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Dilogarithm identities for solutions to Pell's equation in terms of continued fraction convergents

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Abstract
We describe a new connection between the dilogarithm function and the solutions of Pell's equation $x^2 - ny^2 = \pm 1$. For each solution x, y to Pell's equation, we obtain a dilogarithm identity whose terms are given by the continued fraction expansion of the associated unit $x + y\sqrt{n} \in \mathbb{Z}[\sqrt{n}]$. We further show that Ramanujan's dilogarithm value-identities correspond to an identity for the regular ideal hyperbolic hexagon.

Keywords Pells equation · Dilogarithm · Hyperbolic surfaces · Identities

Mathematics Subject Classification 11D09 · 11G55 · 32Q45

1 Dilogarithm and Pell's equation

Dilogarithm The dilogarithm function $\mathrm{Li}_2(z)$ is the integral function

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt.$$

It follows that it has power series

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \qquad \text{for} \qquad |z| \leq 1.$$

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In [16], Rogers introduced the following normalization for the dilogarithm for x real,

$$\mathcal{L}(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log|x|\log(1-x).$$

The dilogarithm function arises naturally in many areas of mathematics, including hyperbolic geometry and number theory (see [17]). In particular, volumes in the Lie group $\mathsf{PSL}(2,\mathbb{R})$ and the symmetric space $\mathbb{H}^3$ can be described in terms of the dilogarithm (see Sect. 5.1 for discussion).

Pell's equation Pell's equation for $n \in \mathbb{N}$ is the Diophantine equation $x^2 - ny^2 = \pm 1$ over $\mathbb{Z}$. Pell's equation has a long and interesting history going back to Archimedes' cattle problem (see [10]). The equation only has solutions for n square-free, so we assume n is square-free. Also, by symmetry, we need only consider solutions with $x, y > 0$. A solution is positive/negative depending on whether $x^2 - ny^2 = 1$, or, $x^2 - ny^2 = -1$. For all square-free n there is always a positive solution but not necessarily a negative solution. Solutions to Pell's equation correspond to units in $\mathbb{Z}[\sqrt{n}]$ by identifying x, y with $x + y\sqrt{n}$ and it is natural to identify the two. The smallest positive unit $u = x + y\sqrt{n}$ is called the *fundamental unit* and a well-known result is that the set of positive units is exactly $\{u^k\}, k \in \mathbb{N}$ (see [15, Theorem 7.26])).

In this paper, we prove a new and surprising connection between the dilogarithm and solutions to Pell's equation. Using earlier work of the author, which gave a dilogarithm identity associated to a hyperbolic surface, we obtain a dilogarithm identity for each solution x, y to Pell's equation whose terms are given by the continued fraction expansion of $x + y\sqrt{n}$.

1.1 Dilogarithm identities

The dilogarithm function satisfies a number of classical identities, see [11] for details. In particular, by adding power series termwise, we have the squaring identity

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(-z) = \frac{1}{2}\mathrm{Li}_2(z^2).$$

It follows by direct computation that this identity holds for the Rogers dilogarithm with

$$\mathcal{L}(x) + \mathcal{L}(-x) = \frac{1}{2}\mathcal{L}(x^2) \qquad \text{(Squaring Identity)}.$$

The other classic identities are Euler's reflection identities

$$\mathcal{L}(x) + \mathcal{L}(1-x) = \frac{\pi^2}{6}, \qquad \mathcal{L}(x) + \mathcal{L}(x^{-1}) = \frac{\pi^2}{6} \qquad \text{(Reflection Identity)},$$

Landen's identity (see [9])

$$\mathcal{L}\left(-\frac{1}{x}\right) = -\mathcal{L}\left(\frac{1}{x+1}\right) \qquad \text{for } x > 0 \qquad \text{(Landen's identity)},$$

and Abel's well-known 5-term identity

$$\mathcal{L}(x) + \mathcal{L}(y) = \mathcal{L}(xy) + \mathcal{L}\left(\frac{x(1-y)}{1-xy}\right) + \mathcal{L}\left(\frac{y(1-x)}{1-xy}\right) \qquad \text{(Abel's Identity)}.$$

It can be easily shown that the reflection identities and Landen's identity follow from Abel's identity.

A closed form for values of $\mathcal{L}$ is only known for a small set of values. These are

$$\begin{aligned} &\mathcal{L}(0) = 0, \qquad \mathcal{L}(1) = \frac{\pi^2}{6}, \qquad \mathcal{L}\left(\frac{1}{2}\right) = \frac{\pi^2}{12}, \\ &\mathcal{L}(\phi^{-1}) = \frac{\pi^2}{10}, \qquad \mathcal{L}(\phi^{-2}) = \frac{\pi^2}{15}, \end{aligned} \tag{1.1}$$

where ϕ is the golden ratio. In [11], Lewin gave the following remarkable infinite identity

$$\sum_{k=2}^{\infty} \mathcal{L}\left(\frac{1}{k^2}\right) = \frac{\pi^2}{6}. \tag{1.2}$$

2 Results

Using earlier work of the author, we first prove the below new infinite identity for $\mathcal{L}$. We prove:

Theorem 2.1 *If $L > 0$ then*

$$\mathcal{L}(e^{-L}) = \sum_{k=2}^{\infty} \mathcal{L}\left(\frac{\sinh^2\left(\frac{L}{2}\right)}{\sinh^2\left(\frac{kL}{2}\right)}\right).$$

One immediate observation is if we let $L \to 0$, we recover the formula of Lewin in Eq. (1.2) above.

We now apply the above identity to solutions of Pell's equation and units in the ring $\mathbb{Z}[\sqrt{n}]$.

Dilogarithm identity for solution to Pell's equation

In order to obtain our identity associated to a given solution $a^2 - nb^2 = \pm 1$ of Pell's equation, we let L satisfy $e^{L/2} = a + b\sqrt{n}$. We then show that the summation terms in Theorem 2.1 above are given in terms of the continued fraction expansion of $a + b\sqrt{n}$. We obtain:

Theorem 2.2 *Let $u = a + b\sqrt{n} \in \mathbb{Z}[\sqrt{n}]$ be a solution to Pell's equation.*

- *If u is a positive solution with continued fraction convergents $r_j = h_j/k_j$, then*

$$\mathcal{L}\left(\frac{1}{u^2}\right) = \sum_{k=1}^{\infty} \mathcal{L}\left(\frac{1}{(h_{2k-1})^2}\right).$$

- *If u is a negative solution and u^2 has convergents $R_j = H_j/K_j$, then*

$$\mathcal{L}\left(\frac{1}{u^2}\right) = \sum_{k=0}^{\infty} \mathcal{L}\left(\frac{1}{b^2 n (2H_{2k-1})^2}\right) + \mathcal{L}\left(\frac{1}{(2H_{2k+1} - H_{2k})^2}\right).$$

Examples

We now consider some examples:

Case of $\mathbb{Z}[\sqrt{2}]$ For $\mathbb{Z}[\sqrt{2}]$ the fundamental unit is $3 + 2\sqrt{2}$ giving

$$\mathcal{L}\left(\frac{1}{(3+2\sqrt{2})^2}\right) = \mathcal{L}\left(\frac{1}{6^2}\right) + \mathcal{L}\left(\frac{1}{35^2}\right) + \mathcal{L}\left(\frac{1}{204^2}\right) + \mathcal{L}\left(\frac{1}{1189^2}\right) + \ldots.$$

We note that $3 + 2\sqrt{2}$ has convergents r_k given by

$$\frac{5}{1}, \frac{6}{1}, \frac{29}{5}, \frac{35}{6}, \frac{169}{29}, \frac{204}{35}, \frac{985}{169}, \frac{1189}{204}.$$

It can be further shown that the units of $\mathbb{Z}[\sqrt{2}]$ are given by $(1+\sqrt{2})^k$. As $u = 1+\sqrt{2}$ is a negative solution to Pell's equation with $u^2 = 3 + 2\sqrt{2}$, we get

$$\mathcal{L}\left(\frac{1}{3+2\sqrt{2}}\right) = \mathcal{L}\left(\frac{1}{2\times(2)^2}\right) + \mathcal{L}\left(\frac{1}{7^2}\right) + \mathcal{L}\left(\frac{1}{2\times(12)^2}\right) + \mathcal{L}\left(\frac{1}{41^2}\right)$$
$$+\mathcal{L}\left(\frac{1}{2\times(70)^2}\right) + \mathcal{L}\left(\frac{1}{239^2}\right) + \mathcal{L}\left(\frac{1}{2\times(408)^2}\right) + \ldots.$$

Case of $\mathbb{Z}[\sqrt{13}]$ An interesting case of a large fundamental solution occurs for $\mathbb{Z}[\sqrt{13}]$. Here $u = 649 + 180\sqrt{13}$ is the fundamental unit, giving

$$\mathcal{L}\left(\frac{1}{842401 + 233640\sqrt{13}}\right) = \mathcal{L}\left(\frac{1}{1298^2}\right) + \mathcal{L}\left(\frac{1}{1684803^2}\right) + \mathcal{L}\left(\frac{1}{2186872996^2}\right) \dots .$$

The continued fraction convergents of u are

$$\frac{1297}{1}, \frac{1298}{1}, \frac{1683505}{1297}, \frac{1684803}{1298}, \frac{2185188193}{1683505}, \frac{2186872996}{1684803} \dots$$

Pell's equation over $\mathbb{Q}$

Similarly, we consider Pell's equation over $\mathbb{Q}$. If $a, b \in \mathbb{Q}$ satisfy Pell's equation $a^2 - nb^2 = \pm 1$, we will identify this with the element $a + b\sqrt{n} \in \mathbb{Q}[\sqrt{n}]$. Applying the identity in Theorem 2.1, we get the following:

Theorem 2.3 *Let* $u = a + b\sqrt{n} \in \mathbb{Q}[\sqrt{n}]$, $a, b > 0$ *satisfy Pell's equation and let* $u^k = a_k + b_k\sqrt{n}$.

If u is a positive solution, then

$$\mathcal{L}\left(\frac{1}{u^2}\right) = \sum_{k=2}^{\infty} \mathcal{L}\left(\frac{1}{(b_k/b)^2}\right).$$

Further if $u \in \mathbb{Z}[\sqrt{n}]$, *then* $b_k/b \in \mathbb{Z}$ *for all* k.

If u is a negative solution, then

$$\mathcal{L}\left(\frac{1}{u^2}\right) = \sum_{k=1}^{\infty} \mathcal{L}\left(\frac{1}{n(b_{2k}/a)^2}\right) + \mathcal{L}\left(\frac{1}{(a_{2k+1}/a)^2}\right).$$

Further, if $u \in \mathbb{Z}[\sqrt{n}]$, *then* $b_{2k}/a, a_{2k+1}/a \in \mathbb{Z}$ *for all* k.

Fibonacci numbers The golden mean $\phi \in \mathbb{Q}[\sqrt{5}]$ corresponds to a negative solution to Pell's equation over $\mathbb{Q}$. Also, we have

$$\phi^k = \frac{g_k + f_k\sqrt{5}}{2}$$

where f_k is the classic Fibonacci sequence $1, 1, 2, 3, 5\dots$ and g_k is the Fibonacci sequence $1, 3, 4, 7, 11, \dots$.

By Eq. (1.1) we have $\mathcal{L}(\phi^{-2}) = \pi^2/15$. Therefore we get the identity,

$$\sum_{k=1}^{\infty}\left(\mathcal{L}\left(\frac{1}{5f_{2n}^2}\right) + \mathcal{L}\left(\frac{1}{g_{2n+1}^2}\right)\right) = \frac{\pi^2}{15}.$$

Chebyshev polynomials, Pell's equation and dilogarithms

Chebyshev polynomials arise in numerous areas of mathematics and have a natural interpretation in terms of Pell's equation. The *Chebyshev polynomial of the first kind* T_n is the unique polynomials satisfying $T_n(\cos(\theta)) = \cos(n\theta)$. The *Chebyshev polynomials of the second kind* U_n is given by

$$U_n(\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}.$$

We obtain the following corollary:

Corollary 2.4 *Let* $x > 1$*, then*

$$\mathcal{L}\left(\frac{1}{\left(x+\sqrt{x^2-1}\right)^2}\right) = \sum_{n=1}^{\infty} \mathcal{L}\left(\frac{1}{U_n(x)^2}\right).$$

The reader interested in knowing more about the dilogarithm function and its generalizations, is referred to the book [11] and the aforementioned article [17].

3 Units in $\mathbb{Z}[\sqrt{n}]$, Pell's equation

We assume n is not a perfect square. If $a+b\sqrt{n} \in \mathbb{Z}[\sqrt{n}]$ is a unit, then $\pm a \pm b\sqrt{n}$ are also units. Therefore, we need only consider solutions $(a, b) \in \mathbb{N}^2$. It follows easily that $a \pm b\sqrt{n} \in \mathbb{Z}[\sqrt{n}]$ is a unit if and only if (a, b) satisfy *Pell's equation* over $\mathbb{Z}$

$$a^2 - nb^2 = \pm 1.$$

We call a solution (a, b) (or the unit $a+b\sqrt{n}$) positive/negative, depending on whether the right-hand side of the Pell equation is positive/negative. Whereas there is always a solution to the positive Pell equation $x^2 - ny^2 = 1$, it can be shown that there are no solutions to $x^2 - ny^2 = -1$ for certain n (see [15, Chapter 7]).

Continued fraction convergents

If $u \in \mathbb{R}_+$, we say u has continued fraction expansion $u = [c_0, c_1, c_2, c_3, \ldots]$ if $c_i \in \mathbb{Z}$ and

$$u = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \ldots}}}$$

By this we mean that if we define $r_n = [c_0, c_1, c_2, \ldots, c_n] \in \mathbb{Q}$ to be the n^{th} *convergent*, then $r_n \to u$ as $n \to \infty$. If the continued fraction coefficients satisfy $c_{n+r} = c_n$ for $n > k$, we say u is *periodic* with period r and write $u = [c_0, c_1, \ldots, c_k, \overline{c_{k+1}, \ldots, c_{k+r}}]$. We have the following standard description of r_n:

Theorem 3.1 [15, Theorems 7.4, 7.5] *Let $u \in \mathbb{R}_+$ with $u = [c_0, c_1, c_2, \ldots]$. Define h_n, k_n by*

$$h_i = c_i h_{i-1} + h_{i-2} \qquad k_i = c_i k_{i-1} + k_{i-2} \qquad i \geq 0$$

with $(h_{-2}, k_{-2}) = (0, 1)$ and $(h_{-1}, k_{-1}) = (1, 0)$. Then $\gcd(h_i, k_i) = 1$ and

$$r_n = [c_0, c_1, c_2, \ldots, c_n] = \frac{h_n}{k_n}.$$

The positive units in $\mathbb{Z}[\sqrt{n}]$ have the following elegant description.

Theorem 3.2 [15, Theorem 7.26] *Let $n \in \mathbb{N}$ not be a perfect square. Then there is a unique solution $(a, b) \in \mathbb{N}^2$ of Pell's equation $x^2 - ny^2 = 1$ such that the set of solutions to $x^2 - ny^2 = 1$ in $\mathbb{N}^2$ is $\{(a_k, b_k)\}_{k=1}^{\infty}$ where*

$$a_k + b_k\sqrt{n} = (a + b\sqrt{n})^k.$$

The pair (a, b) is called the *fundamental solution* of $x^2 - ny^2 = 1$. One consequence of the above is, if u is the fundamental unit, then $\{u^k\}$ gives the set of all positive solutions to Pell's equation. Thus the dilogarithm identity in Theorem 2.2 can be interpreted as a sum over all solutions to Pell's equation.

4 The orthospectrum identity

In a prior paper, the author proved a dilogarithm identity for a hyperbolic surface with geodesic boundary. In [6] the identity was generalized to hyperbolic manifolds by the author and Kahn. The relation to other identities on hyperbolic manifolds, such as the Basmajian identity (see [3]), the McShane–Mirzakhani identity (see [13,14]), and the Luo–Tan identity (see [12]), is discussed in [7].

4.1 Hyperbolic geometry

We will use two models for the hyperbolic plane $\mathbb{H}^2$, the upper half-plane model $\mathbb{H} = \{z \mid \mathrm{Im}(z) > 0\}$, with hyperbolic metric $ds = |dz|/\mathrm{Im}(z)$, and the Poincaré model $\mathsf{D} = \{z \mid |z| < 1\}$ with the hyperbolic metric $ds = 2|dz|/(1 - |z|^2)$. In each model, the group of orientation preserving isometries correspond to the group of conformal automorphisms and is therefore isomorphic to $\mathsf{PSL}(2, \mathbb{R})$.

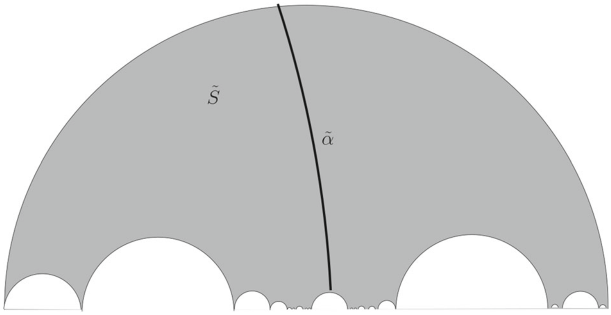

Fig. 1 Lift of orthogeodesic α to universal cover $\tilde{S}$

In $\mathbb{H}$ the geodesics are semi-circles which are orthogonal to the boundary $\partial\mathbb{H} = \overline{\mathbb{R}}$ (including vertical lines). Thus a geodesic can be identified with its endpoints in $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$. Two disjoint geodesics are *ultra parallel* if they do not have a common endpoint in $\overline{\mathbb{R}}$, and *asymptotically parallel* if they have a common endpoint. If g, h are ultra parallel, then by a Móbius transformation $m \in \mathsf{PSL}(2, \mathbb{R})$, g, h can be mapped to geodesics $m(g), m(h)$ where $m(g)$ has endpoints $1, -1$ and $m(h)$ has endpoints $e^l, -e^l$ for some $l > 0$. Then the y-axis is a common perpendicular geodesic to $m(g), m(h)$ in $\mathbb{H}$, showing that g, h have a common perpendicular. Also by simple integration, we have that l is the length of the common perpendicular. If g, h are asymptotically parallel, then there is no common perpendicular and the region between g, h is said to form a *cusp* at the common endpoint in $\overline{\mathbb{R}}$.

4.2 Orthogeodesics and orthospectrum

In order to state the orthospectrum identity, we recall some basic terms.

Let S be a finite area hyperbolic surface with totally geodesic boundary. Then an *orthogeodesic* for S is a proper geodesic arc α which is perpendicular to the boundary ∂S at its endpoints. The set of orthogeodesics of S is denoted $O(S)$. Each boundary component is either a closed geodesic or an infinite geodesic whose endpoints are *boundary cusps* of S. We let $N(S)$ be the number of boundary cusps of S. Further, let $\chi(S)$ be given by $Area(S) = -2\pi\chi(S)$. We note that if there are no boundary cusps, then $\chi(S)$ is the Euler Characteristic of S.

We note that for S a finite area hyperbolic surface with totally geodesic boundary, the universal cover $\tilde{S} \subseteq \mathbb{H}^2$ is a simply connected convex region bounded by a countable collection of geodesics (see Fig. 1). A lift of an orthogeodesic is then a common perpendicular to two boundary components of $\tilde{S}$ that are ultra parallel.

One elementary example of a surface is an ideal hyperbolic n-gon. In this case, $N(S) = n$ and $O(S)$ is a finite set. Also as $Area(S) = (n-2)\pi$, then $\chi(S) = 1 - n/2$. In fact, ideal hyperbolic n-gons are the only surfaces with $O(S)$ finite.

The dilogarithm orthospectrum identity is as follows:

Theorem 4.1 (Dilogarithm Orthospectrum Identity, [5]) *Let S be a finite area hyperbolic surface with totally geodesic boundary $\partial S \neq 0$. Then*

$$\sum_{\alpha \in O(S)} \mathcal{L}\left(\frac{1}{\cosh^2\left(\frac{l(\alpha)}{2}\right)}\right) = -\frac{\pi^2}{12}(6\chi(S) + N(S)),$$

and equivalently,

$$\sum_{\alpha \in O(S)} \mathcal{L}\left(-\frac{1}{\sinh^2\left(\frac{l(\alpha)}{2}\right)}\right) = \frac{\pi^2}{12}(6\chi(S) + N(S)).$$

4.3 A geometric decomposition using orthogeodesics

For completeness, we now give a sketch of the proof of the orthospectrum identity. We will see that it follows from an elementary decomposition of the unit tangent bundle of S.

Let $T_1(S)$ be the unit tangent bundle of S. Given $v \in T_1(S)$, we let α_v be the maximal geodesic with tangent vector v. Generically (except for a set of measure zero), α_v will be a geodesic arc with endpoints on the boundary of S. We define an equivalence relation on $T_1(S)$, by defining $v \sim w$ if the geodesics α_v, α_w are homotopic rel. ∂S. This gives a partition of (a full measure subset of) $T_1(S)$ into equivalence classes of two types, one type corresponding to the orthogeodesics and the other type corresponding to boundary cusps. For each orthogeodesic $\gamma \in O(S)$ we have an equivalance class E_γ corresponding to all $w \in T_1(S)$ such that α_w is homotopic rel. boundary to γ. For each boundary cusp c, we have an equivalence class E_c corresponding to all $w \in T_1(S)$ such that α_w is homotopic rel boundary out the cusp c. Then the equivalence relation gives a volume relation

$$Vol(T_1(S)) = \sum_{\gamma \in O(S)} Vol(E_\gamma) + \sum_{c \text{ boundary cusp}} Vol(E_c).$$

For the left-hand side, we have $Vol(T_1(S)) = 2\pi Area(S) = -4\pi^2\chi(S)$.

Lifting an orthogeodesic γ to $\tilde{\gamma}$ in the universal cover $\tilde{S}$, we have $\tilde{\gamma}$ is the common perpendicular to two geodesic components g, h of $\partial\tilde{S}$. Then E_γ lifts to the set $\tilde{E}_\gamma$ of vectors which are between g and h in the following sense. The vector $v \in T_1(\mathbb{H}^2)$ is *between* g and h, if the unique geodesic α_v tangent to v, intersects both g and h. Thus it follows that $Vol(E_\gamma)$ only depends on $l(\gamma)$ and, by direct calculation (see [5]), we have

$$Vol(E_\gamma) = 8\mathcal{L}\left(\frac{1}{\cosh^2\left(\frac{l(\gamma)}{2}\right)}\right). \tag{4.3}$$

Similarly, the equivalence class corresponding to a cusp E_c lifts to the set of tangent vectors between two geodesics g, h that have a common ideal endpoint. Therefore, as $\mathsf{PSL}(2,\mathbb{R})$ acts transitively on triples on $\overline{\mathbb{R}}$, we can assume the endpoints of g are $0, 1$ and h are $1, 2$. Therefore each E_c are isometric and have the same volume. Then applying the identity to an ideal triangle, which has no orthogeodesics, 3 boundary cusps and area π, we get

$$Vol(E_c) = \frac{2\pi^2}{3}.$$

Substituting these gives the orthospectrum identity,

$$Vol(T_1(S)) = -4\pi^2\chi(S) = \sum_{\gamma\in O(S)} 8\mathcal{L}\left(\frac{1}{\cosh^2\left(\frac{l(\gamma)}{2}\right)}\right) + N(S)\frac{2\pi^2}{3}.$$

In the original paper [5], we showed that the orthospectrum identity above recovers the reflection identities, Landen's identity and Abel's identity, by considering the elementary cases of the ideal quadrilateral and ideal pentagon, respectively.

5 An infinite dilogarithm identity

Given $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ distinct points we define the *cross-ratio* by

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_3)(z_4 - z_2)}.$$

Let $\mathbb{H}$ be the upper half-plane model for the hyperbolic plane and $x_1, x_2, x_3, x_4 \in \partial\mathbb{H} = \overline{\mathbb{R}}$ be distinct points, ordered counterclockwise on $\overline{\mathbb{R}}$. If g is the geodesic with endpoints x_1, x_2, and h is the geodesic with endpoints x_3, x_4, then g, h are disjoint. We let l be the perpendicular distance between g and h. Then we can choose a Möbius transformation $m \in \mathsf{PSL}(2,\mathbb{R})$ such that $m(g)$ has endpoints $-1, 1$ and $m(h)$ has endpoints $-e^l, e^l$. Then by invariance of the cross-ratio under Möbius transformations, we have

$$[x_1, x_2, x_3, x_4] = [-1, 1, e^l, -e^l] = \frac{1}{\cosh^2(l/2)} \tag{5.4}$$

We now prove Theorem 2.1.

Proof of Theorem 2.1 Let S be an annulus with two geodesic boundary components g, h. Let g be a closed geodesic of length L, and h an infinite geodesic with a single boundary cusp (see Fig. 2).

We lift S to the upper half-plane with g lifted to the y-axis. Further let $\lambda = e^L$. Then $\tilde{S}$ is an infinite-sided ideal polygon invariant under multiplication by λ (see Fig. 3). □

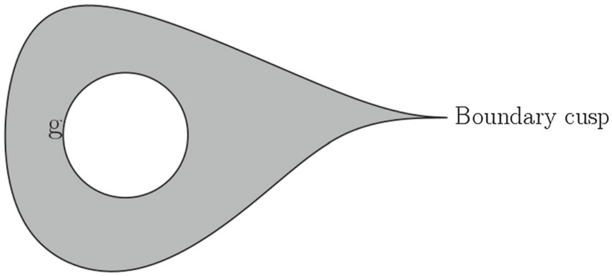

Fig. 2 Surface S

Fig. 3 Universal cover of S

We normalize so that one of the ideal vertices is at $z = 1$. Then the vertices of $\tilde{S}$ are $0, \infty$ and λ^k for $k \in \mathbb{Z}$. The edges of $\tilde{S}$ are the lift of g, denoted $\tilde{g}$, which has vertices $0, \infty$, and the lifts of h, labelled $\tilde{h}_k$, which has vertices λ^k, λ^{k+1}.

We now compute the orthospectrum of S. Every orthogeodesic lifts to a geodesic that is the common perpendicular between two boundary components of $\tilde{S}$. We consider two types.

If α is an orthospectrum with an endpoint on g, then it lifts to $\tilde{\alpha}$ which is a perpendicular between two edges of $\tilde{S}$, with one edge being $\tilde{g}$. By the $\mathbb{Z}$ action, which preserves $\tilde{g}$, we can choose $\tilde{\alpha}$ to have the other endpoint on $\tilde{h}_0$. Therefore α has length l satisfying

$$\frac{1}{\cosh^2(l/2)} = [\infty, 0, 1, \lambda] = \frac{\lambda - 1}{\lambda} = 1 - e^{-L}.$$

Any other orthogeodesic α has both endpoints in h. Therefore α lifts to $\tilde{\alpha}$ which is the perpendicular between $\tilde{h}_j, \tilde{h}_k$ for some $j < k$. By the action of $\mathbb{Z}$, we can assume $j = 0$. Also, as adjacent sides do not have a common perpendicular, we have that

$k \geq 2$. Denoting the length l_k of the perpendicular between $\tilde{h}_0$ and $\tilde{h}_k$, we have

$$\frac{1}{\cosh^2(l_k/2)} = [1, \lambda, \lambda^k, \lambda^{k+1}] = \frac{(1-\lambda)(\lambda^{k+1}-\lambda^k)}{(1-\lambda^k)(\lambda^{k+1}-\lambda)} = \lambda^{k-1}\frac{(\lambda-1)^2}{(\lambda^k-1)^2}$$
$$= \frac{(\lambda^{1/2}-\lambda^{-1/2})^2}{(\lambda^{k/2}-\lambda^{-k/2})^2} = \frac{\sinh^2(L/2)}{\sinh^2(kL/2)}.$$

As $Area(S) = \pi$, then $\chi(S) = -1/2$. Furthermore $N(S) = 1$. Thus by Theorem 4.1, we have the dilogarithm identity for S is

$$\mathcal{L}(1-e^{-L}) + \sum_{k=2}^{\infty} \mathcal{L}\left(\frac{\sinh^2(L/2)}{\sinh^2(kL/2)}\right) = -\frac{\pi^2}{12}(-6(1/2)+1) = \frac{\pi^2}{6}.$$

Using the reflection identity $\mathcal{L}(1-x) + \mathcal{L}(x) = \pi^2/6$, we get

$$\mathcal{L}(e^{-L}) = \sum_{k=2}^{\infty} \mathcal{L}\left(\frac{\sinh^2(L/2)}{\sinh^2(kL/2)}\right).$$

□

5.1 Hyperbolic volume and PSL(2, $\mathbb{R}$) volume

Another important normalization of the dilogarithm is the Bloch–Wigner dilogarithm $D : \mathbb{C} - \{0, 1\} \to \mathbb{R}$ by

$$D(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|.$$

This was introduced by Bloch on his work in K-theory and regulators and by Wigner in his work on Lie groups (see [4]).

The Bloch–Wigner dilogarithm function also arises naturally in the formula for the volume of an ideal hyperbolic tetrahedron. If T is an ideal hyperbolic tetrahedron T in $\mathbb{H}^3$, with ideal vertices $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$, then a classical result (see [8, Equation 4.13] states that the volume of T is given by

$$Vol(T) = D([z_1, z_2, z_3, z_4]).$$

Similarly, in the orthospectrum identity, we see that $\mathcal{L}(x)$ is also a volume. If x_1, x_2, x_3, x_4 are distinct points ordered counterclockwise on $\partial\mathbb{H}^2$, we let g be the geodesic with endpoints x_1, x_2 and h the geodesic with endpoints x_3, x_4. We let T be the set of tangent vectors in $T_1(\mathbb{H}^2)$ between g, h as described in Sect. 4.3. Then, considering the volume measure on $T_1(\mathbb{H}^2)$, by Eq. (4.3), we have

$$Vol(T) = 8\mathcal{L}([x_1, x_2, x_3, x_4]).$$

Interpreting $T_1(\mathbb{H}^2)$ as $\mathsf{PSL}(2,\mathbb{R})$, we see that the volume of an *ideal tetrahedron* in $\mathsf{PSL}(2,\mathbb{R})$ is given by the Rogers dilogarithm of the cross-ratio of its vertices.

6 Proof of identity for solutions to Pell's equation over $\mathbb{Q}$

We now prove the dilogarithm identity for solutions to Pell's equation over $\mathbb{Q}$ given in Theorem 2.3.

Proof of Theorem 2.3 Let $e^{L/2} = u = a + b\sqrt{n}$, then $e^{-L/2} = u^{-1} = \pm(a - b\sqrt{n})$ with the sign depending on if u is a positive or negative unit. If u is a positive unit, then

$$\cosh(L/2) = a \qquad \text{and} \qquad \sinh(L/2) = b\sqrt{n}.$$

If u is a negative unit, then

$$\sinh(L/2) = a \qquad \text{and} \qquad \cosh(L/2) = b\sqrt{n}.$$

In both cases we have

$$u^k = e^{kL/2} = \cosh(kL/2) + \sinh(kL/2).$$

We let $m_k = \sinh(kL/2)$ and $n_k = \cosh(kL/2)$. The dilogarithm identity gives

$$\mathcal{L}\left(\frac{1}{u^2}\right) = \sum_{k=2}^{\infty} \mathcal{L}\left(\frac{\sinh^2(L/2)}{\sinh^2(kL/2)}\right) = \sum_{k=2}^{\infty} \mathcal{L}\left(\frac{m_1^2}{m_k^2}\right).$$

If u is a positive root, then $m_1 = \sinh(L/2) = b\sqrt{n}$ and $n_1 = \cosh(L/2) = a$. Then by the addition formulae, we have

$$m_{k+1} = a.m_k + bn_k\sqrt{n} \qquad \text{and} \qquad n_{k+1} = n_k a + bm_k\sqrt{n}.$$

By induction, we have $n_k = a_k$ and $m_k = b_k\sqrt{n}$, and

$$b_{k+1} = ab_k + ba_k \qquad \text{and} \qquad a_{k+1} = aa_k + nbb_k.$$

Substituting, we get

$$\mathcal{L}\left(\frac{1}{u^2}\right) = \sum_{k=1}^{\infty} \mathcal{L}\left(\frac{b^2}{b_k^2}\right) = \sum_{k=1}^{\infty} \mathcal{L}\left(\frac{1}{(b_k/b)^2}\right).$$

If u is a negative solution, then $m_1 = \sinh(L/2) = a$ and $n_1 = \cosh(L/2) = b\sqrt{n}$. Then, by the addition formulae we have,

$$m_{k+1} = bm_k\sqrt{n} + an_k \qquad \text{and} \qquad n_{k+1} = bn_k\sqrt{n} + am_k.$$

Therefore

$$n_{2k} = a_{2k}, \qquad n_{2k+1} = b_{2k+1}\sqrt{n}, \qquad m_{2k} = b_{2k}\sqrt{n}, \qquad \text{and} \qquad m_{2k+1} = a_{2k+1}.$$

It follows that

$$b_{2k} = ba_{2k-1} + ab_{2k-1} \qquad \text{and} \qquad a_{2k+1} = bb_{2k}n + aa_{2n}.$$

Therefore,

$$\mathcal{L}\left(\frac{1}{u^2}\right) = \sum_{k=1}^{\infty} \mathcal{L}\left(\frac{m_1^2}{m_k^2}\right) = \sum_{k=1}^{\infty} \mathcal{L}\left(\frac{1}{n(b_{2k}/a)^2}\right) + \mathcal{L}\left(\frac{1}{(a_{2k+1}/a)^2}\right).$$

□

We now prove Corollary 2.4 relating the identity to the Chebyshev polynomials U_n of the second kind.

Proof of Corollary 2.4 We have the Chebyshev polynomials $T_n(x), U_n(x) \in \mathbb{R}[x]$. We let $x = \cos(\theta)$, then $\sin(\theta) = \sqrt{1 - x^2}$. Therefore

$$e^{i\theta} = \cos(\theta) + i\,\sin(\theta) = x + i\sqrt{1 - x^2} = x + \sqrt{x^2 - 1}.$$

Thus

$$e^{in\theta} = (x + \sqrt{x^2 - 1})^n \qquad \text{and} \qquad e^{-in\theta} = (x - \sqrt{x^2 - 1})^n.$$

Substituting, we get

$$T_n(x) = \cos(n\theta) = \frac{1}{2}\left((x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n\right)$$

and

$$U_{n-1}(x) = \frac{\sin(n\theta)}{\sin\theta} = \frac{1}{2\sqrt{x^2 - 1}}\left((x + \sqrt{x^2 - 1})^n - (x - \sqrt{x^2 - 1})^n\right).$$

As this holds for $|x| < 1$, it also holds for all $x \in \mathbb{R}$. If $x > 1$, we define $L > 0$ to be given by $x = \cosh(L/2)$. Then $\sqrt{x^2 - 1} = \sinh(L/2)$, giving

$$x + \sqrt{x^2 - 1} = e^{L/2} \qquad \text{and} \qquad x - \sqrt{x^2 - 1} = e^{-L/2}.$$

Therefore, by the above formulae

$$T_k(x) = \frac{e^{kL/2} + e^{-kL/2}}{2} = \cosh(kL/2) \qquad \text{and}$$
$$U_{k-1}(x) = \frac{e^{kL/2} - e^{-kL/2}}{2\sinh(L/2)} = \frac{\sinh(kL/2)}{\sinh(L/2)}.$$

Thus

$$\mathcal{L}\left(\frac{1}{\left(x+\sqrt{x^2-1}\right)^2}\right)=\sum_{k=2}^{\infty}\mathcal{L}\left(\frac{\sinh^2(L/2)}{\sinh^2(kL/2)}\right)=\sum_{k=1}^{\infty}\mathcal{L}\left(\frac{1}{U_k(x)^2}\right).$$

□

7 Identity for continued fraction convergents

We now consider the case where $u \in \mathbb{Z}[\sqrt{n}]$. We prove Theorem 2.2 expressing the above in terms of the convergents $r_j = h_j/k_j$ of their continued fractions expansion. First, we have the following lemma.

Lemma 7.1 *Let $u = a + b\sqrt{n} \in \mathbb{Z}[\sqrt{n}]$ be a solution to Pell's equation with $a, b \in \mathbb{N}$. If u is a positive solution, then $u = [2a-1, \overline{1, 2a-2}]$. If u is a negative solution, then $u = [\overline{2a}]$.*

Proof If u is a negative solution, then $u = a + \sqrt{a^2+1}$. Therefore $u^2 - 2au - 1 = 0$. Therefore

$$u = 2a + \frac{1}{u}.$$

Thus $u = [\overline{2a}]$.

If u is a positive solution, then $u = a + \sqrt{a^2-1}$. Therefore u satisfies the quadratic $u^2 - 2au + 1 = 0$. Rewriting, we have

$$u = 2a - \frac{1}{u} = 2a - 1 + 1 - \frac{1}{u} = 2a - 1 + \frac{u-1}{u}.$$

Now we have

$$\frac{u-1}{u} = \frac{1}{\frac{u}{u-1}} = \frac{1}{1+\frac{1}{u-1}} = \frac{1}{1+\frac{1}{2a-2+\frac{u-1}{u}}}.$$

Therefore $u = [2a-1, \overline{1, 2a-2}]$. □

Using the above description of the continued fraction, we will show the relation between the approximates $r_j = h_j/k_j$ for u and the coefficients a_j, b_j given by $u^j = a_j + b_j\sqrt{n}$. This will allow us to prove Theorem 2.2.

Lemma 7.2 *Let $u = a + b\sqrt{n} \in \mathbb{Z}[\sqrt{n}]$ be a solution to Pell's equation.*

If u is a positive solution and u has continued fraction convergents $r_j = h_j/k_j$, then $k_j = h_{j-2}$ and

$$\mathcal{L}\left(\frac{1}{u^2}\right) = \sum_{j=1}^{\infty}\mathcal{L}\left(\frac{1}{(h_{2j-1})^2}\right).$$

If u is a negative solution and u^2 has continued fraction convergents $R_j = H_j/K_j$, then

$$\mathcal{L}\left(\frac{1}{u^2}\right) = \sum_{j=0}^{\infty}\left(\mathcal{L}\left(\frac{1}{nb^2(2H_{2j-1})^2}\right) + \mathcal{L}\left(\frac{1}{(2H_{2k+1} - H_{2k})^2}\right)\right).$$

Proof Let $u = a + b\sqrt{n} = e^{L/2}$, then $u^k = a_k + b_k\sqrt{n} = \cosh(kL/2) + \sinh(kL/2)$.

If u is a positive solution, then $u = [2a-1, \overline{1, 2a-2}]$. Therefore we have $(h_0, h_{-1}) = (2a-1, 1)$. By Theorem 3.1 describing the continued fraction convergents, for $k > 0$

$$\begin{bmatrix} h_{2k} \\ h_{2k-1} \end{bmatrix} = A^k \begin{bmatrix} 2a-1 \\ 1 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 2a-2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2a-1 & 2a-2 \\ 1 & 1 \end{bmatrix}.$$

The matrix A has characteristic polynomial $x^2 - 2ax + 1$. Therefore A has eigenvalues u, u^{-1} and eigenvectors $(u-1, 1)$, $(1-u, u)$. Diagonalizing, we get

$$\begin{bmatrix} h_{2k} \\ h_{2k-1} \end{bmatrix} = \frac{1}{u^2-1}\begin{bmatrix} u-1 & 1-u \\ 1 & u \end{bmatrix}\begin{bmatrix} u^k & 0 \\ 0 & u^{-k} \end{bmatrix}\begin{bmatrix} u & u-1 \\ -1 & u-1 \end{bmatrix}\begin{bmatrix} 2a-1 \\ 1 \end{bmatrix}.$$

As $u = e^{L/2}$, we have

$$h_{2k} = \frac{(u-1)\left(u^{k+2} + u^{-(k+1)}\right)}{u^2-1} = \frac{\cosh((k+\frac{3}{2})L/2)}{\cosh(L/4)}, \tag{7.5}$$

$$h_{2k-1} = \frac{u^{k+2} - u^{-k}}{u^2-1} = \frac{\sinh((k+1)L/2)}{\sinh(L/2)}. \tag{7.6}$$

It follows that for $k \geq 1$

$$h_{2k-3} = \frac{\sinh(kL/2)}{\sinh(L/2)} = \frac{b_k}{b}. \tag{7.7}$$

Therefore

$$\mathcal{L}\left(\frac{1}{u^2}\right) = \sum_{k=2}^{\infty}\mathcal{L}\left(\frac{1}{(b_k/b)^2}\right) = \sum_{j=1}^{\infty}\mathcal{L}\left(\frac{1}{(h_{2j-1})^2}\right).$$

Similarly, we note that as $(k_0, k_1) = (1, 0)$, then applying the above analysis we get

$$k_{2j} = \frac{\cosh((j+\frac{1}{2})L/2)}{\cosh(L/4)} = h_{2j-2}$$

and

$$k_{2j-1} = \frac{\sinh(jL/2)}{\sinh(L/2)} = h_{2j-3}.$$

Therefore $k_j = h_{j-2}$.

Let u is a negative solution. Then for k odd, $a_k = \sinh(kL/2)$, $b_k\sqrt{n} = \cosh(kL/2)$ and for k even, $b_k\sqrt{n} = \sinh(kL/2)$, $a_k = \cosh(kL/2)$.

As $u = [\overline{2a}]$, by Theorem 3.1 we have the formula

$$h_{j+1} = 2ah_j + h_{j-1} \qquad \text{and} \qquad k_{j+1} = 2ak_j + k_{j+1},$$

with $(h_{-2}, k_{-2}) = (0, 1)$ and $(h_{-1}, k_{-1}) = (1, 0)$. Iterating, we get $h_j = 0, 1, 2a, \ldots$ and $k_j = 1, 0, 1, 2a, \ldots$. Therefore $k_j = h_{j-1}$ for $j \geq -1$. We focus on calculating h_k. As $(h_{-1}, h_{-2}) = (1, 0)$, we have the recursion

$$\begin{bmatrix} h_k \\ h_{k-1} \end{bmatrix} = A^{k+1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \text{where} \qquad A = \begin{bmatrix} 2a & 1 \\ 1 & 0 \end{bmatrix}.$$

The matrix A has characteristic polynomial $x^2 - 2ax - 1 = 0$. Therefore A has eigenvalues u, $-u^{-1}$ and eigenvectors $(u, 1)$, $(1, -u)$. Thus,

$$\begin{bmatrix} h_k \\ h_{k-1} \end{bmatrix} = \frac{1}{u^2+1} \begin{bmatrix} u & 1 \\ 1 & -u \end{bmatrix} \begin{bmatrix} u^{k+1} & 0 \\ 0 & (-u)^{-k-1} \end{bmatrix} \begin{bmatrix} u & 1 \\ 1 & -u \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Multiplying, we get

$$h_k = \frac{1}{u^2+1} \left(u^{k+3} + (-1)^{k+1} u^{-(k+1)} \right) = \frac{1}{u+u^{-1}} \left(u^{k+2} + (-1)^{k+1} u^{-(k+2)} \right).$$

For k odd, we have

$$h_k = \frac{\cosh((k+2)L/2)}{\cosh(L/2)} = \frac{b_{k+2}}{b}.$$

Similarly, for k even, we have

$$h_k = \frac{\sinh((k+2)L/2)}{\cosh(L/2)} = \frac{b_{k+2}}{b}.$$

Thus for all $k \geq 0$

$$\frac{b_k}{b} = h_{k-2}.$$

We let H_j, K_j be the convergents for the continued fraction expansion of u^2. Then $u^2 = e^L$ is a positive solution to Pell's equation. Applying Eqs. 7.5 and 7.6 above we have,

$$H_{2k} = \frac{\cosh((2k+3)L/2}{\cosh(L/2)} = \frac{b_{2k+3}}{b} = h_{2k+1}.$$

$$H_{2k-1} = \frac{\sinh((k+1)L)}{\sinh(L)} = \frac{1}{2}\left(\frac{\sinh(2k+1)L/2}{\sinh(L/2)} + \frac{\cosh(2k+1)L/2}{\cosh(L/2)}\right)$$

$$= \frac{1}{2}\left(\frac{a_{2k+1}}{a} + h_{2k-1}\right).$$

Also, if $(u^2)^k = A_k + B_k\sqrt{n}$ then $A_k = a_{2k}$, $B_k = b_{2k}$. Then by Eq. (7.7),

$$H_{2k-3} = \frac{B_k}{B_1} = \frac{b_{2k}}{2ab} = \frac{h_{2k-2}}{2a}.$$

Therefore

$$h_{2k} = 2aH_{2k-1} \quad \text{and} \quad h_{2k+1} = H_{2k}.$$

Also

$$\frac{b_{2k}}{a} = 2bH_{2k-3} \quad \text{and} \quad \frac{a_{2k+1}}{a} = 2H_{2k-1} - h_{2k-1} = 2H_{2k-1} - H_{2k-2}.$$

Thus, if u is a negative solution to Pell's equation

$$\mathcal{L}\left(\frac{1}{u^2}\right) = \sum_{k=0}^{\infty} \mathcal{L}\left(\frac{1}{n(2bH_{2k-1})^2}\right) + \mathcal{L}\left(\frac{1}{(2H_{2k-1} - H_{2k-2})^2}\right).$$

□

8 Ideal n-gon identities

We now describe the orthospectrum identity for a general ideal hyperbolic n-gon. We show that the case of the regular ideal hyperbolic $(2n+1)$-gon recovers an identity of Richmond and Szekeres (see [11, Equation 2.51]). We also show that the regular ideal hyperbolic hexagon case recovers the following value-identities.

Ramanujan gave the following value-identities for linear combinations of specific values of $\mathcal{L}$ (see [1, Entry 39]):

1. $\mathrm{Li}_2\left(\frac{1}{3}\right) - \frac{1}{6}\mathrm{Li}_2\left(\frac{1}{9}\right) = \frac{\pi^2}{18} - \frac{\log^2 3}{6}$
2. $\mathrm{Li}_2\left(-\frac{1}{2}\right) + \frac{1}{6}\mathrm{Li}_2\left(\frac{1}{9}\right) = -\frac{\pi^2}{18} + \log 2 \log 3 - \frac{\log^2 2}{2} - \frac{\log^2 3}{3}$
3. $\mathrm{Li}_2\left(\frac{1}{4}\right) + \frac{1}{3}\mathrm{Li}_2\left(\frac{1}{9}\right) = \frac{\pi^2}{18} + 2\log 2 \log 3 - 2\log^2 2 - \frac{2\log^2 3}{3}$
4. $\mathrm{Li}_2\left(-\frac{1}{3}\right) - \frac{1}{3}\mathrm{Li}_2\left(\frac{1}{9}\right) = -\frac{\pi^2}{18} - \frac{\log^2 3}{6}$
5. $\mathrm{Li}_2\left(-\frac{1}{8}\right) + \mathrm{Li}_2\left(\frac{1}{9}\right) = -\frac{\log^2(9/8)}{2}$

More recently, in the article [2], Bailey, Borwein and Plouffe gave the identity

$$36\mathrm{Li}_2\left(\frac{1}{2}\right) - 36\mathrm{Li}_2\left(\frac{1}{4}\right) - 12\mathrm{Li}_2\left(\frac{1}{8}\right) + 6\mathrm{Li}_2\left(\frac{1}{64}\right) = \pi^2. \tag{8.8}$$

Applying Landen's identity, we have $\mathcal{L}(-1/3) = -\mathcal{L}(1/4)$ and $\mathcal{L}(-1/8) = -\mathcal{L}(1/9)$. This reduces the value-identities of Ramanujan to the two equations

$$\mathcal{L}\left(\frac{1}{4}\right) + \frac{1}{3}\mathcal{L}\left(\frac{1}{9}\right) = \frac{\pi^2}{18} \quad \text{and} \quad \mathcal{L}\left(\frac{1}{3}\right) - \frac{1}{6}\mathcal{L}\left(\frac{1}{9}\right) = \frac{\pi^2}{18}.$$

We recall the dilogarithm identity in [5] for ideal hyperbolic polygons. Let P be an ideal polygon in $\mathbb{H}^2$ with vertices in counterclockwise order $x_1, \dots, x_n$ about $\partial\mathbb{H}^2$. If l_{ij} is the length of the orthogeodesic joining side $[x_i, x_{i+1}]$ to $[x_j, x_{j+1}]$, then by Eq. (5.4), we have

$$[x_i, x_{i+1}, x_j, x_{j+1}] = \frac{1}{\cosh^2(l_{ij}/2)}.$$

As $Area(P) = (n-2)\pi$, then $\chi(P) = -n/2$. Furthermore, $N(P) = n$.

Applying the orthospectrum identity in Theorem 4.1 to P, we obtain the equation

$$\sum_{|i-j|\geq 2} \mathcal{L}([x_i, x_{i+1}, x_j, x_{j+1}]) = -\frac{\pi^2}{12}\left(-6\left(\frac{n-2}{2}\right) + n\right) = \frac{(n-3)\pi^2}{6}.$$

If P is the regular ideal n-gon, then in the Poincaré disk model for $\mathbb{H}^2$, we can choose P to have vertices $e^{\frac{2\pi i k}{n}}$ for $k = 0, \dots, n-1$. Therefore, taking cross-ratios and grouping terms, we obtain the equation

$$\frac{e_n}{2}\mathcal{L}\left(\sin^2(\pi/n)\right) + \sum_{k=2}^{\lfloor \frac{n}{2} \rfloor} \mathcal{L}\left(\frac{\sin^2(\pi/n)}{\sin^2(k\pi/n)}\right) = \frac{(n-3)\pi^2}{6n} \tag{8.9}$$

where $e_n = 0$ if n is odd and $e_n = 1$ if n is even. In the case of n odd, Eq. (8.9) recovers the identity of Richmond and Szekeres (see [11, Equation 2.51]) which they derived using Rogers–Ramanujan partition identities.

8.1 Ideal hexagons and Ramanujan's value-identities

We now show that Ramanujan's value-identities 1–5, and identity 8.8 of Bailey, Borwein, Plouffe, correspond to identities for the regular ideal hexagon.

For the regular 6-gon H_{reg} the orthospectrum identity gives

$$6\mathcal{L}\left(\frac{1}{3}\right) + 3\mathcal{L}\left(\frac{1}{4}\right) = \frac{\pi^2}{2}.$$

By Landen's identity, $\mathcal{L}(-1/3) = -\mathcal{L}(1/4)$. Therefore applying the squaring identity we get

$$\frac{1}{2}\mathcal{L}\left(\frac{1}{9}\right) = \mathcal{L}\left(\frac{1}{3}\right) + \mathcal{L}\left(-\frac{1}{3}\right) = \mathcal{L}\left(\frac{1}{3}\right) - \mathcal{L}\left(\frac{1}{4}\right).$$

Thus, we obtain

$$\mathcal{L}\left(\frac{1}{3}\right) - \mathcal{L}\left(\frac{1}{4}\right) = \frac{1}{2}\mathcal{L}\left(\frac{1}{9}\right).$$

Combining this and the identity above for the regular hexagon, we obtain Ramanujan's value-identities

$$\mathcal{L}\left(\frac{1}{4}\right) + \frac{1}{3}\mathcal{L}\left(\frac{1}{9}\right) = \frac{\pi^2}{18} \qquad \mathcal{L}\left(\frac{1}{3}\right) - \frac{1}{6}\mathcal{L}\left(\frac{1}{9}\right) = \frac{\pi^2}{18}.$$

To recover the identity 8.8, we note that by Landen's identity $\mathcal{L}(-1/8) = -\mathcal{L}(1/9)$. Then by the squaring identity, we have

$$\frac{1}{2}\mathcal{L}\left(\frac{1}{64}\right) = \mathcal{L}\left(\frac{1}{8}\right) + \mathcal{L}\left(-\frac{1}{8}\right) = \mathcal{L}\left(\frac{1}{8}\right) - \mathcal{L}\left(\frac{1}{9}\right).$$

Therefore, substituting for $\mathcal{L}(1/8)$, we get

$$\begin{aligned} &36\mathcal{L}\left(\frac{1}{2}\right) - 36\mathcal{L}\left(\frac{1}{4}\right) - 12\mathcal{L}\left(\frac{1}{8}\right) + 6\mathcal{L}\left(\frac{1}{64}\right) \\ &= 36\mathcal{L}\left(\frac{1}{2}\right) - 36\mathcal{L}\left(\frac{1}{4}\right) - 12\mathcal{L}\left(\frac{1}{9}\right). \end{aligned}$$

As $\mathcal{L}(1/2) = \pi^2/12$, and applying the hexagon identity $3\mathcal{L}(1/4) + \mathcal{L}(1/9) = \pi^2/6$, we recover identity 8.8.

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