Abstract

Let A be a residually finite dimensional algebra (not necessarily associative) over a field k. Suppose first that k is algebraically closed. We show that if A satisfies a homogeneous almost identity Q, then A has an ideal of finite codimension satisfying the identity Q. Using well known results of Zelmanov, we conclude that, if a residually finite dimensional Lie algebra L over k is almost d-Engel, then L has a nilpotent (resp. locally nilpotent) ideal of finite codimension if char $k = 0$ (resp. char $k > 0$).

Next, suppose that k is finite (so A is residually finite). We prove that, if A satisfies a homogeneous probabilistic identity Q, then Q is a coset identity of A. Moreover, if Q is multilinear, then Q is an identity of some finite index ideal of A.

Along the way we show that, if $Q \in k\langle x_1, \dots, x_n \rangle$ has degree d, and A is a finite k-algebra such that the probability that $Q(a_1, \dots, a_n) = 0$ (where $a_i \in A$ are randomly chosen) is at least $1 - 2^{-d}$, then Q is an identity of A. This solves a ring-theoretic analogue of a (still open) group-theoretic problem posed by Dixon.

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Residually finite dimensional algebras and polynomial almost identities

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1. *Introduction*

In this paper, we prove three theorems concerning residually finite dimensional algebras A and polynomial identities. The common theme is that if a (non-commutative) homogeneous polynomial Q in n variables vanishes on a large enough subset of A^n, then it is actually a *coset identity*, that is, it holds identically on $(a_1 + I) \times \cdots \times (a_n + I)$ for some two-sided ideal I of finite codimension in A and some $a_1, \ldots, a_n \in A$. Under some assumptions we obtain stronger conclusions, namely, that Q is an identity of the ideal I.

Let k be an algebraically closed field, V a k-vector space, possibly of infinite dimension, and n a positive integer. We recall [**LS**] that the *codimension* of a subset $X \subset V^n$ is the smallest integer c for which there exists a direct sum decomposition of k-vector spaces $V^n = V_1 \oplus V_2$, where V_2 is finite dimensional, and an algebraic set X_2 of codimension c in V_2, such that $X \supset V_1 \times X_2$. We say that X is of *infinite codimension* if no such decomposition exists.

Let A be an associative k-algebra, possibly non-unital. Each non-commutative polynomial $Q \in k\langle x_1, \ldots, x_n\rangle$ defines the evaluation map $e_Q\colon A^n \to A$. We define $\mathrm{cd}_Q A$ to be the codimension of $e_Q^{-1}(0)$. We say that Q is an *almost identity* if $\mathrm{cd}_Q A < \infty$. If $\mathrm{cd}_Q A = 0$, or, equivalently, $e_Q(A^n) = 0$, we say Q is an *identity* for A.

We can likewise consider a Lie (resp. Jordan) algebra A over k and a Lie (resp. Jordan) polynomial Q and define the codimension of the zero set of Q in A^n and an almost identity in the analogous way. In fact the same definition applies for an arbitrary algebra A, namely a linear space over k with a bilinear map $A \times A \to A$ as multiplication (possibly but not necessarily satisfying some extra-conditions). In this case the polynomial Q is an element of a free algebra in the respective category. Note that an *ideal* of A will always mean a two-sided ideal, and A *residually finite dimensional* means that the intersection of all ideals of A of finite codimension is the zero ideal.

Our first main result is the following:

Theorem 1. *Let A be a residually finite dimensional algebra over k and Q a homogeneous polynomial as above. Then the following are equivalent:*
(i) *The polynomial Q is an almost identity for A.*
(ii) *The polynomial Q is an identity for some ideal I of A of finite codimension.*

The non-trivial part is that (1) implies (2). The reverse implication follows from the fact that I^n is of finite codimension in A^n.

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We now compare the theorem above with the main result of [**LS**] (see Theorem 1 there and the comments on p. 10). The latter result shows that an associative/Lie/Jordan algebra with an almost identity Q satisfies some identity P (usually different and more complex than Q). Theorem 1 above holds for *all* algebras A, and the identity satisfied by the ideal I is the original almost identity Q.

Our second main result concerns the case that k is a finite field $\mathbb{F}_q$ for a prime power q. Any residually finite algebra A over $\mathbb{F}_q$ is a dense subalgebra of its completion $\overline{A}$, and as an additive group, $\overline{A}$ is profinite. For each n, we endow $\overline{A}^n$ with its Haar measure and consider the condition on a subset $X \subset A^n$ that $\overline{X}$ has non-zero measure. We say that Q is a *probabilistic identity* if the closure of $e_Q^{-1}(0)$ has positive measure.

THEOREM 2. *Let A be a residually finite dimensional algebra over $k = \mathbb{F}_q$ and Q a homogeneous polynomial as above. Then the following are equivalent:*
(i) *The polynomial Q is a probabilistic identity of A.*
(ii) *The polynomial Q is a coset identity of some finite index ideal I of A.*
Furthermore, if Q is multilinear, then these conditions are equivalent to
(3) *Q is an identity of some finite index ideal I of A.*

Again, the non-trivial part is the claim that (1) implies (2).

In [**D**] Dixon asks whether, for every group-word $w \in F_n$ (the free group of rank n) there exists $\epsilon = \epsilon(w) > 0$ such that if G is a finite group, and the word map $w : G^n \to G$ attains the value 1 with probability $\geq 1 - \epsilon$, then w is an identity of G. In spite of some positive results for special words w, Dixon's Problem is still very much open. Here we obtain a general positive solution of an analogous question on finite algebras A. The solution is effective in the sense that ϵ is given explicitly; in fact, if d is the degree of the ambient polynomial map, then $\epsilon = 2^{-d}$ will do.

THEOREM 3. *Let A be any finite-dimensional algebra over a finite field and $e_Q \colon A^n \to A$ a polynomial map associated with a polynomial Q of degree d in n variables in the respective category. If $\frac{|e_Q^{-1}(0)|}{|A|^n} \geq 1 - 2^{-d}$, then Q is identically zero.*

For $d \geq 2$ consider the degree d Engel polynomial $E_{d-1} := [x, y, \ldots, y]$, a left-normed Lie product where y appears $d-1$ times, as an element of the free Lie algebra on x, y over the underlying field k. By [**MM**, 2.1], if L is a finite Lie algebra in which the Engel condition $E_{d-1} = 0$ holds with probability greater than $1 - 2^{-d}$, then E_{d-1} is an identity of L. Theorem 3 above extends this for any finite algebra and any polynomial.

2. *Algebras over algebraically closed fields*

We recall that every subset S of $k^m = \mathbb{A}^m(k)$ defines the ideal $Z(S)$ of elements in $k[x_1, \ldots, x_m]$ which vanish on S and every ideal $I \subset k[x_1, \ldots, x_m]$ defines the algebraic set $V(I)$ of common zeroes of I. By Hilbert's Nullstellensatz, these two maps give a bijection between algebraic sets in k^n and radical ideals I. Radical ideals I are in bijective correspondence with reduced closed k-subschemes Spec $k[x_1, \ldots, x_m]/I$. The bijection between algebraic sets and reduced closed k-subschemes is given in one direction by taking the Zariski closure with its reduced closed subscheme structure and in the other by taking k-points.

Each linear transformation $T \colon V \to W$ of finite dimensional k-vector spaces defines a homomorphism of commutative graded k-algebras $T^* \colon \mathrm{Sym}^* W^* \to \mathrm{Sym}^* V^*$. If $S_V \subset V$, $S_T \subset$

W, and $T(S_V) \subset S_W$, then $T^*(Z(S_W)) \subset Z(S_V)$. Thus, T determines a morphism of affine k-schemes

$$\mathrm{Spec}\ \mathrm{Sym}^* V^*/Z(S_V) \to \mathrm{Spec}\ \mathrm{Sym}^* W^*/Z(S_W)$$

which, at the level of k-points, gives the restriction of T.

We now prove the key proposition:

Proposition 4. *If A is an arbitrary algebra over k, I is an ideal of A of finite codimension, and Q is a homogeneous polynomial as above, then*

$$\mathrm{cd}_Q A \geq \mathrm{cd}_Q I + \mathrm{cd}_Q A/I.$$

Proof. There is something to check only if Q is an almost identity for A. Let $V_1 \oplus V_2$ be a direct sum decomposition of A^n such that $e_Q^{-1}(0)$ contains $V_1 \times X_2$ for some algebraic set X_2 of codimension $\mathrm{cd}_Q A$ in the finite-dimensional space V_2. Let $V_1' = V_1 \cap I^n$, and let V_1'' denote a complementary subspace to V_1' in V_1. Let $V_2' = V_2 \oplus V_1''$ and $X_2' = X_2 \times V_1''$. Then X_2' is of codimension $\mathrm{cd}_Q A$ in V_2', and $V_1' \times X_2' = V_1 \times X_2 \subset e_Q^{-1}(0)$. Replacing V_1, V_2, X_2 by V_1', V_2', X_2', we may therefore assume that $V_1 \subset I^n$.

We identify V_2 with the k-points of the variety $\mathbb{A}^{\dim V_2}$. Let $\underline{X}_2$ denote the Zariski closure of X_2 in $\mathbb{A}^{\dim V_2}$, so we can identify X_2 with $\underline{X}_2(k)$, and $\dim \underline{X}_2 = \dim V_2 - \mathrm{cd}_Q A$. We identify $(A/I)^n$ with the k-points of $\mathbb{A}^{n \dim A/I}$ and denote by $\underline{Y}$ the Zariski closure of the algebraic set

$$Y = \{(\overline{a}_1, \ldots, \overline{a}_n) \in (A/I)^n \mid e_Q(\overline{a}_1, \ldots, \overline{a}_n) = 0\},$$

so the algebraic set is identified with $\underline{Y}(k)$.

The projection map $A^n/V_1 \to A^n/I^n$ maps the algebraic set X_2 to the algebraic set Y, and it follows that the associated projection morphism $\mathbb{A}^{\dim V_2} \to \mathbb{A}^{n \dim A/I}$ defines a morphism $\pi\colon \underline{X}_2 \to \underline{Y}$. As $0 \in \underline{X}_2(k) = X_2$ maps by π to $0 \in \underline{Y}(k) = Y$, the fiber $\underline{Z}$ of π over 0 is non-empty. Now, $\underline{Z}(k)$ is $X_2 \cap (I^n/V_1)$, so $\mathrm{cd}_Q I \leq \dim(I^n/V_1) - \dim \underline{Z}$. By [**Stacks**, Tag 02JS],

$$\dim \underline{X}_2 \leq \dim \underline{Y} + \dim \underline{Z}.$$

Thus,

$$\begin{aligned}\mathrm{cd}_Q A &= \dim A^n/V_1 - \dim \underline{X}_2 = \dim A^n/I^n + \dim I^n/V_1 - \dim \underline{X}_2 \\ &\geq \dim A^n/I^n - \dim \underline{Y} + \dim I^n/V_1 - \dim \underline{Z} \geq \mathrm{cd}_Q(A/I) + \mathrm{cd}_Q I.\end{aligned}$$

□

We can now prove Theorem 1.

Proof. It suffices to prove that, if condition (2) does not hold, then for all $i \geq 0$ there exists an ideal I of finite codimension in A such that $\mathrm{cd}_Q A/I \geq i$. We proceed by induction on i, the statement being trivial for $i = 0$. If the induction hypothesis holds, as Q is not an identity for I, there exists $\alpha \in I^n$ with $e_Q(\alpha) \neq 0$. As A is residually finite dimensional, there exists an ideal J of A of finite codimension such that $e_Q(\alpha) \notin J$. If $\overline{\alpha}$ denotes the image of α in $(I/I \cap J)^n$, then $e_Q(\overline{\alpha}) \neq 0$, so Q is not an identity for $I/I \cap J$. By Proposition 4,

$$\mathrm{cd}_Q A/(I \cap J) \geq \mathrm{cd}_Q A/I + \mathrm{cd}_Q I/(I \cap J) \geq i + 1,$$

and the theorem follows by induction. □

We now discuss some consequences of the theorem.

Corollary 5. *Let k be a field of characteristic 0, A a finitely generated algebra over k, and Q a homogeneous polynomial in n variables defined over k. If V is a subspace of A^n of finite codimension, $\vec{a} = (a_1, \ldots, a_n) \in A^n$, and $e_Q(\vec{a} + V) = 0$, then Q is an identity on an ideal of A of finite codimension.*

Proof. As $A \otimes_k \overline{k}$ is finitely generated over $\overline{k}$, it is residually finite-dimensional. As k is infinite, if Q is an identity on $\vec{a} + V$, then it is an identity on $\vec{a} + V \otimes_k \overline{k}$, which is a subset of finite codimension in $A \otimes_k \overline{k}$. Therefore, there exists an ideal I of finite codimension in $A \otimes_k \overline{k}$ such that Q is an identity on I.

Let $\{a_1, \ldots, a_m\}$ be a generating subset for A over k. Let $\overline{a}_i$ denote the image of $a_i \otimes 1$ under the quotient homomorphism

$$A \otimes_k \overline{k} \to (A \otimes_k \overline{k})/I.$$

The $\overline{k}$-algebra $(A \otimes_k \overline{k})/I$ is finite-dimensional, so fixing any basis, the structure constants lie in a finite extension K of k. We fix a finite-dimensional K-algebra B and an isomorphism of K-algebras $\iota\colon B \otimes_K \overline{k} \to (A \otimes_k \overline{k})/I$. There exists a finite extension L/K such that $\iota^{-1}(\overline{a}_i) \in B \otimes_k L$ for all i. Enlarging L, we may assume L/K is Galois. There is a unique L-algebra homomorphism $\phi\colon A \otimes_k L \to B \otimes_K L$ such that the diagram

$$\begin{tikzcd}
A \otimes_k L \arrow[r] \arrow[d, "\phi"'] & A \otimes_k \overline{k} \arrow[r] & (A \otimes_k \overline{k})/I \\
B \otimes_K L \arrow[rr] & & B \otimes_K \overline{k} \arrow[u, "\iota"']
\end{tikzcd}$$

commutes.

We consider the L-algebra homomorphism

$$A \otimes_k L \to \bigoplus_{\sigma \in \mathrm{Gal}(L/k)} B \otimes_K L$$

which in the σ-coordinate is given by $(\mathrm{Id}_B \otimes \sigma) \circ \phi$. The kernel is invariant under the action of $\mathrm{Gal}(L/k)$ on $A \otimes_k L$, so by Galois descent for vector spaces [**B**, Chap. 5, §10 Prop. 6] it is of the form $W \otimes_k L$, where W is the kernel of the composition of $A \to A \otimes_k L$ and ϕ. Thus W is a finite codimension ideal of A.

□

Corollary 6. *Let A be a residually finite dimensional associative algebra over k. Let $d \geq 1$ and suppose x^d is an almost identity for A. If k has characteristic $p > 0$ suppose also $p > d$. Then A has an ideal I of finite codimension satisfying $I^{f(d)} = 0$, where $f(d)$ is a suitable function of d.*

Proof. By Theorem 1, A has an ideal I of finite codimension satisfying the identity $x^d = 0$. The well known Nagata-Higman Theorem applied for the associative (non-unital) k-algebra I shows that $I^{f(d)} = 0$. See for instance [**DF**, Chapter 6] for the theorem and for explicit bounds on the function f. □

Our next result describes almost d-Engel Lie-algebras.

Theorem 7. *Let L be a residually finite dimensional Lie algebra over k. Let $d \geq 1$ and suppose The Engel polynomial E_d is an almost identity for L. Then*

(i) *If k has characteristic zero then L has a nilpotent ideal of finite codimension.*

(ii) *If k has positive characteristic then L has a locally nilpotent ideal of finite codimension.*

Proof. By Theorem 1, L has an ideal I of finite codimension satisfying the identity $E_d = 0$. The conclusion now follows from well known theorems of Zelmanov on the nilpotency of d-Engel Lie algebras in characteristic zero [**Z1**] and the local nilpotency of d-Engel Lie algebras in positive characteristic [**Z2**, **?**]. □

3. *Algebras over finite fields*

From now on, we assume $k = \mathbb{F}_q$. For every positive integer d, we write $d = m(q-1) + r$, where $0 \le r \le q-2$, and define

$$f_q(d) = \frac{q-r}{q^{m+1}}.$$

Lemma 8. *We have*

$$f_q(d) = \min \prod_{i=1}^{\infty} \frac{q - x_i}{q},$$

where the sum ranges over all infinite real sequences $x_1, x_2, x_3, \ldots \in [0, q-1]$ summing to d.

Proof. As $\log(q-x)$ is concave, the value of the product can only decrease if we replace the sequence with

$$x_1, \ldots, x_{n-1}, \sum_{i=n}^{\infty} x_i, 0, 0, \ldots$$

Thus, we may consider only sequences which are eventually zero. If any two non-zero terms x_i and x_j satisfy $x_i + x_j \le q-1$, then we can decrease the product by replacing x_i and x_j by $x_i + x_j$ and 0 respectively, so we may assume any two non-zero terms sum to more than $q-1$. If $q - 1 > x_i > x_j > 0$ and $x_i + x_j > q-1$, then we can decrease the product by replacing x_i and x_j by $q-1$ and $x_i + x_j - (q-1)$ respectively. Thus, we may assume that there is at most one x_i which is neither 0 nor $q-1$. Without loss of generality, the sequence can be taken to be non-increasing, so the minimum is achieved for

$$x_1 = \cdots = x_m = q-1,\ x_{m+1} = r,\ x_{m+2} = \cdots = 0.$$

□

Lemma 9. *For $1 \le k \le q-1$ and $d \ge k$,*

$$f_q(d) \le \frac{q-k}{q} f_q(d-k).$$

Proof. This follows immediately from the description of f_q in Lemma 8. □

Lemma 10. *For $q \ge 2$, we have $f_q(d) \ge 2^{-d}$.*

Proof. We have $f_q(0) = 1$, and writing $d = m(q-1) + r$, with $0 \le r \le q-2$, we have

$$\frac{f_q(d+1)}{f_q(d)} = \frac{q-r-1}{q-r} \ge \frac{1}{2}.$$

□

THEOREM 11. *Let $\mathbb{F}_q$ be a finite field and $P(x_1,\dots,x_n) \in \mathbb{F}_q[x_1,\dots,x_n]$ a polynomial of degree d. If*

$$N_P := |\{(a_1,\dots,a_n) \in \mathbb{F}_q^n \mid P(a_1,\dots,a_n) \neq 0\}|$$

is non-zero, it satisfies

$$N_P \geq f_q(d)q^n \geq 2^{-d}q^n.$$

Proof.
As a function on $\mathbb{F}_q^n$, P depends only on its residue class modulo $(x_1^q - x_1,\dots,x_n^q - x_n)$. Each such residue class contains a unique element which is of degree $< q$ in each variable separately, and the (total) degree of this representative achieves the minimal degree of all polynomials in the residue class of P. As $f_q(d)$ decreases monotonically in d, we may assume that P is of degree less than q in each variable separately.

We use induction on d, the base case $d = 0$ being trivial. Without loss of generality, we may assume that, as a function on $\mathbb{F}_q^n$, P is not constant in the variable x_n. Since the x_n-degree of P is less than q, this means that the x_n-degree is $l \in [1, q-1]$. We write

$$P(x_1,\dots,x_n) = \sum_{i=0}^{l} P_i(x_1,\dots,x_{n-1})x_n^i.$$

As $l + \deg P_l = \deg x_n^l P_l \leq \deg P$, we have $\deg P_l \leq d - l$. If $(a_1,\dots,a_{n-1}) \in \mathbb{F}_q^{n-1}$ satisfies $P_l(a_1,\dots,a_{l-1}) \neq 0$, there are at least $q - l$ solutions of $P(a_1,\dots,a_{n-1},x_n) \neq 0$. By the induction hypothesis,

$$N_P \geq (q-l)N_{P_l} \geq (q-l)f_q(d-l)q^{n-1} = \frac{q-l}{q} f_q(d-l)q^n \geq f_q(d)q^n.$$

□

We can now prove Theorem 3.

Proof.
We have to show that, if the evaluation map e_Q associated with Q is not identically 0, then $\frac{|e_Q^{-1}(0)|}{|A|^n} < 1 - 2^{-d}$.

If e_Q does not vanish on A, then there exists a linear functional $f \colon A \to \mathbb{F}_q$ such that $f \circ e_Q$ does not vanish on A^n. If Q is of degree d, then $f \circ e_Q$ has degree $\leq d$. Theorem 3 now follows from Theorem 11. □

Let n be a positive integer and consider $\mathbb{F}_q\langle x_1,\dots,x_n\rangle$, the algebra over $\mathbb{F}_q$ of the free magma on n generators. This is a graded $\mathbb{F}_q$-algebra. Let $Q \in \mathbb{F}_q\langle x_1,\dots,x_n\rangle$ denote a non-zero element of degree d. If I is any ideal of A of finite codimension, then Q induces a map $(A/I)^n \to A/I$, which we denote Q_I . Let

$$f(Q,I) := \frac{|Q_I^{-1}(0)|}{|A/I|^n}.$$

Regarding A/I as a finite-dimensional vector space, Q_I is given by a polynomial of degree $\leq d$, so either it maps $(A/I)^n$ to 0, or

$$f(Q,I) \leq 1 - 2^{-d}.$$

We now prove Theorem 2.

Proof. We first prove that condition (1) implies condition (2). We have to show that if Q is not a coset identity of A then for all $\epsilon > 0$, there exists an ideal I of finite codimension such that $f(Q, I) \leq \epsilon$. We first prove that it implies that for any ideal I of finite codimension, there exists an ideal $J \subset I$ of finite codimension such that

$$f(Q, J) \leq (1 - 2^{-d}) f(Q, I).$$

For each element $\alpha \in Q_I^{-1}(0)$, we choose a representative $\tilde{\alpha} = (a_1, \ldots, a_n) \in A^n$ such that $e_Q(a_1, \ldots, a_n) \neq 0$ and an ideal of finite codimension I_α to which $e_Q(a_1, \ldots, a_n) \neq 0$ does not belong. Let

$$J = I \cap \bigcap_\alpha I_\alpha,$$

which, by construction, is of finite codimension. Again by construction, Q_J does not map any n-tuple of cosets of I/J to 0. Therefore, for each such n-tuple, the number of elements mapping to 0 by Q_J is at most $(1 - 2^{-d})|I/J|^n$. If the coset maps to an element of $(A/I)^n$ which is not in $Q_I^{-1}(0)$, then no element of that coset maps to 0 by Q_J. This proves the claim, and the equivalence of conditions (1) and (2) follows immediately.

Now, suppose Q is multilinear. We will show that condition (2) implies condition (3) with the same ideal I. Assuming (2) we have

$$e_Q(a_1 + y_1, \ldots, a_n + y_n) = 0$$

for all $y_1, \ldots, y_n \in I$. By the multilinearity of Q we have, for all $y_1 \in I$,

$$\begin{aligned} 0 = e_Q(a_1 + y_1, a_2, \ldots, a_n) &= e_Q(a_1, a_2, \ldots, a_n) + e_Q(y_1, a_2, \ldots, a_n) \\ &= e_Q(y_1, a_2, \ldots, a_n). \end{aligned}$$

Similarly we have, for all $y_2 \in I$,

$$0 = e_Q(y_1, a_2 + y_2, a_3, \ldots, a_n) = e_Q(y_1, a_2, a_3, \ldots, a_n) + e_Q(y_1, y_2, a_3, \ldots, a_n).$$

Proceeding in this way we obtain

$$e_Q(y_1, \ldots, y_n) = 0$$

for all $y_1, \ldots, y_n \in I$.

This completes the proof. □

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