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Factorization length distribution for affine semigroups IV: a geometric approach to weighted factorization lengths in three-generator numerical semigroups

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ABSTRACT
For numerical semigroups with three generators, we study the asymptotic behavior of weighted factorization lengths, that is, linear functionals of the coefficients in the factorizations of semigroup elements. This work generalizes many previous results, provides more natural and intuitive proofs, and yields a completely explicit error bound.

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1. Introduction

In what follows, $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{>0}$ denote the sets of nonnegative and positive integers, respectively. Let

$$S = \langle n_1, n_2, n_3 \rangle = \{x_1 n_1 + x_2 n_2 + x_3 n_3 : x_1, x_2, x_3 \in \mathbb{Z}_{\geq 0}\}$$

denote a *numerical semigroup* (an additive subsemigroup of $\mathbb{Z}_{\geq 0}$) with three *generators* $n_1, n_2, n_3 \in \mathbb{Z}_{>0}$ [29]. We do not assume that the generators are given in a particular order; on rare occasions, we even let them coincide. Although unconventional, these generous conventions eliminate the need for some special cases and permit a few interesting and unusual applications.

A *factorization* of $n \in S$ is an expression $n = x_1 n_1 + x_2 n_2 + x_3 n_3$ in which $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3_{\geq 0}$. The set of all factorizations of n is denoted

$$\mathsf{Z}_S(n) = \{\mathbf{x} \in \mathbb{Z}^3_{\geq 0} : n = x_1 n_1 + x_2 n_2 + x_3 n_3\}.$$

A *factorization functional* is a linear functional of $\mathbf{x}$. For example, the *length* $x_1 + x_2 + x_3$ of $\mathbf{x}$ is a factorization functional. Other examples are x_1 and $x_1 - 2x_2 + 3x_3$. Values of factorization functionals are *weighted* factorization lengths. Combinatorial descriptions of maximum and

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minimum weighted factorization lengths are obtained in [25]; the present paper pushes this work in new directions.

In this paper, which complements the previous papers in the series [15–17], we answer many questions about the asymptotic behavior of weighted factorization lengths for three-generator numerical semigroups. We recover and extend [17], in which the asymptotic behavior of the mean, median, and mode of (unweighted) factorization lengths are described. Our Theorem 1 is more general and more precise than the main results of [17], and its proof is shorter and more transparent.

The paper [16], which subsumes [17], treats numerical semigroups with arbitrarily many generators. However, the approach uses tools outside the mainstream of numerical semigroup theory, such as algebraic combinatorics, harmonic analysis, measure theory, and functional analysis. In contrast, our results here are geometric and transparent, only invoking analysis (of an elementary sort) at the final stage. The results of this paper, although presented only for three-generator numerical semigroups (but for general weighted factorization lengths), may provide a clearer path to the results of [16] and their generalizations via polyhedral geometry techniques.

As convenience dictates, we denote (column) vectors in boldface, or as ordered pairs or triples. A superscript T denotes the transpose. We let $|X|$ denote the cardinality of a set or multiset X. Here is our main result.

Theorem 1. *Let $n_1, n_2, n_3 \in \mathbb{Z}_{>0}$ have $\gcd(n_1, n_2, n_3) = 1$; let $m_1, m_2, m_3 \in \mathbb{Z}$ be such that*

$$\frac{m_3}{n_3} \leq \frac{m_2}{n_2} \leq \frac{m_1}{n_1},$$

with at least one inequality strict; let $S = \langle n_1, n_2, n_3 \rangle$; and let

$$\lambda(\mathbf{x}) = m_1 x_1 + m_2 x_2 + m_3 x_3$$

for $\mathbf{x} \in \mathbb{Z}^3$. Define the multiset (set with multiplicities)

$$\Lambda[\![n]\!] = \{\!\{ \lambda(\mathbf{x}) : \mathbf{x} \in \mathsf{Z}_S(n) \}\!\}.$$

Then for $\alpha < \beta$ and $n \in \mathbb{Z}_{\geq 0}$,

$$\left| \frac{|\Lambda[\![n]\!] \cap [\alpha n, \beta n]|}{n^2/(2n_1 n_2 n_3)} - \int_\alpha^\beta F(x)\, dx \right| \leq \frac{2n_1 n_2 n_3}{n} \left[\frac{5d}{n_2} + \frac{2d}{n} + \left(\beta - \alpha + \frac{2d}{n} \right) \left(1 + d \max\{n_1, n_3\} \right) \right],$$

in which

$$F(t) = \frac{2n_1 n_2 n_3}{m_1 n_3 - m_3 n_1} \begin{cases} 0 & \text{if } t < \dfrac{m_3}{n_3}, \\ \dfrac{t n_3 - m_3}{m_2 n_3 - m_3 n_2}, & \text{if } \dfrac{m_3}{n_3} \leq t < \dfrac{m_2}{n_2}, \\ \dfrac{m_1 - n_1 t}{m_1 n_2 - m_2 n_1} & \text{if } \dfrac{m_2}{n_2} \leq t < \dfrac{m_1}{n_1}, \\ 0 & \text{if } t \geq \dfrac{m_1}{n_1}, \end{cases}$$

is a (possibly degenerate) triangular probability density function, and

$$d = \gcd(m_2 n_3 - m_3 n_2,\ m_1 n_3 - m_3 n_1,\ m_1 n_2 - m_2 n_1).$$

The rate of convergence to the triangular density is explicit, a huge improvement over [17]. Moreover, the error estimate can be improved at the expense of introducing a more complicated, but still explicit, expression; see Remark 31. Modifications of our Lemma 23 should also permit us to recover the modular results of [15] in the three-generator setting, with the added bonus of explicit bounds on the rate of convergence in [15, Theorem 3a].

The motivation for Theorem 1 stems from its centrality to the study of numerical semigroups. Non-unique factorization has long been studied in commutative algebra, both for more general families of semigroups [1, 12, 13, 22, 23, 30] and for numerical semigroups specifically [7, 8, 24, 28]. The study of length sets (as opposed to multisets) is well-established territory [2, 14, 19, 21, 26] and similar questions have been studied in both number-theoretic [5, 6, 11] and algebraic [3, 4, 20] contexts. Our explicit asymptotic theorem on weighted factorization lengths and multisets breaks new ground in the three-generator setting.

This paper is structured as follows. We first consider examples and applications in Section 2, after which we move into the proof of Theorem 1 in Section 3.

2. Examples and applications

Throughout this section, we consider pairs of vectors $\mathbf{m} = (m_1, m_2, m_3) \in \mathbb{Z}^3$ and $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3_{>0}$ which satisfy the conditions of Theorem 1. In each such context we define $S = \langle n_1, n_2, n_3 \rangle$,

$$\lambda(\mathbf{x}) = m_1x_1 + m_2x_2 + m_3x_3,$$

and

$$\Lambda[\![n]\!] = \{\!\{\lambda(\mathbf{x}) : \mathbf{x} \in \mathsf{Z}_S(n)\}\!\}.$$

as in the statement of Theorem 1. We also define

$$\mathcal{Z}(m, n) = \{\mathbf{x} \in \mathsf{Z}_S(n) : \lambda(\mathbf{x}) = m\}. \tag{2}$$

Our first application of Theorem 1 is to swiftly obtain general weighted versions of the main results of [17], in which the asymptotic mean, median, and mode (unweighted) factorization lengths are computed for three-generator numerical semigroups. In what follows, $f \sim g$ means that $\lim_{n\to\infty} f(n)/g(n) = 1$.

Example 3. Let $S = \langle n_1, n_2, n_3 \rangle$, in which $\gcd(n_1, n_2, n_3) = 1$. Apply Theorem 1 with $\alpha = \frac{m_3}{n_3}$ and $\beta = \frac{m_1}{n_1}$ and obtain [27, Theorem 3.9]:

$$\Lambda[\![n]\!] = |\mathsf{Z}_S(n)| \sim \frac{n^2}{2n_1n_2n_3}. \tag{4}$$

For $\alpha < \beta$, Theorem 1 and (4) ensure that

$$\frac{|\Lambda[\![n]\!] \cap [\alpha n, \beta n]|}{|\Lambda[\![n]\!]|} \sim \int_\alpha^\beta F(x)dx$$

as $n \to \infty$. Since the support of F is $[\frac{m_3}{n_3}, \frac{m_1}{n_1}]$ and its peak is at $\frac{m_2}{n_2}$, we have

$$\text{Min } \Lambda[\![n]\!] \sim \frac{m_3}{n_3}n, \qquad \text{Mode } \Lambda[\![n]\!] \sim \frac{m_2}{n_2}n, \qquad \text{and} \qquad \text{Max } \Lambda[\![n]\!] \sim \frac{m_1}{n_1}n.$$

Symbolic integration and computer algebra reveals the unique $\gamma \in [\frac{m_3}{n_3}, \frac{m_1}{n_1}]$ such that $\int_{-\infty}^{\gamma} F(t)dt = \frac{1}{2}$. This yields the asymptotic median:

$$\text{Median } \Lambda[\![n]\!] \sim n \cdot \begin{cases} \dfrac{m_3}{n_3} + \sqrt{\dfrac{1}{2}\left(\dfrac{m_1}{n_1} - \dfrac{m_3}{n_3}\right)\left(\dfrac{m_2}{n_2} - \dfrac{m_3}{n_3}\right)} & \text{if } \dfrac{m_2}{n_2} \geq \dfrac{1}{2}\left(\dfrac{m_1}{n_1} + \dfrac{m_3}{n_3}\right), \\ \dfrac{m_1}{n_1} - \sqrt{\dfrac{1}{2}\left(\dfrac{m_1}{n_1} - \dfrac{m_3}{n_3}\right)\left(\dfrac{m_1}{n_1} - \dfrac{m_2}{n_2}\right)} & \text{if } \dfrac{m_2}{n_2} < \dfrac{1}{2}\left(\dfrac{m_1}{n_1} + \dfrac{m_3}{n_3}\right). \end{cases}$$

Consider the absolutely continuous probability measure ν defined by

$$\nu([\alpha,\beta]) = \int_\alpha^\beta F(x)\,dx$$

for $\alpha < \beta$. Define the singular probability measures

$$\nu_n = \frac{1}{|\mathsf{Z}_S(n)|}\sum_{\mathbf{x}\in\mathsf{Z}_S(n)} \delta_{\frac{\mu(\mathbf{x})}{n}},$$

in which δ_x is the unit point measure at $x \in \mathbb{R}$. Use (4) to deduce that

$$\lim_{n\to\infty} \nu_n([\alpha,\beta]) = \lim_{n\to\infty} \frac{|\Lambda[\![n]\!] \cap [\alpha n, \beta n]|}{|\mathsf{Z}_S(n)|} = \int_\alpha^\beta F(x)\,dx = \nu([\alpha,\beta]).$$

If $g : \mathbb{R} \to \mathbb{R}$ is bounded and continuous, then [9, Theorem 25.8] ensures that

$$\lim_{n\to\infty} \frac{1}{|\Lambda[\![n]\!]|}\sum_{\mathbf{x}\in\mathsf{Z}_S(n)} g\left(\frac{\lambda(\mathbf{x})}{n}\right) = \lim_{n\to\infty}\int_{\mathbb{R}} g\,d\nu_n = \int_{\mathbb{R}} g(x)F(x)\,dx.$$

The integral on the right-hand side can be evaluated explicitly for $g(x) = x$ and $g(x) = x^2$. From here one obtains the asymptotic mean and variance of $\Lambda[\![n]\!]$:

$$\text{Mean } \Lambda[\![n]\!] \sim \frac{n}{3}\left(\frac{m_1}{n_1}+\frac{m_2}{n_2}+\frac{n_3}{n_3}\right),$$

$$\text{Var } \Lambda[\![n]\!] \sim \frac{\mathrm{n}^2}{18}\left(\frac{m_1^2}{n_1^2}+\frac{m_2^2}{n_2^2}+\frac{m_3^2}{n_3^2}-\frac{m_1m_2}{n_1n_2}-\frac{m_2m_3}{n_2n_3}-\frac{m_3m_1}{n_3n_1}\right).$$

Asymptotic formulas for the higher moments, skewness, harmonic mean, and geometric mean, follow in a similar manner; see [16, Section 2.1] for definitions. For $m_1 = m_2 = m_3 = 1$, we obtain the asymptotic formulas for factorization-length statistics obtained in [17]. Thus, Theorem 1 recaptures the results of [17], generalizes them to the weighted setting, and provides explicit error bounds in some instances.

Example 5. In [16, Table 1, Figure 2], a special case of Theorem 1 was illustrated for factorization lengths in the McNugget semigroup $S = \langle 6, 9, 20\rangle$. Here we explore a different weighted factorization length on S. Table 1 gives the actual and predicted values of several statistics pertaining to $\Lambda[\![n]\!]$ for $\mathbf{m} = (4, 7, 2)$, $\mathbf{n} = (9, 20, 6)$, and $n = 10^5$. The components of $\mathbf{m}$ and $\mathbf{n}$ are ordered to comply with Theorem 1; in particular $4/9 > 7/20 > 2/6$. If one charges \$2 for a box of 6 McNuggets, \$4 for 9 McNuggets, and \$7 for 20 McNuggets, then $\Lambda[\![n]\!]$ is the multiset of prices corresponding to all the ways to fill an order of n McNuggets.

The next example illustrates another use of Theorem 1.

Example 6. Let $S = \langle 6, 9, 20\rangle$ as in the previous example. We now let $\mathbf{n} = (1, 1, 1)$ and $\mathbf{m} = (20, 9, 6)$. Then $|\Lambda[\![n]\!] \cap [\alpha n, \beta n]|$ is the number of possible orders of n boxes of McNuggets that contain between αn and βn McNuggets. For example, when $n = 100$, $\alpha = 8$, and $\beta = 15$, we have $|\Lambda[\![100]\!] \cap [800, 1500]| = 3785$; that is, there are 3785 ways to order between 800 and 1500 McNuggets using 100 boxes. Table 2 illustrates predictions and error bounds afforded by Theorem 1 and (30).

In the following examples, we plot $\frac{|\mathcal{Z}(m,n)|}{dn/(2n_1n_2n_3)}$ versus $\frac{m}{n}$ (in blue) overlaid with $F(x)$ versus x (in red). These make sense to plot together because Lemma 18 and Equation (22) imply that $\frac{|\mathcal{Z}(m,n)|}{dn/(2n_1n_2n_3)}$ is within $\frac{2n_1n_2n_3}{dn}$ of $F\left(\frac{m}{n}\right)$. Since $|\mathcal{Z}(m,n)|$ gives the multiplicity of m in $\Lambda[\![n]\!]$, we refer

Table 1. Actual versus predicted statistics (rounded to two decimal places) for $\Lambda[\![10^5]\!]$ with $\mathbf{n} = (9, 20, 6)$ and $\mathbf{m} = (4, 7, 2)$.

Statistic	Actual	Predicted	Statistic	Actual	Predicted
Mean $\Lambda[\![10^5]\!]$	37,591.84	37,592.59	Mode $\Lambda[\![10^5]\!]$	35,000	35,000
Median $\Lambda[\![10^5]\!]$	37,200	37,200.89	StDev $\Lambda[\![10^5]\!]$	2446.32	2446.27
Min $\Lambda[\![10^5]\!]$	33,334	33,333.33	Max $\Lambda[\![10^5]\!]$	44,440	44,444.44

Table 2. Error analysis (rounded to 6 decimal places) for $\mathbf{m} = (20, 9, 6)$ and $\mathbf{n} = (1, 1, 1)$.

n	α	β	$\frac{\lvert\Lambda[\![n]\!]\cap[\alpha n,\beta n]\rvert}{n^2/2}$	$\int_\alpha^\beta F(x)\,dx$	Error	Theorem 1 bound	Equation (30) bound
100	8	15	0.757	0.742424	0.014576	0.3812	0.151286
1000	8	15	0.743884	0.742424	0.001460	0.038012	0.015056
10000	8	15	0.742570	0.742424	0.000146	0.003800	0.001505
100	7	7.1	0.0058	0.005	0.0008	0.1052	0.01
1000	7	7.1	0.00509	0.005	0.00009	0.010412	0.000927
10,000	7	7.1	0.005009	0.005	0.000009	0.001040	0.000092

to this sort of plot as the *scaled histogram* of $\Lambda[\![n]\!]$. These plots illustrate the convergence of the distribution of $\Lambda[\![n]\!]$ to $F(x)$.

Example 7. Figure 1 gives the scaled histograms of $\Lambda[\![100]\!]$ and $\Lambda[\![1000]\!]$ for $\mathbf{m} = (20, 9, 6)$ and $\mathbf{n} = (1, 1, 1)$.

Example 8. Theorem 1 does not require m_1, m_2, m_3 to be positive. Figure 2 demonstrates the theorem when $m_1 < 0$.

Example 9. The error bound in Theorem 1 and the definition of the scaled histogram involve the quantity $d = \gcd(m_2n_3 - m_3n_2, m_1n_3 - m_3n_1, m_1n_2 - m_2n_1)$. For $d=1$, the scaled histogram of $\Lambda[\![n]\!]$ approximately coincides with the plot of $F(x)$ at each point. For $d \neq 1$, Lemma 15 says that there is a $c = c_n$ such that $\mathcal{Z}(m, n)$ is empty unless $m \equiv c \pmod{d}$. If $\mathcal{Z}(m, n)$ is nonempty, Lemma 18 implies that its cardinality is d times larger than what we would expect for $d=1$. This is accounted for in the definition of the scaled histogram so that $d - 1$ out of every d points of the scaled histogram of $\Lambda[\![n]\!]$ are 0, but the remaining points approximately lie on the plot of $F(x)$; see Figure 3.

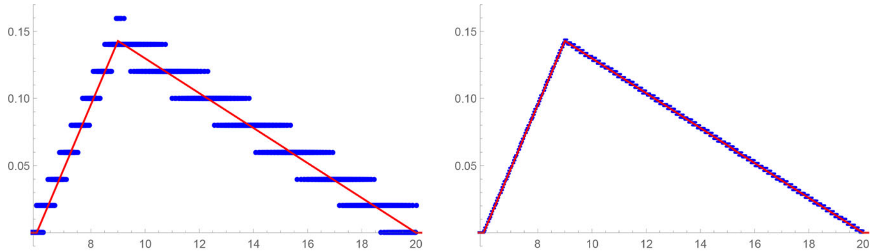
Example 10. The proof of Theorem 1 defines $\rho_1 = m_2n_3 - m_3n_2$ and $\rho_3 = m_1n_2 - m_2n_1$. Although these are denominators in the formula for F, we permit one of them to be 0. Figure 4 illustrates the case $\mathbf{n} = (6, 9, 20)$ and $\mathbf{m} = (1, 0, 0)$, for which $\rho_1 = 0$. Here $\lambda(\mathbf{x}) = x_1$ is the number of 6s in the factorization $6x_1 + 9x_2 + 30x_3 = n$. Since $\rho_1 = 0$, the "left side" of the triangle is degenerate.

Theorem 1 concerns large-n asymptotic behavior. On the other hand, Proposition 12 identifies a curious exact phenomenon even for small n. We first illustrate this with an example.

Example 11. Let $\mathbf{m}_1 = (2, 3, 1)$, $\mathbf{n}_1 = (2, 6, 3)$, $\mathbf{m}_2 = (3, 1, 2)$, and $\mathbf{n}_2 = (3, 2, 6)$; note that $\mathbf{n}_1$ and $\mathbf{n}_2$ generate the same semigroup. Figure 5 shows the scaled histograms of the multisets $\Lambda_1[\![n]\!]$ and $\Lambda_2[\![n]\!]$ corresponding to $\mathbf{m}_1, \mathbf{n}_1$ and to $\mathbf{m}_2, \mathbf{n}_2$, respectively. The histograms are the same up to a horizontal translation. To be specific, there is an r, which depends only upon n, such that the multiplicity of x in $\Lambda_1[\![n]\!]$ equals the multiplicity of $x+r$ in $\Lambda_2[\![n]\!]$. In Figure 5, we have $n=75$ and $r=2$. Observe that the probability density F depends only upon m_1/n_1, m_2/n_2, and m_3/n_2, so Theorem 1 predicts the same asymptotic distribution for $\Lambda_1[\![n]\!]$ and $\Lambda_2[\![n]\!]$ because

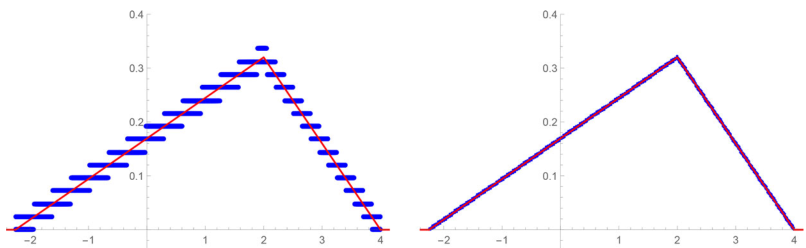
$$\frac{m_1}{n_1} = \frac{2}{2} = \frac{m_1'}{n_1'} = \frac{3}{3} = 1, \qquad \frac{m_2}{n_2} = \frac{3}{6} = \frac{m_2'}{n_2'} = \frac{1}{2}, \quad \text{and} \quad \frac{m_3}{n_3} = \frac{1}{3} = \frac{m_3'}{n_3'} = \frac{2}{6}.$$

However, this only implies that $\Lambda_1[\![n]\!]$ and $\Lambda_2[\![n]\!]$ should appear similar for large n, not that they should be translations of each other.



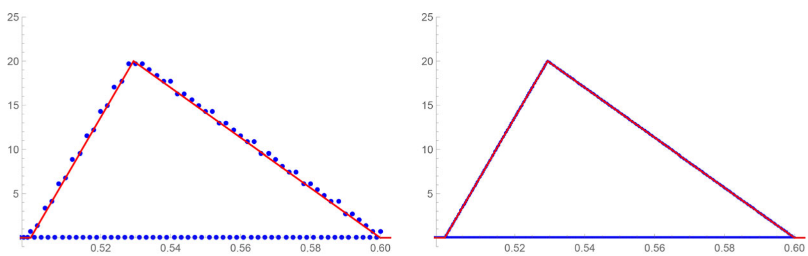
(a) $n = 100$ **(b)** $n = 50000$

Figure 1. Scaled histograms of $\Lambda[\![n]\!]$ with $\mathbf{n} = (1, 1, 1)$ and $\mathbf{m} = (20, 9, 6)$.



(a) $n = 5000$ **(b)** $n = 50000$

Figure 2. Scaled histograms of $\Lambda[\![n]\!]$ with $\mathbf{m} = (-9, 20, 6)$ and $\mathbf{n} = (4, 5, 3)$.



(a) $n = 1000$ **(b)** $n = 30000$

Figure 3. Scaled histograms of $\Lambda[\![n]\!]$ with $\mathbf{n} = (5, 17, 8)$ and $\mathbf{m} = (3, 9, 4)$; here $d = 2$.

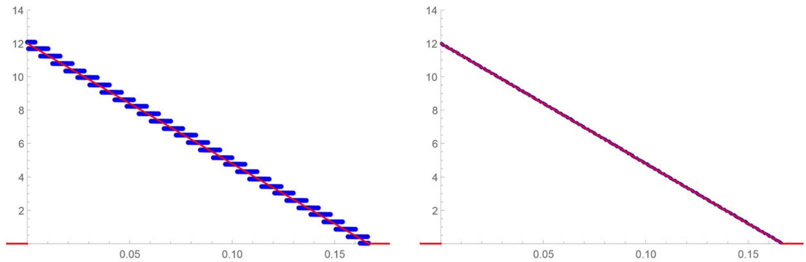
Proposition 12 says that two different weighted lengths on the same numerical semigroup yield nearly the same (translated) statistical behavior. This is consistent with Theorem 1 since

$$\left\{\frac{ac}{a}, \frac{ab}{b}, \frac{bc}{c}\right\} = \{a, b, c\} = \left\{\frac{ac}{c}, \frac{ab}{a}, \frac{bc}{b}\right\},$$

so the asymptotic distribution functions in the two cases are equal. The numerical semigroups in Proposition 12 are called *supersymmetric* [10].

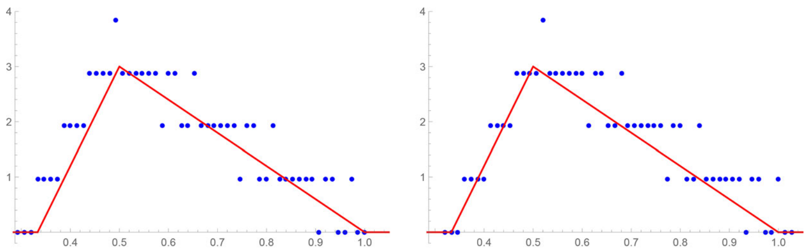
Proposition 12. *Fix distinct, pairwise coprime $a, b, c \in \mathbb{Z}_{>0}$, and let $\mathbf{m}_1 = (b, a, c)$, $\mathbf{m}_2 = (a, c, b)$, and $\mathbf{n} = (ab, ac, bc)$. Define*

$$A_1 = \begin{bmatrix} \mathbf{m}_1^\mathsf{T} \\ \mathbf{n}^\mathsf{T} \end{bmatrix} \in \mathsf{M}_{2\times 3}(\mathbb{Z}) \qquad \text{and} \qquad A_2 = \begin{bmatrix} \mathbf{m}_2^\mathsf{T} \\ \mathbf{n}^\mathsf{T} \end{bmatrix} \in \mathsf{M}_{2\times 3}(\mathbb{Z}),$$



(a) $n = 5000$ **(b)** $n = 50000$

Figure 4. Scaled histograms of $\Lambda[\![n]\!]$ with $\mathbf{n} = (6, 9, 20)$ and $\mathbf{m} = (1, 0, 0)$.



(a) $\mathbf{m}_1 = (2, 3, 1)$, $\mathbf{n}_2 = (2, 6, 3)$ **(b)** $\mathbf{m}_2 = (3, 1, 2)$ and $\mathbf{n}_2 = (3, 2, 6)$

Figure 5. Different values of $\mathbf{m}$ and $\mathbf{n}$ can produce scaled histograms that are translations of each other. In the context of Proposition 12, $(a, b, c) = (1, 2, 3)$.

along with

$$\mathcal{Z}_1(m,n) = \left\{\mathbf{x} \in \mathbb{Z}^3_{\geq 0} : A_1\mathbf{x} = \begin{bmatrix} m \\ n \end{bmatrix}\right\} \quad \text{and} \quad \mathcal{Z}_2(m,n) = \left\{\mathbf{x} \in \mathbb{Z}^3_{\geq 0} : A_2\mathbf{x} = \begin{bmatrix} m \\ n \end{bmatrix}\right\}.$$

For all $n \in \mathbb{Z}_{\geq 0}$, there exists an $r_n \in \mathbb{Z}$ such that for all $m \in \mathbb{Z}$,

$$|\mathcal{Z}_1(m,n)| = |\mathcal{Z}_2(m + r_n, n)|.$$

Moreover, $r_n = r_{n+abc}$ for all $n \in \mathbb{Z}_{\geq 0}$.

Proof. Let $S = \langle ab, ac, bc\rangle$ and fix $n \in S$. We can write $n = qabc + r$ with $r \in S$ and $r - abc \notin S$. By [18, Proposition 1, Theorem 12], we have $|\mathsf{Z}_S(r)| = 1$,

$$\mathsf{Z}_S(qabc) = \{(z_1c, z_2b, z_3a) \in \mathbb{Z}^3_{\geq 0} : z_1 + z_2 + z_3 = q\},$$

and

$$\mathsf{Z}_S(n) = \mathsf{Z}_S(qabc) + \mathsf{Z}_S(r).$$

For any $z_1, z_2, z_3 \in \mathbb{Z}_{\geq 0}$ with $z_1 + z_2 + z_3 = q$, we have

$$(z_1c, z_2b, z_3a) \cdot \mathbf{m}_1 = z_1bc + z_2ab + z_3ac = (z_3c, z_1b, z_2a) \cdot \mathbf{m}_2,$$

which implies $|\mathcal{Z}_1(m, qabc)| = |\mathcal{Z}_2(m, qabc)|$ for all $m \in \mathbb{Z}$. Writing $\mathsf{Z}_S(r) = \{\mathbf{x}\}$, linearity then implies

$$\mathcal{Z}_1(m + (\mathbf{m}_1 \cdot \mathbf{x}), n) = \mathcal{Z}_1(m, qabc) + \mathbf{x} \quad \text{and} \quad \mathcal{Z}_2(m + (\mathbf{m}_2 \cdot \mathbf{x}), n) = \mathcal{Z}_2(m, qabc) + \mathbf{x}$$

for all $m \in \mathbb{Z}$. This yields the desired claim upon letting $r_n = (\mathbf{m}_2 - \mathbf{m}_1) \cdot \mathbf{x}$. $\square$

3. Proof of Theorem 1

The proof of Theorem 1 is geometric: the limiting distribution arises from the projection of a simplex with one vertex on each axis, with each vertical value in the distribution being the volume of a cross section. This yields a piecewise-polynomial function; the transition between each polynomial piece occurs when the cross section contains a vertex. Making this general and precise, with explicit error bounds, adds to the complexity of the argument.

3.1. Setup

Let $\mathbf{m} = (m_1, m_2, m_3)$, $\mathbf{n} = (n_1, n_2, n_3)$, and

$$A = \begin{bmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} = \begin{bmatrix} \mathbf{m}^\mathsf{T} \\ \mathbf{n}^\mathsf{T} \end{bmatrix} \in \mathsf{M}_{2\times 3}(\mathbb{Z}).$$

The hypotheses on the ratios m_i/n_i imply that

$$\rho_1 = m_2n_3 - m_3n_2, \quad \rho_2 = m_1n_3 - m_3n_1, \quad \text{and} \quad \rho_3 = m_1n_2 - m_2n_1 \tag{13}$$

are all non-negative, and moreover that $\rho_2 > 0$. Observe that

$$\rho_1 = 0 \iff \frac{m_3}{n_3} = \frac{m_2}{n_2} \qquad \text{and} \qquad \rho_3 = 0 \iff \frac{m_2}{n_2} = \frac{m_1}{n_1},$$

so at most one of ρ_1, ρ_3 can be zero, since otherwise $\rho_2 = 0$ and $m_3/n_3 = m_2/n_2 = m_1/n_1$. Treat the corresponding interval $[\frac{m_3}{n_3}, \frac{m_2}{n_2}]$ or $[\frac{m_2}{n_2}, \frac{m_1}{n_1}]$ as degenerate in these cases. This also means that at least two of the three inequalities in (13) are strict.

The one-dimensional subspace $\{\mathbf{m}\}^\perp \cap \{\mathbf{n}\}^\perp$ of $\mathbb{R}^3$ is spanned by

$$\mathbf{r} = \mathbf{m} \times \mathbf{n} = \begin{bmatrix} m_2n_3 - m_3n_2 \\ m_3n_1 - m_1n_3 \\ m_1n_2 - m_2n_1 \end{bmatrix} = \begin{bmatrix} \rho_1 \\ -\rho_2 \\ \rho_3 \end{bmatrix} \in \mathbb{Z}^3 \backslash \{\mathbf{0}\}.$$

By construction, $A\mathbf{r} = \mathbf{0}$. Define $\lambda(\mathbf{x})$ and $\Lambda[\![n]\!]$ as in the statement of Theorem 1 and note that $\lambda(\mathbf{x}) = \mathbf{m} \cdot \mathbf{x}$.

3.2. The sets $\mathcal{Z}(b)$ and $\widetilde{\mathcal{Z}}(b)$

We adjust the notation (2) to permit vector arguments: for $\mathbf{b} = (m, n) \in \mathbb{Z}^2$, let

$$\mathcal{Z}(\mathbf{b}) = \{\mathbf{x} \in \mathbb{Z}^3_{\geq 0} : A\mathbf{x} = \mathbf{b}\} = \{\mathbf{x} \in \mathsf{Z}_S(n) : \lambda(\mathbf{x}) = m\}. \tag{14}$$

Similarly, define

$$\widetilde{\mathcal{Z}}(\mathbf{b}) = \{\mathbf{x} \in \mathbb{Z}^3 : A\mathbf{x} = \mathbf{b}\}.$$

We may denote these as $\mathcal{Z}(m,n)$ and $\widetilde{\mathcal{Z}}(m,n)$, respectively, as convenient. Both $\mathcal{Z}(\mathbf{b})$ and $\widetilde{\mathcal{Z}}(\mathbf{b})$ may be empty; the following lemma gives some crucial insight on when $\widetilde{\mathcal{Z}}(\mathbf{b})$ is empty. Although the lemma is a special case of [25, Theorem 3.2b], we provide another proof since the three-dimensional setting permits the use of the cross product and geometric reasoning to simplify the argument.

Lemma 15. *Let $d = \gcd(\rho_1, \rho_2, \rho_3)$. For each $n \in \mathbb{Z}$, there is some $c \in \{0, 1, \ldots, d-1\}$ such that $\widetilde{\mathcal{Z}}(m,n) \neq \varnothing$ if and only if $m \equiv c \pmod{d}$.*

Proof. The definition of d ensures that, $\rho_i \equiv 0 \pmod{d}$ for $i = 1, 2, 3$. Thus,

$$m_in_j \equiv m_jn_i \pmod{d}$$

for $i, j = 1, 2, 3$. For any $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$ and $i \in \{1, 2, 3\}$, it follows that

$$m_i(n_1x_1 + n_2x_2 + n_3x_3) \equiv n_i(m_1x_1 + m_2x_2 + m_3x_3) \pmod{d};$$

that is,

$$m_i(\mathbf{n} \cdot \mathbf{x}) \equiv n_i(\mathbf{m} \cdot \mathbf{x}) \pmod{d}.$$

($\Rightarrow$) Suppose there is an $\mathbf{x} \in \widetilde{\mathcal{Z}}(m, n)$. Since $\gcd(n_1, n_2, n_3) = 1$, Bézout's identity provides $a_1, a_2, a_3 \in \mathbb{Z}$ such that $a_1n_1 + a_2n_2 + a_3n_3 = 1$. Let c denote the least nonnegative residue of $(a_1m_1 + a_2m_2 + a_3m_3)n$ modulo d. Then

$$\begin{aligned} m &= \mathbf{m} \cdot \mathbf{x} \\ &= (a_1n_1 + a_2n_2 + a_3n_3)(\mathbf{m} \cdot \mathbf{x}) \\ &\equiv (a_1m_1 + a_2m_2 + a_3m_3)(\mathbf{n} \cdot \mathbf{x}) \pmod{d} \\ &\equiv (a_1m_1 + a_2m_2 + a_3m_3)n \pmod{d} \\ &\equiv c \pmod{d}. \end{aligned}$$

($\Leftarrow$) Since $d = \gcd(\rho_1, \rho_2, \rho_3)$, Bézout's identity provides a $\mathbf{v} \in \mathbb{Z}^3$ such that

$$\mathbf{r} \cdot \mathbf{v} = (\rho_1, -\rho_2, \rho_3) \cdot \mathbf{v} = d.$$

Let $\mathbf{w} = \mathbf{n} \times \mathbf{v}$ and observe that

$$\mathbf{n} \cdot \mathbf{w} = \mathbf{n} \cdot (\mathbf{n} \times \mathbf{v}) = (\mathbf{n} \times \mathbf{n}) \cdot \mathbf{v} = 0$$

and

$$\mathbf{m} \cdot \mathbf{w} = \mathbf{m} \cdot (\mathbf{n} \times \mathbf{v}) = (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{v} = \mathbf{r} \cdot \mathbf{v} = d.$$

Fix $n \in \mathbb{Z}_{\geq 0}$. Since $\gcd(n_1, n_2, n_3) = 1$, there is a $\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{Z}^3$ such that $\mathbf{n} \cdot \mathbf{z} = n$. Let $s = \mathbf{m} \cdot \mathbf{z}$, so that $\mathbf{z} \in \widetilde{\mathcal{Z}}(s, n)$. The first half of the proof ensures that $s \equiv c \pmod{d}$. If $m \equiv c \pmod{d}$, then $d \mid (m - s)$ and hence

$$\mathbf{m} \cdot \left(\mathbf{z} + \frac{m-s}{d}\mathbf{w}\right) = \mathbf{m} \cdot \mathbf{z} + \frac{m-s}{d}\mathbf{m} \cdot \mathbf{w} = s + \frac{m-s}{d}d = m.$$

Therefore,

$$\mathbf{z} + \frac{m-s}{d}\mathbf{w} \in \widetilde{\mathcal{Z}}\left(s + \frac{m-s}{d}d, n\right) = \widetilde{\mathcal{Z}}(m, n). \qquad \square$$

Lemma 16. *Let* $\mathbf{b} \in \mathbb{Z}^2$. *If* $\mathbf{z} \in \widetilde{\mathcal{Z}}(\mathbf{b})$, *then* $\widetilde{\mathcal{Z}}(\mathbf{b}) = \{\mathbf{z} + s\mathbf{r}/d : s \in \mathbb{Z}\}$ *where* $d = \gcd(\rho_1, \rho_2, \rho_3)$.

Proof. Since $A\mathbf{r} = \mathbf{0}$, we have $A(\mathbf{z} + s\mathbf{r}/d) = \mathbf{b}$. Additionally, $\mathbf{r}/d \in \mathbb{Z}^3$ because $d = \gcd(\rho_1, \rho_2, \rho_3)$. Therefore, $\{\mathbf{z} + s\mathbf{r}/d : s \in \mathbb{Z}\} \subseteq \widetilde{\mathcal{Z}}(\mathbf{b})$. Suppose that $\mathbf{x} \in \widetilde{\mathcal{Z}}(\mathbf{b})$. Then $A(\mathbf{x} - \mathbf{z}) = A\mathbf{x} - A\mathbf{z} = \mathbf{0}$, so $\mathbf{x} - \mathbf{z} = s\mathbf{r}/d$ for some $s \in \mathbb{R}$. Then $s\mathbf{r}/d = \mathbf{x} - \mathbf{z} \in \mathbb{Z}^3$, and hence $s \in \mathbb{Z}$ because $\gcd(\rho_1, \rho_2, \rho_3) = d$. Thus, $\widetilde{\mathcal{Z}}(\mathbf{b}) \subseteq \{\mathbf{z} + s\mathbf{r}/d : s \in \mathbb{Z}\}$.

3.3. Some geometry

For $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$, let $\ell(\mathbf{y})$ denote the length of the line segment

$$\mathcal{L}(\mathbf{y}) = \{\mathbf{x} \in \mathbb{R}^3_{\geq 0} : A\mathbf{x} = \mathbf{y}\}$$

if it is nonempty; let $\ell(\mathbf{y}) = 0$ otherwise. On occasion, we may write $\mathcal{L}(y_1, y_2)$ and $\ell(y_1, y_2)$ instead. The line $\mathcal{L}(\mathbf{y})$ is contained in the plane $\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{n} \cdot \mathbf{x} = y_2\}$ which, owing to the positivity of the components of $\mathbf{n}$, has compact intersection with $\mathbb{R}^3_{\geq 0}$. Thus, $\ell(\mathbf{y})$ is finite. Observe that for $n \in \mathbb{Z}_{n>0}$,

$$n\ell\left(\frac{x}{n},1\right) = \ell(x,n). \tag{17}$$

Lemma 18. *Let $d = \gcd(\rho_1,\rho_2,\rho_3)$ and $\mathbf{b} \in \mathbb{Z}^2$. If $\widetilde{\mathcal{Z}}(\mathbf{b}) \neq \emptyset$, then*

$$\frac{d\ell(\mathbf{b})}{||\mathbf{r}||} - 1 \;\leq\; |\mathcal{Z}(\mathbf{b})| \;\leq\; \frac{d\ell(\mathbf{b})}{||\mathbf{r}||} + 1.$$

Proof. Suppose that $\mathcal{Z}(\mathbf{b}) \neq \emptyset$. Then Lemma 16 provides a $\mathbf{z} \in \mathbb{Z}^3$ such that

$$\mathcal{Z}(\mathbf{b}) = \mathbb{Z}^3_{\geq 0} \cap \{\mathbf{z} + s\mathbf{r}/d : s \in \mathbb{Z}\}.$$

Define

$$a = \inf\{s \in \mathbb{R} : \mathbf{z} + s\mathbf{r}/d \in \mathbb{R}^3_{\geq 0}\} \qquad \text{and} \qquad b = \sup\{s \in \mathbb{R} : \mathbf{z} + s\mathbf{r}/d \in \mathbb{R}^3_{\geq 0}\}.$$

Then $\mathbf{z} + s\mathbf{r}/d \in \mathbb{R}^3_{\geq 0}$ if and only if $s \in [a,b]$. Consequently,

$$|\mathcal{Z}(\mathbf{b})| = |[a,b] \cap \mathbb{Z}| = \lfloor b \rfloor - \lceil a \rceil + 1.$$

Since

$$b - 1 < \lfloor b \rfloor \leq b \qquad \text{and} \qquad -a - 1 \leq -\lceil a \rceil \leq -a,$$

it follows that

$$b - a - 1 \leq |\mathcal{Z}(\mathbf{b})| \leq b - a + 1. \tag{19}$$

The length of $\mathcal{L}(\mathbf{b})$ is

$$\ell(\mathbf{b}) = ||(\mathbf{z} + b\mathbf{r}/d) - (\mathbf{z} + a\mathbf{r}/d)|| = \frac{b-a}{d}||\mathbf{r}||.$$

Substitute $b - a = d\ell(\mathbf{b})/||\mathbf{r}||$ in (19) and obtain the desired inequalities. □

3.4. The triangle emerges

Recall that $f : I \to \mathbb{R}$ is *Lipschitz* on a (possibly infinite) interval I with *Lipschitz constant* C if $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in I$.

Lemma 20. *Suppose that $\rho_1, \rho_3 \neq 0$. For $t \in \mathbb{R}$,*

$$\ell(t,1) = \frac{||\mathbf{r}||}{\rho_2}\begin{cases} 0 & \text{if } t < \dfrac{m_3}{n_3}, \\[2mm] \dfrac{n_3 t - m_3}{\rho_1} & \text{if } \dfrac{m_3}{n_3} \leq t \leq \dfrac{m_2}{n_2}, \\[2mm] \dfrac{m_1 - n_1 t}{\rho_3} & \text{if } \dfrac{m_2}{n_2} \leq t \leq \dfrac{m_1}{n_1}, \\[2mm] 0 & \text{if } t > \dfrac{m_1}{n_1}. \end{cases}$$

is a "triangular" function of t with base $[\frac{m_3}{n_3}, \frac{m_1}{n_1}]$, peak at $t = \frac{m_2}{n_2}$, and height

$$\ell\left(\frac{m_2}{n_2},1\right) = \frac{||\mathbf{r}||}{n_2\rho_2}.$$

Furthermore, $\ell(t,1)$ is Lipschitz with Lipschitz constant

$$\frac{||\mathbf{r}||}{\rho_2}\max\left\{\frac{n_3}{\rho_1},\frac{n_1}{\rho_3}\right\}.$$

Proof. If it is nonempty, the line segment $\mathcal{L}(t,1)$ lies in $\mathbb{R}^3_{\geq 0}$; its endpoints each lie on one of the coordinate planes. Solve the corresponding equations and obtain the points of intersection with the three coordinate planes:

- $\mathbf{p}_1(t)=\rho_1^{-1}(0,n_3t-m_3,m_2-n_2t)$, hence $\mathbf{p}_1(t)\in\mathbb{R}^3_{\geq 0}\iff t\in[\frac{m_3}{n_3},\frac{m_2}{n_2}]$,
- $\mathbf{p}_2(t)=\rho_2^{-1}(n_3t-m_3,0,m_1-n_1t)$, hence $\mathbf{p}_2(t)\in\mathbb{R}^3_{\geq 0}\iff t\in[\frac{m_3}{n_3},\frac{m_1}{n_1}]$,
- $\mathbf{p}_3(t)=\rho_3^{-1}(n_2t-m_2,m_1-n_1t,0)$, hence $\mathbf{p}_3(t)\in\mathbb{R}^3_{\geq 0}\iff t\in[\frac{m_2}{n_2},\frac{m_1}{n_1}]$,

since $\rho_1,\rho_2,\rho_3\geq 0$. In particular, if $\rho_1=0$ or $\rho_3=0$, then $\mathcal{L}(t,1)$ does not meet the corresponding coordinate plane in $\mathbb{R}^3_{\geq 0}$ (recall that $\rho_2>0$).

For $t<\frac{m_3}{n_3}$ or $t>\frac{m_1}{n_1}$, we have $\ell(t,1)=0$. For $t\in[\frac{m_3}{n_3},\frac{m_2}{n_2}]$, we see that $\mathcal{L}(t,1)$ is the line segment from $\mathbf{p}_1(t)$ to $\mathbf{p}_2(t)$. A computation confirms that

$$\ell(t,1)=||\mathbf{p}_1(t)-\mathbf{p}_2(t)||=\frac{n_3t-m_3}{\rho_1\rho_2}||\mathbf{r}||.$$

For $t\in[\frac{m_2}{n_2},\frac{m_1}{n_1}]$, we see that $\mathcal{L}(t,1)$ is the line segment from $\mathbf{p}_2(t)$ to $\mathbf{p}_3(t)$, so

$$\ell(t,1)=||\mathbf{p}_2(t)-\mathbf{p}_3(t)||=\frac{m_1-n_1t}{\rho_2\rho_3}||\mathbf{r}||$$

via another computation. This yields the desired piecewise-linear formula for $\ell(t,1)$. An admissible Lipschitz constant is the maximum of the slopes of $\ell(t,1)$ on $[\frac{m_3}{n_3},\frac{m_2}{n_2}]$ and $[\frac{m_2}{n_2},\frac{m_1}{n_1}]$, so long as the corresponding interval is nondegenerate. Elementary computations confirm the remainder of the lemma. □

Remark 21. If $\rho_1=0$ or $\rho_3=0$ (the conditions are mutually exclusive), then the corresponding interval in the definition of $\ell(t,1)$ and term in the maximum above are omitted. Moreover, $\ell(t,1)$ is Lipschitz on $[\frac{m_3}{n_3},\infty)$ or $(-\infty,\frac{m_1}{n_1}]$, respectively.

Lemma 20 states that $\ell(x,1)$ is a triangular function with base $[\frac{m_3}{n_3},\frac{m_1}{n_1}]$ and height $\frac{||\mathbf{r}||}{n_2\rho_2}$. Since the base width is

$$\frac{m_1}{n_1}-\frac{m_3}{n_3}=\frac{m_1n_3-n_1m_3}{n_1n_3}=\frac{\rho_2}{n_1n_3},$$

the area of the triangle is

$$\int_{\mathbb{R}}\ell(x,1)\ dx=\frac{1}{2}\cdot\frac{\rho_2}{n_1n_3}\cdot\frac{||\mathbf{r}||}{n_2\rho_2}=\frac{||\mathbf{r}||}{2n_1n_2n_3}.$$

In particular,

$$F(t)=2n_1n_2n_3\frac{\ell(t,1)}{||\mathbf{r}||}\tag{22}$$

is the probability density from Theorem 1.

3.5. A technical lemma

The next lemma permits us to approximate a discrete sum by an integral with a completely explicit error estimate.

Lemma 23. *Suppose that*

(a) $g : \mathbb{R} \to \mathbb{R}$ *satisfies* $|g(x)| \le C_1$ *for* $x \in \mathbb{R}$;
(b) g *is Lipschitz on some closed interval* I *with Lipschitz constant* C_2;
(c) $n, c, d \in \mathbb{Z}_{\ge 0}$ *and* $c < d$;
(d) $f : \mathbb{Z} \to \mathbb{Z}$ *satisfies*

$$\left| f(c+kd)/d - ng\left(\frac{c+kd}{n}\right) \right| \le 1$$

for $k \in \mathbb{Z}$; *and*
(e) $f(x) = 0$ *for* $x \not\equiv c \pmod d$.
Then for real $\alpha < \beta$ *such that* $[\alpha, \beta] \subseteq I$,

$$\left| \frac{1}{n^2} \sum_{k \in \mathbb{Z} \cap [\alpha n, \beta n]} f(k) - \int_\alpha^\beta g(x)\ dx \right| \le \frac{\left(\beta - \alpha + \frac{2d}{n}\right)(1 + dC_2) + d\left(5C_1 + \frac{1}{n}\right)}{n}.$$

Proof. Since the proof is somewhat long, we break it up into several pieces.
An auxiliary function. Let $G(x) = \frac{n}{d} f(c + \lfloor nx/d \rfloor d)$. Then

$$\begin{aligned}
\int_{\frac{d}{n}\lceil \frac{\alpha n - c}{d} \rceil}^{\frac{d}{n}\lfloor \frac{\beta n - c}{d} + 1 \rfloor} G(x)\ dx &= \int_{\lceil \frac{\alpha n - c}{d} \rceil}^{\lfloor \frac{\beta n - c}{d} + 1 \rfloor} \frac{d}{n} G\left(\frac{d}{n} u\right) du \\
&= \int_{\lceil \frac{\alpha n - c}{d} \rceil}^{\lfloor \frac{\beta n - c}{d} + 1 \rfloor} f(c + \lfloor u \rfloor d) du \\
&= \sum_{k \in \mathbb{Z} \cap [\frac{\alpha n - c}{d}, \frac{\beta n - c}{d}]} f(c + kd) \\
&= \sum_{k \in \mathbb{Z} \cap [\alpha n, \beta n]} f(k).
\end{aligned} \tag{24}$$

For $k \in \mathbb{Z}$, condition (d) ensures that

$$\begin{aligned}
\left| \frac{G(kd/n)}{n^2} - g\left(\frac{c+kd}{n}\right) \right| &= \left| \frac{\frac{n}{d} f\left(c + \left\lfloor n \frac{kd}{nd} \right\rfloor d\right)}{n^2} - g\left(\frac{c+kd}{n}\right) \right| \\
&= \left| \frac{f(c+kd)/d - ng\left(\frac{c+kd}{n}\right)}{n} \right| \\
&\le \frac{1}{n}.
\end{aligned} \tag{25}$$

We also need a bound afforded by (a) and (d):

$$\left| \frac{G(x)}{n^2} \right| = \frac{|f(c + \lfloor nx/d \rfloor d)|}{dn} \le \frac{1}{n} + \left| g\left(\frac{c + \lfloor nx/d \rfloor d}{n}\right) \right| \le C_1 + \frac{1}{n}. \tag{26}$$

A Lipschitz estimate. Observe that $G(x) = G(\lfloor nx/d \rfloor d/n)$ and

$$\frac{-d}{n} \le \frac{c + nx - d}{n} - x \le \frac{c + \lfloor nx/d \rfloor d}{n} - x \le \frac{c + nx}{n} - x \le \frac{d}{n}.$$

If x and $(c+\lfloor nx/d\rfloor d)/n$ are both in I, condition (b) and (25) imply that

$$\begin{aligned}\left|\frac{G(x)}{n^2}-g(x)\right| &\le \left|\frac{G(\lfloor nx/d\rfloor d/n)}{n^2}-g\left(\frac{c+\lfloor nx/d\rfloor d}{n}\right)\right| \\ &\quad + \left|g\left(\frac{c+\lfloor nx/d\rfloor d}{n}\right)-g(x)\right| \\ &\le \frac{1+dC_2}{n}. \end{aligned} \tag{27}$$

A containment. We claim x and $(c+\lfloor nx/d\rfloor d)/n$ belong to $[\alpha,\beta]\subset I$ whenever

$$\frac{d}{n}\left\lceil\frac{\alpha n-c}{d}+1\right\rceil \;\le\; x \;\le\; \frac{d}{n}\left\lfloor\frac{\beta n-c}{d}\right\rfloor.$$

Suppose that the inequality above holds. Then

$$x\ge\frac{d}{n}\left(\frac{\alpha n-c}{d}+1\right)=\alpha+\frac{d-c}{n}\ge\alpha$$

and

$$x\le\frac{d}{n}\left\lfloor\frac{\beta n-c}{d}\right\rfloor\le\frac{d}{n}\cdot\frac{\beta n-c}{d}=\beta-\frac{c}{n}\le\beta.$$

Next observe that

$$\frac{c+\lfloor\frac{n}{d}x\rfloor d}{n}\ge\frac{c+\left\lfloor\frac{n}{d}\left(\frac{d}{n}\lceil\frac{\alpha n-c}{d}+1\rceil\right)\right\rfloor d}{n}=\frac{c+\lceil\frac{\alpha n-c}{d}+1\rceil d}{n}\ge\frac{c+\alpha n-c+d}{n}\ge\alpha$$

and

$$\frac{c+\lfloor\frac{n}{d}x\rfloor d}{n}\le\frac{c+\left\lfloor\frac{n}{d}\left(\frac{d}{n}\lfloor\frac{\beta n-c}{d}\rfloor\right)\right\rfloor d}{n}=\frac{c+\lfloor\frac{\beta n-c}{d}\rfloor d}{n}\le\frac{c+\beta n-c}{n}=\beta.$$

This completes the proof of the claim.

An observation. Since

$$\frac{d}{n}\left\lfloor\frac{\beta n-c}{d}\right\rfloor-\frac{d}{n}\left\lceil\frac{\alpha n-c}{d}+1\right\rceil\le\beta-\frac{c}{n}-\left(\alpha+\frac{d-c}{n}\right)=\beta-\alpha-\frac{d}{n}$$

and

$$\frac{d}{n}\left\lceil\frac{\alpha n-c}{d}+1\right\rceil-\frac{d}{n}\left\lfloor\frac{\beta n-c}{d}\right\rfloor<\alpha+\frac{2d-c}{n}-\left(\beta-\frac{c}{n}\right)=\alpha-\beta+\frac{2d}{n},$$

we conclude that

$$\left|\frac{d}{n}\left\lfloor\frac{\beta n-c}{d}\right\rfloor-\frac{d}{n}\left\lceil\frac{\alpha n-c}{d}+1\right\rceil\right|\le\max\left\{\beta-\alpha-\frac{d}{n},\ \alpha-\beta+\frac{2d}{n}\right\}. \tag{28}$$

Small intervals. Consider the intervals $\left[\alpha,\frac{d}{n}\lceil\frac{\alpha n-c}{d}+1\rceil\right]$ and $\left[\frac{d}{n}\lfloor\frac{\beta n-c}{d}\rfloor,\beta\right]$. Since

$$\begin{aligned}0<\frac{d-c}{n}&=\frac{d}{n}\left(\frac{\alpha n-c}{d}+1\right)-\alpha\\ &\le\frac{d}{n}\left\lceil\frac{\alpha n-c}{d}+1\right\rceil-\alpha<\frac{d}{n}\left(\frac{\alpha n-c}{d}+2\right)-\alpha\\ &=\frac{2d-c}{n},\end{aligned}$$

the first interval is nonempty with length at most $(2d-c)/n$. Similarly,

$$\begin{aligned}0 \leq \frac{c}{n} = \beta - \frac{d}{n}\left(\frac{\beta n - c}{d}\right) &\leq \beta - \frac{d}{n}\left\lfloor \frac{\beta n - c}{d} \right\rfloor \\ &< \beta - \frac{d}{n}\left(\frac{\beta n - c}{d} - 1\right) = \frac{c+d}{n},\end{aligned}$$

so the second interval has length at most $(c+d)/n$. In summary,

$$0 < \frac{d}{n}\left\lceil \frac{\alpha n - c}{d} + 1 \right\rceil - \alpha < \frac{2d-c}{n} \quad \text{and} \quad 0 \leq \beta - \frac{d}{n}\left\lfloor \frac{\beta n - c}{d} \right\rfloor < \frac{c+d}{n}. \tag{29}$$

Conclusion. We conclude that

$$\begin{aligned}
&\left| \frac{1}{n^2} \sum_{k \in \mathbb{Z} \cap [\alpha n, \beta n]} f(k) - \int_\alpha^\beta g(x)\ dx \right| \\
&= \left| \int_{\frac{d}{n}\lceil \frac{\alpha n - c}{d} \rceil}^{\frac{d}{n}\lfloor \frac{\beta n - c}{d} + 1 \rfloor} \frac{G(x)}{n^2}\ dx - \int_\alpha^\beta g(x)\ dx \right| && \text{by (24)} \\
&= \left| \int_{\frac{d}{n}\lceil \frac{\alpha n - c}{d} \rceil}^{\frac{d}{n}\lceil \frac{\alpha n - c}{d} + 1 \rceil} \frac{G(x)}{n^2}\ dx + \int_{\frac{d}{n}\lceil \frac{\alpha n - c}{d} + 1 \rceil}^{\frac{d}{n}\lfloor \frac{\beta n - c}{d} \rfloor} \frac{G(x)}{n^2}\ dx \right. \\
&\quad \left. + \int_{\frac{d}{n}\lfloor \frac{\beta n - c}{d} \rfloor}^{\frac{d}{n}\lfloor \frac{\beta n - c}{d} + 1 \rfloor} \frac{G(x)}{n^2}\ dx - \int_\alpha^\beta g(x)\ dx \right| \\
&\leq \int_{\frac{d}{n}\lceil \frac{\alpha n - c}{d} \rceil}^{\frac{d}{n}\lceil \frac{\alpha n - c}{d} + 1 \rceil} \left| \frac{G(x)}{n^2} \right|\ dx + \int_{\frac{d}{n}\lfloor \frac{\beta n - c}{d} \rfloor}^{\frac{d}{n}\lfloor \frac{\beta n - c}{d} + 1 \rfloor} \left| \frac{G(x)}{n^2} \right|\ dx \\
&\quad + \left| \int_{\frac{d}{n}\lceil \frac{\alpha n - c}{d} + 1 \rceil}^{\frac{d}{n}\lfloor \frac{\beta n - c}{d} \rfloor} \frac{G(x)}{n^2}\ dx - \int_\alpha^\beta g(x)\ dx \right| \\
&\leq \left| \int_{\frac{d}{n}\lceil \frac{\alpha n - c}{d} + 1 \rceil}^{\frac{d}{n}\lfloor \frac{\beta n - c}{d} \rfloor} \left(\frac{G(x)}{n^2} - g(x) \right)\ dx - \int_\alpha^{\frac{d}{n}\lceil \frac{\alpha n - c}{d} + 1 \rceil} g(x)\ dx \right. \\
&\quad \left. - \int_{\frac{d}{n}\lfloor \frac{\beta n - c}{d} \rfloor}^\beta g(x)\ dx \right| + \frac{2d}{n}\left(C_1 + \frac{1}{n} \right) && \text{by (26)} \\
&\leq \left| \int_{\frac{d}{n}\lceil \frac{\alpha n - c}{d} + 1 \rceil}^{\frac{d}{n}\lfloor \frac{\beta n - c}{d} \rfloor} \left| \frac{G(x)}{n^2} - g(x) \right|\ dx \right| + \int_\alpha^{\frac{d}{n}\lceil \frac{\alpha n - c}{d} + 1 \rceil} |g(x)|\ dx \\
&\quad + \int_{\frac{d}{n}\lfloor \frac{\beta n - c}{d} \rfloor}^\beta |g(x)|\ dx + \frac{2d}{n}\left(C_1 + \frac{1}{n} \right) \\
&\leq \left| \frac{d}{n}\left\lfloor \frac{\beta n - c}{d} \right\rfloor - \frac{d}{n}\left\lceil \frac{\alpha n - c}{d} + 1 \right\rceil \right| \left(\frac{1 + dC_2}{n} \right) && \text{by (27)} \\
&\quad + \frac{(2d-c)C_1}{n} + \frac{(c+d)C_1}{n} + \frac{2d}{n}\left(C_1 + \frac{1}{n} \right) && \text{by (a), (29)} \\
&\leq \max\left\{ \beta - \alpha - \frac{d}{n},\ \alpha - \beta + \frac{2d}{n} \right\} \left(\frac{1 + dC_2}{n} \right) && \text{by (28)} \\
&\quad + \frac{3dC_1}{n} + \frac{2d}{n}\left(C_1 + \frac{1}{n} \right) \\
&\leq \frac{\left(\beta - \alpha + \frac{2d}{n} \right)(1 + dC_2) + d\left(5C_1 + \frac{2}{n} \right)}{n}.
\end{aligned}$$

□

3.6. A simplification

If $\mathbf{x} = (x_1, x_2, x_3)$ and $\lambda(\mathbf{x}) \in \Lambda[\![n]\!]$, then

$$\begin{aligned}\lambda(\mathbf{x}) &= m_1x_1 + m_2x_2 + m_3x_3 = \frac{m_1}{n_1}n_1x_1 + \frac{m_2}{n_2}n_2x_2 + \frac{m_3}{n_3}n_3x_3 \\ &> \frac{m_3}{n_3}(n_1x_1 + n_2x_2 + n_3x_3) = \frac{m_3}{n_3}n.\end{aligned}$$

Thus, $\Lambda[\![n]\!] \cap [-\infty, \frac{m_3}{n_3}n] = \emptyset$ and, in a similar manner, $\Lambda[\![n]\!] \cap [\frac{m_1}{n_1}n, \infty] = \emptyset$. Since F is supported on $[\frac{m_3}{n_3}, \frac{m_1}{n_1}]$, we may assume that $[\alpha, \beta] \in [\frac{m_3}{n_3}, \frac{m_1}{n_1}]$. In particular, we can assume that the function $\ell(x, 1)$ of Lemma 20 is Lipschitz on $[\alpha, \beta]$.

3.7. Conclusion

We now conclude the proof of Theorem 1. Fix $n \in \mathbb{Z}_{\geq 0}$ and let

$$f(m) = |\mathcal{Z}(m, n)| = \{\mathbf{x} \in \mathsf{Z}_S(n) : \lambda(\mathbf{x}) = m\} \qquad \text{and} \qquad g(x) = \frac{\ell(x, 1)}{||\mathbf{r}||}.$$

Let $d = \gcd(\rho_1, \rho_2, \rho_3)$ and deduce from (17) and Lemmas 15 and 18 that there is a $c \in \{0, 1, ..., d-1\}$ such that

$$\begin{aligned}\left|\frac{f(c+kd)}{d} - ng\left(\frac{c+kd}{n}\right)\right| &= \left|\frac{|\mathcal{Z}(c+kd, n)|}{d} - \frac{n\ell\left(\frac{c+kd}{n}, 1\right)}{||\mathbf{r}||}\right| \\ &= \left|\frac{|\mathcal{Z}(c+kd, n)|}{d} - \frac{\ell(c+kd, n)}{||\mathbf{r}||}\right| \\ &\leq 1\end{aligned}$$

for all $k \in \mathbb{Z}$; moreover, $f(x) = 0$ if $x \not\equiv c \pmod d$.

Suppose that $\rho_1, \rho_3 \neq 0$. Apply Lemma 23 to the functions f and g and the parameters c, d, n defined above, and to the constants

$$C_1 = \frac{1}{n_2\rho_2} \qquad \text{and} \qquad C_2 = \frac{1}{\rho_2}\max\left\{\frac{n_3}{\rho_1}, \frac{n_1}{\rho_3}\right\}$$

provided by Lemma 20:

$$\left|\frac{1}{n^2}\sum_{m \in \mathbb{Z}\cap[\alpha n, \beta n]} f(m) - \int_\alpha^\beta g(x)\ dx\right| \leq \frac{\left(\beta - \alpha + \frac{2d}{n}\right)(1 + dC_2) + d\left(5C_1 + \frac{2}{n}\right)}{n}.$$

Since

$$\sum_{m \in \mathbb{Z}\cap[\alpha n, \beta n]} f(m) = |\Lambda[\![n]\!] \cap [\alpha n, \beta n]|$$

and $\rho_1, \rho_2, \rho_3 \geq 1$, it follows that

$$\left| \frac{|\Lambda[\![n]\!] \cap [\alpha n, \beta n]|}{n^2} - \int_\alpha^\beta \frac{\ell(x,1)}{||\mathbf{r}||} \, dx \right|$$
$$\leq \frac{\left(\beta - \alpha + \frac{2d}{n}\right)(1 + dC_2) + d\left(5C_1 + \frac{2}{n}\right)}{n}$$
$$\leq \frac{\left(\beta - \alpha + \frac{2d}{n}\right)\left(1 + \frac{d}{\rho_2} \max\left\{\frac{n_3}{\rho_1}, \frac{n_1}{\rho_3}\right\}\right) + d\left(\frac{5}{n_2\rho_2} + \frac{2}{n}\right)}{n}$$
$$\leq \frac{\left(\beta - \alpha + \frac{2d}{n}\right)(1 + d\max\{n_1, n_3\}) + d\left(\frac{5}{n_2} + \frac{2}{n}\right)}{n}. \tag{30}$$

To complete the proof of Theorem 1 in this case, multiply by $2n_1n_2n_3$ and use (22). If $\rho_1 = 0$ or $\rho_3 = 0$, the corresponding term in the maximum in (30) is omitted by virtue of Remark 21 and the restriction of $[\alpha, \beta]$ in Section 3.6. □

Remark 31. The bound implied by (30) is better, but more complicated, than the bound in Theorem 1. The two bounds are compared in Table 2.

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