



Restrictions on realizable unitary operations imposed by symmetry and locality

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According to a fundamental result in quantum computing, any unitary transformation on a composite system can be generated using so-called 2-local unitaries that act only on two subsystems. Beyond its importance in quantum computing, this result can also be regarded as a statement about the dynamics of systems with local Hamiltonians: although locality puts various constraints on the short-term dynamics, it does not restrict the possible unitary evolutions that a composite system with a general local Hamiltonian can experience after a sufficiently long time. Here we show that this universality does not remain valid in the presence of conservation laws and global continuous symmetries such as $U(1)$ and $SU(2)$. In particular, we show that generic symmetric unitaries cannot be implemented, even approximately, using local symmetric unitaries. Based on this no-go theorem, we propose a method for experimentally probing the locality of interactions in nature. In the context of quantum thermodynamics, our results mean that generic energy-conserving unitary transformations on a composite system cannot be realized solely by combining local energy-conserving unitaries on the components. We show how this can be circumvented via catalysis.

Locality and symmetry are fundamental and ubiquitous properties of physical systems, and their interplay leads to diverse emergent phenomena such as spontaneous symmetry breaking. They also put various constraints on both equilibrium and dynamical properties of physical systems. For instance, symmetry implies conservation laws, as highlighted by Noether's theorem^{1,2}, and locality of interactions implies finite speed of propagation of information, as highlighted by the Lieb–Robinson bound³. Nevertheless, in spite of the restrictions imposed by locality on the short-term dynamics, it turns out that, after a sufficiently long time and in the absence of symmetries, a composite system with a general local (time-dependent) Hamiltonian can experience any arbitrary unitary time evolution. This is related to a fundamental result in quantum computing: any unitary transformation on a composite system can be generated by a sequence of 2-local unitary transformations, that is, those that couple, at most, two subsystems^{4–6}.

Here we study this phenomenon in the presence of conservation laws and global symmetries. In particular, we ask whether this universality remains valid in the presence of symmetries, or whether locality puts additional constraints on the possible unitary evolutions of a composite system. Clearly, if all the local unitaries obey a certain symmetry, then the overall unitary evolution also obeys the same symmetry. The question is whether all symmetric unitaries on a composite system can be generated using local symmetric unitaries on the system. Surprisingly, it turns out that the answer is negative in the case of continuous symmetries such as $SU(2)$ and $U(1)$. In fact, we show that generic symmetric unitaries cannot be implemented, even approximately, using local symmetric unitaries. Furthermore, the difference between the dimensions of the manifold of all symmetric unitaries and the sub-manifold of unitaries generated by k -local symmetric unitaries with a fixed k increases constantly with the system size.

This result implies that, in the presence of locality, symmetries of Hamiltonian impose extra constraints on the time evolution of the system, which are not captured by Noether's theorem. We show how the violation of these constraints can be observed experimentally and, in fact, can be used as a new method for probing the locality

of interactions in nature. These additional constraints can also have interesting implications in the context of quantum chaos and thermalization of many-body systems⁷. We also explain how, in the case of $U(1)$ symmetry, the no-go theorem can be circumvented using ancillary qubits and discuss the implications of these results in the contexts of the resource theory of quantum thermodynamics^{8–15}, quantum reference frames¹⁶ and quantum circuit synthesis.

Preliminaries

Local symmetric quantum circuits. Consider an arbitrary composite system formed from local subsystems or sites (for example, qubits or spins). Here, we focus on systems with finite-dimensional Hilbert spaces. An operator is called k -local if it acts non-trivially on the Hilbert spaces of, at most, k sites. Consider a symmetry described by a general group G . To simplify the following discussion, unless otherwise stated, we assume that all sites in the system have identical Hilbert spaces and carry the same unitary representation of group G (see Supplementary Note 1 for a more general case). In particular, in a system with n sites, assume that each group element $g \in G$ is represented by the unitary $U(g) = u(g)^{\otimes n}$. An operator A acting on the total system is called G -invariant, or symmetric, if it satisfies $U(g)AU^\dagger(g) = A$, for any group element $g \in G$. The set of symmetric unitaries itself forms a group, denoted by

$$\mathcal{V}^G \equiv \{V : VV^\dagger = I, [V, U(g)] = 0, \forall g \in G\}, \quad (1)$$

where I is the identity operator.

As an example, consider a system with n qubits and the $U(1)$ symmetry corresponding to global rotations around the z axis. Then, an operator A is symmetric if $(e^{-i\theta Z})^{\otimes n} A (e^{i\theta Z})^{\otimes n} = A$, for $\theta \in [0, 2\pi)$, or, equivalently, if it commutes with $\sum_{j=1}^n Z_j$, where X_j, Y_j, Z_j denote Pauli operators on the qubit j tensor product with the identity operators on the rest of the qubits. Depending on the context, this symmetry can have different physical interpretations. For instance, if each qubit has a Hamiltonian $\frac{\Delta E}{2} Z_j$, then $\frac{\Delta E}{2} \sum_{j=1}^n Z_j$ is the total Hamiltonian of the system. Then, unitaries that satisfy this symmetry are the energy-conserving unitaries.

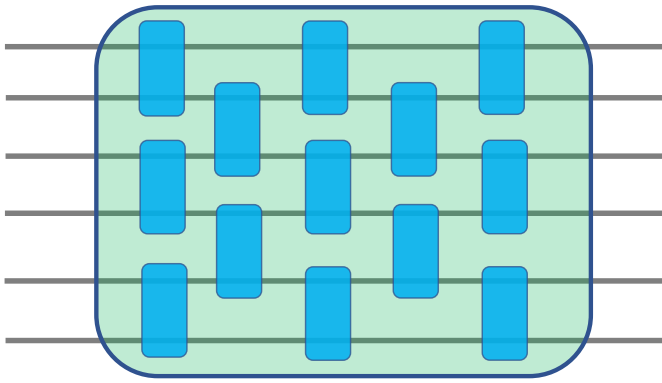


Fig. 1 | LSQCs. A quantum circuit with 2-local unitaries. Here, each line represents a subsystem, for example, a qubit, through time, and each box represents a 2-local unitary transformation. An LSQC only contains local unitaries that respect a certain symmetry. For instance, they are all invariant under rotations around the z axis. Such circuits can model the time evolution of systems with local symmetric Hamiltonians. Conversely, any LSQC corresponds to the time evolution generated by a local symmetric (time-dependent) Hamiltonian. Therefore, by studying LSQCs, we can also characterize general features of time evolution under local symmetric Hamiltonians.

We define $\mathcal{V}_k^G$ to be the set of all unitary transformations that can be implemented with local symmetric quantum circuits (LSQCs) with k -local unitaries (Fig. 1). More formally, $\mathcal{V}_k^G$ is the set of unitaries $V = \prod_{i=1}^m V_i$, generated by composing symmetric k -local unitaries V_i ; $i=1 \dots m$, for a finite m . It can be easily seen that $\mathcal{V}_k^G$ is a subgroup of $\mathcal{V}^G = \mathcal{V}_n^G$, the group of all symmetric unitaries. More generally, for $k \leq l \leq n$, we have $\mathcal{V}_k^G \subseteq \mathcal{V}_l^G \subseteq \mathcal{V}^G$. We are interested in characterizing each subgroup $\mathcal{V}_k^G$ and, in particular, in determining whether there exists $k < n$ such that k -local symmetric unitaries become *universal*, that is, $\mathcal{V}_k^G = \mathcal{V}_n^G = \mathcal{V}^G$. As discussed above, in the absence of symmetries, that is, when G is the trivial group, this holds for $k=2$. To study these questions, we use the Lie algebraic methods of quantum control theory^{17,18}, which have also been previously used to study the universality of 2-local gates in the absence of symmetries^{4,5,19–24}.

It is worth noting that, for composite systems with a given geometry, one can consider the stronger constraint of geometric locality in the above definitions: the k -local symmetric unitaries should act on local neighbourhoods, for example, only on k nearest-neighbour sites. However, provided that the sites lie on a connected graph, for example, on a connected 1D chain, this additional constraint does not change the generated group $\mathcal{V}_k^G$ for $k > 1$. This is true because the swap unitary that exchanges the states of two nearest-neighbour sites is 2-local and respects the symmetry, for all symmetry groups. If the graph is connected, by combining these 2-local permutations on pairs of neighbouring sites, we can generate all permutations and hence change the order of sites arbitrarily. Therefore, any k -local symmetric unitary can be realized by a sequence of k -local symmetric unitaries on k nearest-neighbour sites.

Time evolution under local symmetric Hamiltonians. Next, we consider a slightly different formulation of this problem in terms of the notion of local symmetric Hamiltonians. A generic local Hamiltonian $H(t)$ acts non-trivially on all subsystems in the system, but it has a decomposition as $H(t) = \sum_j h_j(t)$, where each term $h_j(t)$ is k -local for a fixed k , which is often much smaller than the total number of subsystems in the system. The unitary evolution generated by this Hamiltonian is determined by the Schrödinger equation

$$\frac{dV(t)}{dt} = -iH(t)V(t) = -i\left[\sum_j h_j(t)\right]V(t), \quad (2)$$

with the initial condition $V(0)=I$. Suppose, in addition to the above locality constraint, that the Hamiltonian $H(t)$ also respects the symmetry described by the group G , such that $[U(g), H(t)] = 0$, for all $g \in G$, and all $t \geq 0$. Then, it can be shown that the family of unitaries $\{V(t); t \geq 0\}$ generated by any such Hamiltonian belongs to $\mathcal{V}_k^G$, that is, the group of symmetric unitaries that can be implemented by k -local symmetric unitaries (Supplementary Note 1). Conversely, any unitary in this group is generated by a Hamiltonian $H(t)$ satisfying the above locality and symmetry constraints (any quantum circuit can be thought of as the time evolution generated by a time-dependent local Hamiltonian). Therefore, by characterizing $\mathcal{V}_k^G$ and studying its relation with the group of all symmetric unitaries $\mathcal{V}^G$, we can also unveil possible constraints on the time evolution under local symmetric Hamiltonians, which are not captured by the standard conservation laws imposed by Noether's theorem.

Main results

A no-go theorem: non-universality of local unitaries in the presence of symmetries. We show that, in the case of continuous symmetries such as $U(1)$ and $SU(2)$, most symmetric unitaries cannot be implemented, even approximately, using local symmetric unitaries. First, as we prove in Supplementary Note 1, for any group G , the set of symmetric unitaries $\mathcal{V}^G = \mathcal{V}_n^G$ and its subgroup $\mathcal{V}_k^G$ generated by k -local symmetric unitaries are both connected compact Lie groups and hence closed manifolds (Fig. 2). This means that, if a unitary V is not in $\mathcal{V}_k^G$, then there is a neighbourhood of symmetric unitaries around V , none of which can be implemented using k -local symmetric unitaries. On the other hand, if V belongs to $\mathcal{V}_k^G$, then it can be implemented with a uniformly finite number of such unitaries that is upper bounded by a fixed number that is independent of V (ref. 17).

Secondly, we prove that, for any finite or compact Lie group G , the difference between the dimensions of the manifolds associated to all symmetric unitaries $\mathcal{V}^G = \mathcal{V}_n^G$ and its sub-manifold $\mathcal{V}_k^G$ is lower bounded by

$$\dim(\mathcal{V}^G) - \dim(\mathcal{V}_k^G) \geq |\text{Irreps}_G(n)| - |\text{Irreps}_G(k)|, \quad (3)$$

where for any integer l , $|\text{Irreps}_G(l)|$ is the number of inequivalent irreducible representations (irreps) of group G , appearing in the representation $\{u(g)^{\otimes l}; g \in G\}$, that is, in the action of symmetry on l subsystems. We conclude that, unless $|\text{Irreps}_G(n)| = |\text{Irreps}_G(k)|$, there is a family of symmetric unitaries on n subsystems that cannot be implemented with k -local symmetric unitaries. In the case of continuous symmetries such as $U(1)$ and $SU(2)$, $|\text{Irreps}_G(n)|$ grows unboundedly with n . This means that there is no fixed integer k such that k -local symmetric unitaries become universal for all system size n . This is in sharp contrast to the universality of 2-local unitaries in the absence of symmetries. In Methods, we provide a simple proof of the non-universality of local unitaries in the case of continuous symmetries using a technique called ‘charge vectors’. In Supplementary Note 2, we prove equation (3) and present a more refined version of this inequality in the case of connected Lie groups, such as $U(1)$ and $SU(2)$, as well as an extension of the no-go theorem to the case where the subsystems can have different representations of the symmetry. We also discuss more about the nature of the constraints imposed by locality that lead to the bound in equation (3). (Namely, we argue that certain elements of the centre of the Lie algebra of symmetric Hamiltonians cannot be generated using local symmetric Hamiltonians.)

Example: $U(1)$ symmetry for systems of qubits. Recall the example of the $U(1)$ symmetry for a system of n qubits. In this case, the

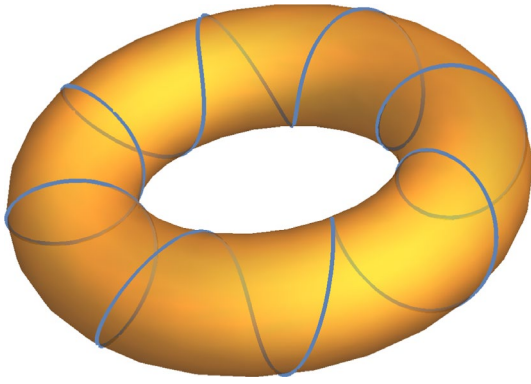


Fig. 2 | Schematic relation between group of all symmetric unitaries (torus) and subgroup generated by LSQCs (blue curve). They are both closed connected Lie groups and hence closed manifolds. Unitary evolution under any local symmetric Hamiltonian is restricted to the sub-manifold corresponding to LSQC. In other words, adding a perturbation to the Hamiltonian can bring the evolution outside this sub-manifold, only if it is non-local or symmetry breaking. In the example of U(1) symmetry, we discuss a more explicit interpretation of this schematic figure.

representation of symmetry on n sites is $(e^{i\theta Z})^{\otimes n} = \exp(i\theta[nI - 2N])$ for $\theta \in [0, 2\pi)$, where $N = \sum_j (I - Z_j)/2$ determines the total ‘charge’ (or, excitations) in the system. It follows that the irreps of U(1) can be labelled by distinct eigenvalues of N , which take integer values $m=0, \dots, n$. Then, equation (3) implies that, for a system with n qubits, the difference between the dimensions of the manifold of all symmetric unitaries and those generated by k -local symmetric unitaries is, at least, $n - k$. Remarkably, it turns out that, in this case, this bound holds as equality. In Methods, we present a full characterization of Hamiltonians that can be generated using k -local U(1)-invariant Hamiltonians. This result, for instance, implies that, even if one can implement all U(1)-invariant unitaries that act on $n - 1$ qubits, still the unitary $\exp(i\phi Z^{\otimes n})$ cannot be implemented for generic values of ϕ .

It is useful to express the constraints imposed by the locality of interactions in terms of experimentally observable quantities. Consider a general U(1)-invariant unitary V on n qubits. For instance, V can be the unitary generated by U(1)-invariant Hamiltonian $H(t)$, from time $t=0$ to T under the Schrödinger equation. Any such unitary has a decomposition as $V = \bigoplus_{m=0}^n V_m$, where V_m is the component of V in the charge sector m , that is, the eigen-subspace of operator $N = \sum_j (I - Z_j)/2$ with eigenvalue m . For any integer $l=0, \dots, n$, define the l -body phase $\Phi_l \in (-\pi, \pi]$ of V as

$$\begin{aligned} \Phi_l &\equiv \sum_{m=0}^n c_l(m) \theta_m \\ &= - \int_0^T dt \sum_{\mathbf{b}: w(\mathbf{b})=l} \text{Tr}(H(t) \mathbf{Z}^{\mathbf{b}}) \pmod{2\pi}, \end{aligned} \quad (4)$$

where $\theta_m = \arg(\det(V_m)) \in (-\pi, \pi]$ is the phase of the determinant of V_m , $c_l(m) = \sum_{s=0}^m (-1)^s \binom{m}{s} \binom{n-m}{l-s}$ is an integer coefficient, and we use the convention that, for integers a and b , the binomial coefficient $\binom{a}{b} = 0$ if $b > a$. In the second equality, the summation is over all bit strings $\mathbf{b} = b_1 \dots b_n \in \{0, 1\}^n$ with Hamming weight $w(\mathbf{b}) \equiv \sum_{j=1}^n b_j$ equal to l , and we have defined $\mathbf{Z}^{\mathbf{b}} \equiv Z_1^{b_1} \dots Z_n^{b_n}$. Note that this equality is satisfied for any U(1)-invariant Hamiltonian $H(t)$ that realizes unitary V . Using this equality, for instance, we can see that, for unitary $V = \exp(i\phi \mathbf{Z}^{\mathbf{b}})$, all l -body phases vanish, except for $l = w(\mathbf{b})$, where $\Phi_{w(\mathbf{b})} = 2^n \phi \pmod{2\pi}$. In Supplementary Note 4 we prove equation (4) and present coefficients $c_l(m)$ for a system with $n=5$ qubits.

The notion of l -body phases provides a useful characterization of the constraints imposed by the locality of interactions. In Supplementary Note 4, we show that: (i) For $l \geq 1$, the l -body phases $\{\Phi_l\}$ of a U(1)-invariant unitary time evolution can be measured experimentally. On the other hand, the phases $\{\theta_m\}$ are not physically observable, because they transform non-trivially under the global phase transformation $V \rightarrow e^{i\alpha} V$. Similarly, $\Phi_0 = \sum_m \theta_m = \arg(\det(V))$ is not observable. (ii) If a unitary is realizable by k -local U(1)-invariant unitaries, then its l -body phases are zero for $l > k$, which can be seen using the second equality in equation (4). This, for instance, implies that, unless ϕ is an integer multiple of $\pi/2^{n-1}$, unitary $\exp(i\phi \mathbf{Z}^{\mathbf{b}})$ cannot be implemented using k -local U(1)-invariant unitaries with $k < w(\mathbf{b})$. (iii) Conversely, for a general U(1)-invariant unitary V , if all l -body phases vanish for $l > k$, then V is realizable using k -local U(1)-invariant unitaries, up to a unitary in a fixed finite subgroup of U(1)-invariant unitaries. Finally, it is worth mentioning that, from a geometrical point of view, the transformation $\{\theta_m\} \rightarrow \{\Phi_l\}$ in equation (4) describes a change of the coordinate system on the $(n+1)$ -torus corresponding to phases $\theta_m = \arg(\det(V_m))$, for charges $m=0, \dots, n$. For instance, when the system evolves under the Hamiltonian $H = \gamma \mathbf{Z}^{\mathbf{b}}$, its trajectory on this torus is a helix described by the equation $\Phi_l(t) = -2^n \gamma t \times \delta_{l, w(\mathbf{b})}$, where δ denotes the Kronecker delta (Fig. 2).

In Sect. 2.6, we discuss an application of this framework for synthesizing phase-insensitive quantum circuits. But first, we start with a rather surprising implication of these ideas.

Application: probing the locality of interactions in nature. Our no-go theorem leads us to a new method for experimentally probing the locality of interactions. According to this theorem, in the presence of symmetries, interactions that couple more subsystems can imprint certain observable effects on the time evolution of the system that cannot be reproduced by those that act on fewer subsystems. Therefore, by probing these effects, we can directly obtain information about the locality of the underlying interactions that govern the process. This is analogous to the fact that, in the presence of symmetries, we can detect a hypothetical symmetry-breaking interaction, just by observing the violation of Noether’s conservation law for the input and output of the process, without knowing the details of the underlying interactions. (In our case, the hypothetical term is not symmetry breaking but rather couples multiple subsystems together.)

As a simple example, consider a system of n qubits evolving for a total time T under an unknown Hamiltonian $H(t)$ that preserves $\sum_j Z_j$. To have a concrete example, one can assume that $H(t)$ models the interactions in a complex scattering process with n particles, and that the states $\{|0\rangle, |1\rangle\}$ of each qubit corresponds to an internal degree of freedom of a particle, for example, its electric charge, whose total value remains conserved in the process. Suppose we want to characterize the locality of interactions that govern this process. For instance, we start with the hypothesis that $H(t) = H_0(t) + \gamma(t) \mathbf{Z}^{\otimes n}$, where H_0 only contains k -local terms with $k < n$ while $\gamma \mathbf{Z}^{\otimes n}$ corresponds to a hypothetical n -body interaction, for example, a correction to the Coulomb law. The goal is to test the hypothesis that the n -body term $\gamma \mathbf{Z}^{\otimes n}$ is non-zero, by probing the output of this process for different input states. Note that, in the absence of symmetries, unless there are further assumptions about the form of H_0 , it is impossible to obtain information about the strength of γ . Indeed, the universality of 2-local unitaries means that, even if $\gamma=0$, the Hamiltonian H_0 with 2-local interactions can generate any arbitrary unitary transformation. Therefore, by probing the outputs of this process for different inputs, we cannot distinguish the cases of $\gamma=0$ and $\gamma \neq 0$.

While this is impossible in the absence of symmetries, our result reveals that symmetries allow us to directly probe the locality of interactions that govern a process, just by observing the inputs

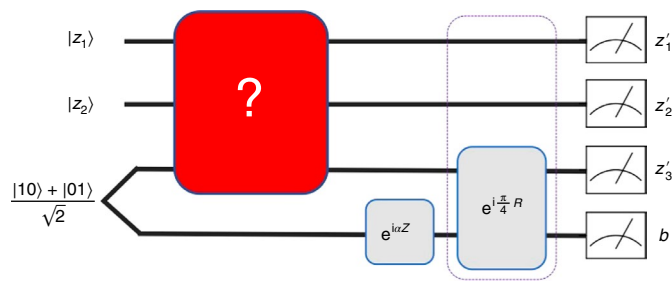


Fig. 3 | Scheme for local symmetric process tomography and measurement of l -body phases. The no-go theorem found in this paper has an immediate useful implication: it gives a new method for detecting the locality of the underlying interactions that govern a charge-conserving unitary process. Specifically, by measuring the l -body phase of the unitary, as defined in equation (4), we can detect l -body interactions. This figure presents a schematic experimental setup that fully characterizes an unknown $U(1)$ -invariant unitary and its l -body phases, using initial states, single-qubit measurements and 2-local unitaries, which all respect the symmetry. In this example, the red box corresponds to an unknown three-qubit charge-conserving unitary V . The goal is to measure the three-body phase $\Phi_3 \in (-\pi, \pi]$. Observing $\Phi_3 \neq 0$ indicates the presence of the three-body interaction $Z^{\otimes 3}$. At the input of V , all the qubits are prepared in the unentangled symmetric states $|z\rangle$ with $z = 0, 1$, except one of them, which is entangled with an ancillary qubit, in the joint state $(|01\rangle + |10\rangle)/\sqrt{2}$. This ancillary qubit plays the role of an internal quantum reference frame¹⁶ and allows us to probe the relative phases between sectors with different charges through an interference experiment. After the unknown unitary V , we apply the single-qubit unitary $\exp(i\alpha Z)$ on the ancillary qubit, then interact it with one of the three qubits in the system via 2-local unitary $\exp(i\frac{\pi R}{4})$, where $R = (XX + YY)/2$, and finally measure all qubits in the $\{|0\rangle, |1\rangle\}$ basis. As we discuss further in Supplementary Note 5, using this scheme we can fully characterize the unknown unitary V , up to a global phase and, in particular, determine the three-body phase Φ_3 .

and outputs of the process. This can be achieved systematically by measuring the l -body phases of the unitary process for $l \geq 1$. For instance, in the above example, by measuring the n -body phase $\Phi_n \in (-\pi, \pi]$ of the unitary V that describes the overall process, we obtain a lower bound on $\gamma_{\max} = \max_{t \in [0, T]} |\gamma(t)|$, which determines the maximum strength of the n -body interaction, namely

$$\gamma_{\max} \geq \frac{|\Phi_n|}{2^n \times T}, \quad (5)$$

where we have applied the second equality in equation (4). Note that, according to the first equality in equation (4), $\Phi_n = \sum_{m=0}^n (-1)^m \theta_m \pmod{2\pi}$.

How can we measure l -body phases of a unitary? More generally, is it possible to characterize a $U(1)$ -invariant unitary transformation and perform process tomography²⁵, using only local symmetric operations? We find that, despite our no-go theorem on realizable unitaries, the answer is affirmative. A general $U(1)$ -invariant unitary can be fully characterized, up to a global phase, using symmetric initial states, symmetric single-qubit measurements and 2-local symmetric unitaries, provided that one can use a single ancillary qubit that is initially entangled with one of the qubits in the system. In particular, the scheme presented in Fig. 3 does not require the preparation of superpositions of states with different charges, which might be impractical due to the superselection rules (see Supplementary Note 5 for further discussion).

Circumventing the no-go theorem with ancillary systems. Interestingly, it turns out that, in the case of $U(1)$ symmetry, our no-go theorem can be circumvented, provided that one is allowed

to interact with an ancillary qubit: for any n -qubit $U(1)$ -invariant unitary V , there exists $(n+1)$ -qubit unitary $\tilde{V}$ that can be implemented using 2-local $U(1)$ -invariant Hamiltonians $XX + YY$ and local Z , and satisfies

$$\tilde{V}(|\psi\rangle \otimes |0\rangle_a) = (V|\psi\rangle) \otimes |0\rangle_a, \quad (6)$$

for all n -qubit states $|\psi\rangle$. This means that, while by applying local symmetric unitaries the ancillary qubit becomes entangled with the qubits in the system, at the end of the process it returns back to its initial state $|0\rangle$, whereas the state of the system transforms as the desired unitary V .

Figure 4 demonstrates a variant of this result that requires two ancillary qubits. In this example, the goal is to implement the unitaries generated by the Hamiltonian $Z^{\otimes n}$. Roughly speaking, in this scheme, a charge is transported through a closed loop that starts from an ancillary qubit, goes through the entire system and finally returns back to the ancilla. As a result, the joint state obtains a phase depending on the parity of the total charge in the system, which corresponds to the observable $Z^{\otimes n}$. The overall effect is equivalent to applying the desired Hamiltonian $Z^{\otimes n}$ on the system. Here, the ancillary qubits can be interpreted as an internal quantum reference frame¹⁶, relative to which the phase shift generated by observable $Z^{\otimes n}$ is measured in a coherent fashion. As we further explain in Supplementary Note 6, this process has also a nice interpretation in the fermionic description of the system, obtained by applying the Jordan–Wigner transform^{26–28}.

Application: quantum thermodynamics with local interactions.

Our surprising no-go theorem also has interesting implications in the context of quantum thermodynamics and, specifically, the operational approach to thermodynamics, which is often called the ‘resource theory’ of quantum thermodynamics^{8–15}. A fundamental assumption in this framework is that all energy-conserving unitaries, that is, those commuting with the intrinsic Hamiltonian of the system, are ‘free’, that is, can be implemented with negligible thermodynamic cost. This is assumed even for composite systems with arbitrarily large number of subsystems. However, our result implies that general energy-conserving unitaries on a composite system cannot be implemented by applying local energy-conserving unitaries on the subsystems. In fact, even by composing energy-conserving unitaries that act on $n-1$ subsystems, one still cannot generate all energy-conserving unitaries on n subsystems. Note that energy-conserving unitaries are those that are invariant under the time-translation symmetry $\{e^{-iH_0 t} : t \in \mathbb{R}\}$ generated by the intrinsic Hamiltonian H_0 ; a continuous symmetry, which is isomorphic to the group $U(1)$ in the case of periodic systems.

Therefore, this no-go theorem suggests that there might be some hidden thermodynamic costs for implementing general energy-conserving unitaries, using local energy-conserving unitaries and, in principle, this additional cost can increase with the system size. The following theorem addresses this concern (see Supplementary Note 7 for a more precise statement).

Theorem: consider a finite set of closed systems with the property that, for each system, the gap between any consecutive pairs of energy levels is ΔE . Then, any global energy-conserving unitary transformation on these systems can be implemented by a finite sequence of 2-local energy-conserving unitaries, provided that the systems can interact with a single ancillary qubit with the energy gap ΔE between its two levels.

To establish this result, we introduce a generalization of the scheme introduced in the previous section for qubit systems with $U(1)$ symmetry. We conclude that the assumption of the resource theory of quantum thermodynamics^{8–14} that all energy-conserving unitaries (and hence all thermal operations) are free, is consistent with the locality of interactions, provided that one allows the use of

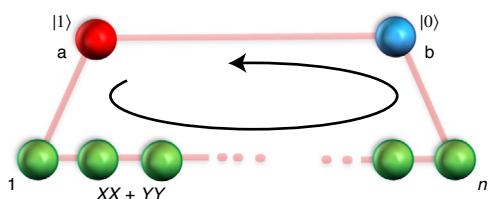


Fig. 4 | Circumventing the no-go theorem with ancillary qubits. Our no-go theorem implies that the family of unitaries generated by the n -qubit Hamiltonian $Z^{\otimes n}$ cannot be implemented using local $U(1)$ -invariant unitaries, even if they act on $n-1$ qubits. This figure presents a scheme for circumventing this no-go result, using two ancillary qubits. This scheme uses the interaction $R = (XX + YY)/2$ between nearest-neighbour qubits on a closed loop. The two ancillary qubits, denoted by a (red ball) and b (blue ball) are initially prepared in states $|1\rangle$ and $|0\rangle$, respectively. First, we show that it is possible to realize the Hamiltonian $K = Z^{\otimes n} \otimes R_{a,b}$ without any direct interaction between the ancillary qubits. This only requires coupling qubit a to qubit $j=1$ in the chain, coupling between nearest-neighbour qubits in the chain (green balls) and coupling between qubit $j=n$ and ancilla b. This Hamiltonian describes the process in which a charge is transported through the chain from one ancillary qubit to the other and obtains a phase depending on the parity of the total charge in the system. As we explain in Supplementary Note 6, this has an intuitive interpretation in the fermionic description of this system, obtained by applying the Jordan–Wigner transform. After evolving the entire system for a short time interval δt under Hamiltonian K , we obtain the joint state $|\psi\rangle |1\rangle_a |0\rangle_b - i\delta t Z^{\otimes n} |\psi\rangle |0\rangle_a |1\rangle_b + \mathcal{O}(\delta t^2)$, where $|\psi\rangle$ is the initial state of n qubits. Next, we directly couple a to b and close the loop, using the 2-local unitary $\exp(i\pi R_{a,b}/4) \exp(i\pi Z_b/4)$ that allows the charge to move back and forth between the ancillary qubits, without going through the chain. Finally, we measure one of the ancillary qubits in the $\{|0\rangle, |1\rangle\}$ basis. This determines the final location of the charge initially located in qubit a. The final state of n qubits is $\exp(\pm i\delta t Z^{\otimes n}) |\psi\rangle + \mathcal{O}(\delta t^2)$, where the sign depends on whether the final location of charge is qubit a or b. Therefore, this process stochastically implements the Hamiltonian $\pm Z^{\otimes n}$. In principle, by choosing an infinitesimal time step δt and repeating this scheme many times, we can implement the desired unitary $\exp(i\phi Z^{\otimes n})$ for arbitrary angle ϕ , with an error approaching zero and probability of success approaching one. We show that a slightly more complicated version of this scheme can be realized deterministically.

ancillary systems. In the context of quantum thermodynamics, such systems can be interpreted as catalysts^{15,29}. It is worth mentioning that the assumption in this theorem on the energy gap ΔE between consecutive levels can be relaxed, provided that one can use larger catalysts with more energy levels.

Application: synthesizing noise-resilient quantum circuits. Another motivation to study LSQCs comes from the field of quantum computing and, specifically, the desire to design fault-tolerant quantum circuits. In both prominent implementations of quantum computers, namely superconducting and trapped-ion computers, the instability of the master clock that determines the timing of the control pulses is a major source of noise^{30,31}. Each qubit in these systems has a non-zero intrinsic Hamiltonian, which corresponds to an energy difference between states $|0\rangle$ and $|1\rangle$. Hence, the state of a qubit is constantly evolving in time. Ideally, using a stable clock, one can keep track of this intrinsic time evolution. In other words, one can assume that quantum computation is performed in a co-rotating frame, where there is no energy difference between $|0\rangle$ and $|1\rangle$. In practice, however, due to the instabilities of the clock, this intrinsic time evolution of qubits causes error and destroys coherence between states with different energies. For instance, if there is a random time delay δt in applying the control pulses that implement a desired

unitary transformation V , then the actual implemented unitary in the co-rotating frame will be $\exp(i\delta t H_0) V \exp(-i\delta t H_0)$, where $H_0 = -\Delta E \sum_j Z_j/2$ is the total intrinsic Hamiltonian of the qubits. In principle, this effect can be suppressed by restricting the state of qubits to an energy eigen-subspace, which is a decoherence-free subspace^{23,24,32}. However, this amounts to sacrificing a fraction of physical qubits. Given the limited number of qubits available in near-term quantum computers, it is crucial to explore other, complementary techniques.

One approach for suppressing this type of noise is to minimize the use of non-energy-conserving unitaries in the circuit. That is, the circuit should be mostly formed from local energy-conserving unitaries. This includes energy-conserving elementary gates, such as single-qubit rotations around z , as well as energy-conserving multi-qubit modules, which may contain non-energy-conserving elementary gates. As long as the entire module can be executed in a sufficiently short time during which the clock fluctuations are negligible, then the energy conservation of the module guarantees its resilience against this type of noise. For example, while the standard Mølmer–Sørensen gate³³ $\exp(i\theta XX)$ on trapped-ion quantum computers is not energy conserving and hence is sensitive to these fluctuations, when it is sandwiched between Hadamards on both qubits, it transforms to $\exp(i\theta ZZ)$, which is energy conserving. Similarly, by combining two Mølmer–Sørensen gates with single-qubit phase gates, we obtain $\exp(i\theta (XX + YY))$, which is again energy conserving.

The tools and ideas introduced in this paper provide a foundation for the systematic synthesis of quantum circuits that are resilient against this type of noise. To minimize the number of non-energy-conserving unitaries, the first step is to determine which unitaries can be efficiently realized using local energy-conserving modules. As an example, consider the family of unitaries generated by the multi-qubit swap Hamiltonian: suppose that a system with $2r$ qubits is partitioned into two subsystems A and B, each with r qubits. Let S_{AB} be the multi-qubit swap operator that exchanges the states of A and B. The family of unitaries $\exp(i\phi S_{AB})$ for $\phi \in [0, 2\pi)$ appears as a subroutine in various quantum algorithms (see, for example, refs. 34–37). It has also found applications in the study of quantum reference frames and quantum thermodynamics^{35,38}. The multi-qubit swap Hamiltonian S_{AB} is not only energy conserving but in fact respects the stronger $SU(2)$ symmetry, that is, $[S_{AB}, U^{\otimes 2r}] = 0$ for all single-qubit unitaries U . Therefore, one may expect that this family of unitaries should be realizable using a sequence of local $SU(2)$ -invariant unitaries or, at least, using local energy-conserving unitaries, which may break the $SU(2)$ symmetry. However, our results refute this conjecture: for generic values of ϕ , all the l -body phases of the unitary $\exp(i\phi S_{AB})$ are non-zero (for example, $\Phi_{2r} = 2^r \phi \bmod 2\pi$), which means that this unitary is not realizable using local energy-conserving unitaries. On the other hand, if one is allowed to use a single ancillary qubit, then this family is realizable using single-qubit rotations around z together with unitaries $\exp(i\theta (XX + YY))$, which, as discussed above, can be obtained from two Mølmer–Sørensen gates. Therefore, to implement a quantum algorithm that employs this subroutine, this part of the circuit can be realized using only energy-conserving modules. This makes the entire circuit more resilient against clock fluctuations.

Discussion

Universality of local unitaries in the absence of symmetries is a profound fact about composite quantum-mechanical systems, with vast applications and implications in different areas of physics. Hence, the failure of universality in the presence of symmetries can also have interesting and unexpected implications in different areas. Here, we saw an example of such surprising implications, namely the possibility of probing the locality of interactions. We end with

a brief discussion about other examples of applications of these results and the related open questions:

Quantum reference frames and covariant codes. Symmetric unitaries naturally appear in the study of quantum reference frames¹⁶. For instance, it is often assumed that, in the absence of a Cartesian reference frame, it is still possible to perform any unitary that respects the $SO(3)$ symmetry group corresponding to rotations in 3D space¹⁶. The no-go theorem found in this paper implies that, if one takes into account the locality of interactions, then there can be further restrictions on the realizable unitaries. It will be interesting to study possible implications of these additional constraints in the context of quantum reference frames. As an example, ref.³⁹ shows that arbitrary symmetry-breaking Hamiltonians on a system can be simulated by coupling the system via rotationally invariant Hamiltonians to $n \gg 1$ spin-half systems aligned in x and z directions. Therefore, in the limit of large n , this quantum reference frame fully lifts the constraint of symmetry. It is interesting to further study the efficiency and complexity of such schemes when the Hamiltonians are restricted to be local.

A similar question also arises in the context of covariant error correction, which has recently attracted attention in the quantum information community (see, for example, refs.^{40–42}). Here, the goal is to understand the limitations and capabilities of quantum error-correcting codes that can be realized by symmetric operations. Then, again, it is crucial to understand whether those codes can be realized via local symmetric unitaries.

Symmetry-protected complexity. Another interesting open question in this area is to understand how the notion of circuit complexity changes under the constraint of symmetry. Recall that the circuit complexity of a unitary transformation or a state is the minimum number of local gates needed to implement the unitary or to prepare the state from a fixed (product) state⁴³. For a symmetric unitary or a symmetric state, we can define a modified notion of complexity, which can be called symmetry-protected complexity (SPC) and is defined as the minimum number of symmetric local unitaries needed to implement a symmetric unitary or to prepare a symmetric state. Certain aspects of this notion of complexity have been studied in the context of symmetry-protected topological phases^{44,45}. In particular, it is known that, for certain families of states, the SPC grow linearly with the number of subsystems, whereas the regular complexity remains constant. Given the conjectured roles of complexity in the context of holography and anti-de Sitter/conformal field theory correspondence^{46–48}, it is interesting to further study the notion of SPC and compare it with the regular complexity.

Analogue quantum simulation. Understanding the constraints imposed by the locality of interactions is also crucial in the context of analogue quantum simulation, which is one of the main applications of near-term quantum technology. In this approach to quantum simulation, the degrees of freedom and the dynamics of the target system are directly mapped to those of the simulator, which is a well-controlled quantum system with a tunable Hamiltonian (see, for example, refs.^{49–51}). As we saw in this work, in the presence of symmetries, the locality of the simulator Hamiltonian severely restrict the set of realizable Hamiltonians. It is interesting to further explore how these restrictions limit the power of analogue quantum simulators in the presence of symmetries, and, in particular, to investigate whether they can be efficiently circumvented.

Proofs. All the results in the paper are rigorously proven in Supplementary Notes 1–7.

Online content

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Methods

Preliminaries: the Lie algebra generated by local symmetric Hamiltonians.

We start with a quick review of a standard result in quantum control theory (see Supplementary Note 1 for more details). Suppose that one can implement the unitary time evolutions generated by Hamiltonians $\pm A$ and $\pm B$ for an arbitrary amount of time $t \geq 0$. That is, one can turn on and off these Hamiltonians at will. Then, combining these time evolutions one can obtain unitaries

$$e^{-iB(c_2\delta t)} e^{-iA(c_1\delta t)} = e^{-i(c_1A+c_2B)\delta t} + \mathcal{O}(\delta t^2) \quad (7a)$$

$$e^{-iA\delta t} e^{-iB\delta t} e^{iA\delta t} e^{iB\delta t} = e^{-[A,B]\delta t^2} + \mathcal{O}(\delta t^3), \quad (7b)$$

for arbitrary coefficients $c_1, c_2 \in \mathbb{R}$, and for sufficiently small δt . This means that, using Hamiltonians $\pm A$ and $\pm B$, one can approximately simulate the time evolutions generated by any Hamiltonian in the linear span of A and B as well as the Hamiltonian $i[A, B]$. Furthermore, by repeating such combinations of unitaries, one can obtain a larger class of unitaries. In fact, it can be proven that, using finite sequences of unitaries generated by the Hamiltonians $\pm A$ and $\pm B$, one obtains all unitary transformations $\{e^{-iHt} : t \in \mathbb{R}\}$ generated by any Hermitian operator H if, and only if, H belongs to the real Lie algebra generated by A and B , that is, can be written as a linear combination of A , B and their (nested) commutators, $i[A, B]$, $[[A, B], A]$, $[[A, B], B]$, $\dots$, with real coefficients. As we explain more in Supplementary Note 1, this result means that, to characterize the group $\mathcal{V}_k^G$ generated by k -local symmetric unitaries, it suffices to characterize the Lie algebra generated by k -local symmetric skew-Hermitian operators. In particular, the dimension of this Lie algebra, as a vector space over $\mathbb{R}$, is equal to $\dim(\mathcal{V}_k^G)$, the dimension of the manifold associated to $\mathcal{V}_k^G$, which is also equal to the number of real parameters needed to specify a general element of $\mathcal{V}_k^G$. Using this relation, we establish an upper bound on $\dim(\mathcal{V}_k^G)$, which is discussed next.

Charge vectors. Next, we introduce the idea of ‘charge vectors’, our main tool for deriving constraints on the unitary evolutions generated by local symmetric Hamiltonians. Recall that $\text{Irreps}_G(n)$ denotes the set of inequivalent irreps of group G that appear in the representation $\{U(g) = u(g)^{\otimes n} : g \in G\}$ and $|\text{Irreps}_G(n)|$ is the number of these inequivalent irreps. Let Π_μ be the projector to the subspace corresponding to irrep $\mu \in \text{Irreps}_G(n)$, also known as the isotypic component of μ . For any operator A , define the charge vector of A as

$$|\chi_A\rangle \equiv \sum_{\mu \in \text{Irreps}_G(n)} \text{Tr}(\Pi_\mu A) |\mu\rangle, \quad (8)$$

where $\{|\mu\rangle : \mu \in \text{Irreps}_G(n)\}$ is a set of orthonormal vectors in an abstract vector space with dimension $|\text{Irreps}_G(n)|$. A general G -invariant Hamiltonian can have any charge vector with real coefficients. In particular, for any set of real numbers $\{h_\mu \in \mathbb{R} : \mu \in \text{Irreps}_G(n)\}$, the Hermitian operator $\sum_{\mu \in \text{Irreps}_G(n)} \frac{h_\mu}{\text{Tr}(\Pi_\mu)} \Pi_\mu$ is G -invariant and has the charge vector $\sum_{\mu \in \text{Irreps}_G(n)} h_\mu |\mu\rangle$. In other words, under the linear map $A \rightarrow |\chi_A\rangle$, the image of the linear space of Hermitian G -invariant operators has dimension $|\text{Irreps}_G(n)|$.

On the other hand, it turns out that, if the unitary evolutions generated by Hamiltonian H can be simulated by k -local G -invariant unitaries, that is, if $\forall t \in \mathbb{R} : e^{-iHt} \in \mathcal{V}_k^G$, then the charge vector of H should satisfy certain constraints. Let $\mathcal{S}_k$ be the set of charge vectors for all such Hamiltonians, that is,

$$\mathcal{S}_k \equiv \{|\chi_H\rangle : e^{-iHt} \in \mathcal{V}_k^G, \forall t \in \mathbb{R}\}. \quad (9)$$

We prove that $\mathcal{S}_k$ is a linear subspace (over the field $\mathbb{R}$) with dimension

$$\dim(\mathcal{S}_k) \leq |\text{Irreps}_G(k)|, \quad (10)$$

and the equality holds if G is a connected Lie group, such as $U(1)$ and $SU(2)$. Therefore, if $|\text{Irreps}_G(k)| < |\text{Irreps}_G(n)|$, then $\dim(\mathcal{S}_k) < |\text{Irreps}_G(n)|$, which means there are G -invariant Hamiltonians whose charge vectors do not belong to $\mathcal{S}_k$, which in turn implies they cannot be simulated using k -local symmetric unitaries. For continuous groups, such as $U(1)$, $|\text{Irreps}_G(n)|$ grows unboundedly with n and, therefore, universality cannot be achieved with k -local symmetric unitaries with a fixed k .

Below we present a simple argument that explains why the dimension of $\mathcal{S}_k$ cannot grow unboundedly with the system size. The specific bound on $\dim(\mathcal{S}_k)$ in equation (10) is proven in Supplementary Note 2, using the Fourier transform of charge vectors. Furthermore, in Supplementary Note 2, we discuss more about charge vectors and their Lie-algebraic interpretation. Briefly, charge vector $|\chi_A\rangle$ of an operator A determines its component in the centre of the Lie algebra of all G -invariant Hamiltonians, that is, the Lie algebra corresponding to the Lie group $\mathcal{V}^G$. Then, the subspace $\mathcal{S}_k$ determines which part of the centre can be generated by k -local G -invariant Hamiltonians. In particular, if $\dim(\mathcal{S}_k)$ is less than $\dim(\mathcal{S}_n) = |\text{Irreps}_G(n)|$, then local symmetric Hamiltonians cannot generate certain elements of the centre, which means such Hamiltonians are not universal. This results in the bound in equation (3).

Next, we explain why $\dim(\mathcal{S}_k)$ cannot grow unboundedly with n . To determine $\mathcal{S}_k$, we use the fact that, if $e^{-iHt} \in \mathcal{V}_k^G$ for all $t \in \mathbb{R}$, then H should be in the Lie algebra generated by k -local G -invariant operators, that is, $H = \sum_j c_j A_j + \sum_{j_1, j_2} c_{j_1 j_2} i[A_{j_1}, A_{j_2}] + \dots$, where A_j are Hermitian k -local G -invariant operators and coefficients $c_j, c_{j_1 j_2}, \dots$ are real numbers. It can be shown that the commutators appearing in this expansion do not

contribute in the charge vector of H , that is, $|\chi_H\rangle = \sum_j c_j |\chi_j\rangle$, where $|\chi_j\rangle \equiv \sum_{\mu \in \text{Irreps}_G(n)} \text{Tr}(\Pi_\mu A_j) |\mu\rangle$ is the charge vector of A_j . To see this, note that, for any irrep $\mu \in \text{Irreps}_G(n)$, $\text{Tr}([A_{j_1}, A_{j_2}] \Pi_\mu) = \text{Tr}(A_{j_1} [A_{j_2}, \Pi_\mu]) = 0$, where the first equality follows from the cyclic property of trace and the second equality follows from the assumption that A_{j_2} is G -invariant and therefore commutes with Π_μ . It follows that the commutator $[A_{j_1}, A_{j_2}]$ and other nested commutators do not contribute in $|\chi_H\rangle$. This implies that $\mathcal{S}_k$ is spanned by the charge vectors of k -local G -invariant Hermitian operators, that is, $\mathcal{S}_k$ is equal to

$$\text{Span}_{\mathbb{R}} \{|\chi_A\rangle : A = A^\dagger, A \text{ is } k\text{-local}, [A, U(g)] = 0 : \forall g \in G\}. \quad (11)$$

Next, note that, for any k -local operator A , by applying a properly chosen permutation operator S which changes the order of sites, we can obtain an operator in the form $SAS^\dagger = \tilde{A} \otimes I_{\text{rest}}$ with the property that $\tilde{A}$ acts on a fixed set of k sites (for example, the first k sites according to a certain ordering) and I_{rest} is the identity operator on the remaining $n-k$ sites. Since charge vectors remain invariant under permutations, operators A and $SAS^\dagger = \tilde{A} \otimes I_{\text{rest}}$ have the same charge vectors. It follows that the subspace in equation (11) is equal to the set of the charge vectors of G -invariant Hermitian operators that act non-trivially only on a fixed set of k sites (for example, the first k sites). Therefore, as the number of total sites n increases, $\dim(\mathcal{S}_k)$ remains bounded by a number independent of n . In other words, even though, using k -local G -invariant unitaries, we can simulate Hamiltonians that are not k -local, they can only have charge vectors which are allowed for k -local G -invariant Hamiltonians. This explains why the upper bound on $\dim(\mathcal{S}_k)$ in equation (10) does not depend on the system size.

Example: $SU(2)$ symmetry with spin- s systems. In the case of $SU(2)$ symmetry, consider n spin- s systems, each with the Hilbert space of dimension $2s+1$. Recall that irreps of $SU(2)$ can be labelled by the eigenvalues of the squared angular momentum operator $J^2 = J_x^2 + J_y^2 + J_z^2$. The eigenvalues have the form of $j(j+1)$, where j is half-integer and takes values $j = 1/2, 3/2, \dots, ns$ if s is not integer and n is odd, and values $j = 0, 1, \dots, ns$, otherwise. In both cases, the total number of distinct irreps is $|\text{Irreps}_{SU(2)}(n)| = \lfloor ns \rfloor + 1$. Because $SU(2)$ is a connected group, the bound in equation (10) holds as equality, that is, $\dim(\mathcal{S}_k) = \lfloor ks \rfloor + 1$. Furthermore, equation (3) implies that the difference between the dimensions of the manifolds of all $SU(2)$ -invariant unitaries and those realizable by k -local $SU(2)$ -invariant unitaries is lower bounded by

$$\dim(\mathcal{V}_n^{SU(2)}) - \dim(\mathcal{V}_k^{SU(2)}) \geq \lfloor ns \rfloor - \lfloor ks \rfloor. \quad (12)$$

For integer spin s , this means that, for any $k < n$, there are $(k+1)$ -local unitaries that cannot be realized using k -local unitaries. Similarly, for non-integer s , there are $(k+2)$ -local unitaries that cannot be realized using k -local unitaries.

Full characterization of realizable $U(1)$ -invariant Hamiltonians for qubits. In Supplementary Note 3, we study the example of $U(1)$ symmetry for qubit systems. Interestingly, it turns out that, in this example, the constraints imposed by the charge vectors fully characterize the set of realizable Hamiltonians. The theorem below states these conditions.

For a system with n qubits, define Hermitian operators $C_l, l=0, \dots, n$ as

$$C_l \equiv \sum_{\mathbf{b} : w(\mathbf{b})=l} \mathbf{Z}^{\mathbf{b}} = \sum_{m=0}^n c_l(m) \Pi_m, \quad (13)$$

where the first summation is over all bit strings $\mathbf{b} = b_1 \dots b_n \in \{0, 1\}^n$ with Hamming weight $w(\mathbf{b}) \equiv \sum_{j=1}^n b_j$ equal to l , and $\mathbf{Z}^{\mathbf{b}} = Z_1^{b_1} \dots Z_n^{b_n}$. In the second term, Π_m is the projector to the eigen-subspace of $N = \sum_{j=1}^n (I - Z_j)/2$ with eigenvalue m , and

$$c_l(m) = \sum_{s=0}^m (-1)^s \binom{m}{s} \binom{n-m}{l-s}, \quad (14)$$

is the eigenvalue of C_l in this subspace (recall that the binomial coefficient $\binom{a}{b} = 0$ for $b > a$). See Supplementary Note 3 for derivation of equation (14)). We prove

Theorem. For any $U(1)$ -invariant Hamiltonian H on n qubits, the family of unitaries $\{e^{-iHt} : t \in \mathbb{R}\}$ can be implemented using k -local $U(1)$ -invariant unitaries for $k \geq 2$, if and only if

$$\text{Tr}(HC_l) = 0 \quad : l = k+1, \dots, n. \quad (15)$$

Note that, using equation (13), these conditions can be rewritten in terms of the charge vector $|\chi_H\rangle = \sum_{m=0}^n \text{Tr}(H \Pi_m) |m\rangle$ of Hamiltonian H , where $\{|m\rangle\}$ is a basis for an abstract $(n+1)$ -dimensional vector space.

Equations (15) impose exactly $n - k$ independent constraints on the set of realizable Hamiltonians. Hence, the difference between the dimension of realizable $U(1)$ -invariant Hamiltonians and all $U(1)$ -invariant Hamiltonians is exactly $n - k$, which means that, in this case, the general bound in equation (3) holds as equality. This theorem is proven in Supplementary Note 3.

Data availability

Data sharing is not applicable to this article, as no datasets were generated or analysed during the current study.

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Author contributions

I.M. was the sole contributor to all aspects of this work.

Competing interests

The author declares no competing interest.

Additional information

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