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First-order Lagrangian and Hamiltonian of Lovelock gravity

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Abstract

Based on the insight gained by many authors over the years on the structure of the Einstein–Hilbert, Gauss–Bonnet and Lovelock gravity Lagrangians, we show how to derive-in an elementary fashion-their first-order, generalized 'Arnowitt–Deser–Misner' Lagrangian and associated Hamiltonian. To do so, we start from the Lovelock Lagrangian supplemented with the Myers boundary term, which guarantees a Dirichlet variational principle with a surface term of the form $\pi^{ij}\delta h_{ij}$, where π^{ij} is the canonical momentum conjugate to the boundary metric h_{ij}. Then, the first-order Lagrangian density is obtained either by integration of π^{ij} over the metric derivative $\partial_w h_{ij}$ normal to the boundary, or by rewriting the Myers term as a bulk term.

Keywords: classical mechanics, modified theories of gravity, general relativity

Introduction

The general relativity (GR), Gauss–Bonnet (GB) and more generally Lovelock [1] Lagrangians, being (quasi) linear in the second derivatives of the metric, yield second-order field equations (see e.g. [2] for a review).

There must hence exist *first-order* Lagrangians, which do not depend on the metric's second derivative normal to a foliation, and which differ from Lovelock's by adding adequate boundary terms, so that they produce the same dynamics but with Dirichlet boundary conditions.

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In GR, a boundary term to be added to the Einstein–Hilbert Lagrangian to yield a Dirichlet variational principle was proposed by Gibbons, Hawking [3] and York [4] (GHY). Its generalization to GB and Lovelock theories was obtained by Myers [5], see also [6–9].

In GR, a well-known first-order Lagrangian is that of Arnowitt–Deser–Misner (ADM), which is written (as well as the corresponding Hamiltonian) in a 1 + 3 form in terms of the extrinsic and intrinsic curvatures of a spacetime foliation [10, 11]. The GB and Lovelock first-order Lagrangians (and corresponding Hamiltonian) generalizing ADM's were found by Teitelboim and Zanelli [12, 13].

In this paper, we will obtain the Teitelboim–Zanelli Lagrangian and Hamiltonian in two different straightforward manners. We shall first illustrate the methods on the (nowadays) simple case of GR, and then generalize the procedure to all Lovelock Lagrangians.

1. The crux of the method

1.1. The example of point mechanics

Consider a particle with position $q(t)$ described by the action

$$I = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}tL \quad \text{with} \quad L(q, \dot{q}, \ddot{q}) = \ell(q, \dot{q}) + \ddot{q} f(q, \dot{q}), \tag{1.1}$$

where a dot denotes a derivative with respect to time t. The variation of I upon an infinitesimal variation $\delta q(t)$ of the path $q(t)$ reads

$$\delta I = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t\, \delta q\, [B(q, \dot{q}) - \ddot{q}A(q, \dot{q})] + \left[\delta q \left(\frac{\partial \ell}{\partial \dot{q}} - \dot{q}\frac{\partial f}{\partial q}\right) + \delta\dot{q} f\right]_{t_\mathrm{i}}^{t_\mathrm{f}}. \tag{1.2}$$

The issue with I is that its variation δI cannot be made to vanish for an arbitrary $\delta q(t)$ between t_i and t_f. Indeed, the vanishing of the boundary terms necessitates fixing 4 constants (to wit the positions and velocities of the particle at t_i and t_f so that $\delta q|_{t_\mathrm{i}} = \delta q|_{t_\mathrm{f}} = \delta\dot{q}|_{t_\mathrm{i}} = \delta\dot{q}_{t_\mathrm{f}} = 0$). These conditions are incompatible with the fact that the solutions of the equation of motion ($B - \ddot{q}A = 0$), which is second order since L is (quasi) linear in the acceleration $\ddot{q}$, depend on 2 integration constants only[4].

Now, it must be possible to build an ordinary, first-order Lagrangian $L_1(q, \dot{q})$ and associated action I_1 which yield a second order equation of motion when imposing $\delta I_1 = 0$ for Dirichlet boundary conditions (that is, by fixing $\delta q|_{t_\mathrm{i}} = \delta q|_{t_\mathrm{f}} = 0$ only). In order to give the same equation of motion as L, $L_1(q, \dot{q})$ is taken to differ from L by the substraction of a total time derivative of some function $F(q, \dot{q})$:

$$L_1(q, \dot{q}) = L - \frac{\mathrm{d}F(q, \dot{q})}{\mathrm{d}t}, \qquad I_1 = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t\, L_1 = I - [F(q, \dot{q})]_{t_\mathrm{i}}^{t_\mathrm{f}}. \tag{1.3}$$

A simple route to obtain L_1 is to compute the surface terms in the variation of the action. We have, on-shell, that is when the equation of motion is satisfied,

$$\delta I_1 = \left[\delta q \left(\frac{\partial \ell}{\partial \dot{q}} - \dot{q}\frac{\partial f}{\partial q} - \frac{\partial F}{\partial q}\right) + \delta\dot{q}\left(f - \frac{\partial F}{\partial \dot{q}}\right)\right]_{t_\mathrm{i}}^{t_\mathrm{f}}, \tag{1.4}$$

where we have used (1.2). The vanishing of the coefficient of $\delta\dot{q}$ in (1.4) gives the function F,

$$F = \int \mathrm{d}\dot{q}\; f(q, \dot{q}). \tag{1.5}$$

[4] For completeness: $A(q, \dot{q}) = \frac{\partial^2 \ell}{\partial \dot{q}^2} - \dot{q}\frac{\partial^2 f}{\partial q \partial \dot{q}} - 2\frac{\partial f}{\partial q}$ and $B(q, \dot{q}) = \frac{\partial}{\partial q}\left(\ell - \dot{q}\frac{\partial \ell}{\partial \dot{q}} + \dot{q}^2\frac{\partial f}{\partial q}\right)$.

If we then identify the coefficient of δq to the canonical momentum (see e.g. [14])

$$p = \frac{\partial L_1}{\partial \dot{q}}, \tag{1.6}$$

L_1 is obtained by a simple integration with respect to the velocity $\dot{q}$:

$$L_1 = \ell(q, \dot{q}) - \dot{q}\frac{\partial F}{\partial q}, \tag{1.7}$$

with F given by equation (1.5).

Another way, even simpler in this case, to obtain L_1 is to lift F to the bulk (a procedure which we shall refer to as *bulkanization* below), and write, using (1.3) and (1.1):

$$\begin{aligned} I_1 &\equiv \int_{t_i}^{t_f} \mathrm{d}t\, L_1(q, \dot{q}) \\ &= \int_{t_i}^{t_f} \mathrm{d}t \left[L - \frac{\mathrm{d}F}{\mathrm{d}t}\right] \\ &= \int_{t_i}^{t_f} \mathrm{d}t \left[\ell(q, \dot{q}) - \dot{q}\frac{\partial F}{\partial q} + \ddot{q}\left(f - \frac{\partial F}{\partial \dot{q}}\right)\right], \end{aligned} \tag{1.8}$$

which yields back (1.7), using (1.5):[5]

1.2. Two routes to the first-order Lagrangian of GR

Let us first recall how the Gibbons–Hawking–York (GHY) boundary term is obtained. Consider, in some coordinate system x^μ labelling the points of a D-dimensional pseudo-Riemannian manifold $\mathcal{M}$ (Greek indices run from 0 to $D-1$; see appendix A for conventions), the GR action

$$I_{\mathrm{GR}} = \int_{\mathcal{M}} \mathrm{d}^D x \sqrt{-g} R\ . \tag{1.9}$$

This action depends linearly on the second derivatives of the field variables $g_{\mu\nu}$, and its variation reads:

$$\delta I_{\mathrm{GR}} = \int_{\mathcal{M}} \mathrm{d}^D x \sqrt{-g}\left(G_{\mu\nu}\delta g^{\mu\nu} + \nabla_\mu V^\mu_{\mathrm{GR}}\right)\ , \tag{1.10}$$

where $G_{\mu\nu}$ is the Einstein tensor. The second term on the rhs of (1.10) is the covariant divergence of the four-vector

$$V^\mu_{\mathrm{GR}} = g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\mu\alpha}\delta\Gamma^\beta_{\alpha\beta}, \tag{1.11}$$

which can be evaluated, using Gauss' theorem, on the $d = D-1$ dimensional boundary $\partial\mathcal{M}$ of $\mathcal{M}$.

[5] It is an exercise to check that the equation of motion derived from L_1 is the same as that derived from L: $\dot{p} - \frac{\partial L_1}{\partial q} = \ddot{q}A - B$, with A and B given in footnote 1. As for the Hamiltonian $H = p\dot{q} - L_1$, it cannot, in general, be written explicitly in terms of q and p unless $p = p(q, \dot{q})$ can be inverted explicitly to give $\dot{q} = \dot{q}(q, p)$. Hence it cannot be shown explicitly that the Hamilton equations yield back the Euler–Lagrange equations derived from L_1.

Let us choose for simplicity a Gaussian coordinate system $x^\mu = \{w, x^i\}$ (Latin indices run from 1 to $d = D - 1$), such that w is constant on $\partial\mathcal{M}$:

$$\mathrm{d}s^2 = \epsilon N(w)^2\,\mathrm{d}w^2 + h_{ij}(w, x^k)\mathrm{d}x^i\,\mathrm{d}x^j, \tag{1.12}$$

with $\epsilon = -1$ if $\partial\mathcal{M}$ is spacelike and $\epsilon = +1$ if it is timelike, where $N(w)$ is a function of w only and h_{ij} are the $d(d+1)/2$ components of the induced metric on $\partial\mathcal{M}$, with extrinsic curvature

$$K_{ij} = \frac{1}{2N}\partial_w h_{ij}. \tag{1.13}$$

From now on Latin indices are lowered and raised with h_{ij} and its inverse h^{ij}. For the gauge-fixed metric (1.12) we have

$$V^w_{\mathrm{GR}} = -\frac{\epsilon}{N}\left(K^{ij}\delta h_{ij} + 2\delta K\right), \tag{1.14}$$

where $K = h^{ij}K_{ij}$, making manifest that the surface term in (1.10) contains variations of the normal derivative of h_{ij} through δK (the latter originates from the components (A.5) of $\delta\Gamma$).

Hence a Dirichlet action principle can be achieved if the GR action is supplemented with the GHY boundary term [3, 4]

$$I_{\mathrm{Dir}}[g] = \int\limits_{\mathcal{M}} \mathrm{d}^D x\sqrt{-g}R + 2\epsilon\int\limits_{\partial\mathcal{M}} \mathrm{d}^d x\sqrt{|h|}K, \tag{1.15}$$

since the variation of this action gives, on-shell (that is, when $G_{\mu\nu} = 0$ in vacuum),

$$\delta I_{\mathrm{Dir}} = \int\limits_{\partial\mathcal{M}} \mathrm{d}^d x\,\pi^{ij}\delta h_{ij}, \tag{1.16}$$

where

$$\pi^{ij} = \epsilon\sqrt{|h|}(Kh^{ij} - K^{ij}), \tag{1.17}$$

and vanishes imposing Dirichlet boundary conditions: $\delta h_{ij}|_{\partial\mathcal{M}} = 0$.

The action principle above can be associated to a first-order bulk functional,

$$I_1 = \int\limits_{\mathcal{M}} \mathrm{d}^D x\,\mathcal{L}_1. \tag{1.18}$$

Indeed, in a Gaussian frame (1.12) which foliates $\mathcal{M}$ with constant-w surfaces Σ_w, $\mathcal{L}_1$ can be obtained by identifying π^{ij} given by equation (1.17) as the canonical momentum density conjugate to h_{ij}, i.e.,

$$\frac{\partial\mathcal{L}_1}{\partial(\partial_w h_{ij})} = \pi^{ij}. \tag{1.19}$$

Integrating π^{ij} with respect to $\partial_w h_{ij} = 2NK_{ij}$ gives

$$\mathcal{L}_1 = N\sqrt{|h|}\left(\epsilon\left(K^2 - K^{ij}K_{ij}\right) + r(h_{ij}, \partial_k h_{ij}, \partial_k\partial_l h_{ij})\right), \tag{1.20}$$

where the integration constant $r(h_{ij}, \partial_k h_{ij}, \partial_k \partial_l h_{ij})$ must identify to the part of the Hilbert Lagrangian which only depends on the intrinsic geometry of the surfaces Σ_w, i.e. $\bar{R}$, where a bar stands for quantities built out of h_{ij} only[6]

$$\mathcal{L}_1 = N\sqrt{|h|}\left(\bar{R} + \epsilon\left(K^2 - K^{ij}K_{ij}\right)\right) = \mathcal{L}_{\mathrm{ADM}}. \tag{1.21}$$

This is the celebrated ADM Lagrangian density [10, 11] written here in Gaussian coordinates.

Let us show now that the same first-order (in the normal derivative) Lagrangian density can be obtained by the bulkanization of the GHY term. Define the closed boundary by the union $\partial\mathcal{M} = \Sigma_{w_\mathrm{i}} \cup \Sigma_{w_f} \cup \mathcal{C}$ of the surfaces $w = w_\mathrm{i}$ and $w = w_\mathrm{f}$ and their complement $\mathcal{C}$, and rewrite the GHY contributions from Σ_{w_i} and Σ_{w_f} in (1.15) as the integral of $2\epsilon\partial_w\left(\sqrt{|h|}K\right)$ over the bulk. Using the Gauss–Codazzi–Mainardi relation (A.13), we then have

$$\sqrt{-g}R + 2\epsilon\partial_w\left(\sqrt{|h|}K\right) = \sqrt{-g}\left[\bar{R} - \epsilon\left(K^2 + K^i_jK^j_i\right)\right] + 2\epsilon\partial_w\left(\sqrt{|h|}\right)K. \tag{1.22}$$

Since moreover $\partial_w\sqrt{|h|} = NK\sqrt{|h|}$, we obtain

$$\sqrt{|h|}R + 2\epsilon\partial_w\left(\sqrt{|h|}K\right) = N\sqrt{|h|}\left(\bar{R} + \epsilon\left(K^2 - K^{ij}K_{ij}\right)\right) = \mathcal{L}_{\mathrm{ADM}}. \tag{1.23}$$

The bulkanized GHY terms on Σ_{w_i} and Σ_{w_f} cancel out with the second normal derivative in equation (1.22) that comes from R^{wi}_{wj}, see (A.9), so that the resulting Lagrangian is of first order. As for the GHY defined on the complement $\mathcal{C}$, it can be discarded for our purposes (but is essential to define the ADM mass [15]).

Finally, the dependence on the $D = d + 1$ extra components of the spacetime metric $g_{\mu\nu}$ can be reinstated using the ADM metric decomposition

$$\mathrm{d}s^2 = \epsilon N^2\,\mathrm{d}w^2 + h_{ij}(\mathrm{d}x^i + N^i\,\mathrm{d}w)(\mathrm{d}x^j + N^j\,\mathrm{d}w), \tag{1.24}$$

where $N(w, x^i)$ is the lapse and $N^i(w, x^j)$ is the shift. The extrinsic curvature is then redefined as

$$K_{ij} = \frac{1}{2N}(\partial_w h_{ij} - \bar{\nabla}_i N_j - \bar{\nabla}_j N_i), \tag{1.25}$$

with $\bar{\nabla}_i$ the covariant derivative associated to h_{ij}.

It can be explicitly checked that variations with respect to N, N^i and h_{ij} of $\mathcal{L}_{\mathrm{ADM}}$ yield respectively the constraints $G^w_w = 0$, $G^i_w = 0$ and the dynamical component $G^i_j = 0$ of the equations of motion written in Gaussian coordinates.

2. The first-order Lagrangian of Lovelock gravity

2.1. Dirichlet principle for Lovelock gravity

As shown by Myers [5], the Dirichlet action for a generic Lovelock theory is given by

$$I_{\mathrm{Dir}} = \sum_{p=0}^{\left[\frac{D-1}{2}\right]} \alpha_p\left(\int_{\mathcal{M}} \mathrm{d}^D x\,\mathcal{L}^{(p)} - \int_{\partial\mathcal{M}} \mathrm{d}^d x\,\beta^{(p)}\right), \tag{2.1}$$

[6] By *intrinsic geometry*, we refer to quantities built out of h_{ij} and its tangential derivatives $\partial_k h_{ij}$ and $\partial_k\partial_l h_{ij}$ only.

where $[(D-1)/2]$ is the integer part of $(D-1)/2$, where[7]

$$\mathcal{L}^{(p)} = \frac{1}{2^p}\sqrt{-g}\delta^{[\mu_1\dots\mu_{2p}]}_{[\nu_1\dots\nu_{2p}]}R^{\nu_1\nu_2}_{\mu_1\mu_2}\dots R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}, \tag{2.2}$$

is of degree p in the curvature, and where

$$\delta^{[\nu_1\dots\nu_{2p}]}_{[\mu_1\dots\mu_{2p}]} \equiv \begin{vmatrix} \delta^{\nu_1}_{\mu_1} & \delta^{\nu_2}_{\mu_1} & \dots & \delta^{\nu_{2p}}_{\mu_1} \\ \delta^{\nu_1}_{\mu_2} & \delta^{\nu_2}_{\mu_2} & & \delta^{\nu_{2p}}_{\mu_2} \\ \vdots & & \ddots & \\ \delta^{\nu_1}_{\mu_{2p}} & \delta^{\nu_2}_{\mu_{2p}} & & \delta^{\nu_{2p}}_{\mu_{2p}} \end{vmatrix}, \tag{2.3}$$

is the generalized Kronecker delta of rank $2p$, which is antisymmetric under exchange of its upper (and lower) indices. In our conventions (see appendix A), the dimension of α_p is $[\text{length}]^{2p-2}$. The corresponding Myers boundary terms are given by [5, 7]

$$\beta^{(p)} = -2\epsilon p\sqrt{|h|}\int_0^1 \mathrm{d}s\, \delta^{[i_1\dots i_{2p-1}]}_{[j_1\dots j_{2p-1}]}K^{j_1}_{i_1}\left(\frac{1}{2}\bar{R}^{j_2j_3}_{i_2i_3} - s^2\epsilon K^{j_2}_{i_2}K^{j_3}_{i_3}\right)\times\cdots$$
$$\cdots\times\left(\frac{1}{2}\bar{R}^{j_{2p-2}j_{2p-1}}_{i_{2p-2}i_{2p-1}} - s^2\epsilon K^{j_{2p-2}}_{i_{2p-2}}K^{j_{2p-1}}_{i_{2p-1}}\right). \tag{2.4}$$

For its rewriting as the covariant derivative of a D-vector, see also [16] or [17] which involve, respectively, the introduction of a background metric or an extra vector field which identifies to the normal n on $\partial\mathcal{M}$. In our conventions we have $\alpha_0 = -2\Lambda$ and $\alpha_1 = 1$.

The variation of equation (2.1) reads

$$\delta I_{\mathrm{Dir}} = \int\limits_{\mathcal{M}} \mathrm{d}^D x\sqrt{-g}\mathcal{E}^{\mu\nu}\delta g_{\mu\nu} + \int\limits_{\partial\mathcal{M}} \mathrm{d}^d x\,\pi^{ij}\delta h_{ij}, \tag{2.5}$$

with

$$\pi^{ij} = \sum_{p=0}^{\left[\frac{D-1}{2}\right]} \alpha_p\pi^{ij}_{(p)}, \tag{2.6}$$

where, from each pth Lovelock density, one obtains

$$\pi^{ij}_{(p)} = p\epsilon\sqrt{|h|}\int_0^1 \mathrm{d}s\, \delta^{[ii_1\dots i_{2p-1}]}_{[kj_1\dots j_{2p-1}]}h^{kj}K^{j_1}_{i_1}\left(\frac{1}{2}\bar{R}^{j_2j_3}_{i_2i_3} - s^2\epsilon K^{j_2}_{i_2}K^{j_3}_{i_3}\right)\times\cdots$$
$$\cdots\times\left(\frac{1}{2}\bar{R}^{j_{2p-2}j_{2p-1}}_{i_{2p-2}i_{2p-1}} - s^2\epsilon K^{j_{2p-2}}_{i_{2p-2}}K^{j_{2p-1}}_{i_{2p-1}}\right). \tag{2.7}$$

As for the Lovelock tensor $\mathcal{E}^{\mu}_{\nu}$, it reads

$$\mathcal{E}^{\mu}_{\nu} = \sum_{p=0}^{\left[\frac{D-1}{2}\right]} \alpha_p\mathcal{E}^{\mu}_{(p)\nu}, \tag{2.8}$$

[7] In even dimensions, the term $p = D/2$ is topological, and it does not contribute to the field equations.

with

$$\mathcal{E}^{\mu}_{(p)\nu} = -\frac{1}{2^{p+1}}\delta^{[\mu\mu_1\ldots\mu_{2p}]}_{[\nu\nu_1\ldots\nu_{2p}]}R^{\nu_1\nu_2}_{\mu_1\mu_2}\ldots R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}. \tag{2.9}$$

Note that in the boundary term of (2.5) we omitted the divergence of a d-vector $\bar{\nabla}_i W^i$ since its integration on the closed boundary $\partial\mathcal{M}$ vanishes (see, e.g., [2]; see also [18] for its explicit expression).

The addition of a topological term in even dimensions cannot induce an associated canonical momentum $\pi^{ij}_{(D/2)}$. This can be seen from the anti-symmetric structure of the indices in the canonical momentum in equation (2.7). In the critical space-time dimension, the canonical momentum is constructed with a Kronecker delta of rank D at the boundary, a fact that makes it identically zero[8].

The action (2.1) yields a Dirichlet variational principle. In other words, the Myers boundary terms are the analogues of the function F, given by (1.5), in the mechanical problem we treated in section 1.1.

2.2. Two routes to the first-order Lagrangian for Lovelock gravity

Integration of π^{ij}. As explicitly worked out above on the example of GR, we can now construct the first-order Lagrangian density by identifying the tensor density (2.7) as the associated canonical momentum:

$$\frac{\partial\mathcal{L}^{(p)}_{\mathrm{ADM}}}{\partial(\partial_w h_{ij})} = \pi^{ij}_{(p)}. \tag{2.10}$$

Substituting $\partial_w h_{ij} = 2NK_{ij}$ above and integrating the canonical momentum as a polynomial of the extrinsic curvature yields the generalization of the ADM Lagrangian density to Lovelock theories, after proper inclusion of the lapse and shift:

$$\begin{aligned}\mathcal{L}^{(p)}_{\mathrm{ADM}} = {} & Nr(h_{ij}, \partial_k h_{ij}, \partial_k\partial_l h_{ij}) + 2p\epsilon N\sqrt{|h|}\int_0^1 \mathrm{d}s(1-s)\delta^{[i_1\ldots i_{2p}]}_{[j_1\ldots j_{2p}]}K^{j_1}_{i_1}\times \\ & \times K^{j_2}_{i_2}\left(\frac{1}{2}\bar{R}^{j_3j_4}_{i_3i_4} - s^2\epsilon K^{j_3}_{i_3}K^{j_4}_{i_4}\right)\times\cdots \\ & \times\left(\frac{1}{2}\bar{R}^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}} - s^2\epsilon K^{j_{2p-1}}_{i_{2p-1}}K^{j_{2p}}_{i_{2p}}\right).\end{aligned} \tag{2.11}$$

where $r(h_{ij}, \partial_k h_{ij}, \partial_k\partial_l h_{ij})$ is a function that does not depend on normal derivatives of the induced metric. In view of the Gauss–Codazzi relations, the only intrinsic quantity coming from a $(d+1)$ decomposition of the Riemann tensor is $\bar{R}^{ij}_{kl}$. In other words, r can only be the pth Lovelock density (2.2) but computed using the induced metric, i.e. $r = \bar{\mathcal{L}}^{(p)}$ with

$$\bar{\mathcal{L}}^{(p)} = \frac{1}{2^p}\sqrt{|h|}\delta^{[i_1\ldots i_{2p}]}_{[j_1\ldots j_{2p}]}\bar{R}^{j_1j_2}_{i_1i_2}\ldots\bar{R}^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}}. \tag{2.12}$$

Bulkanization of the Myers term. When the bulk Lagrangian density $\mathcal{L}^{(p)}$ is re-expressed in the coordinate frame (1.12), a term linear in the acceleration (that is, the normal derivatives

[8] In gravity theories with AdS asymptotics, topological terms do play an essential role in the renormalization of the action and its variation (see, e.g., [19]). The corresponding coupling, however, is not arbitrary, but fixed by the boundary dynamics.

of the extrinsic curvature) arises from R^{wi}_{wj}. On the other hand, lifting $\beta^{(p)}$ to the bulk produces two types of contributions: (i) normal derivatives of the extrinsic curvature, that eliminate the acceleration-dependent part coming from $\mathcal{L}^{(p)}$, (ii) first-order normal derivatives of the induced metric, i.e. powers of the velocity. The latter contain, in particular, a term with an antisymmetric Kronecker delta with an additional pair of indices.

This task is explicitly carried out in appendix B. In doing so, it is useful to employ equation (B.11) to derive the equivalent form of the Dirichlet action (in Gaussian coordinates)

$$\int_{\mathcal{M}} \mathrm{d}^D x \left(\mathcal{L}^{(p)} - \frac{\mathrm{d}}{\mathrm{d}w}\left(\beta^{(p)}\right)\right) = -\int_{\mathcal{M}} \mathrm{d}^D x\, \mathcal{Q}^{(p)} + 2p\epsilon N \int_{\mathcal{M}} \mathrm{d}^D x \sqrt{|h|} \int_0^1 \mathrm{d}s\, \delta^{[i_1\dots i_{2p}]}_{[j_1\dots j_{2p}]} K^{j_1}_{i_1} K^{j_2}_{i_2} \times$$
$$\times \left(\frac{1}{2}\bar{R}^{j_3 j_4}_{i_3 i_4} - s^2 \epsilon K^{j_3}_{i_3} K^{j_4}_{i_4}\right)$$
$$\times \cdots \times \left(\frac{1}{2}\bar{R}^{j_{2p-1} j_{2p}}_{i_{2p-1} i_{2p}} - s^2 \epsilon K^{j_{2p-1}}_{i_{2p-1}} K^{j_{2p}}_{i_{2p}}\right), \tag{2.13}$$

where

$$\mathcal{Q}^{(p)} = -\frac{1}{2^p} N \sqrt{|h|} \delta^{[i_1\dots i_{2p}]}_{[j_1\dots j_{2p}]} R^{j_1 j_2}_{i_1 i_2} \times \cdots \times R^{j_{2p-1} j_{2p}}_{i_{2p-1} i_{2p}}, \tag{2.14}$$

is $-\mathcal{L}^{(p)}$ saturated with intrinsic indices, where R_{ijkl} is understood as a function of $\bar{R}_{ijkl}$ and K_{ij}, see (A.6) (for a different decomposition see [20]). We note that $\mathcal{Q}^{(p)}$ is also proportional to the w-w component of the pth Lovelock tensor $\mathcal{E}^{\mu}_{(p)\nu}$, see (2.9).

Using equation (2.13) and the Gauss–Codazzi relations to express $\mathcal{Q}^{(p)}$ in terms of the intrinsic curvature with the identity

$$(x+y)^p = x^p + 2py \int_0^1 \mathrm{d}s\, s(x + s^2 y)^{p-1}, \tag{2.15}$$

we can rewrite the Dirichlet action (2.1) purely as a functional of h_{ij}, K_{ij} and R_{ijkl} (or $\bar{R}_{ijkl}$) to obtain

$$I_{\mathrm{ADM}}[h, K, \bar{R}] = \int_{\mathcal{M}} \mathrm{d}^D x\, \mathcal{L}_{\mathrm{ADM}} = \int_{\mathcal{M}} \mathrm{d}^D x \sum_{p=0}^{\left[\frac{D-1}{2}\right]} \alpha_p \mathcal{L}^{(p)}_{\mathrm{ADM}}, \tag{2.16}$$

where the pth first-order Lagrangian density $\mathcal{L}^{(p)}_{\mathrm{ADM}}$ can be expressed, once the lapse and shift are reintroduced, as

$$\mathcal{L}^{(p)}_{\mathrm{ADM}} = -\mathcal{Q}^{(p)} + 2p\epsilon N \sqrt{|h|} \int_0^1 \mathrm{d}s\, \delta^{[i_1\dots i_{2p}]}_{[j_1\dots j_{2p}]} K^{j_1}_{i_1} K^{j_2}_{i_2} \left(\frac{1}{2} R^{j_3 j_4}_{i_3 i_4} + \left(1 - s^2\right) \epsilon K^{j_3}_{i_3} K^{j_4}_{i_4}\right)$$
$$\times \cdots \times \left(\frac{1}{2} R^{j_{2p-1} j_{2p}}_{i_{2p-1} i_{2p}} + \left(1 - s^2\right) \epsilon K^{j_{2p-1}}_{i_{2p-1}} K^{j_{2p}}_{i_{2p}}\right), \tag{2.17}$$

$$= N\bar{\mathcal{L}}^{(p)} + 2p\epsilon N \sqrt{|h|} \int_0^1 \mathrm{d}s (1-s) \delta^{[i_1\dots i_{2p}]}_{[j_1\dots j_{2p}]} K^{j_1}_{i_1} K^{j_2}_{i_2}$$
$$\times \left(\frac{1}{2}\bar{R}^{j_3 j_4}_{i_3 i_4} - s^2 \epsilon K^{j_3}_{i_3} K^{j_4}_{i_4}\right) \times \ldots \times \left(\frac{1}{2}\bar{R}^{j_{2p-1} j_{2p}}_{i_{2p-1} i_{2p}} - s^2 \epsilon K^{j_{2p-1}}_{i_{2p-1}} K^{j_{2p}}_{i_{2p}}\right), \tag{2.18}$$

which explicitly eliminates second-order normal derivatives of h_{ij} and where the second equality coincides with (2.11), thus confirming that the intrinsic function r entering it is $\bar{\mathcal{L}}^{(p)}$.

This shows that, just as in the GR case, the Dirichlet action is equivalent to the first-order action when we express all quantities in terms of h_{ij}, K_{ij} and $\bar{R}_{ijkl}$. Thus, $\mathcal{L}_{\mathrm{ADM}} = \sum \alpha_p \mathcal{L}^{(p)}_{\mathrm{ADM}}$ represents the first-order Lagrangian density for a generic Lovelock gravity theory.

In reference [13] the authors obtain the expression

$$\pounds^{(p)} = N\sqrt{|h|}\sum_{i=0}^{p} \tilde{C}_{i(p)}\delta^{[i_1\ldots i_{2p}]}_{[j_1\ldots j_{2p}]}R^{j_1 j_2}_{i_1 i_2}\ldots R^{j_{2i-1}j_{2i}}_{i_{2i-1}i_{2i}}K^{j_{2i+1}}_{j_{2i+1}}\ldots K^{j_{2p}}_{j_{2p}}, \tag{2.19}$$

with coefficients

$$\tilde{C}_{i(p)} = \frac{(-4)^{p-i}}{2i![2(p-i)-1]!!}. \tag{2.20}$$

In order to compare (2.19) to our result $\mathcal{L}^{(p)}_{\mathrm{ADM}}$, we schematically represent $x = R^{ij}_{kl}$ and $y = K^i_j$ in equation (2.17) to obtain

$$\mathcal{L}^{(p)}_{\mathrm{ADM}} = \frac{x^p}{2^p} + 2p\epsilon\int_0^1 \mathrm{d}s y^2\left(\frac{1}{2}x + \left(1-s^2\right)\epsilon y^2\right)^{p-1} = \sum_{i=0}^{p} C_{i(p)}x^i y^{2p-2i}, \tag{2.21}$$

or, equivalently,

$$\mathcal{L}^{(p)}_{\mathrm{ADM}} = N\sqrt{|h|}\sum_{i=0}^{p} C_{i(p)}\delta^{[i_1\ldots i_{2p}]}_{[j_1\ldots j_{2p}]}R^{j_1 j_2}_{i_1 i_2}\ldots R^{j_{2i-1}j_{2i}}_{i_{2i-1}i_{2i}}K^{j_{2i+1}}_{j_{2i+1}}\ldots K^{j_{2p}}_{j_{2p}}, \tag{2.22}$$

where

$$C_{i(p)} = \frac{p!2^{p-2i}\epsilon^{p-i}}{i!(2(p-i)-1)!!}. \tag{2.23}$$

Comparison between $\pounds^{(p)}$ and $\mathcal{L}^{(p)}_{\mathrm{ADM}}$ exhibits agreement up to an overall factor $p!/2^{p-1}$ due to different conventions.

Obtaining the Lovelock first-order Lagrangian densities $\mathcal{L}^{(p)}_{\mathrm{ADM}}$ through two straightforward routes, together with their explicit expressions in terms of K_{ij} and $\bar{R}_{ijkl}$, see (2.18), are the core results of the paper.

The GB action. As an example, consider the GB action supplemented with the Myers boundary term [5–7],

$$I_{\mathrm{Dir}}[g] = \int_{\mathcal{M}} \mathrm{d}^D x\, \mathcal{L}^{(2)} - \int_{\partial\mathcal{M}} \mathrm{d}^d x\, \beta^{(2)}, \tag{2.24}$$

setting $\alpha_2 = 1$ for simplicity, where

$$\mathcal{L}^{(2)} = \sqrt{-g}\left(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2\right) = \sqrt{-g}R^{\mu\nu\rho\sigma}P_{\mu\nu\rho\sigma}, \tag{2.25}$$

is the GB scalar density, and where

$$P^{\mu\nu}_{\rho\sigma} = \frac{1}{4}\delta^{[\mu\nu\alpha_1\alpha_2]}_{[\rho\sigma\beta_1\beta_2]}R^{\beta_1\beta_2}_{\alpha_1\alpha_2} = R^{\mu\nu}_{\rho\sigma} - 2\delta^{\mu}_{[\rho}R^{\nu}_{\sigma]} + 2\delta^{\nu}_{[\rho}R^{\mu}_{\sigma]} + \delta^{\mu}_{[\rho}\delta^{\nu}_{\sigma]}R, \tag{2.26}$$

has the symmetries of the Riemann tensor and is divergenceless ($\nabla_\mu P^{\mu}_{\nu\rho\sigma} = 0$) due to the Bianchi identities. Here brackets denote antisymmetrization, as in $A^{\mu}_{[\rho}B^{\nu}_{\sigma]} = \frac{1}{2}(A^{\mu}_{\rho}B^{\nu}_{\sigma} - A^{\mu}_{\sigma}B^{\nu}_{\rho})$. Finally,

$$\beta^{(2)} = -2\epsilon\sqrt{|h|}\delta^{[i_1i_2i_3]}_{[j_1j_2j_3]}K^{j_1}_{i_1}\left(\bar{R}^{j_2j_3}_{i_2i_3} - \frac{2\epsilon}{3}K^{j_2}_{i_2}K^{j_3}_{i_3}\right) = -4\epsilon\left(J - 2\bar{G}^{i}_{j}K^{j}_{i}\right),$$

$$\text{with} \quad \epsilon J^{i}_{j} = \frac{1}{3}K^{i}_{j}\left(K^{k}_{l}K^{l}_{k} - K^2\right) + \frac{2}{3}KK^{i}_{k}K^{k}_{j} - \frac{2}{3}K^{i}_{k}K^{k}_{l}K^{l}_{j} \quad \text{and} \quad J = J^{k}_{k}. \tag{2.27}$$

This case has been studied in, e.g, references [2, 9, 16] and generalized to Einstein-scalar-GB theories in [21]. In Gaussian coordinates, the variation of (2.24) adopts the form

$$\delta I_{\text{Dir}} = \int_{\mathcal{M}} \mathrm{d}^D x\sqrt{-g}H^{\mu\nu}\delta g_{\mu\nu} + \int_{\partial\mathcal{M}} \mathrm{d}^d x\sqrt{|h|}\pi^{ij}_{(2)}\delta h_{ij}, \tag{2.28}$$

where

$$H^{\mu}_{\nu} = -\frac{1}{8}\delta^{[\mu\mu_1\mu_2\mu_3\mu_4]}_{[\nu\nu_1\nu_2\nu_3\nu_4]}R^{\nu_1\nu_2}_{\mu_1\mu_2}R^{\nu_3\nu_4}_{\mu_3\mu_4}, \tag{2.29}$$

is the Lanczos tensor and where

$$\begin{aligned}\pi^{ij}_{(2)} &= \epsilon\sqrt{|h|}h^{ik}\delta^{[jj_1j_2j_3]}_{[ki_1i_2i_3]}K^{i_1}_{j_1}\left(\bar{R}^{i_2i_3}_{j_2j_3} - \frac{2\epsilon}{3}K^{i_2}_{j_2}K^{i_3}_{j_3}\right)\\ &= 2\epsilon\sqrt{|h|}\,\left(2h^{mj}\bar{P}^{ik}_{ml}K^{l}_{k} - 3J^{ij} + h^{ij}J\right)\right).\end{aligned} \tag{2.30}$$

The tensor density (2.30) is the canonical momentum associated to the first-order action. Hence, solving

$$\frac{\partial\mathcal{L}^{(2)}_{\text{ADM}}}{\partial(\partial_w h_{ij})} = \pi^{ij}_{(2)}, \tag{2.31}$$

we find (after inclusion of the lapse and shift)

$$\begin{aligned}\mathcal{L}^{(2)}_{\text{ADM}} &= N\bar{\mathcal{L}}^{(2)} + N\sqrt{|h|}\delta^{[i_1i_2i_3i_4]}_{[j_1j_2j_3j_4]}\left[\epsilon K^{j_1}_{i_1}K^{j_2}_{i_2}\left(\bar{R}^{j_3j_4}_{i_3i_4} - \frac{\epsilon}{3}K^{j_3}_{i_3}K^{j_4}_{i_4}\right)\right]\\ &= N\bar{\mathcal{L}}^{(2)} + N\sqrt{|h|}\left[4\epsilon\bar{P}^{ij}_{kl}K^{k}_{i}K^{l}_{j} + KJ - 3K^{i}_{j}J^{j}_{i}\right],\end{aligned} \tag{2.32}$$

where the first term is obtained by identifying it to the restriction of the GB Lagrangian density to the surface $w = cst$, that is building it with the intrinsic curvature only:

$$\bar{\mathcal{L}}^{(2)} = \frac{1}{4}\sqrt{|h|}\delta^{[i_1i_2i_3i_4]}_{[j_1j_2j_3j_4]}\bar{R}^{j_1j_2}_{i_1i_2}\bar{R}^{j_3j_4}_{i_3i_4}. \tag{2.33}$$

When $D = 4$ (i.e. $d = 3$), the Lanczos tensor, the momentum and the generalized ADM Lagangian vanish, as evident from their expression in terms of rank-five and rank-four Kronecker deltas, respectively.

On the other hand, in appendix B, the decomposition of $\mathcal{L}^{(2)}$ shows that the same Lagrangian density can be obtained by bulkanization. Using equation (B.9), the Lagrangian density in equation (2.24) can be shown to yield the same result, that is (2.32).

3. Hamiltonian dynamics

In order to define an ordinary Hamiltonian, a first-order Lagrangian density $\mathcal{L}_{\mathrm{ADM}}$ is required. If the induced metric h_{ij} is chosen as the dynamical variable, the Hamiltonian is given by the Legendre transformation

$$H = \int \mathrm{d}^d x \left(\pi^{ij}\partial_w h_{ij} - \mathcal{L}_{\mathrm{ADM}}\right), \tag{3.1}$$

where the canonical momentum π^{ij} is defined as

$$\pi^{ij} \equiv \frac{\partial \mathcal{L}_{\mathrm{ADM}}}{\partial(\partial_w h_{ij})}. \tag{3.2}$$

This functional must be written in terms of h_{ij} and π^{ij}. This is the path chosen by ADM to construct their celebrated Hamiltonian.

The same path can be taken to construct a Hamiltonian from the first-order Lagrangian density of Lovelock gravity found in the previous section. For each pth contribution, the associated Hamiltonian is computed as

$$H^{(p)} = \int \mathrm{d}^d x \left(\pi^{ij}_{(p)}\partial_w h_{ij} - \mathcal{L}^{(p)}_{\mathrm{ADM}}\right). \tag{3.3}$$

From the canonical momentum (2.6), and in Gaussian coordinates, we have

$$\begin{aligned}\pi^{ij}_{(p)}\partial_w h_{ij} &= 2NK^i_j \pi^j_{(p)i} \\ &= 2p\epsilon N\sqrt{|h|}\int_0^1 \mathrm{d}s\, \delta^{[i_1\dots i_{2p}]}_{[j_1\dots j_{2p}]} K^{j_1}_{i_1} K^{j_2}_{i_2}\left(\frac{1}{2}\bar{R}^{j_3 j_4}_{i_3 i_4} - s^2\epsilon K^{j_3}_{i_3}K^{j_4}_{i_4}\right) \\ &\quad \times\cdots\times\left(\frac{1}{2}\bar{R}^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}} - s^2\epsilon K^{j_{2p-1}}_{i_{2p-1}}K^{j_{2p}}_{i_{2p}}\right),\end{aligned} \tag{3.4}$$

which identifies to the last term of the second member of equation (2.13). Therefore the pth Hamiltonian density $\mathscr{H}^{(p)}$ identifies, in the Gaussian gauge (1.12), to the functional $\mathcal{Q}^{(p)}$ (which is proportional to $\mathcal{E}^w_{(p)w}$, as mentioned below (2.14)). The lapse and shift N^i can then be restored using equation (1.25) to find the full Hamiltonian:

$$H = \int \mathrm{d}^d x \left(N\mathscr{H} + N^i\mathscr{H}_i\right), \tag{3.5}$$

where the Hamiltonian constraints take the form

$$\begin{aligned}\mathscr{H} &= \sum_{p=0}^{\left[\frac{D-1}{2}\right]} \alpha_p \mathscr{H}^{(p)}, \\ \mathscr{H}_i &= -2\bar{\nabla}_j \pi^j_i,\end{aligned} \tag{3.6}$$

where $\mathscr{H}^{(p)} = \mathcal{Q}^{(p)}/N$ and π^i_j are given respectively in equations (2.14) and (2.6).

Due to the non linear relation between π^{ij} and K_{ij}, it is not possible in general to write K_{ij} in terms of π^{ij}. Thus, the Hamiltonian above is only given implicitly in terms of the momenta. On the other hand, it is an exercise to check that the components of the Lovelock tensor $\mathcal{E}^\mu_\nu$ defined in (2.8) verify $\mathcal{E}^w_w = \mathscr{H}/2\sqrt{|h|}$ and $\mathcal{E}^w_i = \mathscr{H}_i/2N\sqrt{|h|}$ in Gaussian coordinates, while $\mathcal{E}^i_{(p)j}$ reads

$$\begin{aligned}
\mathcal{E}^i_{(p)j} = &-p\epsilon\int_0^1 \mathrm{d}s\, \delta^{[ii_1\dots i_{2p}]}_{[jj_1\dots j_{2p}]} K^{j_1}_{i_1} K^{j_2}_{i_2}\left(\frac{1}{2}\bar{R}^{j_3 j_4}_{i_3 i_4} - \epsilon s^2 K^{j_3}_{i_3} K^{j_4}_{i_4}\right)\\
&\times\cdots\times\left(\frac{1}{2}\bar{R}^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}} - \epsilon s^2 K^{j_{2p-1}}_{i_{2p-1}} K^{j_{2p}}_{i_{2p}}\right) - p\epsilon\int_0^1 \mathrm{d}s\, \delta^{[ii_2\dots i_{2p}]}_{[j_1\dots j_{2p}]} K^{j_1}_{j} K^{j_2}_{i_2}\\
&\times\left(\frac{1}{2}\bar{R}^{j_3 j_4}_{i_3 i_4} - \epsilon s^2 K^{j_3}_{i_3} K^{j_4}_{j_4}\right)\times\cdots\times\left(\frac{1}{2}\bar{R}^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}} - \epsilon s^2 K^{j_{2p-1}}_{i_{2p-1}} K^{j_{2p}}_{i_{2p}}\right)\\
&-\frac{1}{2^{p+1}}\delta^{[ii_1\dots i_{2p}]}_{[jj_1\dots j_{2p}]} R^{j_1 j_2}_{i_1 i_2}\times\cdots\times R^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}} - \frac{1}{2}\delta^{[ii_1\dots i_{2p-1}]}_{[jj_1\dots j_{2p-1}]}\bar{\nabla}_{i_1}\\
&\times\left(K^{j_2}_{i_2}\bar{\nabla}^{j_1} K^{j_3}_{i_3} R^{j_4 j_5}_{i_4 i_5}\times\cdots\times R^{j_{2p-2}j_{2p-1}}_{i_{2p-2}i_{2p-1}}\right) + \frac{\partial_w\left(\pi^i_j\right)}{N\sqrt{|h|}}
\end{aligned}\tag{3.7}$$

where R_{ijkl} is understood as an implicit function of $\bar{R}_{ijkl}$ and K_{ij}; see equation (B.14) for completeness. Here we gathered terms which are equal to the normal derivative of π^i_j using the tools presented in appendix B (for its explicit expansion in the scalar-GB case, see [21]).

The Lagrangian and Hamiltonian dynamics are equivalent and the correspondence between the field equations is given by

$$\begin{aligned}
&\frac{\delta H}{\delta N} = 0 \Leftrightarrow \mathcal{E}^w_w = 0,\\
&\frac{\delta H}{\delta N^i} = 0 \Leftrightarrow \mathcal{E}^w_i = 0.
\end{aligned}\tag{3.8}$$

In addition, by definition of H we have that

$$\left.\frac{\delta H}{\delta h_{ij}}\right|_{\pi^{ij}} = -\left.\frac{\delta L}{\delta h_{ij}}\right|_{\partial_w h_{ij}} \quad \text{where} \quad L = \int \mathrm{d}^d x\, \mathcal{L}_{\mathrm{ADM}}.\tag{3.9}$$

Hence, it can be checked explicitly using the equation above and (2.18) that

$$\left.\frac{\delta H}{\delta h_{ij}}\right|_{\pi^{ij}} = -\partial_w \pi^{ij} \Leftrightarrow \mathcal{E}^{ij} = 0.\tag{3.10}$$

In the case of GR, we also have that $\frac{\delta H}{\delta \pi^{ij}} = \partial_w h_{ij} \Leftrightarrow K_{ij} = \frac{1}{2N}\partial_w h_{ij}$. This relation cannot be proven in the general Lovelock case, as it requires the invertibility of π^{ij}. However, it does not provide provide extra dynamical information.

The particular case of GB gives

$$\begin{aligned}
H^{(2)} &= -\int \mathrm{d}^d x\, N\bar{\mathcal{L}}^{(2)} + \epsilon\int \mathrm{d}^d x\, N\sqrt{|h|}\,\delta^{[i_1 i_2 i_3 i_4]}_{[j_1 j_2 j_3 j_4]} K^{j_1}_{i_1} K^{j_2}_{i_2}\left(\bar{R}^{j_3 j_4}_{i_3 i_4} - \epsilon K^{j_3}_{i_3} K^{j_4}_{i_4}\right)\\
&= -\int \mathrm{d}^d x\, N\bar{\mathcal{L}}^{(2)} + \int \mathrm{d}^d x\, N\sqrt{|h|}\left(2\epsilon\bar{P}_{ijkl}K^{ik}K^{jl} - \frac{1}{2}K^4 + 3K^2 K^i_j K^j_i\right.\\
&\quad\left. - 4KK^i_j K^j_k K^k_i - \frac{3}{2}K^i_j K^j_i K^k_l K^l_k + 3K^i_j K^j_k K^k_l K^l_i\right),
\end{aligned}\tag{3.11}$$

where in the second line we have just expanded the generalized Kronecker delta.

4. Conclusions

In this paper we investigated the links between the Dirichlet variational principle, and the first-order Lagrangian density and Hamiltonian of Lovelock gravity. Starting from the simple example of a Lagrangian linear in the acceleration in point mechanics, we have identified two methods to compute the associated first-order Lagrangian: integration of the momentum and bulkanization of boundary terms. We then worked out the case of GR to recover the ADM Lagrangian density from the Dirichlet action.

More powerful, however, is the use of the momentum integration and bulkanization methods to obtain the first-order Lagrangian density of Lovelock gravity. Bulkanizing the Myers term explicitly eliminates all second-order normal derivatives in the bulk. In Gaussian coordinates, the resulting Lagrangian density has the form $\mathcal{L}^{(p)}_{\mathrm{ADM}} = \pi^{ij}\partial_w h_{ij} - 2N\sqrt{|h|}\mathcal{E}^w_w$, making manifest the connection with the Hamiltonian formalism. Indeed, a Legendre transformation of the first-order Lagrangian density, directly gives the Hamiltonian density of the system $N\mathscr{H}^{(p)} = 2N\sqrt{|h|}\mathcal{E}^w_w$. In addition, we have that the Lagrangian and Hamiltonian formalisms are equivalent at the level of the dynamics and surface terms. Indeed, the variation of the Hamiltonian action

$$I_H = \int_{\mathcal{M}} \mathrm{d}^D x \, \left(\pi^{ij}\partial_w h_{ij} - \mathcal{L}_{\mathrm{ADM}}\right), \tag{3.12}$$

produces-on-shell-

$$\delta I_H = \int_{\partial\mathcal{M}} \mathrm{d}^d x \, \pi^{ij}\delta h_{ij}. \tag{3.13}$$

This matches the surface term obtained in equation (2.5) from the variation of the first-order Lagrangian. This fact will be employed in future work to define junction conditions for thin shells à la Hamilton for Lovelock gravity.

Our methods should also be useful to generalize the ADM mass formula to Lovelock gravities. In fact, the canonical momentum readily defines a conserved current when contracted with a boundary Killing vector.

For an arbitrary set of couplings in the Lovelock action, some of the components of the metric solution may not be fully determined by the field equations [23]. For instance, the component g_{tt} of any static spherically symmetric ansatz remains arbitrary if the action has non-unique degenerate vacuum. This problem can be avoided by a given choice of the coefficients (e.g., the cases of GR, Chern–Simons, Born–Infeld and pure Lovelock [28–30]). However, the higher curvature terms in the action make the symplectic matrix change the rank for certain backgrounds, generating extra local symmetries and decreasing degrees of freedom in some sectors of the space of solutions [24–27]. This kind of degeneracy in Lovelock gravity also occurs in cosmological solutions [22], where the field equations cannot predict the evolution of the scale factor $a(t)$ because the coefficient of $\ddot{a}(t)$ goes through zero during the evolution. This also renders the Hamiltonian quantization of the system problematic [13].

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Conventions

In this paper we set $16\pi G = c = 1$. Throughout the text g is the determinant of the metric $g_{\mu\nu}$ (with inverse $g^{\mu\nu}$), $R^{\mu}{}_{\nu\rho\sigma} \equiv \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \cdots$ is the Riemann tensor where $\Gamma^{\mu}_{\nu\sigma} \equiv \frac{1}{2} g^{\mu\lambda}(\partial_\nu g_{\sigma\lambda} + \cdots)$ are the Christoffel symbols, $R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu}$ is the Ricci tensor and $R \equiv g^{\mu\nu} R_{\mu\nu}$ is the scalar curvature.

In Gaussian coordinates

$$\mathrm{d}s^2 = \epsilon N^2(w)\mathrm{d}w^2 + h_{ij}\left(w, x^i\right)\mathrm{d}x^i\,\mathrm{d}x^j, \tag{A.1}$$

the non-vanishing components of the Christoffel symbols are $\Gamma^{i}{}_{jk} = \bar{\Gamma}^{i}{}_{jk}(h)$ and

$$\Gamma^{w}_{ij} = -\frac{\epsilon}{2N^2}\partial_w h_{ij}, \qquad \Gamma^{i}_{wj} = \frac{1}{2}h^{ik}\partial_w h_{jk} \qquad \Gamma^{w}_{ww} = \frac{\partial_w N}{N}. \tag{A.2}$$

The normal to a surface Σ_w of constant w is defined as

$$n_\mu = \epsilon N \delta^{w}_{\mu}, \tag{A.3}$$

so that $n_\mu n^\mu = \epsilon$. On the other hand, the extrinsic curvature is defined as

$$K_{ij} = V^{\mu}_{i} V^{\nu}_{j} \nabla_\mu n_\nu,$$

where V^{μ}_{i} are the projectors on the corresponding surface. In Gaussian coordinates $V^{\mu}_{i} = \delta^{\mu}_{i}$ and as a consequence of the normal vector definition (A.3), the extrinsic curvature is given in terms of h_{ij} by

$$K_{ij} = \nabla_i n_j = -\epsilon N \Gamma^{w}_{ij} = \frac{1}{2N}\partial_w h_{ij}. \tag{A.4}$$

Consequently, the Christoffel symbols satisfy

$$\Gamma^{i}_{wj} = N K^{i}_{j}, \qquad \Gamma^{w}_{ij} = -\frac{\epsilon}{N} K_{ij}, \tag{A.5}$$

and the curvature tensors have the form

$$R^{ij}_{kl} = \bar{R}^{ij}_{kl} - \epsilon\left(K^{i}_{k}K^{j}_{l} - K^{i}_{l}K^{j}_{k}\right), \tag{A.6}$$

$$R^{wi}_{jk} = -\frac{\epsilon}{N}\left(\bar{\nabla}_j K^{i}_{k} - \bar{\nabla}_k K^{i}_{j}\right), \tag{A.7}$$

$$R^{ij}_{wk} = -N\left(\bar{\nabla}^i K^j_k - \bar{\nabla}^j K^i_k\right), \tag{A.8}$$

$$R^{wi}_{wj} = -\frac{\epsilon}{N}\partial_w K^i_j - \epsilon K^i_k K^k_j, \tag{A.9}$$

$$R^i_j = \bar{R}^i_j - \epsilon K K^i_j - \frac{\epsilon}{N}\partial_w K^i_j, \tag{A.10}$$

$$R^w_i = -\frac{\epsilon}{N}\bar{\nabla}_j\left(K\delta^j_i - K^j_i\right), \tag{A.11}$$

$$R^w_w = -\frac{\epsilon}{N}\partial_w K - \epsilon K^i_j K^j_i, \tag{A.12}$$

$$R = \bar{R} - \epsilon\left(K^2 + K^i_j K^j_i\right) - \frac{2\epsilon}{N}\partial_w K. \tag{A.13}$$

The equations above are the Gauss–Codazzi–Mainardi relations in tensorial language and in Gaussian coordinates.

Appendix B. Bulkanization of Myers terms

As a warmup exercise, let us consider the integral of $\beta^{(2)}$, see (2.27), on the boundary $\partial\mathcal{M} = \Sigma_{w_\mathrm{i}} \cup \Sigma_{w_\mathrm{f}} \cup \mathcal{C}$, which is the union of the surfaces $w = w_\mathrm{i}$ and $w = w_\mathrm{f}$ and of their complement $\mathcal{C}$. Its bulkanization yields:

$$\int\limits_{\partial\mathcal{M}} \mathrm{d}^d x\, \beta^{(2)} = -2\epsilon \int\limits_{\mathcal{M}} \mathrm{d}^D x\, \partial_w \left[\sqrt{|h|}\delta^{[j_1 j_2 j_3]}_{[i_1 i_2 i_3]} K^{i_1}_{j_1}\left(\bar{R}^{i_2 i_3}_{j_2 j_3} - \frac{2\epsilon}{3}K^{i_2}_{j_2}K^{i_3}_{j_3}\right)\right], \tag{B.1}$$

modulo a contribution on $\mathcal{C}$ which can be discarded for our purposes, see below (1.23).

In order to compute the normal derivatives involved and construct the desired structures, it is useful to rewrite $\partial_w K^{i_1}_{j_1}$ using (A.9) as

$$\partial_w K^{i_1}_{j_1} = -\epsilon N\left(R^{wi_1}_{wj_1} + \epsilon K^{i_1}_l K^l_{j_1}\right), \tag{B.2}$$

and

$$\partial_w\sqrt{|h|} = \frac{1}{2}\sqrt{|h|}h^{ij}\partial_w h_{ij} = NK\sqrt{|h|}, \tag{B.3}$$

where K is the trace of the extrinsic curvature. Since moreover $\partial_w \bar{R}^i_{jkl} = \bar{\nabla}_k(\partial_w\bar{\Gamma}^i_{jl}) - \bar{\nabla}_l(\partial_w\bar{\Gamma}^i_{jk})$ with $\partial_w\bar{\Gamma}^k_{ij} = N\left(\bar{\nabla}_i K^k_j + \bar{\nabla}_j K^k_i - \bar{\nabla}^k K_{ij}\right)$ (which exhibits $\partial_w\bar{\Gamma}^k_{ij}$ as an intrinsic tensor), a short calculation yields

$$\delta^{[j_1 j_2 j_3]}_{[i_1 i_2 i_3]} K^{i_1}_{j_1}\partial_w \bar{R}^{i_2 i_3}_{j_2 j_3} = -2N\delta^{[j_1 j_2 j_3]}_{[i_1 i_2 i_3]} K^{i_1}_{j_1}\left[K^{i_3}_k R^{i_2 k}{}_{j_2 j_3} + 2\epsilon K^{i_3}_k K^{i_2}_{j_2} K^k_{j_3} + 2\bar{\nabla}_{j_2}\bar{\nabla}^{i_2} K^{i_3}_{j_3}\right]. \tag{B.4}$$

Combining the results above, (B.1) can be rewritten as

$$\begin{aligned}\int\limits_{\partial\mathcal{M}} \mathrm{d}^d x\, \beta^{(2)} = -2\epsilon \int\limits_{\mathcal{M}} \mathrm{d}^D x\, N\sqrt{|h|}\delta^{[j_1 j_2 j_3]}_{[i_1 i_2 i_3]} \Big(&-2\, K^{i_1}_{j_1} K^{i_2}_k R^{k i_3}_{j_2 j_3} \\ &- 4\epsilon K^{i_1}_{j_1} K^{i_2}_{j_2} K^{i_3}_k K^k_{j_3} - 4\, K^{i_1}_{j_1}\bar{\nabla}_{j_2}\bar{\nabla}^{i_2} K^{i_3}_{j_3} \\ &- \epsilon\left(R^{wi_1}_{wj_1} + \epsilon K^{i_1}_l K^l_{j_1}\right) R^{i_2 i_3}_{j_2 j_3} + K^{i_1}_{j_1}\left(R^{i_2 i_3}_{j_2 j_3} + \frac{4\epsilon}{3}K^{i_2}_{j_2}K^{i_3}_{j_3}\right)K\Big).\end{aligned} \tag{B.5}$$

At this point, we can use the identities (C.1) and (C.2) to find

$$\int_{\partial\mathcal{M}} \mathrm{d}^d x\, \beta^{(2)} = 2\int_{\mathcal{M}} \mathrm{d}^D x\, N\sqrt{|h|}\delta^{[j_1 j_2 j_3]}_{[i_1 i_2 i_3]} \left(R^{wi_1}_{wj_1} R^{i_2 i_3}_{j_2 j_3} + 4\epsilon K^{i_1}_{j_1} \bar{\nabla}_{j_2} \bar{\nabla}^{i_2} K^{i_3}_{j_3}\right)$$
$$- 2\epsilon \int_{\mathcal{M}} \mathrm{d}^D x\, N\sqrt{|h|}\delta^{[j_1 j_2 j_3 j_4]}_{[i_1 i_2 i_3 i_4]} K^{i_1}_{j_1} K^{i_2}_{j_2} \left(\bar{R}^{i_3 i_4}_{j_3 j_4} - \frac{2\epsilon}{3} K^{i_3}_{j_3} K^{i_4}_{j_4}\right). \quad \text{(B.6)}$$

We notice that $R^{wi_1}_{wj_1}$ in the first term contains normal derivatives of the extrinsic curvature, see (A.9), that will cancel out with those coming from the expanded GB Lagrangian density,

$$\mathcal{L}^{(2)} = 2N\sqrt{|h|}\delta^{[i_1 i_2 i_3]}_{[j_1 j_2 j_3]} \left(R^{wj_1}_{wi_1} R^{j_2 j_3}_{i_2 i_3} + R^{wj_1}_{i_1 i_2} R^{j_2 j_3}_{wi_3}\right)$$
$$+ \frac{1}{4} N\sqrt{|h|}\delta^{[i_1 \dots i_4]}_{[j_1 \dots j_4]} R^{j_1 j_2}_{i_1 i_2} R^{j_3 j_4}_{i_3 i_4}. \quad \text{(B.7)}$$

Using $\delta^{[j_1 j_2 j_3]}_{[i_1 i_2 i_3]} R^{wi_1}_{j_1 j_2} R^{i_2 i_3}_{wj_3} = -4\epsilon \delta^{[j_1 j_2 j_3]}_{[i_1 i_2 i_3]} \bar{\nabla}_{j_2} K^{i_1}_{j_1} \bar{\nabla}^{i_2} K^{i_3}_{j_3}$ and integrating by parts we get

$$\int_{\mathcal{M}} \mathrm{d}^D x\, \mathcal{L}^{(2)} = 2\int_{\mathcal{M}} \mathrm{d}^D x\, N\sqrt{|h|}\delta^{[j_1 j_2 j_3]}_{[i_1 i_2 i_3]} \left(R^{wi_1}_{wj_1} R^{i_2 i_3}_{j_2 j_3} + 4\epsilon \bar{\nabla}^{i_2} K^{i_1}_{j_1} \bar{\nabla}_{j_2} K^{i_3}_{j_3}\right) +$$
$$+ \frac{1}{4}\int_{\mathcal{M}} \mathrm{d}^D x\, N\sqrt{|h|}\delta^{[i_1 \dots i_4]}_{[j_1 \dots j_4]} R^{j_1 j_2}_{i_1 i_2} R^{j_3 j_4}_{i_3 i_4}, \quad \text{(B.8)}$$

where we discarded terms that are total $\bar{\nabla}_i$ derivatives, i.e. terms living on $\mathcal{C}$.

Subtracting (B.8) and (B.6) we finally get

$$\int_{\mathcal{M}} \mathrm{d}^D x \left(\mathcal{L}^{(2)} - \frac{\mathrm{d}}{\mathrm{d}w}\left(\beta^{(2)}\right)\right) = -\int_{\mathcal{M}} \mathrm{d}^D x\, \mathcal{Q}^{(2)} + 2\epsilon \int_{\mathcal{M}} \mathrm{d}^D x\, N\sqrt{|h|}\delta^{[i_1 \dots i_4]}_{[j_1 \dots j_4]} K^{j_1}_{i_1} K^{j_2}_{i_2} \times$$
$$\times \left(\bar{R}^{j_3 j_4}_{i_3 i_4} - \frac{2\epsilon}{3} K^{j_3}_{i_3} K^{j_4}_{i_4}\right). \quad \text{(B.9)}$$

where $\mathcal{Q}^{(2)}$ is obtained by setting $p = 2$ in equation (2.14).

The same bulkanization procedure can be performed for any Lovelock density with its corresponding Myers term. The use of equations (B.2)–(B.4), (C.3) and similar steps to those described above yield

$$\frac{\mathrm{d}}{\mathrm{d}w}\left(\beta^{(p)}\right) = -2p\epsilon N\sqrt{|h|}\int_0^1 \mathrm{d}s\, \delta^{[i_1 \dots i_{2p}]}_{[j_1 \dots j_{2p}]} K^{i_1}_{j_1} K^{i_2}_{j_2} \left(\frac{1}{2}\bar{R}^{i_3 i_4}_{j_3 j_4} - \epsilon s^2 K^{i_3}_{j_3} K^{i_4}_{j_4}\right) \times$$
$$\times \left(\frac{1}{2}\bar{R}^{i_{2p-1} i_{2p}}_{j_{2p-1} j_{2p}} - \epsilon s^2 K^{i_{2p-1}}_{j_{2p-1}} K^{i_{2p}}_{j_{2p}}\right) + \frac{p}{2^{p-2}} \delta^{[i_1 \dots i_{2p-1}]}_{[j_1 \dots j_{2p-1}]}$$
$$\times \left(R^{wj_1}_{wi_2} R^{j_2 j_3}_{i_2 i_3} + (p-1) R^{wj_1}_{i_1 i_2} R^{j_2 j_3}_{wi_3}\right) R^{j_4 j_5}_{i_4 i_5} \times \dots \times R^{j_{2p-2} j_{2p-1}}_{i_{2p-2} i_{2p-1}}.$$

Since the expanded Lagrangian density $\mathcal{L}^{(p)}$ takes the form

$$\mathcal{L}^{(p)} = \frac{p}{2^{p-2}}\sqrt{|h|}\delta^{[i_1\dots i_{2p-1}]}_{[j_1\dots j_{2p-1}]}\left(R^{wj_1}_{wi_1}R^{j_2j_3}_{i_2i_3} + (p-1)R^{wj_1}_{i_1i_2}R^{j_2j_3}_{wi_3}\right)R^{j_4j_5}_{i_4i_5}\times\cdots \times R^{j_{2p-2}j_{2p-1}}_{i_{2p-2}i_{2p-1}} + \frac{1}{2^p}N\sqrt{|h|}\delta^{[i_1\dots i_{2p}]}_{[j_1\dots j_{2p}]}R^{j_1j_2}_{i_1i_2}\times\cdots\times R^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}}, \tag{B.10}$$

we get

$$\int_{\mathcal{M}}\mathrm{d}^Dx\left(\mathcal{L}^{(p)} - \frac{\mathrm{d}}{\mathrm{d}w}\left(\beta^{(p)}\right)\right) = -\int_{\mathcal{M}}\mathrm{d}^Dx\,\mathcal{Q}^{(p)} + 2p\epsilon\int_{\mathcal{M}}\mathrm{d}^Dx\,N\sqrt{|h|} \times\int_0^1\mathrm{d}s\,\delta^{[i_1\dots i_{2p}]}_{[j_1\dots j_{2p}]}K^{j_1}_{i_1}K^{j_2}_{i_2}\times \times\left(\frac{1}{2}\bar{R}^{j_3j_4}_{i_3i_4} - \epsilon s^2K^{j_3}_{i_3}K^{j_4}_{i_4}\right)\times\cdots \times\left(\frac{1}{2}\bar{R}^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}} - \epsilon s^2K^{j_{2p-1}}_{i_{2p-1}}K^{j_{2p}}_{i_{2p}}\right). \tag{B.11}$$

Finally, the same game can be played when projecting the equations of motion $\mathcal{E}^i_j$: we can see that

$$\mathcal{E}^i_{(p)j} = -\frac{1}{2^{p+1}}\delta^{[i\mu_1\dots\mu_{2p}]}_{[j\nu_1\dots\nu_{2p}]}R^{\nu_1\nu_2}_{\mu_1\mu_2}\dots R^{\mu_{2p-1}\mu_{2p}}_{\nu_{2p-1}\nu_{2p}} \tag{B.12}$$

exhibits the same structure as $\mathcal{L}^{(p)}$ except for the extra pair of indices. According to equation (C.4), we will need an extra term when packing the terms in a one-rank-higher delta and get, restoring the lapse and shift,

$$\mathcal{E}^i_{(p)j} = -p\epsilon\int_0^1\mathrm{d}s\,\delta^{[ii_1\dots i_{2p}]}_{[jj_1\dots j_{2p}]}K^{j_1}_{i_1}K^{j_2}_{i_2}\left(\frac{1}{2}\bar{R}^{j_3j_4}_{i_3i_4} - \epsilon s^2K^{j_3}_{i_3}K^{j_4}_{i_4}\right)\times\cdots \times\left(\frac{1}{2}\bar{R}^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}} - \epsilon s^2K^{j_{2p-1}}_{i_{2p-1}}K^{j_{2p}}_{i_{2p}}\right) - p\epsilon\int_0^1\mathrm{d}s\,\delta^{[ii_2\dots i_{2p}]}_{[j_1\dots j_{2p}]}K^{j_1}_{j}K^{j_2}_{i_2} \left(\frac{1}{2}\bar{R}^{j_3j_4}_{i_3i_4} - \epsilon s^2K^{j_3}_{i_3}K^{j_4}_{j_4}\right)\times\cdots\cdots\times\left(\frac{1}{2}\bar{R}^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}} - \epsilon s^2K^{j_{2p-1}}_{i_{2p-1}}K^{j_{2p}}_{i_{2p}}\right) - \frac{1}{2}\delta^{[ii_1\dots i_{2p-1}]}_{[jj_1\dots j_{2p-1}]}\bar{\nabla}_{i_1}\left(K^{j_2}_{i_2}\bar{\nabla}^{j_1}K^{j_3}_{i_3}R^{j_4j_5}_{i_4i_5}\times\cdots\cdots\times R^{j_{2p-2}j_{2p-1}}_{i_{2p-2}i_{2p-1}}\right) - \frac{1}{2^{p+1}}\delta^{[ii_1\dots i_{2p}]}_{[jj_1\dots j_{2p}]}R^{j_1j_2}_{i_1i_2}\times\cdots\times R^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}} + \frac{\partial_w\pi^i_j}{N\sqrt{|h|}}, \tag{B.13}$$

or as a functional of intrinsic quantities as

$$\mathcal{E}^i_{(p)j} = \frac{\partial_w\pi^i_j}{N\sqrt{|h|}} - p\epsilon\int_0^1\mathrm{d}s(1-s)\delta^{[ii_1\dots i_{2p}]}_{[jj_1\dots j_{2p}]}K^{j_1}_{i_1}K^{j_2}_{i_2}\left(\frac{1}{2}\bar{R}^{j_3j_4}_{i_3i_4} - \epsilon s^2K^{j_3}_{i_3}K^{j_4}_{i_4}\right)\times\cdots \dots\times\left(\frac{1}{2}\bar{R}^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}} - \epsilon s^2K^{j_{2p-1}}_{i_{2p-1}}K^{j_{2p}}_{i_{2p}}\right) - p\epsilon\int_0^1\mathrm{d}s\delta^{[ii_2\dots i_{2p}]}_{[j_1\dots j_{2p}]}K^{j_1}_{j}K^{j_2}_{i_2}$$

$$\times\left(\frac{1}{2}\bar{R}^{j_3j_4}_{i_3i_4}-\epsilon s^2K^{j_3}_{i_3}K^{j_4}_{i_4}\right)\times\cdots\times\left(\frac{1}{2}\bar{R}^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}}-\epsilon s^2K^{j_{2p-1}}_{i_{2p-1}}K^{j_{2p}}_{i_{2p}}\right)$$
$$+\bar{\mathcal{E}}^{i}_{(p)j}-\frac{1}{2}\delta^{[ii_1\ldots i_{2p-1}]}_{[jj_1\ldots j_{2p-1}]}\bar{\nabla}_{i_1}\left(K^{j_2}_{i_2}\bar{\nabla}^{j_1}K^{j_3}_{i_3}\left(\bar{R}^{j_4j_5}_{i_4i_5}-2K^{j_4}_{i_4}K^{j_5}_{i_5}\right)\times\cdots\right.$$
$$\left.\ldots\times\left(\bar{R}^{j_{2p-2}j_{2p-1}}_{i_{2p-2}i_{2p-1}}-2K^{j_{2p-2}}_{i_{2p-2}}K^{j_{2p-1}}_{i_{2p-1}}\right)\right). \tag{B.14}$$

Appendix C. Additional identities

We need to relate Kronecker deltas that differ in rank. For a rank-four Kronecker delta, useful identities are

$$\delta^{[j_1\ldots j_4]}_{[i_1\ldots i_4]}K^{i_1}_{j_1}K^{i_2}_{j_2}K^{i_3}_{j_3}K^{i_4}_{j_4}=\delta^{[j_1j_2j_3]}_{[i_1i_2i_3]}\left(KK^{i_1}_{j_1}K^{i_2}_{j_2}K^{i_3}_{j_3}-3K^{i_1}_{j_1}K^{i_2}_{j_2}K^{i_3}_{l}K^{l}_{j_3}\right), \tag{C.1}$$

and

$$\delta^{[j_1\ldots j_4]}_{[i_1\ldots i_4]}K^{i_1}_{j_1}K^{i_2}_{j_2}\bar{R}^{i_3i_4}_{j_3j_4}=\delta^{[j_1j_2j_3]}_{[i_1i_2i_3]}\left(KK^{i_1}_{j_1}\bar{R}^{i_2i_3}_{j_2i_3}-K^{i_1}_{l}K^{l}_{j_1}\bar{R}^{i_2i_3}_{j_2j_3}-2K^{i_1}_{j_1}K^{i_2}_{l}\bar{R}^{li_3}_{j_2j_3}\right). \tag{C.2}$$

Notice that the identity holds for any pair of tensors that share the same symmetries as the extrinsic and intrinsic curvature. The generalization of the relations (C.1) and (C.2) for $2m$ extrinsic curvatures and $n-m$ Riemann tensors is

$$\begin{aligned}&\delta^{[i_1\ldots i_{2n}]}_{[j_1\ldots j_{2n}]}K^{j_1}_{i_1}\ldots K^{j_{2m}}_{i_{2m}}\bar{R}^{j_{2m+1}j_{2m+2}}_{i_{2m+1}i_{2m+2}}\ldots\bar{R}^{j_{2n-1}j_{2n}}_{i_{2n-1}i_{2n}}\\&\quad=\delta^{[i_1\ldots i_{2n-1}]}_{[j_1\ldots j_{2n-1}]}K^{j_1}_{i_1}\ldots K^{j_{2m-2}}_{i_{2m-2}}\bar{R}^{j_{2m-1}j_{2m}}_{i_{2m-1}i_{2m}}\ldots\bar{R}^{j_{2n-5}j_{2n-4}}_{i_{2n-5}i_{2n-4}}\left(KK^{j_{2n-3}}_{i_{2n-3}}\bar{R}^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}}\right.\\&\qquad\left.-(2m-1)K^{j_{2n-3}}_{l}K^{l}_{i_{2n-3}}\bar{R}^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}}-(2m-2j)K^{j_{2n-3}}_{i_{2n-3}}K^{j_{2n-2}}_{l}\bar{R}^{lj_{2n-1}}_{i_{2n-2}i_{2n-1}}\right)\end{aligned} \tag{C.3}$$

where we factored out $2m-2$ extrinsic curvatures and $n-m-1$ Riemann tensors.

In presence of a pair of free indices, we have

$$\begin{aligned}&\delta^{[ii_1\ldots i_{2n}]}_{[jj_1\ldots j_{2n}]}K^{j_1}_{i_1}\ldots K^{j_{2m}}_{i_{2m}}\bar{R}^{j_{2m+1}j_{2m+2}}_{i_{2m+1}i_{2m+2}}\ldots\bar{R}^{j_{2n-1}j_{2n}}_{i_{2n-1}i_{2n}}\\&\quad=\delta^{[ii_1\ldots i_{2n-1}]}_{[jj_1\ldots j_{2n-1}]}K^{j_1}_{i_1}\ldots K^{j_{2m-2}}_{i_{2m-2}}\bar{R}^{j_{2m-1}j_{2m}}_{i_{2m-1}i_{2m}}\ldots\bar{R}^{j_{2n-5}j_{2n-4}}_{i_{2n-5}i_{2n-4}}\left(KK^{j_{2n-3}}_{i_{2n-3}}\bar{R}^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}}\right.\\&\quad\left.-(2m-1)K^{j_{2n-3}}_{l}K^{l}_{i_{2n-3}}\bar{R}^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}}-(2m-2j)K^{j_{2n-3}}_{i_{2n-3}}K^{j_{2n-2}}_{l}\bar{R}^{lj_{2n-1}}_{i_{2n-2}i_{2n-1}}\right)\\&\quad-\delta^{[ii_2\ldots i_{2n}]}_{[j_1j_2\ldots j_{2n}]}K^{j_1}_{j}K^{j_2}_{i_2}\ldots K^{j_{2m}}_{i_{2m}}\bar{R}^{j_{2m+1}j_{2m+2}}_{i_{2m+1}i_{2m+2}}\ldots\bar{R}^{j_{2n-1}j_{2n}}_{i_{2n-1}i_{2n}},\end{aligned} \tag{C.4}$$

that has one extra term-the last one-in comparison to equation (C.3). Notice that we fixed i_1 when taking the trace to lower the degree of the generalized Kronecker symbol.

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