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FACTORIZATION LENGTH DISTRIBUTION FOR AFFINE SEMIGROUPS III: MODULAR EQUIDISTRIBUTION FOR NUMERICAL SEMIGROUPS WITH ARBITRARILY MANY GENERATORS

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Abstract

For numerical semigroups with a specified list of (not necessarily minimal) generators, we describe the asymptotic distribution of factorization lengths with respect to an arbitrary modulus. In particular, we prove that the factorization lengths are equidistributed across all congruence classes that are not trivially ruled out by modular considerations.

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1. Introduction

In what follows, we let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and denote the cardinality of a set X by $|X|$. A *numerical semigroup* $S \subset \mathbb{N}$ is an additive subsemigroup containing 0. Each numerical semigroup S admits a finite generating set, and we write

$$S = \langle n_1, n_2, \ldots, n_k \rangle = \{a_1 n_1 + a_2 n_2 + \cdots + a_k n_k \ : \ a_1, a_2, \ldots, a_k \in \mathbb{N}\}$$

for the numerical semigroup generated by the distinct positive $n_1 < n_2 < \cdots < n_k$. Throughout this document, we always assume S has finite complement in $\mathbb{N}$, or, equivalently, that $\gcd(n_1, n_2, \ldots, n_k) = 1$. However, we do not assume that $n_1, n_2, \ldots, n_k$ form the unique minimal generating set of S under containment [21].

A *factorization* of $n \in S$ is an expression

$$n = a_1 n_1 + a_2 n_2 + \cdots + a_k n_k$$

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of n as a sum of generators of S, denoted by the k-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{N}^k$. The *length* of the factorization $\mathbf{a}$ is

$$||\mathbf{a}|| = a_1 + a_2 + \cdots + a_k.$$

The *length multiset of* n, denoted $\mathsf{L}[\![n]\!]$, is the multiset with a copy of $||\mathbf{a}||$ for each factorization $\mathbf{a}$ of n. Recall that a *multiset* is a set in which repetition is taken into account; that is, its elements can occur multiple times. In particular, the cardinality $|\mathsf{L}[\![n]\!]|$ equals the number of factorizations of n.

Factorizations and their lengths have been studied extensively in numerous contexts, including factorization theory [15–17], additive combinatorics [18, 19], discrete optimization [12, 20], commutative and noncommutative algebra [3, 4], and algebraic geometry [1, 5]. Until recently, results concerning the multiplicities of factorization lengths have been surprisingly absent from the literature. The study of $\mathsf{L}[\![n]\!]$ was initiated in [14], wherein a closed form for the limiting distribution was obtained for three-generator numerical semigroups via careful combinatorial arguments. In a sequel paper [13], measure theory and algebraic combinatorics were used to characterize the distribution for arbitrary numerical semigroups. The present paper, the third in this series, examines the spread of $\mathsf{L}[\![n]\!]$ across congruence classes modulo a fixed positive integer N. More precisely, given $n \in S$, $N \geq 1$, and $i \in \{0, \ldots, N-1\}$, we study the distribution of $\mathsf{L}[\![n]\!] \cap (i + N\mathbb{Z})$.

Before we introduce the main theorem, let us briefly recall the constant

$$\delta = \gcd(n_2 - n_1,\, n_3 - n_2, \ldots,\, n_k - n_{k-1})$$

and its relation to factorization lengths. Given $n \in S$, all lengths $\ell_1, \ell_2 \in \mathsf{L}[\![n]\!]$ must satisfy $\ell_1 \equiv \ell_2 \pmod{\delta}$ (in fact, δ is the largest integer with this property by [8, Proposition 2.9]). In particular, the intersection $\mathsf{L}[\![n]\!] \cap (i + N\mathbb{Z})$ is sometimes empty, even for arbitrarily large n.

Let us consider an example. For the numerical semigroup $S = \langle 7, 19, 25, 31 \rangle$, every factorization length of $n = 434$ is congruent modulo $\delta = 6$. More specifically, since $n = 62\, n_1$, every length in $\mathsf{L}[\![n]\!]$ is congruent to 2 modulo 6. As such, for $N = 4$, only the intersections $\mathsf{L}[\![n]\!] \cap 4\mathbb{Z}$ and $\mathsf{L}[\![n]\!] \cap (2 + 4\mathbb{Z})$ can be nonempty. If, on the other hand, we had chosen n odd, then every element of $\mathsf{L}[\![n]\!]$ must be congruent to either 1 or 3 modulo 4. Among other things, Theorem 1.1 below implies that in this example, for n large and odd, the multisets $\mathsf{L}[\![n]\!] \cap (1 + N\mathbb{Z})$ and $\mathsf{L}[\![n]\!] \cap (3 + N\mathbb{Z})$ have identical distributions.

To state Theorem 1.1, we require some algebraic terminology. The *complete homogeneous symmetric polynomial* of degree p in the k variables $x_1, x_2, \ldots, x_k$ is

$$h_p(x_1, x_2, \ldots, x_k) \quad = \sum_{1 \leq \alpha_1 \leq \cdots \leq \alpha_p \leq k} x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_p},$$

the sum of all degree p monomials in $x_1, x_2, \ldots, x_k$. A *quasipolynomial* of degree d is a function $f : \mathbb{Z} \to \mathbb{C}$ of the form

$$f(n) = c_d(n) n^d + c_{d-1}(n) n^{d-1} + \cdots + c_1(n) n + c_0(n),$$

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in which the coefficients $c_1(n), c_2(n), \dots, c_d(n)$ are periodic functions of $n \in \mathbb{Z}$ [6]. A *quasirational function* is a quotient of two quasipolynomials.

THEOREM 1.1. *Let $S = \langle n_1, n_2, \dots, n_k \rangle$ with $\gcd(n_1, n_2, \dots, n_k) = 1$ and define δ as above. Fix $N \in \mathbb{N}$ and $p, i \in \mathbb{Z}$ with $p \geq 0$ and $0 \leq i < N$; and let $m = \gcd(\delta, N)$. Then*

$$\sum_{\substack{\ell \in \mathsf{L}[\![n]\!] \\ \ell \equiv i \,(\mathrm{mod}\ N)}} \ell^p = \begin{cases} \dfrac{p!\; m\, h_p(1/n_1, 1/n_2, \dots, 1/n_k)}{N(k+p-1)!\,(n_1 \cdots n_k)} n^{k+p-1} + w_i(n) & \text{if } n \equiv i n_1 \,(\mathrm{mod}\ m), \\ 0 & \text{if } n \not\equiv i n_1 \,(\mathrm{mod}\ m), \end{cases}$$

in which $w_i(n)$ is a quasipolynomial of degree at most $k + p - 2$ whose coefficients are rational-valued and have period dividing $N \operatorname{lcm}(n_1, n_2, \dots, n_k)$.

The choice of m in Theorem 1.1 implies

$$n_1 \equiv n_2 \equiv \cdots \equiv n_k \,(\mathrm{mod}\ m). \tag{1-1}$$

Together with the assumption $\gcd(n_1, n_2, \dots, n_k) = 1$, it follows that $n_1, n_2, \dots, n_k$ are invertible modulo m. Since n_1 is invertible modulo m, there is a unique i modulo m such that $n \equiv i n_1 \,(\mathrm{mod}\ m)$. Because $m \mid N$, there are N/m distinct values of i modulo N for which this occurs. As such, Theorem 1.1 yields (depending upon the parameters involved) m/N or 0 times the corresponding result from [13, Theorem 2] on $\sum_{\ell \in \mathsf{L}[\![n]\!]} \ell^p$.

Define a sequence of probability measures on $[0, 1]$ by

$$\nu_n = \frac{1}{|\mathsf{L}[\![n]\!]|} \sum_{\ell \in \mathsf{L}[\![n]\!]} \delta_{\ell/n},$$

in which δ_x denotes the point mass at x (not to be confused with the number δ defined earlier; both notations are standard and unavoidable). As shown in [13, Theorem 1], these measures converge weakly to an absolutely continuous measure whose probability density function is a certain Curry–Schoenberg B-spline. Theorem 1.1 permits us to obtain a result analogous to [13, Theorem 1], with all results scaled by m/N or 0 depending on the relevant congruence class.

THEOREM 1.2. *Let $S = \langle n_1, n_2, \dots, n_k \rangle$, in which $k \geq 3$ and $\gcd(n_1, n_2, \dots, n_k) = 1$, and define δ as above. Fix $N \in \mathbb{N}$ and $i \in \mathbb{Z}$ with $0 \leq i < N$; and let $m = \gcd(\delta, N)$.*

(a) *For real $\alpha < \beta$,*

$$\lim_{n \to \infty} \frac{|\{\ell \in \mathsf{L}[\![n]\!] : \ell \equiv i \,(\mathrm{mod}\ N),\ \ell \in [\alpha n, \beta n]\}|}{|\mathsf{L}[\![n]\!]|} = \begin{cases} \dfrac{m}{N} \displaystyle\int_\alpha^\beta F(t)\, dt & \text{if } n \equiv i n_1 \,(\mathrm{mod}\ m), \\ 0 & \text{if } n \not\equiv i n_1 \,(\mathrm{mod}\ m), \end{cases}$$

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where $F : \mathbb{R} \to \mathbb{R}$ is the probability density function

$$F(x) := \frac{(k-1)n_1 n_2 \cdots n_k}{2} \sum_{r=1}^{k} \frac{|1 - n_r x|(1 - n_r x)^{k-3}}{\prod_{j \neq r}(n_j - n_r)}.$$

The support of F is $[1/n_k, 1/n_1]$.

(b) *For any continuous function $g : (0, 1) \to \mathbb{C}$,*

$$\lim_{n\to\infty} \frac{1}{|\mathsf{L}[\![n]\!]|} \sum_{\substack{\ell \in \mathsf{L}[\![n]\!] \\ \ell \equiv i \,(\mathrm{mod}\, N)}} g\Big(\frac{\ell}{n}\Big) = \begin{cases} \dfrac{m}{N} \displaystyle\int_0^1 g(t)F(t)\,dt & \text{if } n \equiv in_1 \,(\mathrm{mod}\, m), \\ 0 & \text{if } n \not\equiv in_1 \,(\mathrm{mod}\, m), \end{cases}$$

Before proceeding to the proof of Theorem 1.1 in Section 3, we first examine some applications and examples in Section 2. We conclude in Section 4 with an open question that provides a possible avenue for future research.

2. Applications and examples

Theorem 1.1 is remarkable since it applies to all numerical semigroups and all moduli. Consequently, there is a lot to explore and comment on.

EXAMPLE 2.1. The generality of [13, Theorems 1 and 2] yielded asymptotic descriptions of numerous statistics of $\mathsf{L}[\![n]\!]$, including the mean, median, mode, variance, standard deviation, and skewness, as well as the harmonic and geometric means; see [13, Section 2.1] for precise statements. As an example, the mean $m_1(n)$ of $\mathsf{L}[\![n]\!]$ was shown to satisfy

$$m_1(n) = \frac{1}{|\mathsf{L}[\![n]\!]|} \sum_{\ell \in \mathsf{L}[\![n]\!]} \ell \sim \frac{n}{k}\Big(\frac{1}{n_1} + \cdots + \frac{1}{n_k}\Big)$$

as $n \to \infty$. As a consequence of Theorems 1.1 and 1.2, we immediately obtain analogous asymptotic descriptions of each statistic for the intersection of $\mathsf{L}[\![n]\!]$ with any fixed congruence class modulo N. In particular, $\mathsf{L}[\![n]\!] \cap (i + N\mathbb{Z})$ is empty unless $n \equiv in_1 \,(\mathrm{mod}\, m)$, in which case for n large,

$$\frac{1}{|\mathsf{L}[\![n]\!]|} \sum_{\substack{\ell \in \mathsf{L}[\![n]\!] \\ \ell \equiv i \,(\mathrm{mod}\, N)}} \ell \sim \frac{m}{N} \cdot \frac{n}{k}\Big(\frac{1}{n_1} + \cdots + \frac{1}{n_k}\Big).$$

EXAMPLE 2.2. The simplest case of Theorem 1.1 is $N = 1$, which forces $m = 1$. Since $\ell \equiv i \,(\mathrm{mod}\, 1)$ for all integers ℓ and i, we obtain [13, Theorem 2]:

$$\sum_{\ell \in \mathsf{L}[\![n]\!]} \ell^p = \frac{p!\; h_p(1/n_1, 1/n_2, \ldots, 1/n_k)}{(k + p - 1)!\,(n_1 n_2 \cdots n_k)} n^{k+p-1} + w(n), \tag{2-2}$$

in which $w(n)$ is a quasipolynomial of degree at most $k + p - 2$ whose coefficients are rational-valued and have period dividing $N \operatorname{lcm}(n_1, n_2, \ldots, n_k)$.

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EXAMPLE 2.3. Fix a modulus N and suppose that $m = 1$, that is, N is relatively prime to δ. Then the second case in Theorem 1.1 does not apply and hence

$$\sum_{\substack{\ell \in \mathsf{L}[\![n]\!] \\ \ell \equiv i \pmod N}} \ell^p = \frac{p!\ h_p(1/n_1, 1/n_2, \ldots, 1/n_k)}{N(k+p-1)!\,(n_1 n_2 \cdots n_k)} n^{k+p-1} + w_i(n)$$

for $i = 0, 1, 2, \ldots, N-1$, in which $w_i(n)$ is a quasipolynomial of degree at most $k + p - 2$ with rational-valued coefficients and period dividing $N\ \mathrm{lcm}(n_1, n_2, \ldots, n_k)$. In particular, factorization lengths are asymptotically equally distributed across all N equivalence classes.

EXAMPLE 2.4. For a fixed numerical semigroup, this modular equidistribution phenomenon occurs for all moduli $N \geq 1$ if and only if $\delta = 1$, since this ensures there is no prime p such that

$$n_1 \equiv n_2 \equiv \cdots \equiv n_k \pmod p.$$

This holds, for example, for the McNugget semigroup $\langle 6, 9, 20\rangle$. As Table 1 illustrates, the equidistribution across congruence classes modulo N is apparent even for relatively

TABLE 1. The McNugget semigroup $\langle 6, 9, 20\rangle$ has $\delta = 1$. For each modulus N and residue $i \pmod N$, the proportion of lengths $\ell \in \mathsf{L}[\![n]\!]$ with $\ell \equiv i \pmod N$ tends to $1/N$. Even for $n = 1000$, as depicted above, this behavior is evident.

Residue	Count	Proportion	Residue	Count	Proportion
0 (mod 2)	233	0.5011	0 (mod 7)	72	0.1548
1 (mod 2)	232	0.4989	1 (mod 7)	73	0.1570
0 (mod 3)	155	0.3333	2 (mod 7)	59	0.1269
1 (mod 3)	155	0.3333	3 (mod 7)	62	0.1333
2 (mod 3)	155	0.3333	4 (mod 7)	64	0.1376
0 (mod 4)	115	0.2473	5 (mod 7)	66	0.1419
1 (mod 4)	116	0.2495	6 (mod 7)	69	0.1484
2 (mod 4)	118	0.2538	0 (mod 8)	58	0.1247
3 (mod 4)	116	0.2496	1 (mod 8)	58	0.1247
0 (mod 5)	94	0.2022	2 (mod 8)	59	0.1269
1 (mod 5)	93	0.2000	3 (mod 8)	58	0.1247
2 (mod 5)	93	0.2000	4 (mod 8)	57	0.1226
3 (mod 5)	92	0.1978	5 (mod 8)	58	0.1247
4 (mod 5)	93	0.2000	6 (mod 8)	59	0.1269
0 (mod 6)	77	0.1656	7 (mod 8)	58	0.1247
1 (mod 6)	77	0.1656			
2 (mod 6)	78	0.1678			
3 (mod 6)	78	0.1678			
4 (mod 6)	78	0.1677			
5 (mod 6)	77	0.1656			

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TABLE 2. The semigroup $\langle 17, 29, 47, 65\rangle$ has $\delta = 6$. The possible congruence classes for $\ell \in \mathsf{L}[\![n]\!]$ depend on N, but those residues that can be attained are attained evenly as $n \to \infty$. The counts for $n = 5000$ are shown here.

Residue	Count	Proportion	Residue	Count	Proportion
0 (mod 2)	14500	1.0000	0 (mod 7)	2096	0.1446
1 (mod 2)	0	0.0000	1 (mod 7)	2094	0.1444
0 (mod 3)	0	0.0000	2 (mod 7)	2045	0.1411
1 (mod 3)	14500	1.0000	3 (mod 7)	2031	0.1401
2 (mod 3)	0	0.0000	4 (mod 7)	2066	0.1424
0 (mod 4)	7349	0.5068	5 (mod 7)	2084	0.1437
1 (mod 4)	0	0.0000	6 (mod 7)	2084	0.1437
2 (mod 4)	7151	0.4932	0 (mod 8)	3682	0.2539
3 (mod 4)	0	0.0000	1 (mod 8)	0	0.0000
0 (mod 5)	2890	0.1993	2 (mod 8)	3578	0.2468
1 (mod 5)	2910	0.2007	3 (mod 8)	0	0.0000
2 (mod 5)	2909	0.2006	4 (mod 8)	3667	0.2529
3 (mod 5)	2888	0.1992	5 (mod 8)	0	0.0000
4 (mod 5)	2903	0.2002	6 (mod 8)	3573	0.2464
0 (mod 6)	0	0.0000	7 (mod 8)	0	0.0000
1 (mod 6)	0	0.0000			
2 (mod 6)	0	0.0000			
3 (mod 6)	0	0.0000			
4 (mod 6)	14500	1.0000			
5 (mod 6)	0	0.0000			

small values of n. The semigroup $\langle 17, 29, 47, 65\rangle$, on the other hand, has $\delta = 6$, and the distributions are depicted in Table 2.

EXAMPLE 2.5. For each prime p and positive integer k, there are precisely $p^k - 1$ admissible k-tuples $(n_1, n_2, \ldots, n_k) \pmod p$ that occur as $\langle n_1, n_2, \ldots, n_k\rangle$ ranges over all k-generator numerical semigroups; the requirement that $\gcd(n_1, n_2, \ldots, n_k) = 1$ ensures that $n_1 \equiv n_2 \equiv \cdots \equiv n_k \equiv 0 \pmod p$ is not possible. Each of these $p^k - 1$ possible k-tuples occurs with equal likelihood. Of these, there are precisely $p - 1$ 'bad' k-tuples that yield numerical semigroups for which $\delta \neq 1$; these are the k-tuples whose entries are all equal to i for some $1 \leq i \leq p - 1$. Thus, the probability[1] that randomly selected numerical semigroup generators $n_1, n_2, \ldots, n_k$ are not mutually congruent

[1] The fact that $\mathbb{N}^k$ does not admit a uniform probability distribution can be remedied by studying numerical semigroups whose generators are at most a given threshold R, as in [2, 7]. The infinite products should be replaced by products over primes $p \leq f(R)$, in which $f(R)$ is a suitable function that tends to infinity as $R \to \infty$. Letting $R \to \infty$ yields the desired result. Note that other models for randomly selecting numerical semigroups have been studied as well [11].

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TABLE 3. Probability $\zeta(k)/\zeta(k-1)$ that a k-generator numerical semigroup has $\delta = 1$.

k	2	3	4	5	6	7	8	9	10
$\zeta(k)/\zeta(k-1)$	0	0.7308	0.9004	0.9581	0.9811	0.9912	0.9958	0.9979	0.9990

modulo p is

$$1 - \frac{p-1}{p^k-1} = \frac{(p^k-1)-(p-1)}{p^k-1} = \frac{p^k-p}{p^k-1} = \frac{1-1/p^{k-1}}{1-1/p^k}.$$

The Chinese remainder theorem and the Euler product formula imply that the probability that $\langle n_1, n_2, \ldots, n_k\rangle$ fails to satisfy $\delta = 1$ is

$$\prod_p \left(\frac{1-1/p^{k-1}}{1-1/p^k}\right) = \prod_p \left(\frac{1}{1-1/p^k}\right) \Big/ \prod_p \left(\frac{1}{1-1/p^{k-1}}\right) = \frac{\zeta(k)}{\zeta(k-1)},$$

in which

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

denotes the Riemann zeta function and the products run over all prime numbers.

Since $\lim_{k\to\infty} \zeta(k) = 1$, the probability that a k-generator numerical semigroup satisfies $\delta = 1$ tends to 1 as $k \to \infty$; see Table 3. This is intuitively clear, since the more generators a semigroup has, the more unlikely it is that they will be mutually congruent modulo some prime. For $k = 2$, the pole of ζ at 1 ensures that the probability that a 2-generator numerical semigroup has $\delta = 1$ is 0. This makes sense: the only way that $\langle n_1, n_2\rangle$ can have $\delta = 1$ is is $n_2 = n_1 + 1$, and this is highly unlikely.

3. Proof of Theorem 1.1

Let $S = \langle n_1, n_2, \ldots, n_k\rangle$ with $\gcd(n_1, n_2, \ldots, n_k) = 1$, let $N \in \mathbb{N}$, define δ as above, and let $m = \gcd(\delta, N)$ denote the largest divisor of N such that $n_1 \equiv n_2 \equiv \cdots \equiv n_k \pmod m$. Define

$$\Lambda_{i,N}^p(n) := \sum_{\substack{\ell \in \mathsf{L}[\![n]\!] \\ \ell \equiv i \pmod N}} \ell^p. \tag{3-3}$$

The associated generating function is

$$F(z) := \sum_{n=0}^{\infty} z^n \Lambda_{i,N}^p(n). \tag{3-4}$$

In the computations that follow, ζ denotes a primitive Nth root of unity (not to be confused with the Riemann zeta function) and i an index (not to be confused with the imaginary unit).

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3.1. A two-variable generating function. Define

$$f(z,w) := \prod_{i=1}^{k} \frac{1}{1-wz^{n_i}},$$

which satisfies

$$\begin{aligned} f(z,w) &= \prod_{i=1}^{k} (1 + wz^{n_i} + w^2 z^{2n_i} + \cdots) \\ &= \sum_{a_1,a_2,\ldots,a_k \geq 0} w^{a_1+a_2+\cdots+a_k}\, z^{a_1 n_1 + a_2 n_2 + \cdots + a_k n_k} \\ &= \sum_{n=0}^{\infty} z^n \sum_{\ell=0}^{\infty} (\#\text{ of factorizations of } n \text{ of length } \ell) w^{\ell} \\ &= \sum_{n=0}^{\infty} z^n \sum_{\ell \in \mathsf{L}[\![n]\!]} w^{\ell}. \end{aligned}$$

Observe that for $p \in \mathbb{N}$,

$$\left(w\frac{\partial}{\partial w}\right)^p f(z,w) = \sum_{n=0}^{\infty} z^n \sum_{\ell \in \mathsf{L}[n]} w^{\ell} \ell^p. \tag{3-5}$$

3.2. Fourier inversion. We claim that

$$F(z) = \frac{1}{N} \sum_{j=0}^{N-1} \zeta^{ij} F_{j,N}^p(z), \tag{3-6}$$

in which

$$F_{j,N}^p(z) := \left(\left(w\frac{\partial}{\partial w}\right)^p f(z,w)\right)\Big|_{w=\overline{\zeta}^j}. \tag{3-7}$$

To prove this result, let

$$V_{i,N}^p(n) := \sum_{j=0}^{N-1} \zeta^{-ij} \Lambda_{j,N}^p(n) = \sum_{\ell \in \mathsf{L}[\![n]\!]} \zeta^{-i\ell} \ell^p \tag{3-8}$$

and then use Fourier inversion to obtain

$$\Lambda_{i,N}^p(n) = \frac{1}{N} \sum_{j=0}^{N-1} \zeta^{ij} V_{j,N}^p(n). \tag{3-9}$$

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Consequently,

$$\begin{aligned}
F(z) &= \sum_{n=0}^{\infty} z^n \Lambda_{i,N}^p(n) && \text{(by (3-4))}\\
&= \sum_{n=0}^{\infty} z^n \Big(\frac{1}{N}\sum_{j=0}^{N-1} \zeta^{ij} V_{j,N}^p(n)\Big) && \text{(by (3-9))}\\
&= \frac{1}{N}\sum_{j=0}^{N-1} \Big(\zeta^{ij} \sum_{n=0}^{\infty} z^n V_{j,N}^p(n)\Big)\\
&= \frac{1}{N}\sum_{j=0}^{N-1} \Big(\zeta^{ij} \sum_{n=0}^{\infty} z^n \sum_{\ell \in \mathsf{L}[\![n]\!]} \zeta^{-j\ell} \ell^p\Big) && \text{(by (3-8))}\\
&= \frac{1}{N}\sum_{j=0}^{N-1} \zeta^{ij} \Big(\sum_{n=0}^{\infty} z^n \sum_{\ell \in \mathsf{L}[\![n]\!]} (\zeta^{-j})^{\ell} \ell^p\Big)\\
&= \frac{1}{N}\sum_{j=0}^{N-1} \zeta^{ij} \Big(\Big(w\frac{\partial}{\partial w}\Big)^p f(z,w)\Big)\Big|_{w=\overline{\zeta}^j} && \text{(by (3-5))}\\
&= \frac{1}{N}\sum_{j=0}^{N-1} \zeta^{ij} F_{j,N}^p(z) && \text{(by (3-7))}.
\end{aligned}$$

3.3. Rational representation. We now represent $F_{j,N}^p(z)$ as an explicit rational function. First, we require the identity

$$\frac{\partial^p}{\partial w^p} f(z,w) = p!\Big(\prod_{b=1}^{k} \frac{1}{1-wz^{n_b}}\Big) \cdot h_p\Big(\frac{z^{n_1}}{1-wz^{n_1}}, \ldots, \frac{z^{n_k}}{1-wz^{n_k}}\Big), \tag{3-10}$$

which can be verified by induction [13, Lemma 22]. Second, recall that the *Stirling number of the second kind* $\left\{ {n \atop i} \right\}$ counts the number of partitions of $\{1, 2, \ldots, n\}$ into i nonempty subsets. It is known that

$$\Big(x\frac{d}{dx}\Big)^p = \sum_{i=0}^{p} \left\{ {p \atop i} \right\} x^i \frac{d^i}{dx^i} \tag{3-11}$$

for $p \in \mathbb{N}$ [9, 10, 22]. Now compute

$$\begin{aligned}
F_{j,N}^p(z) &= \Big(\Big(w\frac{\partial}{\partial w}\Big)^p f(z,w)\Big)\Big|_{w=\overline{\zeta}^j} && \text{(by (3-7))}\\
&= \sum_{a=0}^{p} \left\{ {p \atop a} \right\} w^a \frac{\partial^a f}{\partial w^a}\Big|_{w=\overline{\zeta}^j} && \text{(by (3-11))}
\end{aligned}$$

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$$= \sum_{a=0}^{p} \left\{ {p \atop a} \right\} a!\, w^a \Big(\prod_{b=1}^{k} \frac{1}{1 - wz^{n_b}} \Big) h_a\Big(\frac{z^{n_1}}{1 - wz^{n_1}}, \dots, \frac{z^{n_k}}{1 - wz^{n_k}} \Big) \Big|_{w=\overline{\zeta}^j} \qquad \text{(by (3-10))}$$

$$= \sum_{a=0}^{p} \left\{ {p \atop a} \right\} a!\, \overline{\zeta}^{aj} \Big(\prod_{b=1}^{k} \frac{1}{1 - \overline{\zeta}^j z^{n_b}} \Big) h_a\Big(\frac{z^{n_1}}{1 - \overline{\zeta}^j z^{n_1}}, \dots, \frac{z^{n_k}}{1 - \overline{\zeta}^j z^{n_k}} \Big).$$

3.4. A crucial subgroup. Consider the subgroup

$$\Gamma = \{ t \in \mathbb{Z}/N\mathbb{Z} : n_1 t \equiv n_2 t \equiv \cdots \equiv n_k t \,(\mathrm{mod}\, N) \} \tag{3-12}$$

of $\mathbb{Z}/N\mathbb{Z}$. Then Γ is cyclic with generator $N/|\Gamma|$ and hence

$$n_1 \equiv n_2 \equiv \cdots \equiv n_k \,(\mathrm{mod}\, |\Gamma|). \tag{3-13}$$

Now $m = \gcd(N, \delta)$ is the largest divisor of N such that

$$n_1 \equiv n_2 \equiv \cdots \equiv n_k \,(\mathrm{mod}\, m). \tag{3-14}$$

Then N/m generates a subgroup Γ' of $\mathbb{Z}/N\mathbb{Z}$ of order m. In particular,

$$n_1 t \equiv n_2 t \equiv \cdots \equiv n_k t \,(\mathrm{mod}\, N) \tag{3-15}$$

for all $t \in \Gamma'$. The maximality of m, (3-13), and (3-14) imply that $|\Gamma| \le |\Gamma'|$. On the other hand, (3-12) and (3-15) ensure that $\Gamma' \subseteq \Gamma$ and hence $|\Gamma'| \le |\Gamma|$. Since Γ, Γ' are subgroups of $\mathbb{Z}/N\mathbb{Z}$ of the same order, $\Gamma = \Gamma'$. In particular, $|\Gamma| = m$.

3.5. An automorphism. The definition (3-12) of the group Γ ensures that multiplication modulo N by any of the numerical semigroup generators $n_1, n_2, \dots, n_k$ yields the same homomorphism $\alpha : \Gamma \to \Gamma$. We claim that α is an automorphism. Since $\gcd(n_1, n_2, \dots, n_k) = 1$, there are $b_1, b_2, \dots, b_k \in \mathbb{Z}$ such that

$$b_1 n_1 + b_2 n_2 + \cdots + b_k n_k = 1.$$

If $r \in \Gamma$ and $\alpha(t) = r$, then

$$n_1 t \equiv n_2 t \equiv \cdots \equiv n_k t \equiv r \,(\mathrm{mod}\, N) \tag{3-16}$$

and hence

$$\begin{aligned} t &= t(b_1 n_1 + b_2 n_2 + \cdots + b_k n_k) \\ &= b_1(n_1 t) + b_2(n_2 t) + \cdots + b_k(n_k t) \\ &\equiv (b_1 + b_2 + \cdots + b_k) r \,(\mathrm{mod}\, N). \end{aligned}$$

In particular, the kernel of α is trivial and thus α is an automorphism. Note that $r, t \in \mathbb{Z}/N\mathbb{Z}$ satisfy (3-16) if and only if $r, t \in \Gamma$ and $\alpha(t) = r$.

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3.6. An exponential sum. Since $\zeta^{N/m}$ is a primitive m th root of unity,

$$\begin{aligned}\sum_{t\in\Gamma}\zeta^{i\alpha(t)-tn} &= \sum_{t\in\Gamma}\zeta^{i(n_1t)-tn} = \sum_{t\in\Gamma}\zeta^{t(in_1-n)}\\ &= \sum_{a=1}^{m}\zeta^{(aN/m)(in_1-n)} = \sum_{a=1}^{m}(\zeta^{N/m})^{a(in_1-n)}\\ &= \begin{cases} m & \text{if } in_1 \equiv n \,(\mathrm{mod}\, m),\\ 0 & \text{if } in_1 \not\equiv n \,(\mathrm{mod}\, m).\end{cases}\end{aligned} \tag{3-17}$$

3.7. Common zeros. For $0 \le r \le N-1$ and $1 \le i \le k$, the polynomial

$$\phi_r^i(z) := 1 - \zeta^r z^{n_i}$$

has zeros ζ^{r+sN/n_i} for $1 \le s \le n_i$. These zeros are distinct because $r + sN \equiv r + s'N \,(\mathrm{mod}\, Nn_i)$ implies $s \equiv s' \,(\mathrm{mod}\, n_i)$, and hence $s = s'$.

LEMMA 3.1. *Fix $r \in \{0, 1, \ldots, N-1\}$. Then ζ^t is a common zero of the polynomials*

$$\phi_r^1(z),\ \phi_r^2(z), \ldots,\ \phi_r^k(z) \tag{3-18}$$

if and only if $r, t \in \Gamma$ and $\alpha(t) = r$.

PROOF. The polynomials (3-18) have a common zero if and only if there are $s_1, s_2, \ldots, s_k \in \mathbb{Z}$ such that

$$\frac{r + s_1N}{Nn_1} = \frac{r + s_2N}{Nn_2} = \cdots = \frac{r + s_kN}{Nn_k}$$

or, equivalently,

$$\frac{L}{n_1}(r + s_1N) = \frac{L}{n_2}(r + s_2N) = \cdots = \frac{L}{n_k}(r + s_kN), \tag{3-19}$$

in which $L := \mathrm{lcm}(n_1, n_2, \ldots, n_k)$. Since $\gcd(n_1, n_2, \ldots, n_k) = 1$,

$$L = \mathrm{lcm}\left(\frac{L}{n_1}, \frac{L}{n_2}, \ldots, \frac{L}{n_k}\right). \tag{3-20}$$

This is because a prime power exactly divides L if and only if it divides at least one, but not all, of $n_1, n_2, \ldots, n_k$. Consequently, it exactly divides the expression on the right-hand side of (3-20). Hence, from (3-20), the integer (3-19) is a multiple of L. Thus, the polynomials (3-18) have a common zero if and only if there is a $t \in \mathbb{Z}$ such that

$$r + s_iN = tn_i \quad \text{for all } i = 1, 2, \ldots, k; \tag{3-21}$$

that is, if and only if (3-16) holds. This is equivalent to $r, t \in \Gamma$ and $\alpha(t) = r$. □

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3.8. A residue computation. The maximum possible order for a pole of

$$F_{j,N}^{p}(z) = \sum_{a=0}^{p} \left\{ {p \atop a} \right\} a!\, \overline{\zeta}^{aj} \Big(\prod_{b=1}^{k} \frac{1}{1-\overline{\zeta}^{j} z^{n_b}} \Big) h_a\Big(\frac{z^{n_1}}{1-\overline{\zeta}^{j} z^{n_1}}, \ldots, \frac{z^{n_k}}{1-\overline{\zeta}^{j} z^{n_k}} \Big)$$

is $k+p$, which can only arise from the summand with $a = p$. Lemma 3.1 ensures that ζ^t is a pole of $F_{r,N}^{p}$ with order $k+p$ if and only if $r, t \in \Gamma$ and $\alpha(t) = r$. In particular,

$$\lim_{z \to \zeta^t} (1-\overline{\zeta}^{t} z)^{k+p} F_{j,N}^{p}(z) = 0 \quad \text{for } j \neq r. \tag{3-22}$$

From (3-6), we see that F has a pole of order $k+p$ at ζ^t for each $t \in \Gamma$, and these are the only poles of F of this (maximal) degree. Write

$$F(z) = \sum_{t \in \Gamma} \Big(\frac{C_t}{(1-\overline{\zeta}^{t} z)^{k+p}} \Big) + G(z), \tag{3-23}$$

in which $C_t \neq 0$ and $G(z)$ is a rational function, all of whose poles are Lth roots of unity with order at most $k+p-1$. In particular,

$$G(z) = \sum_{n=0}^{\infty} u(n) z^n$$

for some quasipolynomial $u(n)$ of degree at most $k+p-2$ whose period divides L. Moreover, for each $t \in \Gamma$,

$$\lim_{z \to \zeta^t} (1-\overline{\zeta}^{t} z)^{k+p} G(z) = 0.$$

With $r = \alpha(t)$,

$$\begin{aligned}
C_t &= \lim_{z \to \zeta^t} (1-\overline{\zeta}^{t} z)^{k+p} F(z) \\
&= \lim_{z \to \zeta^t} (1-\overline{\zeta}^{t} z)^{k+p} \Big(\frac{1}{N} \sum_{j=0}^{N-1} \zeta^{ij} F_{j,N}^{p}(z) \Big) && \text{(by (3-6))} \\
&= \frac{1}{N} \sum_{j=0}^{N-1} \zeta^{ij} \big(\lim_{z \to \zeta^t} (1-\overline{\zeta}^{t} z)^{k+p} F_{j,N}^{p}(z) \big) \\
&= \frac{\zeta^{ir}}{N} \big(\lim_{z \to \zeta^t} (1-\overline{\zeta}^{t} z)^{k+p} F_{r,N}^{p}(z) \big) && \text{(by (3-22))} \\
&= \frac{\zeta^{ir}}{N} \lim_{z \to \zeta^t} \Big[\sum_{a=0}^{p} \left\{ {p \atop a} \right\} a!\, \overline{\zeta}^{ar} (1-\overline{\zeta}^{t} z)^{k+p} \Big(\prod_{b=1}^{k} \frac{1}{1-\overline{\zeta}^{r} z^{n_b}} \Big) \cdot h_a\Big(\frac{z^{n_1}}{1-\overline{\zeta}^{r} z^{n_1}}, \ldots, \frac{z^{n_k}}{1-\overline{\zeta}^{r} z^{n_k}} \Big) \Big] \\
&= \frac{\zeta^{ir} \overline{\zeta}^{pr} p!}{N} \lim_{z \to \zeta^t} (1-\overline{\zeta}^{t} z)^{k+p} \Big(\prod_{b=1}^{k} \frac{1}{1-\overline{\zeta}^{r} z^{n_b}} \Big) \cdot h_p\Big(\frac{z^{n_1}}{1-\overline{\zeta}^{r} z^{n_1}}, \ldots, \frac{z^{n_k}}{1-\overline{\zeta}^{r} z^{n_k}} \Big)
\end{aligned}$$

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$$= \frac{\zeta^{ir}\overline{\zeta}^{pr}p!}{N}\lim_{z\to\zeta^t}\Big(\prod_{b=1}^{k}\frac{1-\overline{\zeta}^t z}{1-\overline{\zeta}^r z^{n_b}}\Big)\cdot h_p\Big(z^{n_1}\frac{1-\overline{\zeta}^t z}{1-\overline{\zeta}^r z^{n_1}},\dots,z^{n_k}\frac{1-\overline{\zeta}^t z}{1-\overline{\zeta}^r z^{n_k}}\Big)$$

$$= \frac{\zeta^{ir}\overline{\zeta}^{pr}p!}{N}\Big(\prod_{b=1}^{k}\frac{\zeta^r}{\zeta^{n_b t}n_b}\Big)h_p\Big(\frac{\zeta^r}{n_1},\dots,\frac{\zeta^r}{n_k}\Big) \qquad \text{(L'Hôpital)}$$

$$= \frac{\zeta^{ir}p!}{N}\Big(\prod_{b=1}^{k}\frac{\zeta^r}{\zeta^{n_b t}n_b}\Big)h_p\Big(\frac{1}{n_1},\dots,\frac{1}{n_k}\Big) \qquad \text{(definition of } h_p\text{)}$$

$$= \frac{\zeta^{i\alpha(t)}p!}{N(n_1n_2\cdots n_k)}h_p\Big(\frac{1}{n_1},\dots,\frac{1}{n_k}\Big) \qquad \text{(by (3-16)).}$$

Here are the details of the somewhat involved L'Hôpital step:

$$\lim_{z\to\zeta^t} z^{n_i}\frac{(1-\overline{\zeta}^t z)}{1-\overline{\zeta}^r z^{n_i}} = \zeta^{n_i t}\lim_{z\to\zeta^t}\frac{1-\overline{\zeta}^t z}{1-\overline{\zeta}^r z^{n_i}} = \zeta^{n_i t}\frac{\overline{\zeta}^t}{n_i\overline{\zeta}^r(\zeta^t)^{n_i-1}} = \zeta^{n_i t}\frac{\overline{\zeta}^t\zeta^t}{n_i\overline{\zeta}^r\zeta^{n_i t}} = \frac{\zeta^r}{n_i}.$$

3.9. Conclusion. Now observe that

$$\begin{aligned}\frac{1}{(1-\overline{\zeta}^t z)^{k+p}} &= \sum_{n=0}^{\infty}\binom{n+k+p-1}{k+p-1}\overline{\zeta}^{tn}z^n\\ &= \sum_{n=0}^{\infty}\frac{(n+k+p-1)\cdots(n+1)}{(k+p-1)!}\overline{\zeta}^{tn}z^n\\ &= \frac{1}{(k+p-1)!}\sum_{n=0}^{\infty}(n^{k+p-1}+v(n))\overline{\zeta}^{tn}z^n,\end{aligned}$$

in which $v(n)$ is a quasipolynomial of degree $k+p-2$ with integer coefficients. Our recent evaluation of C_t and (3-23) imply

$$\begin{aligned}F(z) &= \sum_{t\in\Gamma}\Big(\frac{C_t}{(1-\overline{\zeta}^t z)^{k+p}}\Big)+G(z)\\ &= \sum_{t\in\Gamma}\Big[\frac{\zeta^{i\alpha(t)}p!}{N(n_1n_2\cdots n_k)}h_p\Big(\frac{1}{n_1},\frac{1}{n_2},\dots,\frac{1}{n_k}\Big)\frac{1}{(1-\overline{\zeta}^t z)^{k+p}}\Big]+G(z)\\ &= \frac{p!\,h_p(\frac{1}{n_1},\dots,\frac{1}{n_k})}{N(n_1n_2\cdots n_k)}\Big(\sum_{t\in\Gamma}\frac{\zeta^{i\alpha(t)}}{(1-\overline{\zeta}^t z)^{k+p}}\Big)+G(z)\\ &= \frac{p!\,h_p(\frac{1}{n_1},\dots,\frac{1}{n_k})}{N(n_1n_2\cdots n_k)}\Big(\sum_{t\in\Gamma}\zeta^{i\alpha(t)}\sum_{n=0}^{\infty}(n^{k+p-1}+v(n))\overline{\zeta}^{tn}z^n\Big)+\sum_{n=0}^{\infty}u(n)z^n\\ &= \frac{p!\,h_p(\frac{1}{n_1},\dots,\frac{1}{n_k})}{N(k+p-1)!\,(n_1\cdots n_k)}\sum_{n=0}^{\infty}(n^{k+p-1}+v(n))\Big(\sum_{t\in\Gamma}\zeta^{i\alpha(t)-tn}\Big)z^n+\sum_{r\in\Gamma}u(z)z^n.\end{aligned}$$

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The evaluation (3-17) of the parenthesized exponential sum and the definition (3-4) of F as the generating function for $\Lambda_{i,N}^{p}(n)$ reveal that

$$\Lambda_{i,N}^{p}(n) = \begin{cases} \dfrac{p!\; m\, h_p(1/n_1, 1/n_2, \ldots, 1/n_k)}{N(k+p-1)!\,(n_1 n_2 \cdots n_k)} n^{k+p-1} + w_i(n) & \text{if } n \equiv in_1 \pmod{m}, \\ 0 & \text{if } n \not\equiv in_1 \pmod{m}, \end{cases}$$

in which $w_i(n)$ is a quasipolynomial of degree at most $k+p-2$ whose coefficients have period dividing $N\,\mathrm{lcm}(n_1, n_2, \ldots, n_k)$. Since $u(n)$ and $v(n)$ both have rational coefficients, so must $w(n)$. This concludes the proof. □

4. Conclusion

Although this paper largely settles the matter of asymptotic modular distribution of factorization lengths for elements in numerical semigroups, a related question worthy of further research remains. Can one characterize the rate of convergence in Theorem 1.2? This would, presumably, require a detailed examination of the quasipolynomial error term $w_i(n)$ in Theorem 1.1 and its dependence on $n_1, n_2, \ldots, n_k, N, i$, and δ, along with the congruence class of n modulo N. A careful study of the proof of Theorem 1.1 might yield sufficiently explicit bounds upon the $w_i(n)$ to carry this out.

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