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Hessian of Hausdorff dimension on purely imaginary directions

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Abstract

Bridgeman–Taylor (Math. Ann. **341** (2008), 927–943) and McMullen (Invent. Math. **173** (2008), 365–425) showed that the Weil–Petersson metric on Teichmüller space can be realized by looking at the infinitesimal change of the Hausdorff dimension of certain quasi-Fuchsian deformations. In this article, we give a similar geometric interpretation of the spectral gap pressure metric introduced by Bridgeman–Canary–Labourie–Sambarino (Geom. Dedicata **192** (2018), 57–86) on the Hitchin component for $\mathsf{PSL}_d(\mathbb{R})$. More generally, we investigate the Hessian of the Hausdorff dimension as a function on the space of (1,1,2)-hyperconvex representations, a class introduced in (J. reine angew. Math. **774** (2021), 1–51) which includes small complex deformations of Hitchin representations and of Θ-positive representations. As another application, we prove that the Hessian of the Hausdorff dimension of the limit set at the inclusion $\Gamma \to \mathsf{PO}(n,1) \to \mathsf{PU}(n,1)$ is positive definite when Γ is co-compact in $\mathsf{PO}(n,1)$ (unless $n = 2$ and the deformation is tangent to $\mathfrak{X}(\Gamma, \mathsf{PO}(2,1))$).

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Contents

1 INTRODUCTION

One of the most interesting and well-studied metrics on the Teichmüller space, the parameter space of hyperbolic structures on a closed surface S of genus $g \geqslant 2$, is the Weil–Petersson metric, a non-complete Riemannian metric. A celebrated result by Taylor [17] and McMullen [38] gives a geometric interpretation of this metric in terms of dynamical invariants of quasi-Fuchsian representations.

To describe their result, recall that the holonomy representation realizes the Teichmüller space $\mathcal{T}(S)$ as a connected component of the character variety

$$\mathfrak{X}(\pi_1 S, \mathsf{PSL}_2(\mathbb{R})) := \operatorname{Hom}(\pi_1 S, \mathsf{PSL}_2(\mathbb{R})) // \mathsf{PSL}_2(\mathbb{R}),$$

which, in turn, sits as a totally real submanifold of the complex character variety $\mathfrak{X}(\pi_1 S, \mathsf{PSL}_2(\mathbb{C}))$, endowed with the complex structure J induced by the complex structure of the Lie group $\mathsf{PSL}_2(\mathbb{C})$. A neighborhood of $\mathcal{T}(S)$ in the complex character variety is given by quasi-Fuchsian space $\mathcal{QF}(S)$, the set of conjugacy classes of representations $\rho : \pi_1 S \to \mathsf{PSL}_2(\mathbb{C}) = \operatorname{Isom}_0(\mathbb{H}^3)$ preserving a convex subset of $\mathbb{H}^3$ on which they act cocompactly. Any such ρ is thus a quasi-isometric embedding and admits an injective equivariant boundary map $\xi_\rho : \partial\pi_1 S \to \mathbb{CP}^1$ whose image is a Jordan curve. Given $\rho \in \mathcal{QF}(S)$, we denote by $\mathrm{Hff}(\rho)$ the Hausdorff dimension of this Jordan curve. It is bounded below by 1 and Bowen showed that $\mathrm{Hff}(\rho)$ equals 1 precisely when ρ belongs to the Teichmüller space [10]. The result of Taylor and McMullen realizes the Weil–Petersson metric by looking at the infinitesimal change of the Hausdorff dimension in purely imaginary directions at a representation $\rho \in \mathcal{T}(S) \subset \mathcal{QF}(S)$.

Theorem 1.1 (Taylor [17] McMullen [38]). *For each $\rho \in \mathcal{T}(S)$ and every differentiable curve $(\rho_t)_{t\in(-\varepsilon,\varepsilon)} \subset \mathcal{T}(S)$ with $\rho_0 = \rho$, it holds*

$$\operatorname{Hess}\mathrm{Hff}(J\dot\rho) = \|\dot\rho\|_{WP}.$$

In recent years, convex-cocompactness has been generalized from rank 1 to real-algebraic semisimple Lie groups[†] G of arbitrary rank, via the concept of *Anosov representations* $\rho : \Gamma \to \mathsf{G}_{\mathbb{K}}$, where, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\mathsf{G}_{\mathbb{K}}$ denotes the group of the $\mathbb{K}$-points of G. Specifying a set Θ of simple roots, let $\mathsf{G}_{\mathbb{K}}/\mathsf{P}_\Theta$ be the space of parabolic subgroups of type Θ. Then Θ-Anosov representations are characterized by admitting a continuous, equivariant, transverse boundary map $\xi_\rho^\Theta : \partial\Gamma \to \mathsf{G}_{\mathbb{K}}/\mathsf{P}_\Theta$

[†] (of non-compact type)

with good dynamical properties [6, 27, 29–31, 34]. They form open subsets

$$\mathfrak{X}_\Theta(\Gamma, \mathsf{G}_\mathbb{K}) = \{\rho \in \mathfrak{X}(\Gamma, \mathsf{G}_\mathbb{K}) : \rho \text{ is} \Theta - \text{Anosov}\}$$

of the character variety.

For each $\mathsf{a} \in \Theta$ Canary–Labourie. [14] constructed, using the thermodynamic formalism, an analogue of the Weil–Petersson metric on $\mathfrak{X}_\Theta(\Gamma, \mathsf{G}_\mathbb{K})$, the *spectral radius pressure form* $\mathbf{P}^{\omega_\mathsf{a}}$, where ω_a is the fundamental weight associated to a. We will recall this construction on Section 4.

Of particular interest is the Hitchin component

$$\mathscr{H}(S, \mathsf{G}_\mathbb{R}) \subset \mathfrak{X}(\pi_1 S, \mathsf{G}_\mathbb{R}),$$

introduced by Hitchin when $\mathsf{G}_\mathbb{R}$ is moreover center-free and simple split. When $\mathsf{G}_\mathbb{R} = \mathsf{PSL}_2(\mathbb{R})$, the Hitchin component agrees with Teichmüller space. When $\mathsf{G}_\mathbb{R} = \mathsf{PSL}_2(\mathbb{R})$, the Hitchin component $\mathscr{H}_d(S) \subset \mathfrak{X}(\pi_1 S, \mathsf{PSL}_d(\mathbb{R}))$ can be described as the connected component containing a *Fuchsian representation*, that is, the composition of the holonomy of a hyperbolic structure with the irreducible representation $\pi_1 S \to \mathsf{PSL}_2(\mathbb{R}) \to \mathsf{PSL}_d(\mathbb{R})$. On the $\mathsf{PSL}_d(\mathbb{R})$-Hitchin component, Canary–Labourie [15] defined a different metric, arising from a different pressure form, denoted by $\mathbf{P}^{\mathsf{a}_1}$, to which we will refer here as the *spectral gap pressure form*. They prove that $\mathbf{P}^{\mathsf{a}_1}$ is non-degenerate on $\mathscr{H}_d(S)$ and extends the Weil–Petersson inner product on Teichmüller space, embedded into $\mathscr{H}_d(S)$ as the space of Fuchsian representations.

A corollary of the main result of the paper is a geometric interpretation of this spectral gap pressure form through the Hession of the Hausdorff dimension of appropriate deformations of Anosov representations. This provides a generalization of Theorem 1.1 to the $\mathsf{PSL}_d(\mathbb{R})$-Hitchin components.

To state the result, we denote by Π the set of simple (restricted) roots of $\mathsf{G}_\mathbb{R}$ and consider the Hitchin component $\mathscr{H}(S, \mathsf{G}_\mathbb{R})$ as a subset of $\mathfrak{X}_\Pi(\pi_1 S, \mathsf{G}_\mathbb{C})$, the latter equipped the complex structure J induced by the complex structure of $\mathsf{G}_\mathbb{C}$. For $\mathsf{a} \in \Pi$, denote by

$$\mathrm{Hff}_\mathsf{a}(\rho) = \dim_{\mathrm{Hff}}\left(\xi^\mathsf{a}_\rho(\partial\pi_1 S)\right)$$

the Hausdorff dimension of the (image of the) limit curve $\xi^\mathsf{a}_\rho : \partial\Gamma \to \mathcal{F}_\mathsf{a}(\mathsf{G}_\mathbb{C})$ for a(ny) Riemannian metric on $\mathcal{F}_\mathsf{a}(\mathsf{G}_\mathbb{C})$. It follows from [43] that Hff_a is critical at $\mathscr{H}(S, \mathsf{G}_\mathbb{R})$ and thus its Hessian is well defined.

Corollary A. *For every $v \in \mathsf{T}\mathscr{H}(S, \mathsf{G}_\mathbb{R})$ and every $\mathsf{a} \in \Pi$ one has*

$$\mathit{Hess}\mathit{Hff}_\mathsf{a}(Jv) = \mathbf{P}^\mathsf{a}(v).$$

Moreover, when $\mathsf{G}_\mathbb{R} = \mathsf{PSL}_d(\mathbb{R})$ the Hessian of $\mathit{Hff}_{\mathsf{a}_1} : \mathfrak{X}_\Pi(\pi_1 S, \mathsf{PSL}_d(\mathbb{C})) \to \mathbb{R}$, at a point $\rho \in \mathscr{H}_d(S)$ is strictly positive on every direction except $\mathsf{T}_\rho\mathscr{H}_d(S)$, where it is degenerate. In particular, the Hitchin locus $\mathscr{H}_d(S) \subset \mathfrak{X}_\Pi(\pi_1 S, \mathsf{PSL}_d(\mathbb{C}))$ is an isolated minimum for $\mathit{Hff}_{\mathsf{a}_1}$.

The second statement follows directly from the first, together with the aforementioned non-degeneracy result by Canary–Labourie [15] for the spectral gap pressure form $\mathbf{P}^{\mathsf{a}_1}$.

Corollary A provides a bridge between two different worlds: we can understand the spectral gap pressure metric on the real character variety in terms of the geometry of the limit set of purely imaginary deformations within the complex character variety. It also brings further evidence that even though the spectral radius pressure forms are so far the more prominently studied ones, the spectral gap pressure form is the more geometric one, and shares more similarities to the classical Weil–Petersson metric.

A key ingredient in the proof of Corollary A is the notion of (1,1,2)-hyperconvex representations, studied in [43] (see Theorem 2.7). These are representations $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{C})$ that are Anosov with respect to the first two simple roots and whose boundary maps satisfy an additional transversality condition (see Section 2.2). The main result of [43] then yields that, on the open set

$$\mathfrak{X}^{\pitchfork}_{\{\mathsf{a}_1,\mathsf{a}_2\}}(\Gamma, \mathsf{PGL}_d(\mathbb{C})) = \left\{\rho \in \mathfrak{X}(\Gamma, \mathsf{PGL}_d(\mathbb{C})) \,:\, (1,1,2)\text{-hyperconvex}\right\},$$

the Hausdorff dimension of the limit set $\xi^1_\rho(\partial\Gamma)$ equals the *critical exponent* $h^{\mathsf{a}_1}_\rho$ for the first root (see Section 2.2 for the definition of the critical exponent) and is thus analytic.

Theorem A. *Let Γ be a word hyperbolic group with $\partial\Gamma$ homeomorphic to a circle and let $\rho \in \mathfrak{X}^{\pitchfork}_{\{\mathsf{a}_1,\mathsf{a}_2\}}(\Gamma, \mathsf{PGL}_d(\mathbb{R}))$ be a regular point of the character variety $\mathfrak{X}(\Gamma, \mathsf{PGL}_d(\mathbb{C}))$. Then for every differentiable curve $(\rho_t)_{t\in(-\varepsilon,\varepsilon)} \subset \mathfrak{X}(\Gamma, \mathsf{PGL}_d(\mathbb{R}))$ with $\rho_0 = \rho$, one has*

$$\mathit{HessHff}_{\mathsf{a}_1}(J\dot\rho) = \mathbf{P}^{\mathsf{a}_1}(\dot\rho).$$

Another important class of representations of fundamental groups of surfaces are Θ-positive representations [28], which, as Hitchin representations, constitute connected components of character varieties [5, 26], generalizing Teichmüller space. It was proven in [42, Theorem 10.3] that for any Θ-positive (Anosov) representation $\rho : \Gamma \to \mathsf{SO}_0(p,q)$ and for any $\mathsf{a} \in \Theta$ the representation $\Lambda_{\mathsf{a}}\rho$ is (1,1,2)-hyperconvex. Thus Theorem A applies; note, however, that in these cases, we do not know for which roots a the associated pressure form is non-degenerate.

Corollary B. *Let $\mathcal{P}(\Gamma, \mathsf{SO}_0(p,q))$ denote the set of Θ-positive Anosov representations. Then for every $v \in \mathsf{T}\mathcal{P}(\Gamma, \mathsf{SO}_0(p,q))$ and every $\mathsf{a} \in \{\mathsf{a}_1, \dots, \mathsf{a}_{p-2}\}$, one has*

$$\mathit{HessHff}_{\mathsf{a}}(Jv) = \mathbf{P}^{\mathsf{a}}(v).$$

For the second main result of the paper, we study the Hausdorff dimension of the limit sets of (1,1,2)-hyperconvex representations, without restricting ourselves to virtual surface groups. We generalize a result of Bridgeman [13] and show:

Theorem B. *Let Γ be a word hyperbolic group and let $\rho \in \mathfrak{X}^{\pitchfork}_{\{\mathsf{a}_1,\mathsf{a}_2\}}(\Gamma, \mathsf{PGL}_d(\mathbb{C}))$ be a smooth point. Assume moreover that*

$$\mathit{Hff}_{\mathsf{a}_1} : \mathfrak{X}^{\pitchfork}_{\{\mathsf{a}_1,\mathsf{a}_2\}}(\Gamma, \mathsf{PGL}_d(\mathbb{C})) \to \mathbb{R}_+$$

is critical at ρ. Then $\mathrm{Hess}_\rho\, \mathit{Hff}_{\mathsf{a}_1}$ is positive semidefinite on a subspace of dimension at least half the real dimension of $\mathfrak{X}^{\pitchfork}_{\{\mathsf{a}_1,\mathsf{a}_2\}}(\Gamma, \mathsf{PGL}_d(\mathbb{C}))$. If, furthermore, the pressure form $\mathbf{P}^{\mathsf{a}_1}$ is non-degenerate, then the function $\mathit{Hff}_{\mathsf{a}_1}$ has no local maxima.

This theorem applies in particular to all convex cocompact Kleinian groups in $\mathsf{PGL}_2(\mathbb{C})$, a result not covered by [13]. In general, admitting (1,1,2)-hyperconvex representations is a relatively restrictive assumption on the group Γ, as, for example, it implies that its boundary has dimension smaller than 2. However there are many classes of subgroups of $\mathsf{PGL}_2(\mathbb{C})$ admitting (1,1,2)-hyperconvex representations with Zariski dense image in $\mathsf{PGL}_d(\mathbb{C})$. It is not yet known if the spectral gap pressure form is non-degenerate for such representations, and we hope that Theorem B will encourage research in this direction.

The proof of both Theorems A and B relies on pluriharmonicity of length functions. The mechanism behind both proofs is relatively versatile, and can be applied in many situations. As an example, we use it in Corollary 5.4 to prove that, at a Fuchsian representation, the critical exponent $h^{\omega_1}(\rho)$ relative to the first fundamental weight necessarily increases along purely imaginary deformations (see Section 5.5 for details).

The last result we want to highlight in the introduction is another application of the previous techniques in a rank one situation. Recall that a representation $\rho : \Gamma \to \mathsf{PU}(n,1)$ is convex-cocompact if and only if it is projective Anosov when $\mathsf{PU}(n,1)$ is considered as a subgroup of $\mathsf{PGL}_{n+1}(\mathbb{C})$ through the standard inclusion. Moreover, for $\gamma \in \Gamma$ the real length of the associated closed geodesic is the spectral radius $\omega_1(\lambda(\rho(\gamma)))$.

If $\rho : \Gamma \to \mathsf{PU}(n,1)$ is convex co-compact, then we denote by $\mathrm{Hff}_{\partial\mathbb{CH}^n}(\rho)$ the Hausdorff dimension the limit set of Γ in the visual boundary of the complex-hyperbolic space, with respect to a visual metric. A celebrated result by Sullivan [48] in real-hyperbolic space, and further extended to arbitrary negatively curved manifolds by Yue [51], asserts that if $\rho : \Gamma \to \mathsf{PU}(n,1)$ is convex-co-compact, then

$$\mathrm{Hff}_{\partial\mathbb{CH}^n}(\Gamma) = h^{\omega_1}(\rho).$$

Assume now that Γ is a co-compact lattice of $\mathsf{PSO}(n,1)$ and denote by ι the inclusion

$$\iota : \Gamma \to \mathsf{PSU}(n,1),$$

where the inclusion of $\mathsf{PSO}(n,1) \to \mathsf{PSU}(n,1)$ is given by extending the coefficients. Bourdon [8] proved that ι is a global rigid minima for $\mathrm{Hff}_{\partial\mathbb{CH}^n}$ among convex co-compact representations of Γ in $\mathsf{PU}(n,1)$. In Section 5.4, we prove the following strengthening.

Theorem C. *Assume ι is a regular point of the character variety $\mathfrak{X}(\Gamma, \mathsf{PSU}(n,1))$, then* Hess_ι *$\mathit{Hff}_{\partial\mathbb{CH}^n}$ is positive definite in any direction not tangent to $\mathfrak{X}(\Gamma, \mathsf{PSO}(n,1))$.*

If $n > 2$ then Mostow's classical rigidity result states that ι is an isolated point of $\mathfrak{X}(\Gamma, \mathsf{PSO}(n,1))$ so Theorem C implies that the Hessian of the Hausdorff dimension at ι is positive definite.

Interestingly enough, in the proof of Theorem C we relate the second variation of the Hausdorff dimension of the limit sets of actions of Γ on the complex hyperbolic space $\mathbb{CH}^n$ to the pressure metric at the Fuchsian locus on the space of real convex projective structures on the closed manifold $\mathbb{RH}^n/\Gamma$, thus providing a link between two very different geometries.

Remark 1.2. If ρ is a small deformation of ι in $\mathsf{PSU}(n,1)$ then it is a convex co-compact subgroup of $\mathsf{PSU}(n,1)$ and moreover, by [43, Corollary 8.5], the Hausdorff dimension of the limit set for a *Riemannian metric* on $\partial\mathbb{CH}^n$ also coincides with $h^{\omega_1}(\rho)$, so in Theorem C we can consider $\mathrm{Hff}_{\partial\mathbb{CH}^n}$ either as the Hausdorff dimension for a visual metric or for a Riemannian metric.

Outline of the paper

In Section 2, we discuss the background on Anosov representations needed in the paper: after reviewing the basic definitions we discuss, in Section 2.2, the results of [43] which singled out (1,1,2)-hyperconvex representations. In Section 2.3, we recall the basic facts about higher rank Teichmüller spaces needed to deduce Corollaries A and B from Theorem A; finally in Section 2.4, we discuss an important dynamical viewpoint on Anosov representations: these can be thought of as reparametrizations of the geodesic flow of Γ, and are thus amenable to the thermodynamic formalism. In Section 3, we discuss the thermodynamic formalism needed to define, in Section 4, the pressure forms. We conclude the paper in Section 5 introducing the main technical tool of the paper, pluriharmonicity of dynamical intersection, which is directly used to prove Theorems A–C.

2 | ANOSOV REPRESENTATIONS

In this section, we introduce the necessary background on Anosov representations, and recall how they give rise to reparametrizations of the geodesic flow.

2.1 | Basic notions

We recall the Cartan and the Jordan–Lyapunov projections and the characterization of Anosov representations we are going to use.

Let G be a semisimple real-algebraic Lie group of non-compact type with finite center, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ denote by $\mathsf{G}_{\mathbb{K}}$ the group of the $\mathbb{K}$-points of G. Fix a maximal compact subgroup $\mathsf{K} < \mathsf{G}_{\mathbb{K}}$ with Lie algebra $\mathfrak{k}$. We denote by $\mathsf{E} < \mathfrak{k}^{\perp}$ a Cartan subalgebra, and by $\Delta \subset \mathsf{E}^*$ a choice of simple roots. This corresponds to the choice of a Weyl chamber in E, which we will denote by E^+. In the case $\mathsf{G} = \mathsf{PGL}_d$, we identify E^+ with

$$\left\{(x_1, \dots, x_d) \in \mathbb{R}^d | x_1 \geqslant \dots \geqslant x_d, \sum x_i = 0\right\}.$$

Every element $g \in \mathsf{G}_{\mathbb{K}}$ can be written as a product

$$g = k_1 \exp(\sigma(g)) k_2$$

for $k_1, k_2 \in \mathsf{K}$ and a unique element $\sigma(g) \in \mathsf{E}^+$, the *Cartan projection* of g. If $\mathsf{G}_{\mathbb{R}} = \mathsf{PGL}_d(\mathbb{R})$, the numbers $\sigma_i(g)$ are the logarithms of the square roots of the eigenvalues of the symmetric matrix $g^t g$. If $\mathsf{a} \in \Delta$, then we denote by ω_{a} its associated fundamental weight.

Let $\Theta \subset \Delta$ be a subset of simple roots. We denote by $\mathsf{P}_{\Theta} < \mathsf{G}$ the associated parabolic subgroup, by $\check{\mathsf{P}}_{\Theta}$ the opposite associated parabolic group and by

$$\mathsf{E}_{\Theta} := \bigcap_{\mathsf{a} \in \Delta \setminus \Theta} \ker(\mathsf{a})$$

the Lie algebra of the center of the Levi group $\mathsf{P}_{\Theta} \cap \check{\mathsf{P}}_{\Theta}$. It comes equipped with the unique projection

$$p_{\Theta} : \mathsf{E} \to \mathsf{E}_{\Theta} \tag{1}$$

that is invariant under the subgroup W_Θ of the Weyl group that leaves E_Θ invariant. Finally, let $\mathsf{E}_\Theta^* < \mathsf{E}^*$ be the subspace generated by the fundamental weights associated to elements in Θ

$$\mathsf{E}_\Theta^* := \langle \omega_{\mathsf{a}} | \ \mathsf{a} \in \Theta \rangle = \{\varphi \in \mathsf{E}^* | \ \varphi \circ p_\Theta = \varphi\}.$$

One has that $\mathsf{E}_\Delta = \mathsf{E}$ and P_Δ is a minimal parabolic subgroup.

Let Γ be a finitely generated discrete group and denote by $|\,|$ the word length for a fixed finite symmetric generating set.

Definition 2.1. Let $\Theta \subset \Delta$. A representation $\rho : \Gamma \to \mathsf{G}_{\mathbb{K}}$ is Θ-*Anosov* if there exist positive constants c, μ such that for all $\gamma \in \Gamma$ and $\mathsf{a} \in \Theta$, one has

$$\mathsf{a}(\sigma(\rho(\gamma))) \geqslant \mu|\gamma| - c. \tag{2}$$

A $\{\mathsf{a}_1\}$-Anosov representation $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{K})$ will be called *projective Anosov*.

Note that this is not the original definition given in Labourie and Guichard [27, 34], but a characterization due to Kapovich–Leeb–Porti and Bochi–Potrie [6, 30]. Note also that there is a recent characterization by Kassel–Potrie [31] only in terms of the Jordan–Lyapunov projection (see below for the definition) rather than the Cartan projection.

Anosov representations are quasi-isometric embeddings, thus in particular they are injective and have discrete image. It was proven in [30] (see also [6]) that only word hyperbolic groups admit Anosov representations; we will denote by $\partial\Gamma$ the Gromov boundary of the group Γ.

A key property of Anosov representations is the existence of equivariant boundary maps with good dynamical properties [6, 25, 27, 30, 34]. With our definition, the existence of boundary maps for such representations is a Theorem of [30] and [6]. From now on we will restrict ourselves, without loss of generality, to self-opposite subsets $\Theta \subset \Delta$

Theorem 2.2 (Kapovich–Leeb–Porti [30]). *Let $\rho : \Gamma \to \mathsf{G}_{\mathbb{K}}$ be Θ-Anosov. Then there exist a unique dynamics preserving, continuous, transverse equivariant boundary map*

$$\xi_\rho^\Theta : \partial\Gamma \to \mathsf{G}_{\mathbb{K}}/\mathsf{P}_\Theta.$$

If $\mathsf{G} = \mathsf{PGL}_d$ and $\Theta = \{\mathsf{a}_r\}$, then $\mathsf{G}_{\mathbb{K}}/\mathsf{P}_\Theta = \mathcal{G}_r(\mathbb{K}^d)$ and we write $\xi_\rho^r = \xi_\rho^{\{\mathsf{a}_r\}}$.

It was proven in [27] that it is possible to reduce the study of general $\{\mathsf{a}\}$-Anosov representations to projective Anosov representations. Indeed one can use the following result by Tits, since for the representations Λ_{a} below one has

$$\mathsf{a}_1(\sigma(\Lambda_{\mathsf{a}}(\rho(\gamma)))) = \mathsf{a}(\sigma(\rho(\gamma))).$$

Proposition 2.3 (Tits [49]). *For every $\mathsf{a} \in \Delta$, there exists an irreducible proximal representation $\Lambda_{\mathsf{a}} : \mathsf{G}_{\mathbb{R}} \to \mathsf{PGL}_d(\mathbb{R})$ whose highest restricted weight is $l\omega_{\mathsf{a}}$ for some $l \in \mathbb{N}$.*

Recall that the Jordan decomposition states that every $g \in \mathsf{G}_{\mathbb{K}}$ can be written uniquely as a commuting product $g = g_e g_h g_n$, where g_e is elliptic, g_h is $\mathbb{R}$-split and g_n is unipotent. The *Jordan–Lyapunov* projection $\lambda : \mathsf{G}_{\mathbb{K}} \to \mathsf{E}^+$ is defined by the logarithm of the eigenvalues of g_h with multiplicities and in decreasing order. If $\mathsf{G} = \mathsf{PGL}_d$, this corresponds to the logarithm of the modulus of the roots of the characteristic polynomial of g with multiplicities and in decreasing order, and we denote by

$$\lambda(g) = (\lambda_1(g), \dots, \lambda_d(g)) \in \left\{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_1 \geqslant \dots \geqslant x_d, \sum x_i = 0\right\}$$

its coordinates.

We will denote by $\Lambda_\rho \subset \mathsf{E}^+$ the *limit cone* of the subgroup $\rho(\Gamma) < \mathsf{G}_{\mathbb{K}}$. This is the cone given by

$$\Lambda_\rho := \overline{\{\mathbb{R}_+ \cdot \lambda(\rho(\gamma)) \mid \gamma \in \Gamma\}}.$$

It was proven by Benoist [4] that, provided $\rho(\Gamma)$ is Zariski dense, Λ_ρ is convex and has non-empty interior.

For every functional $\varphi \in \mathsf{E}^*$ that is positive on the limit cone Λ_ρ, we denote by $h^\varphi(\rho)$ the *critical exponent* of the Dirichlet series

$$s \mapsto \sum_{\gamma \in \Gamma} e^{-s\varphi(\sigma(\rho(\gamma)))},$$

it can be computed as

$$h^\varphi(\rho) = \inf\left\{s : \sum_{\gamma\in\Gamma} e^{-s\varphi(\sigma(\rho(\gamma)))} < \infty\right\} = \sup\left\{s : \sum_{\gamma\in\Gamma} e^{-s\varphi(\sigma(\rho(\gamma)))} = \infty\right\}.$$

2.2 | Hyperconvex representations

We begin with the following definition from [43]:

Definition. A $\{\mathsf{a}_1, \mathsf{a}_2\}$-Anosov representation $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{K})$ is called (1,1,2)-*hyperconvex* if for every triple of pairwise distinct points $x, y, z \in \partial\Gamma$ one has

$$\left(\xi^1_\rho(x) \oplus \xi^1_\rho(y)\right) \cap \xi^{d-2}_\rho(z) = \{0\}.$$

The following is a direct consequence of the uniqueness of boundary maps:

Lemma 2.4. *The complexification of a real hyperconvex representation is hyperconvex (over $\mathbb{C}$).*

An important property of (1,1,2)-hyperconvex representations, established in [43] is that, for these representations, the Hausdorff dimension of the limit curve for a Riemannian metric on $\mathbb{P}(\mathbb{K}^d)$ computes the critical exponent for the first simple root. If ρ is $\{\mathsf{a}_1\}$-Anosov, the root a_1 is positive on the limit cone (recall equation (2)) and thus its critical exponent is well defined. We then have the following.

Theorem 2.5 [43]. *Let $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{K})$ be (1,1,2)-hyperconvex, then*

$$\dim_{Hff}\left(\xi^1(\partial\Gamma)\right) = h_\rho^{\mathsf{a}_1}.$$

A second important property of (1,1,2)-hyperconvex representations into $\mathsf{PGL}_d(\mathbb{R})$ was established in [43] (and independently in Zimmer–Zhang [52]) is the following: if Γ is such that $\partial\Gamma$ is homeomorphic to a circle, then the image of the boundary map ξ_ρ^1 is a C^1-curve. As a result, we get

Theorem 2.6 [43]. *Let $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ be (1,1,2)-hyperconvex. If $\partial\Gamma$ is homeomorphic to a circle, then* $\dim_{Hff}(\xi^1(\partial\Gamma)) = 1$.

Thus, for fundamental groups of surfaces, the Hausdorff dimension is constant and minimal on the real (1,1,2)-hyperconvex locus.

2.3 | Higher rank Teichmüller Theory

A higher Teichmüller space is a union of connected components of a character variety $\mathfrak{X}(\pi_1 S, \mathsf{G}_\mathbb{R})$ only consisting of Anosov representations.

Historically, the first family of higher Teichmüller spaces are Hitchin components. They arise whenever $\mathsf{G}_\mathbb{R}$ is a center free real-split simple Lie group. In this case, there is a unique *principal* subalgebra $\mathfrak{sl}_2(\mathbb{R}) < \mathfrak{g}_\mathbb{R}$ characterized by the property that the centralizer of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has minimal dimension (Kostant [33]). The *Hitchin component* $\mathscr{H}(S, \mathsf{G}_\mathbb{R}) \subset \mathfrak{X}(\pi_1 S, \mathsf{G}_\mathbb{R})$ is the connected component containing *Fuchsian representations*: the composition of the holonomy of a hyperbolization $\pi_1 S \to \mathsf{PSL}_2(\mathbb{R})$ and the morphism $\mathsf{PSL}_2(\mathbb{R}) \to \mathsf{G}_\mathbb{R}$ induced by the inclusion of the principal subalgebra. Representations in the Hitchin component are Anosov with respect to the minimal parabolic subgroup [34].[†] Furthermore, representations in the Hitchin component are hyperconvex:

Theorem 2.7 [34, 43, 47]. *Let $\mathsf{G}_\mathbb{R}$ be a simple split center-free real group. For every $\rho \in \mathscr{H}(S, \mathsf{G}_\mathbb{R})$ and $\mathsf{a} \in \Pi$, the representation $\Lambda_\mathsf{a}\rho : \pi_1 S \to \mathsf{PSL}(V, \mathbb{R})$ is (1,1,2)-hyperconvex.*

Proof. This was established, for the groups $\mathsf{G}_\mathbb{R} = \mathsf{PSL}_d(\mathbb{R}), \mathsf{PSp}(2n, \mathbb{R}), \mathsf{PSO}(n, n+1)$ or the split form of the exceptional complex Lie group G_2, by Labourie [34], for $\mathsf{G} = \mathsf{SO}(n, n)$ by [43, Theorem 9.9]. The general case follows from [47, Remark 5.14]. □

The second family of higher Teichmüller spaces are spaces of maximal representations in Hermitian Lie groups $\mathsf{G}_\mathbb{R}$ [19]. Our results here do not apply in this setting. Maximal representations are, in general, only Anosov with respect to one root a, which therefore does not belong to the Levi–Anosov subspace. Even though we know that for maximal representations the critical exponent h_ρ^a is constant and equal to 1 ([42, Theorem 1.2]), it is not clear if a spectral gap pressure metric

[†] In fact, Labourie proved this for $\mathsf{G}_\mathbb{R} = \mathsf{PSL}_d(\mathbb{R})$, which implies the result also for symplectic groups and odd-dimensional orthogonal groups. Fock and Goncharov gave a characterization of representations in the Hitchin component as positive representations, from which the Anosov property can be deduced with a little work.

$\mathbf{P}^{\mathsf{a}}$ can be constructed in this case. Moreover, since maximal representations are, in general, not (1,1,2)-hyperconvex, it is not known if, for complex deformations in $\rho : \Gamma \to \mathsf{G}_{\mathbb{C}}$, the critical exponent h^{a}_{ρ} equals the Hausdorff dimension of the limit set.

There are two further families of higher Teichmüller spaces, given by Θ-positive representations as introduced in [28, 50]. Θ-positive representations exist when $\mathsf{G}_{\mathbb{R}}$ is locally isomorphic to $\mathsf{SO}(p, q)$, $p < q$, or when $\mathsf{G}_{\mathbb{R}}$ belongs to a special family of exceptional Lie groups. Guichard, Labourie and Wienhard [26] prove that Θ-positive representations are Θ-Anosov. They also show that set of Θ-positive representations contain unions of connected components [26]. In the case of $\mathsf{SO}(p, q)$, $p < q$, Beyrer [5] show that the set of Θ-positive representations is in fact a union of connected components, coinciding with the components described using methods from the theory of Higgs bundles by Aparicio-Arroyo, Bradlow, Collier, García-Prada, Gothen, and Oliveira [2, 12]. In particular, in the case of $\mathsf{SO}(p, q)$, $p < q$, Θ-positive representations are Anosov with respect to the first $p - 1$ roots. We will use the following result from [42].

Theorem 2.8 [42, Theorem 10.3]. *Let $\rho : \Gamma \to \mathsf{SO}(p, q)$ be a Θ-positive Θ-Anosov representation. For every $\mathsf{a} \in \{\mathsf{a}_1, \dots, \mathsf{a}_{p-2}\}$, the representation $\Lambda_{\mathsf{a}}\rho : \pi_1 S \to \mathsf{PGL}(V, \mathbb{R})$ is (1,1,2)-hyperconvex.*

2.4 | Reparametrizations of geodesic flows

In this section, we describe a very useful dynamical viewpoint on Anosov representations from [46] and Canary–Labourie [14], which makes them amenable to the thermodynamic formalism: Any Anosov representation gives rise to a reparametrization of the geodesic flow.

Given a hyperbolic group Γ, we denote by $\mathsf{U}\Gamma$ the Gromov geodesic flow; this is a metric space endowed with a topologically transitive flow ϕ whose periodic orbits correspond to conjugacy classes in Γ. If Γ admits an Anosov representation, then ϕ is moreover metric Anosov [14]. Note that, if Γ is the fundamental group of a compact negatively curved manifold M, we can choose $\mathsf{U}\Gamma = \mathsf{U}M$; more generally, whenever Γ admits an Anosov representation, its geodesic flow can be explicitly constructed with the aid of the associated boundary maps [14, Theorem 1.10].

If $\alpha > 0$, we denote by $\mathrm{Hol}_{\alpha}(\mathsf{U}\Gamma, \mathbb{R})$ the space of α-Hölder continuous functions on $\mathsf{U}\Gamma$ and by $\mathrm{Hol}(\mathsf{U}\Gamma, \mathbb{R})$ the space of all Hölder continuous functions. If $f \in \mathrm{Hol}(\mathsf{U}\Gamma, \mathbb{R})$ and $a \in [\Gamma]$ is a conjugacy class, then we define the f-period of a by

$$\ell_a(f) = \int_0^{\ell(a)} f(\phi_t x)dt$$

where $x \in a$. If $f \in \mathrm{Hol}_{\alpha}(\mathsf{U}\Gamma, \mathbb{R}_+)$, we obtain a new flow ϕ^f on $\mathsf{U}\Gamma$ called the *reparametrization* of ϕ by f. The flow ϕ^f is given by the formula

$$\phi_t(x) = \phi^f_{k_f(x,t)}(x), \tag{3}$$

where $k_f(x, t) = \int_0^t f(\phi_s x)ds$ for all $x \in X$ and $t \in \mathbb{R}$. The flow ϕ^f is Hölder orbit equivalent to ϕ and if $a \in [\Gamma]$, then $\ell_a(f)$ is the period of a in the flow ϕ^f.

In [14, Section 4], Canary–Labourie associate to any projective Anosov representation a reparametrization of the geodesic flow $\mathsf{U}\Gamma$. They prove the following statement, the second part is proved in [14, Section 6].

Proposition 2.9 [14, § 6]. *Let $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ be a projective Anosov representation. Then there exists a positive Hölder-continuous function $f_\rho^{\lambda_1} : \mathsf{U}\Gamma \to \mathbb{R}_{>0}$ such that for every conjugacy class $[\gamma] \in [\Gamma]$, one has*

$$\ell_\gamma(f_\rho^{\lambda_1}) = \lambda_1(\rho(\gamma)).$$

Moreover, if D is a small disc and $\{\rho_u\}_{u\in D}$ is an analytic family of such representations, then there exists α such that $f_{\rho_u}^{\omega_1}$ can be chosen in $\mathrm{Hol}_\alpha(\mathsf{U}\Gamma, \mathbb{R})$ and the function $u \mapsto f_{\rho_u}^{\lambda_1}$ is analytic.

Proposition 2.9 together with Tits Proposition 2.3 directly give the following from Potrie [41], where $\mathbb{K} = \mathbb{R}$ case is treated. When $\mathbb{K} = \mathbb{C}$, the result follows at once by considering $\mathsf{G}_\mathbb{C}$ as a real group. Recall equation (1) for the definition of $p_\Theta : \mathsf{E} \to \mathsf{E}_\Theta$.

Corollary 2.10 [41, Corollary 4.5]. *Let $\rho : \Gamma \to \mathsf{G}_\mathbb{K}$ be Θ-Anosov, then there exists a positive Hölder-continuous function $f_\rho^\Theta : \mathsf{U}\Gamma \to \mathsf{E}_\Theta$ such that for every conjugacy class $[\gamma] \in [\Gamma]$, one has*

$$\ell_\gamma(f_\rho^\Theta) = p_\Theta(\lambda(\rho(\gamma))).$$

Moreover, if D is a small disc and $\{\rho_u\}_{u\in D}$ is an analytic family of such representations, then there exists α such that $f_{\rho_u}^\Theta$ can be chosen in $\mathrm{Hol}_\alpha(\mathsf{U}\Gamma, \mathbb{R})$ and the function $u \mapsto f_{\rho_u}^\Theta$ is analytic.

Thus, Corollary 2.10 readily implies that if ρ is Θ-Anosov, then for every $\varphi \in (\mathsf{E}_\Theta)^*$ that is strictly positive on $\Lambda_\rho - \{0\}$ there exists a reparametrization of the geodesic flow of Γ whose periods are given by[†]

$$\varphi(\lambda(\rho(\gamma))).$$

namely, if we denote by $f_\rho^\varphi = \varphi(f_\rho^\Theta)$, then one considers the flow $\phi^{f_\rho^\varphi}$. We will need in the following that, in this situation, the critical exponent $h^\varphi(\rho)$ is also the entropy of the flow $\phi^{f_\rho^\varphi}$. This can be found, for example. in Ledrappier [35], [46] and on Glorieux–Monclair–Tholozan [24] for the general version.

Proposition 2.11. *Let $\rho : \Gamma \to \mathsf{G}_\mathbb{K}$ be Θ-Anosov. For each $\varphi \in \mathsf{E}_\Theta^*$ strictly positive on $\Lambda_\rho - \{0\}$, it holds that*

$$h^\varphi(\rho) = \lim_{T\to\infty} \frac{\log \#\left\{\gamma \in [\Gamma] \mid \varphi(\lambda(\rho(\gamma))) < T\right\}}{T}$$

This applies, in particular, to the root a_1 if a representation ρ is (1,1,2)-hyperconvex.

3 | THERMODYNAMIC FORMALISM

We now briefly describe the thermodynamic formalism introduced by Bowen, Ruelle, Parry, Pollicott (among others), and in particular the pressure function on the space of Hölder observables

[†] Recall that for every $\varphi \in (\mathsf{E}_\Theta)^*$, one has $\varphi \circ p_\Theta = \varphi$.

on a metric space with a Hölder flow (see [45]). This will then be used, in Section 4, to define various pressure forms $\mathbf{P}^{\varphi}$ on subsets of the representation variety $\mathfrak{X}(\Gamma, \mathsf{PGL}_d(\mathbb{R}))$ by assigning to each representation ρ the Hölder function f_ρ^{φ} on the geodesic flow space $\mathsf{U}\Gamma$ of the group.

For a moment we forget about representations and let X be a compact metric space with a Hölder continuous flow $\phi = \{\phi_t : X \to X\}_{t\in\mathbb{R}}$ without fixed points. As in Section 2.4, we denote by $\mathrm{Hol}_\alpha(X, \mathbb{R})$ the space of α-Hölder continuous functions on X for some $\alpha > 0$ endowed with the complete norm

$$\|f\| = |f|_\infty + \sup_{x\neq y\in X} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}.$$

We denote by O the collection of periodic orbits of the flow ϕ. For $a \in O$ and $f \in \mathrm{Hol}_\alpha(X, \mathbb{R})$, we let $\ell(a)$ be the period of the periodic orbit a,

$$\ell_a(f) = \int_0^{\ell(a)} f(\phi_t x)dt$$

be f*-period* of $a \in O$, and δ_a the associated ϕ-invariant probability measure

$$\delta_a(f) = \frac{\ell_a(f)}{\ell(a)}.$$

Two maps $f, g \in \mathrm{Hol}_\alpha(X, \mathbb{R})$ are called *Livšic cohomologuous* if there exists $U : X \to \mathbb{R}$ such that, for all $x \in X$, then

$$f(x) - g(x) = \left.\frac{\partial}{\partial t}\right|_{t=0} U(\phi_t x).$$

It follows that if f and g are Livšic cohomologous, then $\ell_a(f) = \ell_a(g)$ for all $a \in O$. If $f \in \mathrm{Hol}_\alpha(X, \mathbb{R}_+)$, we denote by ϕ^f the *reparametrization* of ϕ by f, which is the flow on X defined by (3).

We let $\mathcal{M}_\phi$ be the set of ϕ-invariant probability measures on X. For $\mu \in \mathcal{M}_\phi$, we denote by $h(\phi, \mu)$ its metric entropy. Then, for $f \in \mathrm{Hol}_\alpha(X, \mathbb{R})$, the *topological pressure* is

$$P(f) = \sup_{m\in\mathcal{M}_\phi} \left\{ h(\phi, m) + \int_X f dm \right\}.$$

Note that the topological pressure P depends on the flow ϕ, but we will omit this in the notation. The *topological entropy* of a flow is given by $h_{\mathrm{top}}(\phi) = P_\phi(0)$. A measure m_f that attains this supremum is called an *equilibrium state* for f and an equilibrium state for the zero function is called a *measure of maximal entropy*.

We note that $P(f)$ only depends on the Livšic cohomology class of f.

Lemma 3.1 [46, Lemma 2.4]. *Let ϕ be a Hölder continuous flow on a compact metric space X and $f \in \mathrm{Hol}_\alpha(X, \mathbb{R}_+)$. Then*

$$P(-hf) = 0$$

if and only if $h = h_{\mathrm{top}}(\phi^f)$. *Moreover, if* m *is an equilibrium state of* $-h_{\mathrm{top}}(\phi)f$, *then* fm *is a positive multiple of a measure of maximal entropy for the flow* ϕ^f.

We now restrict to *transitive metric Anosov flows*. In the manifold setting, and if the flow is differentiable, a metric Anosov flow ϕ corresponds to a standard Anosov flow where the unit tangent bundle of X has a ϕ-invariant decomposition $T_1(X) = E_- \oplus E_0 \oplus E_+$ where E_- is exponentially contracted under the flow, E_0 is the direction of the flow and E_+ is exponentially contracted under the flow reverse flow of ϕ (see [40] for details). We have the following theorem of Livšic.

Theorem 3.2 (Livšic's Theorem, [37]). *Let* ϕ *be a transitive metric Anosov flow. If* $f \in \mathrm{Hol}_\alpha(X, \mathbb{R})$, *then* $\ell_a(f) = 0$ *for all* $a \in O$ *if and only if* f *is Livšic cohomologous to 0.*

It follows that for metric Anosov flows, the Livšic cohomology class of f is determined by its periods.

Given $f \in \mathrm{Hol}_\alpha(X, \mathbb{R})$, we let

$$R_T^\phi(f) = R_T(f) = \{a \in O \mid \ell_a(f) \leqslant T\}.$$

Then we have the following;

Theorem 3.3 [9], [11], [40]. *Let* ϕ *be a transitive metric Anosov flow and* $f \in \mathrm{Hol}_\alpha(X, \mathbb{R}_+)$ *nowhere vanishing. Then*

$$h(f) = \lim_{T\to\infty} \frac{\log \#R_T(f)}{T} = h_{\mathrm{top}}(\phi^f)$$

is finite and positive. Moreover, for all $g \in \mathrm{Hol}_\alpha(X, \mathbb{R})$ *there exists a unique equilibrium state* m_g *for* g. *The measure of maximal entropy* μ_ϕ *for the flow* ϕ *is*

$$\mu_\phi = \lim_{T\to\infty} \frac{1}{\#R_T(1)} \sum_{a\in R_T(1)} \delta_a.$$

Furthermore for Anosov flows the derivatives of the Pressure function satisfy the following.

Proposition 3.4 [39]. *Let* ϕ *be a transitive metric Anosov flow and* $f, g \in \mathrm{Hol}_\alpha(X, \mathbb{R})$. *Then:*

(i) *the function* $t \to P(f + tg)$ *is analytic;*
(ii) *the first derivative satisfies*

$$\left.\frac{\partial P(f+tg)}{\partial t}\right|_{t=0} = \int g\,dm_f,$$

where m_f *is the equilibrium state for* f;
(iii) *if* $\int g\,dm_f = 0$ *(mean-zero), then*

$$\left.\frac{\partial^2 P(f+tg)}{\partial t^2}\right|_{t=0} = \lim_{T\to\infty} \int \left(\int_0^T g(\phi_s(x))ds\right)^2 dm_f(x) = \mathrm{Var}(g, m_f),$$

(iv) *if* $\mathrm{Var}(g, m_f) = 0$, *then* g *is Livšic cohomologous to zero.*

Using the above, in [38] McMullen defined the Pressure semi-norm as follows. We let $\mathcal{P}(X)$ be the space of pressure zero functions, that is,

$$\mathcal{P}(X) = \{F \in \mathrm{Hol}(X, \mathbb{R}) \mid P(F) = 0\}.$$

Then by Proposition 3.4(ii), the tangent space to $\mathcal{P}(X)$ at F can be identified with

$$\mathsf{T}_F(\mathcal{P}(X)) = \left\{ g \in \mathrm{Hol}(X, \mathbb{R}) \mid \int g dm_F = 0 \right\},$$

where m_F is the equilibrium state for F. Then the *pressure semi-norm* of $g \in \mathsf{T}_F(\mathcal{P}(X))$ is

$$\mathbf{P}(g) = -\frac{\mathrm{Var}(g, m_F)}{\int F dm_F}.$$

Observe that $\mathbf{P}$ is a quadratic form, so actually $\sqrt{\mathbf{P}}$ (and not $\mathbf{P}$) is a semi-norm. It follows from Proposition 3.4 (iii) that $\mathbf{P}$ is induced by a bilinear paring, which is positive semi-definite. We call $\mathbf{P}$ the *pressure semi-norm* with a slight abuse of notation to stress the fact that the associated bilinear form is positive semi-definite.

By Proposition 3.4, it follows that $\mathbf{P}(g)$ only depends on the Livšic-cohomology class $[g]$ and is positive definite in the sense that it is zero if and only if $[g] = 0$. Therefore, it can be considered as a (positive-definite) metric on the space of Livšic cohomology classes.

The *dynamical intersection* is defined in [14] as follows; if $f, g \in \mathrm{Hol}_\alpha(X, \mathbb{R})$ are positive, then their dynamical intersection is

$$\mathbf{I}(f, g) = \lim_{T \to \infty} \frac{1}{\# R_T(f)} \sum_{a \in R_T(f)} \frac{\ell_a(g)}{\ell_a(f)} = \frac{\int g \mathbf{d}m_{-h_f f}}{\int f \mathbf{d}m_{-h_f f}} = \int \frac{g}{f} d\mu_{\phi^f}. \tag{4}$$

The last equalities follow from [14, Section 3.4]. Similar definitions have been studied in different situations, for example, by Bonahon [7], Burger [18] and Knieper [32].

The *renormalized dynamical intersection* is

$$\mathbf{J}(f, g) = \frac{h(g)}{h(f)} \mathbf{I}(f, g).$$

It follows from [14, Proposition 3.12] that both $g \mapsto \mathbf{I}(f, g)$ and $g \mapsto h(g)$ are analytic functions, in particular $\mathbf{J}$ is twice differentiable.

Proposition 3.5 (Canary–Labourie [14, Proposition 3.8]). *For every pair of positive Hölder-continuous functions f and g, one has $\mathbf{J}(f, g) \geqslant 1$. In particular, $\mathbf{J}(f, \cdot)$ is critical at f which gives*

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \log h(f_t) = \left.\frac{\partial}{\partial t}\right|_{t=0} \mathbf{I}(f, f_t), \tag{5}$$

where $(f_t)_{t \in (-\varepsilon, \varepsilon)}$ is a C^1 curve of positive Hölder-continuous functions with $f_0 = f$.

Then we have:

Theorem 3.6 (Canary–Labourie [14, Proposition 3.11]). *Let ϕ be a transitive metric Anosov flow on a compact metric space X. If $f_t \in \mathrm{Hol}(X, \mathbb{R}_+)$, $t \in (-1,1)$ is a 1-parameter family and $F_t = -h_{f_t} f_t$, then*

$$\left.\frac{\partial^2}{\partial t^2}\right|_{t=0} \mathbf{J}(f_0, f_t) = \mathbf{P}\big(\dot{F}_0\big)$$

The following proposition characterizes degenerate vectors for the second derivative of $\mathbf{J}$.

Proposition 3.7 (Canary–Labourie [14, Lemma 9.3]). *Let $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ be a C^2 curve of positive Hölder-continuous functions. Then $(\partial^2/\partial t^2)|_{t=0}\mathbf{J}(f_0, f_t) = 0$ if and only if for every periodic orbit τ, one has*

$$\left.\frac{\partial}{\partial t}\right|_{t=0} h(f_t)\ell_\tau(f_t) = 0.$$

4 | PRESSURE FORMS

Now we will apply the thermodynamic formalism to representations. For this we make use of the interpretation of a Θ-Anosov representation as a reparametrization of the geodesic flow as explained in Section 2.4.

Given any functional $\varphi \in \mathsf{E}_\Theta^*$ that is positive on the limit cone, one can associate a reparametrization f_ρ^φ of the geodesic flow on Γ. Here we describe in detail two special cases of this construction which play an important role in the paper

4.1 | Spectral radius pressure form

Let ρ, η be two projective Anosov representations (with possibly different target groups). They both give rise to reparametrizations of the geodesic flow $f_\rho^{\omega_1}$ and $f_\eta^{\omega_1}$, where ω_1 is the first fundamental weight.

We define the *spectral radius dynamical intersection* of the two projective-Anosov representations ρ, η to be the dynamical intersection between $f_\rho^{\omega_1}$ and $f_\eta^{\omega_1}$:

$$\mathbf{I}^{\omega_1}(\rho, \eta) = \mathbf{I}(f_\rho^{\omega_1}, f_\eta^{\omega_1}).$$

Analogously we define $\mathbf{J}^{\omega_1}(\rho, \eta)$. Moreover, given a C^1 curve $(\rho_t)_{t\in(-\varepsilon,\varepsilon)}$ of projective Anosov representations the *spectral radius pressure norm*[†] of $\dot{\rho}_0$ is defined by

$$\mathbf{P}_\rho^{\omega_1}(\dot{\rho}_0) = \left.\frac{\partial^2}{\partial t^2}\right|_{t=0} \mathbf{J}^{\omega_1}(\rho_0, \rho_t) \geqslant 0.$$

[†] Again $\mathbf{P}$ is a quadratic form and not a norm.

The spectral radius pressure norm induces a positive semidefinite symmetric bilinear two form at the smooth points of $\{\mathsf{a}_1\}$-Anosov representations. However, positive semi-definiteness is as far as thermodynamics goes, and one needs geometric arguments to establish non-degeneracy. In [14], Canary–Labourie prove non-degeneracy under some mild assumptions, giving

Theorem 4.1 (Canary–Labourie [14, Theorem 1.4]). *Let Γ be word hyperbolic, and $\mathsf{G}_{\mathbb{R}} < \mathsf{PGL}_d(\mathbb{R})$ be reductive. The spectral radius pressure form is an analytic Riemannian metric on the space $C_g(\Gamma, \mathsf{G}_{\mathbb{R}})$ of conjugacy classes of $\mathsf{G}_{\mathbb{R}}$-generic, regular, irreducible, projective Anosov representations.*

Recall that a representation $\rho : \Gamma \to \mathsf{G}_{\mathbb{R}}$ is $\mathsf{G}_{\mathbb{R}}$-generic if its Zariski closure contains elements whose centralizer is a maximal torus in $\mathsf{G}_{\mathbb{R}}$, and it is regular if it is a smooth point of the algebraic variety $\mathrm{Hom}(\Gamma, \mathsf{G}_{\mathbb{R}})$.

4.2 | Spectral gap pressure form

We now consider two $\{\mathsf{a}_1, \mathsf{a}_2\}$-Anosov representations ρ, η (with possibly different target groups). As explained in Section 2.4, they define reparametrizations $f_\rho^{\mathsf{a}_1}$ and $f_\eta^{\mathsf{a}_1}$ of the geodesic flow.

We define the *spectral gap dynamical intersection* of ρ and η to be the dynamical intersection between $f_\rho^{\mathsf{a}_1}$ and $f_\eta^{\mathsf{a}_1}$:

$$\mathbf{I}^{\mathsf{a}_1}(\rho, \eta) = \mathbf{I}(f_\rho^{\mathsf{a}_1}, f_\eta^{\mathsf{a}_1}),$$

and analogously for $\mathbf{J}^{\mathsf{a}_1}(\rho, \eta)$. Given a C^1 curve $(\rho_t)_{t\in(-\varepsilon,\varepsilon)}$ of such $\{\mathsf{a}_1, \mathsf{a}_2\}$-representations the *spectral gap pressure norm* of $\dot\rho_0$ is defined by

$$\mathbf{P}_\rho^{\mathsf{a}_1}(\dot\rho_0) = \frac{\partial^2}{\partial t^2}\bigg|_{t=0} \mathbf{J}^{\mathsf{a}_1}(\rho_0, \rho_t) \geqslant 0.$$

The spectral gap pressure norm induces a semidefinite symmetric bilinear two form on smooth points of $\{\mathsf{a}_1, \mathsf{a}_2\}$-Anosov representations. This looks very similar to the spectral radius pressure norm. It is, however, in general harder to check when the spectral gap pressure form is non-degenerate. As far as the authors know this has, so far, only been established for the Hitchin component in $\mathsf{PSL}_d(\mathbb{R})$:

Theorem 4.2 (Canary–Labourie [15, Theorem 1.6]). *Let $\mathsf{G}_{\mathbb{R}}$ denote either $\mathsf{PSL}_d(\mathbb{R})$, $\mathsf{PSp}(2n, \mathbb{R})$, $\mathsf{PSO}(n, n+1)$ or the split form of the exceptional complex Lie group G_2. Then the spectral gap pressure form is positive definite on the Hitchin component $\mathscr{H}(S, \mathsf{G}_{\mathbb{R}})$.*

4.3 | Vanishing directions

Complex conjugation of matrices is an external automorphism of $\mathsf{PGL}_d(\mathbb{C})$ and thus induces an involution

$$\tau : \mathfrak{X}(\Gamma, \mathsf{PGL}_d(\mathbb{C})) \to \mathfrak{X}(\Gamma, \mathsf{PGL}_d(\mathbb{C}))$$

whose fixed point set contains $\mathfrak{X}(\Gamma, \mathsf{PGL}_d(\mathbb{R}))$. If $\rho \in \mathfrak{X}(\Gamma, \mathsf{PGL}_d(\mathbb{R}))$ is a regular point, then the differential $d_\rho \tau$ splits the tangent space as a sum of *purely imaginary vectors* and the tangent space to the real characters:

$$\mathsf{T}_\rho \mathfrak{X}(\Gamma, \mathsf{PGL}_d(\mathbb{C})) = \ker(d_\rho \tau + \mathrm{id}) \oplus \mathsf{T}_\rho \mathfrak{X}(\Gamma, \mathsf{PGL}_d(\mathbb{R}));$$

the almost complex structure J of $\mathfrak{X}(\Gamma, \mathsf{PGL}_d(\mathbb{C}))$ interchanges this splitting.

With a standard symmetry argument (see, for example, Canary [16, Section 5.8]), we get:

Lemma 4.3. *Let $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ be $\{\mathsf{a}_1\}$-Anosov and let v be a purely imaginary direction at ρ. Then $\mathbf{P}^{\omega_1}(v) = 0$. If ρ is moreover $\{\mathsf{a}_2\}$-Anosov, then $\mathbf{P}^{\mathsf{a}_1}(v) = 0$.*

Proof. Let us prove on the second statement, the first one being analogous. Consider a differentiable curve $(\rho_t)_{t\in(-\varepsilon,\varepsilon)} \subset \mathfrak{X}_{\{\mathsf{a}_1,\mathsf{a}_2\}}(\Gamma, \mathsf{PGL}_d(\mathbb{C}))$ such that $\rho_0 = \rho$, $\dot\rho_0 = v$ and $\tau\rho_t = \rho_{-t}$. For every conjugacy class $[\gamma] \in [\Gamma]$, the functions

$$t \mapsto \ell_\gamma(f^{\mathsf{a}_1}_{\rho_t}) = (\lambda_1 - \lambda_2)(\rho_t(\gamma)) \quad \text{and} \quad t \mapsto h(f^{\mathsf{a}_1}_{\rho_t})$$

are invariant under $t \mapsto -t$ and are thus critical at 0. Consequently, for every conjugacy class, the function $t \mapsto h(f^{\mathsf{a}_1}_{\rho_t})\ell_\gamma(f^{\mathsf{a}_1}_{\rho_t})$ is critical at 0 and hence Proposition 3.7 implies $\mathbf{P}^{\mathsf{a}_1}(v) = 0$. □

5 | PLURIHARMONICITY OF LENGTH FUNCTIONS AND ITS CONSEQUENCES

In this section, we prove the main results stated in the Introduction.

5.1 | Pluriharmonic length functions

Recall that we denote by $\mathfrak{X}_\Theta(\Gamma, \mathsf{G}_\mathbb{C})$ the set of Θ Anosov representations $\Gamma \to \mathsf{G}_\mathbb{C}$ and, for $\rho \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G}_\mathbb{C})$, we denote by $\Lambda_\rho \subset \mathsf{E}^+$ the limit cone of the subgroup $\rho(\Gamma)$.

If $\rho, \eta \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G}_\mathbb{C})$ and $\varphi \in (\mathsf{E}_\Theta)^*$ is strictly positive on $(\Lambda_\rho \cup \Lambda_\eta) - \{0\}$, then one can define their *φ-dynamical intersection* by

$$\mathbf{I}^\varphi(\rho, \eta) = \mathbf{I}(f^\varphi_\rho, f^\varphi_\eta) = \lim_{T\to\infty} \frac{1}{\#R_T(f^\varphi_\rho)} \sum_{[\gamma]\in R_T(f^\varphi_\rho)} \frac{\varphi(\lambda(\eta(\gamma)))}{\varphi(\lambda(\rho(\gamma)))}, \tag{6}$$

where $f^\varphi_\rho = \varphi(f^\Theta_\rho)$ is given by Corollary 2.10.

Recall that a function is *pluriharmonic* if it is locally the real part of a holomorphic function. The argument from Taylor [17, Section 5] applies directly and one has the following result.

Proposition 5.1. *Consider $\rho \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G}_\mathbb{C})$ and $\varphi \in (\mathsf{E}_\Theta)^*$ that is strictly positive in $\Lambda_\rho - \{0\}$. Then the function*

$$\mathbf{I}^\varphi_\rho = \mathbf{I}^\varphi(\rho, \cdot) : \mathfrak{X}_\Theta(\Gamma, \mathsf{G}_\mathbb{C}) \to \mathbb{R}$$

is pluriharmonic when defined (that is, on the open set consisting of representations $\eta \in \mathfrak{X}_\Theta(\Gamma, \mathsf{G}_\mathbb{C})$ such that $\varphi|\Lambda_\eta - \{0\}$ is strictly positive).

Recall from Potrie [41, Corollary 4.9] that the map $\eta \mapsto \mathbb{P}(p_\Theta(\Lambda_\eta))$ is continuous on $\mathfrak{X}_\Theta(\Gamma, \mathsf{G}_\mathbb{K})$, when considering the Hausdorff topology on compact subsets of $\mathbb{P}((\mathsf{E}_\Theta)^*)$. Thus the domain of definition of $\mathbf{I}^\varphi_\rho$ is an open subset of $\mathfrak{X}_\Theta(\Gamma, \mathsf{G}_\mathbb{C})$ that contains, in particular, ρ. The proposition implies then that $\mathbf{I}^\varphi_\rho$ is (defined and) pluriharmonic on a neighborhood of ρ.

Proof. In order to simplify notation, we denote by $\psi = \phi^{f^\varphi_\rho}$ the reparametrized flow and, for a periodic orbit $a \in O$, let δ^ψ_a be the associated ψ-invariant probability measure. A computation gives (as in Abramov [1], see, for example, [46, § 2]), for every continuous $g : \mathsf{U}\Gamma \to \mathbb{R}$,

$$\delta^\psi_a(g) = \frac{1}{\ell_a(f^\varphi_\rho)} \int_0^{\ell_a(f^\varphi_\rho)} g(\psi_s x) ds = \frac{1}{\ell_a(f^\varphi_\rho)} \int_0^{\ell_a} g(\phi_s x) f^\varphi_\rho(\phi_s x) ds.$$

Thus, for $T > 0$ one has $R^\psi_T(1) = R_T(f^\varphi_\rho)$. We consider the ψ-invariant probability measure

$$\mu_T = \frac{1}{\#R^\psi_T(1)} \sum_{a \in R^\psi_T(1)} \delta^\psi_a.$$

Using the last equality in equation (4) together with Bowen's Theorem 3.3 applied to ψ, one has

$$\mathbf{I}^\varphi_\rho(\eta) = \int \frac{f^\varphi_\eta}{f^\varphi_\rho} d\mu_\psi = \lim_{T \to \infty} \mu_T\left(\frac{f^\varphi_\eta}{f^\varphi_\rho}\right). \tag{7}$$

We now justify that the weak-* convergence in equation (7) is uniform when η varies on compact subsets of the domain of definition of $\mathbf{I}^\varphi_\rho$. Indeed, Corollary 2.10 states that the map $\eta \mapsto f^\varphi_\eta \in \mathrm{Hol}_\alpha(\mathsf{U}\Gamma, \mathbb{R})$ is continuous and thus, when η varies on a compact set $K \subset \mathfrak{X}_\Theta(\Gamma, \mathsf{G}_\mathbb{C})$, the family of Hölder functions with fixed exponent $\{f^\varphi_\eta : \eta \in K\}$ is bounded and equicontinuous. Applying then, for example, Rao [44, Theorem 3.1], one has

$$\lim_{T \to \infty} \sup_{\eta \in K} \left| \mu_T\left(\frac{f^\varphi_\eta}{f^\varphi_\rho}\right) - \mathbf{I}^\varphi_\rho(\eta) \right| = 0,$$

giving the desired uniformity.

Since for each $T > 0$, the map

$$\eta \mapsto \mu_T\left(\frac{f^\varphi_\eta}{f^\varphi_\rho}\right) = \frac{1}{\#R_T(f^\varphi_\rho)} \sum_{[\gamma] \in R_T(f^\varphi_\rho)} \frac{\varphi(\lambda(\eta(\gamma)))}{\varphi(\lambda(\rho(\gamma)))}$$

is the real part of a holomorphic function, the result follows from Axler–Bourdon–Ramey [3, Theorem 1.23]. □

5.2 | Proof of Theorem A

Let $\rho \in \mathfrak{X}(\pi_1 S, \mathsf{PGL}_d(\mathbb{R}))$ be (1,1,2)-hyperconvex and assume that it is a regular point of the character variety $\mathfrak{X}(\pi_1 S, \mathsf{PGL}_d(\mathbb{R}))$. Consider a tangent vector $v \in \mathsf{T}_\rho \mathfrak{X}(\pi_1 S, \mathsf{PGL}_d(\mathbb{R}))$. Note that then Jv is a purely imaginary tangent direction in $\mathsf{T}_\rho \mathfrak{X}(\pi_1 S, \mathsf{PGL}_d(\mathbb{C}))$. Thus, Lemma 4.3 implies that for any C^1 curve $(\rho_t)_{t\in(-\varepsilon,\varepsilon)}$ with $\rho_0 = \rho$, $\dot\rho_0 = Jv$ and $\tau\rho_t = \rho_{-t}$ we have

$$0 = \mathbf{P}^{\mathsf{a}_1}(Jv) = \left.\frac{\partial^2}{\partial t^2}\right|_{t=0} \mathbf{J}^{\mathsf{a}_1}(\rho_0, \rho_t). \tag{8}$$

If ρ is (1,1,2)-hyperconvex then Theorem 2.6 states that $h^{\mathsf{a}_1}(\rho) = 1$, moreover, as observed in the proof of Lemma 4.3, $\dot h^{\mathsf{a}_1}(\dot\rho_0) = 0$ and hence by equation (5) $\dot{\mathbf{I}}^{\mathsf{a}_1}_\rho(\dot\rho_0) = 0$. Moreover, from (4) one has $\mathbf{I}^{\mathsf{a}_1}(\rho,\rho) = 1$; so developing the last term of equation (8), one obtains

$$0 = \mathrm{Hess}_\rho(h^{\mathsf{a}_1})(Jv) + \mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho(Jv).$$

Proposition 5.1 states that $\mathbf{I}^{\mathsf{a}_1}_\rho$ is pluriharmonic, so $\mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho(Jv) = -\mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho(v)$ and thus

$$\mathrm{Hess}_\rho h^{\mathsf{a}_1}(Jv) = \mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho(v).$$

Since being (1,1,2)-hyperconvex is an open condition in the character variety [43, Proposition 6.2], Lemma 2.4 implies that, at least for small t, the representation ρ_t is (1,1,2)-hyperconvex (over $\mathbb{C}$) and thus Theorem 2.5 yields $h^{\mathsf{a}_1}(\rho_t) = \mathrm{Hff}_{\mathsf{a}_1}(\rho_t)$. Finally, since $h^{\mathsf{a}_1} \equiv 1$ in a neighborhood of ρ in $\mathfrak{X}(\pi_1 S, \mathsf{PGL}_d(\mathbb{R}))$, one has

$$\mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho(v) = \mathbf{P}^{\mathsf{a}_1}(v).$$

The result follows from Theorem 2.5.

5.3 | Proof of Theorem B

By Theorem 2.5, $\mathrm{Hff}_1 = h^{\mathsf{a}_1}$ in a neighborhood of ρ, and thus by assumption, the latter is critical at ρ. Since $\mathbf{J}^{\mathsf{a}_1}(\rho,\cdot)$ is also critical at ρ (Proposition 3.5), one concludes that $\mathbf{I}^{\mathsf{a}_1}_\rho$ is critical at ρ and thus its Hessian is well defined.

By Proposition 5.1, $\mathbf{I}^{\mathsf{a}_1}_\rho$ is pluriharmonic and thus one has (as before) that for every $v \in \mathsf{T}_\rho \mathfrak{X}(\Gamma, \mathsf{PGL}_d(\mathbb{C}))$

$$\mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho(Jv) = -\mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho(v).$$

One concludes that the $(+,0,-)$ signature of $\mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho$ is of the form $(p, 2k, p)$ for some $p \leq$ half $\dim_{\mathbb{R}} \mathfrak{X}(\Gamma, \mathsf{PGL}_d(\mathbb{C}))$. Moreover, by Theorem 3.6, one has

$$0 \leqslant \mathbf{P}^{\mathsf{a}_1}(Jv) = \mathrm{Hess}_\rho h^{\mathsf{a}_1}(Jv) - h^{\mathsf{a}_1}_\rho \mathrm{Hess}_\rho \mathbf{I}^{\mathsf{a}_1}_\rho(v),$$

so that $\mathrm{Hess}_\rho\, \mathbf{I}^{\mathsf{a}_1}_\rho(v) \geqslant 0$ implies $\mathrm{Hess}_\rho\, h^{\mathsf{a}_1}(Jv) \geqslant 0$. In particular, $\mathrm{Hess}_\rho\, h^{\mathsf{a}_1}$ is positive semidefinite on a subspace of dimension at least

$$\dim_{\mathbb{R}} \mathfrak{X}(\Gamma, \mathsf{PGL}_d(\mathbb{C})) - p \geqslant \frac{1}{2} \dim_{\mathbb{R}} \mathfrak{X}(\Gamma, \mathsf{PGL}_d(\mathbb{C}))$$

and the theorem is proven.

5.4 | Proof of Theorem C

Let Γ be a co-compact lattice in $\mathsf{PSO}(n,1)$ such that the inclusion $\iota : \Gamma \to \mathsf{PSO}(n,1)$ defines, after extending coefficients, a regular point of the character variety $\mathfrak{X}(\Gamma, \mathsf{PSU}(n,1))$. Theorem 2.2 (and the Remark following it) in Cooper–Long–Thistlethwaite [20] assert that ι is then a regular point of the $\mathsf{PSL}_{n+1}(\mathbb{R})$ character variety $\mathfrak{X}(\Gamma, \mathsf{PSL}_{n+1}(\mathbb{R}))$.

Moreover, since $\mathfrak{so}(n,1)$ is the fixed point set of an involution in $\mathfrak{sl}_{n+1}(\mathbb{R})$, one has the decomposition $\mathfrak{sl}_{n+1}(\mathbb{R}) = \mathfrak{so}(n,1) \oplus \mathfrak{s}$ with $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{so}(n,1)$. One readily sees that

$$\mathfrak{su}(n,1) = \mathfrak{so}(n,1) \oplus i\mathfrak{s} \subset \mathfrak{sl}_{n+1}(\mathbb{C}). \tag{9}$$

The twisted cohomology $H^1_\iota(\Gamma, \mathfrak{sl}_{n+1}(\mathbb{R}))$ splits as

$$H^1_\iota\left(\Gamma, \mathfrak{sl}_{n+1}(\mathbb{R})\right) = H^1_\iota(\Gamma, \mathfrak{so}(n,1)) \oplus H^1_\iota(\Gamma, \mathfrak{s}).$$

Consequently, by equation (9) the subspace $H^1_\iota(\Gamma, \mathfrak{s}) \subset H^1_\iota(\Gamma, \mathfrak{sl}_{n+1}(\mathbb{C}))$ is sent bijectively to $H^1_\iota(\Gamma, i\mathfrak{s})$ when multiplied by the complex structure J, that is,

$$J \cdot H^1_\iota(\Gamma, \mathfrak{s}) = H^1_\iota(\Gamma, i\mathfrak{s}). \tag{10}$$

We will need the following generalization of Crampon [21].

Theorem 5.2 (Potrie [41, Theorem 7.2]). *Assume $\rho \in \mathfrak{X}(\Gamma, \mathsf{PSL}_{n+1}(\mathbb{R}))$ has finite kernel and divides a proper open convex set of $\mathbb{P}(\mathbb{R}^{n+1})$. Then the entropy*

$$h^{\omega_1}(\rho) \leqslant n - 1$$

and equality holds only if ρ has values in $\mathsf{PSO}(n,1)$.

This has the following useful consequence.

Corollary 5.3. *The spectral radius pressure form $\mathbf{P}^{\omega_1}$ on $\mathfrak{X}(\Gamma, \mathsf{PSL}_{n+1}(\mathbb{R}))$ is non-degenerate at ι.*

Proof. When $n = 2$, this follows directly from Theorem 4.1, but if $n > 2$, the embedding $\mathfrak{so}(n,1) \subset \mathfrak{sl}_{n+1}(\mathbb{R})$ is not $\mathsf{PSL}_{n+1}(\mathbb{R})$-generic so, even though $\iota(\Gamma)$ is irreducible, we need additional arguments. Nevertheless, by Theorem 5.2, the entropy function $\rho \mapsto h^{\omega_1}(\rho)$ is critical at ι, so by Proposition 3.7 one only needs to verify that the set

$$\left\{d_\iota \omega_1^\gamma : [\gamma] \in [\Gamma]\right\}$$

spans the cotangent space $\mathsf{T}_\iota^* \mathfrak{X}(\Gamma, \mathsf{PSL}_{n+1}(\mathbb{R}))$, where $\omega_1^\gamma : \mathfrak{X}(\Gamma, \mathsf{PGL}_{n+1}(\mathbb{R})) \to \mathbb{R}$ is the function

$$\rho \mapsto \omega_1(\lambda(\rho(\gamma))).$$

As ι is irreducible and projective Anosov, this is the content of Canary–Labourie [14, Proposition 10.1]. $\square$

Consider then $v \in H_\iota^1(\Gamma, \mathfrak{s}) \subset \mathsf{T}_\iota \mathfrak{X}(\Gamma, \mathsf{PSL}_{n+1}(\mathbb{R}))$, by equation (10), the purely imaginary vector $J \cdot v \in \mathsf{T}_\iota \mathfrak{X}(\Gamma, \mathsf{PSL}_{n+1}(\mathbb{C}))$ belongs to $H_\iota^1(\Gamma, \mathsf{PSU}(n, 1))$ and represents thus a non-trivial infinitesimal deformation of ι inside $\mathsf{PSU}(n, 1)$. As in Lemma 4.3, we choose a differentiable curve $(\rho_t)_{t\in(-\varepsilon,\varepsilon)} \subset \mathfrak{X}(\Gamma, \mathsf{PSU}(n, 1))$ with $\rho_0 = \iota$ and $\dot{\rho}_0 = Jv$ and $\tau\rho_t = \rho_{-t}$.

By Lemma 4.3, we have that

$$0 = \mathbf{P}_\iota^{\omega_1}(Jv) = \left.\frac{\partial^2}{\partial t^2}\right|_{t=0} \mathbf{J}^{\omega_1}(\iota, \rho_t). \tag{11}$$

Expanding the second term, and using that both $h^{\omega_1}(\rho_t)$ and $\mathbf{I}_\iota^{\omega_1}(\rho_t)$ are critical at $t = 0$ (as in the proof of Lemma 4.3) and that $\mathbf{I}_\iota^{\omega_1}$ is pluriharmonic, we get

$$0 = \operatorname{Hess}_\iota(h^{\omega_1})(Jv) - (n-1)\operatorname{Hess}_\iota(\mathbf{I}_\iota^{\omega_1})(v).$$

On the other hand

$$\mathbf{P}_\iota^{\omega_1}(v) = \operatorname{Hess}_\iota(h^{\omega_1})(v) + (n-1)\operatorname{Hess}_\iota(\mathbf{I}_\iota^{\omega_1})(v),$$

which in turn gives

$$\operatorname{Hess}_\iota(h^{\omega_1})(Jv) = \mathbf{P}_\iota^{\omega_1}(v) - \operatorname{Hess}_\iota(h^{\omega_1})(v) > 0,$$

since $\mathbf{P}_\iota^{\omega_1}(v) > 0$ by Corollary 5.3, and $-\operatorname{Hess}_\iota(h^{\omega_1})(v) \geqslant 0$ since by Theorem 5.2 ι is a global maxima of h^{ω_1} among deformations in $\mathsf{PSL}_{n+1}(\mathbb{R})$. The result then follows.

5.5 | The Hessian of the entropy at the Fuchsian locus of the Hitchin component

Applying the same techniques as in the last section, we can also show the following result on the Hitchin component.

Corollary 5.4. *Let $\iota \in \mathscr{H}_d(S)$ be a representation $\pi_1 S \to \mathsf{PSL}_2(\mathbb{R}) \to \mathsf{PSL}_d(\mathbb{R})$ in the embedded Teichhmüller space. Then* $\operatorname{Hess}(h_\iota^{\omega_1})$ *is positive definite on purely imaginary directions of* $\mathsf{T}_\iota \mathfrak{X}(\pi_1 S, \mathsf{PSL}_d(\mathbb{C}))$.

Proof. We mimic the last paragraph. In this case, the pressure form $\mathbf{P}^{\omega_1}$ is positive definite on $\mathsf{T}_\iota\mathscr{H}(S, \mathsf{PSL}_d(\mathbb{R}))$ directly by Theorem 4.1. One gets, through the same arguments, that

$$\mathrm{Hess}_\rho(h^{\omega_1})(Jv) = \mathbf{P}^{\omega_1}(v) - \mathrm{Hess}(h^{\omega_1})(v).$$

As we already observed, the first term on the right-hand side is positive by Theorem 4.1, while $\mathrm{Hess}(h^{\omega_1})(v) \leqslant 0$ since, by Potrie [41, Theorem A], Fuchsian representations are maxima for the entropy within the Hitchin locus. The corollary follows. □

We refer the reader to Dey–Kapovich [23] (see also Ledrappier [35] and Link [36]) for an interpretation of the critical exponent $h^{\omega_1}(\rho)$ as the Hausdorff dimension of the limit set with respect to a visual metric, that is, a metric with respect to which the group action is conformal.

Finally, it would be interesting to relate Corollary 5.4, or an analog of it, to the recent work by Dai–Li [22] studying the translation lengths on the symmetric space of $\mathsf{PSL}_d(\mathbb{C})$, when one deforms a Fuchsian representation along its Hitchin fiber.

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REFERENCES

1. L. M. Abramov, *On the entropy of a flow*, Dokl. Akad. Nauk. SSSR **128** (1959), 873–875.
2. M. Aparicio-Arroyo, S. Bradlow, B. Collier, O. García-Prada, P. Gothen, and A. Oliveira, $\mathrm{SO}(p,q)$-*Higgs bundles and higher Teichmüller components*, Invent. Math. **218** (2019), no. 1, 197–299.
3. S. Axler, P. Bourdon, and W. Ramey, *Harmonic function theory*, Grad. Texts Math., vol. 137, Springer, Berlin, 1992.
4. Y. Benoist, *Propriétés asymptotiques des groupes linéaires*, Geom. Funct. Anal. **7** (1997), no. 1, 1–47.
5. J. Beyrer and B. Pozzetti, *Positive surface group representations in* $\mathsf{PO}(p,q)$, https://arxiv.org/pdf/2106.14725.pdf, 2021.
6. J. Bochi, R. Potrie, and A. Sambarino, *Anosov representations and dominated splittings*, J. Eur. Math. Soc. **21** (2019), no. 11, 3343–3414.
7. F. Bonahon, *The geometry of Teichmüller space via geodesic currents*, Invent. Math. **92** (1988), 139–162.
8. M. Bourdon, *Sur le biraport au bord des CAT*(-1)*-espaces*, Publ. Math. Inst. Hautes Études Sci. **83** (1996), 95–104.
9. R. Bowen, *Periodic orbits of hyperbolic flows*, Amer. J. Math. **94** (1972), 1–30.
10. R. Bowen, *Hausdorff dimension of quasi-circles*, Publ. Math. Inst. Hautes Études Sci. **50** (1979), no. 1, 11–25.
11. R. Bowen and D. Ruelle, *The ergodic theory of Axiom A flows*, Invent. Math. **29** (1975), 181–202.

12. S. Bradlow, B. Collier, O. Garcia-Prada, P. Gothen, and A. Oliveira, *A general Cayley correspondence and higher Teichmüller spaces*, Preprint, https://arxiv.org/pdf/2101.09377.pdf, 2021.
13. M. Bridgeman, *Hausdorff dimension and the Weil-Petersson extension to quasifuchsian space*, Geom. Topol. **14** (2010), no. 2, 799–831.
14. M. Bridgeman, R. Canary, F. Labourie, and A. Sambarino, *The pressure metric for Anosov representations*, Geom. Funct. Anal. **25** (2015), no. 4, 1089–1179.
15. M. Bridgeman, R. Canary, F. Labourie, and A. Sambarino, *Simple roots flows for Hitchin representations*, Geom. Dedicata **192** (2018), 57–86.
16. M. Bridgeman, R. Canary, and A. Sambarino, *An introduction to pressure metrics on higher Teichmüller spaces*, Ergod. Theory Dynam. Syst. **38** (2018), no. 6, 2001–2035.
17. M. Bridgeman and E. Taylor, *An extension of the Weil-Petersson metric to quasi-Fuchsian space*, Math. Ann. **341** (2008), 927–943.
18. M. Burger, *Intersection, the Manhattan curve, and Patterson-Sullivan Theory in rank 2*, Inter. Math. Res. Not. **7** (1993).
19. M. Burger, A. Iozzi, and A. Wienhard, *Surface group representations with maximal Toledo invariant*, Ann. of Math. (2) **172** (2010), no. 1, 517–566.
20. D. Cooper, D. D. Long, and M. Thistlethwaite, *Flexing closed hyperbolic manifolds*, Geom. Topol. **11** (2007), 2413–2440.
21. M. Crampon, *Entropies of compact strictly convex projective manifolds*, J. Mod. Dyn. **3** (2009), 511–547.
22. S. Dai and Q. Li, *Domination results in n-Fuchsian fibers in the moduli space of Higgs bundles*, Proc. Lond. Math. Soc. (2022). https://doi.org/10.1112/plms.12431.
23. S. Dey and M. Kapovich, *Patterson-Sullivan theory for Anosov subgroups*, To appear on Trans. Amer. Math. Soc., arXiv:1904.10196, 2019.
24. O. Glorieux, D. Monclair, and N. Tholozan, *Hausdorff dimension of limit sets for projective Anosov representations*, Preprint, https://arxiv.org/pdf/1902.01844.pdf, 2019.
25. F. Guéritaud, O. Guichard, F. Kassel, and A. Wienhard, *Anosov representations and proper actions*, Geom. Topol. **21** (2017), 485–584.
26. O. Guichard, F. Labourie, and A. Wienhard, *Positivity and representations of surface groups*, https://arxiv.org/pdf/2106.14584.pdf, 2021.
27. O. Guichard and A. Wienhard, *Anosov representations: domains of discontinuity and applications*, Invent. Math. **190** (2012), 357–438.
28. O. Guichard and A. Wienhard, *Positivity and higher Teichmüller theory*, 2018, pp. 289–310.
29. M. Kapovich, B. Leeb, and J. Porti, *Anosov subgroups: dynamical and geometric characterizations*, Eur. J. Math. **3** (2017), no. 4, 808–898.
30. M. Kapovich, B. Leeb, and J. Porti, *A Morse Lemma for quasigeodesics in symmetric spaces and Euclidean buildings*, Geom. Topol. **22** (2018), 3827–3923.
31. F. Kassel and R. Potrie, *Eigenvalue gaps for hyperbolic groups and semigroups*, To appear in J. Mod. Dyn., arXiv:2002.07015, 2020.
32. G. Knieper, *Volume growth, entropy and the geodesic stretch*, Math. Res. Lett. **2** (1995), 39–58.
33. B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. **81** (1959), 973–1032.
34. F. Labourie, *Anosov flows, surface groups and curves in projective space*, Invent. Math. **165** (2006), 51–114.
35. F. Ledrappier, *Structure au bord des variétés à courbure négative*, Séminaire de théorie spectrale et géométrie de Grenoble **71** (1994-1995), 97–122.
36. G. Link, *Hausdorff dimension of limit sets of discrete subgroups of higher rank Lie groups*, Geom. Funct. Anal. **14** (2004), 400–432.
37. A.N. Livšic, *Cohomology of dynamical systems*, Math. USSR Izvestija **6** (1972), 1278–1301.
38. C. McMullen, *Thermodynamics, dimension and the Weil-Petersson metric*, Invent. Math. **173** (2008), 365–425.
39. W. Parry and M. Pollicott, *Zeta Functions and the periodic orbit structure of hyperbolic dynamics*, Astérisque (1990), 187–188.
40. M. Pollicott, *Symbolic dynamics for Smale flows*, Amer. J. Math. **109** (1987), no. 1, 183–200.
41. R. Potrie and A. Sambarino, *Eigenvalues and entropy of a Hitchin representation*, Invent. Math. **209** (2017), 885–925.

42. B. Pozzetti, A. Sambarino, and A. Wienhard, *Anosov representations with Lipschitz limit set*, To appear in Geom. & Top., https://arxiv.org/abs/1910.06627, 2019.
43. B. Pozzetti, A. Sambarino, and A. Wienhard, *Conformality for a robust class of non-conformal attractors*, J. reine angew. Math. **774** (2021), 1–51.
44. R. Ranga Rao, *Relations between weak and uniform convergence of measures with applications*, Ann. Math. Stat. **33** (1962), 659–680.
45. D. Ruelle, *Thermodynamic formalism*, Addison-Wesley, London, 1978.
46. A. Sambarino, *Quantitative properties of convex representations*, Comment. Math. Helv. **89** (2014), no. 2, 443–488.
47. A. Sambarino, *Infinitesimal Zariski closures of positive representations*, https://arxiv.org/pdf/2012.10276.pdf, 2020.
48. D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Publ. Math. Inst. Hautes Études Sci. **50** (1979), 171–202.
49. J. Tits, *Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconqe*, J. reine angew. Math. **247** (1971), 196–220.
50. A. Wienhard, *An invitation to higher Teichmüller theory*. In Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 1013–1039.
51. C. Yue, *The ergodic theory of discrete isometry groups on manifolds of variable negative curvature*, Trans. Amer. Math. Soc. **348** (1996), no. 12, 4965–5005.
52. T. Zhang and A. Zimmer, *Regularity of limit sets of Anosov representations*, https://arxiv.org/pdf/1903.11021.pdf, 2019.