the proof of Theorems 13.12 and 13.25 and Sect. 14 that of Theorems 14.14 and 14.16.

1.1 Acknowledgments

This paper is a result of a long journey, and a great many people helped along the way. The project began in the spring of 2016 during an extended visit by Mike Hopkins to Northwestern University, as part of his residency after winning the Nemmers Prize in Mathematics. The initial question we set ourselves was to find some equivariant explanation for the results of Behrens and Bobkova mentioned above; that is, why 44? We quickly understood how to prove Theorem 11.16 and then, after blithely assuming the Linearization Hypothesis and some geometric topology, we began making the calculations found in the final sections of this paper. Thus, we first have to acknowledge and thank Northwestern University and the Nemmers Committee for this opportunity for an extended period of collaborative research.

Coming to terms with the Linearization Hypothesis took some time, and some of that intellectual pilgrimage was documented in a lecture by Hopkins at the conference in July 2017 “Homotopy Theory: Tools and Applications” at the University of Illinois Champaign–Urbana. This lecture, now known as the “Mean Streets of Evanston” talk (“After 10PM, after the Whole Foods closes, when only sinners are on the streets...”) has been widely viewed on YouTube [43].

Lannes theory and the cohomology of the classifying space for sphere bundles first arose in conversation between Mike Hopkins and Vesna Stojanoska at Harvard University. We picked up these ideas at another NSF supported conference “Chromatic Homotopy Theory: Journey to the Frontier” in 2018 at the University of Colorado. Thus we also heartily thank the Massachusetts Institute of Technology, which sponsored Stojanoska’s visit to Cambridge MA,

the Universities of Illinois and Colorado, the other organizers of the various conferences, and the National Science Foundation.

However, the use of Lannes Theory went from speculation to concrete reality only after a sequence of pivotal conversations with Jesper Grodal at the Isaac Newton Institute in 2018 during the semester “Homotopy Harnessing Higher Structures”. Jesper has unique and deep knowledge of group actions, Lannes Theory, and classifying spaces – all crucial to this part of the project – and we are extremely grateful that he was there to share his expertise.

The Newton Institute program brought together a remarkable range of mathematicians in homotopy theory, and we had very fruitful conversations also with Tobias Barthel, Clark Barwick, Anna Marie Bohmann, John Greenlees, Hans-Werner Henn, and Michael A. Hill. Thanks are due to all of them, for the inspiring conversations as well as email correspondence well after the program ended. We added thanks as well to the organizers of the semester program, and to the Newton Institute and the staff there.

As a profession, we would be nowhere without the anonymous and careful work of diligent referees; we are especially grateful to the main referee of this paper, who gave the paper a very close and careful reading, and who had a number of insightful comments and questions.

The Linearization Hypothesis is not new with this paper, as we explained above, and we are grateful to Dustin Clausen for sharing and explaining his ideas.

Finally, when it became apparent that we would make heavy use of Steenrod squares in some of our arguments, someone repeated a hoary old joke about “Squeenrod stares” and Vesna Stojanoska suggested to Anna Marie Bohmann, who has a long track-record of writing quality light verse, that she could use “Steenrod stairs” as the basis of a poem with a good many words beginning st- and sq-. The result is below. We are touched and grateful that Anna Marie has granted us permission to reproduce it here.

Derivation

To a stolid squalid attic
Full of strident squeaking squirrels
Dr. Squeenrod staggered squiffy up the stairs

When a tumble most dramatic
Put his letters in a whirl:
Squeenrod’s stumble gave the storied Steenrod squares!

Anna Marie Bohmann

2 Basics on compact p -adic analytic groups

While our main applications of the results here are for the Morava stabilizer group and the $K(n)$ -local category, the initial analysis of the dualizing sphere applies to a much wider class of groups $\mathcal{G}$. The key feature is that $\mathcal{G}$ has a finite index subgroup which is a pro- p group of a particularly nice type. Thus we begin by fixing a prime p , and reviewing some basic definitions.

The following material is from Sections I.4 and II.8 of [25]. If G is a topological group, let $G^n \subseteq G$ be the closed normal subgroup obtained by taking the closure of the subgroup generated by the n th powers. Similarly, if $H \subseteq G$ is a closed subgroup, we let $[H, G] \subseteq G$ be the closure of the commutator subgroup of H with G , and if H and K are two subgroups, let HK be the closure of the product subgroup.

In all the statements below, there are slight modifications for the prime 2.

Definition 2.1 A pro- p group G is *uniformly powerful* if

- (a) G/G^p (or G/G^4 if $p = 2$) is abelian;
- (b) G is topologically finitely generated;
- (c) The lower p -series

$$G = G_1 \supseteq G_2 \supseteq \dots \supseteq G_i \supseteq G_{i+1} = G_i^p [G_i, G] \supseteq \dots$$

has the properties that the p th power map induces an isomorphism

$$G_i/G_{i+1} \xrightarrow[\cong]{(-)^p} G_{i+1}/G_{i+2}$$

and it is exhaustive: $\bigcap_i G_i = \{e\}$.

If G is a uniformly powerful pro- p -group and $G/G^p \cong (\mathbb{Z}/p)^d$, then G has rank d . Note that each of the subgroups G_j is normal in G , and hence normal in G_i with $i \leq j$.

Remark 2.2 If G is uniformly powerful and topologically generated by $a_1, \dots, a_d$

$$\begin{aligned} G/G^p &\cong \mathbb{Z}/p\{a_1, \dots, a_d\} \\ G_i/G_{i+j} &\cong \mathbb{Z}/p^j\{a_1^{p^{i-1}}, \dots, a_d^{p^{i-1}}\} \quad \text{for } j \leq i. \end{aligned}$$

In §II.8.2. of [25], the authors give an intrinsic definition of a compact p -adic analytic group. For our purposes it is enough to have the following characterization. See Corollary 8.34 of [25].

Theorem 2.3 *Let $\mathcal{G}$ be a topological group. Then the following are equivalent:*

- (1) $\mathcal{G}$ is a compact p -adic analytic group;
- (2) $\mathcal{G}$ is a profinite group with an open subgroup which is a topologically finitely generated pro- p group;
- (3) $\mathcal{G}$ has an open normal subgroup of finite index which is a uniformly powerful pro- p group of finite rank.

For such a group $\mathcal{G}$, the *rank* of $\mathcal{G}$ is defined to be the rank of G where G is any open normal subgroup of finite index which is a uniformly powerful pro- p group. The rank is independent of this choice.

Definition 2.4 Let $\mathcal{G}$ be a compact p -adic analytic group and let $G \subseteq \mathcal{G}$ be an open uniformly powerful finite index normal subgroup of rank d . Then, from Definition 2.1 we have that $(-)^p : G_i \rightarrow G_{i+1}$ is a continuous bijection of sets. This allows us to form the *Lie algebra* $\mathfrak{g}$ of $\mathcal{G}$. This is the set G with addition given by

$$x + y = \lim_n (x^{p^n} y^{p^n})^{p^{-n}}$$

and Lie bracket by

$$[x, y] = \lim_n [x^{p^n}, y^{p^n}]^{p^{-2n}}.$$

The Lie algebra $\mathfrak{g}$ is independent of the choice of G ; see the beginning of §II.9.5 of [25]. A chosen set of topological generators for G defines a continuous isomorphism $\mathbb{Z}_p^d \cong \mathfrak{g}$ of compact abelian groups. See Theorem 4.17 of [25].

Remark 2.5 (The adjoint representation) Again, let $G \subseteq \mathcal{G}$ be an open uniformly powerful finite index normal subgroup of rank d . Since G is normal in $\mathcal{G}$, there is a conjugation action of $\mathcal{G}$ on G . This gives a $\mathbb{Z}_p$ -linear action of $\mathcal{G}$ on $\mathfrak{g}$ called the adjoint action, and $\mathfrak{g}$ with this action is the *adjoint representation* of $\mathcal{G}$. For later purposes we define

$$\mathfrak{g}_i = p^{i-1} \mathfrak{g}.$$

These form a nested sequence of sub-representations and the equality of sets $\mathfrak{g} = G$ induces an isomorphism of abelian groups

$$\mathfrak{g}_i / \mathfrak{g}_{i+j} \cong G_i / G_{i+j}, \quad j \leq i, \quad (2.6)$$

when $i \geq 1$ if $p > 2$ and $i \geq 2$ for $p = 2$. This becomes an isomorphism of $\mathcal{G}$ -modules if we act on G_i / G_{i+j} by conjugation.

Example 2.7 The simplest example is $\mathcal{G} = \mathbb{Z}_p^d$. Then $G = \mathcal{G} = \mathfrak{g}$ and $\Gamma_i = (p^{i-1} \mathbb{Z}_p)^d$.

2.1 Groups of units in normed algebras

We discuss two instances of compact p -adic analytic groups; the general linear group $\mathrm{GL}_n(\mathbb{Z}_p)$, in Example 2.12, and the Morava stabilizer group, in Example 2.15. In both cases the group $\mathcal{G}$ arises as the group of units in a sub-algebra of a complete normed $\mathbb{Q}_p$ -algebra $(A, \|\cdot\|)$. The norm extends the standard norm on $\mathbb{Q}_p$; hence, $\|p^i\| = p^{-i}$. If A is of finite rank over $\mathbb{Q}_p$, we define

$$A_i = \{x \in A \mid \|x\| \leq p^{-i}\}.$$

Then $A_0 \subseteq A$ is a sub- $\mathbb{Z}_p$ -algebra, each A_i is an ideal in A_0 , and $pA_i \subseteq A_{i+1}$. In our examples, this inclusion is an equality, so for convenience we assume this. The most basic example is $A = \mathbb{Q}_p$, $A_0 = \mathbb{Z}_p$ and $A_i = p^i \mathbb{Z}_p$.

We let $\mathcal{G} = A^\times$ be the group of units in A_0 and define closed normal subgroups

$$\Gamma_i = 1 + p^i A_0 \subseteq \mathcal{G}.$$

In both our examples $\mathcal{G}$ is a compact p -adic analytic group with a uniformly powerful subgroup

$$\begin{aligned} \Gamma_1 &= 1 + A_1 = 1 + pA_0 \text{ if } p > 2 \text{ or} \\ \Gamma_2 &= 1 + A_2 = 1 + 4A_0 \text{ if } p = 2. \end{aligned}$$

In fact, more is true. The following is an exercise in definitions.

Lemma 2.8 *Let A be a complete normed $\mathbb{Q}_p$ -algebra. With the notation established above and if $pA_i = A_{i+1}$ we have the following conclusions.*

- (1) *If $p > 2$, the subgroup $\Gamma_{i+1} \subseteq \Gamma_1$ is the i th term in the lower p -series for Γ_1 .*
- (2) *If $p = 2$, the subgroup $\Gamma_{i+2} \subseteq \Gamma_2$ is the i th term in the lower p -series of Γ_2 .*

It follows from Lemma 2.8. that the rank of $\mathcal{G}$ is the $\mathbb{Q}_p$ -rank of A . The standard exponential map

$$\exp : A_i \longrightarrow 1 + A_i = \Gamma_i \tag{2.9}$$

converges for $i \geq 1$ if $p > 2$ and $i \geq 2$ if $p = 2$. If we give A_0 the structure of a $\mathbb{Z}_p$ -Lie algebra with bracket $[x, y] = xy - yx$, then this induces an isomorphism of Lie algebras

$$\exp : A_i = p^i A_0 \xrightarrow{\cong} \mathfrak{g}_i \tag{2.10}$$

for $i \geq 1$ if $p > 2$ and $i \geq 2$ if $p = 2$. This is the content of Corollary II.7.14 of [25].

The group $\mathcal{G}$ acts on A_0 by conjugation and the exponential map also gives an isomorphism of $\mathcal{G}$ representations

$$\exp : A_i/A_{i+j} \xrightarrow{\cong} (1 + A_i)/(1 + A_{i+j}) = \Gamma_i/\Gamma_{i+j} \quad (2.11)$$

for $i \geq 1$ if $p > 2$ (or $i \geq 2$ if $p = 2$) and $j \leq i$. Note that since $A_i A_j \subseteq A_{i+j}$, this exponential function has a very simple form: if $x \in A_i$, then

$$\exp(x) = 1 + x \quad \text{modulo } A_{2i}.$$

In combination with (2.6) this establishes an isomorphism of representations

$$A_i/A_{i+j} \cong \mathfrak{g}_i/\mathfrak{g}_{i+j}$$

for $i \geq 1$ if $p > 2$ (or $i \geq 2$ if $p = 2$) and $j \leq i$.

2.2 The main examples

We now give our two main examples.

Example 2.12 (The general linear group) Let $\mathcal{G} = \mathrm{GL}_n(\mathbb{Z}_p)$ be the group of invertible $n \times n$ matrices with entries in the p -adic integers. Then $\mathrm{GL}_n(\mathbb{Z}_p)$ exactly fits the rubric of Sect. 2.1: we can let $A = M_n(\mathbb{Q}_p)$, the $\mathbb{Q}_p$ -algebra of $n \times n$ matrices. Then $A_0 = M_n(\mathbb{Z}_p)$ and $\mathrm{GL}_n(\mathbb{Z}_p) = M_n(\mathbb{Z}_p)^\times$. Further, $A_i = p^i M_n(\mathbb{Z}_p)$.

For the moment assume $p > 2$. Then $\mathrm{GL}_n(\mathbb{Z}_p)$ is a compact p -adic analytic group with uniformly powerful subgroup $\Gamma_1 = 1 + pM_n(\mathbb{Z}_p)$. The group Γ_i is the kernel of the map $\mathrm{GL}_n(\mathbb{Z}_p) \rightarrow \mathrm{GL}_n(\mathbb{Z}/p^i)$; thus,

$$\Gamma_i = 1 + p^i M_n(\mathbb{Z}_p).$$

The group Γ_1 is of rank n^2 . Furthermore, by Remark 2.2 we see that if $j \leq i$ then Γ_i/Γ_{i+j} is a free $\mathbb{Z}/p^j$ -module of rank n^2 .

If $\mathfrak{g} = \mathfrak{gl}_n$ is the Lie algebra of $\mathrm{GL}_n(\mathbb{Z}_p)$ then Sect. 2.1 gives an isomorphism of Lie algebras

$$M_n(\mathbb{Z}_p) \xrightarrow{\cong} \mathfrak{gl}_n. \quad (2.13)$$

If we let $\mathrm{GL}_n(\mathbb{Z}_p)$ act by conjugation on $M_n(\mathbb{Z}_p)$, this gives an isomorphism of representations from $M_n(\mathbb{Z}_p)$ to the adjoint representation. As in Remark 2.5,

we can filter $\mathfrak{g}$ by powers of p and the isomorphism of (2.13) descends to an isomorphism of $\mathrm{Gl}_n(\mathbb{Z}_p)$ -modules

$$p^i M_n(\mathbb{Z}_p) / p^{i+j} M_n(\mathbb{Z}_p) \cong \mathfrak{g}_i / \mathfrak{g}_{i+j} \cong \Gamma_i / \Gamma_{i+j}, \quad j \leq i. \quad (2.14)$$

Here the action of $\mathrm{Gl}_n(\mathbb{Z}_p)$ on Γ_i / Γ_{i+j} is again by conjugation.

If $p = 2$, $\mathrm{Gl}_n(\mathbb{Z}_2)$ is a compact p -adic analytic group with uniformly powerful subgroup $\Gamma_2 = 1 + 4M_n(\mathbb{Z}_2)$. The rest of the remarks go through, with the evident changes on bounds.

This example extends without much change to $\mathrm{Gl}_n(\mathbb{W})$ with $\mathbb{W} = W(k)$ the p -typical Witt vectors of a finite algebraic extension k of $\mathbb{F}_p$. The rank of $\mathrm{Gl}_n(\mathbb{W})$ is now $n^2[k : \mathbb{F}_p]$.

Example 2.15 (The Morava stabilizer group) This is our main example, and it has a number of variants, all of which fit the setup of Sect. 2.1.

To be concrete let $F = F(x, y)$ be the Honda formal group law of height $n \geq 1$ over $\mathbb{F}_p$ and let $\mathcal{O}_n$ be the endomorphism ring of F over $\mathbb{F}_{p^n}$; thus, an element of $\mathcal{O}_n$ is a power series $\varphi(x) \in \mathbb{F}_{p^n}[[x]]$ so that $\varphi(F(x, y)) = F(\varphi(x), \varphi(y))$. Since F is defined over $\mathbb{F}_p$, the power series $S = x^p$ is an endomorphism in $\mathcal{O}_n$, and there is an isomorphism of $\mathbb{Z}_p$ -algebras

$$\mathbb{W}\langle S \rangle / (S^n - p) \rightarrow \mathcal{O}_n$$

where $\mathbb{W} = W(\mathbb{F}_{p^n})$ is the Witt vectors. The source is a non-commuting truncated polynomial ring: if $a \in \mathbb{W}$ then $Sa = a^\sigma S$ where we write σ in exponent to indicate the action of the Frobenius on $\mathbb{W}$.

Note that $\mathcal{O}_n$ is free of rank n^2 over $\mathbb{Z}_p$. The $\mathbb{Q}_p$ -algebra $A = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_n$ is a complete normed $\mathbb{Q}_p$ -algebra with $\|S\| = p^{-1/n}$. Then $A_0 = \mathcal{O}_n$ and $A_i = p^i \mathcal{O}_n$. Define

$$\mathbb{S}_n = \mathcal{O}_n^\times.$$

This is the small Morava stabilizer group. The Galois group $\mathrm{Gal}(\mathbb{F}_{p^n} / \mathbb{F}_p)$ acts on $\mathcal{O}_n$ through the action on $\mathbb{W}$; the full Morava stabilizer group is the semi-direct product

$$\mathbb{G}_n = \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n} / \mathbb{F}_p).$$

Let $p > 2$. The groups $\mathbb{S}_n$ and $\mathbb{G}_n$ are compact p -adic analytic groups with uniformly powerful subgroup $\Gamma_1 = 1 + p\mathcal{O}_n$. Then

$$\Gamma_i = 1 + p^i \mathcal{O}_n.$$

Section 2.1 again applies and we have an isomorphism of Lie algebras

$$\mathcal{O}_n \xrightarrow{\cong} \mathfrak{g}. \quad (2.16)$$

Furthermore $\mathbb{S}_n$ acts by conjugation on $\mathcal{O}_n$ and we have an isomorphism of representations from $\mathcal{O}_n$ to the adjoint representation. This isomorphism extends to one of $\mathbb{G}_n$ -modules. As in Remark 2.5, we can filter $\mathfrak{g}$ by powers of p and the isomorphism of (2.13) descends to an isomorphism of $\mathbb{G}_n$ -modules

$$p^i \mathcal{O}_n / p^{i+j} \mathcal{O}_n \cong \mathfrak{g}_i / \mathfrak{g}_{i+j} \cong \Gamma_i / \Gamma_{i+j}, \quad j \leq i. \quad (2.17)$$

Here the action of $\mathbb{G}_n$ on Γ_i / Γ_{i+j} is again by conjugation.

If $p = 2$, the group $\mathbb{G}_n$ is a compact p -adic analytic group with uniformly powerful subgroup $\Gamma_2 = 1 + 4\mathcal{O}_n$. The rest of the remarks go through, with the evident changes on bounds.

Again this could be generalized to the example where F is a formal group of height n over an algebraic extension of $\mathbb{F}_p$. We can go further. The algebra $A = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_n$ is a central division algebra of Hasse invariant $1/n$ over $\mathbb{Q}_p$. The paradigm of Sect. 2.1 extends to the case when $\mathcal{G}$ is the group of units in the maximal order of any finite dimensional central division algebra over $\mathbb{Q}_p$.

Notation 2.18 Driven by these examples, we will adopt the following conventions for the rest of the paper. We will equip our p -adic analytic groups $\mathcal{G}$ with a nested sequence of open normal subgroups of finite index

$$\mathcal{G} \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_i \supseteq \Gamma_{i+1} \supseteq \cdots$$

so that either Γ_1 (if $p > 2$) or Γ_2 (if $p = 2$) is a uniformly powerful pro- p group. If $p > 2$ and $i \geq 1$ then

$$\Gamma_{i+1} = \Gamma_i^p[\Gamma_i, \Gamma_1]. \quad (2.19)$$

If $p = 2$, then $\Gamma_{i+1} = \Gamma_i^p[\Gamma_i, \Gamma_2]$ for $i \geq 2$. Thus the remaining terms for the sequence are the lower p -series for Γ_1 or Γ_2 , depending on the prime. See Lemma 2.8.

Note that if $p = 2$, there is no theoretical stipulation on Γ_1 , although in practice it will be very concrete.

The rank of $\mathcal{G}$ will be the rank Γ_i for i large.

3 Some group cohomology, with applications

Notation 3.1 In this section as well as the remainder of the paper, we are concerned with continuous group cohomology. *We will not decorate the coho-*

mology notation to specify continuity; it is to be understood that all cohomology is continuous whenever that makes sense. If G is any compact p -analytic group (or simply a finite group), we write $H_1(G, \mathbb{F}_p)$ for G/G_1 , where G_1 is the closure of $G^p[G, G]$. We have that $H^1(G, \mathbb{F}_p) \cong H_1(G, \mathbb{F}_p)^*$, where $(-)^*$ denotes the $\mathbb{F}_p$ -linear dual. If we write $H^*(-)$ we mean $H^*(-, \mathbb{F}_p)$; we use a similar convention for homology.

Following [64] we could define

$$H_*(G, M) = \mathrm{Tor}_*^{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, M)$$

where $\mathbb{Z}_p[[G]]$ is the completed group ring, M lies in some category of continuous G -modules, and Tor_* denotes the derived functors of a completed tensor product. The equation $H_1(G, \mathbb{F}_p) = G/G_1$ is then a lemma, rather a definition. We won't use the greater generality, so we won't need to make these ideas precise.

Now let $\mathcal{G}$ be a fixed compact p -adic analytic group with uniformly powerful subgroup Γ_1 if $p > 2$ or perhaps Γ_2 if $p = 2$. See Notation 2.18. We are interested in the continuous cohomology $H^*(\Gamma_i, M)$ for various i and various coefficients M .

For any i and all $j > i$, the quotient maps $\Gamma_i \rightarrow \Gamma_i/\Gamma_j \rightarrow \Gamma_i/\Gamma_{i+1}$ induce isomorphisms

$$H_1(\Gamma_i) \cong H_1(\Gamma_i/\Gamma_j) \cong \Gamma_i/\Gamma_{i+1}.$$

Let $V_i = H^1(\Gamma_i) \cong (\Gamma_i/\Gamma_{i+1})^*$. Then

$$H^1(\Gamma_i/\Gamma_j) \cong H^1(\Gamma_i) = V_i, \quad j > i. \quad (3.2)$$

If V is a vector space over a field k , let $\Lambda(V)$ denote the graded exterior algebra on V , and write $\Lambda^r(V) \subseteq \Lambda(V)$ for the homogeneous elements of degree r . Below we will also have a symmetric (polynomial) algebra $P(W)$ on a vector space W . The following basic result is the key to much of what follows.

Theorem 3.3 (1) *The natural inclusion $V_i = H^1(\Gamma_i) \hookrightarrow H^*(\Gamma_i)$ induces an isomorphism of graded commutative algebras*

$$\Lambda(V_i) \longrightarrow H^*(\Gamma_i).$$

(2) *The inclusion $\Gamma_{i+1} \rightarrow \Gamma_i$ induces the zero map on $H^k(-)$ for $k > 0$.*

Proof The first statement is proved by combining Theorem 5.1.5 of [64] with Theorem 3.6 of [25]. The second statement follows from the first and the fact that

$$H_1(\Gamma_{i+1}, \mathbb{F}_p) \cong \Gamma_{i+1}/\Gamma_{i+2} \rightarrow \Gamma_i/\Gamma_{i+1} \cong H_1(\Gamma_i, \mathbb{F}_p)$$

is the zero map. $\square$

We now turn to the cohomology of Γ_i/Γ_j , with $j > i$, working our way to Theorem 3.6. To establish some notation we contemplate the exact sequence of groups

$$1 \longrightarrow \Gamma_{j-1}/\Gamma_j \xrightarrow{f} \Gamma_i/\Gamma_j \xrightarrow{q} \Gamma_i/\Gamma_{j-1} \longrightarrow 1, \quad (3.4)$$

where f is the inclusion and q is the projection. Let $W_{j-1} \subseteq H^2(\Gamma_{j-1}/\Gamma_j)$ be the image of the Bockstein operation on $V_{j-1} \cong H^1(\Gamma_{j-1}/\Gamma_j)$. By Remark 2.2, the group Γ_{j-1}/Γ_j is an elementary abelian p -group; therefore, there are isomorphisms

$$\begin{aligned} \Lambda(V_{j-1}) \otimes P(W_{j-1}) &\cong H^*(\Gamma_{j-1}/\Gamma_j), & p > 2; \\ P(V_{j-1}) &\cong H^*(\Gamma_{j-1}/\Gamma_j), & p = 2. \end{aligned} \quad (3.5)$$

In the case $p = 2$, $W_{j-1} \subseteq H^2(\Gamma_{j-1}/\Gamma_j)$ is the subvector space given by squares of the elements in degree 1.

In Theorem 3.6, we extend (3.5) to $H^*(\Gamma_i/\Gamma_j)$. The bounds on i and j in this result are not optimal, but certainly are good enough for later applications and relieve us of the duty of making special statements at $p = 2$. Note that part (1) gives a splitting result, but some care is needed in interpreting that statement. We will add further comments below in Remark 3.9.

Theorem 3.6 (1) *For all $i \geq 3$ and $j > i + 1$, there is an exact sequence of vector spaces*

$$0 \longrightarrow \Lambda^2 V_i \xrightarrow{q^*} H^2(\Gamma_i/\Gamma_j) \xrightarrow{f^*} W_{j-1} \longrightarrow 0$$

and any splitting of this short exact sequence defines an isomorphism of algebras

$$\Lambda(V_i) \otimes P(W_{j-1}) \cong H^*(\Gamma_i/\Gamma_j). \quad (3.7)$$

(2) *For all $i \geq 3$ and $j > i + 1$, the image of $H^*(\Gamma_i/\Gamma_j) \rightarrow H^*(\Gamma_i/\Gamma_{j+1})$ is $\Lambda(V_i)$.*

Proof We will do the case when $p > 2$. There are evident modifications needed for $p = 2$ to account for the fact that $H^*(\Gamma_j/\Gamma_{j+1})$ is a polynomial algebra if $p = 2$.

Part (1) is by induction on j , using that Γ_i/Γ_{i+1} is elementary abelian for the base case. Compare (3.8). The induction step is completed using the Lyndon-Hochschild-Serre Spectral Sequence for the exact sequence of (3.4). To compute the differentials we extend that exact sequence to a diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \Gamma_{j-1}/\Gamma_j & \longrightarrow & \Gamma_{j-2}/\Gamma_j & \longrightarrow & \Gamma_{j-2}/\Gamma_{j-1} \longrightarrow 1 \\ & & \downarrow = & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \Gamma_{j-1}/\Gamma_j & \xrightarrow{f} & \Gamma_i/\Gamma_j & \xrightarrow{q} & \Gamma_i/\Gamma_{j-1} \longrightarrow 1. \end{array} \quad (3.8)$$

Since $j > 4$, Remark 2.2 implies that the top exact sequence is non-canonically isomorphic to

$$0 \longrightarrow (\mathbb{Z}/p)^d \xrightarrow{\times p} (\mathbb{Z}/p^2)^d \longrightarrow (\mathbb{Z}/p)^d \longrightarrow 0.$$

The diagram of (3.8) gives a diagram of Lyndon-Hochschild-Serre Spectral Sequences

$$\begin{array}{ccc} \overline{E}_2^{p,q} \cong H^p(\Gamma_i/\Gamma_{j-1}, H^q(\Gamma_{j-1}/\Gamma_j)) & \Longrightarrow & H^p(\Gamma_i/\Gamma_j) \\ \downarrow & & \downarrow \\ E_2^{p,q} \cong H^p(\Gamma_{j-2}/\Gamma_{j-1}, H^q(\Gamma_{j-1}/\Gamma_j)) & \Longrightarrow & H^p(\Gamma_{j-2}/\Gamma_j). \end{array}$$

Note that we have decorated the top spectral sequence with an overbar to later help distinguish between the two.

Since Γ_1 (or Γ_2 if $p = 2$) is uniformly p -powerful, we have by Definition 2.1 that $[\Gamma_i, \Gamma_j] \subseteq \Gamma_{i+j}$ for all $j \geq i \geq 1$ (or $i \geq 2$ if $p = 2$). Thus in both spectral sequences the action of the base group on $H^*(\Gamma_{j-1}/\Gamma_j)$ is trivial.

By Remark 2.2 the bottom of these spectral sequences is completely known: indeed, there are isomorphisms

$$\begin{aligned} E_2^{*,0} &\cong H^*(\Gamma_{j-2}/\Gamma_{j-1}) \cong \Lambda(V_{j-2}) \otimes P(W_{j-2}), \\ E_2^{0,*} &\cong H^*(\Gamma_{j-1}/\Gamma_j) \cong \Lambda(V_{j-1}) \otimes P(W_{j-1}), \end{aligned}$$

and $E_2^{*,*} \cong E_2^{*,0} \otimes E_2^{0,*}$. The only non-zero differential is d_2 . This is determined by the isomorphism $d_2: V_{j-1} \cong W_{j-2} \subseteq E_2^{2,0}$ and the multiplicative

structure of the spectral sequence. We conclude

$$E_{\infty}^{*,*} \cong E_3^{*,*} \cong \Lambda(V_{j-2}) \otimes P(W_{j-1})$$

with $V_{j-2} = E_{\infty}^{1,0}$ and $W_{j-1} = E_{\infty}^{0,2}$.

We now turn to the top spectral spectral sequence. The base case of $j = i + 2$ is covered by the above. By the induction hypothesis we have a short exact sequence

$$0 \longrightarrow \Lambda^2 V_i \xrightarrow{q^*} H^2(\Gamma_i / \Gamma_{j-1}) \xrightarrow{f^*} W_{j-2} \longrightarrow 0.$$

We will make a useful choice of splitting of this exact sequence to obtain an isomorphism $\Lambda(V_i) \otimes P(W_{j-2}) \cong H^*(\Gamma_i / \Gamma_{j-1})$ to aid in completing the induction step.

We have isomorphisms

$$\begin{aligned} \overline{E}_2^{*,0} &\cong H^*(\Gamma_i / \Gamma_{j-1}) \\ \overline{E}_2^{0,*} &\cong H^*(\Gamma_{j-1} / \Gamma_j) \cong \Lambda(V_{j-1}) \otimes P(W_{j-1}) \end{aligned}$$

and $\overline{E}_2^{*,*} \cong \overline{E}_2^{*,0} \otimes \overline{E}_2^{0,*}$. By the induction hypothesis we have an isomorphism

$$\overline{E}_2^{*,0} \cong H^*(\Gamma_i / \Gamma_{j-1}) \cong \Lambda(V_i) \otimes P(W_{j-2}).$$

By the naturality of the spectral sequences and the calculation just completed we have that the composition

$$V_{j-1} = H^1(\Gamma_{j-1} / \Gamma_j) \cong \overline{E}_2^{0,1} \xrightarrow{d_2} \overline{E}_2^{2,0}$$

$$\cong H^2(\Gamma_i / \Gamma_{j-1}) \longrightarrow E_2^{2,0} \cong H^2(\Gamma_{j-2} / \Gamma_{j-1})$$

has image exactly $W_{j-2} \subseteq E_2^{2,0}$. So d_2 maps $V_{j-1} = \overline{E}_2^{0,1}$ isomorphically onto $W_{j-2} \subseteq \overline{E}_2^{2,0}$. We then have

$$\overline{E}_3^{*,*} \cong \Lambda(V_i) \otimes P(W_{j-1})$$

with $V_i = \overline{E}_3^{1,0}$ and $W_{j-1} = \overline{E}_3^{0,2}$.

It remains to show the spectral sequence collapses at $\overline{E}_3$. For this it is sufficient to show d_3 vanishes on $W_{j-1} = \overline{E}_2^{0,2}$. The target of this differential is $\Lambda^3(V_i) \cong \overline{E}_3^{3,0}$. However, we know from Theorem 3.3 that the composition

$$\Lambda(V_i) \rightarrow H^*(\Gamma_i/\Gamma_j) \rightarrow H^*(\Gamma_i)$$

is an isomorphism, so the map induced by the edge homomorphism

$$\Lambda^3(V_i) \rightarrow \overline{E}_\infty^{3,0} \rightarrow H^3(\Gamma_i/\Gamma_j)$$

must be an injection. Thus $d_3(W_{j-1}) = 0$ as needed.

For part (2) we examine the diagram from part (1)

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Lambda^2 V_i & \xrightarrow{q^*} & H^2(\Gamma_i/\Gamma_j) & \xrightarrow{f^*} & W_{j-1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \Lambda^2 V_i & \xrightarrow{q^*} & H^2(\Gamma_i/\Gamma_{j+1}) & \xrightarrow{f^*} & W_j \longrightarrow 0 \end{array}$$

Since the map $H^2(\Gamma_{j-1}/\Gamma_j) \rightarrow H^2(\Gamma_j/\Gamma_{j+1})$ is zero, the map $W_{j-1} \rightarrow W_j$ is zero. It follows that the kernel of the map $H^2(\Gamma_i/\Gamma_j) \rightarrow H^2(\Gamma_i/\Gamma_{j+1})$ is isomorphic to W_{j-1} and, in fact, this gives a splitting of the top f^* . The result follows. $\square$

Remark 3.9 Part (1) of Theorem 3.6 says that we can choose an isomorphism of graded algebras (3.7)

$$\Lambda(V_i) \otimes P(W_{j-1}) \cong H^*(\Gamma_i/\Gamma_j).$$

If $j \leq 2i$ then $\Gamma_i/\Gamma_j \cong (\mathbb{Z}/p^{j-i})^d$ and we can choose this isomorphism to be an isomorphism of unstable algebras over the Steenrod algebra, and we can even add the further stipulation that the appropriate higher Bockstein defines an isomorphism from V_i to W_{j-1} . If $j > 2i$, it is far less clear how the higher Bocksteins behave, and it is no longer possible to be so explicit.

For example, if $\mathcal{G} = \mathbb{S}_n$ is the Morava stabilizer group of Example 2.15 then we know by work of Lazard and Morava (see Remark 2.2.5 of [56]) that there is an isomorphism

$$\Lambda_{\mathbb{Q}_p}(x_1, x_3, \dots, x_{2n-1}) \cong H^*(\mathbb{S}_n, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

where x_{2k-1} is in degree $2k-1$. This implies that the structure of the higher order Bocksteins must be rich.

As a further remark, note that if we choose a splitting as in (3.7), then the map

$$\Lambda(V_i) \otimes P(W_{j-1}) \cong H^*(\Gamma_i/\Gamma_j) \rightarrow H^*\Gamma_i$$

gives an isomorphism $\Lambda(V_i) \cong H^*\Gamma_i$. However, we have not proved that W_{j-1} maps to zero.

If G is a profinite group, it is not immediately obvious how to define its classifying space. In particular, there is a long history, going back to Artin and Mazur, of regarding the classifying space as a pro-object. We will need nothing that sophisticated. For our purposes, the following will suffice.

Definition 3.10 Let $G = \lim_j G_j$ be a profinite group. Then the classifying space BG is defined by

$$BG = \operatorname{holim}_j BG_j.$$

In particular, if $\mathcal{G}$ is our compact p -adic analytic group with uniformly powerful subgroup Γ_1 we have $B\Gamma_i = \operatorname{holim}_j B(\Gamma_i/\Gamma_{i+j})$. It is immediate that

$$\pi_n B\Gamma_i \cong \begin{cases} \Gamma_i, & n = 1; \\ 0, & n \neq 1. \end{cases}$$

We remark, however, that this isomorphism does not recover the topology on Γ_i .

The next result now follows from Theorem 3.3, Theorem 3.6, Proposition 3.12 below, and the fact that if G is a finite p -group, then $\Sigma^\infty BG$ is already p -complete. It is slightly surprising, since suspension and limits to do not formally commute.

Proposition 3.11 *The natural map*

$$\Sigma_+^\infty B\Gamma_i \rightarrow \operatorname{holim}_j \Sigma_+^\infty B(\Gamma_i/\Gamma_{i+j})$$

is an equivalence after p -completion. Furthermore, we have isomorphisms

$$\Lambda(V_i) \cong \operatorname{colim}_j H^*(\Gamma_i/\Gamma_{i+j}) \cong H^*(\Gamma_i) \cong H^*(B\Gamma_i).$$

The proof of Proposition 3.11 is completed with this general result.

Proposition 3.12 *Let X be a bounded below spectrum and let*

$$X \twoheadrightarrow \cdots \twoheadrightarrow X_k \twoheadrightarrow X_{k-1} \twoheadrightarrow \cdots \twoheadrightarrow X_1$$

be a tower of bounded below spectra under X with the property that $H_*X \rightarrow H_*X_k$ induces an isomorphism

$$H_*X \cong \text{Image}\{H_*X_{k+1} \rightarrow H_*X_k\}.$$

Then $X_p \rightarrow \text{holim}_k (X_k)_p^\wedge$ is an equivalence and the induced maps

$$\begin{aligned} H_*X &\longrightarrow \lim_k H_*X_k \\ \text{colim}_k H^*X_k &\longrightarrow H^*X \end{aligned}$$

are isomorphisms.

Proof If Z is any spectrum, write $H\mathbb{F}_p^{\bullet+1} \wedge Z$ for the standard cosimplicial cobar complex defining the Adams Spectral Sequence. This spectral sequence reads

$$\text{Ext}_{A_*}^s(\Sigma^t \mathbb{F}_p, H_*Z) \cong \pi^s \pi_t H\mathbb{F}_p^{\bullet+1} \wedge Z \implies \pi_{t-s} \text{holim}_\Delta H\mathbb{F}_p^{\bullet+1} \wedge Z.$$

Here A_* is the dual Steenrod algebra. If Z is bounded below then the map

$$Z \longrightarrow \text{holim}_\Delta H\mathbb{F}_p^{\bullet+1} \wedge Z$$

is p -completion. By construction, there is an isomorphism, natural in Z ,

$$\pi_* H\mathbb{F}_p^{s+1} \wedge Z \cong A_*^{\otimes s} \otimes H_*Z.$$

Now turn to the tower under X . The hypothesis on the homology of the tower implies that for all s the natural map

$$\pi_* H\mathbb{F}_p^{s+1} \wedge X \rightarrow \pi_* \text{holim}_k H\mathbb{F}_p^{s+1} \wedge X_k \cong \lim_k \pi_* H\mathbb{F}_p^{s+1} \wedge X_k$$

is an isomorphism. Thus we have equivalences

$$\begin{aligned} X_p &\simeq \text{holim}_\Delta H\mathbb{F}_p^{\bullet+1} \wedge X \simeq \text{holim}_\Delta \text{holim}_k H\mathbb{F}_p^{s+1} \wedge X_k \\ &\simeq \text{holim}_k \text{holim}_\Delta H\mathbb{F}_p^{s+1} \wedge X_k \\ &\simeq \text{holim}_k (X_k)_p^\wedge. \end{aligned}$$

4 Duality and Frobenius reciprocity

This section collects a great deal of relatively standard material about group cohomology, with the wrinkle that we need these results for compact p -adic analytic groups. Much of this can be collected from [62] and [64].

Define the completed group ring of $\mathcal{G}$ to be

$$\mathbb{Z}_p[[\mathcal{G}]] = \lim_{n,i} \mathbb{Z}/p^n[[G_i]] \quad (4.1)$$

where $\mathcal{G} \cong \lim G_i$ is any presentation of $\mathcal{G}$ as an inverse limit of finite groups.

Notation 4.2 In this section $\mathcal{G}$ will be a compact p -adic analytic group unless otherwise stated. All modules we consider will be continuous $\mathbb{Z}_p$ -modules and most examples will be either p -profinite or discrete p -torsion. The following conventions and definitions hold or continue to hold.

- (1) Hom always means continuous homomorphisms and tensor products between profinite modules are always completed, and taken over $\mathbb{Z}_p$ if unadorned.
- (2) If M is a continuous right $\mathcal{G}$ -module and N is a continuous left $\mathcal{G}$ -module we will also write $M \otimes_{\mathcal{G}} N$ for $M \otimes_{\mathbb{Z}_p[[\mathcal{G}]]} N$. and we will write $\mathrm{Hom}_{\mathcal{G}}(M, N)$ for $\mathrm{Hom}_{\mathbb{Z}_p[[\mathcal{G}]]}(M, N)$. We will similarly abbreviate the Tor and Ext decorations.
- (3) If M and N are two continuous left $\mathcal{G}$ -modules we give $M \otimes N$ the structure of a left $\mathcal{G}$ -module with the diagonal $\mathcal{G}$ -action: $g(x \otimes y) = gx \otimes gy$.
- (4) Define $\mathrm{Hom}^{\Delta}(M, N)$ to be the abelian group of continuous homomorphisms with the left $\mathcal{G}$ -action given $(g\varphi)(a) = g\varphi(g^{-1}a)$.

The action on Hom^{Δ} has the twin features that $\mathrm{Hom}_{\mathcal{G}}(A, B) = \mathrm{Hom}^{\Delta}(A, B)^{\mathcal{G}}$ and the evaluation map

$$\mathrm{Hom}^{\Delta}(A, B) \otimes A \longrightarrow B$$

sending $\varphi \otimes a \rightarrow \varphi(a)$ is a morphism of left $\mathcal{G}$ -modules.

4.1 Group cohomology basics

Let $\mathcal{G}$ be a compact p -adic analytic group and M a continuous $\mathcal{G}$ -module. Following Serre [62], we let $C^s(\mathcal{G}, M)$ be the set of continuous maps

$$\phi : \mathcal{G}^s \longrightarrow M.$$

If $s = 0$, then $C^s(\mathcal{G}, M) = M$. We will also call these continuous cochains. The collection $C^{\bullet}(\mathcal{G}, M)$ is a cosimplicial abelian group with coface operators given by

$$d^i \phi(x_1, \dots, x_{s+1}) = \begin{cases} x_1 \phi(x_1, \dots, x_{s+1}), & i = 0; \\ \phi(x_0, \dots, x_i x_{i+1}, \dots, x_{s+1}), & 1 \leq i \leq s; \\ \phi(x_1, \dots, x_s), & i = s + 1. \end{cases}$$

Codegeneracy operators are obtained by inserting identities; we won't need them. We then have

$$H^s(\mathcal{G}, M) = H^s(C^\bullet(\mathcal{G}, M), \partial)$$

where $\partial = \sum_{i=0}^s (-1)^i d^i$. As explained in Sections 3.1 and 3.2 of [64], these are the right derived functors of $H^0(\mathcal{G}, M) = M^{\mathcal{G}}$; indeed, there is a natural isomorphism

$$H^s(\mathcal{G}, M) \cong \text{Ext}_{\mathcal{G}}^s(\mathbb{Z}_p, M)$$

and the cochain complex above is the standard cobar construction for calculating the Ext-groups. We also define

$$H_s(\mathcal{G}, M) = \text{Tor}_s^{\mathcal{G}}(\mathbb{Z}_p, M). \quad (4.3)$$

Let $f : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ be a continuous map of compact p -adic analytic groups and M a $\mathcal{G}_2$ -module. Then M becomes a $\mathcal{G}_1$ -module by restriction: if $x \in \mathcal{G}_1$ and $a \in M$, then $x \cdot a = f(x)a$. Write this module as f_*M . Then we get a map

$$f^* : C^s(\mathcal{G}_2, M) \rightarrow C^s(\mathcal{G}_1, f_*M)$$

given by sending $\phi : \mathcal{G}_2^s \rightarrow M$ to a cochain $f^*\phi : \mathcal{G}_1^s \rightarrow f_*M$ with

$$(f^*\phi)(x_1, \dots, x_s) = \phi(f(x_0), \dots, f(x_s)).$$

The new action on M is needed so that $f^*(d^0\phi) = d^0 f^*\phi$. We then get a map

$$f^* : H^*(\mathcal{G}_2, M) \rightarrow H^*(\mathcal{G}_1, f_*M).$$

Example 4.4 For example, suppose $f = c_g : \mathcal{G} \rightarrow \mathcal{G}$ is given by conjugation and M is a left $\mathcal{G}$ -module; then we write gM for f_*M and we get a map

$$c_g^* : H^*(\mathcal{G}, M) \rightarrow H^*(\mathcal{G}, {}^gM).$$

The action on gM is given by $x \cdot a = (gxg^{-1})a$.

More generally, let $\Gamma \subseteq \mathcal{G}$ be a closed subgroup. If $g \in \mathcal{G}$ let ${}^g\Gamma$ denote the conjugate subgroup $g^{-1}\Gamma g$ and write $c_g(-) = g(-)g^{-1} : {}^g\Gamma \rightarrow \Gamma$ for the conjugation homomorphism. The conventions are chosen so that $c_{gh} = c_g \circ c_h$. If M is a $\mathcal{G}$ -module, we then get a homomorphism

$$c_g^* : H^*(\Gamma, M) \rightarrow H^*({}^g\Gamma, {}^gM).$$

4.2 Induced modules and transfer

Let $\mathcal{G}$ be a compact p -adic analytic group and let $\Gamma \subseteq \mathcal{G}$ be a closed subgroup. If M is a continuous Γ -module, define the *coinduced* $\mathcal{G}$ -module to be the module

$$M_{\Gamma}^{\mathcal{G}} = \text{map}_{\Gamma}(\mathcal{G}, M)$$

of *continuous* Γ -equivariant functions $\varphi : \mathcal{G} \rightarrow M$ with $\mathcal{G}$ action given by the formula

$$(g\varphi)(x) = \varphi(xg).$$

The functor $M \mapsto M_{\Gamma}^{\mathcal{G}}$ is right adjoint to the forgetful functor to Γ -modules. The following result is *Schapiro's Lemma*.

Proposition 4.5 *Let M be a Γ -module which is either discrete p -torsion or profinite. Then we have a natural isomorphism*

$$H^*(\mathcal{G}, M_{\Gamma}^{\mathcal{G}}) \cong H^*(\Gamma, M).$$

Proof The discrete case is covered in §I.2.5 of [62]. The profinite case follows from the discrete case and the fact that if $M = \varprojlim M_i$, then

$$\text{map}_{\Gamma}(\mathcal{G}, M) \cong \varprojlim \text{map}_{\Gamma}(\mathcal{G}, M_i). \quad \square$$

If M is a continuous $\mathcal{G}$ -module, then there is a map

$$\eta_M : M \rightarrow M_{\Gamma}^{\mathcal{G}} = \text{map}_{\Gamma}(\mathcal{G}, M)$$

adjoint to the identity $M \rightarrow M$ regarded as a Γ -module map. The induced map

$$\text{res} = \text{res}_{\Gamma}^{\mathcal{G}} : H^*(\mathcal{G}, M) \longrightarrow H^*(\mathcal{G}, M_{\Gamma}^{\mathcal{G}}) \cong H^*(\Gamma, M)$$

is the restriction map.

Now we specialize to the case where $\Gamma \subseteq \mathcal{G}$ is open and, hence, closed and of finite index. There is a $\mathcal{G}$ -equivariant averaging map

$$\begin{aligned} M_{\Gamma}^{\mathcal{G}} = \text{map}_{\Gamma}(\mathcal{G}, M) &\longrightarrow \text{map}_{\mathcal{G}}(\mathcal{G}, M) \cong M \\ \varphi(-) &\longmapsto \sum_{g\Gamma \in \mathcal{G}/\Gamma} g\varphi(g^{-1}-). \end{aligned}$$

The induced map

$$\mathrm{tr} = \mathrm{tr}_\Gamma^\mathcal{G} : H^*(\Gamma, M) \cong H^*(\mathcal{G}, M_\Gamma^\mathcal{G}) \longrightarrow H^*(\mathcal{G}, M) \quad (4.6)$$

is the *transfer map*. The next result follows from the definitions.

Lemma 4.7 *We have the following naturality statements for restriction and transfer.*

- (1) *Let $f : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ be a continuous homomorphism of compact p -adic analytic groups. Suppose $\Gamma_i \subseteq \mathcal{G}_i$ is a closed subgroups and suppose $f(\Gamma_1) \subseteq \Gamma_2$. Let M be a $\mathcal{G}_2$ -module. Then the following diagram commutes.*

$$\begin{array}{ccc} H^*(\mathcal{G}_2, M) & \xrightarrow{\mathrm{res}} & H^*(\Gamma_2, M) \\ f^* \downarrow & & \downarrow f^* \\ H^*(\mathcal{G}_1, f_*M) & \xrightarrow{\mathrm{res}} & H^*(\Gamma_1, f_*M). \end{array}$$

- (2) *Suppose further that Γ_2 is of finite index in $\mathcal{G}_2$ and f induces an isomorphism $\mathcal{G}_1/\Gamma_1 \cong \mathcal{G}_2/\Gamma_2$. Then the following diagram commutes.*

$$\begin{array}{ccc} H^*(\Gamma_2, M) & \xrightarrow{\mathrm{tr}} & H^*(\mathcal{G}_2, M) \\ f^* \downarrow & & \downarrow f^* \\ H^*(\Gamma_1, f_*M) & \xrightarrow{\mathrm{tr}} & H^*(\mathcal{G}_1, f_*M). \end{array}$$

Still assuming $\Gamma \subseteq \mathcal{G}$ is open, we have that the functor $M \mapsto M_\Gamma^\mathcal{G}$ is also isomorphic to the left adjoint to the forgetful functor. The natural map of Γ -modules

$$M_\Gamma^\mathcal{G} = \mathrm{map}_\Gamma^\mathcal{G}(\mathcal{G}, M) \longrightarrow M$$

sending φ to $\varphi(e)$ has a natural Γ -module splitting that extends to an isomorphism of $\mathcal{G}$ -modules

$$\mathbb{Z}_p[[\mathcal{G}]] \otimes_{\mathbb{Z}_p[[\Gamma]]} M \cong M_\Gamma^\mathcal{G}. \quad (4.8)$$

Thus we equally call $M_\Gamma^\mathcal{G}$ the *induced $\mathcal{G}$ -module*.

We have a commutative diagram of $\mathcal{G}$ -modules

$$\begin{array}{ccccc}
 M & \xrightarrow{\psi} & \mathbb{Z}_p[[\mathcal{G}]] \otimes_{\Gamma} M & \xrightarrow{m} & M \\
 \downarrow = & & \downarrow \cong & & \downarrow = \\
 M & \xrightarrow{\eta_M} & M_{\Gamma}^{\mathcal{G}} & \xrightarrow{\text{tr}} & M
 \end{array} \tag{4.9}$$

where $\psi(a) = \sum_{g \in \mathcal{G}/\Gamma} g \otimes g^{-1}a$ and m is induced by the action of $\mathcal{G}$ on M . The map ψ induces the transfer map in homology

$$\text{tr}_*: H_s(\mathcal{G}, M) \longrightarrow H_s(\Gamma, M).$$

More generally, if N is a right $\mathcal{G}$ -module we can use the isomorphism

$$N \otimes_{\mathbb{Z}_p[[\mathcal{G}]]} \mathbb{Z}_p[[\mathcal{G}]] \otimes_{\mathbb{Z}_p[[\Gamma]]} M \cong N \otimes_{\mathbb{Z}_p[[\Gamma]]} M$$

and the map ψ to obtain a transfer map

$$\text{tr}_*: \text{Tor}_s^{\mathcal{G}}(N, M) \longrightarrow \text{Tor}_s^{\Gamma}(N, M).$$

Likewise, m induces the map $\text{res}_*: \text{Tor}_s^{\Gamma}(N, M) \longrightarrow \text{Tor}_s^{\mathcal{G}}(N, M)$ arising from the ring homomorphism $\mathbb{Z}_p[[\Gamma]] \rightarrow \mathbb{Z}_p[[\mathcal{G}]]$. There is a naturality statement analogous to Lemma 4.7.

4.3 Frobenius reciprocity and cohomology

Let M and N be two continuous $\mathcal{G}$ -modules and let $M \otimes N$ be their tensor product with the diagonal action. Then we have a cup product pairing

$$H^m(\mathcal{G}, M) \otimes H^n(\mathcal{G}, N) \longrightarrow H^{m+n}(\mathcal{G}, M \otimes N).$$

If $\phi \in C^m(\mathcal{G}, M)$ and $\psi \in C^n(\mathcal{G}, N)$, then

$$(\phi \cdot \psi)(x_1, \dots, x_{n+m}) = \phi(x_1, \dots, x_m) \otimes (x_1 \cdots x_m)\psi(x_{m+1}, \dots, x_n).$$

The following formulas (1) and (2) are in §I.2.6a of [62]. Following Serre we leave them as an exercise.

- (1) Restriction preserves products: for all $x \in H^m(\mathcal{G}, M)$ and all $y \in H^n(\mathcal{G}, N)$

$$\text{res}_{\Gamma}^{\mathcal{G}}(x \cdot y) = \text{res}_{\Gamma}^{\mathcal{G}}(x) \cdot \text{res}_{\Gamma}^{\mathcal{G}}(y);$$

(2) Frobenius Reciprocity holds: for $x \in H^m(\Gamma, M)$ and $y \in H^n(\mathcal{G}, N)$

$$\mathrm{tr}_\Gamma^\mathcal{G}(x) \cdot y = \mathrm{tr}_\Gamma^\mathcal{G}(x \cdot \mathrm{res}_\Gamma^\mathcal{G}(y))$$

and for all $x \in H^m(\mathcal{G}, M)$ and $y \in H^n(\Gamma, N)$

$$x \cdot \mathrm{tr}_\Gamma^\mathcal{G}(y) = \mathrm{tr}_\Gamma^\mathcal{G}(\mathrm{res}_\Gamma^\mathcal{G}(x) \cdot y).$$

(3) Let $\Gamma \subseteq \mathcal{G}$ be a closed subgroup and $g \in \mathcal{G}$. Then for all $x \in H^m(\Gamma, M)$ and $y \in H^n(\Gamma, N)$

$$c_g^*(x \cdot y) = c_g^*(x) \cdot c_g^*(y) \in H^{m+n}({}^g\Gamma, {}^gM \otimes {}^gN).$$

Note ${}^gM \otimes {}^gN \cong {}^g(M \otimes N)$.

4.4 Conjugation actions

As above, let $c_g(x) = g^{-1}xg$ denote conjugation by $g \in \mathcal{G}$. Let M be a $\mathcal{G}$ -bimodule; that is, M is both a left and right $\mathcal{G}$ -module and the two actions commute. Our main example is $M = \mathbb{Z}_p[[\mathcal{G}]]$ of (4.1). As M is a left $\mathcal{G}$ -module, we have a map $c_g^* : H^*(\mathcal{G}, M) \rightarrow H^*(\mathcal{G}, {}^gM)$. But the right multiplication of $\mathcal{G}$ on M defines a map $r_g^* : H^*(\mathcal{G}, M) \rightarrow H^*(\mathcal{G}, M)$. We relate these two maps.

If $\phi : \mathcal{G}^s \rightarrow M$ is a continuous cochain, define

$$\chi_g(\phi) : \mathcal{G}^s \rightarrow M$$

by

$$\chi_g(\phi)(x_1, \dots, x_s) = g^{-1}\phi(c_g(x_1), \dots, c_g(x_s))g. \quad (4.10)$$

We check that this induces a map on cochain complexes and hence a map

$$\chi_g^* : H^*(\mathcal{G}, M) \rightarrow H^*(\mathcal{G}, M).$$

Note we have $\chi_{gh}^* = \chi_h^* \circ \chi_g^*$, so the assignment $g \mapsto \chi_g^*$ defines a right action on $H^*(\mathcal{G}, M)$. The following result says the action is very simple.

Proposition 4.11 *The map $\chi_g^* : H^*(\mathcal{G}, M) \rightarrow H^*(\mathcal{G}, M)$ is given by the right action of $g \in \mathcal{G}$ on M .*

Proof The right action of $\mathcal{G}$ on $H^*(\mathcal{G}, M)$ is given by multiplication of $\mathcal{G}$ on the right on M ; on cochains we have

$$r_g(\phi)(x_1, \dots, x_s) = \phi(x_1, \dots, x_s)g.$$

We construct a cochain homotopy $T_g : C^{s+1}(\mathcal{G}, M) \rightarrow C^s(\mathcal{G}, M)$ with

$$\partial T_g + T_g \partial = \chi_g - r_g.$$

The result will follow.

Define $T_g^i : C^{s+1}(\mathcal{G}, \mathbb{Z}_p[[\mathcal{G}]]) \rightarrow C^s(\mathcal{G}, \mathbb{Z}_p[[\mathcal{G}]])$, $0 \leq i \leq s$, by

$$T_g^i(\phi)(x_1, \dots, x_s) = \phi(x_1, \dots, x_i, g^{-1}, c_g(x_{i+1}), \dots, c_g(x_s))g.$$

Thus g^{-1} is in the $(i+1)$ st-slot. Note

$$T_g^0(d^0\phi)(x_1, \dots, x_s) = \chi_g(\phi)(x_1, \dots, x_s)$$

and

$$T_g^s(d^{s+1}\phi)(x_1, \dots, x_s) = r_g(\phi)(x_1, \dots, x_s).$$

and also

$$T_g^j d^i = \begin{cases} d^i T_g^{j-1}, & i < j; \\ T_g^{i-1} d^i, & i = j \neq 0; \\ d^{i-1} T_g^j, & i > j + 1. \end{cases}$$

Then we set

$$T_g = \sum_{i=0}^s (-1)^i T_g^i : C^{s+1}(\mathcal{G}, M) \rightarrow C^s(\mathcal{G}, M). \quad \square$$

Remark 4.12 In particular if M is a trivial module, we can make it a bimodule by giving it the trivial right module structure. Then we get the standard argument that conjugation of g induces the trivial action on $H^*(\mathcal{G}, M)$.

If M is bimodule then we get an isomorphism $M \cong {}^g M$ of left $\mathcal{G}$ -modules that sends a to gag^{-1} . This gives a commutative diagram

$$\begin{array}{ccc}
 & \xrightarrow{\chi_g^* = r_g^*} & H^*(\mathcal{G}, M) \\
 H^*(\mathcal{G}, M) & & \downarrow \cong \\
 & \xrightarrow{c_g^*} & H^*(\mathcal{G}, {}^g M).
 \end{array} \tag{4.13}$$

Thus all possible ways of defining the conjugation action determine each other.

4.5 Dualizing modules and duality

We review the concepts of Poincaré duality, relying especially on Section 4.4. of [64]. Our emphasis will be on p -adic analytic groups.

Definition 4.14 Let $\mathcal{G}$ be a profinite group. Then we have the following concepts.

- (1) The group $\mathcal{G}$ has *finite cohomological p -dimension* $cd_p(\mathcal{G})$ if there is some integer m so that for all p -torsion $\mathcal{G}$ -modules M and all $s > m$

$$H^s(\mathcal{G}, M) = 0.$$

Then $cd_p(\mathcal{G}) = n$ if n is minimal among the integers m for which this condition holds.

- (2) The group $\mathcal{G}$ is of *type p -FP* if the trivial $\mathcal{G}$ -module $\mathbb{Z}_p$ has a finite resolutions

$$0 \longrightarrow P_k \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z}_p$$

where each P_i is a finite direct sum of copies of $\mathbb{Z}_p[[\mathcal{G}]]$. Call k the *length* of the resolution.

- (3) The group $\mathcal{G}$ is a *Poincaré duality group* of dimension n if $cd_p(\mathcal{G}) = n$, $\mathcal{G}$ is of type p -FP and

$$H^s(\mathcal{G}, \mathbb{Z}_p[[\mathcal{G}]]) \cong \begin{cases} \mathbb{Z}_p, & s = n; \\ 0, & s \neq n. \end{cases}$$

The following is a consequence of Proposition 4.1.1 of [64].

Lemma 4.15 Let $\mathcal{G}$ be a Poincaré duality group of dimension n . Then

- (1) $H^s(\mathcal{G}, M) = 0$ for $s > n$ for all p -profinite $\mathcal{G}$ -modules M ; and
- (2) the trivial $\mathcal{G}$ -module $\mathbb{Z}_p$ has a resolution as in Definition 4.14.2 of exactly length n .

Let $\mathcal{G}$ be a compact p -adic analytic group of rank d with an exhaustive system

$$\cdots \subseteq \Gamma_{i+1} \subseteq \Gamma_i \subseteq \cdots \subseteq \Gamma_1 \subseteq \mathcal{G}$$

of uniformly powerful open normal subgroups. We then have.

Proposition 4.16 *For any compact p -adic analytic group $\mathcal{G}$ of rank d we have*

$$H^s(\mathcal{G}, \mathbb{Z}_p[[\mathcal{G}]]) \cong \begin{cases} \mathbb{Z}_p, & s = d; \\ 0, & s \neq d. \end{cases}$$

Furthermore, for all $i \geq 1$ (or $i \geq 2$ if $p = 2$), the uniformly powerful open normal subgroup Γ_i is a Poincaré duality group of dimension d .

Proof It is a theorem, going back to Serre (Proposition I.4.5 of [62]) and Lazard (Théorème V. 2.5.8 of [49]) that Γ_i is a Poincaré duality group of dimension d . The statement about $H^s(\mathcal{G}, \mathbb{Z}_p[[\mathcal{G}]])$ follows from Schapiro's Lemma Proposition 4.5 and that fact if $\Gamma \subseteq \mathcal{G}$ is of finite index then $\mathbb{Z}_p[[\Gamma]]_{\Gamma}^{\mathcal{G}} \cong \mathbb{Z}_p[[\mathcal{G}]]$. See (4.8). $\square$

The group ring $\mathbb{Z}_p[[\mathcal{G}]]$ is a $\mathcal{G}$ -bimodule, using the left and right actions of $\mathcal{G}$ on itself.

Remark 4.17 Our main examples of compact p -adic analytic groups, such as $\mathrm{GL}_n(\mathbb{Z}_p)$ or $\mathbb{G}_n$, need not be of finite cohomological dimension.

In calculating $H^s(\mathcal{G}, \mathbb{Z}_p[[\mathcal{G}]])$ we use the left action of $\mathcal{G}$ on the group ring. This continuous cohomology group retains a residual action on the right by $\mathcal{G}$.

Definition 4.18 Let $\mathcal{G}$ be a compact p -adic analytic group of rank d . Then the compact dualizing module $D_p(\mathcal{G})$ of $\mathcal{G}$ is the right- $\mathcal{G}$ module

$$D_p(\mathcal{G}) = H^d(\mathcal{G}, \mathbb{Z}_p[[\mathcal{G}]])$$

The following version of Poincaré Duality can be found as Proposition 4.5.1 of [64].

Theorem 4.19 *Let $\mathcal{G}$ be a compact p -adic analytic group of rank d with dualizing module $D_p(\mathcal{G})$. Suppose further that $\mathcal{G}$ is a Poincaré duality group. Then there is a natural homomorphism*

$$\mathrm{Tor}_s^{\mathcal{G}}(D_p(\mathcal{G}), M) \longrightarrow H^{d-s}(\mathcal{G}, M).$$

This homomorphism is an isomorphism if M is either a p -profinite module or a discrete p -torsion module.

Proof The paper [64] covers a great deal of ground and it takes a while to pull together the proof of Theorem 4.19 from that source. Here is a summary.

For any continuous left $\mathcal{G}$ -module P let $\text{Hom}_{\mathcal{G}}(P, \mathbb{Z}_p[[\mathcal{G}]])$ be the module of left $\mathcal{G}$ -module maps. This is a right $\mathcal{G}$ -module using the right $\mathcal{G}$ -module structure on $\mathbb{Z}_p[[\mathcal{G}]]$. Choose a projective resolution $P_{\bullet} \rightarrow \mathbb{Z}_p$ of the trivial $\mathbb{Z}_p[[\mathcal{G}]]$ -module $\mathbb{Z}_p$. Let M be a continuous left $\mathcal{G}$ -module. We then we have a pairing

$$\text{Hom}_{\mathcal{G}}(P_s, \mathbb{Z}_p[[\mathcal{G}]]) \otimes_{\mathcal{G}} M \longrightarrow \text{Hom}_{\mathcal{G}}(P_s, M)$$

sending $\phi \otimes a$ to the function $x \mapsto \phi(x)a$. This passes to an induced pairing

$$D_p(\mathcal{G}) \otimes_{\mathcal{G}} M = H^d(\mathcal{G}, \mathbb{Z}_p[[\mathcal{G}]]) \otimes_{\mathcal{G}} M \longrightarrow H^d(\mathcal{G}, M).$$

By Lemma 4.15, $\mathcal{G}$ has the property that $H^s(\mathcal{G}, M) = 0$ for $s > d$ for all p -profinite modules M ; thus, we find this is a natural transformation between right exact functors in M . Thus we get a natural transformation of left derived functors and, since the s th left derived functor of $H^d(\mathcal{G}, -)$ is $H^{d-s}(\mathcal{G}, -)$ we have the homomorphism

$$\text{Tor}_s^{\mathcal{G}}(D_p(\mathcal{G}), M) \longrightarrow H^{d-s}(\mathcal{G}, M). \quad (4.20)$$

If $M \cong \mathbb{Z}_p[[\mathcal{G}]]$ this map is an isomorphism when $s = 0$ and both source and target of this homomorphism vanish if $s \neq 0$. From this we deduce that (4.20) is an isomorphism when M is p -profinite. In particular, it is true when M is finite. Since any discrete p -torsion module is the colimit of its finite submodules, we have the result in that case as well. $\square$

Remark 4.21 In [64] the authors define $D_p(\mathcal{G})$ to be the left $\mathcal{G}$ -module obtained from $H^d(\mathcal{G}, \mathbb{Z}_p[[\mathcal{G}]])$ by $g \cdot x = xg^{-1}$. This allows them to write the Poincaré Duality isomorphism of Theorem 4.19 using homology. If we write $D_p(\mathcal{G})^{\ell}$ for this left module structure, then we have, under the hypotheses of Theorem 4.19

$$H_s(\mathcal{G}, D_p(\mathcal{G})^{\ell} \otimes M) \cong \text{Tor}_s^{\mathcal{G}}(D_p(\mathcal{G}), M) \cong H^{d-s}(\mathcal{G}, M).$$

There are times when this point of view is convenient. See the proof of Proposition 4.38.

Remark 4.22 The Poincaré duality group $\mathcal{G}$ is called *orientable* if the action on $D_p(\mathcal{G})$ is trivial. Then Theorem 4.19 implies

$$H_s(\mathcal{G}, M) \cong H^{d-s}(\mathcal{G}, M)$$

and $H_d(\mathcal{G}, \mathbb{Z}_p) \cong \mathbb{Z}_p$.

Remark 4.23 Our main examples are orientable, using the following argument adapted from the proof of Proposition 5 of [63].

By Corollary 5.2.5 of [64] (see Proposition 4.40 for more explanation) there is an isomorphism of right $\mathcal{G}$ -modules

$$D_p(\mathcal{G}) \cong \Lambda^d \mathfrak{g}^*$$

where $\Lambda^d \mathfrak{g}^*$ is the top exterior power of the dual of the adjoint representation. In the case where $\mathcal{G} = \mathrm{GL}_n(\mathbb{Z}_p)$ or $\mathbb{G}_n$, the group $\mathcal{G}$ is the group of units in a sub-algebra A_0 of a complete normed $\mathbb{Q}_p$ -algebra and $\mathfrak{g}$ is A_0 with the conjugation action of $\mathcal{G}$. See Example 2.12 and Example 2.15. If $f : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is any linear transformation, then $\Lambda^d f$ is multiplication by the determinant of f . Thus if $g \in \mathcal{G}$ acts by conjugation on $\mathfrak{g} = A_0$, it must act trivially $\Lambda^d \mathfrak{g}^*$.

There is a variant of Poincaré Duality which is closer to the Serre-Grothendieck duality of algebraic geometry. Let $\mathbb{Z}/p^\infty = \mathrm{colim} \mathbb{Z}/p^n$; this is an injective abelian group. Define the *discrete dualizing module* for $\mathcal{G}$ as the left $\mathcal{G}$ -module

$$I_p(\mathcal{G}) = \mathrm{Hom}(D_p(\mathcal{G}), \mathbb{Z}/p^\infty). \quad (4.24)$$

The action of $\mathcal{G}$ is given by the formula $g\phi(x) = \phi(xg)$. Evaluation gives an isomorphism

$$D_p(\mathcal{G}) \otimes_{\mathcal{G}} I_p(\mathcal{G}) \cong \mathbb{Z}/p^\infty.$$

If $H^s(\mathcal{G}, -) = 0$ for $s > d$, then Theorem 4.19 gives an isomorphism

$$\epsilon : H^d(\mathcal{G}, I_p(\mathcal{G})) \xrightarrow{\cong} \mathbb{Z}/p^\infty.$$

If M is a continuous $\mathcal{G}$ module define

$$M^\vee = \mathrm{Hom}^\Delta(M, I_p(\mathcal{G}))$$

to be the abelian group of continuous homomorphisms with the left $\mathcal{G}$ -action given $(g\varphi)(a) = g\varphi(g^{-1}a)$. See (4) of Notation 4.2. The evaluation map $M^\vee \otimes M \rightarrow I_p(\mathcal{G})$ and cup product give us a pairing

$$H^s(\mathcal{G}, M^\vee) \otimes H^{d-s}(\mathcal{G}, M) \longrightarrow H^d(\mathcal{G}, I_p(\mathcal{G})) \xrightarrow{\epsilon} \mathbb{Z}/p^\infty. \quad (4.25)$$

Then we have, as in §I.3.4 of [62]:

Theorem 4.26 *Let $\mathcal{G}$ be a Poincaré duality group of dimension d and suppose M is either p -profinite or p -torsion. Then (4.25) is a perfect pairing which identifies $H^s(\mathcal{G}, M^\vee)$ with the Pontryagin dual of $H^{d-s}(\mathcal{G}, M)$.*

4.6 Frobenius reciprocity and duality

Poincaré Duality flips transfer and restriction. To make this precise we begin with the following result.

Proposition 4.27 *Let $\mathcal{G}$ be a compact analytic group of rank d and $\Gamma \subseteq \mathcal{G}$ an open subgroup. Then the compact dualizing module $D_p(\Gamma)$ for Γ is isomorphic to the compact dualizing module $D_p(\mathcal{G})$ for $\mathcal{G}$ with action restricted to Γ .*

Proof Since we have an isomorphism of right Γ -modules

$$\mathbb{Z}_p[[\Gamma]]_\Gamma^\mathcal{G} = \mathbb{Z}_p[[\mathcal{G}]] \otimes_\Gamma \mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[\mathcal{G}]]$$

Schapiro's Lemma Proposition 4.5 gives

$$D_p(\Gamma) = H^d(\Gamma, \mathbb{Z}_p[[\Gamma]]) \cong H^d(\mathcal{G}, \mathbb{Z}_p[[\mathcal{G}]]) = D_p(\mathcal{G})$$

and these isomorphisms respect the right action by Γ . $\square$

Note that it follows that the discrete dualizing module $I_p(\Gamma)$ for Γ is isomorphic to the discrete dualizing module $I_p(\mathcal{G})$ for $\mathcal{G}$ with action restricted to Γ . This is Proposition 18 of §I.3.5 of [62].

We now have the following result. Note that $D_p(\mathcal{G})$ has a canonical structure as Γ -module by Proposition 4.27.

Proposition 4.28 *Let $\mathcal{G}$ be a compact p -adic analytic group of rank d and suppose $H^s(\mathcal{G}, -) = 0$ for $s > d$. Then we have commutative diagrams*

$$\begin{array}{ccc} \mathrm{Tor}_s^\mathcal{G}(D_p(\mathcal{G}), M) & \longrightarrow & H^{d-s}(\mathcal{G}, M) \\ \mathrm{tr}_* \downarrow & & \downarrow \mathrm{res} \\ \mathrm{Tor}_s^\Gamma(D_p(\mathcal{G}), M) & \longrightarrow & H^{d-s}(\Gamma, M). \end{array} \quad (4.29)$$

and

$$\begin{array}{ccc} \mathrm{Tor}_s^\Gamma(D_p(\mathcal{G}), M) & \longrightarrow & H^{d-s}(\Gamma, M) \\ \mathrm{res}_* \downarrow & & \downarrow \mathrm{tr} \\ \mathrm{Tor}_s^\mathcal{G}(D_p(\mathcal{G}), M) & \longrightarrow & H^{d-s}(\mathcal{G}, M). \end{array} \quad (4.30)$$

Proof We give an argument only for (4.29), as that is what we will use later. The argument for (4.30) is very similar.

The map $\mathrm{Tor}_s^{\mathcal{G}}(D_p(\mathcal{G}), M) \rightarrow H^{d-s}(\mathcal{G}, M)$ was constructed in the proof of Theorem 4.19 by first defining it for $s = 0$ and then using naturality to extend to the higher left derived functors. Built into this construction is the assertion that if

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

is a short exact sequence of continuous $\mathcal{G}$ -modules then we have a commutative diagram of long exact sequences

$$\begin{array}{ccccccc} \mathrm{Tor}_{s+1}^{\mathcal{G}}(D_p(\mathcal{G}), N) & \longrightarrow & \mathrm{Tor}_s^{\mathcal{G}}(D_p(\mathcal{G}), K) & \longrightarrow & \mathrm{Tor}_s^{\mathcal{G}}(D_p(\mathcal{G}), M) & \longrightarrow & \mathrm{Tor}_s^{\mathcal{G}}(D_p(\mathcal{G}), N) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{d-s-1}(\mathcal{G}, N) & \longrightarrow & H^{d-s}(\mathcal{G}, K) & \longrightarrow & H^{d-s}(\mathcal{G}, M) & \longrightarrow & H^{d-s}(\mathcal{G}, N) \end{array}$$

Thus if we can prove that (4.29) commutes at $s = 0$ then it will follow for all $s \geq 1$ as well.

Choose a projective resolution $P_{\bullet} \rightarrow \mathbb{Z}_p$ of the trivial $\mathbb{Z}_p[[\mathcal{G}]]$ -module $\mathbb{Z}_p$. The map $D_p(\mathcal{G}) \otimes_{\mathcal{G}} M \rightarrow H^d(\mathcal{G}, M)$ is defined at the chain level by the pairing

$$\mathrm{Hom}_{\mathcal{G}}(P_s, \mathbb{Z}_p[[\mathcal{G}]]) \otimes_{\mathcal{G}} M \longrightarrow \mathrm{Hom}_{\mathcal{G}}(P_s, M)$$

sending $\phi \otimes a$ to the function $x \mapsto \phi(x)a$. Let

$$\begin{aligned} \psi : M &\longrightarrow \mathbb{Z}_p[[\mathcal{G}]] \otimes_{\Gamma} M \\ \psi(a) &= \sum_{g \in \Gamma} g \otimes g^{-1}a \end{aligned}$$

be the map of (4.9) used to define tr_* . Since $\mathbb{Z}_p[[\mathcal{G}]] \cong (\mathbb{Z}_p[[\Gamma]])_{\Gamma}^{\mathcal{G}}$ we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{G}}(P, \mathbb{Z}_p[[\mathcal{G}]]) \otimes_{\mathcal{G}} M & \longrightarrow & \mathrm{Hom}_{\mathcal{G}}(P, M) \\ \downarrow 1 \otimes \psi & & \downarrow \\ \mathrm{Hom}_{\mathcal{G}}(P, \mathbb{Z}_p[[\mathcal{G}]]) \otimes_{\mathcal{G}} \mathbb{Z}_p[[\mathcal{G}]] \otimes_{\Gamma} M & & \\ \downarrow \cong & & \downarrow \\ \mathrm{Hom}_{\Gamma}(P, \mathbb{Z}_p[[\Gamma]]) \otimes_{\Gamma} M & \longrightarrow & \mathrm{Hom}_{\Gamma}(P, M) \end{array}$$

where the right vertical map is inclusion. Passing to cohomology gives the needed commutative diagram

$$\begin{array}{ccc} H^d(\mathcal{G}, \mathbb{Z}_p[[\mathcal{G}]]) \otimes_{\mathcal{G}} M & \longrightarrow & H^d(\mathcal{G}, M) \\ \text{tr}_* \downarrow & & \downarrow \text{res} \\ H^d(\Gamma, \mathbb{Z}_p[[\Gamma]]) \otimes_{\Gamma} M & \longrightarrow & H^d(\Gamma, M). \end{array} \quad \square$$

4.7 Restricting to subgroups

The compact dualizing module $D_p(\Gamma)$ for Γ is isomorphic to the compact dualizing module $D_p(\mathcal{G})$ for $\mathcal{G}$ with action restricted to Γ . See Proposition 4.27. We now discuss how to recover the right- $\mathcal{G}$ module structure on $D_p(\mathcal{G})$ from $D_p(\Gamma)$ using the subgroup structure of $\mathcal{G}$.

If $g \in \mathcal{G}$ and $\Gamma \subseteq \mathcal{G}$ is a subgroup, let ${}^g\Gamma$ denote the conjugate subgroup $g^{-1}\Gamma g$ and $c_g(-) = g(-)g^{-1} : {}^g\Gamma \rightarrow \Gamma$ the conjugation homomorphism. If $\phi : \Gamma^{s+1} \rightarrow \mathbb{Z}_p[[\Gamma]]$ is a continuous cochain, define $\chi_g(\phi) : {}^g\Gamma^{s+1} \rightarrow \mathbb{Z}_p[[{}^g\Gamma]]$ by

$$\chi_g(\phi)(x_0, x_1, \dots, x_s) = g^{-1}\phi(c_g(x_0), \dots, c_g(x_s))g.$$

This extends to subgroups the definition given for $\mathcal{G}$ itself in (4.10). We check that this induces a map on cochain complexes and hence a map

$$\chi_g^* : H^*(\Gamma, \mathbb{Z}_p[[\Gamma]]) \rightarrow H^*({}^g\Gamma, \mathbb{Z}_p[[{}^g\Gamma]]).$$

Proposition 4.31 *Suppose $\mathcal{G}$ is a compact p -adic analytic group of rank d and Γ is an open subgroup. Suppose $H^s(\Gamma, -) = 0$ for $s > d$. The map*

$$D_p(\Gamma) = H^d(\Gamma, \mathbb{Z}_p[[\Gamma]]) \xrightarrow{\chi_g^*} D_p({}^g\Gamma) = H^d({}^g\Gamma, \mathbb{Z}_p[[{}^g\Gamma]])$$

is isomorphic to the map

$$r_g = (-)g : D_p(\mathcal{G}) \longrightarrow D_p(\mathcal{G}).$$

Proof A chain level calculation shows that the following diagram commutes

$$\begin{array}{ccc} H^d(\Gamma, \mathbb{Z}_p[[\Gamma]]) & \xrightarrow{\chi_g^*} & H^d({}^g\Gamma, \mathbb{Z}_p[[{}^g\Gamma]]) \\ \cong \downarrow & & \downarrow \cong \\ H^d(\mathcal{G}, \mathbb{Z}_p[[\mathcal{G}]]) & \xrightarrow{\chi_g^*} & H^d(\mathcal{G}, \mathbb{Z}_p[[\mathcal{G}]]) \end{array}$$

By Proposition 4.11 the bottom map is given by the standard right $\mathcal{G}$ -action on $H^*(\mathcal{G}, \mathbb{Z}_p[[\mathcal{G}]])$. $\square$

Remark 4.32 In particular, if Γ is normal in $\mathcal{G}$, the conjugation action of $\mathcal{G}$ on Γ and $\mathbb{Z}_p[[\Gamma]]$ recovers the right action of $\mathcal{G}$ on $H^d(\mathcal{G}, \mathbb{Z}_p[[\mathcal{G}]])$ from the conjugation action on $H^d(\Gamma, \mathbb{Z}_p[[\Gamma]])$.

Remark 4.33 The conjugation map c_g defines an isomorphism of left $\mathbb{Z}_p[[{}^g\Gamma]]$ -modules $\mathbb{Z}_p[[{}^g\Gamma]] \cong {}^g\mathbb{Z}_p[[\Gamma]]$ and we have a diagram mapping to the diagram of (4.13)

$$\begin{array}{ccc} & & H^*({}^g\Gamma, \mathbb{Z}_p[[{}^g\Gamma]]) \\ & \nearrow \chi_g^* & \downarrow \cong \\ H^*(\Gamma, \mathbb{Z}_p[[\Gamma]]) & & \\ & \searrow c_g^* & H^*({}^g\Gamma, {}^g\mathbb{Z}_p[[\Gamma]]) \end{array}$$

Thus we could have worked with c_g^* (Example 4.4) as well.

Proposition 4.28 described how duality interacted with transfer and restriction. We'd also like to know how duality interacts with conjugation. We gain conceptual clarity by proving a more general result; we then specialize. The main result is Proposition 4.36

Let $\mathcal{G}$ be a compact p -adic analytic group, N a continuous left $\mathcal{G}$ -module and M a continuous $\mathcal{G}$ -bimodule. We have a pairing

$$\langle -, - \rangle : C^s(\mathcal{G}, M) \otimes_{\mathcal{G}} N \longrightarrow C^s(\mathcal{G}, M \otimes_{\mathcal{G}} N)$$

sending $\phi \otimes a$ to ψ where

$$\psi(x_1, \dots, x_s) = \phi(x_1, \dots, x_s) \otimes a.$$

We now explore the naturality of this pairing. Let $f : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ be a continuous homomorphism of compact p -adic analytic groups, N a continuous left $\mathcal{G}_1$ -module and M a continuous $\mathcal{G}_2$ -bimodule. For simplicity we assume that f is

an injection onto a subgroup of finite index, so that we can form the induced module

$$N_{\mathcal{G}_1}^{\mathcal{G}_2} \cong \mathbb{Z}_p[[\mathcal{G}_2]] \otimes_{\mathbb{Z}_p[[\mathcal{G}_1]]} N =: \mathcal{G}_2 \times_{\mathcal{G}_1} N.$$

If M is a right $\mathcal{G}_2$ module, let M^f denote the *right* $\mathcal{G}_1$ -module obtained by restriction. Recall, from Section 4.1, that f_*M denotes the *left* $\mathcal{G}_1$ -module obtained by restriction. The following can be checked on cochains.

Lemma 4.34 *Let N be a continuous left $\mathcal{G}_1$ -module, M a continuous $\mathcal{G}_2$ -bimodule and $f: \mathcal{G}_1 \rightarrow \mathcal{G}_2$ a continuous homomorphism of compact p -adic analytic groups which is an injection onto a subgroup of finite index. Then the pairing $\langle -, - \rangle$ induces a commutative diagram*

$$\begin{array}{ccc} H^s(\mathcal{G}_2, M) \otimes_{\mathcal{G}_2} (\mathcal{G}_2 \times_{\mathcal{G}_1} N) & \longrightarrow & H^s(\mathcal{G}_2, M \otimes_{\mathcal{G}_2} (\mathcal{G}_2 \times_{\mathcal{G}_1} N)) \\ \cong \downarrow & & \downarrow \cong \\ H^s(\mathcal{G}_2, M^f) \otimes_{\mathcal{G}_1} N & \longrightarrow & H^s(\mathcal{G}_2, M^f \otimes_{\mathcal{G}_1} N) \\ f_* \downarrow & & \downarrow f_* \\ H^s(\mathcal{G}_1, f_*M^f) \otimes_{\mathcal{G}_1} N & \longrightarrow & H^s(\mathcal{G}_1, f_*M^f \otimes_{\mathcal{G}_1} N). \end{array}$$

We now specialize. Let $\Gamma \subseteq \mathcal{G}$ be an open subgroup and $g \in \mathcal{G}$. We let f be the conjugation map $c_g: {}^g\Gamma \rightarrow \Gamma$. For M a Γ -bimodule, let ${}^gM^g$ be f_*M^f , the ${}^g\Gamma$ -bimodule obtained by restriction along $f = c_g$ for both the right and left actions. Then c_g defines an isomorphism of ${}^g\Gamma$ -modules

$$\mathbb{Z}_p[[{}^g\Gamma]] \xrightarrow{\cong} {}^g\mathbb{Z}_p[[\Gamma]]^g$$

and if N is a ${}^g\Gamma$ -module

$$\Gamma \times_{{}^g\Gamma} N \cong {}^{g^{-1}}N$$

where ${}^{g^{-1}}N$ is the module obtained by restriction along $c_{g^{-1}}: \Gamma \rightarrow {}^g\Gamma$. Thus if we replace N by gN in Lemma 4.34 we have a diagram

$$\begin{array}{ccc} H^s(\Gamma, \mathbb{Z}_p[[\Gamma]]) \otimes_{\Gamma} N & \longrightarrow & H^s(\Gamma, \mathbb{Z}_p[[\Gamma]] \otimes_{\Gamma} N) \\ c_g^* \downarrow & & \downarrow c_g^* \\ H^s({}^g\Gamma, \mathbb{Z}_p[[{}^g\Gamma]]) \otimes_{{}^g\Gamma} {}^gN & \longrightarrow & H^s({}^g\Gamma, \mathbb{Z}_p[[{}^g\Gamma]] \otimes_{{}^g\Gamma} {}^gN) \end{array}$$

After using the module multiplications on the coefficients, we get a diagram

$$\begin{array}{ccc} H^s(\Gamma, \mathbb{Z}_p[[\Gamma]]) \otimes_{\Gamma} N & \longrightarrow & H^s(\Gamma, N) \\ c_g^* \otimes N \downarrow & & \downarrow c_g^* \\ H^s({}^g\Gamma, \mathbb{Z}_p[[{}^g\Gamma]]) \otimes_{{}^g\Gamma} N & \longrightarrow & H^s({}^g\Gamma, {}^gN) \end{array} \quad (4.35)$$

The right hand morphism c_g^* sends $\phi \otimes a$ to the cocycle

$$(x_1, \dots, x_s) \mapsto g^{-1}\phi(c_g(x_1), \dots, c_g(x_s))g \otimes a.$$

In the next result we use that the compact dualizing module $D_p(\Gamma)$ for $\Gamma \subseteq \mathcal{G}$ open is isomorphic to the compact dualizing module $D_p(\mathcal{G})$ for $\mathcal{G}$ with action restricted to Γ . See Proposition 4.27.

Proposition 4.36 *Suppose $\mathcal{G}$ is a compact p -adic analytic group of rank d and Γ is an open subgroup and a Poincaré duality group. Let M be a left Γ -module. Then the map*

$$D_p(\Gamma) = H^d(\Gamma, \mathbb{Z}_p[[\Gamma]]) \xrightarrow{\chi_g^*} D_p({}^g\Gamma) = H^d({}^g\Gamma, \mathbb{Z}_p[[{}^g\Gamma]])$$

is isomorphic to the map

$$r_g = (-)g : D_p(\mathcal{G}) \longrightarrow D_p(\mathcal{G})$$

and we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Tor}_s^{\Gamma}(D_p(\mathcal{G}), M) & \xrightarrow{\cong} & H^{d-s}(\Gamma, M) \\ (r_g)_* \downarrow & & \downarrow c_g^* \\ \mathrm{Tor}_s^{{}^g\Gamma}(D_p(\mathcal{G}), {}^gM) & \xrightarrow{\cong} & H^{d-s}({}^g\Gamma, {}^gM). \end{array}$$

Proof The first statement is a repeat of Proposition 4.31. For the second, we can proceed as in the proof of Proposition 4.28 and note it is sufficient to check the case $s = 0$. This is exactly (4.35). The horizontal isomorphisms are Poincaré Duality. See Theorem 4.19. $\square$

Corollary 4.37 *Suppose $\mathcal{G}$ is a compact p -adic analytic group of rank d and Γ is an open subgroup and a Poincaré duality group. Suppose M is a trivial*

$\mathcal{G}$ -module and $\Gamma \subseteq \mathcal{G}$ is normal. Then Proposition 4.36 gives a diagram

$$\begin{array}{ccc} D_p(\mathcal{G}) \otimes_{\Gamma} M & \xrightarrow{\cong} & H^d(\Gamma, M) \\ (r_g)_* \otimes M \downarrow & & \downarrow c_g^* \\ D_p(\mathcal{G}) \otimes_{\Gamma} M & \xrightarrow{\cong} & H^d(\Gamma, M) \end{array}$$

where the left column is given by the right action of g on $D_p(\mathcal{G})$ and the right column is given by the conjugation action of g on Γ .

4.8 The dualizing module revisited

We can now give a formula for the discrete dualizing module $I_p(\mathcal{G})$ due to Serre. The module $I_p(\mathcal{G})$ was defined in (4.24). Let $\mathcal{G}$ be a compact p -adic analytic group of rank d with an exhaustive system

$$\cdots \subseteq \Gamma_{i+1} \subseteq \Gamma_i \subseteq \cdots \subseteq \Gamma_1 \subseteq \mathcal{G}$$

of uniformly powerful open normal subgroups. The following gives exactly Serre's formula in §3.5 of [62].

Proposition 4.38 *There is an isomorphism of left $\mathcal{G}$ -modules*

$$I_p(\mathcal{G}) \cong \operatorname{colim}_{\operatorname{tr}} H_d(\Gamma_i, \mathbb{Z}/p^\infty) \quad (4.39)$$

where $\mathcal{G}$ acts on $H_*(\Gamma_i, \mathbb{Z}/p^\infty)$ through conjugation on Γ_i .

Proof Let $D_p(\mathcal{G})^\ell$ be left- $\mathcal{G}$ -module obtained by transposing the right action; that is $g \cdot a = ag^{-1}$. Then if M is any continuous $\mathcal{G}$ -module we have a natural isomorphism

$$D_p(\mathcal{G}) \otimes_{\mathcal{G}} M \cong \mathbb{Z}_p \otimes_{\mathcal{G}} (D_p(\mathcal{G})^\ell \otimes M)$$

where $D_p(\mathcal{G})^\ell \otimes M$ has the diagonal action. We thus get a natural isomorphism

$$H_s(\mathcal{G}, D_p(\mathcal{G})^\ell \otimes M) \cong \operatorname{Tor}_s^{\mathcal{G}}(D_p(\mathcal{G}), M).$$

There is an isomorphism of left $\mathcal{G}$ -modules

$$D_p(\mathcal{G})^\ell \otimes I_p(\mathcal{G}) \cong \mathbb{Z}/p^\infty$$

where the tensor product has the diagonal $\mathcal{G}$ -structure and the action on $\mathbb{Z}/p^\infty$ is trivial. For i large Frobenius reciprocity (4.29) and Poincaré Duality Theorem 4.19 give a commutative square

$$\begin{array}{ccc} H_s(\Gamma_i, \mathbb{Z}/p^\infty) & \xrightarrow{\cong} & H^{d-s}(\Gamma_i, I_p(\mathcal{G})) \\ \text{tr}_* \downarrow & & \downarrow \text{res} \\ H_s(\Gamma_{i+1}, \mathbb{Z}/p^\infty) & \xrightarrow{\cong} & H^{d-s}(\Gamma_{i+1}, I_p(\mathcal{G})) \end{array}.$$

Thus, if we set $s = d$ and $M = \mathbb{Z}/p^\infty$ we get an isomorphism

$$[I_p(\mathcal{G})]^{\Gamma_i} \cong H_d(\Gamma_i, \mathbb{Z}/p^\infty).$$

This is a $\mathcal{G}$ -equivariant isomorphism by Proposition 4.36. If M is a discrete $\mathcal{G}$ -module then the natural map

$$\text{colim}_i M^{\Gamma_i} \rightarrow M$$

is an isomorphism; now using the above commutative diagram, we get an isomorphism of $\mathcal{G}$ -modules

$$I_p(\mathcal{G}) \cong \text{colim}_{\text{tr}_*} H_d(\Gamma_i, \mathbb{Z}/p^\infty).$$

The following is a direct consequence of Corollary 5.2.5 of [64], although it would take some translation to get from that statement to ours. The adjoint representation was defined in Remark 2.5

Proposition 4.40 *Let $\mathcal{G}$ be a compact p -adic analytic group of dimension d and let $\mathfrak{g}$ be the adjoint representation. Then there is a natural isomorphism of right $\mathcal{G}$ -modules*

$$\Lambda^d \mathfrak{g}^* \cong D_p(\mathcal{G}).$$

where

$$\mathfrak{g}^* = \text{Hom}(\mathfrak{g}, \mathbb{Z}_p)$$

with right $\mathcal{G}$ -action given by $(\phi g)(x) = \phi(gx)$ and $\Lambda^d \mathfrak{g}^*$ is the top exterior power.

Proof Since $\Lambda^d \mathfrak{g}^*$ and $D_p(\mathcal{G})$ are free modules of rank 1 over $\mathbb{Z}_p$ it is sufficient to produce a surjective map of $\mathcal{G}$ -modules

$$\Lambda^d \mathfrak{g}^* \longrightarrow D_p(\mathcal{G})/p^j$$

for all $j \geq 1$. Recall from Remark 2.5 that $\mathfrak{g}_i = p^{i-1}\mathfrak{g}$ and, since $\mathfrak{g}$ is p -torsion free, that $p^{i-1}: \mathfrak{g} \rightarrow \mathfrak{g}_i$ is an isomorphism.

Fix j . Then $D_p(\mathcal{G})/p^j$ is a finite $\mathcal{G}$ -module, so we may choose $i \geq j$ so that Γ_i acts trivially on $D_p(\mathcal{G})/p^j$. Then, using (2.6), we have a map of left $\mathcal{G}$ -modules

$$\mathfrak{g} \xrightarrow{p^{i-1}} \mathfrak{g}_i/\mathfrak{g}_{i+j} \xrightarrow{\cong} \Gamma_i/\Gamma_{i+j}$$

where $\mathcal{G}$ acts on the target by conjugation. This gives an isomorphism of right $\mathcal{G}$ -modules

$$H^1(\Gamma_i/\Gamma_{i+j}, \mathbb{Z}/p^j) \cong \text{Hom}(\mathfrak{g}, \mathbb{Z}/p^j).$$

We then obtain a map of right $\mathcal{G}$ -modules

$$\begin{aligned} \mathfrak{g}^* &= \text{Hom}(\mathfrak{g}, \mathbb{Z}_p) \longrightarrow \text{Hom}(\mathfrak{g}, \mathbb{Z}/p^j) \\ &\cong H^1(\Gamma_i/\Gamma_{i+j}, \mathbb{Z}/p^j) \cong H^1(\Gamma_i, \mathbb{Z}/p^j). \end{aligned}$$

Using the cup product we get a map of right $\mathcal{G}$ -modules

$$\Lambda^d \mathfrak{g}^* \longrightarrow H^d(\Gamma_i, \mathbb{Z}/p^j)$$

which is surjective by Theorem 3.3. By Poincaré Duality and the fact that $D_p(\Gamma_i) \cong D_p(\mathcal{G})$ with action restricted to Γ_i Corollary 4.37 gives an isomorphism of right $\mathcal{G}$ -modules

$$H^d(\Gamma_i, \mathbb{Z}/p^j) \cong H^d(\mathcal{G}, \mathbb{Z}_p[[\mathcal{G}]]) \otimes_{\Gamma_i} \mathbb{Z}/p^j \cong H^d(\mathcal{G}, \mathbb{Z}_p[[\mathcal{G}]])/p^j.$$

The last isomorphism follows from the fact we have chosen i so that Γ_i acts trivially on

$$D_p(\mathcal{G})/p^j = H^d(\mathcal{G}, \mathbb{Z}_p[[\mathcal{G}]])/p^j.$$

The composition

$$\Lambda^d \mathfrak{g}^* \longrightarrow H^d(\Gamma_i, \mathbb{Z}/p^j) \cong D_p(\mathcal{G})/p^j$$

is the surjection we need. □

Remark 4.41 We can combine Proposition 4.38 and Proposition 4.40 to obtain an isomorphism of left $\mathcal{G}$ -modules

$$I_p(\mathcal{G})/p^j \cong \text{colim}_{\text{tr}^*} H^d(\Gamma_i, \mathbb{Z}/p^j) \cong \Lambda^d \mathfrak{g}/p^j.$$

This follows from the fact that $I_p(\mathcal{G}) = \text{Hom}(D_p(\mathcal{G}), \mathbb{Z}/p^\infty)$ and that the dual of $\Lambda^d \mathfrak{g}$ is $\Lambda^d \mathfrak{g}^*$.

5 Two $\mathcal{G}$ -spheres

In this section, we produce the two p -complete $\mathcal{G}$ -spheres which are the main characters of this paper. They are produced by the same method, so we will give complete details for only one. Here as usual $\mathcal{G}$ is a compact p -adic analytic group with an exhaustive system $\cdots \subseteq \Gamma_{i+1} \subseteq \Gamma_i \subseteq \cdots \Gamma_1 \subseteq \mathcal{G}$ of uniformly powerful open normal subgroups.

Notation 5.1 Here and hereafter we will work in the category of p -complete spectra unless we specify otherwise. In particular we will write

$$\Sigma_+^\infty B\Gamma_i \quad \text{for} \quad (\Sigma_+^\infty B\Gamma_i)_p.$$

This is necessary as we will want to use Proposition 3.11.

The spectra $\Sigma_+^\infty B\Gamma_i / \Gamma_j$ and $\Sigma_+^\infty B\Gamma_i$ have a $\mathcal{G}$ -action induced from the conjugation action of $\mathcal{G}$ on Γ_i . Let $j > 1$; then we have an inclusion of a normal subgroup of finite index $\Gamma_{i+1} / \Gamma_{i+j} \rightarrow \Gamma_i / \Gamma_{i+j}$ and hence a $\mathcal{G}$ -equivariant transfer map

$$\Sigma_+^\infty B(\Gamma_i / \Gamma_{i+j}) \longrightarrow \Sigma_+^\infty B(\Gamma_{i+1} / \Gamma_{i+j}).$$

By taking homotopy inverse limits, we then get an induced $\mathcal{G}$ -equivariant transfer map

$$\text{tr} : \Sigma_+^\infty B\Gamma_i \longrightarrow \Sigma_+^\infty B\Gamma_{i+1}. \quad (5.2)$$

Remark 5.3 To produce the transfer map of (5.2) we use the following naturality property of the transfer. Let $K \subseteq H \subseteq G$ be a nested sequence of subgroups with all three inclusions normal. Then the following diagram commutes

$$\begin{array}{ccc} \Sigma_+^\infty BG & \xrightarrow{\text{tr}} & \Sigma_+^\infty BH \\ \downarrow & & \downarrow \\ \Sigma_+^\infty B(G/K) & \xrightarrow{\text{tr}} & \Sigma_+^\infty B(H/K). \end{array}$$

The following construction is motivated by Serre's formula from Proposition 4.38.

Definition 5.4 Let $I_{\mathcal{G}}$ be the $\mathcal{G}$ -spectrum

$$I_{\mathcal{G}} = \operatorname{hocolim} \Sigma_+^{\infty} B\Gamma_i \quad (5.5)$$

where the colimit is over the transfer maps and in the category of p -complete spectra; that is, we are taking the p -completion of the ordinary homotopy colimit.

Proposition 5.6 *We have the following calculations in homology.*

(1) *The map induced in homology by the transfer*

$$\operatorname{tr}_* : H_k(B\Gamma_i, \mathbb{F}_p) \longrightarrow H_k(B\Gamma_{i+1}, \mathbb{F}_p)$$

is zero if $k \neq d$ and an isomorphism if $k = d$.

(2) *$H_k I_{\mathcal{G}} = 0$ unless $k = d$ and $H_d I_{\mathcal{G}} \cong \mathbb{F}_p$. In particular, $I_{\mathcal{G}}$ has the homotopy type of a p -complete d -sphere.*

Proof Using Proposition 4.28 the map induced by the transfer is isomorphic, via Poincaré Duality, to the map induced in cohomology by restriction

$$H^{k-d}(B\Gamma_i, \mathbb{F}_p) \longrightarrow H^{k-d}(B\Gamma_{i+1}, \mathbb{F}_p).$$

By Part (2) of Theorem 3.3 the map $H^*(B\Gamma_i, \mathbb{F}_p) \rightarrow H^*(B\Gamma_{i+1}, \mathbb{F}_p)$ is zero in positive degrees, and the result follows from the Whitehead Theorem for connective spectra. $\square$

We can use these same methods to build a p -complete $\mathcal{G}$ sphere from the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. (The Lie algebra and its properties were discussed in Definition 2.4 and material following that definition.) Specifically, we use the abelian groups $p^i \mathfrak{g} \subseteq \mathfrak{g}$ in place of Γ_i , and form the p -complete spectrum

$$S^{\mathfrak{g}} = \operatorname{hocolim} \Sigma_+^{\infty} B(p^i \mathfrak{g}). \quad (5.7)$$

As in Definition 5.4, the colimit is taken over transfers in the category of p -complete spectra. The action of $\mathcal{G}$ on $S^{\mathfrak{g}}$ is through the adjoint representation. See Remark 2.5.

The following is the analog of Proposition 5.6.

Lemma 5.8 *We have the following calculations in homology.*

(1) *The map induced in homology by the transfer*

$$\operatorname{tr}_* : H_k(Bp^i \mathfrak{g}, \mathbb{F}_p) \longrightarrow H_k(Bp^{i+1} \mathfrak{g}, \mathbb{F}_p)$$

is zero if $k \neq d$ and an isomorphism if $k = d$.

(2) $H_k S^{\mathfrak{g}} = 0$ unless $k = d$ and $H_d S^{\mathfrak{g}} \cong \mathbb{F}_p$. In particular, $S^{\mathfrak{g}}$ has the homotopy type of a p -complete d -sphere.

The next result identifies the $\mathcal{G}$ action on $H_*(I_{\mathcal{G}}, \mathbb{Z}_p)$ and $H_*(S^{\mathfrak{g}}, \mathbb{Z}_p)$. As these are both p -complete d -spheres, we have $H_*(I_{\mathcal{G}}, \mathbb{Z}_p) \cong \mathbb{Z}_p \cong H_*(S^{\mathfrak{g}}, \mathbb{Z}_p)$.

Proposition 5.9 *There are canonical isomorphisms of $\mathcal{G}$ -modules*

$$H_d(I_{\mathcal{G}}, \mathbb{Z}_p) \cong \Lambda^d \mathfrak{g} \cong H_d(S^{\mathfrak{g}}, \mathbb{Z}_p),$$

where $\Lambda^d \mathfrak{g}$ is the top exterior power of the adjoint representation.

Proof For $I_{\mathcal{G}}$ we have

$$H_d(I_{\mathcal{G}}, \mathbb{Z}_p) \cong \lim_j H_d(I_{\mathcal{G}}, \mathbb{Z}/p^j) \cong \lim_j \operatorname{colim}_{\operatorname{tr}} H_d(\Gamma_i, \mathbb{Z}/p^j).$$

Now apply Remark 4.41.

For $S^{\mathfrak{g}}$ we argue slightly differently. By Lemma 5.8 we have

$$H_d(S^{\mathfrak{g}}, \mathbb{Z}_p) \cong H_d(B\mathfrak{g}, \mathbb{Z}_p) \cong \Lambda^d \mathfrak{g},$$

as $H_1(B\mathfrak{g}, \mathbb{Z}_p) \cong \mathfrak{g}$. □

Proposition 5.9 suggests the following conjecture.

Linearization hypothesis

There is a $\mathcal{G}$ -equivariant equivalence $S^{\mathfrak{g}} \simeq I_{\mathcal{G}}$.

This conjecture appeared in the work of Clausen [20, §6.4]. Furthermore, [21] contains an outline of an argument for showing this conjecture in full generality, including a discussion of what category is a natural home for this equivalence.

If we restrict to a finite subgroup $F \subseteq \mathcal{G}$, we can regard $I_{\mathcal{G}}$ and $S^{\mathfrak{g}}$ as objects in some standard category of spectra with an F -action, such as the functor category from the classifying space BF to spectra, or the localization of (any model of) genuine F -spectra at the underlying equivalences. Then we have the following conjectural statement we can attack with neoclassical techniques, as we do in the next few sections.

Finite linearization hypothesis

For every finite subgroup $F \subseteq \mathcal{G}$ there is an F -equivariant equivalence $S^{\mathfrak{g}} \simeq I_{\mathcal{G}}$.

6 Restricting to finite subgroups

In this section we reduce the problem of producing an F -equivariant equivalence $I_{\mathcal{G}} \rightarrow S^{\mathfrak{g}}$ for a finite subgroup of $\mathcal{G}$ to a calculation of modules over the Steenrod algebra. The main result is Theorem 6.25.

Notation 6.1 Both p -complete and non- p -complete spectra will appear in this section, so we will be suitably decorating the p -complete objects. For spaces X and Y , we will write $\mathbf{map}(X, Y)$ for the unpointed mapping space, and $\mathbf{map}_*(X, Y)$ for the pointed mapping space (if X, Y are based). The mapping space of spectra will be denoted by $\mathbf{map}(-, -)$. For spaces we will write

$$[X, Y] = \pi_0 \mathbf{map}_*(X, Y)$$

for *based* homotopy classes of maps.

Remark 6.2 Recall that if X and Y are pointed spaces then evaluation at the basepoint of Y gives a fiber sequence

$$\mathbf{map}_*(X, Y) \longrightarrow \mathbf{map}(X, Y) \longrightarrow Y$$

with a section given by the constant maps. When Y is a loop space, this gives a splitting $\mathbf{map}(X, Y) \simeq \mathbf{map}_*(X, Y) \times Y$. Therefore, if Y is connected, then

$$[X, Y] = \pi_0 \mathbf{map}_*(X, Y) \cong \pi_0 \mathbf{map}(X, Y).$$

6.1 F -spheres

Let F be a finite group. An F -sphere is an F -spectrum X so that the underlying spectrum is equivalent to a p -complete sphere S_p^k , $k \in \mathbb{Z}$. We call k the virtual dimension of X . An F -equivariant map $X \rightarrow Y$ of two F -spheres is an F -equivalence if the underlying map of spectra is an equivalence. Let $\mathbf{Sph}_F$ be the set of F -equivariant equivalence classes of F -spheres. This a group under smash product in the p -complete stable category.

We have the following classical result.

Lemma 6.3 *Let $\mathrm{Gl}_1(S_p^0)$ be the space of self homotopy equivalence of the p -complete sphere spectrum S_p^0 . There is an isomorphism of abelian groups*

$$\mathbf{Sph}_F \cong \pi_0 \mathbf{map}(BF, \mathbb{Z} \times B\mathrm{Gl}_1(S_p^0))$$

so that an F -sphere corresponds to an unbased map $BF \rightarrow \mathbb{Z} \times B\mathrm{Gl}_1(S_p^0)$, which in turn classifies a stable, fiberwise p -complete spherical fibration over

BF . We also have

$$\begin{aligned}\mathbf{Sph}_F &\cong [BF_+, \mathbb{Z} \times B\mathrm{Gl}_1(S_p^0)] \\ &\cong \mathbb{Z} \times [BF, B\mathrm{Gl}_1(S_p^0)].\end{aligned}$$

Proof The first isomorphism goes back to Theorem 1.1 of [23] and has been extensively explored and generalized in the follow-up literature. See, for example, Theorem 1.2 of [24].¹

For the isomorphisms of the second equation use Remark 6.2 and the fact that $\mathbb{Z} \times B\mathrm{Gl}_1(S_p^0)$ is an infinite loop space. $\square$

Remark 6.4 The isomorphism of Lemma 6.3 allows for very explicit constructions. For example, if $X \in \mathbf{Sph}_F$, then the Thom spectrum of the associated spherical fibration is equivalent to the homotopy orbits $EF_+ \wedge_F X$.

As just noted, $\mathbb{Z} \times B\mathrm{Gl}_1(S_p^0)$ is an infinite loop space; indeed, it is weakly equivalent to the Picard space $\mathrm{Pic}(S_p^0)$ of invertible p -complete spectra. Thus, there is a spectrum $\mathrm{pic}(S_p^0)$ and an equivalence $\mathbb{Z} \times B\mathrm{Gl}_1(S_p^0) \simeq \Omega^\infty \mathrm{pic}(S_p^0)$, yielding an isomorphism

$$\mathbf{Sph}_F \cong [\Sigma_+^\infty BF, \mathrm{pic}(S_p^0)].$$

Compare Remark 12.1.

The spectrum $\mathrm{pic}(S_p^0)$ is not p -complete, but is nearly so, and it will be useful to have some control over the difference. Note that

$$\pi_0 \mathrm{Gl}_1(S_p^0) \cong \mathbb{Z}_p^\times \cong \begin{cases} \{1 + p\mathbb{Z}_p\} \times C_{p-1} & \text{if } p > 2; \\ \{1 + 4\mathbb{Z}_2\} \times \{\pm 1\} & \text{if } p = 2. \end{cases}$$

where C_{p-1} is a cyclic group of order $p-1$. There are isomorphisms $1 + p\mathbb{Z}_p \cong \mathbb{Z}_p$ and $1 + 4\mathbb{Z}_2 \cong \mathbb{Z}_2$ using a logarithm.

Lemma 6.5 *If $p = 2$, then $B\mathrm{Gl}_1(S_2^0)$ is 2-complete. If $p > 2$ the canonical map*

$$B\mathrm{Gl}_1(S_p^0) \longrightarrow BC_{p-1}$$

has p -complete homotopy fiber.

¹ In later developments, an action of F on X is *defined* to be a map from BF to the classifying space of self-equivalences of X . The connection to our definition can be found in Proposition 4.2.4.4 of [51]; see also Remark 1.2.6.2 of the same source.

Proof By definition $\mathrm{Gl}_1(X)$ is a set of components in the mapping space $\mathrm{map}(X, X)$ and, hence, for any basepoint and any $k \geq 1$ we have

$$\pi_k \mathrm{Gl}_1(X) \cong \pi_k \mathrm{map}(X, X) \cong [\Sigma^k X, X].$$

From this we can conclude that for $k \geq 1$ the map

$$\pi_k \mathrm{Gl}_1(S^0) \rightarrow \pi_k \mathrm{Gl}_1(S_p^0)$$

is p -completion. The result follows. $\square$

If X is an F -sphere of virtual dimension k , then F acts on $H_k(X, \mathbb{Z}) \cong \mathbb{Z}_p$. We write

$$\phi_X : F \longrightarrow \mathbb{Z}_p^\times$$

for the resulting character. The map

$$f_X : BF \longrightarrow \{k\} \times B\mathrm{Gl}_1(S_p^0) \subseteq \mathbb{Z} \times B\mathrm{Gl}_1(S_p^0)$$

induces ϕ_X on the fundamental group. There is also an action on $H_k(X, \mathbb{F}_p) \cong \mathbb{F}_p$. If $p = 2$, this action is necessarily trivial, and if $p > 2$ we write

$$\psi_X : F \longrightarrow \mathbb{F}_p^\times = C_{p-1}$$

for this character. Note that ψ_X is obtained from ϕ_X by composing with the quotient map $\mathbb{Z}_p^\times \rightarrow \mathbb{F}_p^\times$.

Lemma 6.6 *Let F be a finite group and $F_0 \subseteq F$ a p -Sylow subgroup. Then two F -spheres X and Y are F -equivalent if and only if*

- (1) *X and Y are equivalent as F_0 -spheres and,*
- (2) *if $p > 2$, $\psi_X = \psi_Y : F \rightarrow C_{p-1}$.*

Proof One implication is clear: if X is equivalent to Y as an F -sphere, then the two listed conditions must hold. We work on the other implication.

Since X and Y are equivalent as F_0 spheres, they have the same virtual dimension. Because $\mathbb{Z} \times B\mathrm{Gl}_1(S_p^0)$ is an infinite loop space, we can form the difference map

$$g := f_X - f_Y : BF \longrightarrow B\mathrm{Gl}_1(S_p^0) = \{0\} \times B\mathrm{Gl}_1(S_p^0) \subseteq \mathbb{Z} \times B\mathrm{Gl}_1(S_p^0)$$

and we need only show this map is null-homotopic. As in Lemma 6.5, let $C = C_{p-1}$ if $p > 2$; let C be trivial if $p = 2$. Let A_0 be the homotopy fiber of

the map $B\mathrm{Gl}_1(S_p^0) \rightarrow BC$. Then (2) implies that there is a (unique) factoring of g as in the diagram

$$\begin{array}{ccccc} & & BF & & \\ & \swarrow g_1 & \downarrow g & & \\ A_0 & \longrightarrow & B\mathrm{Gl}_1(S_p^0) & \longrightarrow & BC. \end{array}$$

We will show g_1 is null-homotopic. Since the order of C is prime to p , we have $[BF_0, A_0] \cong [BF_0, B\mathrm{Gl}_1(S_p^0)]$. Hence (1) implies that the composition

$$BF_0 \longrightarrow BF \xrightarrow{g_1} A_0$$

is null-homotopic.

Let Σgl_1 be the zero connected cover of $\mathrm{pic}(S_p^0)$; thus, $\Omega^\infty \Sigma gl_1 \simeq B\mathrm{Gl}_1(S_p^0)$. Let A be the fiber of the map $\Sigma gl_1 \rightarrow \Sigma HC$; then $\Omega^\infty A \simeq A_0$ and g_1 is adjoint to a map $h_1 : \Sigma_+^\infty BF \rightarrow A$. By Lemma 6.5, A_0 is p -complete; hence A is p -complete as well. Since F_0 is a p -Sylow subgroup of F and A is p -complete, the restriction map

$$[\Sigma_+^\infty BF, A] \longrightarrow [\Sigma_+^\infty BF_0, A]$$

is an injection; the transfer for the inclusion $F_0 \subseteq F$ gives a splitting. Since h_1 maps to zero under this map, we must have $h_1 = 0$, and hence g_1 is null as well. $\square$

Remark 6.7 Our main examples of F -spheres are the two spheres constructed in Sect. 5. When trying to compare I_G with S^g , the condition (2) of Lemma 6.6 holds automatically. See Proposition 5.9.

We now begin to specialize F . We let $\mathrm{Sl}_1(X)$ be the component of the identity in $\mathrm{Gl}_1(X)$.

Lemma 6.8 *Let F be a finite p -group. Then the map $\mathrm{Gl}_1(S^0) \rightarrow \mathrm{Gl}_1(S_p^0)$ induced from the completion $S^0 \rightarrow S_p^0$ induces a weak equivalence*

$$\mathbf{map}_*(BF, B\mathrm{Gl}_1(S^0)) \simeq \mathbf{map}_*(BF, B\mathrm{Gl}_1(S_p^0)).$$

Proof For $k \geq 1$ the map of homotopy groups based at the identity

$$\pi_k \mathrm{Gl}_1(S^0) \rightarrow \pi_k \mathrm{Gl}_1(S_p^0)$$

is p -completion. If we set $C_2 = \{\pm 1\}$, there is a short exact sequence

$$0 \rightarrow \pi_0 \mathrm{Gl}_1(S^0) \cong C_2 \rightarrow \pi_0 \mathrm{Gl}_1(S_0^p) \cong \mathbb{Z}_p^\times \rightarrow \mathbb{Z}_p^\times / C_2 \rightarrow 0.$$

We have a diagram of fiber sequences

$$\begin{array}{ccccc} B\mathrm{Sl}_1(S^0) & \longrightarrow & B\mathrm{Gl}_1(S^0) & \longrightarrow & BC_2 \\ i \downarrow & & j \downarrow & (\star) & \downarrow \\ B\mathrm{Sl}_1(S_p^0) & \longrightarrow & B\mathrm{Gl}_1(S_p^0) & \longrightarrow & B\mathbb{Z}_p^\times. \end{array}$$

This is a diagram of infinite loop spaces so we still have an analogous diagram of fiber sequences after p -completion. The space $B\mathrm{Sl}_1(S_p^0)$ is already p complete so the map i defines an equivalence

$$B\mathrm{Sl}_1(S^0)_p \simeq B\mathrm{Sl}_1(S_p^0).$$

Hence, upon p completion the square $(\star)$ becomes the homotopy pullback square

$$\begin{array}{ccc} B\mathrm{Gl}_1(S^0)_p & \longrightarrow & (BC_2)_p \\ j \downarrow & (\star\star) & \downarrow \\ B\mathrm{Gl}_1(S_p^0)_p & \longrightarrow & (B\mathbb{Z}_p^\times)_p. \end{array}$$

At odd primes, $(BC_2)_p \simeq *$ and $B(\mathbb{Z}_p^\times)_p \simeq B\mathbb{Z}_p$, so $(\star\star)$ reduces to a fiber sequence

$$B\mathrm{Gl}_1(S^0)_p \longrightarrow B\mathrm{Gl}_1(S_p^0)_p \longrightarrow B\mathbb{Z}_p. \quad (6.9)$$

When $p = 2$, the spaces BC_2 and $B\mathbb{Z}_2^\times$ are already 2-complete. Since $(\star\star)$ is a pullback square, we get a fiber sequence (6.9) at the prime 2 as well.

At all primes, we have $\mathbf{map}_*(BF, B\mathbb{Z}_p) \simeq \mathrm{holim} \mathbf{map}_*(BF, B\mathbb{Z}/p^k)$. Hence, for $i \geq 0$,

$$\pi_i \mathbf{map}_*(BF, B\mathbb{Z}_p) \cong \lim_k \tilde{H}^{1-i}(BF, \mathbb{Z}/p^k) = 0,$$

so $\mathbf{map}_*(BF, B\mathbb{Z}_p)$ is contractible.

We now take pointed maps from BF to (6.9) to get an equivalence

$$\mathbf{map}_*(BF, B\mathrm{Gl}_1(S^0)_p) \rightarrow \mathbf{map}_*(BF, B\mathrm{Gl}_1(S_p^0)_p).$$

Finally, to get the claim, notice that since BF has no $\mathbb{F}_\ell$ homology for $\ell \neq p$, and no rational homology, we also have

$$\mathbf{map}_*(BF, X) \simeq \mathbf{map}_*(BF, X_p)$$

for any space X . $\square$

Remark 6.10 As an addendum to Lemma 6.8 we examine the role of the fundamental group of $B\mathrm{Gl}_1(S^0)$. There is a split fiber sequence

$$BS\mathrm{I}_1(S^0) \longrightarrow B\mathrm{Gl}_1(S^0) \longrightarrow BC_2.$$

Since $B\mathrm{Gl}_1(S^0)$ is an infinite loop space, we obtain a weak equivalence of spaces

$$BS\mathrm{I}_1(S^0) \times BC_2 \simeq B\mathrm{Gl}_1(S^0).$$

Hence, for any pointed connected space X , we have an equivalence

$$\mathbf{map}_*(X, BS\mathrm{I}_1(S^0)) \times \mathrm{Hom}(\pi_1 X, C_2) \simeq \mathbf{map}_*(X, B\mathrm{Gl}_1(S^0)). \quad (6.11)$$

Here we use that $\mathbf{map}_*(X, BC_2)$ has contractible components and

$$\pi_0 \mathbf{map}_*(X, BC_2) \cong \mathrm{Hom}(\pi_1 X, C_2).$$

If F is an elementary abelian p -group and $p > 2$ we then obtain an equivalence

$$\mathbf{map}_*(BF, BS\mathrm{I}_1(S^0)) \simeq \mathbf{map}_*(BF, B\mathrm{Gl}_1(S^0)).$$

If $p = 2$ then (6.11) becomes

$$\mathbf{map}_*(BF, BS\mathrm{I}_1(S^0)) \times \mathrm{Hom}(F, C_2) \simeq \mathbf{map}_*(BF, B\mathrm{Gl}_1(S^0)).$$

There is a similar decomposition for maps into BO and, in fact, the map $BO \rightarrow B\mathrm{Gl}_1(S^0)$ induces a commutative diagram

$$\begin{array}{ccc} \mathbf{map}_*(BF, BS\mathrm{O}) \times \mathrm{Hom}(F, C_2) & \xrightarrow{\simeq} & \mathbf{map}_*(BF, BO) \\ \downarrow & & \downarrow \\ \mathbf{map}_*(BF, BS\mathrm{I}_1(S^0)) \times \mathrm{Hom}(F, C_2) & \xrightarrow{\simeq} & \mathbf{map}_*(BF, B\mathrm{Gl}_1(S^0)). \end{array}$$

In particular, we get that for a p -group F ,

$$\mathbf{Sph}_F \cong \mathbb{Z} \times \mathrm{Hom}(F, C_2) \times [BF, BS\mathrm{I}_1(S^0)]. \quad (6.12)$$

Thus we need to concentrate on computing $[BF, B\mathrm{Sl}_1(S^0)]$, which is isomorphic to the unpointed homotopy classes of maps $\pi_0 \mathbf{map}(BF, B\mathrm{Sl}_1(S^0))$. See Remark 6.2.

6.2 Interlude on characteristic classes and Steenrod operations

The comparison of F -spheres and vector bundles on BF will play a crucial role below, when we specialize F even further to an elementary abelian p -group and reduce the calculation of F -spheres to a cohomology calculation. We will use the theory developed by Lannes [48], giving us that it is sufficient to understand $H^*(B\mathrm{Sl}_1(S^0))/J$ where J is the ideal of nilpotent elements. As a prelude we pull together what we need from the literature on characteristic classes.

Let ξ be a stable real vector bundle of virtual dimension n over a space X and let $M\xi$ be the Thom spectrum. If $p > 2$, we assume ξ is oriented. Let $U \in H^n(M\xi, \mathbb{F}_p)$ be the Thom class; then the Thom isomorphism is given by the cup product

$$U \smile (-) = U \cdot (-) : H^k(X, \mathbb{F}_p) \rightarrow H^{k+n}(M\xi, \mathbb{F}_p).$$

If $p = 2$ one can define the Steifel-Whitney classes $w_i(\xi)$ using Steenrod operations

$$U \cdot w_i(\xi) = \mathrm{Sq}^i U. \quad (6.13)$$

This is the approach in [57, §4]. Then we have $H^*(BO, \mathbb{F}_2) \cong \mathbb{F}_2[w_1, w_2, \dots]$ where $w_i = w_i(\gamma)$, with γ the universal stable bundle over BO .

If $p > 2$, the relationship between Pontrjagin classes and Steenrod operations is more complicated. Recall that if ξ is a real vector bundle over X , then the Pontrjagin classes are defined in terms of Chern classes:

$$p_i(\xi) = (-1)^i c_{2i}(\xi \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(X, \mathbb{Z}_{(p)}).$$

Then $H^*(BSO, \mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}[p_1, p_2, \dots]$, where p_i is the Pontrjagin class of the universal bundle over BSO .

If γ_1 over $BU(1) = BSO(2) = \mathbb{CP}^\infty$ is the universal complex line bundle, then $H^*(BU(1), \mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}[c]$, where $c = c_1(\gamma_1)$. We have $p_1(\gamma_1) = c^2$, and we write

$$t = c^2 \in H^4(BSO(2), \mathbb{Z}_{(p)}).$$

We then get a map $BSO(2)^n \rightarrow BSO$ classifying $\gamma_1^{\times n}$, which on cohomology factors as

$$H^*(BSO, \mathbb{Z}_{(p)}) \longrightarrow \mathbb{Z}_{(p)}[t_1, \dots, t_n]^{\Sigma_n} \xrightarrow{\subseteq} H^*(BSO(2)^n, \mathbb{Z}_{(p)}).$$

The first of these maps is an isomorphism in degrees $* \leq 4n$; this initiates the classical analysis of Pontrjagin classes using symmetric polynomials. By the Cartan formula for Pontrjagin classes, each p_i maps to the i th elementary symmetric polynomial in the t_j .

If ξ is an oriented real n -bundle over a space X , we define a cohomology class

$$q_n(\xi) \in H^{2n(p-1)}(X, \mathbb{F}_p)$$

by

$$U \cdot q_n(\xi) = P^n U \quad (6.14)$$

where P^n is the n th odd-primary Steenrod operation. By considering the universal case over $X = BSO$, we see that $q_n(\xi)$ must be a polynomial in the Pontrjagin classes $p_i(\xi)$. More specifically, we have the following result.

Proposition 6.15 *Let $p = 2r + 1$, so $4r = 2(p - 1)$. Then there is a congruence*

$$q_n(\xi) \equiv (-1)^{n(r+1)} r p_{rn}(\xi) \quad \text{modulo} \quad I_{rn} \quad (6.16)$$

where I_{rn} is the ideal generated by the Pontrjagin classes $p_i(\xi)$, $i < rn$.

Proof This is originally due to Wu [67]. However the result can be deduced by putting together various ideas from Milnor-Stasheff [57]; we now go through this exercise. As usual, it is sufficient to do the universal example.

First, we assert

$$1 + q_1 + q_2 + \dots + q_n + \dots = (1 + t_1^r) \cdots (1 + t_{rn}^r) \quad (6.17)$$

in $\mathbb{F}_p[t_1, \dots, t_{rn}]^{\Sigma_{rn}} = \mathbb{F}_p[p_1, \dots, p_{rn}]$. This is in the proof of [57, Theorem 19.7], explicitly credited there to Wu. It can also be seen by observing that since the Steenrod operations have a Cartan formula, we have that

$$q_i(\xi \times \zeta) = \sum_{j+k=i} q_j(\xi) \times q_k(\zeta).$$

Then to prove (6.17) we need only note that $q_1(\gamma_1) = t^r \in H^{4r}(BSO(2), \mathbb{F}_p)$ and that $q_i(\gamma_1) = 0$ for $i > 1$.

Note that the right hand side of (6.17) is the reduction of an obvious integral polynomial and we define integral lifts $\bar{q}_i$ of the classes q_i by the formula

$$1 + \bar{q}_1 + \bar{q}_2 + \cdots + \bar{q}_n + \cdots = (1 + t_1^r) \cdots (1 + t_{rn}^r) \in \mathbb{Z}_{(p)}[t_1, \dots, t_{rn}]^{\Sigma_{rn}}.$$

We may equally regard these as elements in $R := \mathbb{Q}_{(p)}[t_1, \dots, t_{rn}]^{\Sigma_{rn}}$.

To finish the argument, we use a variant of Girard's formula, as in [57, Problem 16A]. Apply the logarithm to the formula

$$1 + p_1 + p_2 + \cdots + p_n = (1 + t_1) \cdots (1 + t_n)$$

to get that $p_k \equiv (-1)^{k+1} \sum t_i^k / k$ modulo decomposables in the p_i in R . If we then apply the logarithm to the formula

$$1 + \bar{q}_1 + \bar{q}_2 + \cdots + \bar{q}_n + \cdots = (1 + t_1^r) \cdots (1 + t_{rn}^r)$$

we get $\bar{q}_k \equiv (-1)^{k+1} \sum_{i=1}^{rn} t_i^{rk} / k$ modulo decomposables in the $\bar{q}_i$, so also modulo decomposables in the p_i . This then gives

$$(-1)^{k(r+1)} r p_{rk} \equiv (-1)^{k(r+1)} (-1)^{rk+1} r \sum_{l=1}^{rn} t_l^{rk} / rk \equiv \bar{q}_k.$$

This is an integral formula, so we can reduce modulo p to get (6.16) □

We now give a construction to realize the classes q_i as generators for the cohomology of a space. This is a variation on the discussion of [1, Lecture 4]. After we complete at an odd prime p , we have stable Adams operations

$$\psi^k: BU_p \longrightarrow BU_p, \quad k \in \mathbb{Z}_p^\times.$$

The Adams summand of BU_p can be realized as the homotopy fixed points $BU_p^{hC_{p-1}}$ of the cyclic group $C_{p-1} = \mathbb{F}_p^\times \subseteq \mathbb{Z}_p^\times$. There is an equivalence

$$BU_p^{hC_{p-1}} \simeq BX(p-1)$$

where $BX(p-1)$ is the colimit of classifying spaces of certain p -compact groups. This space is discussed in detail by Castellana [19].

Proposition 6.18 *We have the following calculations in cohomology.*

(1) The map $BX(p-1) \simeq BU_p^{hC_{p-1}} \rightarrow BU_p$ induces an isomorphism

$$H^*(BU, \mathbb{F}_p)/I \cong H^*BX(p-1) \cong \mathbb{F}_p[c_{(p-1)}, c_{2(p-1)}, \dots]$$

where $I \subseteq H^*(BU)$ is the ideal generated by the Chern classes c_n with $n \not\equiv 0$ modulo $(p-1)$.

(2) Let $f : BSO \rightarrow BU$ classify the complexification of the universal bundle. Then

$$f^* : H^*(BU, \mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}[c_1, c_2, \dots] \rightarrow H^*BSO \cong \mathbb{Z}_{(p)}[p_1, p_2, \dots]$$

sends c_{2n+1} to zero and c_{2n} to $(-1)^n p_n$. Furthermore f induces an equivalence

$$BSO_p \simeq BU_p^{hC_2}.$$

(3) There is an equivalence $BX(p-1) \simeq BSO_p^{hC_{(p-1)/2}}$ and an isomorphism of unstable algebras over the Steenrod algebra

$$\mathbb{F}_p[q_1, q_2, \dots] \cong H^*BX(p-1)$$

where $q_i \in H^{2i(p-1)}BSO$ is the class of (6.14) defined using Steenrod operations.

Proof If X is a connected infinite loop space and G is a finite group of order prime to p acting on X , then the map $X^{hG} \rightarrow X$ induces an isomorphism

$$H^*(X, \mathbb{F}_p)/I(G) \cong H^*(X^{hG}, \mathbb{F}_p) \quad (6.19)$$

where $I(G)$ is the ideal generated by elements of the form $g_*x - x$ with $x \in H^*(X, \mathbb{F}_p)$. If found nowhere else, this can be deduced using the techniques of [34].²

If L is a line bundle, then $\psi^k(L) = L^{\otimes k}$. Hence, $\psi_*^k c_1(L) = k c_1(L)$. The splitting principle then implies $\psi_*^k c_n = k^n c_n$. Thus if k is any integer which generates $\mathbb{F}_p^\times$, the ideal generated by $(\psi_*^k - 1)H^*BU$ in H^*BU is generated by c_n with $n \not\equiv 0$ modulo $(p-1)$. The first statement follows.

For $k = -1$, the ideal $(\psi_*^{-1} - 1)H^*BU$ in H^*BU is generated by the odd-dimensional Chern classes c_{2n+1} . Since $p_n = (-1)^n c_{2n}$, the second statement follows from (6.19).

The final statement is the iteration of homotopy fixed points and (6.16). $\square$

² What can be proved easily from [34] is that $D_*(H_*X^{hG}) \cong [D_*H_*X]^G$ where $D_*(-)$ is the graded Dieudonné module functor. The assertion here can be deduced from that.

6.3 Characteristic classes and $H^*BGL_1(S^0)$

The following result is crucial for us. At $p = 2$ it can be found in [55]; for $p > 2$ it can be found in [65]. Definitive references for both results are [22, Theorems II.5.1 and II.5.2].

Theorem 6.20 *Let $J \subseteq H^*(BGL_1(S^0), \mathbb{F}_p)$ be the ideal of nilpotent elements.*

(1) *If $p = 2$, the map $BSO \rightarrow BSL_1(S^0)$ induces an isomorphism*

$$H^*(BSL_1(S^0), \mathbb{F}_2)/J \cong H^*(BSO, \mathbb{F}_2).$$

(2) *If $p > 2$, the composition*

$$BX(p-1)_p = BSO_p^{hC_{(p-1)/2}} \rightarrow BSO_p \rightarrow BSL_1(S^0)_p$$

induces an isomorphism

$$H^*(BSL_1(S^0), \mathbb{F}_p)/J \longrightarrow H^*(BX(p-1), \mathbb{F}_p).$$

Below, we will use Theorem 6.20 as input along with the following result. Let $\mathcal{U}$ be the category of unstable modules over the Steenrod algebra, and $\mathcal{K}$ the category of unstable algebras over the Steenrod algebra.

Lemma 6.21 *Let $g: A^* \rightarrow B^*$ be a surjective homomorphism of connected unstable algebras of finite type over the Steenrod algebra and let I^* be the kernel of g . If I^* consists of nilpotent elements, then g induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{K}}(B^*, H^*BF) \cong \mathrm{Hom}_{\mathcal{K}}(A^*, H^*BF)$$

for all elementary abelian p -groups F .

Proof It is sufficient to show that $\mathrm{Hom}_{\mathcal{U}}(I^*, H^*BF) = 0$. At the prime $p = 2$ this is relatively easy to prove. Indeed, we have that the top Steenrod operation

$$\mathrm{Sq}_0 = \mathrm{Sq}^n = (-)^2: H^n BF \rightarrow H^{2n} BF$$

is an injection. On the other hand, for all $x \in I^*$ there is a k so that $\mathrm{Sq}_0^k(x) = x^{2^k} = 0$.

If $p > 2$ the argument takes more technology because H^*BF itself contains nilpotent elements. First we have the Carlsson-Miller theorem that H^*BF is an injective object in the category $\mathcal{U}$. See [18], [54], and [52, Appendix A]. Second, $\widetilde{\Sigma} H^*BF = 0$; see the proof of [52, Corollaire 7.2]. Combining these two facts with [52, Proposition 6.1.4] gives the result. $\square$

We now have the following remarkable facts; see [19, Corollary 4.5]. In sum, we conclude that if F is an elementary abelian p -group, then F -spheres are in one-to-one correspondence with stable vector bundles over BF and that any such vector bundle is completely determined by its Stiefel-Whitney classes or Pontraygin classes depending on the prime. See Remark 6.2.

Theorem 6.22 *Let $p = 2$ and let F be an elementary abelian 2-group. Then there is a commutative diagram with all maps isomorphisms of sets*

$$\begin{array}{ccc} [BF, BO] & \longrightarrow & \mathrm{Hom}_{\mathcal{K}}(H^*BO, H^*BF) \\ \downarrow & & \downarrow \\ [BF, B\mathrm{Gl}_1(S^0)] & \longrightarrow & \mathrm{Hom}_{\mathcal{K}}(H^*B\mathrm{Gl}_1(S^0), H^*BF). \end{array}$$

Theorem 6.23 *Let $p > 2$ and let F be an elementary abelian p -group. Then there is a commutative diagram with all maps isomorphisms of sets*

$$\begin{array}{ccc} [BF, BX(p-1)] & \longrightarrow & \mathrm{Hom}_{\mathcal{K}}(H^*BX(p-1), H^*BF) \\ \downarrow & & \downarrow \\ [BF, B\mathrm{Gl}_1(S^0)] & \longrightarrow & \mathrm{Hom}_{\mathcal{K}}(H^*B\mathrm{Gl}_1(S^0), H^*BF). \end{array}$$

Proof The same argument, with variations, works for both Theorem 6.22 and Theorem 6.23. We use [48, Théorème 0.4]. This result says that if Y is a simply connected space with $H^*Y = H^*(Y, \mathbb{F}_p)$ finite in each degree then the set of unbased homotopy classes of maps from BF to Y is in bijection with $\mathrm{Hom}_{\mathcal{K}}(H^*Y, H^*BF)$. So, using Remark 6.2, we have that for a simply connected infinite loop space Y , the natural map

$$[BF, Y] \longrightarrow \mathrm{Hom}_{\mathcal{K}}(H^*Y, H^*BF)$$

is a bijection. The spaces in Theorem 6.22 and Theorem 6.23 are not simply connected, so a little more care is required.

When p is odd, we use [48, Théorème 0.4] with $BS\mathrm{L}_1(S^0)$ in place of $B\mathrm{Gl}_1(S^0)$. Recall Remark 6.10 and note that the natural map $BS\mathrm{L}_1(S^0) \rightarrow B\mathrm{Gl}_1(S^0)$ induces an isomorphism in mod p cohomology, and as $BX(p-1)$ is p -complete, any map $BX(p-1) \rightarrow B\mathrm{Gl}_1(S^0)$ factors through $BS\mathrm{L}_1(S^0)$. We get a commutative diagram

$$\begin{array}{ccc} [BF, BX(p-1)] & \longrightarrow & \mathrm{Hom}_{\mathcal{K}}(H^*BX(p-1), H^*BF) \\ \downarrow & & \downarrow \\ [BF, BS\mathrm{L}_1(S^0)] & \longrightarrow & \mathrm{Hom}_{\mathcal{K}}(H^*BS\mathrm{L}_1(S^0), H^*BF), \end{array}$$

with all maps isomorphisms, due to Theorem 6.20 and Lemma 6.21, giving the claim.

When $p = 2$, we again appeal to [48, Théorème 0.4], with BSO in place of BO as well as $BSl_1(S^0)$ in place of $BGl_1(S^0)$. Using Theorem 6.20 and Lemma 6.21 we get a commutative diagram with all maps isomorphisms of sets

$$\begin{array}{ccc} [BF, BSO] & \longrightarrow & \mathrm{Hom}_{\mathcal{K}}(H^*BSO, H^*BF) \\ \downarrow & & \downarrow \\ [BF, BSl_1(S^0)] & \longrightarrow & \mathrm{Hom}_{\mathcal{K}}(H^*BSl_1(S^0), H^*BF). \end{array}$$

To complete the proof, again use Remark 6.10 and the observation that

$$\mathrm{Hom}(F, C_2) \cong \mathrm{Hom}_{\mathcal{K}}(H^*BC_2, H^*BF). \quad \square$$

Theorem 6.24 *Let F be an elementary abelian p -group and let X be a p -complete F -sphere of virtual dimension k . Then there is a stable vector bundle ξ over BF of virtual dimension k and a p -equivalence of spectra*

$$M\xi \simeq EF_+ \wedge_F X.$$

Furthermore there is an F -equivalence $X \simeq Y$ of p -complete F -spheres if and only if there is an isomorphism of modules over the Steenrod algebra

$$H^*(EF_+ \wedge_F X) \cong H^*(EF_+ \wedge_F Y).$$

Such an isomorphism uniquely determines the F -equivalence up to F -homotopy.

Proof We have phrased this as a result about modules over the Steenrod algebra, but there is more structure here that can be of use. Let ξ be a spherical fibration over a base space B ; we assume ξ is oriented if $p > 2$. Then the cohomology $H^*M\xi$ is a free module of rank 1 over H^*B on the Thom class U . Thus the Steenrod algebra structure on $H^*M\xi$ is completely determined by the Cartan formula and the action of the Steenrod operations on U . Note that if p is odd, the Bockstein vanishes on U , because U is the reduction of an integral class. This action is, in turn, completely determined by the Stiefel-Whitney classes, as in (6.13), or the modified Pontrjagin classes of (6.14) and (6.16).

Our result now follows by combining this observation with Lemma 6.8, Proposition 6.18, and Theorem 6.22 or Theorem 6.23, as needed. $\square$

The key result of this section is the following application of Theorem 6.24. Let $\mathcal{G}$ be a compact p -adic analytic group. For any group H let $Z(H)$ denote its center. Note that $Z(\mathcal{G})$ acts trivially on both $S^{\mathfrak{g}}$ and $I_{\mathcal{G}}$, since both actions are defined by conjugation.

Theorem 6.25 *Let $\mathcal{G}$ be a compact p -adic analytic group and let H be a closed subgroup of $\mathcal{G}$ such that $H/H \cap Z(\mathcal{G})$ is finite. Suppose the p -Sylow subgroup F of $H/H \cap Z(\mathcal{G})$ is an elementary abelian p -group. Then there is an H -equivalence $S^{\mathfrak{g}} \simeq I_{\mathcal{G}}$ if and only if there is an isomorphism of modules over the Steenrod algebra*

$$H^*(EF_+ \wedge_F S^{\mathfrak{g}}) \cong H^*(EF_+ \wedge_F I_{\mathcal{G}}). \quad (6.26)$$

Proof Suppose we are given the isomorphism of (6.26). Then Theorem 6.24 gives an F -equivalence $S^{\mathfrak{g}} \simeq I_{\mathcal{G}}$. We apply Lemma 6.6 and Remark 6.7 to get an equivalence of $H/H \cap Z(\mathcal{G})$ -spectra. This is automatically an H -equivalence.

For the converse, if $S^{\mathfrak{g}} \rightarrow I_{\mathcal{G}}$ is an H -equivalence then, since the center acts trivially, it is an $H/H \cap Z(\mathcal{G})$ -equivalence and, by restriction, an F -equivalence. Theorem 6.24 again applies, giving the claim. $\square$

Remark 6.27 In our main examples, the center of $\mathcal{G}$ is small. For example, if $\mathcal{G} = \mathbb{S}_n$ is the Morava stabilizer group and H is a finite subgroup, then $Z(\mathbb{S}_n) = \mathbb{Z}_p^{\times}$, so $H \cap Z(\mathbb{S}_n)$ is a cyclic group of order at most $p - 1$ if $p > 2$. In the more interesting case of $p = 2$, $H \cap Z(\mathbb{S}_n)$ is either trivial or $\{\pm 1\}$. Note that H can be a quaternionic group if $p = 2$. For more on the finite subgroups of $\mathbb{S}_n$, see [17, 38].

Similarly, for $\mathcal{G} = \mathrm{Gl}_n(\mathbb{Z}_p)$, the center is the scalar matrices: diagonal matrices with all entries equal. Thus the center is again $\mathbb{Z}_p^{\times}$. Of course, any finite group is isomorphic to a subgroup $\mathrm{Gl}_n(\mathbb{Z}_p)$ for some n .

7 Two algebraic preliminaries

In this section we collect two technical remarks about spectral sequences needed for the proof of the main theorem of the next section. This should perhaps not be read until after skimming the next section to see why on earth we need such things.

We begin with a first quadrant cohomology spectral sequences

$$E_2^{p,q} \Longrightarrow A^{p+q}$$

with differentials $d_r: E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$. The E_2 -term is bounded below in p and q ; that is, $E_2^{p,q} = 0$ for $p < 0$ and $q < 0$. This means, in particular, that

$E_r^{p,q} \cong E_\infty^{p,q}$ if $r > \max(p, q + 1)$. The motivating example is the homotopy orbit spectral sequence

$$H^p(F, H^q X) \Longrightarrow H^{p+q}(EF_+ \wedge_F X)$$

for some F -spectrum X with the property that $H^q X = 0$ for $q < 0$.

Here is the set-up. By the very nature of our project, we will have a diagram of such spectral sequences for positive integers j bigger than some fixed integer N

$$\begin{array}{ccc} E_2^{p,q}(j) & \Longrightarrow & A^{p+q}(j) \\ g \downarrow & & \downarrow g \\ E_2^{p,q}(j+1) & \Longrightarrow & A^{p+q}(j+1) \\ f \downarrow & & \downarrow f \\ E_2^{p,q} & \Longrightarrow & A^{p+q} \end{array} \quad (7.1)$$

We make this diagram specific in (8.4).

Let $K_r^{*,*}(j)$ be the kernel of $g: E_r^{*,*}(j) \rightarrow E_r^{*,*}(j+1)$.

Lemma 7.2 *Suppose the map induced by f*

$$E_2^{*,*}(j)/K_2^{*,*}(j) \longrightarrow E_2^{*,*}$$

is an isomorphism for $j \geq N$. Then for $j > N + m - 1$ the map

$$A^*(j)/\text{Ker}\{A^*(j) \rightarrow A^*(j+m+1)\} \rightarrow A^*$$

is an isomorphism in degrees $n \leq m$.

Proof We break this proof into a number of steps. In the first two steps we show, by induction, that the map $f: E_r^{*,*}(j) \rightarrow E_r^{*,*}$ induces an isomorphism for $j \geq N + r - 2$

$$E_r^{*,*}(j)/K_r^{*,*}(j) \longrightarrow E_r^{*,*}. \quad (7.3)$$

We have assumed the base case of $r = 2$. The third step will complete the argument.

Step 1. Here we show that

$$f: E_{r+1}^{*,*}(j) \longrightarrow E_{r+1}^{*,*}$$

is onto for $j \geq N + (r + 1) - 2 = N + r - 1$. It is sufficient to do the case where $j = N + r - 1$.

If we choose an r -cycle in $E_r^{*,*}$, then the induction hypothesis implies there is a class $y \in E_r^{*,*}(N + r - 2)$ with $f(y) = x$. We show that $g(y) \in E_r^{*,*}(N + r - 1)$ is an r -cycle; this will complete this step.

Since the induced map

$$f : E_r^{*,*}(N + r - 2)/K_r^{*,*}(N + r - 2) \rightarrow E_r^{*,*}$$

is injective and $d_r(f(y)) = d_r(x) = 0$, we have that

$$d_r(y) \in K_r^{*,*}(N + r - 2)$$

and hence that $d_r(g(y)) = 0$ as needed.

Step 2. We now show that

$$f : E_{r+1}^{*,*}(j)/K_{r+1}^{*,*}(j) \longrightarrow E_{r+1}^{*,*}$$

is injective for $j \geq N + r - 1$.

It is equivalent to show that if a class in $E_{r+1}^{*,*}(j)$ maps to zero in $E_{r+1}^{*,*}$, then it maps to zero in $E_{r+1}^{*,*}(j + 1)$. Suppose we have $y \in E_r^{*,*}(j)$ and

$$f(y) = d_r(w) \in E_r^{*,*}.$$

By the induction hypothesis we may choose a class $z \in E_r^{*,*}(j)$ so that $f(z) = w$. Then

$$f(y - d_r(z)) = f(y) - d_r f(z) = f(y) - d_r(w) = 0 \in E_r^{*,*}$$

so, again by the induction hypotheses $y - d_r(z) \in K_r^{*,*}(j)$. Hence $y - d_r(z)$ maps to zero in $E_r^{*,*}(j + 1)$, or

$$d_r(g(z)) = g(y)$$

as needed.

Step 3. Here we complete the argument with another inductive procedure. It is sufficient to show that for $j > N + m - 1$ the map

$$A^m(j)/\text{Ker}\{A^m(j) \rightarrow A^*(j + m + 1)\} \rightarrow A^m$$

is an isomorphism, for in degrees $n < m$, we have $j > N + m - 1 > N + n - 1$.

Note that in a particular bidegree (p, q) , all our spectral sequences satisfy $E_r^{p,q} = E_\infty^{p,q}$ for $r > \max\{p, q + 1\}$. By the first two steps we have that (7.3) is an isomorphism for $j \geq N + r - 2$, hence

$$f : E_\infty^{p,q}(j)/K_\infty^{p,q}(j) \longrightarrow E_\infty^{p,q} \quad (7.4)$$

is an isomorphism for

$$j > N + \max\{p, q + 1\} - 2.$$

Since we have first quadrant cohomological spectral sequences, we have a diagram of filtrations

$$\begin{array}{ccccccc} 0 = F^{-1}A^m(j) & \xrightarrow{\subseteq} & F^0A^m(j) & \xrightarrow{\subseteq} & \dots & \xrightarrow{\subseteq} & F^{m-1}A^m(j) \xrightarrow{\subseteq} F^mA^m(j) = A^m(j) \\ \downarrow & & \downarrow & & & & \downarrow \\ 0 = F^{-1}A^m & \xrightarrow{\subseteq} & F^0A^m & \xrightarrow{\subseteq} & \dots & \xrightarrow{\subseteq} & F^{m-1}A^m \xrightarrow{\subseteq} F^mA^m = A^m, \end{array}$$

and diagrams of short exact sequences

$$\begin{array}{ccccccc} 0 \longrightarrow & F^{q-1}A^m(j) & \longrightarrow & F^qA^m(j) & \longrightarrow & E_\infty^{m-q,q}(j) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & F^{q-1}A^m & \longrightarrow & F^qA^m & \longrightarrow & E_\infty^{m-q,q} & \longrightarrow 0. \end{array} \quad (7.5)$$

We will use induction on q to show

$$f^q : F^qA^m(j)/\ker\{F^qA^m(j) \rightarrow F^qA^m(j+q+1)\} \rightarrow F^qA^m \quad (7.6)$$

is an isomorphism for $j > N + m - 1$. The case $q = m$ is what we need to prove; the case $q = 0$ is the assertion that we have an isomorphism

$$E_\infty^{m,0}(j)/K_\infty^{m,0}(j) \xrightarrow{\cong} E_2^{m,0},$$

which has already been proved and in fact requires only that $j > N + m - 2$.

Now assume inductively that f^{q-1} is an isomorphism for $j > N + m - 1$. To complete the induction step we examine the exact sequence of (7.5). The inductive hypothesis gives that $F^{q-1}A^m(j) \rightarrow F^{q-1}A^m$ is onto for $j > N + m - 1$, hence by the snake lemma we have a short exact sequence

$$0 \rightarrow \kappa_1 \rightarrow \kappa_2 \rightarrow \kappa_3 \rightarrow 0,$$

where the κ_i are the kernels of the three vertical maps in (7.5). Note that since also the outer two vertical map in (7.5) are onto, the snake lemma implies that the middle map is onto as well.

It remains to identify $\kappa_2 = \ker\{F^q A^m(j) \rightarrow F^q A^m\}$ with

$$\kappa'_2 = \ker\{F^q A^m(j) \rightarrow F^q A^m(j+q+1)\}.$$

By construction, κ'_2 is contained in κ_2 ; thus, we just need to show the reverse containment. Let x be an element of $\kappa_2 \subseteq F^q A^m(j)$. Its image in $E_\infty^{m-q,q}$ is zero, and since $\kappa_3 \cong \ker\{E_\infty^{m-q,q}(j) \rightarrow E_\infty^{m-q,q}(j+1)\}$, its image in $E_\infty^{m-q,q}(j+1)$ is zero. Hence, from the exact sequence

$$0 \rightarrow F^{q-1} A^m(j+1) \rightarrow F^q A^m(j+1) \rightarrow E_\infty^{m-q,q}(j+1) \rightarrow 0,$$

it lifts to an element $y \in F^{q-1} A^m(j+1)$, with the property that y maps to zero in $F^{q-1} A^m$; that is, it is in $\ker\{F^{q-1} A^m(j+1) \rightarrow F^{q-1} A^m\}$. By the inductive hypothesis, this kernel is $\ker\{F^{q-1} A^m(j+1) \rightarrow F^{q-1} A^m(j+1+q)\}$. In particular, the image of y in $F^q A^m(j+q+1)$ is zero; but this is the same as the image of x in $F^q A^m(j+q+1)$; hence, x is in the kernel κ'_2 . $\square$

Our second algebraic result is about building isomorphisms from arrays of graded abelian groups.

Remark 7.7 Suppose we are given a commutative diagram of graded abelian groups

$$\begin{array}{ccccc} B_{i+1,j} & \longrightarrow & B_{i,j} & \longrightarrow & B_{i-1,j} \\ \downarrow & & \downarrow & & \downarrow \\ B_{i+1,j+1} & \longrightarrow & B_{i,j+1} & \longrightarrow & B_{i-1,j+1} \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & A_{i+1} & \longrightarrow & A_i & \longrightarrow & A_{i-1} \end{array}$$

This is an infinite array, with $i \geq 1$ and $j > i$, but we haven't put in the dots because it makes for a very cluttered diagram. In our main examples, we will have

$$\begin{aligned} A &= H^*(EF_+ \wedge_F I\mathcal{G}), \\ A_i &= H^*(EF_+ \wedge_F \Sigma_+^\infty B\Gamma_i), \\ B_{i,j} &= H^*(EF_+ \wedge_F \Sigma_+^\infty B(\Gamma_i / \Gamma_j)). \end{aligned}$$

For integers i, j, d , and n , let

$$\begin{aligned} I(i, d) &= \text{Ker}\{A_i \rightarrow A_{i-d}\}, \\ J(i, j, m) &= \text{Ker}\{B_{i,j} \rightarrow B_{i,j+m}\}, \end{aligned}$$

We make the following assumptions, which will have to be justified in our main examples. See Lemma 8.5 and Lemma 8.6. The stipulation that $i > d + 1$ arises there, and d will be the rank of $\mathcal{G}$.

- (1) For all $i > d + 1$, the composition $A \rightarrow A_i \rightarrow A_i/I(i, d)$ is an isomorphism of graded abelian groups; and,
- (2) there is an integer N , so that for all $i > d + 1$ and all $j > N + m - 1$ the induced map $B_{i,j}/J(i, j, m) \rightarrow A_i$ is an isomorphism in degrees $n \leq m$.

If we fix i and j , let J_m be the kernel of either of the two ways around the diagram

$$\begin{array}{ccc} B_{i,j} & \longrightarrow & B_{i-d,j} \\ \downarrow & & \downarrow \\ B_{i,j+m} & \longrightarrow & B_{i-d,j+m}. \end{array}$$

Note we could write $J(i, j, d, m)$ for J_m but m will be the crucial index. Then we can conclude

- (3) For all $i > d + 1$ and all $j > N + m - 1$ the map $B_{i,j}/J_m \longrightarrow A_i/I(i, d)$ is an isomorphism in degrees $n \leq m$.

The proof is a diagram chase. We then can conclude we have maps

$$A \xrightarrow{\cong} A_i/I(i, d) \longleftarrow B_{i,j}/J_m. \quad (7.8)$$

If $i > d + 1$ and $j > N + m - 1$ the first map is an isomorphism and the second map is an isomorphism in degrees $n \leq m$.

8 Equivalences of $\mathcal{G}$ -spheres for finite subgroups

The goal of this section is to show that for suitable closed subgroups H of our group $\mathcal{G}$ we have an equivalence of H -spectra $I_{\mathcal{G}} \simeq S^{\mathfrak{g}}$. This is accomplished in Theorem 8.11, as corollary of the main calculation of this section: we will show in Theorem 8.10 that for any finite subgroup $F \subseteq \mathcal{G}/Z(\mathcal{G})$, we have an isomorphism of modules over the Steenrod algebra

$$H^*(EF_+ \wedge_F I_{\mathcal{G}}) \simeq H^*(EF_+ \wedge_F S^{\mathfrak{g}}).$$

Then we appeal to Theorem 6.25 to prove Theorem 8.11.

The strategy is as follows. Fix a finite subgroup $F \subseteq \mathcal{G}/Z(\mathcal{G})$. Since $\mathcal{G}$ acts by conjugation on Γ_i and $\mathfrak{g}_i$, F acts on all of the spaces $B\Gamma_i$, $B(\Gamma_i/\Gamma_j)$, $B\mathfrak{g}_i$, and so on.

Recall we have compatible transfer maps $\mathrm{tr} : \Sigma_+^\infty B\Gamma_i \rightarrow \Sigma_+^\infty B\Gamma_{i+t}$, for $t > 0$ by (5.2), and hence maps

$$r : \Sigma_+^\infty B\Gamma_i \longrightarrow \mathrm{colim}_{\mathrm{tr}} \Sigma_+^\infty B\Gamma_{i+t} = I_{\mathcal{G}}.$$

We then have a diagram of F -spectra

$$\Sigma_+^\infty B(\Gamma_i/\Gamma_j) \xleftarrow{q} \Sigma_+^\infty B\Gamma_i \xrightarrow{r} I_{\mathcal{G}}.$$

There is a corresponding diagram for $S^{\mathfrak{g}}$

$$\Sigma_+^\infty B(\mathfrak{g}_i/\mathfrak{g}_j) \xleftarrow{q} \Sigma_+^\infty B\mathfrak{g}_i \xrightarrow{r} S^{\mathfrak{g}}.$$

Finally, there is a group isomorphism $\Gamma_i/\Gamma_j \cong \mathfrak{g}_i/\mathfrak{g}_j$ for $j \leq 2i$, as per (2.6). Using the techniques of Sect. 7, we can put together the comparison we need. The key intermediate results are Proposition 8.7 and Proposition 8.8.

In the final applications we will only need i large, so while more generality is possible, all our preliminary lemmas will set $i > d + 1$, where $d \geq 1$ is the rank of $\mathcal{G}$ and, hence, of Γ_i . This will avoid having to spell out special cases, especially when $p = 2$. As usual, all homology and cohomology in this section is with $\mathbb{F}_p$ -coefficients.

Notation 8.1 In this section we will be working heavily with homotopy orbits and it is convenient to shorten the notation. If Y is a G -spectrum for some finite group G we will often write

$$Y_{hG} = EG_+ \wedge_G Y.$$

Hence, if X is a G -space

$$[\Sigma_+^\infty X]_{hG} = EG_+ \wedge_G \Sigma_+^\infty X.$$

Notation 8.2 In what follows we will need to have names for the kernels of various restriction and transfer maps. We will recall the definitions as needed, but we collect them here to put the confusion in a place where it can be organized. Compare Remark 7.7 as well.

First, there are the kernels of maps before taking homotopy orbits; they are decorated with subscript 0:

$$I_0(i) = \ker\{\mathrm{tr}^* : H^* B\Gamma_i \longrightarrow H^* B\Gamma_{i-1}\},$$

$$J_0(i, j) = \ker\{\mathrm{res}^* : H^* B\Gamma_i / \Gamma_j \longrightarrow H^* B\Gamma_i / \Gamma_{j+1}\}.$$

Next, there are the kernels of maps after taking homotopy orbits; they do not have a subscript, but acquire a new index because we shift more than one step:

$$I(i, m) = \ker\{\mathrm{tr}^* : H^*[\Sigma_+^\infty B\Gamma_i]_{hF} \longrightarrow H^*[\Sigma_+^\infty B\Gamma_{i-m}]_{hF}\},$$

$$J(i, j, m) = \ker\{\mathrm{res}^* : H^*[\Sigma_+^\infty B\Gamma_i / \Gamma_j]_{hF} \longrightarrow H^*[\Sigma_+^\infty B\Gamma_i / \Gamma_{j+m}]_{hF}\}.$$

There is one more kernel, which combines restrictions and transfers. Fixing i and j , let J_m be the kernel of either of the two ways of composing around the commutative diagram

$$\begin{array}{ccc} H^*[\Sigma_+^\infty B\Gamma_i / \Gamma_j]_{hF} & \xrightarrow{\mathrm{tr}^*} & H^*[\Sigma_+^\infty B\Gamma_{i-d} / \Gamma_j]_{hF} \\ \downarrow \mathrm{res}^* & & \downarrow \mathrm{res}^* \\ H^*[\Sigma_+^\infty B\Gamma_i / \Gamma_{j+m}]_{hF} & \xrightarrow{\mathrm{tr}^*} & H^*[\Sigma_+^\infty B\Gamma_{i-d} / \Gamma_{j+m}]_{hF} . \end{array}$$

We could write $J(i, j, d, m)$ for J_m but m will be the crucial index. Finally, there are the kernels $K_r^{*,*}(j)$ in Lemma 7.2. These are distinct, but related, and will also be recalled as needed.

We begin with the following algebraic result.

Lemma 8.3 (1) *Let $I_0(i)$ be the kernel of the map $\mathrm{tr}^* : H^* B\Gamma_i \longrightarrow H^* B\Gamma_{i-1}$. For all $i > d + 1$ the composition*

$$H^* I_{\mathcal{G}} \xrightarrow{r^*} H^* B\Gamma_i \longrightarrow H^* B\Gamma_i / I_0(i)$$

is an isomorphism of modules over the Steenrod algebra.

(2) *Let $J_0(i, j)$ be the kernel of $\mathrm{res}^* : H^*(B\Gamma_i / \Gamma_j) \rightarrow H^*(B\Gamma_i / \Gamma_{j+1})$. Then for all $i > d + 1$ and $j \geq i + 1$ the map of modules over the Steenrod algebra*

$$H^*(B\Gamma_i / \Gamma_j) / J_0(i, j) \longrightarrow H^* B\Gamma_i$$

is an isomorphism.

Proof Part (1) follows from part (1) of Proposition 5.6 and part (2) is a consequence of part (2) of Theorem 3.6. $\square$

Here is the input for an application of Lemma 7.2 and Remark 7.7. The following diagram makes (7.1) concrete; all the spectral sequences are homotopy orbit spectral sequences.

$$\begin{array}{ccc}
 H^p(F, H^q B\Gamma_i / \Gamma_j) & \Longrightarrow & H^{p+q}[\Sigma_+^\infty B\Gamma_i / \Gamma_j]_{hF} \\
 \downarrow g & & \downarrow g \\
 H^p(F, H^q B\Gamma_i / \Gamma_{j+1}) & \Longrightarrow & H^{p+q}[\Sigma_+^\infty B\Gamma_i / \Gamma_{j+1}]_{hF} \\
 \downarrow f & & \downarrow f \\
 H^p(F, H^q B\Gamma_i) & \Longrightarrow & H^{p+q}[\Sigma_+^\infty B\Gamma_i]_{hF}
 \end{array} \quad (8.4)$$

We continue to write d for the rank of $\mathcal{G}$ and, hence, of Γ_i . See also Fig. 1 below.

Lemma 8.5 *Let $I(i, d)$ be the kernel of the map induced by the transfer*

$$H^*([\Sigma_+^\infty B\Gamma_i]_{hF}) \longrightarrow H^*([\Sigma_+^\infty B\Gamma_{i-d}]_{hF}).$$

For all $i > d + 1$ the composition

$$H^*([I_{\mathcal{G}}]_{hF}) \longrightarrow H^*([\Sigma_+^\infty B\Gamma_i]_{hF}) \longrightarrow H^*([\Sigma_+^\infty B\Gamma_i]_{hF})/I(i, d)$$

is an isomorphism of modules over the Steenrod algebra.

Proof By part (1) of Lemma 8.3 we have that the maps

$$H^*(I_{\mathcal{G}}) \longrightarrow \text{Im}\{\text{tr}^*: H^* B\Gamma_{i+1} \rightarrow H^* B\Gamma_i\} \longrightarrow H^* B\Gamma_i / I_0(i)$$

are isomorphisms. Furthermore $H^*(I_{\mathcal{G}}) \cong \mathbb{F}_p$ is concentrated in degree d , the map $H^d(I_{\mathcal{G}}) \rightarrow H^d B\Gamma_i$ is an isomorphism, and $H^n B\Gamma_i = 0$ if $n > d$ by Theorem 3.3.

The spectral sequence

$$E_2^{p,q}(I_{\mathcal{G}}) = H^p(F, H^q(I_{\mathcal{G}})) \Longrightarrow H^{p+q}([I_{\mathcal{G}}]_{hF})$$

has the property that $E_2^{p,q} = 0$ unless $q = d$. Thus it collapses at E_2 .

The map of spectral sequences

$$\begin{array}{ccc}
 E_2^{p,q}(I_{\mathcal{G}}) = H^p(F, H^q(I_{\mathcal{G}})) & \Longrightarrow & H^{p+q}([I_{\mathcal{G}}]_{hF}) \\
 \downarrow & & \downarrow \\
 E_2^{p,q}(\Sigma_+^\infty B\Gamma_i) = H^p(F, H^q \Sigma_+^\infty B\Gamma_i) & \Longrightarrow & H^{p+q}([\Sigma_+^\infty B\Gamma_i]_{hF})
 \end{array}$$

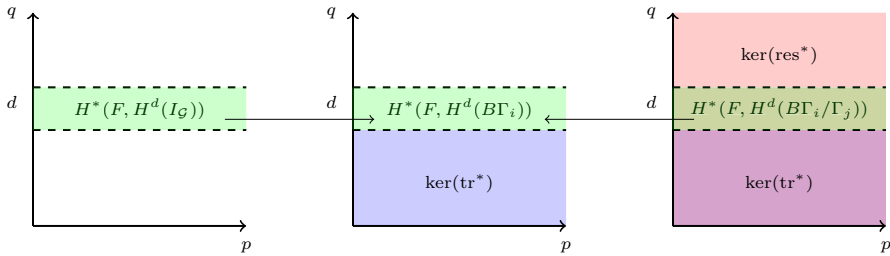


Fig. 1 Comparison of homotopy orbit spectral sequences

induces isomorphisms

$$E_{\infty}^{p,d}(I_G) \cong E_{\infty}^{p,d}(\Sigma_+^{\infty} B\Gamma_i)$$

as $E_2^{p,q}(\Sigma_+^{\infty} B\Gamma_i) = 0$ for $q > d$. In addition if $x \in H^*([\Sigma_+^{\infty} B\Gamma_i]_{hF})$ is detected by $a \in E_{\infty}^{p,q}(\Sigma_+^{\infty} B\Gamma_i)$ with $q < d$, then

$$\text{tr}^*(a) = 0 \in E_{\infty}^{p,q}(\Sigma_+^{\infty} B\Gamma_{i-1});$$

hence,

$$\text{tr}^*(x) \in H^*([\Sigma_+^{\infty} B\Gamma_{i-1}]_{hF})$$

is detected by a class in $E_{\infty}^{p+s,q-s}(B\Gamma_{i-1})$ with $s > 0$. Thus if we apply the transfer d times the class x will be sent to zero, as needed. $\square$

Lemma 8.6 *Let $J(i, j, m)$ be the kernel of the restriction map*

$$H^*([\Sigma_+^{\infty} B\Gamma_i / \Gamma_j]_{hF}) \rightarrow H^*([\Sigma_+^{\infty} B\Gamma_i / \Gamma_{j+m}]_{hF}).$$

Let $i > d + 1$ and let $j > i + m$. Then the homomorphism of modules over the Steenrod algebra

$$H^*([\Sigma_+^{\infty} B\Gamma_i / \Gamma_{i+j}]_{hF}) / J(i, j, m) \longrightarrow H^*([\Sigma_+^{\infty} B\Gamma_i]_{hF})$$

is an isomorphism in degrees $n \leq m$.

Proof This is a direct application of Lemma 7.2, with $N = i + 1$, and part (2) of Lemma 8.3.

To use Lemma 7.2 we must justify the hypotheses of that result. We let

$$E_2^{*,*}(j) = H^*(F, H^*(B\Gamma_i / \Gamma_j)) \implies H^*([\Sigma_+^{\infty} B\Gamma_i / \Gamma_j]_{hF}) = A^*(j),$$

$$E_2^{*,*} = H^*(F, H^*(B\Gamma_i)) \implies H^*([\Sigma_+^\infty B\Gamma_i]_{hF}) = A^*, \text{ and} \\ K_2^{*,*}(j) = \text{Ker}\{E_2^{*,*}(j) \longrightarrow E_2^{*,*}(j+1)\}.$$

We need to check that we have an isomorphism

$$E_2^{*,*}(j)/K_2^{*,*}(j) \longrightarrow E_2^{*,*}.$$

Suppose $j \geq i+1 = N$. From Lemma 8.3, we have exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & J_0(i, j) & \longrightarrow & H^*(B\Gamma_i/\Gamma_j) & \xrightarrow{\quad \quad} & H^*B\Gamma_i \longrightarrow 0 \\ & & \downarrow =0 & & \downarrow & & \downarrow \cong \\ 0 & \longrightarrow & J_0(i, j+1) & \longrightarrow & H^*(B\Gamma_i/\Gamma_{j+1}) & \xrightarrow{\quad \quad} & H^*B\Gamma_i \longrightarrow 0. \end{array}$$

By Theorem 3.3 and Theorem 3.6 we have algebra splittings, as indicated, and these splitting respect the F -action. Indeed, $H^*(B\Gamma_i) \cong \Lambda(V_i)$, where

$$V_i \cong H^1(B\Gamma_i) \cong H^1(B\Gamma_i/\Gamma_j).$$

The action of F preserves V_i for degree reasons and thus $\Lambda(V_i)$ since F acts by algebra homomorphisms. These short exact sequences would normally give long exact sequences in group cohomology, but the splittings reduce these to the following short exact sequences:

$$\begin{array}{ccccccc} 0 \rightarrow H^*(F, J_0(i, j)) & \longrightarrow & H^*(F, H^*(B\Gamma_i/\Gamma_j)) & \longrightarrow & H^*(F, H^*B\Gamma_i) & \rightarrow 0 \\ & & \downarrow & & \downarrow \cong & \\ 0 \rightarrow H^*(F, J_0(i, j+1)) & \longrightarrow & H^*(F, H^*(B\Gamma_i/\Gamma_{j+1})) & \longrightarrow & H^*(F, H^*B\Gamma_i) & \rightarrow 0 \end{array}$$

The left most vertical map is zero as the map

$$J_0(i, j) = \text{ker}\{\text{res}^* : H^*B\Gamma_i/\Gamma_j \longrightarrow H^*B\Gamma_i/\Gamma_{j+1}\} \longrightarrow H^*B\Gamma_i/\Gamma_{j+1}$$

is zero. An easy diagram chase implies that $K_2^{*,*}(j)$ is the kernel of

$$H^*(F, H^*(B\Gamma_i/\Gamma_j)) \rightarrow H^*(F, H^*B\Gamma_i).$$

The splitting above implies that this map is surjective. $\square$

Applying Lemma 8.5, Lemma 8.6, and the isomorphism of (7.8) we get the following result. Fix i and j and let J_m be the kernel of either of the two ways around the diagram

$$\begin{array}{ccc}
H^*[\Sigma_+^\infty B\Gamma_i/\Gamma_j]_{hF} & \xrightarrow{\text{tr}^*} & H^*[\Sigma_+^\infty B\Gamma_{i-d}/\Gamma_j]_{hF} \\
\downarrow \text{res}^* & & \downarrow \text{res}^* \\
H^*[\Sigma_+^\infty B\Gamma_i/\Gamma_{j+m}]_{hF} & \xrightarrow{\text{tr}^*} & H^*[\Sigma_+^\infty B\Gamma_{i-d}/\Gamma_{j+m}]_{hF}.
\end{array}$$

Proposition 8.7 *Let $i > d + 1$ and $j > i + m$. Let $I(i, d)$ be the kernel of the map induced by the transfer*

$$H^*([\Sigma_+^\infty B\Gamma_i]_{hF}) \longrightarrow H^*([\Sigma_+^\infty B\Gamma_{i-d}]_{hF}).$$

Then the maps

$$\Sigma_+^\infty B(\Gamma_i/\Gamma_{i+j}) \xleftarrow{q} \Sigma_+^\infty B(\Gamma_i) \xrightarrow{r} I_{\mathcal{G}}$$

define homomorphisms of modules over the Steenrod algebra

$$H^*([I_{\mathcal{G}}]_{hF}) \xrightarrow{\cong} H^*([\Sigma_+^\infty B\Gamma_i]_{hF})/I(i, d) \longleftarrow H^*([\Sigma_+^\infty B\Gamma_i/\Gamma_j]_{hF})/J_m.$$

The first map is an isomorphism and the second map is an isomorphism in cohomological degree n with $n \leq m$.

The same argument which proved Proposition 8.7 can be immediately adapted to prove the following.

Proposition 8.8 *Let $i > d + 1$ and $j > i + m$. Let $I^a(i, d)$ be the kernel of the map induced by the transfer*

$$H^*([\Sigma_+^\infty B\mathfrak{g}_i]_{hF}) \longrightarrow H^*([\Sigma_+^\infty B\mathfrak{g}_{i-d}]_{hF}).$$

Let J_m^a denote the kernel of the map

$$\text{res}^* \text{tr}^* = \text{tr}^* \text{res}^* : H^*([\Sigma_+^\infty B\mathfrak{g}_i/\mathfrak{g}_j]_{hF}) \rightarrow H^*([\Sigma_+^\infty B\mathfrak{g}_{i-d}/\mathfrak{g}_{j+m}]_{hF}).$$

Then the maps

$$\Sigma_+^\infty B(\mathfrak{g}_i/\mathfrak{g}_{i+j}) \xleftarrow{q} \Sigma_+^\infty B(\mathfrak{g}_i) \xrightarrow{r} S^{\mathfrak{g}}$$

define homomorphisms of modules over the Steenrod algebra

$$H^*([S^{\mathfrak{g}}]_{hF}) \xrightarrow{\cong} H^*([\Sigma_+^\infty B\mathfrak{g}_i]_{hF})/I^a(i, d) \longleftarrow H^*([\Sigma_+^\infty B\mathfrak{g}_i/\mathfrak{g}_j]_{hF})/J_m^a.$$

The first map is an isomorphism and the second map is an isomorphism in cohomological degree n with $n \leq m$.

We now come to our main calculation. Consider the following diagram of modules over the Steenrod algebra with $i < j \leq 2i$ and $j > i + m$.

$$\begin{array}{ccc}
 H^*([B\Gamma_i/\Gamma_j]_{hF})/J_m & \xrightarrow{\cong} & H^*([B\mathfrak{g}_i/\mathfrak{g}_j]_{hF})/J_m^a \\
 \downarrow & & \downarrow \\
 H^*([B\Gamma_i]_{hF})/I(i, j) & & H^*([B\mathfrak{g}_i]_{hF})/I^a(i, j) \\
 \uparrow \cong & & \uparrow \cong \\
 H^*([I_{\mathcal{G}}]_{hF}) & & H^*([S^{\mathfrak{g}}]_{hF})
 \end{array} \quad (8.9)$$

The horizontal isomorphism comes from the isomorphism of groups

$$\Gamma_i/\Gamma_j \cong \mathfrak{g}_i/\mathfrak{g}_j$$

discussed in (2.6). The upwards vertical maps are isomorphisms by Proposition 8.7 and Proposition 8.8. The same results show that the downward vertical maps are isomorphisms in degrees n with $n \leq m$.

Theorem 8.10 *Let $F \subseteq \mathcal{G}/Z(\mathcal{G})$ be a finite subgroup. Then the maps of (8.9) define an isomorphism of modules over the Steenrod algebra*

$$H^*(EF_+ \wedge_F I_{\mathcal{G}}) \cong H^*(EF_+ \wedge_F S^{\mathfrak{g}}).$$

Proof We look at (8.9). In any given range of degrees up to an integer m , we may choose i and j so that $i < j \leq 2i$ and $j > i + m$. $\square$

Combining Theorem 6.25 and Theorem 8.10 immediately implies our key result.

Theorem 8.11 *Let $\mathcal{G}$ be a compact p -adic analytic group and let H be a closed subgroup of $\mathcal{G}$ such that $H/H \cap Z(\mathcal{G})$ is finite. Suppose the p -Sylow subgroup of $H/H \cap Z(\mathcal{G})$ is an elementary abelian p -group. Then there is an H -equivariant equivalence*

$$I_{\mathcal{G}} \simeq S^{\mathfrak{g}}.$$

9 Analyzing the linear action

In this section we write down a general result that allows us to use linear algebra to analyze the equivariant homotopy type of $S^{\mathfrak{g}}$ for finite subgroups. The main

result is Proposition 9.8, but before getting there we need an intermediate result about the transfer for finite covering spaces of equivariant manifolds, namely Proposition 9.6. This can be proved by putting together ideas from standard sources such as [16] and [50], but it is easy enough and clearer to be completely explicit.

9.1 Equivariant geometric transfer

Fix a finite group F and let

$$q : M \longrightarrow N$$

be a F -equivariant differentiable finite-sheeted cover of a closed F -manifold N . To be clear:

- (1) M and N are closed C^∞ -manifolds of dimension d with a differentiable F -action;
- (2) q is an F -equivariant differentiable map and a finite covering map.

If M is a differentiable manifold, let TM denote the tangent bundle and $T_m M$ the fiber of TM at $m \in M$.

Remark 9.1 (Equivariant geometric transfers) Let $\mathcal{U} = \{U_i | i \in A\}$ be a finite open cover of M by open subsets $U_i \subseteq M$ with the following properties:

- (1) for all i the restriction of the covering map to $q : U_i \rightarrow N$ is an open embedding defining a diffeomorphism onto its image; and,
- (2) for all $g \in F$ and all $i \in A$, there is a $k \in A$ so the action by g on M restricts to a diffeomorphism $g : U_i \rightarrow U_k$.

Note that part (2) defines an action of F on A , and hence a permutation representation $W = \mathbb{R}^A$.

Next, let ϕ_i be a partition of unity subordinate to the cover $\mathcal{U}$ with the following equivariance property: if $g \in F$ and $g : U_i \rightarrow U_k$ is as in Part (2), then

$$\phi_k(gx) = \phi_i(x).$$

The existence of such partitions of unity can be found in §III.6 of [16]. The map

$$\begin{aligned} j : M &\longrightarrow W \times N \\ x &\mapsto ((\phi_i(x)), q(x)) \end{aligned}$$

is an F -equivariant differentiable embedding and we have a commutative diagram of F -maps

$$\begin{array}{ccc} M & \xrightarrow{j} & W \times N \\ & \searrow q & \downarrow p_2 \\ & & N. \end{array}$$

We have an isomorphism of the F -equivariant tangent bundles $q^*T_N \cong T_M$; this is a property of covering spaces. For any base F -space Y let θ_W be the trivial F -bundle of Y with total space $Y \times W$ and diagonal F -action. Then we have an isomorphism of F -bundles $T_{W \times N} \cong \theta_W \oplus p_2^*T_N$. Thus we can conclude that

$$j^*T_{W \times N} \cong \theta_W \oplus T_M.$$

It follows that the normal bundle of M in $W \times N$ is isomorphic to the trivial bundle θ_W over M . Choose an equivariant tubular neighborhood ν of M in $W \times N \subseteq S^W \times N$. (See Theorem IV.2.2 of [16].) Then one model for the transfer is the Thom collapse map

$$\begin{aligned} S^W \wedge N_+ &\cong (S^W \times N)/(\{\infty\} \times N) \\ &\longrightarrow (S^W \times N)/((S^W \times N) - \nu) \cong \bar{\nu}/\partial\nu = S^W \wedge M_+. \end{aligned} \quad (9.2)$$

This last isomorphism uses that the normal bundle is trivial, so there is an equivariant diffeomorphism

$$W \times M \cong \nu. \quad (9.3)$$

The collapse map of (9.2) is an unstable model for the transfer, so we write $\mathrm{tr}_W : S^W \wedge N_+ \rightarrow S^W \wedge M_+$ for this map. After further suspension we can cancel the representation sphere S^W and get the stable equivariant transfer map

$$\mathrm{tr} : \Sigma_+^\infty N \longrightarrow \Sigma_+^\infty M.$$

This is independent of the choices.

The next result is a special case of the equivariant tubular neighborhood theorem. See Theorem VI.2.2 of [16].

Lemma 9.4 *Let M be a smooth manifold with a differentiable action by a finite group F . Let $m \in M$ be a fixed point for the action. Then there exists an*

F -invariant open neighborhood U of m and an F -equivariant diffeomorphism $f : U \cong T_m M$ taking m to 0.

Furthermore, the closure $\overline{U}$ of U is homeomorphic to a closed ball and the map f extends to an F -equivariant collapse map

$$f_U : M \rightarrow M/(M - U) \cong \overline{U}/\partial\overline{U} \cong S^{T_m M}. \quad (9.5)$$

If $q : M \rightarrow N$ is our covering map, then $q(m) \in N$ is also an F -fixed point. We may suppose further that q maps U and $\overline{U}$ diffeomorphically onto the images V and $\overline{V}$ in N . Note that $q : M \rightarrow N$ defines an isomorphism of representations $T_m M \cong T_{q(m)} N$.

Proposition 9.6 *There is an F -equivariant commutative diagram of spectra*

$$\begin{array}{ccc} \Sigma_+^\infty N & \xrightarrow{\text{tr}} & \Sigma_+^\infty M \\ \Sigma_+^\infty \wedge f_V \downarrow & & \downarrow \Sigma_+^\infty \wedge f_U \\ S^{T_{q(m)} N} & \xrightarrow[\simeq]{} & S^{T_m M} \end{array}$$

where f_U and f_V are the equivariant collapse maps of (9.5).

Proof We claim there is a commutative diagram of F -spaces

$$\begin{array}{ccc} S^W \wedge N_+ & \xrightarrow{\text{tr}_W} & S^W \wedge M_+ \\ S^W \wedge f_V \downarrow & & \downarrow S^W \wedge f_U \\ S^{W+T_{q(m)} N} & \xrightarrow[\simeq]{} & S^{W+T_m M}. \end{array} \quad (9.7)$$

To see this, let $v_U \subseteq v$ be the image of $W \times U$ under the equivariant diffeomorphism $W \times M \cong v$ of (9.3). Then a model for the square (9.7) is

$$\begin{array}{ccc} \frac{S^W \times N}{\{\infty\} \times N} & \xrightarrow{\quad} & \frac{S^W \times N}{(S^W \times N) - v} \\ \downarrow & & \downarrow \\ \frac{S^W \times N}{\{\infty\} \times N \cup S^W \times (N - V)} & \xrightarrow{\quad} & \frac{S^W \times N}{(S^W \times N) - v_U}. \end{array}$$

We can now take the diagram of (9.7), stabilize, and cancel the representation sphere S^W to obtain the result. $\square$

9.2 Linear spheres

Let $\mathcal{G}$ be a compact p -adic analytic group of rank d and let $\mathfrak{g}$ be the adjoint representation.

Proposition 9.8 *Let $F \subseteq \mathcal{G}$ be a finite subgroup. Suppose there is a finitely generated free abelian group $L \subseteq \mathfrak{g}$ with the properties that*

- (1) *L is stable under the adjoint action of F on $\mathfrak{g}$, and*
- (2) *$L/pL \cong \mathfrak{g}/p\mathfrak{g}$.*

Let $V = \mathbb{R} \otimes L$ and let S^V be the one-point compactification of V . Then there is an F -equivariant map $S^V \rightarrow S^{\mathfrak{g}}$ which becomes a weak equivalence after completion at p .

The proof is below, after Proposition 9.11. The argument we have in mind proceeds geometrically, using that $V/L = (\mathbb{R} \otimes L)/L$ is an F -parallelizable manifold, as shown in Lemma 9.9.

As before, if Y is any F -space and W is a real representation of F , then θ_W denotes the trivial bundle with total space $Y \times W$. If M has an action by a finite group F and $m \in M$ is a fixed point, then $T_m M$ is real representation of F . Note also that $0 = 0 + L \in V/L$ is fixed under the action of F .

Lemma 9.9 *Let F be a finite group acting on a finitely generated free abelian group L . Let $V = \mathbb{R} \otimes L$. Then there is an equivariant isomorphism of bundles over V/L*

$$T(V/L) \cong \theta_V.$$

In particular there is an isomorphism of F -representations $T_0(V/L) \cong V$.

Proof The standard linear trivialization $\theta_V \cong TV$ over V descends to the needed trivialization over V/L . Specifically, the trivialization over V is the map

$$V \times V \longrightarrow TV$$

sending (v, w) to $(v, \gamma'(0))$ where $\gamma(t) = v + tw$. □

Lemma 9.9 and Lemma 9.4 immediately imply the following.

Proposition 9.10 *Let F be a finite group acting on a finitely generated free abelian group L . Let $V = \mathbb{R} \otimes L$. Then there is a choice of F -equivariant neighborhood U of $0 + L \in V/L$ that gives an equivariant collapse map*

$$f_U : V/L \rightarrow S^V.$$

This map sends $0 + L$ to $0 \in V \subseteq S^V$.

The following result is an immediate consequence of Proposition 9.6 and Proposition 9.10.

Proposition 9.11 *Let F be a finite group acting on a finitely generated free abelian group L . Let $V = \mathbb{R} \otimes L$. Then there is a stable F -equivariant equivalence*

$$\mathrm{hocolim} \Sigma_+^\infty (V/p^i L) \simeq S^V$$

where the colimit is over the transfers for the covering maps $V/p^{i+1}L \rightarrow V/p^i L$.

We can now prove the main result of this section.

Proof of Proposition 9.8 Let $\Gamma_i \subseteq \mathcal{G}$ be our preferred set of open uniformly powerful subgroups of the compact p -adic analytic group $\mathcal{G}$. Recall from Definition 3.10 that we have defined

$$B(p^i \mathfrak{g}) = \mathrm{holim} B(p^i \mathfrak{g}/p^{i+j} \mathfrak{g})$$

and we proved in Proposition 3.11 that there is an equivalence after p -completion

$$\Sigma_+^\infty B(p^i \mathfrak{g}) \simeq \mathrm{holim} \Sigma_+^\infty B(p^i \mathfrak{g}/p^{i+j} \mathfrak{g}). \quad (9.12)$$

Finally, $S^{\mathfrak{g}} \simeq \mathrm{hocolim} \Sigma_+^\infty B(p^i \mathfrak{g})$, again after p -completion.

Our assumption that $L/pL \cong \mathfrak{g}/p\mathfrak{g}$ allows us to conclude that for all i the map

$$B(p^i L) \longrightarrow B(p^i \mathfrak{g})$$

induces p -completion on the fundamental group. The map $V/p^i L \rightarrow B(p^i L)$ classifying the principal $p^i L$ bundle $V \rightarrow V/p^i L$ is a weak equivalence and we also have that the composition

$$V/p^i L \longrightarrow B(p^i L) \longrightarrow B(p^i \mathfrak{g})$$

induces p -completion on the fundamental group. Since both spaces have no non-zero higher homotopy groups it is then an isomorphism on $H_*(-, \mathbb{F}_p)$. Then

$$\Sigma_+^\infty V/p^i L \longrightarrow \Sigma_+^\infty B(p^i \mathfrak{g})$$

is also an isomorphism on $H_*(-, \mathbb{F}_p)$. Using (5.7) and Proposition 9.11 we can now conclude that

$$S^V \simeq \operatorname{hocolim} \Sigma_+^\infty V/p^i L \longrightarrow \operatorname{hocolim} \Sigma_+^\infty B(p^i \mathfrak{g}) = S^{\mathfrak{g}}$$

is an isomorphism on $H_*(-, \mathbb{F}_p)$. This implies the result. $\square$

In our examples the natural choice of a finitely generated free abelian group $L \subseteq \mathfrak{g}$ may not have the property that $L/pL \cong \mathfrak{g}/p\mathfrak{g}$. The following result will let us relax this hypothesis.

Lemma 9.13 *Let M be a free $\mathbb{Z}_p$ -module of finite rank and let $L_0 \subseteq M$ be a finitely generated free abelian group so that $\mathbb{Q}_p \otimes L_0 \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} M$. Define*

$$L = \{x \in M \mid p^k x \in L_0 \text{ for some } k \geq 0\} \subseteq M.$$

Then L is a finitely generated free abelian group, $\mathbb{Q}_p \otimes L \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} M$, $L_0 \subseteq L$ has finite index, and

$$L/pL \cong M/pM.$$

Furthermore, if $f : M \rightarrow M$ is any continuous $\mathbb{Z}_p$ -linear isomorphism such that $f(L_0) = L_0$, then $f(L) = L$.

Proof Filter M by powers of p ; that is, set $F_s M = p^s M$. Let $E_* M$ be the associated graded module. Then $E_* M = \mathbb{F}_p[x] \otimes M/pM$ where M/pM has filtration 0 and x has filtration 1 and is the residue class of p in $E_* \mathbb{Z}_p$. Since M/pM is finite, $E_* M$ is a finitely generated $\mathbb{F}_p[x]$ -module.

We have induced filtrations on L_0 and L ; for example, $F_s L = F_s M \cap L$. The hypothesis that $\mathbb{Q}_p \otimes L_0 \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} M$ implies there is an integer n so that we have inclusions of graded $\mathbb{F}_p[x]$ -modules

$$x^n E_* M \subseteq E_* L_0 \subseteq E_* L \subseteq E_* M.$$

We first show the last inclusion is an equality.

Suppose $a \in E_s M$ is represented by $\alpha \in F_s M$. Then $x^n a \in E_* L_0$ so there is a $\beta \in F_{s+n} L_0$ and $z \in p^{s+n+1} M = F_{s+n+1} M$ so that

$$p^n \alpha = \beta + z.$$

Since $p : F_s M \rightarrow F_{s+1} M$ is an isomorphism, we may choose $y \in F_{s+1} M$ so that $p^n y = z$. Set $\gamma = \alpha - y$. Then $\gamma \equiv \alpha$ modulo $F_{s+1} M$ and hence γ also represents a . Furthermore

$$p^n \gamma = \beta \in L_0.$$

Thus $\gamma \in L$ as needed.

We can now prove the result. Since $p^n L \subseteq L_0$ and $p^n : L \rightarrow p^n L$ is an isomorphism, we have that L is finitely generated and free. The inclusions $p^n L \subseteq L_0 \subseteq L$ imply L_0 is of finite index in L , that L_0 and L have the same rank, and that

$$\mathbb{Q}_p \otimes L_0 \cong \mathbb{Q}_p \otimes L \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} M.$$

The equality $E_* L = E_* M$ shows $L/pL = M/pM$.

Let $f : M \rightarrow M$ be any isomorphism, and let $x \in L$ so that $p^k x \in L_0$ for some k . Since L_0 is invariant under f , we have $p^k f(x) \in L_0$, so $f(x) \in L$. That every $x \in L$ is in $f(L)$ is an easy exercise. $\square$

Using this we have the following useful variant of Proposition 9.8.

Proposition 9.14 *Let $F \subseteq \mathcal{G}$ be a finite subgroup. Suppose there is a finitely generated free abelian group $L_0 \subseteq \mathfrak{g}$ with the properties that*

- (1) L_0 is stable under the adjoint action of F on $\mathfrak{g}$, and
- (2) $\mathbb{Q}_p \otimes L_0 \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathfrak{g}$.

Let $V = \mathbb{R} \otimes L_0$ and let S^V be the one-point compactification of V . Then there is an F -equivariant map $S^V \rightarrow S^{\mathfrak{g}}$ which becomes a weak equivalence after completion at p .

Proof Use Lemma 9.13 to produce a finitely generated free abelian group L so that $L_0 \subseteq L \subseteq \mathfrak{g}$, L is F -invariant, and $L/pL \cong \mathfrak{g}/p\mathfrak{g}$. Since $V = \mathbb{R} \otimes L_0 \cong \mathbb{R} \otimes L$, Proposition 9.8 now applies. $\square$

10 Lubin–Tate theory and its fixed points

We fix p and $n \geq 1$. If F is a formal group law of height n over a finite algebraic extension k of $\mathbb{F}_p$, let $\mathbf{E} = E(k, F)$ be the Lubin–Tate spectrum – aka the Morava E -theory – associated to the pair (k, F) . The spectrum $\mathbf{E}$ is an E_∞ -ring spectrum with an action, through E_∞ -ring maps, of the group $\mathbb{G} = \text{Aut}(k, F)$ of the automorphisms of pair (k, Γ) [30, 60]. Namely, the elements of $\mathbb{G}$ are pairs (f, φ) where $\varphi \in \text{Gal}(k/\mathbb{F}_p)$ and $f : \varphi^* F \xrightarrow{\cong} F$ is an isomorphism; an explicit description of $\mathbb{G}$ for the Honda formal group law is given in Example 2.15. A general theory is possible here, but we will focus on two examples. For both, the pair (k, F) will have the property that for any algebraic extension $k \subseteq k'$ the inclusion $\text{Aut}(k, F) \subseteq \text{Aut}(k', F)$ is an isomorphism. We will say that the pair (k, F) has all automorphisms at k .

Example 10.1 The classical example is the Honda formal group law F of height n over $\mathbb{F}_{p^n}$. In this case F is the unique p -typical formal group law

defined over $\mathbb{F}_p$ with p -series x^{p^n} . Not all automorphisms of F are defined over $\mathbb{F}_p$; for this we need to pass to $\mathbb{F}_{p^n}$.

Example 10.2 The other case of interest is specific to $n = 2$, and we only concentrate on $p = 2$ or 3 . Then F is the formal group law of a supersingular elliptic curve defined over $\mathbb{F}_p$. We can and will choose the curves so that once we pass to $\mathbb{F}_{p^2}$ we get all automorphisms.

From now on we assume we are working with one of these two examples. In both cases $k = \mathbb{F}_{p^n}$, where n is the height of the formal group.

Remark 10.3 (Structure of the Morava stabilizer group) Because the formal group law F is defined over $\mathbb{F}_p$, there is a split surjective homomorphism

$$\mathbb{G} = \text{Aut}(\mathbb{F}_{p^n}, F) \longrightarrow \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$$

with kernel the group $\mathbb{S} := \text{Aut}(F/\mathbb{F}_{p^n})$, the group of automorphisms of F over $\mathbb{F}_{p^n}$. Both $\mathbb{G}$ and its subgroup $\mathbb{S} \subseteq \mathbb{G}$ are compact p -adic analytic groups. We write

$$\cdots \subseteq \Gamma_{i+1} \subseteq \Gamma_i \subseteq \cdots \subseteq \Gamma_1 \subseteq \mathbb{S} \quad (10.4)$$

for the filtration of Example 2.15.

There is a non-canonical isomorphism

$$W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]] [u^{\pm 1}] \cong \mathbf{E}_*$$

where the power series ring is in degree 0 and u is in degree -2 . The power series ring $\mathbf{E}_0$ is a complete local Noetherian ring with maximal ideal $\mathfrak{m} = (p, u_1, \dots, u_{n-1})$ and residue field $\mathbb{F}_{p^n}$. Define $\mathbf{K} = K(n)$ to be the version of Morava K -theory with $\mathbf{K}_* = \mathbb{F}_{p^n}[u^{\pm 1}]$ and formal group law F . There is map of complex oriented ring spectra $\mathbf{E} \rightarrow \mathbf{K}$ which, on coefficients, is the quotient by $\mathfrak{m}$. The category of $\mathbf{K}$ -local spectra is the standard $K(n)$ -local category.

The $\mathbf{K}$ -local category is a closed symmetric monoidal category with smash product $L_{\mathbf{K}}(X \wedge Y)$. As is commonly done, we define

$$\mathbf{E}_* X \cong \pi_* L_{\mathbf{K}}(\mathbf{E} \wedge X).$$

This is not quite a homology theory, so the notation is slightly abusive. It has very nice algebraic properties; see for example Remark 10.10. The action of $\mathbb{G}$ on $\mathbf{E}$ gives a continuous action of $\mathbb{G}$ on $\mathbf{E}_* X$ making $\mathbf{E}_* X$ a *Morava module* (see [10, Section 1.3] for a definition).

Remark 10.5 Recall that L_n -localization is localization with respect to the homology theory $K(0) \vee \dots \vee K(n)$ or, equivalently, with respect to $\mathbf{E}$. Here $K(0)$ is rational homology.

Using the periodicity results of Hopkins and Smith [44], Hovey and Strickland produce a sequence of ideals $J(i) \subseteq \mathfrak{m} \subseteq \mathbf{E}_0$ and finite type n spectra $\mathbf{M}_{J(i)}$ with the following properties:

- (1) $J(i+1) \subseteq J(i)$ and $\bigcap_i J(i) = 0$;
- (2) $\mathbf{E}_0/J(i)$ is finite;
- (3) $\mathbf{E}_*(\mathbf{M}_{J(i)}) \cong \mathbf{E}_*/J(i)$ and there are spectrum maps $q: \mathbf{M}_{J(i+1)} \rightarrow \mathbf{M}_{J(i)}$ realizing the quotient $\mathbf{E}_*/J(i+1) \rightarrow \mathbf{E}_*/J(i)$;
- (4) There are maps $\eta = \eta_i: S^0 \rightarrow \mathbf{M}_{J(i)}$ inducing the quotient map $\mathbf{E}_0 \rightarrow \mathbf{E}_0/J(i)$ and $q\eta_{i+1} = \eta_i: S^0 \rightarrow \mathbf{M}_{J(i)}$;
- (5) If X is a finite type n spectrum, then the map $X \rightarrow \text{holim}_i (X \wedge \mathbf{M}_{J(i)})$ induced by the maps η is an equivalence.
- (6) If X is any L_n -local spectrum, then $L_{\mathbf{K}}X \simeq \text{holim}_i X \wedge \mathbf{M}_{J(i)}$. In particular we have $\mathbf{E} \simeq \text{holim}_i \mathbf{E} \wedge \mathbf{M}_{J(i)}$.

Most of this is proved in [45, Section 4], while (6) is proved in [45, Proposition 7.10]. In the same source, Hovey and Strickland also prove that items (1)–(5) characterize the tower $\{\mathbf{M}_{J(i)}\}$ up to equivalence in the pro-category of towers under S^0 ; see [45, Proposition 4.22]. Note that the topology on $\mathbf{E}_0$ defined by the sequence $\{J(i)\}$ is same as the $\mathfrak{m}$ -adic topology and that $\mathbb{G}$ acts on $\mathbf{E}_*/J(i)$ through a finite quotient of $\mathbb{G}$.

We can now prove the following useful recognition lemma.

Lemma 10.6 *Let $f: X \rightarrow Y$ be a map of L_n -local spectra. Then $L_{\mathbf{K}}f: L_{\mathbf{K}}X \rightarrow L_{\mathbf{K}}Y$ is an equivalence if and only if there is a finite type n complex T so that*

$$f \wedge T: X \wedge T \longrightarrow Y \wedge T$$

is an equivalence.

Proof The collection of type n complexes forms a thick subcategory $\mathcal{C}_n$ of finite spectra, so if $f \wedge T$ is an equivalence for any $T \in \mathcal{C}_n$, it is an equivalence for all $T \in \mathcal{C}_n$. Thus one implication follows from item (6) of Remark 10.5. The other implication follows from the fact that if X is L_n -local and T is of type n , the map $X \wedge T \rightarrow L_{\mathbf{K}}(X \wedge T)$ is an equivalence. $\square$

Remark 10.7 (Fixed point spectra and transfers) According to the theory of Devinatz and Hopkins [27], for any closed subgroup $K \subseteq \mathbb{G}$ there is a continuous homotopy fixed point spectrum $\mathbf{E}^{hK}$. These are again E_∞ -ring spectra, and there is a fixed point spectral sequence

$$H^s(K, \mathbf{E}_t) \implies \pi_{t-s} \mathbf{E}^{hK}. \quad (10.8)$$

The E_2 -term is continuous group cohomology. We have $\mathbf{E}^{h\mathbb{G}} \simeq L_{\mathbf{K}}S^0$ and the spectral sequence of (10.8) with $K = G$ is the $\mathbf{E}$ -based Adams-Novikov spectral sequence in the $\mathbf{K}$ -local category.

The spectral sequence of (10.8) has a very rigid convergence properties. There is an integer N , independent of K , so that $E_{\infty}^{s,*} = 0$ for $s \geq N$. If $K \subseteq \Gamma_1$ (or $K \subseteq \Gamma_2$ if $p > 2$), then

$$H^s(K, \mathbf{E}_t) = 0$$

for $s > n^2$, as K is then a Poincaré duality group of dimension $\leq n^2$.

These fixed point spectra of $\mathbf{E}$ also have the property that if $K_1 \subseteq K_2$ is normal and of finite index, then $\mathbf{E}^{hK_1}$ has an action by K_2/K_1 and there is a weak equivalence

$$\mathbf{E}^{hK_2} \simeq (\mathbf{E}^{hK_1})^{hK_2/K_1}. \quad (10.9)$$

This is proved as Theorem 4 of [27]. In particular, we have a transfer map $\mathrm{tr} : \mathbf{E}^{hK_1} \rightarrow \mathbf{E}^{hK_2}$. If $K \subseteq \mathbb{G}$ is open (and hence closed), then K is of finite index in $\mathbb{G}$ and we have a transfer map

$$\mathrm{tr} : \mathbf{E}^{hK} \rightarrow L_{\mathbf{K}}S^0 \simeq \mathbf{E}^{h\mathbb{G}}.$$

Remark 10.10 Let $H \subseteq \mathbb{G}$ be a closed subgroup. A crucial property of the fixed point spectrum $\mathbf{E}^{hH}$, from [27, Theorem 2], is that there are isomorphisms

$$\begin{aligned} \mathbf{K}_* \mathbf{E}^{hH} &\cong \pi_* L_{\mathbf{K}}(\mathbf{K} \wedge \mathbf{E}^{hH}) \cong \mathrm{map}(\mathbb{G}/H, \mathbf{K}_*) \\ \mathbf{E}_* \mathbf{E}^{hH} &\cong \pi_* L_{\mathbf{K}}(\mathbf{E} \wedge \mathbf{E}^{hH}) \cong \mathrm{map}(\mathbb{G}/H, \mathbf{E}_*) \end{aligned} \quad (10.11)$$

where map denotes continuous set maps. The action of $\mathbb{G}$ on the left factor of $\mathbf{E}$ defines the Morava module structure on $\mathbf{E}_* \mathbf{E}^{hH}$. The isomorphism of (10.11) becomes an isomorphism of Morava modules if we give the module of continuous maps the conjugation action

$$(g\varphi)(x) = g\varphi(g^{-1}x).$$

If H is normal, then $\mathbb{G}$ acts on $\mathbf{E}^{hH}$. The isomorphism of (10.11) becomes equivariant if we define an action on the module of continuous maps by

$$(g \star \varphi)(x) = \varphi(xg).$$

Remark 10.12 (Continuous actions and fixed points) For one of our main results we will need some details about how the continuous fixed points of [27] are constructed. See Lemma 11.12.

Let $\mathcal{G} = \lim_i G_i$ be a profinite group. We are thinking of $\mathcal{G} = \mathbb{G}$, $\mathbb{S}$, or $\Gamma_i \subseteq \mathbb{S}$. Let X be a spectrum presented as a homotopy inverse limit

$$X \simeq \operatorname{holim} X_j$$

with $\pi_t X_j$ **finite** for all j and t . We are thinking of the $\mathbf{E} \simeq \operatorname{holim}_j \mathbf{E} \wedge \mathbf{M}_{J(j)}$, as in point (6) of Remark 10.5. Define the spectrum of continuous map by

$$F_c(\Sigma_+^\infty \mathcal{G}, X) = \operatorname{holim}_j \operatorname{hocolim}_i F(\Sigma_+^\infty G_i, X_j). \quad (10.13)$$

This definition is built so that

$$\pi_* F_c(\Sigma_+^\infty \mathcal{G}, X) \cong \operatorname{map}(\mathcal{G}, \pi_* X), \quad (10.14)$$

where the target is the group of continuous maps. We can extend this definition of the continuous function spectrum to

$$F_c(\Sigma_+^\infty \mathcal{G}^{1+s}, X) = \operatorname{holim}_j \operatorname{hocolim}_i F(\Sigma_+^\infty \mathcal{G}_i^{1+s}, X_j),$$

where $\mathcal{G}^m$ is the m -fold Cartesian product of $\mathcal{G}$ with the product topology. Then the data of a continuous action of $\mathcal{G}$ on X is a map $X \rightarrow F_c(\Sigma_+^\infty \mathcal{G}, X)$ which extends (in the obvious way) to a map of cosimplicial spectra

$$X \longrightarrow F_c(\Sigma_+^\infty \mathcal{G}^{1+\bullet}, X).$$

One of the main theorems of [27] is that the action of $\mathbb{G}$ in $\mathbf{E}$ can be refined to a continuous action.

If $\Gamma \subseteq \mathcal{G}$ is open, then we define

$$X^{h\Gamma} = \operatorname{holim}_\Delta F_c(\Sigma_+^\infty \mathcal{G}^{1+\bullet}, X)^\Gamma = \operatorname{holim}_\Delta F_c(\Sigma_+^\infty (\mathcal{G}/\Gamma \times \mathcal{G}^\bullet), X).$$

The Bousfield-Kan Spectral Sequence of this cosimplicial space then gives the homotopy fixed point spectral sequence

$$H^s(\Gamma, \pi_t X) \implies \pi_{t-s} X^{h\Gamma},$$

where the E_2 -term is continuous cohomology.

This notion of homotopy fixed points commutes with various types of inverse limits; the following will be sufficient for our purposes. If X is a spectrum, let $P_n X$ be its n th Postnikov section.

Lemma 10.15 *Let $\mathcal{G}$ be a profinite group and let $\Gamma \subseteq \mathcal{G}$ be an open subgroup. Suppose X is presented as a homotopy inverse limit*

$$X \simeq \operatorname{holim} X_j$$

with $\pi_t X_j$ finite for all j and t and that, with this presentation, X has a continuous $\mathcal{G}$ -action. Then $P_n X \simeq \operatorname{holim} P_n X_j$ and the natural map

$$X^{h\Gamma} \longrightarrow \operatorname{holim}_n (P_n X)^{h\Gamma}$$

is an equivalence.

Proof Note that by the definition (10.13) and (10.14)

$$\pi_t F_c(\Sigma_+^\infty \mathcal{G}^\bullet, X) \longrightarrow \pi_t F_c(\Sigma_+^\infty \mathcal{G}^\bullet, P_n X)$$

is an isomorphism for $t \leq n$. Thus the natural map

$$F_c(\Sigma_+^\infty \mathcal{G}^{1+\bullet}, X) \longrightarrow \operatorname{holim}_n F_c(\Sigma_+^\infty \mathcal{G}^{1+\bullet}, P_n X)$$

is an equivalence of cosimplicial spectra. The result follows. $\square$

11 Dualizing the Lubin–Tate spectrum

In this section we prove one of our main theorems, identifying the equivariant homotopy type of the Spanier–Whitehead dual of $\mathbf{E}$. See Theorem 11.16 and Corollary 11.18.

Let $D(-) = F(-, L_{\mathbf{K}} S^0)$ denote Spanier–Whitehead duality in the $\mathbf{K}$ -local category. By definition, a $\mathbf{K}$ -local spectrum X is dualizable if the natural map

$$L_{\mathbf{K}}(DX \wedge Y) \rightarrow F(X, Y)$$

is an equivalence for all $\mathbf{K}$ -local spectra Y .

Lemma 11.1 (1) *Let $\Gamma \subseteq \mathbb{G}$ be an open subgroup. Then $\mathbf{E}^{h\Gamma}$ is dualizable in the $\mathbf{K}$ -local category. Furthermore, $\mathbf{E}_* D\mathbf{E}^{h\Gamma}$ is zero in odd degrees and there is a $\mathbb{G}$ -equivariant isomorphism*

$$\mathbf{E}_0 D\mathbf{E}^{h\Gamma} \cong \mathbf{E}^0 \mathbf{E}^{h\Gamma} \cong \mathbf{E}_0[\mathbb{G}/\Gamma]$$

where $\mathbb{G}$ acts on $\mathbf{E}_0[\mathbb{G}/\Gamma]$ by $g(\sum a_i x_i \Gamma) = \sum g(a_i) g x_i \Gamma$.

(2) Let $\Gamma_i \subseteq \mathbb{S} \subseteq \mathbb{G}$ be the open subgroups of (10.4). The natural maps $\mathbf{E}^{h\Gamma_i} \rightarrow \mathbf{E}$ induce a $\mathbb{G}$ -equivariant equivalence in the $\mathbf{K}$ -local category

$$\mathrm{hocolim} \mathbf{E}^{h\Gamma_i} \xrightarrow{\sim} \mathbf{E}.$$

Proof Since Γ is open, it is of finite index in $\mathbb{G}$; therefore, (10.11) implies that $\mathbf{E}_* \mathbf{E}^{h\Gamma}$ is finitely generated as an $\mathbf{E}_*$ -module. Then Theorem 8.6 of [45] implies that $\mathbf{E}^{h\Gamma}$ is dualizable. To finish the proof of part (1) we use the standard Spanier–Whitehead duality isomorphism, the Universal Coefficient Spectral Sequence

$$\mathbf{E}_0 D \mathbf{E}^{h\Gamma} \cong \mathbf{E}^0 \mathbf{E}^{h\Gamma} \cong \mathrm{Hom}_{\mathbf{E}_0}(\mathbf{E}_0 \mathbf{E}^{h\Gamma}, \mathbf{E}_0)$$

and (10.11). Note that to make this isomorphism $\mathbb{G}$ -equivariant we must act by conjugation on the target; that is, $(g\psi)(a) = g\psi(g^{-1}a)$.

For (2) we need to check that

$$\mathrm{hocolim} \mathbf{K}_* \mathbf{E}^{h\Gamma_i} \cong \mathbf{K}_* \mathbf{E}^{h\Gamma_j}.$$

By (10.11), this is equivalent to showing the map

$$\mathrm{hocolim} \mathrm{map}(\mathbb{G}/\Gamma_i, \mathbf{K}_*) \rightarrow \mathrm{map}(\mathbb{G}, \mathbf{K}_*)$$

is an isomorphism, where map continues to denote the continuous set maps. Since the topology on $\mathbf{K}_*$ is discrete, this is clear. $\square$

In order to assemble the Spanier–Whitehead duals of $\mathbf{E}^{h\Gamma_i}$ for various i we need a version of Frobenius reciprocity. Let R be an E_∞ -ring spectrum and suppose G is a finite group that acts on R through E_∞ -ring maps. Then the spectrum $R^h := F(EG_+, R)$ is an E_∞ -ring in genuine G -equivariant spectra [41].³ The notation is chosen so that for any subgroup $H \subseteq G$, the categorical fixed points $(R^h)^H$ agree with the homotopy fixed points of the original H -action on R . Suppose that $K \subseteq H \subseteq G$ are subgroups. Then we have the following maps

- (a) the inclusion (restriction) map $r : R^{hH} \rightarrow R^{hK}$;
- (b) the transfer (induction) map $\mathrm{tr} : R^{hK} \rightarrow R^{hH}$; and,
- (c) conjugation maps $c_g : R^{hH} \rightarrow R^{h(g^{-1}Hg)}$.

These maps are in fact defined for any genuine G -spectrum, and are at the core of the identification of genuine G -spectra with spectral Mackey functors [3, 33]. Because it will be important for the next Lemma, let us recall

³ In fact, the same source explains that R^h has more equivariant multiplicative structure, but we don't need that here.

the provenance of these maps. The restriction is induced by the map of G -sets $p : G/K \rightarrow G/H$; then $r = F(p_+, R^h)^G$. The conjugation maps are similarly induced by maps of G -sets, namely the conjugation by $g \in G$ maps $G/g^{-1}Hg \rightarrow G/H$. The transfer is not induced by a map of G -sets, but rather by the Spanier–Whitehead dual of p_+ in G -spectra [53].

Lemma 11.2 *The inclusion, transfer, and conjugation maps have following properties.*

- (1) *The inclusion $r : R^{hH} \rightarrow R^{hK}$ and conjugation $c_g : R^{hH} \rightarrow R^{h(g^{-1}Hg)}$ are maps of E_∞ -ring spectra.*
- (2) *We have Frobenius Reciprocity commutative diagrams such as*

$$\begin{array}{ccccc}
 & & R^{hK} \wedge R^{hK} & \xrightarrow{m} & R^{hK} \\
 & \nearrow r \wedge 1 & & & \downarrow \text{tr} \\
 R^{hH} \wedge R^{hK} & & & & \\
 & \searrow 1 \wedge \text{tr} & R^{hH} \wedge R^{hH} & \xrightarrow{m} & R^{hH}
 \end{array}$$

Proof We use ideas from [28]. More generally, this result holds for any E_∞ -ring A in genuine G -spectra, not just those of form $A = R^h$.

For such an A , we have that the restriction and conjugations are maps of E_∞ -rings, since they are obtained by taking G -fixed points (a lax monoidal functor) of the E_∞ -ring maps

$$F(G/H_+, R) \rightarrow F(G/K_+, R) \quad \text{and} \quad F(G/H_+, R) \rightarrow F(G/g^{-1}Hg_+, R).$$

For the transfer, it suffices to restrict to H , so that tr will be the H -fixed points of the A -module map $F(H/K_+, A) \rightarrow A$, obtained by mapping into A the Spanier–Whitehead dual in H -spectra of the map of H -sets $H/K_+ \rightarrow H/H_+$. Upon taking H -fixed points, we get that $A^K \rightarrow A^H$ is an A^H -module map, where A^K has the A^H -algebra structure given by the restriction map. Frobenius Reciprocity is simply the diagrammatic form of the statement that tr is an A^H -module map. $\square$

Much more sophisticated results are possible, using the language of spectral Mackey functors and spectral Green functors. See [3] and [11].

From Lemma 11.2 we have pairings

$$R^{hH} \wedge R^{hH} \xrightarrow{m} R^{hH} \xrightarrow{\text{tr}} R^{hG}$$

and get maps

$$R^{hH} \rightarrow F(R^{hH}, R^{hG})$$

that switch restriction and transfer; for example, the following diagram commutes

$$\begin{array}{ccc} R^{hK} & \longrightarrow & F(R^{hK}, R^{hG}) \\ \text{tr} \downarrow & & \downarrow F(r, 1) \\ R^{hH} & \longrightarrow & F(R^{hH}, R^{hG}). \end{array} \quad (11.3)$$

Lemma 11.4 *Let $r_j : \mathbf{E}^{h\Gamma_j} \rightarrow \mathbf{E}^{h\Gamma_{j+1}}$ be the restriction maps. The maps f_j fit into a $\mathbb{G}$ -equivariant commutative diagram*

$$\begin{array}{ccc} \mathbf{E}^{h\Gamma_{j+1}} & \xrightarrow{f_{j+1}} & F(\mathbf{E}^{h\Gamma_{j+1}}, L_{\mathbf{K}}S^0) \\ \text{tr} \downarrow & & \downarrow F(r_j, 1) \\ \mathbf{E}^{h\Gamma_j} & \xrightarrow{f_j} & F(\mathbf{E}^{h\Gamma_j}, L_{\mathbf{K}}S^0). \end{array}$$

Furthermore, we have a $\mathbf{K}$ -local $\mathbb{G}$ -equivariant equivalence

$$\operatorname{holim}_{\text{tr}} \mathbf{E}^{h\Gamma_j} \simeq D\mathbf{E}.$$

Proof The commutative square can be obtained from (11.3) by taking $R = E^{h\Gamma_{i+1}}$, $G = \mathbb{G}/\Gamma_{i+1}$, $H = \Gamma_i/\Gamma_{i+1}$ and $K = \{e\} = \Gamma_{i+1}/\Gamma_{i+1}$. Note that $R^{hG} = L_{\mathbf{K}}S^0$.

The last statement follows from Part (2) of Lemma 11.1. $\square$

Remark 11.5 It is an observation due to the third author (see [63]), that there is an underlying (that is, non-equivariant) equivalence of spectra $D\mathbf{E} \simeq \Sigma^{-n^2}\mathbf{E}$. This included a calculation the $\mathbb{G}$ -action on $\pi_*D\mathbf{E}$. This older result is a corollary of our theorem Theorem 11.16, but we will end up giving a proof of this fact as we go; the argument is essentially the same as in [63]. In summary, we will show certain spectral sequences that appear in the proof of Lemma 11.9 have only one non-zero line and the shift of n^2 is due to the degree of that line.

We now add an algebraic result which will be used in several arguments to follow. There is nothing special about the group $\mathbb{G}$; the argument works equally well for any compact p -adic analytic group $\mathcal{G}$. We fix a nested sequence of open subgroups $\Gamma_i \subseteq \mathcal{G}$, as in Notation 2.18. An abelian group A is finite p -torsion if A is finite and there is an integer n so that $p^n A = 0$.

Lemma 11.6 *Let $\mathcal{G}$ be a compact p -adic analytic group of rank d and A be a finite p -torsion discrete $\mathcal{G}$ -module.*

- (1) There is an integer N so that for all $i \geq N$ the subgroup Γ_i acts trivially on A .
- (2) Taking the limit over transfers gives an isomorphism

$$\lim_j H^s(\Gamma_j, A) \cong \begin{cases} A, & s = d; \\ 0, & s \neq d. \end{cases}$$

If $i \geq N$, then the natural map $\lim_j H^d(\Gamma_j, A) \rightarrow H^d(\Gamma_i, A) \cong A$ is an isomorphism.

- (3) Taking the colimits over restriction gives an isomorphism

$$\operatorname{colim}_i H^s(\Gamma_i, A) \cong \begin{cases} A & s = 0; \\ 0 & s > 0. \end{cases}$$

If $j \geq N$, then natural map $A \cong H^d(\Gamma_j, A) \rightarrow \operatorname{colim}_i H^d(\Gamma_i, A)$ is an isomorphism.

Proof To prove (1) we use that if A is discrete the orbit of any element $a \in A$ is finite. Let $H_a \subseteq \mathbb{G}$ be the isotropy subgroups of a , then H_a is open, so $\cap_a H_a$ is also open, since this is a finite intersection. Choose N so that $\Gamma_N \subseteq \cap_a H_a$.

We next prove part (2). Since the limit depends only on large j , we make take $j \geq N$ and assume the action is trivial. Furthermore, since $H^k(\Gamma_i, \mathbb{F}_p)$ is finite for all k , by Theorem 3.3, the functor

$$A \mapsto \lim_j H^s(\Gamma_j, A)$$

sends short exact sequences in A to long exact sequences. If $A = \mathbb{Z}/p$ the result can be found in Proposition 5.6; more precisely, that result is for homology, but in this case cohomology is dual to homology. The result is then immediate if $pA = 0$. Now use induction on k and the long exact sequence in cohomology obtained from the short exact sequence

$$0 \longrightarrow A/pA \xrightarrow{p^{k-1}} A \longrightarrow A/p^{k-1}A \longrightarrow 0.$$

Part (3) is proved the same way, now using part (2) of Theorem 3.3 □

The next step is to decompose the homotopy inverse limit of Lemma 11.4 even further, using part (2) of Lemma 11.1.

Lemma 11.7 *The $\mathbb{G}$ -equivariant $\mathbf{K}$ -local equivalence $\operatorname{hocolim}_i \mathbf{E}^{h\Gamma_i} \simeq \mathbf{E}$ induces a $\mathbb{G}$ -equivariant $\mathbf{K}$ -local equivalence*

$$\operatorname{hocolim}_i [(\mathbf{E}^{h\Gamma_i})^{h\Gamma_j}] \xrightarrow{\simeq} [\operatorname{hocolim}_i \mathbf{E}^{h\Gamma_i}]^{h\Gamma_j} \simeq \mathbf{E}^{h\Gamma_j}$$

and hence a $\mathbb{G}$ -equivariant $\mathbf{K}$ -local equivalence

$$\mathrm{holim}_j \mathrm{hocolim}_i [(\mathbf{E}^{h\Gamma_i})^{h\Gamma_j}] \simeq \mathrm{holim}_j \mathbf{E}^{h\Gamma_j}$$

Proof We apply Lemma 10.6: to prove it is a $\mathbf{K}$ -local equivalence, it is sufficient to check we have an equivalence after smashing with some finite type n complex T . Since T is finite, $(-) \wedge T$ commutes with homotopy fixed points and the map becomes

$$\mathrm{hocolim}_i [(\mathbf{E} \wedge T)^{h\Gamma_i}]^{h\Gamma_j} \longrightarrow (\mathbf{E} \wedge T)^{h\Gamma_j} \quad (11.8)$$

We may assume that we have chosen T so that for all t the $\mathbb{G}$ -module $\mathbf{E}_t T$ is discrete, finite, and annihilated by $\mathfrak{m}^k$ for some k . Such T appeared in Remark 10.5. Write $X_i = (\mathbf{E} \wedge T)^{h\Gamma_i}$.

In the next diagram we use the following basic fact: If $G = \varprojlim G_s$ is profinite and M is finite and discrete, then $\mathrm{colim} H^*(G_s, M) = H^*(G, M)$.

We now have a diagram of spectral sequences

$$\begin{array}{ccc} \mathrm{colim}_i H^s(\Gamma_j, \pi_t X_i) & \longrightarrow & H^s(\Gamma_j, \mathbf{E}_t T) \\ \Downarrow & & \Downarrow \\ \mathrm{colim}_i \pi_{t-s}(X_i^{h\Gamma_j}) & \longrightarrow & \pi_{t-s}(\mathbf{E} \wedge T)^{h\Gamma_j}. \end{array}$$

Since $\mathrm{colim} \pi_* X_i = \mathbf{E}_* T$ and $\pi_t X_i$ is finite and discrete for each t , the map across the top is an isomorphism. Since each of the spectral sequences

$$H^s(\Gamma_j, \pi_t X_i) \Longrightarrow \pi_{t-s}(X_i^{h\Gamma_j})$$

strongly converge and colim is an exact functor, the spectral sequence on the left strongly converges. Since the spectral sequence on the right strongly converges as well, we are done. $\square$

The next step is to switch the limit and colimit.

Lemma 11.9 *The natural map*

$$\mathrm{colim}_i \mathrm{holim}_j (\mathbf{E}^{h\Gamma_i})^{h\Gamma_j} \longrightarrow \mathrm{holim}_j \mathrm{colim}_i (\mathbf{E}^{h\Gamma_i})^{h\Gamma_j} \simeq D\mathbf{E} \quad (11.10)$$

is a $\mathbb{G}$ -equivariant $\mathbf{K}$ -local equivalence.

Proof As a key to the upcoming argument, we make the following convention. If we use the index i we will be taking a colimit along maps induced by restrictions; thus for example

$$\dots \longrightarrow \mathbf{E}^{h\Gamma_{i-1}} \xrightarrow{r} \mathbf{E}^{h\Gamma_i} \xrightarrow{r} \mathbf{E}^{h\Gamma_{i+1}} \longrightarrow \dots$$

If we use the index j we will be taking the limit along maps induced by transfer; thus for example,

$$\dots \longrightarrow \mathbf{E}^{h\Gamma_{j+1}} \xrightarrow{\text{tr}} \mathbf{E}^{h\Gamma_j} \xrightarrow{\text{tr}} \mathbf{E}^{h\Gamma_{j-1}} \longrightarrow \dots$$

The spectra $\mathbf{E}^{h\Gamma_i}$ are $\mathbf{K}$ -local, hence L_n -local. Since L_n -localization is smashing, the category of L_n -local spectra is closed under homotopy colimits and homotopy limits; hence the map of (11.10) is a map of L_n -local spectra. We now apply Lemma 10.6: to prove it is a $\mathbf{K}$ -local equivalence, it is sufficient to check it is an equivalence after smashing with some finite type n complex T . Since T is finite the map may be rewritten

$$\text{colim}_i \text{holim}_j ((\mathbf{E} \wedge T)^{h\Gamma_i})^{h\Gamma_j} \longrightarrow \text{holim}_j \text{colim}_i ((\mathbf{E} \wedge T)^{h\Gamma_i})^{h\Gamma_j}. \quad (11.11)$$

As before we assume that for all t , the $\mathbb{G}$ -module $\mathbf{E}_t T$ is discrete, finite and annihilated by $\mathfrak{m}^k$ for some k ; see Remark 10.5. In particular, $\mathbf{E}_t T$ satisfies the hypotheses of Lemma 11.6.

By Lemma 10.6, the fixed point spectral sequences

$$H^s(\Gamma_i, \mathbf{E}_t T) \implies \pi_{t-s}(\mathbf{E} \wedge T)^{h\Gamma_i}$$

have a horizontal vanishing line at $s = n^2$ at E_2 for all $i \geq 2$. For all t , the cohomology groups are finite and p -torsion; hence, $\pi_t(\mathbf{E} \wedge T)^{h\Gamma_i}$ is finite and of bounded p -power order for all t . A similar argument shows $\pi_t((\mathbf{E} \wedge T)^{h\Gamma_i})^{h\Gamma_j}$ is also finite and of bounded p -power order for all t . Thus, to prove the result, it's enough to prove that

$$\text{colim}_i \lim_j \pi_*((\mathbf{E} \wedge T)^{h\Gamma_i})^{h\Gamma_j} \longrightarrow \lim_j \text{colim}_i \pi_*((\mathbf{E} \wedge T)^{h\Gamma_i})^{h\Gamma_j}$$

is an isomorphism as all $\lim^1$ -terms vanish.

By assembling the Γ_j fixed point spectral sequences we get a diagram of spectral sequences

$$\begin{array}{ccc} \text{colim}_i \lim_j H^s(\Gamma_j, \pi_t(\mathbf{E} \wedge T)^{h\Gamma_i}) & \implies & \text{colim}_i \lim_j \pi_{t-s}((\mathbf{E} \wedge T)^{h\Gamma_i})^{h\Gamma_j} \\ \downarrow & & \downarrow \\ \lim_j \text{colim}_i H^s(\Gamma_j, \pi_t(\mathbf{E} \wedge T)^{h\Gamma_i}) & \implies & \lim_j \text{colim}_i \pi_{t-s}((\mathbf{E} \wedge T)^{h\Gamma_i})^{h\Gamma_j}. \end{array}$$

There is an issue here: while colim is exact on directed systems of abelian groups, $\lim$ is not, so applying $\lim$ to a system of spectral sequences does

not necessarily yield a spectral sequence. Here again we use that the terms $H^s(\Gamma_j, \pi_t(\mathbf{E} \wedge T)^{h\Gamma_i})$ are all finite, so that there will be no $\lim^1$ -terms and we do get a diagram of strongly convergent spectral sequences.

It is now sufficient to show that the right vertical map on E_2 -terms is an isomorphism. We will show much more: it will turn out that both the source and target of the map on E_2 -terms are zero unless $s = n^2$; this directs our attention to $s = n^2$.

We now use Lemma 11.6. Choose j large enough that the action of Γ_j on $\pi_t(\mathbf{E} \wedge T)$ is trivial. Then we have a diagram

$$\begin{array}{ccc} \operatorname{colim}_i \lim_j H^{n^2}(\Gamma_j, \pi_t(\mathbf{E} \wedge T)^{h\Gamma_i}) & \longrightarrow & \operatorname{colim}_i H^{n^2}(\Gamma_j, \pi_t(\mathbf{E} \wedge T)^{h\Gamma_i}) \\ \downarrow & & \downarrow \\ \lim_j \operatorname{colim}_i H^{n^2}(\Gamma_j, \pi_t(\mathbf{E} \wedge T)^{h\Gamma_i}) & & \\ \downarrow & & \\ \lim_j H^{n^2}(\Gamma_j, \pi_t(\mathbf{E} \wedge T)) & \longrightarrow & H^{n^2}(\Gamma_j, \pi_t(\mathbf{E} \wedge T)). \end{array}$$

By Lemma 11.6, we know $H^{n^2}(\Gamma_j, \pi_t(\mathbf{E} \wedge T)) \cong \pi_t(\mathbf{E} \wedge T)$ and that the natural map $\lim_j H^{n^2}(\Gamma_j, \pi_t(\mathbf{E} \wedge T)) \rightarrow H^{n^2}(\Gamma_j, \pi_t(\mathbf{E} \wedge T))$ is an isomorphism. Thus we have a commutative triangle

$$\begin{array}{ccc} \operatorname{colim}_i \lim_j H^{n^2}(\Gamma_j, \pi_t(\mathbf{E} \wedge T)^{h\Gamma_i}) & \longrightarrow & \lim_j \operatorname{colim}_i H^{n^2}(\Gamma_j, \pi_t(\mathbf{E} \wedge T)^{h\Gamma_i}) \\ & \searrow f & \swarrow g \\ & \pi_t(\mathbf{E} \wedge T) & \end{array}$$

We show both the maps f and g are isomorphisms.

We begin with the map f . Using part (2) of Lemma 11.6 we have

$$\lim_j H^s(\Gamma_j, \pi_t(\mathbf{E} \wedge T)^{h\Gamma_i}) \cong \begin{cases} \pi_t(\mathbf{E} \wedge T)^{h\Gamma_i}, & s = n^2; \\ 0, & s \neq n^2. \end{cases}$$

Then, using part (2) of Lemma 11.1, we have (as needed) that

$$\operatorname{colim}_i \lim_j H^s(\Gamma_j, \pi_t(\mathbf{E} \wedge T)^{h\Gamma_i}) \cong \begin{cases} \pi_t(\mathbf{E} \wedge T), & s = n^2; \\ 0, & s \neq n^2. \end{cases}$$

We complete the argument by analyzing the map g . We claim the maps

$$(\mathbf{E} \wedge T)^{h\Gamma_i} \simeq \mathbf{E}^{h\Gamma_i} \wedge T \rightarrow \mathbf{E} \wedge T$$

induce an isomorphism

$$\operatorname{colim}_i H^s(\Gamma_j, \pi_t(\mathbf{E} \wedge T)^{h\Gamma_i}) \cong H^s(\Gamma_j, \pi_t(\mathbf{E} \wedge T))$$

and hence, by Lemma 11.6, isomorphisms

$$\lim_j \operatorname{colim}_i H^s(\Gamma_j, \pi_t(\mathbf{E} \wedge T)^{h\Gamma_i}) \cong \begin{cases} \pi_t(\mathbf{E} \wedge T), & s = n^2; \\ 0, & s \neq n^2. \end{cases}$$

If M is any finite and discrete $\mathbb{G}$ -module, then Γ_k acts trivially on M for large k . It follows that for all j we have

$$\operatorname{colim}_k H^*(\Gamma_j / \Gamma_{j+k}, M) \cong H^*(\Gamma_j, M).$$

This applies to $M = \pi_t(\mathbf{E} \wedge T)^{h\Gamma_i}$ and $M = \pi_t(\mathbf{E} \wedge T)$. Note also that by part (2) of Lemma 11.1 we have $\operatorname{hocolim}_i \mathbf{E}^{h\Gamma_i} \wedge T \simeq \mathbf{E} \wedge T$. We now have

$$\begin{aligned} \operatorname{colim}_i H^s(\Gamma_j, \pi_t(\mathbf{E} \wedge T)^{h\Gamma_i}) &\cong \operatorname{colim}_i \operatorname{colim}_k H^s(\Gamma_j / \Gamma_{j+k}, \pi_t(\mathbf{E} \wedge T)^{h\Gamma_i}) \\ &\cong \operatorname{colim}_k \operatorname{colim}_i H^s(\Gamma_j / \Gamma_{j+k}, \pi_t(\mathbf{E} \wedge T)^{h\Gamma_i}) \\ &\cong \operatorname{colim}_k H^s(\Gamma_j / \Gamma_{j+k}, \operatorname{colim}_i \pi_t(\mathbf{E} \wedge T)^{h\Gamma_i}) \\ &\cong \operatorname{colim}_k H^s(\Gamma_j / \Gamma_{j+k}, \pi_t(\mathbf{E} \wedge T)) \\ &\cong H^s(\Gamma_j, \pi_t(\mathbf{E} \wedge T)). \end{aligned}$$

The second to last isomorphism uses part (2) of Lemma 11.1. □

In light of Lemma 11.9, the project now is to compute $\operatorname{hocolim}_i \operatorname{holim}_j (\mathbf{E}^{h\Gamma_i})^{h\Gamma_j}$; see Lemma 11.15. The next result is the first step.

Lemma 11.12 *Let $j > i$ so that the action Γ_j on $\mathbf{E}^{h\Gamma_i}$ is trivial. Then there is a $\mathbb{G}$ -equivariant $\mathbf{K}$ -local equivalence*

$$(\mathbf{E}^{h\Gamma_i})^{h\Gamma_j} \xrightarrow{\sim} F(\Sigma_+^\infty B\Gamma_j, \mathbf{E}^{h\Gamma_i}).$$

where $\mathbb{G}$ acts on $\Sigma_+^\infty B\Gamma_i$ through conjugation and diagonally on the function spectrum.

Proof We use the following basic fact about fixed point spectra. Let G be a finite group, K a normal subgroup of G , and X a G spectrum on which K acts trivially. Then there is a natural equivalence of G -spectra

$$X^{hK} \xrightarrow{\sim} F(\Sigma_+^\infty BK, X)$$

where G acts on $\Sigma_+^\infty BK$ through conjugation and diagonally on the function spectrum. The difficulty here is that $\mathbb{G}$ is not finite.

Since we seek a $\mathbf{K}$ -local equivalence we need only prove this after smashing with a type n -complex T . See Lemma 10.6. Thus we assume we have a spectrum X with a trivial $\Gamma = \Gamma_j$ action and with the property that $\pi_* X$ finite, discrete, and annihilated by some power of p . We will show there is a natural $\mathbb{G}$ -equivariant equivalence

$$X^{h\Gamma_j} \simeq F(\Sigma_+^\infty B\Gamma_j, X).$$

In practice, we will take $X = \mathbf{E}^{h\Gamma_i} \wedge T \simeq (\mathbf{E} \wedge T)^{h\Gamma_i}$.

Let $\Gamma = \Gamma_j$. Since Γ acts trivially on X , we have a trivial action of the finite group Γ/Γ_k on X for all $k \geq j$ and the maps $X^{h\Gamma/\Gamma_k} \rightarrow X^{h\Gamma}$ induced by the quotient map on groups give us a natural equivariant diagram

$$\begin{array}{ccc} \operatorname{hocolim} X^{h\Gamma/\Gamma_k} & \xrightarrow{\simeq} & \operatorname{hocolim} F(\Sigma_+^\infty B(\Gamma/\Gamma_k), X) \\ f \downarrow & & \downarrow g \\ X^{h\Gamma} & & F(\Sigma_+^\infty B\Gamma, X). \end{array}$$

Our strategy is now as follows. We will show that the maps f and g are equivalences when X is bounded above in homotopy; that is, there is an integer n so that $\pi_t X = 0$ for $t > n$. For general X , this will then provide us a natural equivalence

$$(P_n X)^{h\Gamma} \simeq F(\Sigma_+^\infty B\Gamma, P_n X).$$

for all n . We now then can use Lemma 10.15 to get an equivalence

$$X^{h\Gamma} \simeq \operatorname{holim}_n (P_n X)^{h\Gamma} \simeq \operatorname{holim}_n F(\Sigma_+^\infty B\Gamma, P_n X) \simeq F(\Sigma_+^\infty B\Gamma, X).$$

Thus, for the rest of the argument, assume $\pi_t X = 0$ if $t > n$.

We first show g is an equivalence. The basic fact we will use is that if A is a finite discrete abelian group with the property that $p^k A = 0$ for some k with trivial Γ -action, then

$$\begin{aligned} H^*(\Gamma, A) &= \operatorname{colim} H^*(\Gamma/\Gamma_k, A) \\ &\cong \operatorname{colim} H^*(\Sigma_+^\infty B(\Gamma/\Gamma_k), A) \cong H^*(\Sigma_+^\infty B\Gamma, A). \end{aligned} \quad (11.13)$$

See Proposition 3.11.

We are asserting that the natural map

$$\operatorname{hocolim} F(\Sigma_+^\infty B(\Gamma/\Gamma_k), X) \longrightarrow F(\operatorname{holim} \Sigma_+^\infty B(\Gamma/\Gamma_k), X) \quad (11.14)$$

is an equivalence. This follows from (11.13) and an Atiyah-Hirzebruch Spectral Sequence argument. We use that if Y is a space, then

$$E_2^{p,q}(Y) = H^p(Y, \pi_{-q}X) \implies \pi_{-p-q}F(\Sigma_+^\infty Y, X) = X^{p+q}(Y)$$

is zero if $p < 0$ or $q < -n$. This implies that for any pair (p, q) , there is an r so that $E_\infty^{p,q}(Y) = E_r^{p,q}(Y)$ independent of Y and, hence, that the spectral sequences converge. Combined with part (2) of Theorem 3.6 this is sufficient to show (11.14) is an isomorphism.

We now show that f is an equivalence. We have a diagram of spectral sequences

$$\begin{array}{ccc} \operatorname{colim} H^s(\Gamma/\Gamma_k, \pi_t X) & \Rightarrow & \operatorname{colim} \pi_{t-s} X^{h\Gamma/\Gamma_k} \\ \cong \downarrow & & \downarrow f_* \\ H^s(\Gamma, \pi_t X) & \Longrightarrow & \pi_{t-s} X^{h\Gamma}. \end{array}$$

The question then remains whether the upper spectral sequence converges. However, for all k the homotopy fixed point spectral sequence

$$E_2^{s,t} = H^s(\Gamma/\Gamma_k, \pi_t X) \implies \pi_{t-s} X^{h\Gamma/\Gamma_k}$$

has the property that $E_2^{s,t} = 0$ if $t > n$ so for a fixed pair (s, t) there is an r , independent of k , such that $E_\infty^{s,t} = E_r^{s,t}$. This is enough to give convergence. $\square$

Lemma 11.15 *There is a $\mathbb{G}$ -equivariant $\mathbf{K}$ -local equivalence*

$$\operatorname{colim}_i \operatorname{holim}_j (\mathbf{E}^{h\Gamma_i})^{h\Gamma_j} \simeq F(I_{\mathbb{G}}, \mathbf{E}).$$

Proof Lemma 11.12 and Frobenius Reciprocity imply that

$$\begin{aligned} \operatorname{holim}_j (\mathbf{E}^{h\Gamma_i})^{h\Gamma_j} &\simeq \operatorname{holim}_j F(\Sigma_+^\infty B\Gamma_j, \mathbf{E}^{h\Gamma_i}) \\ &\simeq F(\operatorname{colim}_j \Sigma_+^\infty B\Gamma_j, \mathbf{E}^{h\Gamma_i}) \\ &\simeq F(I_{\mathbb{G}}, \mathbf{E}^{h\Gamma_i}) \end{aligned}$$

since the (co-)limit is taken over transfer maps. Since $I_{\mathbb{G}}$ is a p -complete sphere, it is dualizable; hence, by part (2) of Lemma 11.1

$$\operatorname{hocolim}_i F(I_{\mathbb{G}}, \mathbf{E}^{h\Gamma_i}) \simeq F(I_{\mathbb{G}}, \mathbf{E}).$$

We now come to the main theorem of the section. If X , Y and Z are left G -spectra the diagonal G -action on $F(Y, Z)$ is the action for which the standard adjunction between smash product and function spectra restricts to an adjunction isomorphism

$$F_G(X \wedge Y, Z) \cong F_G(X, F(Y, Z))$$

where F_G denotes the spectrum of G -maps and G acts diagonally on $X \wedge Y$. If we were allowed to use functional notation and $\phi \in F(Y, Z)$, then

$$(g\phi)(y) = g\phi(g^{-1}y).$$

Recall that $I_{\mathbb{G}} = \operatorname{hocolim}_j \Sigma_+^{\infty} B\Gamma_j$ where the colimit is in the category of p -complete spectra and taken over the transfer maps. The $\mathbb{G}$ action is given by conjugation on the subgroups Γ_j . See Definition 5.4.

Theorem 11.16 *There is a $\mathbb{G}$ -equivariant $\mathbf{K}$ -local equivalence*

$$D\mathbf{E} \simeq F(I_{\mathbb{G}}, \mathbf{E}) \simeq I_{\mathbb{G}}^{-1} \wedge \mathbf{E}$$

where $\mathbb{G}$ acts diagonally on both $F(I_{\mathbb{G}}, \mathbf{E})$ and the smash product.

Proof This follows by combining Lemma 11.7, Lemma 11.9, and Lemma 11.15. $\square$

Remark 11.17 In Theorem 7.3.1 of [7], Behrens and Davis give an entirely different expression of the Spanier–Whitehead dual of $\mathbf{E}$ as a homotopy fixed point spectrum. It would be interesting to make a detailed comparison of that result with Theorem 11.16.

The following is now a consequence of Theorem 8.11 and Theorem 11.16.

Corollary 11.18 *Let $F \subseteq \mathbb{G}$ be a finite subgroup with p -Sylow subgroup F_0 . Suppose that $F_0/(F_0 \cap Z(\mathbb{G}))$ is an elementary abelian p -group. Then there is an F -equivariant $\mathbf{K}$ -local equivalence*

$$D\mathbf{E} \simeq S^{-\mathfrak{g}} \wedge \mathbf{E}.$$

12 The Spanier–Whitehead duals of $\mathbf{E}^{hF}$: some theory

As an application of the theory developed so far, we will give some calculations of $D(\mathbf{E}^{hF})$ where $F \subseteq \mathbb{G}$ is a finite subgroup whose p -Sylow subgroup F_0 has the property that $F_0/F_0 \cap Z(\mathbb{G})$ is an elementary abelian p -group. Recall that $Z(\mathbb{G})$ is the center of $\mathbb{G}$. This recovers in a coherent way all the known calculations of this kind in the literature. We also produce a new example.

This section provides background and set-up. The precise calculations, which involve the representation theory of specific groups, are in Sect. 13 and Sect. 14.

12.1 Generalities on Picard groups

Because Tate spectra vanish in the $K(n)$ -local category (see [36]), the norm map $\mathbf{E}_{hF} \rightarrow \mathbf{E}^{hF}$ is a weak equivalence; it follows immediately that

$$D(\mathbf{E}^{hF}) = F(\mathbf{E}^{hF}, L_{\mathbf{K}}S^0) \simeq F(\mathbf{E}_{hF}, L_{\mathbf{K}}S^0) \simeq (DE)^{hF}.$$

By Corollary 11.18 we have, for F satisfying our hypothesis, that there is an F -equivariant equivalence

$$DE \simeq S^{-\mathfrak{g}} \wedge \mathbf{E}$$

where $S^{\mathfrak{g}}$ is the $\mathbb{G}$ -sphere obtained from the adjoint representation. We will assume that the F action on $S^{\mathfrak{g}}$ satisfies the hypotheses of Proposition 9.8, so that we have a finite dimensional real representation V of F and an F -equivariant equivalence of p -complete F -spheres $S^V \simeq S^{\mathfrak{g}}$. The project then is to calculate the homotopy type of $(S^{-V} \wedge \mathbf{E})^{hF}$.

We begin with some nomenclature. If F is a finite group, let $RO(F)$ be the real representation ring of F . If $V \in RO(F)$, we write S^V for the stable one-point compactification of V as an F -sphere, $|V|$ for the dimension of V , and $S^{|V|}$ for the stable sphere of dimension $|V|$. Then $S^{|V|}$ is the underlying non-equivariant spectrum of S^V .

Remark 12.1 (Picard spectra) We begin with such basic theory as we need. A good summary and references to the classical literature can be found in Section 2 of [58]. We work in the category of local spectra for some ambient homology theory, but leave this assumption implicit.

Let R be an E_{∞} -ring spectrum. The space $\mathrm{Gl}_1(R) \subseteq \Omega^{\infty}R$ is defined by the pull-back diagram

$$\begin{array}{ccc} \mathrm{Gl}_1(R) & \longrightarrow & \Omega^{\infty}R \\ \downarrow & & \downarrow \\ (\pi_0 R)^{\times} & \longrightarrow & \pi_0 \Omega^{\infty}R = \pi_0 R. \end{array}$$

The E_{∞} -multiplication on R gives $\mathrm{Gl}_1(R)$ the structure of an infinite loop space.

Let $\mathrm{Pic}(R)$ be the category of invertible R -modules and R -module equivalences; in a slight abuse of notation we will also write $\mathrm{Pic}(R)$ for the nerve of the category. Then

$$\pi_0 \mathcal{P}\mathrm{ic}(R) \cong \mathrm{Pic}(R)$$

where $\mathrm{Pic}(R)$ is now the group of invertible R -module spectra up to equivalence. The space $\mathcal{P}\mathrm{ic}(R)$ is an infinite loop space; indeed, there is a connective spectrum $\mathrm{pic}(R)$ with

$$\Omega^\infty \mathrm{pic}(R) \simeq \mathcal{P}\mathrm{ic}(R) \simeq \mathrm{Pic}(R) \times B\mathrm{GL}_1(R).$$

Remark 12.2 (Group actions on Picard spectra) Let R be an algebra in G -spectra over the terminal N_∞ -operad (as in [12]). In particular, R has all multiplicative norms. We assume that R is cofree, so that $R \rightarrow F(EG_+, R)$ is an equivalence in G -spectra. We also assume that the natural map $R^{hG} \rightarrow R$ is a faithful Galois extension in the sense of Rognes [61]. With appropriate care and using [12, 13, 41], in particular, [12, Theorem 6.23], we may assume these assumptions hold for $\mathbb{E}$ with its action of a finite subgroup of $\mathbb{G}$; see Remark 2.1 of [5].

Let $H \subseteq G$ be a subgroup. Define $\mathcal{P}\mathrm{ic}_H(R)$ to be the category of invertible R -modules P with a compatible H -action; that is, the module multiplication map $R \wedge P \rightarrow P$ is an H -map, where H acts diagonally on the smash product. Let $\mathrm{pic}_H(R)$ be the associated spectrum. There is a functor $\mathcal{P}\mathrm{ic}(R^{hH}) \rightarrow \mathcal{P}\mathrm{ic}_H(R)$ sending Q to $R \wedge_{R^{hH}} Q$. Under our assumptions, this is an equivalence of categories. Hence $\mathrm{Pic}(R^{hH}) = \mathrm{Pic}_H(R)$ or, more generally, there is an equivalence of spectra

$$\mathrm{pic}(R^{hH}) \simeq \mathrm{pic}_H(R). \quad (12.3)$$

See Proposition 3.1 of [5].

If $K \subseteq H$ is a subgroup, there is a restriction map $\mathcal{P}\mathrm{ic}_H(R) \rightarrow \mathcal{P}\mathrm{ic}_K(R)$. There is also a transfer $\mathrm{tr} : \mathcal{P}\mathrm{ic}_K(R) \rightarrow \mathcal{P}\mathrm{ic}_H(R)$ defined using the Hill-Hopkins-Ravenel norm functor N_K^H . The assignment

$$G/H \mapsto \mathrm{Pic}_H(R) = \pi_0 \mathrm{pic}_H(R)$$

is then a Mackey functor, which we write $\underline{\mathrm{Pic}}(R)$, with G understood. See Corollary 3.12 of [5]. All of this uses technology developed in [13].

Remark 12.4 (Homotopy fixed point spectral sequences) Let R and G be as in Remark 12.2. Then G acts on the category of invertible R -modules: if $g \in G$, and P is an invertible R -module spectrum, then

$$gP = R \wedge_R^g P$$

where we have extended scalars along the map $g: R \rightarrow R$. If $P \in \mathcal{P}\mathrm{ic}_G(R)$, then multiplication by g defines an R -module equivalence

$$R \wedge_R^g P \xrightarrow{1 \wedge g} R \wedge_R P \xrightarrow{\simeq} P.$$

This yields a map of spectra

$$\mathrm{pic}_G(R) \longrightarrow \mathrm{pic}(R)^{hG} \quad (12.5)$$

which is an equivalence on (-1) -connected covers. See Theorem 3.3.1 of [58] and the further references there. In particular we have isomorphisms

$$\mathrm{Pic}(R^{hG}) \cong \mathrm{Pic}_G(R) \cong \pi_0 \mathrm{pic}_G(R) \cong \pi_0 \mathrm{pic}(R)^{hG}.$$

There is then a homotopy fixed point spectral sequence

$$E_2^{s,t}(R, G) = H^s(G, \pi_t \mathrm{pic}(R)) \implies \pi_{t-s} \mathrm{pic}(R)^{hG}. \quad (12.6)$$

Remark 12.7 (The J -homomorphism) Let R and G be as in Remark 12.2 and $H \subseteq G$ a subgroup. If V is a real H -representation, then $R \wedge S^V \in \mathrm{Pic}_H(R)$, where we give $R \wedge S^V$ the diagonal H action. As in Proposition 3.13 of [5] this extends to a morphism of Mackey functors

$$J_R: \underline{RO} \rightarrow \underline{\mathrm{Pic}}(R)$$

where $\underline{RO}$ is the Mackey functor $G/H \mapsto RO(H)$.

In fact, more is true. If we let $\mathrm{Rep}(H)$ be the category of real representations and isomorphisms; this is a symmetric monoidal category under direct sum. Then we have a symmetric monoidal functor $J_R^H: \mathrm{Rep}(H) \rightarrow \mathcal{P}\mathrm{ic}_H(R)$ and hence a map of spectra $\mathrm{ko}_H \rightarrow \mathrm{pic}_H(R)$. Here ko_H is the spectrum of H -equivariant real K -theory. This is surely part of a morphism between spectral Mackey functors in sense of [3], but we won't need that much structure. We will use that we have a commutative diagram for all $H \subseteq G$

$$\begin{array}{ccc} \mathrm{ko}_H & \longrightarrow & \mathrm{pic}_H(R) \\ \downarrow & & \downarrow \\ \mathrm{ko}^{hH} & \longrightarrow & \mathrm{pic}(R)^{hH}. \end{array}$$

The action of H on ko is trivial, so there is a weak equivalence

$$F(\Sigma_+^\infty BH, \mathrm{ko}) \simeq \mathrm{ko}^{hH}$$

and the homotopy fixed point spectral sequence for ko^{hH} is the Atiyah–Hirzebruch Spectral Sequence for ko^*BH .

Proposition 12.8 *Let R be as in Remark 12.2.*

- (1) *Let $f: S^V \rightarrow R$ be a G -equivariant map so that the underlying map of spectra $S^{|V|} \rightarrow R$ is a unit in π_*R . Then $J_R^G(V) = R^{hG} \in \mathrm{Pic}(R^{hG})$.*
- (2) *Let $K \subset H$ be a subgroup. If $V \in RO(K)$ is in the kernel of J_R^K then $W = \mathrm{ind}_K^G V$ is in the kernel of J_R^G .*

Proof For part (1) we extend f to a map $g: R \wedge S^V \rightarrow R$ of G -equivariant R -modules. This is an underlying equivalence by our assumption on f , and so a G -equivalence since R is cofree. (Note that if R is cofree, so is the R -module $R \wedge S^V$). The claim follows.

Part (2) follows from the fact that $J_R: RO \rightarrow \underline{\mathrm{Pic}}(R)$ is a morphism of Mackey functors and induction of representations is the transfer in RO . $\square$

When $R = \mathbf{E}$, the Lubin–Tate spectrum and G is a finite subgroup of the Morava stabilizer group, the fixed point spectral sequence of (12.6) vanishes in high degree at E_∞ . This inspires the following result. If X is a spectrum, let $f: X\langle n \rangle \rightarrow X$ denote the $(n-1)$ -connected cover of X ; thus $\pi_k f$ is an isomorphism for $k \geq n$ and $\pi_k X\langle n \rangle = 0$ if $k < n$.

Proposition 12.9 *Let R be as in Remark 12.2. Let ℓ be an integer such that $E_\infty^{s,s}(R) = 0$ for $s \geq \ell$ in the homotopy fixed point spectral sequence*

$$E_2^{s,t}(R) = H^s(G, \pi_t \mathrm{pic}(R)) \Longrightarrow \pi_{t-s} \mathrm{pic}(R)^{hG}$$

Then the composite mapping

$$[\Sigma_+^\infty BG, \mathrm{ko}\langle \ell \rangle] \rightarrow [\Sigma_+^\infty BG, \mathrm{ko}] \rightarrow \pi_0 \mathrm{pic}(R)^{hG}$$

is zero.

Proof The G -equivariant maps $\mathrm{ko}\langle \ell \rangle \rightarrow \mathrm{ko} \rightarrow \mathrm{pic}(R)$ induces a diagram of homotopy fixed point spectral sequences

$$\begin{array}{ccc} E_2^{s,t}(\mathrm{ko}\langle \ell \rangle) = H^s(G, \pi_t \mathrm{ko}\langle \ell \rangle) & \Longrightarrow & \pi_{t-s} F(\Sigma_+^\infty BG, \mathrm{ko}\langle \ell \rangle) \\ \downarrow & & \downarrow \\ E_2^{s,t}(\mathrm{ko}) = H^s(G, \pi_t \mathrm{ko}) & \Longrightarrow & \pi_{t-s} F(\Sigma_+^\infty BG, \mathrm{ko}) \\ \downarrow & & \downarrow \\ E_2^{s,t}(R) = H^s(G, \pi_t \mathrm{pic} R) & \Longrightarrow & \pi_{t-s} (\mathrm{pic} R)^{hG}. \end{array}$$

By construction $E_2^{s,s}(\mathrm{ko}(\ell)) = 0$ if $s < \ell$ and by hypothesis $E_\infty^{s,s}(R) = 0$ is $s \geq \ell$. The result follows. $\square$

Remark 12.10 The question now is how to check the hypotheses of Proposition 12.9. One technique, which we will employ below, is to access the ideas and techniques of [58] to relate the differentials in the homotopy fixed point spectral sequence for $\mathrm{pic}(R)$ to the differentials in the homotopy fixed point spectral sequence for R itself.

12.2 Applications to the Lubin–Tate spectrum

We now consider the case where $R = \mathbf{E}$ for some prime p and some height n formal group. We established conventions and notation at the beginning of Sect. 10. Let $F \subseteq \mathbb{G}$ be a finite subgroup. We are interested in calculating the image of specific representations under the map $J_{\mathbf{E}}^F : RO(F) \rightarrow \mathrm{Pic}(\mathbf{E}^{hF})$. We have the following useful preliminary result.

Proposition 12.11 *Let $F \subset \mathbb{S}$ be a finite subgroup that contains the central subgroup $C_2 = \{\pm 1\}$. The regular representation $\rho_F \in RO(F)$ maps to the trivial element of $\mathrm{Pic}(\mathbf{E}^{hF})$ under $J_{\mathbf{E}}^F$.*

Proof As describe in Remark 12.2, we can replace $\mathbf{E}$ by a cofree genuine N_∞ ring G -spectrum. In particular, it admits all norm maps. The key for this argument is to refine a unit $x \in \pi_2 i_e^* \mathbf{E}$ to an equivariant map $\bar{x} : S^{\rho_2} \rightarrow i_{C_2}^* \mathbf{E}$ where $i_H^* \mathbf{E}$ is the restriction of $\mathbf{E}$ to an H -spectrum. With this established, one applies Proposition 12.8.

At the prime 2 the question is very subtle, but has been accomplished in [46].

If p is odd this is much easier. Since $\mathbf{E}$ is p -local and p is odd, we have that

$$\pi_{\rho_2}^{C_2} i_e^* \mathbf{E} \cong \pi_0(S^{-\rho_2} \wedge \mathbf{E})^{hC_2} \cong (\pi_0 i_e^*(S^{-\rho_2} \wedge \mathbf{E}))^{C_2} \cong (\pi_2 i_{C_2}^*(S^{1-\sigma} \wedge \mathbf{E}))^{C_2}.$$

As a C_2 -module,

$$\pi_* i_e^*(S^{1-\sigma} \wedge \mathbf{E}) \cong \pi_* i_e^* \mathbf{E} \otimes \mathbb{Z}(-1)$$

where $\mathbb{Z}(-1)$ is the sign representation of C_2 on $\mathbb{Z}$. Choose a complex orientation $x \in \pi_2 i_e^* \mathbf{E}$, and note that $\gamma x = -x$ for γ a generator of C_2 . Therefore, $x \otimes 1$ gives an element of $(\pi_2 i_e^* \mathbf{E} \otimes \mathbb{Z}(-1))^{C_2}$ which corresponds to a C_2 -equivariant map $\bar{x} : S^{\rho_2} \rightarrow i_{C_2}^* \mathbf{E}$ that refines x . $\square$

Remark 12.12 We can now outline the general strategy we use below. Fix a finite subgroup $F \subseteq \mathbb{G}$ with p -Sylow subgroup F_0 . Suppose F contains the central C_2 and that $F_0/F_0 \cap Z(\mathbb{G})$ is an elementary abelian p -group, so that

Theorem 6.25 applies. Finally suppose we have identified a finite dimensional real representation V of F so we can apply Proposition 9.8 and write $S^{\mathfrak{g}} \cong S^V$ as an F -equivariant sphere.

It follows from [27] that the homotopy fixed point spectral sequence

$$H^s(F, \mathbf{E}_t) \implies \pi_{t-s} \mathbf{E}^{hF}$$

has a horizontal vanishing line at E_{∞} . Applying the ideas from [58] one can then conclude that the spectral sequence (12.6)

$$E_2^{s,t}(\mathbf{E}) = H^s(F, \pi_t \text{pic}(\mathbf{E})) \implies \pi_{t-s} \text{pic}(\mathbf{E})^{hF}$$

has the property that $E_{\infty}^{s,s} = 0$ for large s . So we can apply Proposition 12.9; that is, there will be an integer ℓ so that the composition

$$[\Sigma_+^{\infty} BF, \text{ko}\langle \ell \rangle] \rightarrow [\Sigma_+^{\infty} BF, \text{ko}] \rightarrow \pi_0 \text{pic}(\mathbf{E})^{hF} = \pi_0 \text{pic}(\mathbf{E}^{hF})$$

is zero. In specific examples, we can be very explicit about the integer ℓ . For example, if $n = 2$ and $p = 2$ and F is the automorphism groups of our supersingular curve, then we take $\ell = 8$. See [58], especially Figure 9 and the surrounding narrative.

Now let

$$I\langle \ell \rangle = RO(F) \cap \text{Im}\{[\Sigma_+^{\infty} BF, \text{ko}\langle \ell \rangle] \rightarrow [\Sigma_+^{\infty} BF, \text{ko}]\}.$$

Since $C_2 \subseteq F$ Proposition 12.11 implies that we have a map

$$RO(F)/I\langle \ell \rangle + \rho_F \rightarrow \text{Pic}(\mathbf{E}^{hF}).$$

The source here is the quotient group of $RO(F)$ by the subgroups generated by $I\langle \ell \rangle$ and the regular representation ρ_F . Since F is a finite group,

$$\text{Im}\{[\Sigma_+^{\infty} BF, \text{ko}\langle \ell \rangle] \rightarrow [\Sigma_+^{\infty} BF, \text{ko}]\} \subseteq [\Sigma_+^{\infty} BF, \text{ko}]$$

is of finite index, so $RO(F)/I\langle \ell \rangle + \rho_F$ is a finite abelian group. Our project is then to calculate the image of a given representation W in this group as well as its image in $\text{Pic}(\mathbf{E}^{hF})$.

Remark 12.13 We will use classical characteristic class arguments to calculate $RO(F)/I\langle \ell \rangle$. Let $P_n X$ denote the n th Postnikov section of X . Then there is an injection (which is often an isomorphism)

$$RO(F)/I\langle \ell \rangle \longrightarrow [\Sigma_+^{\infty} BF, P_{\ell-1} \text{ko}] \cong [BF, \mathbb{Z} \times P_{\ell-1} BO].$$

For example, suppose $\ell = 8$. We have a tower of fibrations

$$\begin{array}{ccccccc} BO\langle 8 \rangle & \longrightarrow & BSpin & \longrightarrow & BSO & \longrightarrow & BO \\ & & \lambda \downarrow & & \downarrow w_2 & & \downarrow w_1 \\ & & K(\mathbb{Z}, 4) & & K(\mathbb{Z}/2, 2) & & K(\mathbb{Z}/2, 1) \end{array}$$

with w_1 and w_2 the first and second Stiefel-Whitney classes of the universal bundles and $\lambda \in H^4(BSpin, \mathbb{Z})$ a class so that that 2λ is the first Pontrjagin class. In particular, $P_7BO \simeq P_7BSpin$ is a three stage Postnikov tower. Thus, we have a filtration of $RO(F)/I\langle 8 \rangle$

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_4 & \xrightarrow{\subseteq} & A_2 & \xrightarrow{\subseteq} & A_1 & \xrightarrow{\subseteq} & RO(F)/I\langle 8 \rangle \\ & & \lambda \downarrow & & \downarrow w_2 & & \downarrow w_1 & & \downarrow \dim \\ & & H^4(BF, \mathbb{Z}) & & H^2(BF, \mathbb{Z}/2) & & H^1(BF, \mathbb{Z}/2) & & \mathbb{Z} \end{array}$$

where $\dim$ assigns to any virtual representation its rank. If the Atiyah-Hirzebruch Spectral Sequence for $ko^*(\Sigma_+^\infty BF)$ collapses, the vertical maps in this filtration will be surjective.

The class λ lies outside the standard list of characteristic classes, but we do have the following result. Let c_i denote the Chern classes.

Lemma 12.14 *Let ξ be a stable complex bundle over X with the property that $c_1(\xi) \equiv 0 \in H^2(X, \mathbb{F}_2)$. Then we can choose a $Spin$ structure on ξ and a fixed choice of $Spin$ structure determines a class $d(\xi) \in H^2(X, \mathbb{Z})$ with the property that $2d(\xi) = c_1(\xi)$. For this $Spin$ structure on ξ we have*

$$\lambda(\xi) = d(\xi)c_1(\xi) - c_2(\xi).$$

Furthermore, for such bundles, the characteristic class λ is additive; that is,

$$\lambda(\xi_1 \oplus \xi_2) = \lambda(\xi_1) + \lambda(\xi_2).$$

Proof The first statement follows from examining what happens in integral cohomology in the fiber sequence

$$B \longrightarrow BU \xrightarrow{c_1} K(\mathbb{Z}/2, 2).$$

We can obtain the equation for λ by studying the universal example ξ_0 over B . Note that $H^4(B, \mathbb{Z}) \cong \mathbb{Z}^2$ generated by $d(\xi_0)^2$ and $c_2(\xi_0)$. Now use that if ξ is any complex bundle, then

$$p_1(\xi) = -c_2(\xi \otimes \mathbb{C}) = -c_2(\xi \oplus \bar{\xi}) = c_1(\xi)^2 - 2c_2(\xi)$$

where $\bar{\xi}$ is ξ with its conjugate complex structure. Additivity follows from this same formula by considering the universal example over $B \times B$. $\square$

Remark 12.15 There is a p -complete variant on the constructions of Remark 12.12. The unit map $S^0 \rightarrow \mathbf{E}$ extends to a unit map $S_p^0 \rightarrow \mathbf{E}$, so the map $F(\Sigma_+^\infty BF, \text{ko}) \rightarrow \text{pic}(\mathbf{E}^{hF})$ factors as a map

$$F(\Sigma_+^\infty BF, \text{ko}) \rightarrow F(\Sigma_+^\infty BF, \text{pic}(S_p^0)) \rightarrow \text{pic}(\mathbf{E}^{hF}).$$

Note that by combining (12.3) and (12.5) we have that the natural map

$$\text{pic}(\mathbf{E}^{hF}) \rightarrow \text{pic}(\mathbf{E})^{hF}$$

induces an equivalence on (-1) -connected covers. If X is any spectrum, define $L_p^{\geq 2} X$ by the homotopy push-out diagram

$$\begin{array}{ccc} X\langle 2 \rangle & \longrightarrow & X\langle 2 \rangle_p \\ \downarrow & & \downarrow \\ X & \longrightarrow & L_p^{\geq 2} X \end{array}$$

where Y_p is the p -completion of Y . Note that $L_p^{\geq 2} X$ has homotopy groups

$$\pi_t L_p^{\geq 2} X \cong \begin{cases} \pi_t X & t = 0, 1 \\ (\pi_t X)_p & t \geq 2. \end{cases}$$

Since the homotopy groups of $\text{pic}(S_p^0)$ are p -complete above dimension 1, then $L_p^{\geq 2} \text{pic}(S_p^0) \simeq \text{pic}(S_p^0)$ and the map $\text{ko} \rightarrow \text{pic}(S_p^0)$ factors through a map $L_p^{\geq 2} \text{ko} \rightarrow \text{pic}(S_p^0)$. The direct analog of Proposition 12.9 is still true, with the same proof, and with ko and $\text{ko}\langle \ell \rangle$ replaced by $L_p^{\geq 2} \text{ko}$ and $L_p^{\geq 2} \text{ko}\langle \ell \rangle$ as needed. Thus there will be an integer ℓ so that the composition

$$[\Sigma_+^\infty BF, L_p^{\geq 2} \text{ko}\langle \ell \rangle] \rightarrow [\Sigma_+^\infty BF, L_p^{\geq 2} \text{ko}] \rightarrow \pi_0 \text{pic}(\mathbf{E})^{hF} = \pi_0 \text{pic}(\mathbf{E}^{hF})$$

is zero. Indeed, we can choose the same integer ℓ as in the uncompleted case. In our examples, we will have $\ell \geq 2$ and, in that case, $L_p^{\geq 2} \text{ko}\langle \ell \rangle = \text{ko}\langle \ell \rangle_p$.

If we let

$$I_p\langle \ell \rangle = RO(F) \cap \text{Im}\{[\Sigma_+^\infty BF, L_p^{\geq 2} \text{ko}\langle \ell \rangle] \rightarrow [\Sigma_+^\infty BF, L_p^{\geq 2} \text{ko}]\},$$

then our variant of Proposition 12.9 gives a map

$$RO(F)/(I_p\langle\ell\rangle + \rho_F) \rightarrow \pi_0\mathrm{pic}(E^{hF})$$

and we have an injection

$$RO(F)/I_p\langle\ell\rangle \longrightarrow [\Sigma_+^\infty BF, P_{\ell-1}L_p^{\geq 2}\mathrm{ko}].$$

We again get a filtration of $RO(F)/I_p\langle\ell\rangle$. For example, if $\ell = 8$ and $p = 2$ we have only a slight change:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_4 & \xrightarrow{\subseteq} & A_2 & \xrightarrow{\subseteq} & A_1 \xrightarrow{\subseteq} RO(F)/I_2\langle 8 \rangle \\ & & \downarrow \lambda & & \downarrow w_2 & & \downarrow w_1 \\ & & H^4(BF, \mathbb{Z}_2) & & H^2(BF, \mathbb{Z}/2) & & H^1(BF, \mathbb{Z}/2) \end{array}$$

$\downarrow \dim$
 $\mathbb{Z}$

where now $H^4(BF, \mathbb{Z}_2) = \lim H^4(BF, \mathbb{Z}/2^n)$. If $p > 2$ and $\ell = 8$ we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_4 & \xrightarrow{\subseteq} & A_1 & \xrightarrow{\subseteq} & RO(F)/I_p\langle 8 \rangle \\ & & \downarrow \lambda & & \downarrow w_1 & & \downarrow \dim \\ & & H^4(BF, \mathbb{Z}_p) & & H^1(BF, \mathbb{Z}/2) & & \mathbb{Z} \end{array}$$

In all of these examples, the vertical maps will be surjective if the Atiyah-Hirzebruch Spectral Sequence for $(L_p^{\geq 2}\mathrm{ko})^*(BF)$ collapses.

13 The Spanier-Whitehead duals of E^{hF} : examples from elliptic curves

We now focus our attention at height $n = 2$ and the primes $p = 2$ and $p = 3$. In both cases we take a formal group of height 2 obtained from a supersingular elliptic curve. We wish to give a concrete calculation of the Spanier-Whitehead dual $D(E^{hF})$ where F is a finite subgroup of $\mathbb{G}_2$. Our main interest is when F is actually the automorphisms of the chosen elliptic curve. The results are in Theorem 13.12 and Theorem 13.25.

At either prime, the basic case will be when $F \subseteq \mathbb{S}_2 = \mathcal{O}_2^\times$ is a subgroup containing a maximal finite p -torsion subgroup F_0 . At $p = 2$, F_0 is isomorphic to the quaternion group of order 8 and at $p = 3$ we have F_0 is cyclic of order 3. In both cases Corollary 11.18 applies and we have an F -equivariant $\mathbf{K}$ -equivalence $D\mathbf{E} = S^{-\mathfrak{g}} \wedge \mathbf{E}$ where $S^{\mathfrak{g}}$ is the linear dualizing sphere. By Tate vanishing [36] we know that

$$D(E^{hF}) \simeq D(\mathbf{E})^{hF} \simeq (S^{-\mathfrak{g}} \wedge \mathbf{E})^{hF}.$$

We will be able to use Proposition 9.8 to write S^g as the p -completion of a representation sphere S^V and hence we have

$$D(\mathbf{E}^{hF}) \simeq D(\mathbf{E})^{hF} \simeq (S^{-V} \wedge \mathbf{E})^{hF}. \quad (13.1)$$

The strategy then developed in Remark 12.15 applies to complete the calculation. In our examples, the subgroups F are such that $\text{Pic}(\mathbf{E}^{hF})$ is cyclic generated by $\Sigma \mathbf{E}^{hF}$; therefore, there is an integer k so that

$$(S^{-V} \wedge \mathbf{E})^{hF} \simeq \Sigma^k \mathbf{E}^{hF}.$$

In both cases we will prove $k = 44$.

We can extend these results to subgroups of the larger Morava stabilizer group $\mathbb{G}_2 = \mathbb{S}_2 \rtimes \text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$. At both the prime 2 and 3 there is a finite subgroup $G \subseteq \mathbb{G}_2$ so that

$$\mathbf{E}^{hG} \simeq L_{\mathbf{K}} \mathbf{tmf}$$

where $\mathbf{tmf}$ is the Hopkins-Miller spectrum of topological modular forms. We will then have

$$D(L_{\mathbf{K}} \mathbf{tmf}) \simeq \Sigma^{44} L_{\mathbf{K}} \mathbf{tmf}.$$

This recovers results of Behrens [9] and Bobkova [14] at $p = 3$ and $p = 2$ respectively.

13.1 The case $n = 2$ and $p = 3$

We first consider the case $p = 3$. There is a supersingular elliptic curve C with Weierstrass equation

$$y^2 = x^3 - x. \quad (13.2)$$

While defined over $\mathbb{F}_3$, we work over $\mathbb{F}_9$, and the formal group F_C of this curve is a formal group of height 2 over that field. Because C is supersingular, the endomorphism ring $\mathcal{E}$ of C over $\mathbb{F}_9$ is a maximal order in a quaternion algebra ramified only at 3 and ∞ . The completion of $\mathcal{E}$ at $p = 3$ is the endomorphism ring $\mathcal{O}_2$ of F_C . Thus $\mathcal{E} \subseteq \mathcal{O}_2$ is a lattice and $\mathcal{E}/3\mathcal{E} \cong \mathcal{O}_2/3\mathcal{O}_2$.

Let $i \in \mathbb{F}_9$ be a fourth root of unity, so $i^2 = -1$. This induces an automorphism

$$(x, y) \mapsto (i^2 x, i^3 y) = (-x, -iy)$$

of C which we will also call i . There are two automorphisms of exact order 3 given by

$$(x, y) \mapsto (x \pm 1, y).$$

We will fix one in a moment, after giving a bit more of the structure of $\mathcal{E}$.

The Frobenius ϕ given by

$$(x, y) \mapsto (x^3, y^3)$$

defines an endomorphism of C as well. Since C has four points over $\mathbb{F}_3$ (including the point at ∞) we have that

$$\phi^2 = -3$$

as an endomorphism of C . See Theorem 4.10 of [66], for example. Note that $\phi i = -i\phi$. The element of order 3 in the automorphism group of C can be chosen to be

$$\sigma = -\frac{1}{2}(1 + \phi).$$

In fact, if σ is any element of exact order 3 then $\sigma^2 + \sigma + 1 = 0$, so $(1 + 2\sigma)^2 = -3$. Thus $1 + 2\sigma = \pm\phi$. The automorphisms of C are generated by i and σ and the subgroup C_3 of the automorphism group generated by σ is normal. The group

$$G_{12} = \text{Aut}(C) \cong C_3 \rtimes C_4 \subseteq \mathcal{E}^\times \subseteq \mathcal{O}_2^\times$$

defines a maximal finite subgroup of $\mathcal{O}_2$ which contains 3 torsion.

For our calculations we will follow the outline of Remark 12.15, and we will use the notation established there. Let w_1 be the first Stiefel-Whitney class and λ the characteristic class for spin bundles discussed in Lemma 12.14; recall that 2λ is the first Pontrjagin class.

The inclusion $C_4 \rightarrow G_{12}$ induces an isomorphism

$$E(x) \otimes \mathbb{F}_2[y] \cong H^*(C_4, \mathbb{F}_2) \cong H^*(G_{12}, \mathbb{F}_2)$$

where x is in degree 1 and y is the second-order Bockstein on x . In addition, the inclusion $C_3 \rightarrow G_{12}$ defines an isomorphism

$$\mathbb{Z}_3[z]/(3z) \cong H^*(G_{12}, \mathbb{Z}_3) \cong H^*(C_3, \mathbb{Z}_3)^{C_4}$$

where we chose z in degree 4 to be the square of either generator of $H^2(C_3, \mathbb{Z}_3)$.

Thus to calculate the characteristic classes w_1 and λ of representations of G_{12} we can restrict those representations to the subgroups C_3 and C_4 . Note that every non-trivial irreducible real representation of C_3 is the restriction of a complex representation and, hence, Lemma 12.14 can be used to calculate λ .

Remark 13.3 We now calculate characteristic classes of the needed real representations of C_3 , C_4 , and G_{12} .

We begin with the regular representation $\rho_{G_{12}}$. Restricted to C_4 we have an isomorphism

$$\rho_{G_{12}} \cong 3\rho_{C_4} = 3(\mathbf{1}_{\mathbb{R}} + \sigma_{\mathbb{R}} + \gamma_4)$$

where $\sigma_{\mathbb{R}}$ is the real sign representation and γ_4 is the 2-dimensional real representation given by rotation by 90 degrees. The latter is the restriction of a complex representation so $w_1(\gamma_4) = 0$ and we have

$$w_1(\rho_{G_{12}}) = x \in H^1(G_{12}, \mathbb{F}_2). \quad (13.4)$$

Restricted to C_3 we have an isomorphism

$$\rho_{G_{12}} \cong 4\rho_{C_3} \cong 4(\mathbf{1}_{\mathbb{R}} + \gamma_3)$$

where γ_3 is the unique non-trivial 2-dimensional real representation of C_3 . This is the restriction of a one-dimensional complex representation with non-zero first Chern class in $H^2(C_3, \mathbb{Z}) \cong \mathbb{Z}/3$. Hence by Lemma 12.14 we have

$$\lambda(\rho_{G_{12}}) = -z^2 \in H^4(C_3, \mathbb{Z}_3) \cong \mathbb{Z}/3. \quad (13.5)$$

Next we examine the conjugation action of G_{12} on $\mathcal{E}$, the endomorphism ring of C . Let

$$\mathcal{E}_0 = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}\phi \oplus \mathbb{Z}i\phi \subseteq \mathcal{E}.$$

This inclusion is not equality, as $\sigma \notin \mathcal{E}_0$, but since $\mathcal{E}$ is of rank 4 over $\mathbb{Z}$ we have that $\mathbb{R} \otimes \mathcal{E}_0 = \mathbb{R} \otimes \mathcal{E}$ and we can use the conjugation action of G_{12} on $\mathcal{E}_0$ to determine the real representation $\mathbb{R} \otimes \mathcal{E}$.

Restricted to $C_4 \subseteq G_{12}$, the subgroup generated by i , there is an isomorphism of C_4 -representations

$$\mathbb{R} \otimes \mathcal{E} \cong \mathbf{1}_{\mathbb{C}} \oplus \sigma_{\mathbb{C}}$$

where $\sigma_{\mathbb{C}}$ is the complex sign representation. Thus

$$w_1(\mathbb{R} \otimes \mathcal{E}) = 0. \quad (13.6)$$

If we restrict to the subgroup $C_3 \subseteq G_{12}$ generated σ then there is an isomorphism of C_3 -representations

$$\mathbb{R} \otimes \mathcal{E} \cong \mathbf{1}_{\mathbb{C}} \oplus \gamma_3$$

where $\mathbf{1}_{\mathbb{C}}$ is the trivial 2-dimensional real representation and γ_3 is the unique non-trivial 2-dimensional real representation. Both are restrictions of complex representations. Thus, again using that γ_3 has non-trivial first Chern class and using Lemma 12.14, we have

$$\lambda(\mathbb{R} \otimes \mathcal{E}) = -z^2 \in H^3(C_3, \mathbb{Z}_3) \cong \mathbb{Z}/3. \quad (13.7)$$

Proposition 13.8 (1) *Let $G_{12} \subseteq \mathbb{S}_2$ be the automorphism group of the super-singular elliptic curve $y^2 = x^3 - x$ over $\mathbb{F}_9$. The composite mapping*

$$[\Sigma_+^\infty BG_{12}, L_3^{\geq 2} \mathrm{ko}\langle 8 \rangle] \rightarrow [\Sigma_+^\infty BG_{12}, L_3^{\geq 2} \mathrm{ko}] \rightarrow \pi_0 \mathrm{pic}(\mathbf{E})^{hG_{12}}$$

is zero.

(2) *There is an isomorphism*

$$\psi: RO(G_{12})/I_3\langle 8 \rangle \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$$

sending a representation V to $(\dim(V), a, b)$ with

$$w_1(V) = ax \in H^1(C_4, \mathbb{F}_2)$$

$$\lambda(V) = bz \in H^4(C_3, \mathbb{Z}_3).$$

Proof For Part (1) we use (the evident variant of) Proposition 12.9. See Remark 12.15. By Theorem 8.1.3 (see also Figure 6) of [58] we have that in the spectral sequence

$$H^s(G_{12}, \pi_t \mathrm{pic}(\mathbf{E})) \Longrightarrow \pi_{t-s} \mathrm{pic}(\mathbf{E})^{hG_{12}}$$

$E_\infty^{s,s} = 0$ for $s \geq 6$. Note also that $\mathrm{ko}\langle 8 \rangle = \mathrm{ko}\langle 6 \rangle$.

Part (2) follows from the filtration of $RO(G_{12})/I_3\langle 8 \rangle$ given in Remark 12.15. Note that the composite mappings

$$RO(G_{12}) \longrightarrow RO(C_4) \xrightarrow{w_1} H^1(C_4, \mathbb{Z}/2) \cong \mathbb{Z}/2$$

$$RO(G_{12}) \longrightarrow RO(C_3) \xrightarrow{\lambda} H^1(C_3, \mathbb{Z}_3) \cong \mathbb{Z}/3$$

are both onto by (13.6) and (13.7). □

We now give a more specific calculation. Let ψ be the map of part (2) of Proposition 13.8.

Proposition 13.9 (1) *If $\rho_{G_{12}} \in RO(G_{12})$ is the regular representation, then*

$$\psi(\rho_{G_{12}}) = (12, 1, -1).$$

(2) *The group $RO(G_{12})/(I_3\langle 8 \rangle + \rho_{G_{12}})$ is generated by the trivial 1-dimensional real representation; this choice of generator determines an isomorphism*

$$\mathbb{Z}/72 \cong RO(G_{12})/(I_3\langle 8 \rangle + \rho_{G_{12}}). \quad (13.10)$$

Furthermore, the J -homomorphism

$$J_{\mathbf{E}}^{G_{12}}: RO(G_{12})/(I_3\langle 8 \rangle + \rho_{G_{12}}) \longrightarrow \text{Pic}(\mathbf{E}^{hG_{12}})$$

is an isomorphism.

Proof Part (1) follows from (13.4) and (13.5). The isomorphism (13.10) is then immediate. The fact the J -homomorphism is an isomorphism then follows from the fact that $\mathbf{E}^{hG_{12}}$ has periodicity 72; that is $\text{Pic}(\mathbf{E}^{hG_{12}}) \cong \mathbb{Z}/72$ generated by $\Sigma \mathbf{E}^{hG_{12}}$. See [58]. $\square$

Proposition 13.11 *Let $\mathcal{E}$ be the endomorphism ring of C and give $\mathbb{R} \otimes \mathcal{E}$ the conjugation action by G_{12} . Then*

$$\mathbb{R} \otimes \mathcal{E} = -44 \cdot \mathbf{1}_{\mathbb{R}}$$

in $RO(G_{12})/(I_3\langle 8 \rangle + \rho_{G_{12}})$.

Proof By (13.6) and (13.7) we have

$$\psi(\mathbb{R} \otimes \mathcal{E}) = (4, 0, -1) \in \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3.$$

By Proposition 13.9, part (1) we have $\psi(\rho_{G_{12}}) = (12, 1, -1)$. Since

$$(4, 0, -1) - 4(12, 1, -1) = (-44, 0, 0) \in \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$$

the result follows. $\square$

We now have a calculation of the Spanier–Whitehead duals to $\mathbf{E}^{hF}$. Let

$$G_{24} = G_{12} \rtimes \text{Gal}(\mathbb{F}_9/\mathbb{F}_3) \subset \mathcal{O}_2 \rtimes \text{Gal}(\mathbb{F}_9/\mathbb{F}_3) = \mathbb{G}_2.$$

We have, here at $p = 3$,

$$\mathbf{E}^{hG_{24}} \simeq L_{\mathbf{K}}\mathbf{tmf}$$

where $\mathbf{tmf}$ is the spectrum of topological modular forms.

Theorem 13.12 *Let $p = 3$ and $F \subseteq G_{24} \subseteq \mathbb{G}_2$ for $\mathbb{G}_2 = \text{Aut}(\mathbb{F}_9, F_C)$ the stabilizer group associated to the formal group law F_C of a supersingular elliptic curve C with Weierstrass equation (13.2). Then*

$$D(\mathbf{E}^{hF}) \simeq \Sigma^{44}\mathbf{E}^{hF}.$$

Proof First suppose $F \subseteq G_{12}$. Then, as in (13.1), we have

$$D(\mathbf{E}^{hF}) \simeq (S^{-(\mathbb{R} \otimes \mathcal{E})} \wedge \mathbf{E})^{hF}$$

and the result follows from Proposition 13.9 and Proposition 13.11.

The other possibility is that the composition $F \rightarrow G_{24} \rightarrow \text{Gal}(\mathbb{F}_9/\mathbb{F}_3)$ is onto. Let F_0 be the kernel of this map and write Gal for $\text{Gal}(\mathbb{F}_9/\mathbb{F}_3)$. We know from [10, Lemma 1.37] that there is a $\text{Gal}(\mathbb{F}_9/\mathbb{F}_3)$ equivariant equivalence

$$\Sigma_+^\infty \text{Gal} \wedge \mathbf{E}^{hF} \simeq \mathbf{E}^{hF_0}.$$

We now have

$$\begin{aligned} D(\mathbf{E}^{hF}) &\simeq D(\mathbf{E})^{hF} \\ &\simeq [D(\mathbf{E})^{hF_0}]^{h\text{Gal}} \simeq (\Sigma^{44}E^{hF_0})^{h\text{Gal}} \\ &\simeq [\Sigma_+^\infty \text{Gal} \wedge \Sigma^{44}\mathbf{E}^{hF}]^{h\text{Gal}} \\ &\simeq \Sigma^{44}\mathbf{E}^{hF}. \end{aligned}$$

□

13.2 The case $n = 2$ and $p = 2$

We proceed exactly as in the case of $n = 2$ and $p = 3$, working with the endomorphism ring of a supersingular elliptic curve over $\mathbb{F}_4$. This is a recapitulation of ideas already laid out by the first author in [8]; see in particular Lemma 2.4.3 of that paper.

Over $\mathbb{F}_4$, there is a supersingular elliptic curve C with Weierstrass equation

$$y^2 + y = x^3. \quad (13.13)$$

The endomorphism ring $\mathcal{E}$ of C over $\mathbb{F}_4$ is a maximal order in a quaternion algebra ramified only at 2 and ∞ . It is possible to be quite explicit.

Choose $\omega \in \mathbb{F}_4$ with $\omega^2 + \omega + 1 = 0$; that is, ω is a primitive third root of unity. Then C has an automorphisms ω and i with

$$\begin{aligned}\omega(x, y) &= (\omega x, y) \\ i(x, y) &= (x + 1, y + x + \omega).\end{aligned}$$

There is a slight abuse of notation here with the symbol ω . Set $j = \omega i \omega^2$ and $k = \omega^2 i \omega$. These elements generate a normal subgroup of $\text{Aut}(C)$ isomorphic to the quaternion group Q_8 of order 8. The automorphism ω defines a cyclic subgroup $C_3 \subseteq \text{Aut}(C)$ of order 3 and there is an isomorphism

$$G_{24} = Q_8 \rtimes C_3 \cong \text{Aut}(C).$$

The group C_3 acts on Q_8 by cyclicly permuting i, j, k . The element $\omega \in \mathcal{E}$ can be written

$$\omega = -\frac{1 + i + j + k}{2}$$

and we have

$$\mathcal{E} = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}\frac{1 + i + j + k}{2}.$$

The completion of $\mathcal{E}$ at 2 is the endomorphism ring $\mathcal{O}_2$ of the formal group associated to C , which is necessarily of height 2. Then $G_{24} \subset \mathcal{O}_2$ is a choice a maximal finite subgroup containing a subgroup isomorphic to Q_8 .

Remark 13.14 Let

$$\mathbb{H} = \mathbb{R}\{1, i, j, k\}/(i^2 = j^2 = k^2 = -1, ij = k = -ji)$$

be the quaternion algebra. Up to isomorphism, this is the unique 4-dimensional associative division algebra over the real numbers. The group Q_8 is a group of units in $\mathbb{H}$ and left multiplication by Q_8 on $\mathbb{H}$ gives, up to isomorphism, the unique irreducible 4-dimensional representation of Q_8 .

We have an evident inclusion

$$\mathcal{E} \subseteq \mathbb{H}$$

which is closed under both the left action and the conjugation action by G_{24} . Furthermore

$$\mathbb{R} \otimes \mathcal{E} = \mathbb{H}.$$

Since $\mathcal{E}/2\mathcal{E} \cong \mathcal{O}_2/2\mathcal{O}_2$, we find that we are exactly in the situation of Proposition 9.8 with $V = \mathbb{H}_{ad}$ where $\mathbb{H}_{ad}$ is $\mathbb{H}$ with its conjugation action by G_{24} . Thus the main goal is to analyze the homotopy type of

$$(S^{-\mathbb{H}_{ad}} \wedge \mathbf{E})^{hG_{24}} \in \text{Pic}(\mathbf{E}^{hG_{24}}).$$

Remark 13.15 We will need to know the cohomology of Q_8 and G_{24} . We have that $Q_8/[Q_8, Q_8] \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ where we choose the residue classes of i and j as the generators. Then

$$H^*(Q_8, \mathbb{F}_2) \cong A \otimes \mathbb{F}_2[P]$$

where $P \in H^4(Q_8, \mathbb{F}_2)$ and A is the 3-dimensional Poincaré duality algebra

$$A = \mathbb{F}_2[a, b]/(a^2 + ab + b^2, a^2b + ab^2).$$

generated by classes a and b of degree 1 dual to i and j respectively. A generator of the group C_3 acts on $H^1(Q_8, \mathbb{F}_2)$ by sending a to b and y to $a + b$, so there is an isomorphism

$$H^*(G_{24}, \mathbb{F}_2) \cong H^*(Q_8, \mathbb{F}_2)^{C_3} \cong E(Q) \otimes \mathbb{F}_2[P]$$

where $Q \in H^3(Q_8, \mathbb{F}_2)$ is the top class in A . We also have

$$H^*(G_{24}, \mathbb{Z}_{(2)}) \cong \mathbb{Z}[P]/(8P)$$

where, by abuse of notation, $P \in H^4(G_{24}, \mathbb{Z}_{(2)})$ is an integral class which reduces to $P \in H^4(G_{24}, \mathbb{F}_2)$. We will give a more specific generator for $H^4(G_{24}, \mathbb{Z}_{(2)})$ below in Lemma 13.21.

Remark 13.16 We review the representation theory of Q_8 and G_{24} .

We have defined two real representations $\mathbb{H}$ and $\mathbb{H}_{ad}$. There are also three isomorphism classes of non-trivial 1-dimensional real representations of Q_8 . Each of the elements i, j, k generates a subgroup of order 4 in Q_8 ; taking the quotient by these subgroups in turn defines homomorphisms $Q_8 \rightarrow \{\pm 1\} = C_2$ and representations χ_i, χ_j , and χ_k by restricting the sign representation of C_2 . If we write $\mathbf{1}_{\mathbb{R}}$ for the trivial representation, then the regular representation of Q_8 decomposes as

$$\rho_{Q_8} \cong \mathbf{1}_{\mathbb{R}} \oplus \chi_i \oplus \chi_j \oplus \chi_k \oplus \mathbb{H}.$$

As a real representation of Q_8 we have

$$\mathbb{H}_{ad} \cong \mathbf{1}_{\mathbb{R}} \oplus \chi_i \oplus \chi_j \oplus \chi_k.$$

and hence

$$\rho_{Q_8} \cong \mathbb{H}_{ad} \oplus \mathbb{H}. \quad (13.17)$$

Because of the symmetries in Q_8 , the representation $\chi_i \oplus \chi_j \oplus \chi_k$ and $\mathbb{H}_{ad}$ can be given the structure of Spin representations. To see this, let $w(\xi) = 1 + w_1(\xi) + w_2(\xi) + \cdots$ be the total Stiefel-Whitney class. Then, using the notation of Remark 13.15, we have that

$$\begin{aligned} w(\mathbb{H}_{ad}) &= w(\chi_i \oplus \chi_j \oplus \chi_k) \\ &= (1 + a)(1 + b)(1 + (a + b)) = 1 \in H^*(Q_8, \mathbb{F}_2). \end{aligned}$$

The representation $\mathbb{H}$ of Q_8 is the restriction of an irreducible complex representation. If we use the right action of $\mathbb{C}$ on $\mathbb{H}$ to give $\mathbb{H}$ the structure of a complex vector space, then the action Q_8 on $\mathbb{H}$ is through complex linear transformations given by the matrices

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \quad (13.18)$$

This defines an inclusion $h: Q_8 \rightarrow SU(2)$.

The representation $\mathbb{H}_{ad}$ is a stabilization of the restriction of the adjoint representation of $SU(2)$ along the inclusion h . Concretely, the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ is the real vector space of 2×2 skew Hermitian complex matrices A of trace 0; thus a 2×2 complex matrix A is in $\mathfrak{su}(2)$ if $\bar{A} + A^t = 0$ and $\text{trace}(A) = 0$. Thus

$$\mathfrak{su}(2) \cong \left\{ \begin{pmatrix} bi & -\bar{z} \\ z & -bi \end{pmatrix} \mid b \in \mathbb{R}, z \in \mathbb{C} \right\}.$$

The group $SU(2)$ acts on the Lie algebra by conjugation; this is the adjoint representation of $SU(2)$. There is an isomorphism of real representations of Q_8

$$\mathbf{1}_{\mathbb{R}} \oplus \mathfrak{su}(2) \cong \mathbb{H}_{ad}.$$

Remark 13.19 Let V be a G -representation for some compact Lie group G and let ξ_V be the bundle

$$EG \times_G V \longrightarrow BG.$$

Thus if V is a complex representation, we get Chern classes $c_i(V) = c_i(\xi_V) \in H^*(BG, \mathbb{Z})$. For example, if $V = \mathbb{C}^n$ with its standard left action by $U(n)$ then this bundle is the dual of the tautological bundle over $BU(n)$. Hence

$$c_i(\xi_{\mathbb{C}^n}) = (-1)^i c_i$$

where c_i denotes the universal Chern class. Note that if a homomorphism $\varphi: G \rightarrow U(n)$ defines the representation V , then there is an isomorphism of bundles over BG

$$\xi_V \cong B\varphi^* \xi_{\mathbb{C}^n}.$$

As preparation for our calculations in Proposition 13.22 and Proposition 13.24 we next calculate some characteristic classes. Since $BSU(2)$ is 3-connected every vector bundle over $BSU(2)$ has a unique Spin structure and the characteristic class λ of Lemma 12.14 is unambiguously defined.

Lemma 13.20 (1) *We have an isomorphism*

$$\mathbb{Z}[y] \cong H^*(BSU(2), \mathbb{Z})$$

where

$$y = \lambda(\xi_{\mathbb{C}^2}) = -c_2(\xi_{\mathbb{C}^2}) \in H^4(BSU(2), \mathbb{Z})$$

is the characteristic class determined by the unique Spin structure on the bundle $\xi_{\mathbb{C}^2}$ associated to the standard left action of $SU(2)$ on $\mathbb{C}^2$.

(2) *Let $\mathfrak{su}(2)$ be the adjoint representation of $SU(2)$. Then*

$$\lambda(\mathfrak{su}(2)) = \lambda(\mathbf{1}_{\mathbb{R}} \oplus \mathfrak{su}(2)) = 2y$$

Proof Since $H^*(BSU(2), \mathbb{Z}) \cong \mathbb{Z}[c_2]$ part (1) follows from Remark 13.19 and an application of the formula

$$\lambda(\xi) = d(\xi)c_1(\xi) - c_2(\xi)$$

of Lemma 12.14.

For Part (2), let $g: BU(1) \rightarrow BSU(2)$ be the map defined by the inclusion of Lie groups $U(1) \rightarrow SU(2)$

$$z \mapsto \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}.$$

The map g classifies the bundle $\xi = \overline{\gamma}_1 \oplus \gamma_1$, where γ_1 is the tautological line bundle. Since $c_2(\xi) = -c_1(\gamma_1)^2$, we have

$$g^*y = \lambda(\overline{\gamma}_1 \oplus \gamma_1) = c_1^2.$$

In addition, $g^*: H^4(BSU(2), \mathbb{Z}) \rightarrow H^4(BU(1), \mathbb{Z})$ is an isomorphism.⁴ Thus to calculate $\lambda(\mathbf{1}_{\mathbb{R}} \oplus \mathfrak{su}(2))$ we can restrict to $U(1)$.

If we identify $\mathbf{1}_{\mathbb{R}}$ with the 2×2 diagonal matrices

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R} \right\}$$

we can then identify the $U(1)$ -representation $\mathbf{1}_{\mathbb{R}} \oplus \mathfrak{su}(2)$ as the direct sum $\mathcal{X} \oplus \mathcal{Y}$ of two 1-dimensional complex representations with

$$\mathcal{X} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} : \alpha \in \mathbb{C} \right\} \quad \text{and} \quad \mathcal{Y} = \left\{ \begin{pmatrix} 0 & -\bar{\alpha} \\ \alpha & 0 \end{pmatrix} : \alpha \in \mathbb{C} \right\}.$$

The conjugation action of $U(1)$ on $\mathcal{X}$ is trivial and the conjugation action of $z \in U(1)$ on $\mathcal{Y}$ is by multiplication by z^2 . Thus if ζ is the bundle over $BU(2)$ defined by $\mathbf{1}_{\mathbb{R}} \oplus \mathfrak{su}(2)$ we have $g^*\zeta = \mathbf{1}_{\mathbb{R}} \oplus \gamma_1^{-\otimes 2}$. Thus

$$g^*\lambda(\mathbf{1}_{\mathbb{R}} \oplus \mathfrak{su}(2)) = (1/2)(-c_2(\gamma_1))^2 = 2g^*y.$$

Since $\lambda(\xi \oplus \zeta) = \lambda(\xi) + \lambda(\zeta)$ we also get the formula for $\lambda(\mathfrak{su}(2))$. □

Lemma 13.21 *Let $\mathbb{H}$ be the real representation of G_{24} extending the left action of Q_8 on the quaternions. Then*

$$H^*(G_{24}, \mathbb{Z}_2) \cong \mathbb{Z}_2[\lambda(\mathbb{H})]/8\lambda(\mathbb{H}).$$

Furthermore $\lambda(\mathbb{H}_{ad}) = 2\lambda(\mathbb{H})$.

Proof By Remark 13.15 we have that $H^*(G_{24}, \mathbb{Z}_2) \cong \mathbb{Z}_2[P]/(8P)$, where P has degree 4. Thus we need only show $\lambda(\mathbb{H})$ generates $H^4(G_{24}, \mathbb{Z}_2)$. Let $C_i \subseteq G_{24}$ be the subgroup generated by i . Since the restriction map

$$r^*: H^4(G_{24}, \mathbb{Z}_2) \rightarrow H^4(C_i, \mathbb{Z}_2) \cong \mathbb{Z}/4$$

is onto, we need only check that $\lambda(\mathbb{H})$ generates $H^4(C_i, \mathbb{Z}_2)$.

We can use the constructions of Remark 13.16 to produce a commutative diagram

$$\begin{array}{ccc} BC_i & \xrightarrow{f} & BU(1) \\ r \downarrow & & \downarrow g \\ BQ_8 & \xrightarrow{h} & BSU(2). \end{array}$$

⁴ The map $U(1) \rightarrow SU(2)$ is the inclusion of the maximal torus. The Weyl group is C_2 and we have explicitly written down the canonical isomorphism $H^*(BSU(2), \mathbb{Z}) \cong H^*(BU(1), \mathbb{Z})^{C_2}$.

We have $h^*\xi_{\mathbb{C}^2} = \xi_{\mathbb{H}}$, $g^*\xi_{\mathbb{C}^2} = \overline{\gamma}_1 \oplus \gamma_1$, and f is induced by the representation γ_i of a cyclic group of order 4 on $\mathbb{C}$ given by multiplication by i . If $y = \lambda(\xi_{\mathbb{C}^2}) \in H^4(BSU(2), \mathbb{Z}_2)$, then we have

$$\begin{aligned}\lambda(\mathbb{H}) &= h^*y \in H^4(Q_8, \mathbb{Z}_2) \\ g^*y &= c_1(\gamma_1)^2 \in H^4(BU(1), \mathbb{Z}_2).\end{aligned}$$

Since $H^2(BC_i, \mathbb{Z}_2) \cong \mathbb{Z}/4$ generated by $c_1(\gamma_i)$ we have that

$$r^*\lambda(\mathbb{H}) = f^*g^*y = c_1(\gamma_i)^2 \in H^4(C_i, \mathbb{Z}_2)$$

is a generator, as needed.

The calculation of $\lambda(\mathbb{H}_{ad})$ follows from Part (2) of Lemma 13.20. $\square$

We are now ready to give our calculations. We will follow the outline of Remark 12.15, and we will use the notation established there.

Proposition 13.22 (1) *Let $G_{24} \subseteq \mathbb{S}_2$ be the automorphism group of the super-singular elliptic curve $y^2 + y = x^3$ over $\mathbb{F}_4$. The composite mapping*

$$[\Sigma_+^\infty BG_{24}, L_2^{\geq 2}\mathrm{ko}\langle 8 \rangle] \rightarrow [\Sigma_+^\infty BG_{24}, L_2^{\geq 2}\mathrm{ko}] \rightarrow \pi_0\mathrm{pic}(\mathbf{E})^{hG_{24}}$$

is zero.

(2) *There is an isomorphism*

$$\psi: RO(G_{24})/I_2\langle 8 \rangle \cong \mathbb{Z} \oplus \mathbb{Z}/8$$

sending a representation W to $(\dim(W), k)$ where $\lambda(W) = k\lambda(\mathbb{H}) \in H^4(G_{24}, \mathbb{Z}_2)$.

Proof For Part (1) we again use the evident variant of Proposition 12.9. See Remark 12.15. By Theorem 8.2.2 (see also Figure 7) of [58] we have that in the spectral sequence

$$H^s(G_{24}, \pi_t\mathrm{pic}(\mathbf{E})) \implies \pi_{t-s}\mathrm{pic}(\mathbf{E})^{hG_{24}}$$

$E_\infty^{s,s} = 0$ for $s \geq 8$.

Part (2) follows from the filtration of $RO(G_{24})/I_2\langle 8 \rangle$ given in Remark 12.15. Note that by Lemma 13.21

$$H^1(G_{24}, \mathbb{Z}/2) = 0 = H^2(G_{24}, \mathbb{Z}/2)$$

and $\lambda: RO(G_{24}) \rightarrow H^4(G_{24}, \mathbb{Z}_2) \cong \mathbb{Z}/8$ is onto. $\square$

Proposition 13.23 (1) If $\rho_{G_{24}} \in RO(G_{24})$ is the regular representation, then

$$\psi(\rho_{G_{24}}) = (24, 1).$$

(2) The group $RO(G_{24})/(I_2\langle 8 \rangle + \rho_{G_{24}})$ is generated by the 1-dimensional real representation $\mathbf{1}_{\mathbb{R}}$; this choice of generator determines an isomorphism

$$\mathbb{Z}/192 \cong RO(G_{24})/(I_2\langle 8 \rangle + \rho_{G_{24}}).$$

Furthermore, the J -homomorphism

$$J_{\mathbf{E}}^{G_{24}} : RO(G_{24})/(I_2\langle 8 \rangle + \rho_{G_{24}}) \longrightarrow \text{Pic}(\mathbf{E}^{hG_{24}})$$

is an isomorphism.

Proof We must calculate $\lambda(\rho_{G_{24}})$. The inclusion $Q_8 \subseteq G_{24}$ defines an isomorphism

$$H^4(G_{24}, \mathbb{Z}_2) \cong H^4(Q_8, \mathbb{Z}_2) \cong \mathbb{Z}/8.$$

Restricted to Q_8 , we have $\rho_{G_{24}} = \rho_{Q_8}^{\oplus 3}$. We have, by (13.17) and Lemma 13.21

$$\lambda(\rho_{Q_8}) = \lambda(\mathbb{H}_{ad}) + \lambda(\mathbb{H}) = 3\lambda(\mathbb{H}),$$

whence

$$\psi(\rho_{G_{24}}) = (24, 9) = (24, 1).$$

Thus $\mathbb{Z}/192 \cong RO(G_{24})/(I_2\langle 8 \rangle + \rho_{G_{24}})$ generated by $\mathbf{1}_{\mathbb{R}}$.

The fact the J -homomorphism is an isomorphism then follows from the fact that $\mathbf{E}^{hG_{24}}$ has periodicity 192; that is $\text{Pic}(\mathbf{E}^{hG_{24}}) \cong \mathbb{Z}/192$ generated by $\Sigma \mathbf{E}^{hG_{24}}$. See [58]. $\square$

Proposition 13.24 We have an equation

$$\mathbb{H}_{ad} \equiv -44 \cdot \mathbf{1}_{\mathbb{R}}$$

in $RO(G_{24})/(I_2\langle 8 \rangle + \rho_{G_{24}})$.

Proof By Lemma 13.21 we have

$$\psi(\mathbb{H}_{ad}) = (4, 2) \in \mathbb{Z} \oplus \mathbb{Z}/8.$$

Since

$$(4, 2) - 2(24, 1) = (-44, 0)$$

the result follows from Proposition 13.23. $\square$

We now have a calculation of the Spanier–Whitehead duals to $\mathbf{E}^{hF}$. If $G_{48} = G_{24} \rtimes \text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$, we have

$$\mathbf{E}^{hG_{48}} \simeq L_{\mathbf{K}}\mathbf{tmf}.$$

Theorem 13.25 *Let $F \subseteq G_{48} \subseteq \mathbb{G}_2$ for $\mathbb{G}_2 = \text{Aut}(\mathbb{F}_4, F_C)$ the stabilizer group associated to the formal group law F_C of a supersingular elliptic curve C with Weierstrass equation (13.13). Then*

$$D(\mathbf{E}^{hF}) \simeq \Sigma^{44}\mathbf{E}^{hF}.$$

Proof First suppose $F \subseteq G_{24}$. Then by Remark 13.14 we have

$$D(\mathbf{E}^{hF}) \simeq (S^{-\mathbb{H}_{ad}} \wedge \mathbf{E})^{hF}$$

and the result follows from Proposition 13.24.

The other possibility is that the composition $F \rightarrow G_{48} \rightarrow \text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ is onto. Let F_0 be the kernel of this map and write Gal for $\text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. We know from Lemma 1.37 of [10] that there is a $\text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ equivariant equivalence

$$\Sigma_+^\infty \text{Gal} \wedge \mathbf{E}^{hF} \simeq \mathbf{E}^{hF_0}.$$

We now have

$$\begin{aligned} D(\mathbf{E}^{hF}) &\simeq D(\mathbf{E})^{hF} \\ &\simeq [D(\mathbf{E})^{hF_0}]^{h\text{Gal}} \simeq (\Sigma^{44}E^{hF_0})^{h\text{Gal}} \\ &\simeq [\Sigma_+^\infty \text{Gal} \wedge \Sigma^{44}\mathbf{E}^{hF}]^{h\text{Gal}} \\ &\simeq \Sigma^{44}\mathbf{E}^{hF}. \end{aligned} \quad \square$$

Remark 13.26 (Using the String orientation) In Proposition 13.8 and Proposition 13.22 we showed that (roughly) the restriction of the J -homomorphism

$$J : [\Sigma_+^\infty BF, \text{ko}\langle 8 \rangle] \rightarrow \pi_0 \text{pic}(E^{hF}) = \pi_0 \text{pic}_F(E)$$

is the zero map for various finite subgroups F of the Morava Stabilizer Group $\mathbb{G}_2$. We used fixed point spectral sequence technology, but this can also be deduced from the existence of the String orientation of $\mathbf{tmf}$ given in [2].

Suppose $V : BF \rightarrow B\text{Gl}_1(S^0)$ defines an action of a finite group on the 0-sphere S^0 . Write S^V for this F -sphere. For any ring spectrum R , the composition

$$BF \xrightarrow{V} B\text{Gl}_1(S^0) \longrightarrow B\text{Gl}_1(R)$$

defines an action of F on R ; this F -spectrum is equivalent to $R \wedge S^V$ with the trivial action on R .

Now let F be a finite subgroup of $\mathbb{G}_2$ and let $R = \mathbf{E}^{hF}$. Then we can write $J(V) \in \mathcal{P}\mathrm{ic}_F(\mathbf{E})$ as

$$J(V) = \mathbf{E} \wedge S^V \simeq \mathbf{E} \wedge_{\mathbf{E}^{hF}} \mathbf{E}^{hF} \wedge S^V.$$

We have the diagonal action on both sides of this equation, although F acts trivially on $\mathbf{E}^{hF}$.

Next suppose the map $BF \rightarrow B\mathrm{Gl}_1(\mathbf{E}^{hF})$ is null-homotopic. Then we have an equivalence of F -spectra $\mathbf{E}^{hF} \wedge S^V \simeq \mathbf{E}^{hF}$, and hence of elements

$$J(V) = \mathbf{E} \wedge S^V \simeq \mathbf{E}$$

in $\mathcal{P}\mathrm{ic}(R)$. Put another way, $J(V) = 0 \in \mathrm{Pic}(R) = \pi_0 \mathrm{pic}_F(E)$.

The existence of a String orientation $MO\langle 8 \rangle \rightarrow \mathbf{tmf}$ is proved by showing that the composition

$$\mathrm{ko}\langle 8 \rangle \rightarrow \mathrm{ko} \rightarrow b\mathrm{gl}_1(S^0) \rightarrow b\mathrm{gl}_1(\mathbf{tmf})$$

is null-homotopic. Now let $n = p = 2$ and $F \subseteq G_{48}$. Then we have a map of ring spectra

$$L_{\mathbf{K}}\mathbf{tmf} \simeq \mathbf{E}^{hG_{48}} \longrightarrow \mathbf{E}^{hF},$$

so we may conclude the composition

$$\mathrm{ko}\langle 8 \rangle \rightarrow \mathrm{ko} \rightarrow b\mathrm{gl}_1(S^0) \rightarrow b\mathrm{gl}_1(\mathbf{E}^{hF})$$

is null-homotopic and that the map

$$[\Sigma_+^\infty BF, \mathrm{ko}\langle 8 \rangle] \longrightarrow \pi_0 \mathrm{pic}_F(E)$$

is zero. A similar statement holds at the prime 3.

This approach, using the String orientation, works only at height 2 because, ultimately, it depends on the geometry of elliptic curves. In Sect. 14 we will present a higher height example and, by necessity, return to homotopy fixed point techniques.

14 The Spanier–Whitehead duals of $\mathbf{E}^{hF}$: examples from higher heights

In this section, we fix a prime $p \geq 3$. We will work with the Honda formal group law F_n of height $n = p - 1$ over $\mathbb{F}_{p^n}$. Our intention is to use the

theory we have developed to calculate $D(\mathbf{E}^{hF})$ for certain finite subgroups of $\mathbb{G}_n = \text{Aut}(\mathbb{F}_{p^n}, F_n)$. We will then use that calculation to make some remarks about exotic elements in the Picard group of the $K(n)$ -local category. The main results are Theorem 14.14 and Theorem 14.16.

The group $\mathbb{S}_n = \text{Aut}(F_n/\mathbb{F}_{p^n})$ contains a maximal finite subgroup

$$G \cong C_p \rtimes C_{n^2}.$$

There is also an extension of subgroups of $\mathbb{G}_n$

$$1 \rightarrow G \rightarrow H \rightarrow \text{Gal} \rightarrow 1 \quad (14.1)$$

where $\text{Gal} = \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. This is discussed, for example, in [37, 3.6.3.1]. We review some of these facts here.

Let $\mathbb{W} = W(\mathbb{F}_{p^n})$ be the Witt vectors on $\mathbb{F}_{p^n}$ and let $\mathcal{O}_n$ be the endomorphism ring of F_n ; see Example 2.15. We have an isomorphism

$$\mathbb{W}\langle S \rangle / (S^n = p, Sa = a^\sigma S) \cong \mathcal{O}_n$$

where $a \in \mathbb{W}$ and $\sigma \in \text{Gal}$ is the Frobenius. Let

$$\omega \in \mathbb{F}_{p^n}^\times \subset \mathbb{W}^\times \subset \mathbb{S}_n = \mathcal{O}_n^\times$$

be a primitive $p^n - 1$ root of unity. We then define elements of $\mathcal{O}_n$ by

$$\tau = \omega^{\frac{p^n-1}{n^2}} \quad \text{and} \quad X = \omega^{n/2} S.$$

Then $X^n = -p$ and the element τ has order n^2 . In particular, $\tau^n \in \mathbb{F}_p^\times$ is a primitive $(p-1)$ st root of unity. By Lemma 19 of [37] the subfield

$$\mathbb{Q}_p(X) \subseteq \mathbb{D}_n = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_n$$

contains a primitive p th root of unity; we choose one such and call it ζ_p . Since any root of unity must have norm 1, $\zeta_p \in \mathbb{S}_n$.

Since $X^n = -p$, $\mathbb{Q}_p(X)$ has degree $n = p-1$ over $\mathbb{Q}_p$ and it then follows that as subfields of $\mathbb{D}_n$

$$\mathbb{Q}_p(\zeta_p) = \mathbb{Q}_p(X).$$

Let $C_p = \langle \zeta_p \rangle$ be the subgroup of $\mathbb{S}_n$ generated by ζ_p . Conjugation by τ induces an automorphism of C_p of order $p-1$, so there is a primitive root of unity in $e \in (\mathbb{Z}/p)^\times$ such that

$$\tau \zeta_p \tau^{-1} = \zeta_p^e.$$

If we let $G \subseteq \mathbb{S}_n$ be the subgroup generated by ζ_p and τ , then we have an isomorphism

$$G \cong C_p \rtimes C_{n^2} = \langle \zeta_p, \tau \mid \tau \zeta_p \tau^{-1} = \zeta_p^e \rangle.$$

The extension H is more subtle to describe and we won't need any of the details here.

Let $\mathbb{Z}(\zeta_p) \subseteq \mathcal{O}_n$ be the subring generated ζ_p ; there is an isomorphism $\mathbb{Z}[x]/\Phi_p(x) \cong \mathbb{Z}(\zeta_p)$ where $\Phi_p(x) = (x^p - 1)/(x - 1)$ is the p th cyclotomic polynomial.

Lemma 14.2 *Let*

$$\mathcal{E} = \mathbb{Z}(\zeta_p)\{1, \tau, \tau^2, \dots, \tau^{n-1}\} \subseteq \mathcal{O}_n$$

be the sub- $\mathbb{Z}(\zeta_p)$ -module generated by τ^i , $0 \leq i \leq n-1$.

- (1) $\mathcal{E}$ is stable under the conjugation action of G , and
 (2) $\mathbb{Q}_p \otimes \mathcal{E} \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_n$.

Proof Part (1) follows from the facts that $\tau \zeta_p \tau^{-1} = \zeta_p^e$ and $\zeta_p \tau^j \zeta_p^{-1} = \zeta_p^{1-e^j} \tau^j$.

For (2) let $K = \mathbb{Q}_p \otimes \mathcal{E}$. Since $\tau^n \in \mathbb{Z}_p^\times$, K is the sub-algebra of $\mathbb{D}_n$ generated by ζ_p and τ . By construction, τ is an n^2 root of unity over $\mathbb{Z}_p$. We examine τ in the extension $\mathbb{Z}_p \subseteq W(\mathbb{F}_{p^d})$.

If the field $\mathbb{F}_{p^d}$ contains an n^2 root of unity, then $n \mid (p^d - 1)/n$, which implies that $n \mid d$. Consider the field extensions

$$\mathbb{Q}_p \subset \mathbb{Q}_p(\tau) \subseteq \mathbb{Q}(\omega)$$

where $\omega \in W(\mathbb{F}_{p^n})$ is our chosen $(p^n - 1)$ st root of unity. Both $\mathbb{Q}_p(\tau)$ and $\mathbb{Q}(\omega)$ are unramified extensions of $\mathbb{Q}_p$ of degree n and, hence, $\mathbb{Q}_p(\tau) = \mathbb{Q}_p(\omega)$ and it follows that $\omega \in K$. However, since $X \in K$, $\omega \in K$ and $S \in K$, we have that $K = \mathbb{D}_n = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_n$. $\square$

Remark 14.3 We find that we are exactly in the situation of Proposition 9.14. We let

$$V = \mathbb{R} \otimes \mathcal{E}$$

with action induced by the conjugation action of G on $\mathcal{E}$. Thus our goal is to analyze the homotopy type of

$$(S^{-V} \wedge \mathbf{E})^{hG} \in \text{Pic}(\mathbf{E}^{hG}).$$

Remark 14.4 Here we describe the additive structure of the real orthogonal representation ring $RO(G)$.

Let k be an integer, C_k the cyclic of order k , and $\gamma = e^{2\pi i/k} \in \mathbb{C}$. Multiplication by γ^m on $\mathbb{C}$ defines a 1-dimensional complex representation $\mathbb{C}(m)$ of C_k . As γ^m and γ^{k-m} are complex conjugate, the representations $\mathbb{C}(m)$ and $\mathbb{C}(k-m)$ become isomorphic as 2-dimensional real representations.

Suppose k is odd. Let $m \neq 0$ and let α_m be $\mathbb{C}(m)$ regarded as a real representation. Then α_m is irreducible as a real representation and $RO(C_k)$ is generated by the trivial one-dimensional representation $\mathbf{1}_{\mathbb{R}}$ and α_m , $1 \leq m \leq (k-1)/2$. If k is even and $1 \leq m < k/2$, write λ_m for $\mathbb{C}(m)$ regarded as a real representation. These are again irreducible. We also have the 1-dimensional sign representation σ obtained by restriction along the unique quotient map $C_k \rightarrow C_2$. Then $RO(C_k)$ is generated by $\mathbf{1}_{\mathbb{R}}$, σ , and λ_m , $1 \leq m \leq (k-2)/2$.

Let $G = C_p \rtimes C_{n^2}$, with $n = p-1$. We induce α_m along the inclusion $C_p \subseteq G$ to obtain $n/2$ irreducible $2n^2$ -dimensional real representations

$$\Lambda_m := \text{Ind}_{C_p}^G(\alpha_m), \quad 1 \leq m \leq n/2.$$

The representations Λ_m are also restrictions of complex representations.

There is an embedding $RO(C_{n^2}) \rightarrow RO(G)$ given by restriction along the quotient map $G \rightarrow C_{n^2}$. From this, we conclude that

$$RO(G) \cong RO(C_{n^2}) \oplus \mathbb{Z}\{\Lambda_1, \dots, \Lambda_{n/2}\}.$$

This gives

$$RO(G) \cong \mathbb{Z}\{\mathbf{1}_{\mathbb{R}}, \sigma, \lambda_1, \dots, \lambda_{(n^2-2)/2}, \Lambda_1, \dots, \Lambda_{n/2}\}.$$

Except for $\mathbf{1}_{\mathbb{R}}$ and σ , the listed generators of $RO(G)$ are all restrictions of complex representations. A dimension count shows that we have a decomposition of the regular representation

$$\rho_G \cong \mathbf{1}_{\mathbb{R}} \oplus \sigma \oplus \lambda_1 \oplus \dots \oplus \lambda_{(n^2-2)/2} \oplus \Lambda_1 \oplus \dots \oplus \Lambda_{n/2}. \quad (14.5)$$

Note there are no repeated summands.

Remark 14.6 We now fix some generators for the relevant cohomology groups. Let $z_0 \in H^2(C_p, \mathbb{Z}_{(p)})$ be such that $z_0 = i^*c_1$ where $c_1 \in H^1(BU, \mathbb{Z}_{(p)})$ is the first Chern class and $i: BC_p \rightarrow BU$ is the canonical map induced by the inclusion of $C_p \subseteq U(1)$ which maps the generator ζ_p to $e^{2\pi i/p}$.

Let $z = z_0^{p-1} \in H^*(C_p, \mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}[z_0]/pz_0$. Since $\tau\zeta_p\tau^{-1} = \zeta_p^e$ where e generates $\mathbb{Z}/p^\times$, the action of τ on z_0 is by multiplication by a generator of

$\mathbb{Z}/p^\times$ and so

$$H^*(G, \mathbb{Z}_{(p)}) \cong H^*(C_p, \mathbb{Z}_{(p)})^{C_{n^2}} \cong \mathbb{Z}_{(p)}[z]/pz.$$

We also let y be a generator of (recall $n = p - 1$)

$$H^1(G, \mathbb{Z}_p^\times) \cong \mathbb{Z}/n\{y\}.$$

Next, we gather information about some characteristic classes which will be used below.

Remark 14.7 If X is a space, let $K^0(X)$ denote the complex K -theory of X in degree 0. Recall that the Chern character

$$ch: K^0(X) \rightarrow H^*(X, \mathbb{Q})$$

is the unique ring homomorphism defined using the splitting principle and the formula $ch(L) = \exp(c_1(L))$ when L is a line bundle. If we write $ch_k \in H^k(X, \mathbb{Q})$ for the k th homogeneous component of ch , then if L is a line bundle

$$ch_k(L) = \frac{c_1(L)^k}{k!}$$

and in general

$$ch_k = \frac{s_k(c_1, \dots, c_k)}{k!}$$

where $s_k(c_1, \dots, c_k)$ is k th Newton polynomial in the Chern classes. In particular, modulo decomposables,

$$ch_k \equiv \alpha c_k, \quad \alpha \in \mathbb{Q}, \quad \alpha \neq 0.$$

We can evaluate ch_k on the universal bundle over BU and obtain a cohomology class $ch_k \in H^{2k}(BU, \mathbb{Q})$. The classes ch_k are algebraic generators and primitives for the Hopf algebra structure on $H^*(BU, \mathbb{Q})$. If we choose the Bott class $v \in \pi_2 BU = H_2(BU, \mathbb{Z})$ so that $\langle c_1, v \rangle = 1$, then the multiplicative properties of the Chern character imply

$$\langle ch_k, v^k \rangle = 1 \tag{14.8}$$

and ch_k is the unique primitive with this property.

The map $BU \rightarrow BO$ classifying the underlying real bundle of the universal complex line bundle defines an isomorphism

$$H^*(BO, \mathbb{Q}) \xrightarrow{\cong} H^*(BU, \mathbb{Q})^{C_2}$$

where C_2 acts via complex conjugation. If L is a line bundle with conjugate $\overline{L}$, then $c_1(\overline{L}) = -c_1(L)$, and it then follows that for any bundle $ch_k(\overline{\xi}) = (-1)^k ch_k(\xi)$. Hence $ch_{2k} \in H^*(BO, \mathbb{Q})$ and $ch_{2k}(\xi)$ is defined for any real bundle ξ . Note that if $\xi_{\mathbb{R}}$ is the real bundle underlying some complex vector bundle ξ , then $ch_{2k}(\xi_{\mathbb{R}}) = ch_{2k}(\xi)$.

If $1 \leq k < p$, the defining expression for ch_k makes sense over $\mathbb{Z}_{(p)}$ and in fact there is a unique lift of ch_k to a class $ch_k \in H^{2k}(BU, \mathbb{Z}_{(p)})$. This gives a characteristic class $ch_k(\xi) \in H^{2k}(X; \mathbb{Z}_{(p)})$ for any complex vector bundle ξ over X . Furthermore, the additivity of ch_k over $\mathbb{Q}$ and the fact that

$$H^{2k}(BU \times BU; \mathbb{Z}_{(p)}) \rightarrow H^{2k}(BU \times BU; \mathbb{Q})$$

is injective implies that ch_k is additive, that is,

$$ch_k(\xi_1 \oplus \xi_2) = ch_k(\xi_1) + ch_k(\xi_2) \in H^{2k}(X, \mathbb{Z}_{(p)})$$

for any bundles ξ_1, ξ_2 over X .

If p is odd, k is even, and $k < p$, then we have

$$ch_k \in H^*(BO, \mathbb{Z}_{(p)}) = H^*(BU, \mathbb{Z}_{(p)})^{C_2}$$

and we can define characteristic classes $ch_k(\xi)$ for any virtual real bundle as well.

Remark 14.9 Suppose $1 \leq k < p$. Let ξ be a stable complex bundle of virtual dimension 0 and suppose the classifying map $\xi: X \rightarrow BU$ lifts to a map

$$\xi: X \rightarrow BU\langle 2k \rangle.$$

If ku is the connective complex K -theory spectrum, then ξ is detected in the Atiyah-Hirzebruch spectral sequence for $ku_{(p)}^*(X)$ by the cohomology class $ch_k(\xi) \in H^{2k}(X, \pi_{2k}ku_{(p)})$; that is, the class given by the composition

$$X \rightarrow BU\langle 2k \rangle \xrightarrow{ch_k} K(\mathbb{Z}_{(p)}, 2k).$$

This follows from (14.8).

If p is odd, k is even and $1 < k < p$ we can make a similar observation about a stable real bundle ξ of virtual dimension 0. Suppose the classifying

map $\xi: X \rightarrow BO$ lifts to $BO\langle 2k \rangle$. If ko is the connective real K -theory spectrum, then ξ is detected in the Atiyah-Hirzebruch spectral sequence for $ko_{(p)}^*(X)$ by the cohomology class $ch_k(\xi) \in H^{2k}(X, \pi_{2k}ko_{(p)})$.

Note that since p is odd $ko_{(p)} = ku_{(p)}^{hC_2}$ and

$$\pi_*ko_{(p)} = \mathbb{Z}_{(p)}[v^2] = (\pi_*ku_{(p)})^{C_2}.$$

Remark 14.10 We can now relate the characteristic classes of Remark 14.7 to the representations of Remark 14.4. Suppose p is odd, $n = p - 1$ and $G = C_p \rtimes C_{n^2}$. Since n is even, ch_n is defined for real vector bundles and we get a homomorphism

$$ch_n: RO(G) \rightarrow H^{2n}(G, \mathbb{Z}_{(p)}).$$

We will be most interested in representations W which are the restriction of a complex representation and $ch_n(W)$ can then be computed using complex characteristic classes. Note that

$$2ch_n(\mathbf{1}_{\mathbb{R}}) = ch_n(2 \cdot \mathbf{1}_{\mathbb{R}}) = ch_n(\mathbf{1}_{\mathbb{C}}) = 0,$$

so $ch_n(\mathbf{1}_{\mathbb{R}}) = 0$. If $\sigma \in RO(G)$ is the sign representation, then σ is obtained by restriction along the unique quotient map $q: G \rightarrow \mathbb{Z}/2$, so $ch_n(\sigma) = 0$ as $H^{2n}(\mathbb{Z}/2, \mathbb{Z}_{(p)}) = 0$.

We are ready to work with the J -homomorphism.

Proposition 14.11 (1) *The composite mapping*

$$[\Sigma_+^\infty BG, L_p^{\geq 2}ko\langle 2p \rangle] \rightarrow [\Sigma_+^\infty BG, L_p^{\geq 2}ko] \rightarrow \pi_0\text{pic}(\mathbf{E})^{hG}$$

is zero.

(2) *There is a homomorphism*

$$\psi: RO(G) \rightarrow \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/p$$

which maps W to $(\dim W, a, b)$ where

$$w_1(W) = ay$$

$$ch_n(W) = bz$$

where y, z are as in Remark 14.6.

Proof It is shown in [42] that in the spectral sequence

$$H^s(G, \pi_t \text{pic}(\mathbf{E})) \implies \pi_{t-s} \text{pic}(\mathbf{E})^{hG}$$

$E_\infty^{s,s} = 0$ for $s \geq 2p$. This gives (1).

For (2), we use the techniques of Remark 12.15. From Remark 14.6 we know $H^s(G, \mathbb{Z}_{(p)}) = 0$ for $0 < s < 2n$. Then Remark 14.9 give us a filtration

$$\begin{array}{ccccc} 0 & \longrightarrow & A_{2n} & \xrightarrow{\subseteq} & A_1 & \xrightarrow{\subseteq} & RO(G)/I_p \langle 2p \rangle \\ & & \downarrow ch_{2n} & & \downarrow w_1 & & \downarrow \dim \\ & & H^n(BG, \mathbb{Z}_{(p)}) & & H^1(BG, \mathbb{Z}/2) & & \mathbb{Z} \end{array}$$

which we use to define the desired homomorphism. $\square$

Proposition 14.12 (1) *If ρ_G is the regular representation of G , then*

$$\psi(\rho_G) = (pn^2, 1, -n/2).$$

(2) *The group $RO(G)/(I_p \langle 2p \rangle + \rho_G)$ is generated by the 1-dimensional real representation $\mathbf{1}_{\mathbb{R}}$; this choice of generator determines an isomorphism*

$$\mathbb{Z}/2p^2n^2 \cong RO(G)/(I_p \langle 2p \rangle + \rho_G).$$

Furthermore, the J -homomorphism

$$J_{\mathbf{E}}^G: RO(G)/(I_p \langle 2p \rangle + \rho_G) \longrightarrow \text{Pic}(\mathbf{E}^{hG})$$

is an isomorphism.

Proof To prove (1), we use (14.5) to see that ρ_G contains a single copy of the sign representation σ and no other non-orientable direct summands. Therefore, $w_1(\rho_G) = 1$.

To compute $ch_n(\rho_G)$, note that

$$\begin{aligned} ch_n(\text{res}^* \rho_G) &= n^2 ch_n(\rho_{C_p}) \\ &= n^2 (ch_n(\alpha_1) \oplus \dots \oplus ch_n(\alpha_{n/2})) \end{aligned}$$

However,

$$ch_n(\alpha_m) = \frac{c_1(\alpha_m)^n}{n!} = -z.$$

The first equality follows since α_m is one dimensional. The second equality follows since $c_1(\alpha_m)$ is non-zero, and any unit in $\mathbb{Z}/p$ raised to the power n is equal to 1. We have also used the fact that $n! \equiv -1 \pmod{p}$. Finally, since $n^2 \equiv 1 \pmod{p}$, we have

$$\begin{aligned} ch_n(\rho_G) &= n^2 (ch_{2n}(\alpha_1) + \dots + ch_{2n}(\alpha_{n/2})) \\ &= -\frac{n}{2}z. \end{aligned}$$

For (2), the fact that $RO(G)/(I_p\langle 2p \rangle + \rho_G) \cong \mathbb{Z}/2p^2n^2$ generated by $\psi(\mathbf{1}_{\mathbb{R}}) = (1, 0, 0)$ is a computation using part (1) and part (2) of Proposition 14.11. Furthermore, by [42], $\text{Pic}(\mathbf{E}^{hG}) \cong \mathbb{Z}/2p^2n^2$ generated by $\Sigma \mathbf{E}^{hG} = \psi(\mathbf{1}_{\mathbb{R}})$. $\square$

Finally, let $V = \mathbb{R} \otimes \mathcal{E}$ be as in Remark 14.3. We need to identify the image of V under the J -homomorphism.

Proposition 14.13 *In $RO(G)/(I_p\langle 2p \rangle + \rho_G)$*

$$V \equiv n^2(1 + 2p) \cdot \mathbf{1}_{\mathbb{R}}.$$

Proof We show that

$$\psi(V) = (n^2, 0, -(n-1)n/2)$$

which implies that

$$\psi(V) = (n^3p(n-1) + n^2) \cdot \mathbf{1}_{\mathbb{R}} \equiv n^2(1 + 2p)\mathbf{1}_{\mathbb{R}} \pmod{2n^2p^2 \cdot \mathbf{1}_{\mathbb{R}}}.$$

First, $\dim V = n^2$ gives the first coordinate. To determine if V is orientable, it suffices to restrict to the action of τ . The vector space underlying V has basis $\{\zeta_p^i \tau^j : 0 \leq i, j \leq n-1\}$ and

$$\tau \zeta_p^i \tau^j \tau^{-1} = \zeta_p^{ie} \tau^j.$$

So, as a representation of C_{n^2} , V is n copies of the same representation. Since n is even, V is orientable and so $w_1(V) = 0$.

To compute $ch_{2n}(V)$, we note that after restricting to C_p , there is an isomorphism

$$\text{res}^* V \cong \mathbf{1}_{\mathbb{R}} \oplus (n-1)\rho_{C_p}.$$

The action is given by

$$\zeta_p \tau^j \zeta_p^{-1} = \zeta_p^{1-e^j} \tau^j.$$

Note that $1 - e^j \neq 0$ for $1 \leq j \leq n - 1$ as e generates $\mathbb{Z}/p^\times$ and

$$1 + \zeta_p + \dots + \zeta_p^n = 0.$$

So,

$$\mathbb{R}\{\zeta_p^i \tau^j : 0 \leq i \leq n - 1\} \cong \begin{cases} n \cdot \mathbf{1}_{\mathbb{R}} & j = 0 \\ \bar{\rho}_{C_p} & 1 \leq j \leq n - 1 \end{cases}$$

where $\bar{\rho}_{C_p}$ is the reduced regular representation. Noting that $\rho_{C_p} \cong \mathbf{1}_{\mathbb{R}} \oplus \bar{\rho}_{C_p}$ proves the claim. From this, we conclude that

$$ch_{2n}(V) = (n - 1)ch_{2n}(\rho_{C_p}) = -(n - 1)n/2. \quad \square$$

We can now have a calculation of the Spanier–Whitehead duals to $\mathbf{E}^{hF}$ for various finite subgroups F .

Theorem 14.14 *Let $n = p - 1$ and $F \subseteq H \subseteq \mathbb{G}_n$ for $\mathbb{G}_n = \text{Aut}(\mathbb{F}_{p^n}, F_n)$ the stabilizer group of the Honda formal group law F_n and for H is as in (14.1). Then*

$$D(\mathbf{E}^{hF}) \simeq \Sigma^{-n^2(2p+1)} \mathbf{E}^{hF}.$$

Proof First suppose $F \subseteq G$. By Remark 14.3, we have that

$$D(\mathbf{E}^{hF}) \simeq (S^{-V} \wedge \mathbf{E})^{hF}$$

and the result follows from Proposition 14.13.

The other possibility is that the composition $F \rightarrow H \rightarrow \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ is non-trivial. Let F_0 be the kernel of this map and write $\widetilde{\text{Gal}}$ for the image of the composite in $\text{Gal} = \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ so that

$$1 \rightarrow F_0 \rightarrow F \rightarrow \widetilde{\text{Gal}} \rightarrow 1$$

is exact. From Lemma 1.37 of [10], we can deduce that there is a $\widetilde{\text{Gal}}$ equivariant equivalence

$$\Sigma_+^\infty \widetilde{\text{Gal}} \wedge \mathbf{E}^{hF} \simeq \mathbf{E}^{hF_0}.$$

Letting $r = -n^2(2p + 1)$, we now have

$$\begin{aligned} D(\mathbf{E}^{hF}) &\simeq D(\mathbf{E})^{hF} \\ &\simeq [D(\mathbf{E})^{hF_0}]^{h\widetilde{\text{Gal}}} \simeq (\Sigma^r E^{hF_0})^{h\widetilde{\text{Gal}}} \end{aligned}$$

$$\begin{aligned} &\simeq [\Sigma_+^\infty \widetilde{\text{Gal}} \wedge \Sigma^r \mathbf{E}^{hF}]^{h\widetilde{\text{Gal}}} \\ &\simeq \Sigma^r \mathbf{E}^{hF}. \end{aligned}$$

□

Remark 14.15 We note that $\mathbf{E}^{hF}$ is always periodic of period $2p^2n^2$, though depending on $F \subseteq H$, the period could be shorter. For example, the period of G and H is exactly $2p^2n^2$. That of C_p is $2p^2$.

Also note that if $p = 3$, $\mathbf{E}^{hF}$ is 72-periodic, and

$$\Sigma^{-n^2(2p+1)} \mathbf{E}^{hF} = \Sigma^{-28} \mathbf{E}^{hF} \simeq \Sigma^{44} \mathbf{E}^{hF}$$

Hence Theorem 13.12 and Theorem 14.14 produce the same shift. Note, however, that the formal group of the supersingular elliptic curve used in Theorem 13.12 is not isomorphic over $\mathbb{F}_9$ to the Honda formal group.

We end with a simple but interesting application of Theorem 14.14 to the study of the Picard group of the $\mathbf{K}$ -local category. We refer the reader to [39, 45], and Section 2.4 of [32] for more background, but recall some of the key ideas here. We let Pic_n be the Picard group of the homotopy category of $\mathbf{K}$ -local spectra. For X an invertible $\mathbf{K}$ -local spectrum, $\mathbf{E}_*X$ is an invertible Morava module. If $\mathbf{E}_*X \cong \mathbf{E}_*S^0$ as Morava modules, we say that X is *exotic* and denote the subgroup of exotic elements in Pic_n by κ_n .

For a $\mathbf{K}$ -local spectrum X , let $I_n(X)$ be the Gross–Hopkins dual of X . Gross–Hopkins duality and Spanier–Whitehead duality are related by the equation

$$I_n(X) \simeq I_n \wedge D(X).$$

Furthermore, if $I_n = I_n(S^0)$, then the work of Gross and Hopkins implies that there is a p -adic $\mathbb{G}$ -sphere $S\langle \det \rangle$ and an element $P_n \in \kappa_n$ such that

$$I_n \simeq S^{n^2-n} \wedge S\langle \det \rangle \wedge P_n.$$

The invertible $\mathbf{K}$ -local spectrum $S\langle \det \rangle$ is described in great detail in [4] and the spectrum P_n is in fact defined by this equation.

As a consequence of [6, Theorem 1.1] which analyzes $I_n(\mathbf{E}^{hF})$ and Theorem 14.14 above, we have the following result.

Theorem 14.16 *Let $n = p - 1$ and $F = C_p$. Then,*

$$P_n \wedge \mathbf{E}^{hF} \simeq \Sigma^{p^2+p} \mathbf{E}^{hF}$$

In particular, P_n is a non-trivial element of κ_n .

Proof By the discussion above, we have a $\mathbf{K}$ -local equivalence

$$I_n(\mathbf{E}^{hF}) \simeq I_n \wedge D(\mathbf{E}^{hF}) \simeq S^{n^2-n} \wedge S\langle \det \rangle \wedge P_n \wedge D(\mathbf{E}^{hF}). \quad (14.17)$$

The spectrum $\mathbf{E}^{hF}$ is periodic with minimal periodicity $2p^2$. Using this, Theorem 14.14 simplifies to

$$D(\mathbf{E}^{hF}) \simeq \Sigma^{-n^2(1+2p)} \mathbf{E}^{hF} \simeq \Sigma^{-(p^2+1)} \mathbf{E}^{hF}.$$

Note that $F \subseteq \ker(\det)$. In [4], it is shown that this implies that

$$\mathbf{E}^{hF} \wedge S\langle \det \rangle \simeq \mathbf{E}^{hF}.$$

Furthermore, [6, Theorem 1.1] states that $I_n(\mathbf{E}^{hF}) \simeq \Sigma^{n^2} \mathbf{E}^{hF}$.

These facts together with (14.17) imply that

$$\Sigma^{n^2} \mathbf{E}^{hF} \simeq S^{n^2-n} \wedge P_n \wedge \Sigma^{-(p^2+1)} \mathbf{E}^{hF},$$

from which the first claim follows. Furthermore, since $p^2 + p < 2p^2$, P_n cannot be equivalent to $L_{\mathbf{K}} S^0$ otherwise $\mathbf{E}^{hF}$ would have a shorter periodicity. $\square$

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