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Tracking Control of UAVs with Uncertainty and Input Constraints

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***Abstract*— This paper considers the position and attitude tracking control problem of a vertical take-off and landing unmanned aerial vehicle with uncertainty and input constraints. Considering the parametric and non-parametric uncertainties in the dynamics of systems, a robust adaptive tracking controller is proposed with the aid of the special structure of the dynamics of the system. Considering the uncertainty and input constraints, a robust adaptive saturation controller is proposed with the aid of an auxiliary compensated system. Simulation results show the effectiveness of the proposed algorithms.**

I. INTRODUCTION

Control of vertical take-off and landing (VTOL) unmanned aerial vehicles (UAVs) has been an active research area in the past decades due to its wide applications in the areas such as surveillance, search and rescue missions, monitoring, etc. A VTOL UAV can operate in cluttered environments and hover for a long time in the air. A VTOL UAV is an underactuated system because it has six degrees of freedom (DOFs) (i.e., three DOFs for the position and three DOFs for the orientation) but has only four inputs. The underactuated nature of the system makes its control problems challenging.

In order to deal with its underactuated property, different control methods have been proposed in [1], [2], [3], [4], [5]. In control of VTOL UAVs, there is always uncertainty in the dynamics of VTOL UAVs in practice. In [6], the position and attitude tracking control of a quadrotor with inertia parameter uncertainty was considered. Adaptive tracking controllers were proposed with the aid of the cascade structure of the dynamics of the system and the immersion and invariance technique. In [7], [8], the sliding mode control technique was applied to estimate disturbance and a sliding mode based tracking controller was proposed. In [9], [10], the sliding mode technique was applied to compensate unmodeled dynamics and adaptive robust tracking controllers were proposed. In [11], composite learning controllers were proposed by using the terminal sliding mode for a quadrotor with unknown dynamics and time-varying disturbances. In [12], [13], [14], the immersion and invariance technique was applied to design adaptive controllers for a quadrotor with parametric uncertainty. In [15], the adaptive backstepping technique and command-filter compensation were applied and adaptive tracking controllers were proposed without computation of derivatives of signals. In [16], [17], controllers were proposed with the aid of model predictive control. In [18], robust adaptive controllers were proposed in the presence of wind disturbance with the aid of the singular-perturbation technique.

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For a VTOL UAV, there is always constraint on its inputs. For the tracking control of VTOL UAVs with input constraints, a nested tracking controller was proposed with the aid of the nested saturation control in [19] if there is no uncertainty and disturbance. However, the selection of the control parameters are intricate and the stability of the closed-loop system cannot be guaranteed if there is uncertainty in the model of the system. To overcome this, the tracking control problem of a quadrotor with parametric uncertainty and input constraints was studied in [20] and an adaptive tracking controller was proposed. However, the closed-loop tracking error system is semi-globally stable and the non-parametric uncertainty was not considered. In [21], the trajectory tracking control was studied for a VTOL aircraft with a simplified model under an input constraint. Since the proposed controller is based on a 3 DOF model, it cannot be extended to deal with a 6 DOF model of VTOL vehicles. In [22], the trajectory tracking control of a 6-DOF quadrotor UAV with an input constraint was studied. Position tracking controllers were proposed with the aid of backstepping techniques and a Nussbaum function under the assumptions that the inertia parameters are exactly known and the disturbances in the dynamics are constants. However, in practice these assumptions are not true.

In this paper we study two control problems of 6-DOF VTOL UAVs under the conditions that there is uncertainty and input constraint. One problem is the position and attitude tracking control of VTOL UAVs with both parametric and non-parametric uncertainty. The other problem is the position and attitude tracking control of VTOL UAVs with both uncertainty and input constraint. For the first problem, with the aid of backstepping techniques and saturation control a new quaternion-based robust adaptive controller is proposed such that the position and the attitude converge to their desired values, respectively. For the second problem, by introducing an auxiliary compensated system a new robust adaptive saturation controller is proposed such that the tracking errors of the position and the attitude are uniformly ultimately bounded (UUB).

II. PROBLEM STATEMENT AND SOME PRELIMINARIES

A. Problem Statement

Consider a rigid VTOL UAV. The well-known rigid body model can be written as [23]:

$$\dot{p} = v \tag{1}$$

$$\dot{v} = -ge_3 + \frac{1}{m} fRe_3 + d_1 \tag{2}$$

$$\dot{R} = RS(\omega) \tag{3}$$

$$J\dot{\omega} \;=\; S(J\omega)\omega + \tau + d_2 \qquad (4)$$

where p and v are the position and velocity of the mass center of the UAV in the inertia frame, respectively, g is the gravitational acceleration, $e_3 = [0,0,1]^\top$, $f \in \Re$ is the total thrust, $R = [b_1, b_2, b_3]$ is the rotation matrix of the body frame with respect to the inertia frame, $\omega = [\omega_1, \omega_2, \omega_3]^\top$ is the angular velocity of the UAV in its body frame, J is the inertia moment of the UAV, d_1 and d_2 denote non-parametric uncertainty which include un-modeled dynamics, friction, and disturbance, $S(\cdot)$ is a skew-symmetric matrix, and $\tau = [\tau_1, \tau_2, \tau_3]^\top$ is the torque input of the system. The dynamics of the thrust f and the torque τ are omitted for simplicity.

The model in (1)-(4) is a 6-DOF model. Due to the coupling between the position and attitude, four DOFs can be controlled independently.

In this paper, we consider the following tracking control problems.

Tracking control with uncertainty: It is given a desired trajectory $p^d(t)$ and a desired unit vector $b_2^d(t)$. If m, J, d_1, and d_2 are unknown, the control problem is to design a state feedback controller (f, τ) such that

$$\lim_{t\to\infty}(p(t) - p^d(t)) = 0 \qquad (5)$$

$$\lim_{t\to\infty}(b_2(t) - b_2^d(t)) = 0. \qquad (6)$$

Tracking control with uncertainty and input saturation: It is given a desired trajectory $p^d(t)$ and a desired unit vector $b_2^d(t)$. If m, J, d_1, and d_2 are unknown, the control problem is to design a state feedback controller (f, τ) such that (5)-(6) are satisfied and

$$0 < f \le M_f, \quad |\tau_j| \le M_\tau, \quad 1 \le j \le 3 \qquad (7)$$

where M_f and M_τ are appropriate positive constants.

In order to solve the above problems, the following assumptions are made.

Assumption 1: The mass m is an unknown constant and $m \in [\underline{m}, \bar{m}]$ where $\underline{m}$ and $\bar{m}$ are known constants.

Assumption 2: The inertia matrix J is an unknown diagonal constant matrix (i.e., $J = \text{diag}([J_1, J_2, J_3])$) and $J_i \in [\underline{J}, \bar{J}]$ for $1 \le i \le 3$, where $\underline{J}$ and $\bar{J}$ are known constants.

Assumption 3: d_1 and d_2 are bounded and $|d_{1j}| \le D_1$ and $|d_{2j}| \le D_2$ for $1 \le j \le 3$, where D_1 and D_2 are known constants, and d_{1j} and d_{2j} are the j-th elements of d_1 and d_2, respectively.

Assumption 4: $p^d(t) = [p_1^d(t), p_2^d(t), p_3^d(t)]^\top$ is smooth, $|\ddot{p}_j^d(t)| \le M_p$ $(1 \le j \le 3)$ for any time.

Assumption 5: $b_2^d(t)$ is smooth. $\dot{b}_2^d$ and $\ddot{b}_2^d$ are bounded. $b_2^d(t) \times b_3^d(t) = 0$ for any time where $b_3^d(t) = \frac{\ddot{p}^d(t) + ge_3}{\|\ddot{p}^d(t) + ge_3\|}$ and $\times$ denotes the cross product of two vectors.

Assumption 6: $M_f > \frac{\sqrt{3}(D_1 + M_p) + g}{\underline{m}}$.

In (7), M_f and M_τ should be large enough such that there exist controllers which make (5)-(6) satisfied.

B. Quaternions

The attitude of a VTOL UAV can be defined by a unit quaternion $q = \left[\eta, \epsilon^\top\right]^\top$ where $\eta \in \Re$ and $\epsilon \in \Re^3$. The relation between q and R is defined by $R = \mathcal{R}(q) = I + 2\eta S(\epsilon) + 2S^2(\epsilon)$.

With the aid of the unit quaternion, (3) can be written as

$$\dot{q} \;=\; \frac{1}{2}A(q)\omega \qquad (8)$$

where $A(q) = \left[-\epsilon, \eta I + S^\top(\epsilon)\right]^\top$.

In this paper, some saturation functions will be applied. It is given a positive constant M, a function $\sigma : \Re \to \Re$ is said to be a smooth monotonically increasing saturation with M if it is a smooth function satisfying: (a). $s\sigma(s) > 0$ for all $s \neq 0$; (b). $|\sigma(s)| \le M$ for all $s \in \Re$; and (c). $\sigma(s)$ is monotonically increasing.

For given $M_i > 0$ $(1 \le i \le 3)$, the smooth function $\sigma(s)$ with M_i is denoted as $\sigma_i(s)$. If $s = [s_1, \ldots, s_n]^\top$, $\sigma_i(s) = [\sigma_i(s_1), \ldots, \sigma_i(s_n)]^\top$ for $1 \le i \le 3$.

III. Controller Design With Uncertainty

We design a controller such that (5)-(6) hold when there is uncertainty. Considering the structure of the system in (1)-(4), a modified backstepping approach will be proposed as follows:

Step 1: Let $e_p = p - p^d$ and $e_v = v - \dot{p}^d$, we have

$$\dot{e}_p \;=\; e_v \qquad (9)$$

$$\dot{e}_v \;=\; -ge_3 - \ddot{p}^d + \frac{1}{m}fRe_3 + d_1. \qquad (10)$$

Consider fRe_3 as a virtual control input, we design it such that (5) is satisfied. Noting the special structure of the system in (9)-(10), we choose a Lyapunov function

$$V_1(\Lambda_1, \Lambda_2, \tilde{\beta}) \;=\; \int_{\mathbf{0}}^{\Lambda_1} \sigma_1(s)^\top ds + \int_{\mathbf{0}}^{\Lambda_2} \sigma_2(s)^\top ds + \frac{k_1}{2} e_v^\top e_v + \frac{\gamma_1^{-1}}{2}\tilde{\beta}^2$$

where $\mathbf{0} = [0,0,0]^\top$, $k_1 > 0$, $k_2 > 0$, γ_1 is a positive constant, and

$$\Lambda_1 = k_1 e_p + k_2 e_v, \quad \Lambda_2 = k_2 e_v, \quad \tilde{\beta} = \frac{1}{m} - \beta$$

where β is an estimate of $\frac{1}{m}$. It can be proved that V_1 is a positive definite function of e_p, e_v, and $\tilde{\beta}$. Furthermore, $V_1 = 0$ if $e_p = \mathbf{0}$, $e_v = \mathbf{0}$, and $\tilde{\beta} = 0$. The derivative of V_1 is

$$\dot{V}_1 = k_1\sigma_1(\Lambda_1)^\top e_v + (k_2\sigma_1(\Lambda_1) + k_2\sigma_2(\Lambda_2) + k_1 e_v)^\top \times (-ge_3 - \ddot{p}^d + \beta fRe_3 + d_1) - \gamma_1^{-1}\left(\frac{1}{m} - \beta\right)(\dot{\beta} - \gamma_1(k_2\sigma_1(\Lambda_1) + k_2\sigma_2(\Lambda_2) + k_1 e_v)^\top fRe_3).$$

To make $\dot{V}_1$ as small as possible, we choose the virtual control input for fRe_3 and update law of β as

$$\begin{aligned}\alpha &= [\alpha_1, \alpha_2, \alpha_3]^\top = \frac{1}{\beta}(-\sigma_1(\Lambda_1) - \sigma_2(\Lambda_2) - D_1 h(G,\delta) \\ &\quad + ge_3 + \ddot{p}^d) \quad (11)\\ \dot{\beta} &= \text{Proj}_{\Omega_m}\left(\gamma_1 G^\top fRe_3\right) = \text{Proj}_{\Omega_m}(H) \\ &= \begin{cases} & \text{if } \beta \in (\frac{1}{\bar{m}}, \frac{1}{\underline{m}}), \\ H, & \text{or if } \beta = \frac{1}{\bar{m}} \text{ and } H > 0, \\ & \text{or if } \beta = \frac{1}{\underline{m}} \text{ and } H < 0 \\ 0, & \text{otherwise} \end{cases} \quad (12)\end{aligned}$$

where

$$\begin{aligned}\Omega_m &= \left[\frac{1}{\bar{m}}, \frac{1}{\underline{m}}\right] \\ H &= \gamma_1 G^\top fRe_3 \quad (13)\\ G &= [G_1, G_2, G_3]^\top = k_2\sigma_1(\Lambda_1) + k_2\sigma_2(\Lambda_2) \\ &\quad + k_1 e_v \quad (14)\\ h(G,\delta) &= \left[\frac{G_1}{\sqrt{G_1^2+\delta(t)^2}}, \frac{G_2}{\sqrt{G_2^2+\delta(t)^2}}, \frac{G_3}{\sqrt{G_3^2+\delta(t)^2}}\right]^\top \quad (15)\end{aligned}$$

$\delta(t) = e^{-\delta_1 t}$ and $\delta_1 > 0$. Then, if $fRe_3 = \alpha$, we have

$$\begin{aligned}\dot{V}_1 &\leq -k_2(\sigma_1(\Lambda_1) + \sigma_2(\Lambda_2))^\top(\sigma_1(\Lambda_1) + \sigma_2(\Lambda_2)) \\ &\quad - \frac{k_1}{k_2}\sigma_2(\Lambda_2)^\top \sigma_2(\Lambda_2) + 3D_1\delta(t). \quad (16)\end{aligned}$$

It should be noted that α is well defined since $\beta \geq \frac{1}{\bar{m}}$. For the defined α, we have the following lemma by simple calculation.

Lemma 1: If M_1 and M_2 are chosen such that

$$g - M_p - M_1 - M_2 - D_1 > 0 \quad (17)$$

then $\|\alpha\| > 0$ and

$$\|\alpha\| \leq \frac{\sqrt{3}(M_1 + M_2 + D_1 + M_p) + g}{\underline{m}}. \quad (18)$$

Step 2: We find f and a virtual control input q_d for the unit quaternion q. We choose

$$f = \|\alpha\|. \quad (19)$$

It is obvious that $f > 0$ for any time with the aid of Lemma 1.

Let $r_3 = \frac{\alpha}{\|\alpha\|}$, we define

$$\begin{aligned}r_2 &= \frac{(r_3^\top b_3^d)b_2^d - (r_3^\top b_2^d)b_3^d}{\|(r_3^\top b_3^d)b_2^d - (r_3^\top b_2^d)b_3^d\|} \quad (20)\\ r_1 &= \frac{r_2 \times r_3}{\|r_2 \times r_3\|}. \quad (21)\end{aligned}$$

The desired attitude of R is chosen as

$$R_d = [r_1, r_2, r_3] \quad (22)$$

and the desired quaternion $q_d = [\eta_d, \epsilon_d^\top]^\top$ of q is calculated from (22) by the equations (166)-(168) in [24] which are omitted here. The desired angular velocity is calculated by

$$\omega_d = 2A(q_d)^\top \frac{dq_d}{dt}. \quad (23)$$

It should be noted that q_d and ω_d are well defined because f is always positive.

With the aid of the virtual control input q_d,

$$fR_d e_3 = \alpha$$

and equation (10) can be written as

$$\begin{aligned}\dot{e}_v &= -\sigma(\Lambda_1) - \sigma(\Lambda_2) - D_1 h + d_1 + \left(\frac{1}{m} - \beta\right) fRe_3 \\ &\quad + \beta\|\alpha\| R_d(R_d^\top R - I_3)e_3. \quad (24)\end{aligned}$$

Step 3: We assume ω is a virtual control input and design a virtual controller for ω such that (5)-(6) satisfied. Let the difference between q and q_d be

$$\tilde{q} = q_d^{-1} \otimes q = [\tilde{\eta}, \tilde{\epsilon}^\top]^\top, \quad (25)$$

The derivative of $\tilde{q}$ is

$$\dot{\tilde{q}} = \frac{1}{2}A(\tilde{q})(\omega - \tilde{R}^\top \omega_d) \quad (26)$$

where $\tilde{R} = R_d^\top R$. (24) can be written as

$$\begin{aligned}\dot{e}_v &= -\sigma_1(\Lambda_1) - \sigma_2(\Lambda_2) - D_1 h + d_1 + \Big(\frac{1}{m} \\ &\quad - \beta\Big) fRe_3 - 2\beta\|\alpha\| R_d(\tilde{\eta} I_3 + S(\tilde{\epsilon}))S(e_3)\tilde{\epsilon}. \quad (27)\end{aligned}$$

Choose a Lyapunov function

$$V_2 = V_1 + 2(1 - \tilde{\eta}) = V_1 + \tilde{\epsilon}^\top \tilde{\epsilon} + (1 - \tilde{\eta})^2 \quad (28)$$

It can be proved that V_2 is a positive definite function of $(\Lambda_1, \Lambda_2, \tilde{\beta}, 1 - \tilde{\eta})$ and $V_2 = 0$ if $(\Lambda_1, \Lambda_2, \tilde{\beta}, 1 - \tilde{\eta}) = (0, 0, 0, 0)$. The derivative of V_2 is

$$\begin{aligned}\dot{V}_2 &\leq -k_2(\sigma_1(\Lambda_1) + \sigma_2(\Lambda_2))^\top(\sigma_1(\Lambda_1) + \sigma_2(\Lambda_2)) \\ &\quad - \frac{k_1}{k_2}\sigma_2(\Lambda_2)^\top \sigma_2(\Lambda_2)\delta + 3D_1 + \tilde{\epsilon}^\top(\omega - \tilde{R}^\top \omega_d \\ &\quad - 2\beta\|\alpha\| S(e_3)^\top(\tilde{\eta} I_3 + S(\tilde{\epsilon}))^\top R_d^\top G)\end{aligned}$$

To make $\dot{V}_2$ as small as possible, a virtual controller μ for ω can be chosen as

$$\mu = -k_3\tilde{\epsilon} + \tilde{R}^\top \omega_d + 2\beta\|\alpha\| S(e_3)^\top(\tilde{\eta} I_3 + S(\tilde{\epsilon}))^\top R_d^\top G$$

where k_3 is a positive constant. If ω were a real control input, i.e., $\omega = \mu$, then

$$\begin{aligned}\dot{V}_2 &\leq -k_2(\sigma_1(\Lambda_1) + \sigma_2(\Lambda_2))^\top(\sigma_1(\Lambda_1) + \sigma_2(\Lambda_2)) \\ &\quad - \frac{k_1}{k_2}\sigma_2(\Lambda_2)^\top \sigma_2(\Lambda_2) + 3D_1\delta - k_3\tilde{\epsilon}^\top \tilde{\epsilon}.\end{aligned}$$

Step 4: Since ω is not a real control input, ω cannot be μ. Let

$$\tilde{\omega} = [\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3]^\top = \omega - \mu,$$

then,

$$\begin{aligned}\dot{\tilde{q}} &= \frac{1}{2}A(\tilde{\eta}, \tilde{\epsilon})(-k_3\tilde{\epsilon} + \tilde{\omega} + 2\beta\|\alpha\| S(e_3)^\top(\tilde{\eta} I_3 \\ &\quad + S(\tilde{\epsilon}))^\top R_d^\top G) \quad (29)\\ J\dot{\tilde{\omega}} &= \tau - (S(\omega)\Gamma(\omega) + \Gamma(\dot{\mu}))a + d_2 \quad (30)\end{aligned}$$

where $\Gamma(\omega)$ denotes a diagonal matrix with its diagonal elements being the vector ω and

$$a = [a_1, a_2, a_3]^\top = [J_1, J_2, J_3]^\top.$$

Since a and d_2 are unknown, an adaptive robust control law will be proposed such that (5)-(6) are satisfied. To this end, we choose a Lyapunov function

$$V_3 = V_2 + \frac{1}{2}\tilde{\omega}^\top J\tilde{\omega} + \frac{\gamma_2^{-1}}{2}(a-\hat{a})^\top(a-\hat{a})$$

where γ_2 is a positive constant and $\hat{a}$ is an estimate of a which will be designed later. The derivative of V_3 along the solution of (27), (29) and (30) is

$$\begin{aligned}\dot{V}_3 \leq\ & -k_2(\sigma_1(\Lambda_1)+\sigma_2(\Lambda_2))^\top(\sigma_1(\Lambda_1)+\sigma_2(\Lambda_2)) \\ & -\frac{k_1}{k_2}\sigma_2(\Lambda_2)^\top\sigma_2(\Lambda_2) + 3D_1\delta - k_3\tilde{\epsilon}^\top\tilde{\epsilon} + \tilde{\epsilon}^\top\tilde{\omega} \\ & +\tilde{\omega}^\top(\tau - (S(\omega)\Gamma(\omega)+\Gamma(\dot{\mu}))\hat{a} + d_2) \\ & -\gamma_2^{-1}(a-\hat{a})^\top\left(\dot{\hat{a}} + \gamma_2(S(\omega)\Gamma(\omega)+\Gamma(\dot{\mu}))^\top\tilde{\omega}\right).\end{aligned}$$

To make $\dot{V}_3$ as small as possible, we choose the control law τ and the update law $\hat{a}$ as follows:

$$\tau = -k_4\tilde{\omega} - \tilde{\epsilon} + (S(\omega)\Gamma(\omega)+\Gamma(\dot{\mu}))\hat{a} - D_2h(\tilde{\omega},\delta) \quad (31)$$

$$\dot{\hat{a}} = \mathrm{Proj}_{\Omega_a}(-\gamma_2(S(\omega)\Gamma(\omega)+\Gamma(\dot{\mu}))^\top\tilde{\omega}) \quad (32)$$

where $\Omega_a = \left[\underline{J}, \bar{J}\right]$ and k_4 is a positive constant. Then,

$$\begin{aligned}\dot{V}_3 \leq\ & -k_2(\sigma_1(\Lambda_1)+\sigma_2(\Lambda_2))^\top(\sigma_1(\Lambda_1)+\sigma_2(\Lambda_2)) \\ & -\frac{k_1}{k_2}\sigma_2(\Lambda_2)^\top\sigma_2(\Lambda_2) + 3D_1\delta + 3D_2\delta - k_3\tilde{\epsilon}^\top\tilde{\epsilon} \\ & -k_4\tilde{\omega}^\top\tilde{\omega}. \end{aligned} \quad (33)$$

Based on the above controller design procedure, we have the following results.

Theorem 1: For the system in (1)-(4) and given desired trajectories p^d and b_2^d, the control inputs (f,τ) in (19) and (31) with the update laws in (12) and (32) ensure that (5)-(6) are satisfied and $(\beta,\hat{a})$ are bounded. Furthermore, the thrust force f is larger than zero at any time and is bounded.

IV. Controller Design with Uncertainty and Input Saturation

Assume the input τ is subject to a saturation constraint, i.e.,

$$\tau = \sigma_3(\tau^d) \quad (34)$$

where $\tau^d \in R^3$ is a new control input without constraint and will be designed later. In order to compensate the effect of the input saturation, the following auxiliary compensated system is defined:

$$\dot{q}_a = \frac{1}{2}A(q_a)\left(\tilde{R}_a\omega_a - k_5\epsilon_a\right) \quad (35)$$

$$\dot{\omega}_a = -k_4\omega_a + \Gamma(\hat{\theta})(\tau-\tau^d) \quad (36)$$

where $q_a = [\eta_a, \epsilon_a^\top]^\top$ is an auxiliary unit quaternion, $\omega_a \in R^3$ is an auxiliary angular velocity, $\hat{\theta} = [\hat{\theta}_j, \hat{\theta}_2, \hat{\theta}_3]^\top$ is an estimate of

$$\theta = [\theta_1,\theta_2,\theta_3]^\top = \left[\frac{1}{J_1},\frac{1}{J_2},\frac{1}{J_3}\right]^\top$$

and will be designed later, k_4 and k_5 are positive constants, and

$$R_a = I + 2\eta_a S(\epsilon_a) + 2S^2(\epsilon_a) \quad (37)$$

$$\tilde{R}_a = R_a^\top R. \quad (38)$$

Define

$$\tilde{q}_a = q_a^{-1}\otimes q \quad (39)$$

then,

$$\dot{\tilde{q}}_a = \frac{1}{2}A(\tilde{q}_a)\left(\omega - \omega_a + k_5\tilde{R}_a^\top\epsilon_a\right). \quad (40)$$

Let r_2 and r_1 be defined as in (20) and (21) and

$$\bar{\alpha} = R_a^\top\alpha, \quad r_3 = \frac{\bar{\alpha}}{\|\bar{\alpha}\|} \quad (41)$$

where α is defined in (11). Define R_d as in (22). The unit quaternion corresponding to R_d is denoted as q_d. By (23), we can calculate ω_d.

Define $\tilde{q} = [\tilde{\eta}, \tilde{\epsilon}^\top]^\top = q_d^{-1}\otimes\tilde{q}_a$, then,

$$\dot{\tilde{q}} = \frac{1}{2}A(\tilde{q})(\omega - \omega_a - \tilde{R}^\top\omega_d + k_5\tilde{R}_a^\top\epsilon_a) \quad (42)$$

where $\tilde{R} = R_d^\top R_a^\top R$. We choose f as in (19). It can be shown that $f = \|\bar{\alpha}\|$. So, $fR_de_3 = \bar{\alpha} = R_a^\top\alpha$ and equation (10) can be written as

$$\begin{aligned}\dot{e}_v =\ & -\sigma(\Lambda_1) - \sigma(\Lambda_2) - D_1h + d_1 + \left(\frac{1}{m}-\beta\right)fRe_3 \\ & -2\beta\|\bar{\alpha}\|R_aR_d(\tilde{\eta}I_3 + S(\tilde{\epsilon}))S(e_3)\tilde{\epsilon}. \end{aligned} \quad (43)$$

Choose a Lyapunov function

$$V_4 = V_1 + 2(1-\tilde{\eta}).$$

It can be proved that V_4 is a nonnegative function of Λ_1, Λ_2, and $\tilde{\eta}$. The derivative of V_4 is

$$\begin{aligned}\dot{V}_4 \leq\ & -k_2(\sigma_1(\Lambda_1)+\sigma_2(\Lambda_2))^\top(\sigma_1(\Lambda_1)+\sigma_2(\Lambda_2)) \\ & -\frac{k_1}{k_2}\sigma_2(\Lambda_2)^\top\sigma_2(\Lambda_2) + 3D_1\delta + \tilde{\epsilon}^\top(\omega - \tilde{R}^\top\omega_d \\ & -\omega_a - 2\beta\|\alpha\|S(e_3)^\top(\tilde{\eta}I_3 + S(\tilde{\epsilon}))^\top(R_aR_d)^\top G \\ & +k_5\tilde{R}_a^\top\epsilon_a).\end{aligned}$$

To make $\dot{V}_4$ as small as possible, a virtual controller μ for ω can be chosen as

$$\begin{aligned}\mu =\ & 2\beta\|\alpha\|S(e_3)^\top(\tilde{\eta}I_3 + S(\tilde{\epsilon}))^\top(R_aR_d)^\top G - k_3\tilde{\epsilon} \\ & +\tilde{R}^\top\omega_d - k_5\tilde{R}_a^\top\epsilon_a \end{aligned} \quad (44)$$

where k_3 is a positive constant. If ω were the real control input, i.e., $\omega = \mu$, then

$$\begin{aligned}\dot{V}_4 \leq\ & -k_2(\sigma_1(\Lambda_1)+\sigma_2(\Lambda_2))^\top(\sigma_1(\Lambda_1)+\sigma_2(\Lambda_2)) \\ & -\frac{k_1}{k_2}\sigma_2(\Lambda_2)^\top\sigma_2(\Lambda_2) + 3D_1\delta + \tilde{\epsilon}^\top(-k_3\tilde{\epsilon} - \omega_a).\end{aligned}$$

Since ω is not the real control input, ω cannot be μ. Let

$$\tilde{\omega} = [\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3]^\top = \omega - \mu - \omega_a$$

then,

$$\begin{aligned}
\dot{\tilde{q}} &= \frac{1}{2}A(\tilde{q})(-k_3\tilde{\epsilon} + \tilde{\omega} + 2\beta\|\alpha\|S(e_3)^\top(\tilde{\eta}I_3 \\
&\quad +S(\tilde{\epsilon}))^\top(R_aR_d)^\top G) \\
\dot{\tilde{\omega}} &= \Gamma(\Pi)B + \Gamma(\theta)(\tau + d_2) - \dot{\mu} + k_4\omega_a - \Gamma(\hat{\theta})(\tau - \tau^d) \\
&= \Gamma(\hat{\theta})\tau^d + \Gamma(\theta - \hat{\theta})\tau + k_4\omega_a + \Gamma(\Pi)B + \Gamma(\theta)d_2 - \dot{\mu}
\end{aligned}$$

where

$$\begin{aligned}
B &= \left[\ B_1, B_2, B_3\ \right]^\top = \left[\ \omega_2\omega_3, \omega_1\omega_3, \omega_1\omega_2\ \right]^\top \\
\Pi &= \left[\ \Pi_1, \Pi_2, \Pi_3\ \right]^\top \\
&= \left[\ J_1^{-1}(J_2 - J_3), J_2^{-1}(J_3 - J_1), J_3^{-1}(J_1 - J_2)\ \right]^\top.
\end{aligned}$$

To make $\tilde{q}$ converge to the identity quaternion and $\tilde{\omega}$ converge to zero, we choose a Lyapunov function

$$\begin{aligned}
V_5 &= V_4 + \frac{1}{2}\tilde{\omega}^\top\tilde{\omega} + \frac{\gamma_2^{-1}}{2}(\theta - \hat{\theta})^\top(\theta - \hat{\theta}) \\
&\quad + \frac{\gamma_3^{-1}}{2}(\Pi - \hat{\Pi})^\top(\Pi - \hat{\Pi})
\end{aligned}$$

where $\hat{\Pi} = [\hat{\Pi}_1, \hat{\Pi}_2, \hat{\Pi}_3]^\top$ is an estimate of Π, γ_2 and γ_3 are positive constants. The derivative of V_5 is

$$\begin{aligned}
\dot{V}_5 &\leq -k_2(\sigma_1(\Lambda_1) + \sigma_2(\Lambda_2))^\top(\sigma_1(\Lambda_1) + \sigma_2(\Lambda_2)) \\
&\quad -\frac{k_1}{k_2}\sigma_2(\Lambda_2)^\top\sigma_2(\Lambda_2) + 3D_1\delta - k_3\tilde{\epsilon}^\top\tilde{\epsilon} + \tilde{\omega}^\top(\Gamma(\hat{\theta})\tau^d \\
&\quad +\tilde{\epsilon} + \Gamma(\hat{\Pi})B + k_4\omega_a + \Gamma(\theta)d_2 - \dot{\mu}) \\
&\quad -\gamma_2^{-1}(\theta - \hat{\theta})^\top\left(\dot{\hat{\theta}} - \gamma_2\Gamma(\tau)\tilde{\omega}\right) \\
&\quad -\gamma_3^{-1}(\Pi - \hat{\Pi})^\top\left(\dot{\hat{\Pi}} - \gamma_3\Gamma(B)\tilde{\omega}\right).
\end{aligned}$$

To make $\dot{V}_5$ as small as possible, we choose

$$\begin{aligned}
\tau^d &= \Gamma^{-1}(\hat{\theta})(-k_4(\omega - \mu) - \tilde{\epsilon} - \Gamma(\hat{\Pi})B + \dot{\mu} \\
&\quad -\frac{D_2}{\underline{J}}h(\tilde{\omega}, \delta)) &(45) \\
\dot{\hat{\theta}}_j &= \mathrm{Proj}_{\Omega_\theta}(\gamma_2\tau_j\tilde{\omega}_j), \quad j = 1, 2, 3 &(46) \\
\dot{\hat{\Pi}}_j &= \gamma_3B_j\tilde{\omega}_j, \quad j = 1, 2, 3 &(47)
\end{aligned}$$

where $\Omega_\theta = \left[\frac{1}{\bar{J}}, \frac{1}{\underline{J}}\right]$. Then,

$$\begin{aligned}
\dot{V}_5 &\leq -k_2(\sigma_1(\Lambda_1) + \sigma_2(\Lambda_2))^\top(\sigma_1(\Lambda_1) + \sigma_2(\Lambda_2)) \\
&\quad -\frac{k_1}{k_2}\sigma_2(\Lambda_2)^\top\sigma_2(\Lambda_2) + 3D_1\delta - k_3\tilde{\epsilon}^\top\tilde{\epsilon} \\
&\quad -k_4\tilde{\omega}^\top\tilde{\omega} + \frac{3D_2\delta}{\underline{J}}. &(48)
\end{aligned}$$

With the aid of the above procedure, we have the following results.

Theorem 2: For the system in (1)-(4) and given desired trajectories p^d and b_2^d, the control inputs (f, τ) in (19) and (34) with τ^d in (45) and the update laws in (12) and (46)-(47) ensure that

1) β, $\hat{\Pi}$, and $\hat{\theta}$ are bounded,
2) e_p, e_v, $\tilde{\epsilon}$, and $\tilde{\omega}$ converge to zero, and
3) $b_2 - b_2^d$ is uniformly ultimate boundedness (UUB).

Furthermore, f is larger than zero at any time and (7) are satisfied if M_1, M_2, and M_3 are chosen such that

$$M_1 + M_2 < \frac{\underline{m}M_f - g}{\sqrt{3}} - D_1 - M_p, M_3 = M_\tau.$$

V. Simulation

Simulation results are presented to illustrate the effectiveness of the proposed controllers. We consider a VTOL UAV modeled as a rigid body with mass $m = 0.85$kg and inertia tensor $J = \mathrm{diag}([4.856, 4.856, 9.801])^{-2}$kg m^2 (see [14]). In the controllers, m and J are unknown. However, it is known that $m \in [0.7, 1]$kg, i.e., $\underline{m} = 0.7$kg and $\bar{m} = 1$kg. For disturbance, it is assumed that d_1 and d_2 are white noise with magnitudes $D_1 = D_2 = 0.05$.

In the simulation, the desired trajectory p^d and b_2^d are chosen as $p^d(t) = [100\cos(0.05t), 100\sin(0.05t), 10 - 10\exp(-0.1t)]$ and $b_2^d = [\sin(0.05t), -\cos(0.05t), 0]^\top$. If there is no input constraint, the robust adaptive controller is (19) and (31) with the aid of Theorem 1. In the control law, we chose $\sigma_i(x) = M_i\tanh(x)$ where $M_i = 4$. It can be verified that (17) is satisfied. Simulation was done for one group of control parameters. The time response of the tracking errors of $p_1 - p_1^d$, $p_2 - p_2^d$, and $p_3 - p_3^d$ are shown in Fig. 1 which shows they converge to zero. Fig. 2 depicts the response of the tracking error $\tilde{q}$. It shows that $\tilde{\eta}$ asymptotically converge to one and $\tilde{\epsilon}$ asymptotically converges to zero. The simulation results show the effectiveness of the results in Theorem 1.

If there are uncertainty and input constraints. The control laws can be obtained in (19) and (34) with the aid of Theorem 2. In the simulation, the bounds on the force and the toque are chosen as $M_f = 12$ N and $M_\tau = 0.05$. It can be shown that Assumption 6 is satisfied. Simulation was done for a set of chosen control parameters. The time response of the tracking errors of $p_1 - p_1^d$, $p_2 - p_2^d$, and $p_3 - p_3^d$ are shown in Fig. 3. Fig. 4 shows the response of the tracking error $q_d^{-1} \otimes q$. Fig. 5 shows the total force f. It is obvious that f is bounded and is larger than zero at any time. Fig. 6 shows the input torque τ. It is bounded by 0.05. The simulation results show the effectiveness of the results in Theorem 2.

VI. Conclusion

This paper considered the tracking control problems of a VTOL UAV with uncertainty and input constraints. Considering the uncertainty in the dynamics of the system, a robust adaptive tracking controller was proposed such that the position and the attitude of a VTOL UAV asymptotically converge to their desired value with the aid of the backstepping technique. Considering the uncertainty and input constraints, a saturation robust adaptive controller was proposed with the aid of an auxiliary compensated system. Simulation results show the effectiveness of the proposed controllers.

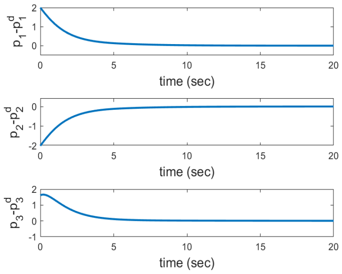

Fig. 1. Time response of $p - p_d$

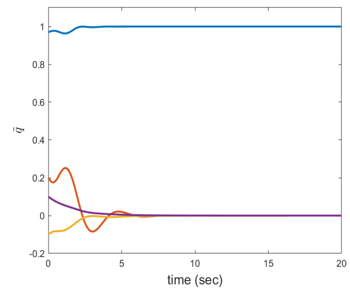

Fig. 2. Time response of $\tilde{q}$

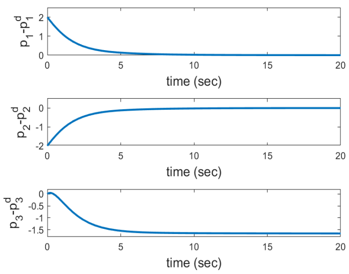

Fig. 3. Time response of $p - p_d$

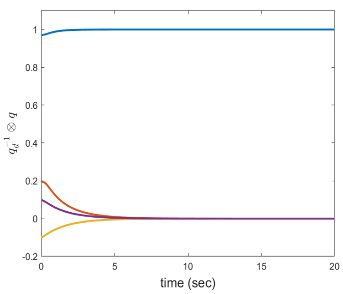

Fig. 4. Time response of $\tilde{q}$

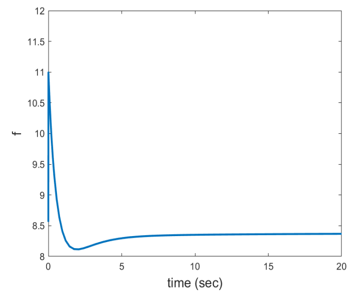

Fig. 5. Time response of f

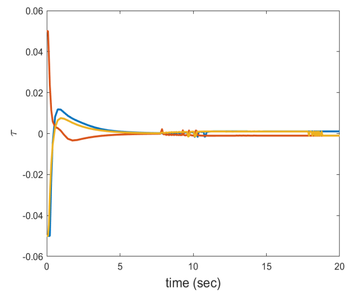

Fig. 6. Time response of τ

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