DETERMINANTAL FORMULAS FOR SEM EXPANSIONS OF SCHUBERT POLYNOMIALS

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Abstract. We show that for any permutation w that avoids a certain set of 13 patterns of length 5 and 6, the Schubert polynomial $\mathfrak{S}_w$ can be expressed as the determinant of a matrix of elementary symmetric polynomials in a manner similar to the Jacobi-Trudi identity. For such w, this determinantal formula is equivalent to a (signed) subtraction-free expansion of $\mathfrak{S}_w$ in the basis of standard elementary monomials.

1. Introduction

The Schubert polynomials $\mathfrak{S}_w$ form an important basis of the polynomial ring $\mathbf{Z}[x_1, x_2, \dots]$, primarily due to their role as representatives for the classes of Schubert varieties in the cohomology of the flag variety. In this paper, we consider the expansion of Schubert polynomials in the SEM basis consisting of *standard elementary monomials*

$$e_{j_1 j_2 \cdots} = e_{j_1}(x_1)e_{j_2}(x_1, x_2)e_{j_3}(x_1, x_2, x_3)\cdots,$$

where e_k is the kth elementary symmetric polynomial and only finitely many of the j_i are nonzero. Such SEM expansions of Schubert polynomials have been studied previously in [8, 11, 13, 16, 17]. In particular, it was shown by Fomin, Gelfand, and Postnikov [8] that these expansions are important for the construction of *quantum Schubert polynomials*, which can be used to compute Gromov-Witten invariants for the small quantum cohomology ring of the flag variety. Additionally, Postnikov and Stanley [16] noted that the problem of finding the SEM expansion of Schubert polynomials is equivalent to the problem of computing the *inverse Schubert-Kostka matrix*—that is, the expansion of monomials in the Schubert basis.

One special case of Schubert polynomials are the Schur polynomials $s_\lambda(x_1, \dots, x_n)$, which have a determinantal formula in terms of elementary symmetric polynomials via the famous Jacobi-Trudi identity. It was observed by Kirillov [11] that this identity can be slightly modified to give a determinantal formula that, when expanded, gives the SEM expansion for Schur polynomials. (See Corollary 4.12 below.) A similar determinantal formula was given in [16] for $\mathfrak{S}_w$ when w is a 213-avoiding permutation. Such determinants can be interpreted via a nonintersecting lattice path model using the Lindström-Gessel-Viennot lemma.

Our main focus will be to study which Schubert polynomials $\mathfrak{S}_w$ can be expressed as a Jacobi-Trudi-like determinant that yields its SEM expansion (and can therefore be described by a nonintersecting lattice path model). Such determinantal formulas are particularly notable because any coefficient appearing in such an SEM expansion has absolute value at most 1. Our main result will be to show that such a determinantal

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formula exists when w avoids the following 13 patterns of length 5 and 6:

$$51324, \quad 15324, \quad 52413, \quad 25413, \quad 53142, \quad 35142, \quad 31542,$$

$$143265, \quad 143625, \quad 143652, \quad 146352, \quad 413265, \quad 413625.$$

(This is not a necessary condition—see §5 for further discussion.)

Our approach will utilize the fact that certain operations such as divided difference operators can be seen to act on the generating functions for nonintersecting lattice paths by moving the endpoints in a simple combinatorial way. A similar observation was also used in [5, 6] to give lattice path interpretations for certain flagged double Schur functions and flagged skew Schubert polynomials (though the interpretations there primarily yield formulas in terms of complete homogeneous symmetric polynomials rather than elementary symmetric polynomials).

The organization of this paper is as follows: In §2, we will discuss background information on permutations, Schubert polynomials, and standard elementary monomials, as well as define lattice path representations for polynomials. We will also discuss how these lattice path models apply to the context of quantum Schubert polynomials. In §3, we will discuss various operations for manipulating lattice path representations. In §4, we will use the operations in §3 to first prove a special case regarding 1324-avoiding separable permutations and then build on this case to prove our main result in Theorem 4.13. We will conclude in §5 with some remaining open questions.

2. Background

In this section, we will introduce necessary background about permutations, Schubert polynomials, standard elementary monomials, and nonintersecting lattice paths. For more information, see, for instance, [15].

2.1. **Permutations.** Let S_n denote the symmetric group of permutations on $[n] = \{1, \dots, n\}$. We will often denote a permutation $w \in S_n$ in one-line notation $w = w_1 w_2 \cdots w_n$.

The simple transpositions $s_i = (i \;\; i+1)$ for $i = 1, \dots, n-1$ generate the group S_n. For a permutation $w \in S_n$, its *length* $\ell(w)$ is the length of the shortest expression for w as a product of simple transpositions $s_{i_1} \dots s_{i_\ell}$ (called a *reduced expression*). Alternatively, $\ell(w)$ is the number of *inversions* of w, where an inversion is an ordered pair (w_i, w_j) satisfying $j > i$ and $w_j < w_i$. We denote by 1_n the identity permutation in S_n, and we denote by $w_0 = w_0^{(n)}$ the permutation $n(n-1)\cdots 1 \in S_n$ of maximum length in S_n.

The *(Lehmer) code* of a permutation $w \in S_n$ is the sequence $c = c(w) = (c_1, \dots, c_n)$, where $c_i = \#\{j > i \mid w_j < w_i\}$. The map from $w \in S_n$ to its code c is a bijection from S_n to the set of integer vectors $(c_1, \dots, c_n)$ satisfying $0 \le c_i \le n - i$ for all i.

For a permutation w, we say that w_i is a *left-to-right maximum* of w if $w_j < w_i$ for all $j < i$.

Sometimes it will be convenient to consider the direct limit S_∞ of symmetric groups under the natural embeddings $\iota\colon S_n \hookrightarrow S_{n+1}$ in which S_n acts on the first n letters. Equivalently, any element $w \in S_\infty$ is a permutation of $\mathbf{N} = \{1, 2, \dots\}$ that fixes all but finitely many elements.

2.1.1. *Pattern avoidance.* Given a permutation (or *pattern*) $p = p_1 \cdots p_k \in S_k$, we say that a permutation $w \in S_n$ *contains* the pattern p if w has a subsequence in the same relative order as p, that is, if there exist $i_1 < i_2 < \cdots < i_k$ such that $w_{i_a} < w_{i_b}$ if and only if $p_{i_a} < p_{i_b}$. We say that w *avoids* p if w does not contain the pattern p. We will sometimes abuse terminology and refer to either $p \in S_k$ or $w_{i_1} \cdots w_{i_k}$ as being a pattern of w.

A permutation $w \in S_n$ is called *dominant* if it avoids the pattern 132. Equivalently, a permutation is dominant if and only if its code is nonincreasing, that is, $c_1 \geq c_2 \geq \cdots \geq c_n$.

2.1.2. *Direct and skew sum.* The following two operations can be used to combine permutations.

Definition 2.1. The *direct sum* of permutations $u \in S_m$ and $v \in S_n$ is the permutation $u \oplus v \in S_{m+n}$ defined by

$$(u \oplus v)(i) = \begin{cases} u(i) & \text{if } i \leq m, \\ v(i-m)+m & \text{if } i > m. \end{cases}$$

The *skew sum* of $u \in S_m$ and $v \in S_n$ is the permutation $u \ominus v \in S_{m+n}$ defined by

$$(u \ominus v)(i) = \begin{cases} u(i)+n & \text{if } i \leq m, \\ v(i-m) & \text{if } i > m. \end{cases}$$

Definition 2.2. A permutation is called *separable* if it can be built from copies of the permutation $1 \in S_1$ using only direct sum and skew sum operations.

In [4], it was shown that separable permutations can alternatively be described as those that avoid the patterns 2413 and 3142.

2.2. **Schubert polynomials.** The symmetric group S_n acts on $\mathbf{Z}[x_1, ..., x_n]$ in a natural way by permuting variables. For instance, if $f \in \mathbf{Z}[x_1, \ldots, x_n]$, then $s_i f$ is the polynomial obtained by switching x_i and x_{i+1} in f.

For $i = 1, \ldots, n-1$, the *divided difference operator* ∂_i is defined by

$$\partial_i f = \frac{1 - s_i}{x_i - x_{i+1}} f = \frac{f - s_i f}{x_i - x_{i+1}}$$

for all $f \in \mathbf{Z}[x_1, \ldots, x_n]$. If $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced expression, then we define $\partial_w = \partial_{i_1} \cdots \partial_{i_\ell}$ (which is independent of the reduced expression).

The *Schubert polynomials* $\mathfrak{S}_w$ for $w \in S_n$ can be defined recursively as follows: for the long word $w_0 \in S_n$, $\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$. Otherwise,

$$\mathfrak{S}_{ws_i} = \partial_i(\mathfrak{S}_w) \qquad \text{if } \ell(ws_i) < \ell(w)$$

(while $\partial_i(\mathfrak{S}_w) = 0$ if $\ell(ws_i) > \ell(w)$). Equivalently, $\mathfrak{S}_w = \partial_{w^{-1}w_0}(x_1^{n-1} x_2^{n-2} \cdots x_{n-1})$ for all $w \in S_n$.

Schubert polynomials are stable under the natural embeddings $\iota \colon S_n \hookrightarrow S_{n+1}$, which implies that $\mathfrak{S}_w$ is well-defined for any $w \in S_\infty$. The set $\{\mathfrak{S}_w \mid w \in S_\infty\}$ forms a basis for the polynomial ring $\mathbf{Z}[x_1, x_2, \ldots]$ called the *Schubert basis.*

The expansion of any Schubert polynomial in terms of monomials has nonnegative coefficients. One combinatorial interpretation for these coefficients is as follows (see [1, 3, 9] for more details).

A *pipe dream* (or *rc-graph*) is a type of wiring diagram in which each box (i, j) with $i, j \geq 1$ (indexed using matrix conventions) contains either a cross or a pair of

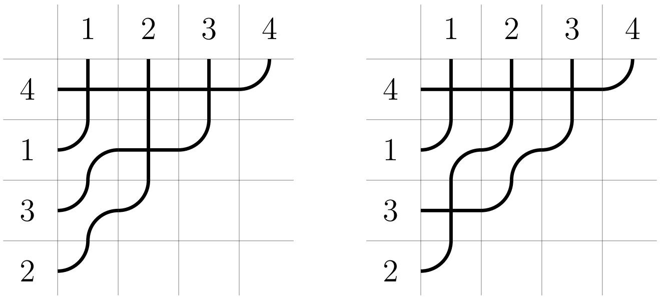

FIGURE 1. The two reduced pipe dreams for 4132. (In this diagram, only the first four pipes are drawn; all other pipes consist only of elbows.)

elbows. (See Figure 1.) A pipe dream corresponds to the permutation $w \in S_\infty$ if the wire that enters at the left of row i exits at the top of column w_i. A pipe dream is called *reduced* if no two wires cross more than once.

Every reduced pipe dream for w contains exactly $\ell(w)$ crosses. Assign to each cross the weight x_i if it occurs in row i, and define the weight of the pipe dream to be the product of the weights of its crosses. Then $\mathfrak{S}_w$ is the sum of the weights of all reduced pipe dreams for w.

Example 2.3. Let $w = 4132$. Figure 1 shows the two reduced pipe dream corresponding to w. Hence $\mathfrak{S}_{4132} = x_1^3 x_2 + x_1^3 x_3$.

One special case of Schubert polynomials occurs when w is a dominant (132-avoiding) permutation. In this case, $\mathfrak{S}_w$ is the monomial $x_1^{c_1} x_2^{c_2} \cdots$, where $(c_1, c_2, \dots)$ is the code of w.

Another special case occurs when w is a *Grassmannian permutation* satisfying $w_1 < w_2 < \cdots < w_r$ and $w_{r+1} < w_{r+2} < \cdots < w_n$ for some r. In this case, $\mathfrak{S}_w$ is a symmetric polynomial in $x_1, \dots, x_r$ called a *Schur polynomial* $s_\lambda(x_1, \dots, x_r)$, where λ is the partition $(w_r - r, w_{r-1} - (r-1), \dots, w_1 - 1)$. A more common combinatorial description for Schur polynomials is given by semistandard tableaux—see [10] for the connection to lattice paths discussed later.

The following proposition describes how Schubert polynomials behave under direct sum and skew sum (see also, for instance, [3, 12]).

Proposition 2.4. *Let $u \in S_m$ and $v \in S_n$. Then:*

(a) $\mathfrak{S}_{u \oplus v} = \mathfrak{S}_u \cdot \mathfrak{S}_{1_m \oplus v}$, *and*
(b) $\mathfrak{S}_{u \ominus v} = \mathfrak{S}_u \cdot \mathfrak{S}_{1_m \ominus v} = \mathfrak{S}_u \cdot (x_1 \cdots x_m)^n \cdot \mathfrak{S}_v(x_{m+1}, \dots, x_{m+n})$.

Proof. For (a), any reduced pipe dream for $u \oplus v$ must have the first m pipes lying strictly above the last n pipes. Thus such a pipe dream can be factored uniquely into a reduced pipe dream for u and (by replacing the first m pipes with the identity pipe dream containing only elbows) a reduced pipe dream for $1_m \oplus v$.

For (b), any reduced pipe dream for $u \ominus v$ must have crosses in the first n boxes of the first m rows. The remaining part consists of a reduced pipe dream for u (shifted to the right by n) and a reduced pipe dream for v (shifted down by m). The result follows easily. $\square$

2.3. **Standard elementary monomials.** For integers j and k with $k \geq 0$, denote by

$$e_j^{(k)} = \sum_{1 \leq i_1 < \cdots < i_j \leq k} x_{i_1} \cdots x_{i_j}$$

the jth *elementary symmetric polynomial* in $x_1, \dots, x_k$. (By convention, $e_j^{(k)} = 1$ for $j = 0$, while $e_j^{(k)} = 0$ if $j > k$ or $j < 0$.) Note that $e_j^{(k)}$ is symmetric in x_i and x_{i+1} for all $i \neq k$.

Let L be the set of sequences of integers $(j_1, j_2, \dots)$ satisfying $0 \leq j_k \leq k$ for which all but finitely many of the j_k vanish. (We will sometimes omit trailing zeroes from such sequences for convenience.) Then for any $(j_1, j_2, \dots) \in L$ we define the *standard elementary monomial* $e_{j_1 j_2 \cdots}$ to be the polynomial

$$e_{j_1 j_2 \cdots} = \prod_{k \geq 1} e_{j_k}^{(k)}.$$

(Note that all but finitely many terms in the product are 1.)

It was shown in [8] that as $(j_1, j_2, \dots)$ ranges over all sequences in L, the standard elementary monomials $e_{j_1 j_2 \cdots}$ form a basis for the polynomial ring $\mathbf{Z}[x_1, x_2, \dots]$, which we call the *SEM basis.* (Though we will not need it here, each standard elementary monomial has nonnegative coefficients when expanded in the Schubert basis, as determined by the Pieri rule for Schubert polynomials—see, for instance, [13].)

Given a permutation $w \in S_n$, consider the expansion of the corresponding Schubert polynomial in the SEM basis

$$\mathfrak{S}_w = \sum \alpha_{j_1 j_2 \cdots j_{n-1}} e_{j_1 j_2 \cdots j_{n-1}}.$$

Most notably, this expansion appears in the study of quantum Schubert calculus: Fomin, Gelfand, and Postnikov [8] define the *quantum Schubert polynomial* as

$$(\dagger) \qquad \mathfrak{S}_w^q = \sum \alpha_{j_1 j_2 \cdots j_{n-1}} E_{j_1 j_2 \cdots j_{n-1}},$$

where $E_{j_1 j_2 \cdots j_{n-1}} = \prod_k E_{j_k}^{(k)}$ is a product of *quantum elementary polynomials* $E_j^{(k)} = E_j(x_1, \dots, x_k)$ defined by

$$(\ddagger) \qquad \det(I + \lambda G_k) = \sum_{j=0}^{k} E_j^{(k)} \lambda^j, \quad \text{where} \quad G_k = \begin{bmatrix} x_1 & q_1 & 0 & \cdots & 0 \\ -1 & x_2 & q_2 & \cdots & 0 \\ 0 & -1 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_k \end{bmatrix}.$$

Hence any formula for the SEM expansion of Schubert polynomials may also be thought of as a formula for quantum Schubert polynomials. See [8] for further background on quantum Schubert polynomials and their role in the quantum cohomology of the flag variety.

In [16, Section 17], it is shown that the coefficients $\alpha_{j_1 j_2 \cdots}$ are also the entries in the *inverse Schubert-Kostka matrix* expressing monomials in terms of the Schubert basis. In general, computational evidence suggests that most of the $\alpha_{j_1 j_2 \cdots}$ are small in absolute value. For instance, all such coefficients have absolute value at most 1 when $n \leq 6$—see Winkel [17] for more observation and discussion about these coefficients.

2.4. **Nonintersecting lattice paths.** A key result for finding determinantal formulas is the following Lindström-Gessel-Viennot lemma [10, 14].

Let $G = (V, E)$ be a locally finite acyclic directed graph, and suppose that each edge $e \in E$ is assigned an edge weight w_e (lying in some commutative ring). For any path in G, we define its weight to be the product of the weights of all edges in the path. For any two vertices a and b, we will write $e(a, b)$ for the total weight of all directed paths from a to b.

Let $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ be subsets of V. A collection of *nonintersecting paths* $P = (P_1, \ldots, P_k)$ from A to B is a sequence of vertex-disjoint paths such that, for some permutation $\sigma \in S_k$, P_i is a directed path from a_i to $b_{\sigma(i)}$ for all i. Denote by $\mathscr{P}(A, B)$ the set of all such P. For any $P \in \mathscr{P}(A, B)$, we will write $\sigma(P)$ for the corresponding permutation σ and $w(P)$ for the product of the weights of paths in P.

Lemma 2.5 (Lindström-Gessel-Viennot)**.** *Let G be a locally finite acyclic directed graph, and let $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ be subsets of vertices of G. Then*

$$\sum_{P \in \mathscr{P}(A,B)} \operatorname{sgn}(\sigma(P)) \cdot w(P) = \det(e(a_i, b_j))_{i,j=1}^k.$$

In particular, if $\sigma(P)$ is the identity permutation for all P, then the left hand side is just the sum of the weights of all collections of nonintersecting paths.

One standard application of Lemma 2.5 is the (dual) *Jacobi-Trudi identity.*

Proposition 2.6 (Dual Jacobi-Trudi)**.** *Let λ be a partition with largest part r. Then the Schur polynomial $s_\lambda(x_1, \ldots, x_n)$ is given by the determinant*

$$s_\lambda(x_1, \ldots, x_n) = \det(e^{(n)}_{\lambda'_i + j - i})_{i,j=1}^r,$$

where each entry is an elementary symmetric polynomial in $x_1, \ldots, x_n$.

(Here, λ' is the conjugate partition to λ, so that for any positive integer i, $\lambda'_i = \#\{j \mid \lambda_j \geq i\}$.)

The proof of this result involves applying Lemma 2.5 on the following graph. Let G have vertex set $\mathbf{Z} \times \mathbf{Z}_{\geq 0}$—by convention, we will draw the positive x-axis to the east and the positive y-axis to the north. Whenever both endpoints lie in G, add a directed edge from (a, b) to $(a, b + 1)$ of weight x_{b+1} (which we call an “upstep”), as well as a directed edge from (a, b) to $(a - 1, b + 1)$ of weight 1 (which we call a “diagonal step”). See Figure 2.

Observe that any directed path from $(a, 0)$ to (b, c) must use $a - b$ diagonal steps and $c + b - a$ upsteps. Moreover, each upstep must occur at a different one of the c possible heights. It follows that $e((a, 0), (b, c)) = e^{(c)}_{c+b-a}$. Applying Lemma 2.5 then immediately implies the following result.

Proposition 2.7. *Let G be defined as above, and let*

$$A = \{(a_1, 0), (a_2, 0), \ldots, (a_k, 0)\},$$
$$B = \{(b_1, c_1), (b_2, c_2), \ldots (b_k, c_k)\}.$$

Then

$$\sum_{P \in \mathscr{P}(A,B)} \operatorname{sgn}(\sigma(P)) \cdot w(P) = \det(e^{(c_j)}_{c_j + b_j - a_i})_{i,j=1}^k.$$

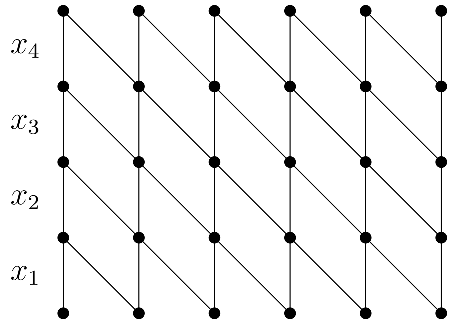

FIGURE 2. An induced subgraph of G on $\mathbf{Z}\times\mathbf{Z}_{\geq 0}$. All vertical edges are directed up, weighted according to height as shown, while all diagonal edges are directed up with weight 1.

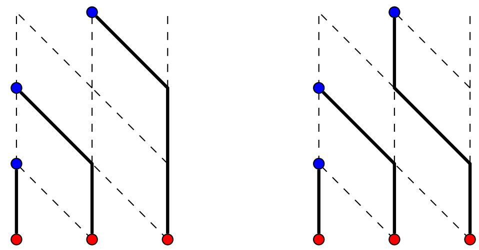

FIGURE 3. Nonintersecting lattice paths for a lattice path representation of $\mathfrak{S}_{4132} = x_1^3x_2 + x_1^3x_3$. (The points in A and B are given by red and blue nodes, respectively.)

Definition 2.8. A polynomial F has a *lattice path representation* (A, B) if

$$A = \{(a_1, 0), (a_2, 0), \ldots, (a_k, 0)\},$$
$$B = \{(b_1, c_1), (b_2, c_2), \ldots (b_k, c_k)\},$$

and

$$(*) \qquad F = \sum_{P \in \mathscr{P}(A,B)} \operatorname{sgn}(\sigma(P)) \cdot w(P) = \det(e^{(c_j)}_{c_j+b_j-a_i})_{i,j=1}^k.$$

Example 2.9. Consider the Schubert polynomial $\mathfrak{S}_{4132}$ as in Example 2.3. One can verify that

$$\mathfrak{S}_{4132} = x_1^3x_2 + x_1^3x_3 = e_{112} - e_{103} - e_{022} = \begin{vmatrix} e_1^{(1)} & e_2^{(2)} & 0 \\ e_0^{(1)} & e_1^{(2)} & e_3^{(3)} \\ 0 & e_0^{(2)} & e_2^{(3)} \end{vmatrix}.$$

This corresponds to the lattice path representation (A, B) with

$$A = \{(0, 0), (1, 0), (2, 0)\}, \qquad B = \{(0, 1), (0, 2), (1, 3)\}$$

whose nonintersecting paths are depicted in Figure 3.

The order of the labelings of the points in A and B only affects F up to a sign. Therefore we will often abuse notation slightly by considering A and B as unordered sets for ease of exposition. In most of the situations that we will consider, each collection P of nonintersecting paths will have the same $\sigma(P)$, and so we can label the elements of B so that $\sigma(P)$ is the identity.

The lattice path representation of a polynomial is not unique: for example, the constant polynomial 1 can be represented by any pair (A, B) such that $a_i = b_i + c_i$ for all i (as the corresponding matrix will be upper triangular with 1's on the diagonal).

Given a determinantal expression whose entries are elementary symmetric polynomials that vary as in $(*)$, it is straightforward to find corresponding sets A and B.

Example 2.10. The dual Jacobi-Trudi identity involves a determinant whose (i, j)th entry is given by $e^{(n)}_{\lambda'_i+j-i}$. This can be obtained from Proposition 2.7 by setting, for instance, $a_i = n + i - \lambda'_i$, $b_j = j$, and $c_j = n$.

One can then give a weight-preserving bijection between $\mathscr{P}(A, B)$ and, for instance, semistandard Young tableaux of shape λ to deduce the dual Jacobi-Trudi identity: see [10].

A particular case of interest is when the points in B all lie at different heights.

Definition 2.11. Let (A, B) be a lattice path representation with

$$\begin{aligned} A &= \{(a_1, 0), (a_2, 0), \ldots, (a_k, 0)\}, \\ B &= \{(b_1, c_1), (b_2, c_2), \ldots, (b_k, c_k)\}. \end{aligned}$$

We say that the lattice path representation (A, B) is *proper* if the c_i are distinct. (We call $\{c_1, \ldots, c_k\}$ the *multiset of heights* of (A, B).)

Observe that if (A, B) is proper, then in the expansion of the determinant in $(*)$, each term either vanishes or equals, up to sign, a standard elementary monomial. In addition, all of the nonzero terms obtained in this way will necessarily be distinct. Therefore when this occurs, this determinant can be thought of as a concise representation of the SEM expansion of the resulting polynomial. Our goal for most of the remainder of this paper is to investigate which Schubert polynomials have a proper lattice path representation.

2.5. **Quantization.** As a brief digression, we will first discuss a slight modification of these lattice path representations for computing quantum Schubert polynomials. (This section will not be needed for the remainder of this paper.)

The quantum Schubert polynomials $\mathfrak{S}^q_w$ are defined by computing the SEM expansion of $\mathfrak{S}_w$ and replacing each elementary polynomial $e^{(k)}_j$ with the quantum elementary polynomial $E^{(k)}_j$—see equations (†) and (‡) in §2.3. In the event that $\mathfrak{S}_w$ has a proper lattice path representation and hence a determinantal formula for its SEM expansion by Proposition 2.7, it follows that $\mathfrak{S}^q_w$ is also expressible as a determinant whose entries are of the form $E^{(k)}_j$. In fact, there exists a simple modification to our underlying graph G on $\mathbf{Z} \times \mathbf{Z}_{\geq 0}$ that yields the quantum elementary polynomials as weights.

Let G^q be the graph on $\mathbf{Z} \times \mathbf{Z}_{\geq 0}$ with the same edges as G as before but with additional edges from (a, b) to $(a, b + 2)$ of weight q_{b+1}. (Thus if we set all $q_i = 0$, then the graph G^q essentially reverts to the original graph G.)

Proposition 2.12. *The total weight $e((a, 0), (b, c))$ of all paths from $(a, 0)$ to (b, c) in G^q is $E^{(c)}_{c+b-a}$.*

Proof. In (‡), expanding the determinant along the last column of $I + \lambda G_k$ gives

$$E^{(k)}_j = E^{(k-1)}_j + x_k E^{(k-1)}_{j-1} + q_{k-1} E^{(k-2)}_{j-2}.$$

Similarly, any path in G^q from $(a,0)$ ending at (b,c) must come from $(b+1,c-1)$ with an edge of weight 1, from $(b,c-1)$ with an edge of weight x_c, or from $(b,c-2)$ with an edge of weight q_{c-1}. Hence $e((a,0),(b,c))$ equals

$$e((a,0),(b+1,c-1)) + x_c e((a,0),(b,c-1)) + q_{c-1} e((a,0),(b,c-2)).$$

Since $e((a,0),(b,c))$ and $E^{(c)}_{c+b-a}$ satisfy the same base cases (equaling 1 if $c+b-a=0$ and 0 if $c+b-a<0$), the result follows easily by induction. □

The following corollary is then immediate.

Corollary 2.13. *Suppose $\mathfrak{S}_w$ has a proper lattice path representation (A,B). Then*

$$\mathfrak{S}^q_w = \sum_{P \in \mathscr{P}^q(A,B)} \operatorname{sgn}(\sigma(P)) \cdot w(P),$$

where $\mathscr{P}^q(A,B)$ is the set of all collections of nonintersecting paths from A to B in the graph G^q.

Proof. By Proposition 2.7 and (†), $\mathfrak{S}^q_w$ is given by a determinant of quantum elementary polynomials. This determinant is precisely the one given by applying Lemma 2.5 to G^q and (A,B) by Proposition 2.12. □

As we will see, a large class of permutations w to which this corollary applies will be described by our main result Theorem 4.13.

3. Operations

In this section, we will describe several operations on lattice path representations that act predictably on the corresponding polynomials.

Proposition 3.1. *Let (A,B) be a lattice path representation of a polynomial F, and suppose $(b,c),(b+1,c) \in B$. Then (A,B') is a lattice path representation for F, where B' is formed by replacing $(b+1,c)$ with $(b,c+1)$ in B.*

Proof 1. Any path that ends at $(b,c+1)$ that does not pass through (b,c) must end with a diagonal step from $(b+1,c)$. Removing this last diagonal step (which has weight 1) then gives a weight-preserving bijection from $\mathscr{P}(A,B')$ to $\mathscr{P}(A,B)$. □

An alternative proof can also be obtained by manipulating the determinantal formula for F.

Proof 2. Let E be the matrix given in (∗). Note that E contains two columns whose entries in each row i have the form $e^{(c)}_{c+b-a_i}$ and $e^{(c)}_{c+b+1-a_i}$. Adding x_{c+1} times the first column to the second does not change the value of the determinant. The entries in the second column then become

$$x_{c+1} e^{(c)}_{c+b-a_i} + e^{(c)}_{c+b+1-a_i} = e^{(c+1)}_{c+b+1-a_i},$$

so that the resulting matrix corresponds to the new representation (A,B'). □

Our next operation concerns the action of the divided difference operators ∂_i. For a similar result, see [5, Lemma 4.4].

Proposition 3.2. *Let (A,B) be a lattice path representation of a polynomial F, and suppose that B has a unique point (b,c) at height c. Then (A,B') is a lattice path representation for $\partial_c(F)$, where B' is formed by replacing (b,c) with $(b,c-1)$ in B.*

If instead B has no point at height c, then $\partial_c(F) = 0$.

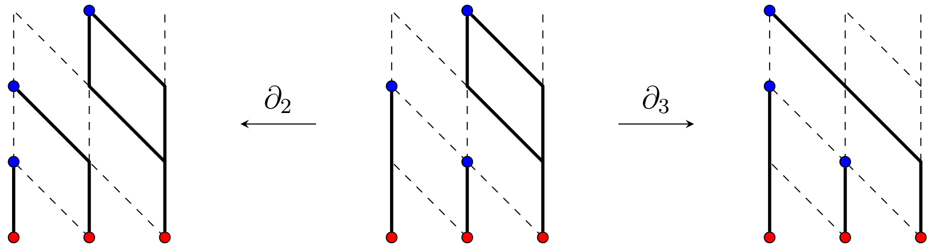

FIGURE 4. Application of Proposition 3.3. The center picture gives a lattice path representation for $F = x_1^3x_2^2 + x_1^3x_2x_3$. (Both sets of nonintersecting paths are overlaid for conciseness.) The left and right pictures represent $\partial_2 F = x_1^3x_2 + x_1^3x_3$ and $\partial_3 F = x_1^3x_2$, respectively.

Proof. From $(*)$, F is the determinant of a matrix $(e^{(c_j)}_{c_j+b_j-a_i})_{i,j=1}^k$. Each entry of this matrix is symmetric in x_c and x_{c+1} unless $c = c_j$, which occurs in a unique column (since B has a unique point at height c). Then in the Laplace expansion of the determinant along this column, each term has the form $e^{(c)}_{c+b-a_i} \cdot g$ for some minor g that is symmetric in x_c and x_{c+1}. Applying ∂_c then gives

$$\partial_c(e^{(c)}_{c+b-a_i} \cdot g) = \partial_c(e^{(c)}_{c+b-a_i}) \cdot g = e^{(c-1)}_{(c-1)+b-a_i} \cdot g.$$

Thus ∂_c has the effect of replacing $e^{(c)}_{c+b-a_i}$ with $e^{(c-1)}_{(c-1)+b-a_i}$ in the determinant. By Proposition 2.7, this new determinant for $\partial_c(F)$ corresponds to the lattice path representation (A, B'), as desired.

If instead B has no point at height c, then every entry of the determinant for F is symmetric in x_c and x_{c+1}, so $\partial_c(F) = 0$. $\square$

By combining Propositions 3.1 and 3.2, we arrive at the following operation that preserves heights.

Proposition 3.3. *Let (A, B) be a lattice path representation for F, and suppose that B has a unique point (b, c) at height c.*

(a) If $(b + 1, c - 1) \in B$, then (A, B') is a lattice path representation for $-\partial_c(F)$, where B' is formed by replacing $(b + 1, c - 1)$ by $(b, c - 1)$ in B.
(b) If $(b-1, c-1) \in B$, then (A, B'') is a lattice path representation for $\partial_c(F)$, where B'' is formed by replacing (b, c) by $(b - 1, c)$ in B.

Proof. Apply Proposition 3.2 to the point (b, c), and then apply Proposition 3.1 to the two points at height $c - 1$. $\square$

Example 3.4. Let $F = x_1^3x_2^2 + x_1^3x_2x_3$, which has lattice path representation

$$A = \{(0, 0), (1, 0), (2, 0)\}, \qquad B = \{(0, 2), (1, 1), (1, 3)\}$$

as shown in the middle diagram of Figure 4.

Applying Proposition 3.3(a) with $c = 2$ shows that we can obtain a representation for $\partial_2 F = x_1^3x_2 + x_1^3x_3$ by moving the endpoint $(1, 1)$ to $(0, 1)$ (and permuting the set B appropriately to get rid of the sign), as shown on the left of Figure 4.

Alternatively, applying Proposition 3.3(b) with $c = 3$ shows that we can obtain a representation for $\partial_3 F = x_1^3x_2$ by moving the endpoint $(1, 3)$ to $(0, 3)$, as shown on the right of Figure 4.

Our last operation concerns products of polynomials. Observe that there exists a directed path from $(a, 0)$ to (b, c) if and only if $b \le a \le b + c$.

Proposition 3.5. *Let (A, B) and (A', B') be lattice path representations for polynomials F and G, respectively, such that there do not exist any directed paths from a point in A to a point in B'. Then $(A \cup A', B \cup B')$ is a lattice path representation for the product FG.*

Proof. By the given condition, the only points of $A \cup A'$ that points in B' can be connected to are those in A'. No paths from A to B intersect any paths from A' to B' (or else there would be a path from A to B'), so the elements of $\mathscr{P}(A \cup A', B \cup B')$ are formed by pairing an element of $\mathscr{P}(A, B)$ with an element of $\mathscr{P}(A', B')$. $\square$

Note that one can always translate (A, B) horizontally to make the condition of Proposition 3.5 true while leaving weights unchanged.

As a special case, we can derive the following result that allows us to delete (or add) certain points from a lattice path representation.

Proposition 3.6. *Let (A, B) be a lattice path representation of a polynomial F, and suppose that for some $s \geq 0$,*

$$A' = \{(a, 0), (a + 1, 0), \ldots, (a + s, 0)\} \subseteq A,$$
$$B' = \{(a, 0), (a, 1), \ldots, (a, s)\} \subseteq B.$$

Then $(A \setminus A', B \setminus B')$ is also a lattice path representation of F.

Proof. There are no directed paths from any point in $A \setminus A'$ to any point in B'. Since there is a unique collection of nonintersecting paths from A' to B', and these paths use only diagonal steps, (A', B') is a lattice path representation of 1. The result then follows from Proposition 3.5. $\square$

4. Representing Schubert polynomials

In this section, we will use the operations described in §3 to construct lattice path representations for a large pattern avoidance class of Schubert polynomials.

4.1. **Compact representations.**

We will first investigate a special type of lattice path representation.

Definition 4.1. A lattice path representation (A, B) is *compact* if $\{a_1, \ldots, a_k\} = \{c_1, \ldots, c_k\} = \{0, \ldots, k - 1\}$, and $0 \leq b_i \leq k - 1$ for all i, where $k = |A| = |B|$.

In other words, a compact lattice path representation is proper, and all endpoints fit within a square of side length $k - 1$, where $k = |A| = |B|$. Note that in order for there to exist at least one set of nonintersecting lattice paths, we must have that at least s of the b_i are less than s (so that the paths starting at the first s points of A have endpoints), that is, the sequence of b_i must be a *parking function*.

Our main result of this section will be the following theorem.

Theorem 4.2. *Let $w \in S_n$ be a permutation that avoids* 1324, 2413, *and* 3142. *Then $\mathfrak{S}_w$ has a compact lattice path representation.*

Recall that a permutation is called separable if it avoids 2413 and 3142. Hence the permutations in the theorem above are the 1324-avoiding separable permutations.

To prove this theorem, we first consider the special cases of dominant (132-avoiding) permutations and 213-avoiding permutations.

Lemma 4.3. *Let $w \in S_n$ be a 132-avoiding permutation. Then $(-1)^{\binom{n}{2}-\ell(w)}\mathfrak{S}_w$ has a compact lattice path representation (A, B), where*

$$A = \{(n-1, 0), (n-2, 0), \dots, (0, 0)\},$$
$$B = \{(b_1, 0), (b_2, 1), \dots (b_n, n-1)\},$$

where w has code $c(w) = (b_1, b_2, \dots, b_n)$

Proof. We will induct on $\binom{n}{2} - \ell(w)$. When $w = w_0$, $b_i = n - i$, and the only way to connect A and B with nonintersecting lattice paths is via vertical paths, which have combined weight $\mathfrak{S}_{w_0} = x_1^{n-1}x_2^{n-2} \cdots x_{n-1}$.

Suppose $w \neq w_0$ and let $c(w) = (b_1, \dots, b_n)$. Since w is dominant, we must have $n - 1 \geq b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$. Since $w \neq w_0$, there exists a minimum index i such that $b_i = b_{i+1}$, so that $w_i < w_{i+1}$. Then $w' = ws_i$ has length $\ell(w') = \ell(w)+1$ and has code $c(w') = (b_1, \dots, b_{i-1}, b_i + 1, b_{i+1}, \dots b_n)$. Since $c(w')$ is still weakly decreasing, w' is also dominant and therefore by induction $(-1)^{\binom{n}{2}-\ell(w')}\mathfrak{S}_{w'}$ has a corresponding lattice path representation (A, B').

Note that B' contains the two points $(b_i + 1, i - 1)$ and $(b_{i+1}, i) = (b_i, i)$. We may then construct B from B' by replacing $(b_i+1, i-1)$ with $(b_i, i-1)$, so that by Proposition 3.3(a), $(-1)^{\binom{n}{2}-\ell(w)}\partial_i(\mathfrak{S}_{w'}) = (-1)^{\binom{n}{2}-\ell(w)}\mathfrak{S}_w$ has lattice path representation (A, B). □

Alternatively, since w is dominant, $\mathfrak{S}_w$ is a monomial. Hence one can also prove Lemma 4.3 by verifying that there exists a unique collection of nonintersecting lattice paths from A to B of the appropriate weight.

One can similarly prove the following result for 213-avoiding permutations. (Note that w is 213-avoiding if and only if w_0ww_0 is dominant.)

Lemma 4.4. *Let $w \in S_n$ be a 213-avoiding permutation. Then $\mathfrak{S}_w$ has compact lattice path representation (A, B), where*

$$A = \{(n-1, 0), (n-2, 0), \dots, (0, 0)\},$$
$$B = \{(b_1, n-1), (b_2, n-2), \dots, (b_n, 0)\},$$

where $c(w_0ww_0) = (b_1, b_2, \dots, b_n)$.

Proof. We induct on $\binom{n}{2} - \ell(w)$. When $w = w_0$, $b_i = n - i$, and there is a unique set of nonintersecting paths from A to B with weight $\mathfrak{S}_{w_0}$.

Suppose $w \neq w_0$, and let $u = w_0ww_0$. Since u is dominant, we can define $u' = us_i$ such that u' is dominant as in Lemma 4.3. Then $w' = w_0u'w_0 = w_0uw_0s_{n-i} = ws_{n-i}$ is also 213-avoiding with $\ell(w') = \ell(w)+1$. Hence by induction $\mathfrak{S}_{w'}$ has a corresponding lattice path representation (A, B').

Since (as in Lemma 4.3) $c(u') = (b_1, \dots, b_{i-1}, b_i + 1, b_{i+1}, \dots, b_n)$, B' contains the two points $(b_i + 1, n - i)$ and $(b_{i+1}, n - i - 1) = (b_i, n - i - 1)$. Then if we construct B from B' by replacing $(b_i + 1, n - i)$ with $(b_i, n - i)$, by Proposition 3.3(b), (A, B) is a lattice path representation for $\partial_{n-i}\mathfrak{S}_{w'} = \mathfrak{S}_w$. □

Applying Proposition 2.7 to the lattice path representation in Lemma 4.4 yields a determinantal formula that gives the SEM expansion for $\mathfrak{S}_w$ when w is 213-avoiding as in Corollary 17.12 of [16].

We are now ready to prove that any 1324-avoiding separable permutation has a compact lattice path representation.

Proof of Theorem 4.2. We proceed by induction on n. The case $n = 1$ is trivial. For $n > 1$, since w avoids 2413 and 3142, it is separable. Hence we can either write $w = u \ominus v$ or $w = u \oplus v$ for separable permutations $u \in S_m$ and $v \in S_{n-m}$ that avoid 1324.

Suppose first that $w = u \ominus v$. By induction, u and v have compact lattice point representations (A_u, B_u) and (A_v, B_v), respectively. Using addition to indicate translation, we claim that if

$$\begin{aligned} A_w &= (A_u + (n-m, 0)) \cup A_v = \{(n-1, 0), (n-2, 0), \ldots, (0, 0)\}, \\ B_w &= (B_u + (n-m, 0)) \cup (B_v + (0, m)), \end{aligned}$$

then (A_w, B_w) is a lattice point representation for $\mathfrak{S}_w$. Note that if A_v and $B_v + (0, m)$ are connected by nonintersecting lattice paths, then all paths must start with m upsteps by compactness. It follows that $(A_v, B_v + (0, m))$ represents the polynomial $(x_1 \cdots x_m)^{n-m} \cdot \mathfrak{S}_v(x_{m+1}, \ldots, x_n) = \mathfrak{S}_{1_m \ominus v}$ as in Proposition 2.4. Also $(A_u + (n - m, 0), B_u + (n-m, 0))$ represents $\mathfrak{S}_u$ as before. Since there are no directed paths from A_v to $B_u + (n - m, 0)$, Proposition 3.5 implies that (A_w, B_w) represents the product $\mathfrak{S}_u \cdot \mathfrak{S}_{1_m \ominus v}$, which equals $\mathfrak{S}_w$ by Proposition 2.4.

Suppose instead that $w = u \oplus v$. Since w avoids 1324, u must avoid 132 and v must avoid 213. Hence $v' = 1_m \oplus v$ must also avoid 213. We can then construct a lattice path representation $(A_{v'}, B_{v'})$ of v' using Lemma 4.4. Note that the code of $w_0^{(n)} v' w_0^{(n)} = w_0^{(n-m)} v w_0^{(n-m)} \oplus 1_m$ ends with m zeroes. Thus $(0, 0), (0, 1), \ldots, (0, m - 1) \in B_{v'}$. By Proposition 3.6, it follows that $(A'_{v'}, B'_{v'})$ is also a lattice path representation of v', where

$$\begin{aligned} A'_{v'} &= A_{v'} \setminus \{(0, 0), (1, 0), \ldots, (m - 1, 0)\}, \\ B'_{v'} &= B_{v'} \setminus \{(0, 0), (0, 1), \ldots, (0, m - 1)\}. \end{aligned}$$

(In fact, if B_v is constructed for v using Lemma 4.4, then $B'_{v'}$ is the translation $B_v + (0, m)$.)

Now consider the lattice path representation (A_u, B_u) as constructed by Lemma 4.3. If $(b_i, i-1) \in B_u$, then $b_i \leq m - i$ by the definition of the code of u. Hence there does not exist a directed path from any point of $A'_{v'}$ to any point in B_u. By Proposition 3.5, it follows that $(A_w, B_w) = (A_u \cup A'_{v'}, B_u \cup B'_{v'})$ is a lattice path representation of $\mathfrak{S}_u \cdot \mathfrak{S}_{1_m \oplus v}$, which equals $\mathfrak{S}_w$ by Proposition 2.4. □

Example 4.5. Let $w = 87321564 = 21 \ominus v$, where $v = 321564$.

Since $v = 321 \oplus 231$, $\mathfrak{S}_v = \mathfrak{S}_{321} \cdot \mathfrak{S}_{123564}$ by Proposition 2.4(a). Now 321 is 132-avoiding with code $(2, 1, 0)$ (see Lemma 4.3), while 123564 is 213-avoiding with

$$c(w_0 \cdot 123564 \cdot w_0) = c(312456) = (2, 0, 0, 0, 0, 0)$$

(see Lemma 4.4). Reversing this second code and combining with the first gives $(2, 1, 0, 0, 0, 2)$, so following the last case of the proof of Theorem 4.2, 321564 has compact lattice path representation (A_v, B_v) (up to sign) with

$$\begin{aligned} A_v &= \{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (5, 0)\}, \\ B_v &= \{(2, 0), (1, 1), (0, 2), (0, 3), (0, 4), (2, 5)\}. \end{aligned}$$

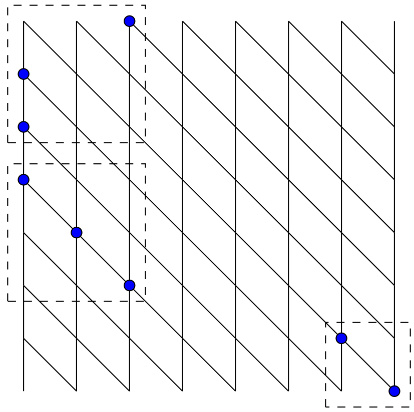

FIGURE 5. The set B_w for a compact lattice path representation for $\mathfrak{S}_w$, where $w = 87321564 = 21 \ominus (321 \oplus 231)$. The dashed squares from bottom to top are translations of B-sets for 21, 321, and 231.

Now $\mathfrak{S}_w = \mathfrak{S}_{21} \cdot (x_1 \cdots x_6)^2 \cdot \mathfrak{S}_v$ by Proposition 2.4(b). Shifting B_v up by 2 and placing a representation for $\mathfrak{S}_{21}$ to its right as in the first case of Theorem 4.2 gives

$$A_w = \{(0,0),(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(7,0)\},$$
$$B_w = \{(7,0),(6,1),(2,2),(1,3),(0,4),(0,5),(0,6),(2,7)\}.$$

Then (A_w, B_w) is a compact lattice path representation (up to sign) for $\mathfrak{S}_w$. To see how the representations for $\mathfrak{S}_{21}$, $\mathfrak{S}_{321}$, and $\mathfrak{S}_{231}$ fit together geometrically to give the representation for $\mathfrak{S}_w$, see Figure 5.

4.2. **Lowering points.** Given a lattice path representation, one can use Proposition 3.6 to remove lattice points at height 0 and Proposition 3.2 to shift lattice points downward into empty rows, thereby generating additional representations. In this section, we will use these two operations on the collection of compact lattice path representations to construct representations for a large pattern avoidance class of Schubert polynomials.

Application of these two operations can be described succinctly in the following way.

Definition 4.6. We say a permutation $v \in S_n$ is a *lowering permutation* if v satisfies

$$v^{-1}(1) > v^{-1}(2) > \cdots > v^{-1}(k) = 1 < v^{-1}(k+1) < \cdots < v^{-1}(n)$$

for some integer k. In other words, in one-line notation, v contains $k(k-1)\cdots 21$ and $k(k+1)\cdots(n-1)n$ as subsequences.

Equivalently, v avoids the patterns 132 and 312. If we let $p_i = v^{-1}(i)$ for $1 \le i \le k$, then v has the reduced expression

$$v = (s_1 s_2 \cdots s_{p_1-1}) \cdot (s_1 s_2 \cdots s_{p_2-1}) \cdots (s_1 s_2 \cdots s_{p_{k-1}-1}).$$

Put another way, the effect of multiplying a permutation u on the right by v is to shuffle $u_k \cdots u_1$ and $u_{k+1} \cdots u_n$ by placing $u_k, \ldots, u_1$ in positions $p_k, \ldots, p_1$.

The significance of these permutations to our current study lies in the following proposition.

Proposition 4.7. *Let* (A, B) *be a lattice path representation of a polynomial* F *of the form*

$$A = \{(a_1, 0), (a_2, 0), \ldots, (a_n, 0)\},$$
$$B = \{(b_1, 0), (b_2, 1), \ldots (b_n, n-1)\}.$$

Suppose further that v *is a lowering permutation with* $v_1 = k$*, and that* $a_i = b_i$ *for* $i = 1, \ldots, k$*. Then* $\partial_{v^{-1}} F$ *has lattice path representation* (A', B')*, where*

$$A' = \{(a_{k+1}, 0), (a_{k+2}, 0), \ldots, (a_n, 0)\},$$
$$B' = \{(b_{k+1}, v^{-1}(k+1) - 1), (b_{k+2}, v^{-1}(k+2) - 1), \ldots, (b_n, v^{-1}(n) - 1)\}.$$

Proof. Let $p_i = v^{-1}(i)$. By Proposition 3.6, removing $(a_1, 0) = (b_1, 0)$ from A and B yields a lattice path representation for F. Then by Proposition 3.2, lowering each of the points $(b_i, i-1)$ to $(b_i, i-2)$ for $i = 2, \ldots, p_1$ yields a lattice path representation for $\partial_{p_1 - 1} \cdots \partial_2 \partial_1 F$ with points at heights $\{0, 1, \ldots, n-1\} \setminus \{p_1 - 1\}$.

We can then repeat this process by removing $(a_2, 0) = (b_2, 0)$ from both A and B and then lowering the points $(b_i, i-2)$ to $(b_i, i-3)$ for $i = 3, \ldots, p_2$, giving a lattice path representation for $(\partial_{p_2-1} \cdots \partial_2 \partial_1)(\partial_{p_1-1} \cdots \partial_2 \partial_1) F$ with points at heights $\{0, 1, \ldots, n-1\} \setminus \{p_1 - 1, p_2 - 1\}$. Continuing in this manner, we arrive at a lattice path representation for $\partial_{v^{-1}} F$ with points at heights $\{0, 1, \ldots, n-1\} \setminus \{p_1 - 1, \ldots, p_k - 1\} = \{v^{-1}(k+1) - 1, \ldots, v^{-1}(n) - 1\}$, as desired. □

For an illustration, see Figure 6 as well as Example 4.10 below.

Note that any compact lattice path representation (up to reordering the elements of A and B) has the form required in Proposition 4.7. Therefore, combining Proposition 4.7 with Theorem 4.2 gives the following result.

Theorem 4.8. *Let* $u, v \in S_n$ *be permutations such that* u *avoids the patterns* 1324, 2413*, and* 3142*,* v *avoids the patterns* 132 *and* 312*, and* $\ell(uv) = \ell(u) - \ell(v)$*. Then* $\mathfrak{S}_{uv}$ *has a proper lattice path representation.*

Proof. By Theorem 4.2, $\mathfrak{S}_u$ has a compact lattice path representation. Since v is a lowering permutation, Proposition 4.7 implies that $\partial_{v^{-1}} \mathfrak{S}_u$ has a proper lattice path representation. The length condition then implies $\partial_{v^{-1}} \mathfrak{S}_u = \mathfrak{S}_{uv}$. □

The following proposition gives an explicit description of when the length condition in Theorem 4.8 holds.

Proposition 4.9. *Let* $u, v \in S_n$ *be permutations such that* v *is a lowering permutation. Suppose* $v_1 = k$ *and let* $p_i = v^{-1}(i)$*. If* $w = uv$*, then* $\ell(w) = \ell(u) - \ell(v)$ *if and only if* w_{p_i} *is a left-to-right maximum of* w *for all* $i = 1, \ldots, k$*.*

Proof. The effect of multiplying u by

$$v = (s_1 s_2 \cdots s_{p_1 - 1}) \cdot (s_1 s_2 \cdots s_{p_2 - 1}) \cdots (s_1 s_2 \cdots s_{p_{k-1} - 1})$$

is to shift u_1 to position p_1, then shift u_2 to position p_2, and so forth. The length condition will then be satisfied if and only if while shifting u_i, it only moves past smaller letters. This occurs exactly when w_{p_i} is a left-to-right maximum of w. □

Note that the w_{p_i} need only be a subset of the left-to-right maxima of w, not the entire set of them.

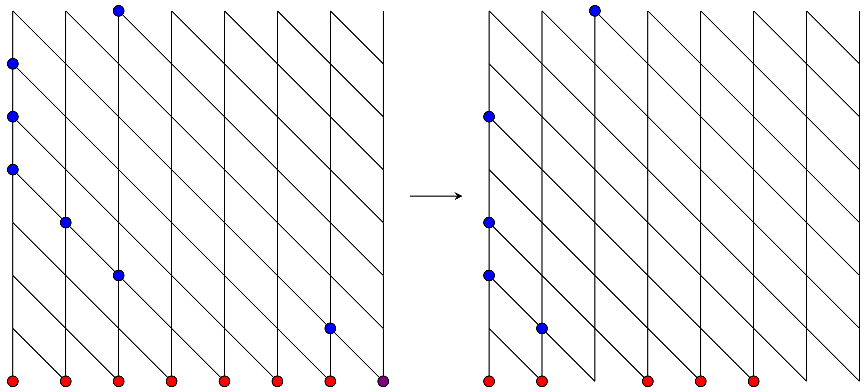

FIGURE 6. Illustration of Proposition 4.7. On the left is a proper lattice path representation for $\mathfrak{S}_u$ (where $u = 87321564$), and on the right is a lattice path representation for $\mathfrak{S}_{uv}$, where $v = 34562718$.

Example 4.10. Let $u = 87321564$ as in Example 4.5, and let $v = 34562718$, so that $p_1 = 7$, $p_2 = 5$, and $p_3 = 1$. Then $w = uv = 32157684$. Since $w_1 = 3$, $w_5 = 7$, and $w_7 = 8$ are left-to-right maxima, $\ell(w) = \ell(u) - \ell(v)$.

From Example 4.5, $\mathfrak{S}_u$ has compact lattice path representation

$$A_u = \{(0,0),(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(7,0)\},$$
$$B_u = \{(7,0),(6,1),(2,2),(1,3),(0,4),(0,5),(0,6),(2,7)\}.$$

By Proposition 4.7, $\mathfrak{S}_w$ then has lattice path representation

$$A_w = \{(0,0),(1,0),(3,0),(4,0),(5,0)\},$$
$$B_w = \{(1,1),(0,2),(0,3),(0,5),(2,7)\}.$$

See Figure 6 for an illustration.

As another illustrative example, we consider the case of 321-avoiding permutations, whose Schubert polynomials are known to be flagged skew Schur polynomials [3].

Corollary 4.11. *Let $w \in S_n$ be a 321-avoiding permutation. Let $\bar{q}_1 < \cdots < \bar{q}_{n-k}$ be the elements of $[n]$ that are* not *left-to-right maxima of w, and let $\bar{p}_i = w^{-1}(\bar{q}_i)$. Then $\mathfrak{S}_w$ has lattice path representation (A, B), where*

$$A = \{(\bar{q}_1 - 1, 0), (\bar{q}_2 - 1, 0), \ldots, (\bar{q}_{n-k} - 1, 0)\},$$
$$B = \{(0, \bar{p}_1 - 1), (0, \bar{p}_2 - 1), \ldots, (0, \bar{p}_{n-k} - 1)\},$$

and therefore

$$\mathfrak{S}_w = \det(e^{(\bar{p}_j - 1)}_{\bar{p}_j - \bar{q}_i})^k_{i,j=1}.$$

Proof. Let $w \in S_n$ have left-to-right maxima in positions $p_1 > p_2 > \cdots > p_k = 1$, and let $q_i = w_{p_i}$ be the values of these maxima (so that $q_1 > q_2 > \cdots > q_k$).

Since w is 321-avoiding, the letters $\bar{q}_1, \ldots, \bar{q}_{n-k}$ must appear in increasing order in w, so $\bar{p}_1 < \cdots < \bar{p}_{n-k}$. Let v be the lowering permutation with $v^{-1} = p_1 \cdots p_k \bar{p}_1 \ldots \bar{p}_{n-k}$. If we let $u = wv^{-1}$, then $u = q_1 \cdots q_k \bar{q}_1 \cdots \bar{q}_{n-k}$ and $\ell(w) = \ell(u) - \ell(v)$ by Proposition 4.9.

Now observe that u is 132-avoiding and $c(u) = (q_1 - 1, \ldots, q_k - 1, 0, \ldots, 0)$, so by Lemma 4.3, $\pm\mathfrak{S}_u$ has lattice path representation (A', B'), where

$$A' = \{(n-1,0), (n-2,0), \ldots, (0,0)\},$$
$$B' = \{(q_1-1,0), (q_2-1,1), \ldots, (q_k-1,k-1), (0,k), \ldots, (0,n-1)\}.$$

Applying Proposition 4.7, we find that $\mathfrak{S}_w$ has lattice path representation (A, B), as desired. (The sign is easily seen to be positive.) The determinantal formula then follows from Proposition 2.7. □

Since Grassmannian permutations are special cases of 321-avoiding permutations, Corollary 4.11 specializes to a formula for Schur polynomials akin to the dual Jacobi-Trudi identity, as also shown in [11, 17]. (Compare the following to Proposition 2.6.)

Corollary 4.12. *Let λ be a partition with largest part r. Then the Schur polynomial $s_\lambda(x_1, \ldots, x_n)$ is given by the determinant*

$$s_\lambda(x_1, \ldots, x_n) = \det(e^{(n+j-1)}_{\lambda'_i+j-i})^r_{i,j=1}.$$

Proof. The Schur polynomial $s_\lambda(x_1, \ldots, x_n)$ is equal to the Schubert polynomial $\mathfrak{S}_w$, where $w \in S_{n+r}$ is the Grassmannian permutation $q_1q_2\cdots q_n\bar{q}_1\bar{q}_2\cdots\bar{q}_r$, where $q_i = \lambda_{n+1-i} + i$ and $\bar{q}_i = n - \lambda'_i + i$. Since the left-to-right maxima are precisely $q_1, \ldots, q_n$ and $\bar{p}_i = w^{-1}(\bar{q}_i) = n + i$, the result follows from Corollary 4.11. □

One can also deduce Corollary 4.12 by interpreting the usual dual Jacobi-Trudi identity (Proposition 2.6) as a lattice point representation (albeit not a proper one) and applying Proposition 3.1 repeatedly to turn it into a proper representation.

4.3. **Pattern avoidance criterion.** In this section, we will give an explicit description of the permutations to which Theorem 4.8 applies via the following theorem.

Theorem 4.13. *A permutation w has a factorization of the form $w = uv$ as in Theorem 4.8 if and only if it avoids the following 13 patterns:*

51324, 15324, 52413, 25413, 53142, 35142, 31542,

143265, 143625, 143652, 146352, 413265, 413625.

Therefore, for any such permutation w, $\mathfrak{S}_w$ has a proper lattice path representation.

For example, there are 569 permutations of length 6, 2932 permutations of length 7, and 15226 permutations of length 8 avoiding these 13 patterns. Note that this theorem gives a sufficient, but not necessary, condition for $\mathfrak{S}_w$ to have a proper lattice path representation. For further discussion, see §5.

While the proof of the forward direction of Theorem 4.13 will be relatively straightforward, for the reverse direction we will need to describe for each permutation w avoiding the given 13 patterns how to construct the corresponding permutations u and v. By Proposition 4.9, we will choose v by choosing a certain subset of the left-to-right maxima of w. We will then verify that $u = wv^{-1}$ avoids 2413, 3142, and 1324 as required.

Fix a permutation $w \in S_n$ that avoids the 13 patterns in Theorem 4.13. We construct a set $Q \subseteq [n]$ as follows. Consider the left-to-right maxima of w from largest to smallest (i.e., from right to left). For each such q, add it to Q unless w has an occurrence of the pattern 1342 consisting of letters $aqq'b$, where $q' \notin Q$.

Example 4.14. Let $w = 32157684$, which avoids the 13 patterns in Theorem 4.13. The left-to-right maxima of w are 8, 7, 5, and 3.

- We first add 8 to Q since it cannot be the second letter in a 1342 pattern.
- Although 7 is the second letter of several 1342 patterns, the third letter in such patterns is always $8 \in Q$, so we add 7 to Q.
- Now 5 occurs in 1564 and $6 \notin Q$, so we do not add 5 to Q.
- Finally, we add 3 to Q, so that $Q = \{3, 7, 8\}$.

The elements of Q occur at positions 1, 5, and 7. Note that if we let v be the lowering permutation 34562718, then the permutation $u = wv^{-1} = 87321564$ obtained by shifting the elements of Q to the left in decreasing order is a 1324-avoiding separable permutation.

We will also need some technical lemmas about the structure of the permutations in Theorem 4.13.

Lemma 4.15. *Let w be a permutation that avoids the 13 patterns in Theorem 4.13. If w has a subsequence $abcde$ that forms a* 13542 *pattern, then any letter that occurs between b and d in w must be greater than b.*

Proof. Suppose x lies between b and d in w. If $x < e$, then w must contain either the 31542 pattern $bxcde$ or the 35142 pattern $bcxde$. If instead $e < x < b$, then w contains either the 143652 pattern $abxcde$ or the 146352 pattern $abcxde$. Since all of these patterns are forbidden, we must have $x > b$. □

Lemma 4.16. *Let w be a permutation that avoids the 13 patterns in Theorem 4.13, and fix a left-to-right maximum $b \notin Q$. Let c be the rightmost letter of w such that $c \notin Q$ and w contains a* 1342 *pattern $abcd$. Then either w contains a* 13542 *pattern $abxcd$, or w contains a* 2413 *pattern $bcde$.*

Proof. Since $c \notin Q$, there are two possibilities.

- If c is not a left-to-right maximum, then there must be a larger letter x to its left. Since b is a left-to-right maximum and $x > c > b$, w must have the 13542 pattern $abxcd$.
- If c is a left-to-right maximum, then since $c \notin Q$, it must be part of a 1342 pattern $fcge$ with $g \notin Q$. By our choice of c to be rightmost, we must have that g lies to the right of d (or else $abgd$ would be a 1342 pattern). Then:
  - If $a < e < b$, then $abge$ would be a 1342 pattern that contradicts our choice of c.
  - If $e < a$, then f cannot lie to the left of b or else $fbge$ would be a 1342 pattern that contradicts our choice of c. Hence f has to lie to the right of b, but then w would contain the 35142 pattern $abfde$, which is a contradiction.
  - The only remaining possibility is that $e > b$, which implies that w has the 2413 pattern $bcde$, as desired. □

Using Lemmas 4.15 and 4.16, we can now prove most of the pattern conditions that we will need for Theorem 4.13.

Lemma 4.17. *Let w be a permutation that avoids the 13 patterns in Theorem 4.13.*

(a) Suppose w contains the 2413 *pattern $abcd$. Then $b \in Q$.*
(b) Suppose w contains the 3142 *pattern $abcd$. Then $c \in Q$.*
(c) Suppose w contains the 1324 *pattern $abcd$ with $d \notin Q$. Then $b \in Q$.*

(d) Suppose w contains the 1342 *pattern $abcd$ with $c \notin Q$. Then $b \notin Q$.*

Proof. For (a), note that b must be a left-to-right maximum, for if it were not, then some letter to the left of b would be greater than b, which would cause w to contain a forbidden 52413 or 25413 pattern.

Suppose the claim does not hold, and let us take b to be the rightmost left-to-right maximum in a 2413 pattern $abcd$ with $b \notin Q$. By Lemma 4.16, either w has some 2413 pattern $befg$ with $e \notin Q$, which contradicts our choice of b, or w contains a 13542 pattern $ebfgh$. In the latter case, by Lemma 4.15, since c and d are both less than b, they must lie to the right of g. But then w contains the forbidden 25413 pattern $afgcd$, completing the proof of (a).

Note that (a) implies that the second possibility in Lemma 4.16 can never hold. In other words, any left-to-right maximum that does not lie in Q must appear second in a 13542 pattern.

For (b), note that c must be a left-to-right maximum or else w would contain a forbidden 53142, 35142, or 31542 pattern. Suppose $c \notin Q$. Then by Lemma 4.16 (as per the discussion above) there exists a 13542 pattern $ecfgh$. By Lemma 4.15, $d < c$ cannot lie between c and g, so d must lie to the right of g. But then $abfgd$ is a forbidden 31542 pattern in w. So $c \in Q$.

For (c), for a fixed b, let us choose $d \notin Q$ to be rightmost. If d were a left-to-right maximum, then there would have to be a 1342 pattern $edfg$ with $f \notin Q$. But then the 1324 pattern $abcf$ would contradict the choice of d. Hence d is not a left-to-right maximum. Therefore, there exists some $h > d$ to the left of d. If h lies to the left of b, then w would either contain the 51324 pattern $habcd$ or the 15324 pattern $ahbcd$, which are both forbidden. Thus h lies to the right of b (and to the left of d).

Suppose b is not a left-to-right maximum. Then there exists some $i > b$ to the left of b. But we cannot have $i > d$ for then w would contain $iabcd$ or $aibcd$, which would be a 51324 or 15324 pattern, nor can we have $i < d$ for then w would contain one of $iabhcd$, $iabchd$, $aibhcd$, or $aibchd$, which would be a 413625, 413265, 143625, or 143265 pattern. Thus b must be a left-to-right maximum.

Now suppose for the sake of contradiction that $b \notin Q$. By Lemma 4.16, there exists a 13542 pattern $jbklm$. By Lemma 4.15, $c < b$ cannot appear between b and l, so it must appear after l. If $d < l$, then $bklcd$ would be a forbidden 25413 pattern. If $l < d < k$, then $aklcd$ would be a forbidden 15324 pattern. Hence $d > k$.

Recall that $h > d$ lies to the right of b. If h lies to the left of l, then $ahlcd$ would be a forbidden 15324 pattern. Then h must lie to the right of l, but now w must contain either the 143265 pattern $aklchd$ or the 143625 pattern $aklhcd$, which are forbidden. It follows that we must have $b \in Q$, as desired.

Finally, (d) follows immediately from the construction of Q. □

It is now straightforward to deduce our main result.

Proof of Theorem 4.13. We first verify that any permutation w with a factorization $w = uv$ as in Theorem 4.8 must avoid the given 13 patterns. Note that if w' is a pattern of w, then there exist patterns u' of u and v' of v such that $w' = u'v'$. Any pattern v' contained in the lowering permutation v is again a lowering permutation. By Proposition 4.9, the length condition $\ell(w) = \ell(u) - \ell(v)$ implies that multiplying u by v has the effect of shifting the first k letters in u to become left-to-right maxima of w. But any left-to-right maximum of w chosen to appear in w' will still be a left-to-right maximum. It follows that $\ell(w') = \ell(u') - \ell(v')$, so w' must also satisfy the conditions of Theorem 4.8.

Therefore, we need only verify that none of the 13 patterns w' have such a factorization $u'v'$. To see this, observe that each pattern other than 143652 and 146352 contains a 1324, 2413, or 3142 pattern that does not involve any left-to-right maxima except for possibly the first letter. Since these would necessarily remain in the same order in u', u' cannot avoid these three patterns. For the last two patterns 143652 and 146352, depending on whether the left-to-right maximum 4 is moved, u' must contain either the 1324-pattern 1435 or the 3142-pattern 4152.

For the reverse direction, we need to verify that any permutation w that avoids the given 13 patterns has the requisite factorization $w = uv$. Defining the set Q as described, let u be the permutation obtained from w by shifting the elements of Q to the left and placing them in decreasing order, so that $u = wv^{-1}$ for some lowering permutation v with $\ell(w) = \ell(u) - \ell(v)$ as in Proposition 4.9. If u were to contain one of the patterns 2413, 3142, or 1324, then there are only four possibilities for how these letters could be ordered in w:

(a) w contains the 2413 pattern $abcd$ and $b \notin Q$, so that $abcd$ occurs in u;
(b) w contains the 3142 pattern $abcd$ and $c \notin Q$, so that $abcd$ occurs in u;
(c) w contains the 1324 pattern $abcd$ and $b, d \notin Q$, so that $abcd$ occurs in u;
(d) w contains the 1342 pattern $abcd$ with $b \in Q$ and $c \notin Q$, so that the 3142 pattern $bacd$ occurs in u.

However, all of these are impossible by Lemma 4.17, which completes the proof. □

5. Conclusion

Although Theorem 4.13 gives a determinantal formula for a wide class of Schubert polynomials, the precise characterization of which Schubert polynomials admit such a formula remains open.

Question 5.1. For which permutations $w \in S_\infty$ does $\mathfrak{S}_w$ admit a proper lattice path representation (and hence a determinantal formula for its SEM expansion)? Is the set of such permutations closed under pattern containment?

We note in particular that the condition in Theorem 4.13 is sufficient but not necessary. For example, although 413625 is a forbidden pattern,

$$\mathfrak{S}_{413625} = \begin{vmatrix} e_1^{(1)} & e_2^{(2)} & 0 & 0 \\ e_0^{(1)} & e_1^{(2)} & e_4^{(4)} & e_5^{(5)} \\ 0 & e_0^{(2)} & e_3^{(4)} & e_4^{(5)} \\ 0 & 0 & e_0^{(4)} & e_1^{(5)} \end{vmatrix}$$

has the proper lattice path representation shown in Figure 7. From this, one can then use Proposition 3.3 to derive representations for $\mathfrak{S}_{413265}$, $\mathfrak{S}_{143625}$, and $\mathfrak{S}_{143265}$. (The Schubert polynomials for the remaining nine forbidden patterns, including all of the ones of length 5, do not have proper lattice path representations.)

Recall that any polynomial with a proper lattice path representation also has the property that its SEM expansion only has coefficients of absolute value at most 1. One can then ask similar questions about the class of Schubert polynomials satisfying this weaker property. (See Winkel [17] for some discussion, as well as [2, 7] for some similar studies.)

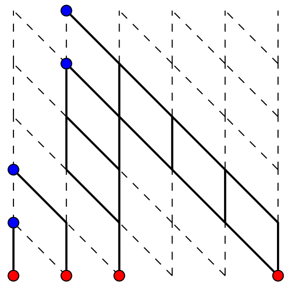

FIGURE 7. Lattice path representation for $\mathfrak{S}_{413625}$. (Edges that appear in at least one collection of nonintersecting lattice paths are solid.)

Question 5.2. For which permutations $w \in S_\infty$ does the SEM expansion of $\mathfrak{S}_w$ have only coefficients of absolute value at most 1? Is the set of such permutations closed under pattern containment?

Our proof of Theorem 4.13 is algebraic as opposed to combinatorial. A bijective proof certainly exists for certain subclasses of permutations (for instance, Grassmannian permutations), and to some extent one can use the operations of §3 to generate bijections for other cases covered by Theorem 4.13. However, it is unclear whether a uniform bijection exists in general, particularly in cases not covered by Theorem 4.13.

Question 5.3. When $\mathfrak{S}_w$ has a proper lattice path representation, is there a natural bijection between the corresponding collections of nonintersecting lattice paths and other known combinatorial interpretations for $\mathfrak{S}_w$ (such as reduced pipe dreams)?

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