SUPPORT FOR INTEGRABLE HOPF ALGEBRAS VIA NONCOMMUTATIVE HYPERSURFACES

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Abstract. We consider finite-dimensional Hopf algebras u which admit a smooth deformation $U \to u$ by a Noetherian Hopf algebra U of finite global dimension. Examples of such Hopf algebras include small quantum groups over the complex numbers, restricted enveloping algebras in finite characteristic, and Drinfeld doubles of height 1 group schemes. We provide a means of analyzing (cohomological) support for representations over such u, via the singularity categories of the hypersurfaces $U/(f)$ associated to functions f on the corresponding parametrization space. We use this hypersurface approach to establish the tensor product property for cohomological support, for the following examples: functions on a finite group scheme, Drinfeld doubles of certain height 1 solvable finite group schemes, bosonized quantum complete intersections, and the small quantum Borel in type A.

1. Introduction

The present paper is dedicated to analyses of cohomology and support for finite-dimensional Hopf algebras. Given a Hopf algebra u over a field k, we are particularly interested in its associated tensor category of finite-dimensional representations $\mathrm{rep}(\mathsf{u})$. At a basic level, cohomological support theory proposes that one can use a certain geometry, namely the geometry of the spectrum of the cohomology algebra $\mathrm{Ext}^*_{\mathsf{u}}(k,k)$ to understand the representation theory $\mathrm{rep}(\mathsf{u})$ *as a tensor category*. We study support for distinguished classes of Hopf algebras via the production of certain rank varieties, which we refer to as *hypersurface support*.

Support theory has its foundations in the modular representation theory of finite group (schemes), that is, the representation theory of finite group schemes in finite characteristic. This geometric approach to modular representation theory arguably began with work of Quillen in the 70's [73] and continues into the present with strong contributions of many authors. For a finite group scheme $\mathcal{G}$ one assigns to any finite-dimensional $\mathcal{G}$-representation V a closed subvariety $\mathrm{supp}(V)$ in the spectrum of the cohomology ring $H^*(\mathcal{G},k) = \mathrm{Ext}^*_{\mathcal{G}}(k,k)$. The subvariety $\mathrm{supp}(V)$ is defined as the support of the $H^*(\mathcal{G},k)$-module $\mathrm{Ext}^*_{\mathcal{G}}(V,V)$, with action defined via the tensor structure on $\mathrm{rep}(\mathcal{G})$ and Yoneda product. In the studies [10, 14], for example, supports of objects are employed to provide strong analyses of the global structure on the (stable) category of $\mathcal{G}$-representations. The fundamental property which allows one to access the tensor structure on $\mathrm{rep}(\mathcal{G})$ via support is the so-called tensor product property, which appears as the equality

$$\mathrm{supp}(V \otimes W) = \mathrm{supp}(V) \cap \mathrm{supp}(W). \tag{1}$$

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One proves the above equality by identifying the spectrum of cohomology with a certain moduli space of maps from $\mathrm{rep}(\mathcal{G})$ to $\mathrm{rep}(\mathbb{Z}/p)$ called the *rank variety* or the $\pi-$support for $\mathcal{G}$ [27, 77, 42, 43].

For a given finite-dimensional Hopf algebra u one can still define support varieties $\mathrm{supp}(V)$ of finite-dimensional u-representation via cohomology as above. However, for almost all examples over the complex numbers, it is not known how these support varieties acknowledge, or do not acknowledge, the tensor structure on $\mathrm{rep}(\mathsf{u})$. Indeed, it is known that the precise relation (1) fails to hold for various classes of Hopf algebras which have non-symmetric representation categories [17, 72].

One conceptual ambition of this paper is to develop an analog of the relation (1) which *can* hold in general, and to test the viability of this new relation by studying a number of nontrivial examples in detail (see §10.4). As suggested above, an understanding of how cohomology and the tensor structure on $\mathrm{rep}(\mathsf{u})$ intertwine is essential, for example, in understanding the global structure of the derived (or stable) category of u-representations, for a given Hopf algebra u. As far as the present study is concerned, results of this paper are used to classify thick ideals in the derived categories of representations for quantum groups, and derived categories of sheaves on finite group schemes, in subsequent works [66, 65, 67]. So, baseline analyses of cohomology and support for Hopf algebras provide the necessary foundations for more homotopical analysis of their derived categories. This principle is well-understood in the classical case of representations $\mathrm{rep}(\mathcal{G})$ of a finite group scheme (see for example [10, 43, 15, 14]).

In this work we consider a class of finite-dimensional Hopf algebras which are *smoothly integrable* (see Definition 2.1). These include many familiar objects such as small quantum groups, restricted enveloping algebras, and Drinfeld doubles of certain finite group schemes. For our tensor triangular aspirations, smooth integrability turns out to be the right noncommutative analogue of the complete intersection property for commutative algebras. Building on the ideas of Avramov-Buchweitz [5] and Avramov-Iyengar [9], we introduce an alternative notion of a rank variety formulated in terms of hypersurfaces on the smooth integration of the algebra u (see Definition 6.7). In some fundamental cases we are able to identify this new hypersurface support with the cohomological support. As an application, we show that the tensor product property holds in various examples related to solvable (quantum) groups (see Section 1.1). Precise results for the full quantum group $u_q(\mathfrak{g})$ should appear in a subsequent paper [67].

Let us now describe the technical content of this paper in more detail. Throughout k is an algebraically closed base field of varying characteristic. Consider u a finite-dimensional Hopf algebra, over k. We say that u is *smoothly integrable*, or just *integrable*, if u admits a deformation $U \to \mathsf{u}$ parametrized by a smooth, augmented, central subalgebra $Z \subset U$ such that (a) U is a Noetherian Hopf algebra of finite global dimension, and $U \to \mathsf{u}$ is a Hopf map, and (b) Z is a coideal subalgebra in U. By smoothness we mean that Z is smooth over our given base field k, and in particular of finite type over k.

For example, the De Concini-Kac algebra $U_q^{DK}(\mathfrak{g})$ integrates the quantum group $u_q(\mathfrak{g})$, and we have a similar deformation for the quantum Borel $U_q^{DK}(\mathfrak{b}) \to u_q(\mathfrak{b})$. A restricted enveloping algebra $u^{\mathrm{res}}(\mathfrak{g})$, in finite characteristic, is smoothly integrated by the associated universal enveloping algebra $U(\mathfrak{g}) \to u^{\mathrm{res}}(\mathfrak{g})$. For $\mathcal{G}$ a finite group scheme, any choice of embedding $\mathcal{G} \to \mathcal{H}$ into a smooth connected algebraic

group $\mathcal{H}$ provides a smooth integration $\mathscr{O}(\mathcal{H}) \to \mathscr{O}(\mathcal{G})$ of functions on $\mathcal{G}$. The last two examples can be combined to produce a smooth integration $\widetilde{D}(\mathcal{G}) \to D(\mathcal{G})$ of the Drinfeld double of an infinitesimal height 1 group scheme which admits a *normal* embedding into a smooth algebraic group. See Section 2 for more details.

Starting with a smooth integration $Z \to \mathsf{U} \to \mathsf{u}$ we develop a dg version of Koszul duality for the algebras $\mathrm{Sym}((m_Z/m_Z^2)^*)$ and $\wedge(m_Z/m_Z^2)$, where m_Z is the kernel of the augmentation on Z, inspired by the ideas of Bezrukavnikov-Ginzburg [21]. This culminates in the main general technical result of the paper, Theorem 4.3. The first application, which is an immediate corollary, is the finite generation of cohomology of u. We adopt the following terminology (cf. [68]): A finite-dimensional Hopf algebra u, over a field k, is said to have finitely generated cohomology if the self-extensions $\mathrm{Ext}^*_{\mathsf{u}}(k,k)$ are a finitely generated algebra and, for any finite-dimensional u-representation V, the extensions $\mathrm{Ext}^*_{\mathsf{u}}(k,V)$ are a finite module over $\mathrm{Ext}^*_{\mathsf{u}}(k,k)$.

Theorem (4.8)**.** *If a Hopf algebra* u *is integrable, then* u *has finitely generated cohomology.*

The above theorem recovers finite generation for all of the examples above [39, 47, 37], although it does not give information on the spectrum of cohomology beyond the embedding dimension. (Our results for height 1 doubles are slightly stronger than those of [37].) We hope that the existence of such a general proof, which covers a lot of known cases of finite generation of cohomology in one swipe, illustrates that the notion of smooth integrability captures, in a uniform manner, some fundamental properties of these Hopf algebras.

Once we establish that $\mathrm{Ext}^*_{\mathsf{u}}(k,k)$ is of finite type, we introduce the cohomological support for a finite-dimensional u-representation V:

$$\mathrm{supp}_{\mathsf{Y}}(V) := \mathrm{Supp}_{\mathsf{Y}} \mathrm{Ext}^*_{\mathsf{u}}(V,V)^{\sim}, \tag{2}$$

where Y is the reduced projective spectrum $\mathrm{Proj}(\mathrm{Ext}^*_{\mathsf{u}}(k,k))_{\mathrm{red}}$ and $(-)^{\sim}$ denotes the sheaf on Y associated to a given graded $\mathrm{Ext}^*_{\mathsf{u}}(k,k)$-module.

The existence of smooth integration $Z \to \mathsf{U} \to \mathsf{u}$ also supplies a map of varieties

$$\kappa : \mathsf{Y} = \mathrm{Proj}(\mathrm{Ext}^*_{\mathsf{u}}(k,k))_{\mathrm{red}} \to \mathbb{P}(m_Z/m_Z^2), \tag{3}$$

with finite fibers. Moreover, in a number of important examples [39, 47, 37], κ is a closed embedding. The second application of Theorem 4.3 is to show that the map κ identifies cohomological support $\mathrm{supp}_{\mathsf{Y}}(V)$ with the *hypersurface* support which we now describe (see also Definition 6.7).

Let $Z \to \mathsf{U} \to \mathsf{u}$ be a smooth integration. Let c be a point in $m_Z/m_Z^2 - \{0\}$. Any choice of section $\tau : m_Z/m_Z^2 \to m_Z$ specifies a corresponding "noncommutative hypersurface" $U_c = U/(\tau(c))$. As the algebra U_c depends only on c up to a scaling, we may instead consider the U_c as associated to points in the projectivization $c \in \mathbb{P}(m_Z/m_Z^2)$. In what follows we say that M, a finitely generated module for U_c, is *perfect* if it has finite projective dimension. For V a u-representation we are interested in whether V is, or is not, perfect when restricted to a given hypersurface U_c. The following theorem is a noncommutative variant of a result of Avramov and Buchweitz [5].

Theorem (7.2)**.** *Suppose that* u *is smoothly integrable, with corresponding deformation* $U \to \mathsf{u}$ *and parametrizing subalgebra* Z*. Then for any* V *in* $\mathrm{rep}(\mathsf{u})$*, and any*

choice of section $\tau : m_Z/m_Z^2 \to m_Z$*, we have*

$$\kappa\big(\operatorname{supp}_{\mathsf{Y}}(V)\big) = \{c \in \mathbb{P}(m_Z/m_Z^2) : V \text{ is not perfect when restricted to } U_c\},$$

where κ *is as in* (3)*. In particular, when* κ *is a closed embedding there is an equality*

$$\operatorname{supp}_{\mathsf{Y}}(V) = \{c \in \mathbb{P}(m_Z/m_Z^2) : V \text{ is not perfect when restricted to } U_c\}. \tag{4}$$

Implicit in the statement above is the fact that perfection of a given u-representation over U_c, at some $c \in \mathbb{P}(m_Z/m_Z^2)$, depends only on the point c, not on the choice of section $\tau : m_Z/m_Z^2 \to m_Z$. This fact is proved independently in Corollary 5.3.

We use the hypersurface expression (4) to provide a stronger analysis of support for certain "solvable" Hopf algebras. As discussed above, we are particularly interested in the so-called *tensor product property* for support,

$$\operatorname{supp}_{\mathsf{Y}}(V \otimes W) \overset{\text{question}}{=} \operatorname{supp}_{\mathsf{Y}}(V) \cap \operatorname{supp}_{\mathsf{Y}}(W). \tag{5}$$

This relation suggests, in the cases in which it applies, that cohomology actually "understands" something about the tensor structure on the category $\operatorname{rep}(\mathsf{u})$. As we also mentioned above, it is known via work of Witherspoon and coauthors, that for Hopf algebras which are sufficiently noncocommutative, so that $\operatorname{rep}(\mathsf{u})$ is sufficiently non-symmetric, the tensor product property will *not* hold [17, 72].

In the present work we consider certain centralizing hypotheses on objects, and (co)normality hypotheses on the integration $U \to \mathsf{u}$, when studying support for tensor products of objects (see Section 10.4). We discuss some of our examples below.

1.1. **Applications.** We first consider rings of regular functions $\mathscr{O}(\mathcal{G})$ on finite group schemes in finite characteristic, where we have $\operatorname{rep}(\mathscr{O}(\mathcal{G})) = \operatorname{Coh}(\mathcal{G})$.

Theorem (10.5)**.** *Let* $\mathcal{G}$ *be a connected finite group scheme, in finite characteristic. Then cohomological support for* $\operatorname{Coh}(\mathcal{G})$ *satisfies the tensor product property*

$$\operatorname{supp}_{\mathsf{Y}}(V \otimes W) = \operatorname{supp}_{\mathsf{Y}}(V) \cap \operatorname{supp}_{\mathsf{Y}}(W).$$

There are some referential points to take into account here, as we explain in the preamble to Section 10 and Remark 10.7.

It is clear, from work of Plavnik and Witherspoon [72] (see also Example 10.2), that the precise tensor product property (5) does not hold for coherent sheaves on general non-connected $\mathcal{G}$. However, we provide a version of the tensor product property for arbitrary finite group schemes by considering certain centralizing structures on objects in $\operatorname{Coh}(\mathcal{G})$ (cf. 10.4).

Theorem (10.8)**.** *Let* $\mathcal{G}$ *be an arbitrary finite group scheme with* $\pi = \pi_0(\mathcal{G})$ *its subgroup of connected components. For* W *in* $\operatorname{Coh}(\mathcal{G})$*, and* V *in the Drinfeld centralizer* $Z^{\operatorname{Coh}(\pi)}(\operatorname{Coh}(\mathcal{G}))$ *of* $\operatorname{Coh}(\pi)$*, we have*

$$\operatorname{supp}_{\mathsf{Y}}(F(V) \otimes W) = \operatorname{supp}_{\mathsf{Y}}(F(V)) \cap \operatorname{supp}_{\mathsf{Y}}(W).$$

Here $Z^{\operatorname{Coh}(\pi)}(\operatorname{Coh}(\mathcal{G}))$ is the category of objects in $\operatorname{Coh}(\mathcal{G})$ which centralize the fusion subcategory $\operatorname{Coh}(\pi) \subset \operatorname{Coh}(\mathcal{G})$ of semisimple objects. More specifically, $Z^{\operatorname{Coh}(\pi)}(\operatorname{Coh}(\mathcal{G}))$ is the category of objects which admit a *centralizing structure* against this distinguished subcategory, and $F : Z^{\operatorname{Coh}(\pi)}(\operatorname{Coh}(\mathcal{G})) \to \operatorname{Coh}(\mathcal{G})$ is the functor forgetting the specific choice of centralizing structure (see Definition 8.1).

In a non-classical setting, over $\mathbb{C}$, one can consider a truncated skew polynomial ring

$$\mathsf{a}_q^+ = \mathsf{a}_q^+(P) = \mathbb{C}\langle x_1, \dots, x_n\rangle/(x_i x_j - q_{ij} x_j x_i,\ x_i^l),$$

which is often referred to as a quantum complete intersection, or quantum linear space, in the literature. Here q is a root of unity of odd order l and $P = [a_{ij}]$ is a skew-symmetric matrix specifying the parameters $q_{ij} = q^{a_{ij}}$. The algebra a_q^+ has the natural structure of a Hopf algebra in a certain braided fusion category $\mathrm{rep}(\Lambda)$, where Λ is the group ring of a finite abelian group. One then obtains, via a standard bosonization procedure, a usual Hopf algebra $\mathsf{a}_q := \mathsf{a}_q^+ \rtimes \Lambda$. We refer to the Hopf algebra a_q as a *bosonized quantum complete intersection.*

Theorem (11.3)**.** *Consider $\mathsf{a}_q = \mathsf{a}_q(P)$ a bosonized quantum complete intersection, with generic parameters q and P. Cohomological support for $\mathrm{rep}(\mathsf{a}_q)$ satisfies the tensor product property*

$$\mathrm{supp}_{\mathsf{Y}}(V \otimes W) = \mathrm{supp}_{\mathsf{Y}}(V) \cap \mathrm{supp}_{\mathsf{Y}}(W).$$

By "generic" parameters we mean precisely that the determinant of the matrix $P = [a_{ij}]$ is a unit mod l. We prove in Theorem 11.6 that, at *arbitrary* q and P, representation of a_q still satisfy a version of the tensor product property where one again employs centralizers, exactly as in the case of sheaves $\mathrm{Coh}(\mathcal{G})$ on a non-connected group scheme $\mathcal{G}$.

At Theorem 13.2, we reprove the tensor product property for $\mathrm{rep}(u^{\mathrm{res}}(\mathfrak{n}))$ via hypersurfaces, where $\mathfrak{n}$ is a nilpotent restricted Lie algebra in large characteristic. This is a fundamental result of Friedlander and Parshall [41, Theorem 2.7], which precedes the π-point approach to support for finite group schemes [40, 77, 43]. We consider also Drinfeld doubles $D(B_{(1)})$ for $B_{(1)}$ the first Frobenius kernel of a Borel $B \subset \mathbb{G}$ in an almost-simple algebraic group $\mathbb{G}$.

Theorem (13.3)**.** *Consider $\mathbb{G}$ an almost-simple algebraic group over $\overline{\mathbb{F}}_p$, and let B be a Borel subgroup in $\mathbb{G}$. If $p > \dim(B) + 1$, then cohomological support for the Drinfeld double $D(B_{(1)})$ satisfies the tensor product property.*

We consider, finally, the small quantum Borel $u_q(\mathfrak{b})$ in a quantum group $u_q(\mathfrak{g})$, over $\mathbb{C}$. This algebra, and its full counterpart $u_q(\mathfrak{g})$, are the most important characteristic 0 examples for which cohomological support is not understood at this point. We address the quantum Borel in type A.

Theorem (13.8)**.** *Consider $u_q(\mathfrak{b})$ the quantum Borel in type A_n, at q of odd order $> h$. Cohomological support for $u_q(\mathfrak{b})$ satisfies the tensor product property.*

In the above statement n is arbitrary and h denotes the Coxeter number for A_n, which is just $n+1$. We observe in Corollary 7.6 that, for the quantum Borel $u_q(\mathfrak{b})$ in *any* Dynkin type, there is at least a containment

$$\mathrm{supp}_{\mathsf{Y}}(V \otimes W) \subset (\mathrm{supp}_{\mathsf{Y}}(V) \cap \mathrm{supp}_{\mathsf{Y}}(W))\,.$$

We conjecture, at Conjecture 13.11 below, that this containment is an equality in all Dynkin types.

1.2. **Snashall-Solberg support for associative algebras.** There is a theory of support for finite-dimensional (non-Hopf) algebras R, introduced by Snashall and Solberg [74, 34], which utilizes the map from Hochschild cohomology to the center of the derived category $- \otimes^{\mathrm{L}}_R id : HH^*(R) \to Z(D^b(R))$. The materials of

Sections 4 and 6 are presented in the general setting of a finite-dimensional algebra R with a Noetherian deformation $Q \to R$ of finite global dimension. We observe at Theorem 6.11 that the Snashall-Solberg type support in this case is identified with the corresponding support defined via hypersurfaces. That is to say, we show that the proper analog of equation (4) holds in this context.

1.3. **Related works.** In [22] Boe, Kujawa, and Nakano propose an extremal version of the tensor product property for (possibly infinite-dimensional) modules over $u_q(\mathfrak{b})$, at odd order parameter with $\mathrm{ord}(q) > h$. Specifically, the authors suggest that for modules M and N over $u_q(\mathfrak{b})$, the support of the product $M \otimes N$ vanishes if and only if the supports for M and N have trivial intersection. The approach of [22] involves reductions to an associated graded algebra $\mathrm{gr}\, u_q(\mathfrak{g})$ with respect to a root vector filtration of De Concini and Kac [29].

In [19] and [70, 71], Benson-Erdmann-Holloway and Pevtsova-Witherspoon, respectively, consider support for the (bosonized) quantum complete intersections a_q with constant parameter $q_{ij} = q_{12}$, for all $i < j$. (To be clear, in [19] the tensor structure is not considered, and in [70, 71] the skewing parameter is taken to be of constant value 1.) In [19] certain rank varieties are also constructed for a_q with arbitrary parameters by reducing, via a sequence of $\mathbb{C}$-linear functors, i.e. non-tensor functors, to the constant parameter case [19, Definition 4.4].

One can view the materials of this paper as an approach to support theory which is an "inversion" of the canonical π-point approach to support for finite group schemes [77, 43, 32, 16]. A strictly categorical reinterpretation of the π-point approach to support can be found in work of Balmer, Krause, and Stevenson [12, 11].

1.4. **A basic outline.** The paper essentially has two interlocking halves. In Sections 3–7 we construct a dg-version of Koszul duality for the smooth integration $Z \to \mathsf{U} \to \mathsf{u}$ leading to the main technical result in Theorem 4.3, then show how it implies strong finiteness conditions on cohomology in Theorem 4.8. We then develop the fundamentals regarding hypersurface support. Sections 8–13 are dedicated to examples and applications. Addressing the tensor product property for these examples does require the cultivation of certain additional tools, and in particular requires an analysis of derived categories for noncommutative local hypersurface rings (see Lemma 9.1).

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Contents

2. Examples of integrable Hopf algebras

Throughout k is an algebraically closed field of varying characteristic, and all algebras are algebras over k. We repeat the definition from the introduction before giving a long list of examples of smoothly integrable Hopf algebras.

Recall that a deformation of an algebra u, parametrized by an affine scheme $\mathrm{Spec}(Z)$ with choice of distinguished closed point $x \in \mathrm{Spec}(Z)$, is the information of a flat Z-algebra U equipped with an algebra map $U \to \mathsf{u}$ which reduces to an isomorphism $k \otimes_Z U \cong \mathsf{u}$. Here we reduce along the given point $x : Z \to k$. Such a deformation can be represented via the corresponding sequence of algebra maps $Z \to U \to \mathsf{u}$, which we refer to informally as a deformation sequence.

Definition 2.1. A finite-dimensional Hopf algebra u is said to be *smoothly integrable*, or just *integrable*, if u admits a deformation $U \to \mathsf{u}$ parametrized by a smooth central subalgebra $Z \subset U$ such that

(a) U is a Noetherian Hopf algebra of finite global dimension, and $U \to \mathsf{u}$ is a map of Hopf algebras.
(b) Z is a coideal subalgebra in U.

Under these conditions, U is called a *smooth integration* of u parametrized by Z.

We suppose specifically that Z is a *right* coideal subalgebra in U, so that the comultiplication on U restricts to a coaction $\Delta : Z \to Z \otimes U$.

Remark 2.2. Our choice of right versus left is irrelevant. If $U \to \mathsf{u}$ is a smooth integration with parametrizing algebra Z a right coideal subalgebra, then U is also flat over the left coideal subalgebra $S(Z)$ and $k \otimes_{S(Z)} U = \mathsf{u}$.

Implicit in Definition 2.1 is the assumption that Z is of finite type over k. Although all of our examples arrive in such a finite type flavor, it is convenient to

complete and work with a formal version of integrability. We elaborate on the formal setting in Section 2.2 below, after giving our examples.

We note that for an integration $f : U \to \mathsf{u}$ of a finite-dimensional Hopf algebra u, as in Definition 2.1, Z is contained in the u-coinvariants in U under the left coaction $(f\otimes 1)\Delta : U \to \mathsf{u}\otimes U$. Indeed, Z must be *equal* to the u-coinvariants $Z = U^{\mathsf{u}}$ in this case [79, Theorem 1]. So the parametrization space $\mathrm{Spec}(Z)$ is determined uniquely by the Hopf surjection $U \to \mathsf{u}$.

2.1. Examples of smooth integration, and an anti-example.

Example 2.3 (Finite abelian groups)**.** Any finite abelian group A admits a surjective group map from a free abelian group of finite rank, $p : \mathbb{Z}^r \to A$. Let A' be the kernel of p. Note that A' is a free abelian group of finite rank as well. We take group rings $kA' \to k(\mathbb{Z}^r) \to kA$, over an arbitrary field, to obtain a smooth integration of kA.

Example 2.4 (Rings of regular functions on a finite group scheme)**.** Take $k = \overline{\mathbb{F}}_p$, and let $\mathbb{G}$ be a smooth connected algebraic group. Denote by $\mathbb{G}^{(r)}$ the r^{th} Frobenius twist of $\mathbb{G}$ and let $\mathcal{G} = \mathbb{G}_{(r)}$ be the r^{th} Frobenius kernel of $\mathbb{G}$, that is, the kernel of the r^{th} iteration of the Frobenius map $F^r : \mathbb{G} \to \mathbb{G}^{(r)}$. The algebra of functions $\mathscr{O}(\mathcal{G})$ is a complete intersection, and in this instance we get a deformation sequence

$$\mathscr{O}(\mathbb{G}^{(r)}) \to \mathscr{O}(\mathbb{G}) \to \mathscr{O}(\mathcal{G}),$$

so that $\mathscr{O}(\mathcal{G})$ is smoothly integrated by $\mathscr{O}(\mathbb{G})$.

Frobenius kernels are examples of finite connected group schemes. More generally, for $\mathcal{G}$ *any* finite group scheme, we choose an embedding $\mathcal{G} \to \mathcal{H}$ into a smooth connected algebraic group. Any faithful representation V for $\mathcal{G}$ provides an embedding into $\mathcal{H} = \mathrm{GL}(V)$, for example. Then we have the deformation sequence $\mathscr{O}(\mathcal{H}/\mathcal{G}) \to \mathscr{O}(\mathcal{H}) \to \mathscr{O}(\mathcal{G})$, and find that $\mathscr{O}(\mathcal{H})$ integrates $\mathscr{O}(\mathcal{G})$. (Recall that $\mathcal{H}/\mathcal{G}$ is smooth [60, Corollary 5.26].) When $\mathcal{G}$ is not normal in $\mathcal{H}$, $\mathscr{O}(\mathcal{H}/\mathcal{G})$ is not a Hopf subalgebra in $\mathscr{O}(\mathcal{H})$, but a $\mathscr{O}(\mathcal{H})$-coideal subalgebra. Indeed, functions on $\mathcal{H}/\mathcal{G}$ are, essentially by definition, the $\mathscr{O}(\mathcal{G})$-coinvariants in $\mathscr{O}(\mathcal{H})$.

Example 2.5 (Restricted enveloping algebras)**.** Take $k = \overline{\mathbb{F}}_p$ and $\mathfrak{g}$ a restricted Lie algebra. We have the surjection $U(\mathfrak{g}) \to u^{\mathrm{res}}(\mathfrak{g})$ onto the restricted enveloping algebra, and consider the Zassenhaus subalgebra, or p-center, $Z_0(\mathfrak{g})$ of $U(\mathfrak{g})$. This is the central subalgebra generated by the differences $x^p - x^{[p]}$ at arbitrary $x \in \mathfrak{g}$ [82, 51]. The algebra $Z_0(\mathfrak{g})$ is a Hopf subalgebra in $U(\mathfrak{g})$ which is isomorphic to a polynomial ring, and $k\otimes_{Z_0(\mathfrak{g})} U(\mathfrak{g}) = u^{\mathrm{res}}(\mathfrak{g})$, so that $u^{\mathrm{res}}(\mathfrak{g})$ is seen to be smoothly integrated by the standard universal enveloping algebra.

Example 2.6 (Height 1 doubles)**.** Consider again $\mathcal{H}$ a smooth connected algebraic group over $\overline{\mathbb{F}}_p$, and $\mathcal{G} \subset \mathcal{H}$ a normally embedded, finite, connected subgroup scheme of height 1. Recall that a connected finite group scheme $\mathcal{G}$ is of height 1 if the p^{th} power of any element in the augmentation ideal of $\mathscr{O}(\mathcal{G})$ vanishes. Equivalently, $\mathcal{G}$ is a subgroup scheme of $\mathcal{H}_{(1)}$, the first Frobenius kernel of $\mathcal{H}$ (see [52]). Let $\mathfrak{g}$ denote the restricted Lie algebra for $\mathcal{G}$. We denote by $k\mathcal{G}$ the *group algebra* of $\mathcal{G}$, which is the linear dual of the algebra of regular functions $\mathscr{O}(\mathcal{G})$. Since $\mathcal{G}$ is of height 1, there is an identification $k\mathcal{G} = u^{\mathrm{res}}(\mathfrak{g})$ ([52, I.9.6]). Then $k\mathcal{G}$ acts on $\mathscr{O}(\mathcal{G})$ via the adjoint representation and we consider the Drinfeld double

$$D(\mathcal{G}) = \mathscr{O}(\mathcal{G}) \rtimes_{\mathrm{ad}} k\mathcal{G}.$$

This algebra is integrated by the smash product $\mathscr{O}(\mathcal{H}) \rtimes_{\mathrm{ad}} U(\mathfrak{g})$, where $U(\mathfrak{g})$ acts on $\mathscr{O}(\mathcal{H})$ via the projection $U(\mathfrak{g}) \to k\mathcal{G}$ (using the isomorphism $k\mathcal{G} \cong u^{\mathrm{res}}(\mathfrak{g})$) and adjoint action of $\mathcal{G}$ on $\mathcal{H}$. The parametrization space in this case is provided by the product $\mathscr{O}(\mathcal{H}/\mathcal{G}) \otimes Z_0(\mathfrak{g})$. The relative double $D(\mathcal{H}_1, \mathcal{G}) := \mathscr{O}(\mathcal{H}_1) \rtimes_{\mathrm{ad}} k\mathcal{G}$ is similarly integrated by $\mathscr{O}(\mathcal{H}) \rtimes_{\mathrm{ad}} U(\mathfrak{g})$.

Example 2.7 (Small quantum groups)**.** Consider the De Concini-Kac quantum enveloping algebra $U_q^{DK}(\mathfrak{g})$ at q a root of unity of odd order l. We have the standard sequence

$$Z_0 \to U_q^{DK}(\mathfrak{g}) \to u_q(\mathfrak{g}),$$

where Z_0 is the subalgebra generated by the l-th powers E_γ^l, F_γ^l, K_α^l [29, Section 3], [30, Proposition 5.6]. By filtering by a normal ordering on the root vectors and considering the associated graded algebra (cf. [29]), we find that $U_q^{DK}(\mathfrak{g})$ is of finite global dimension [28, Corollary 3.1.2]. Hence $u_q(\mathfrak{g})$ is smoothly integrated by $U_q^{DK}(\mathfrak{g})$. Similarly, the quantum Borel $u_q(\mathfrak{b})$ is integrated by $U_q^{DK}(\mathfrak{b})$. For the quantum Borel, however, we consider the De Concini-Kac algebra $U_q^{DK}(\mathfrak{b})$ with the same (torsion) group of grouplikes as its small counterpart $u_q(\mathfrak{b})$.

In each of the above examples, faithful flatness of the extension $Z \to U$ is well-known. However, it is also the case that any extension $Z \to U$ of a commutative coideal subalgebra Z whatsoever is faithfully flat [4, Proposition 3.12]. (There is a problem with the proof given in [4], but the result is nonetheless correct [45].)

Example 2.8 (Anti-example)**.** The group ring $k(\mathrm{GL}_n)_{(2)}$ of the second Frobenius kernel in GL_n, over $\overline{\mathbb{F}}_p$, should not be integrable. This is due to the presence of divided powers in the algebra.

2.2. **Formality and deformations.** In considering deformations of Hopf algebras, it is often advantageous to suppose that the parametrizing subalgebra Z is not smooth, but *formally* smooth.

Definition 2.9. A finite-dimensional Hopf algebra u is said to be *formally smoothly integrable* if u admits a deformation $\mathsf{U} \to \mathsf{u}$ parametrized by a central subalgebra $Z \subset \mathsf{U}$ such that

(z) Z is isomorphic to a power series algebra $k[\![y_1, \ldots, y_n]\!]$.
(a) U is a Noetherian Hopf algebra of finite global dimension, and $\mathsf{U} \to \mathsf{u}$ is a map of Hopf algebras.
(b) Z is a (right) coideal subalgebra in U.

In this case we call U a *formally smooth integration* of u.

Abstractly, a commutative algebra Z is isomorphic to such a power series provided Z is formally smooth and complete local, with finite-dimensional cotangent space m_Z/m_Z^2 at the unique maximal ideal m_Z. Such a formally smooth integration U, which is necessarily a finite Z-module, is complete with respect to the m_Z-adic topology. We therefore consider U as a complete linear topological algebra. By a "Hopf algebra" in this case we mean a Hopf algebra in the symmetric category of complete, linear topological, vector spaces. This simply means that the comultiplication on U is a map to the completed tensor product $\Delta : \mathsf{U} \to \mathsf{U}\widehat{\otimes}\mathsf{U}$.

To distinguish between the finite type and formal setting we generally employ a sans serif font U when we intend for our deformation to be complete. We note that the formal situation is preferable simply because the parametrizing subalgebra

Z is local in this case. As we explain below, any finite type integration $U \to \mathsf{u}$ determines a formal integration $\mathsf{U} \to \mathsf{u}$. Unless confusion will arise, we drop the modifier "formal" in our presentation and speak only of integrable Hopf algebras and (implied, formally smooth) integrations $\mathsf{U} \to \mathsf{u}$.

Now, given a deformation $U \to \mathsf{u}$ parameterized by a smooth, finite type, k-algebra Z, one produces a corresponding formal deformation by completing at the distinguished maximal ideal m_Z,

$$\mathsf{U} \to \mathsf{u}, \quad \mathsf{U} := \widehat{Z} \otimes_Z U.$$

Since $m_Z U$ is the kernel of the Hopf map $U \to \mathsf{u}$ it is a Hopf ideal in U, and the completion $\mathsf{U} = \varprojlim_n U/m_Z^n U$ is a complete, linear topological Hopf algebra.

Lemma 2.10. *Suppose that* u *is (finite-dimensional and) integrable,* $Z \to U \to \mathsf{u}$ *is a smooth integration with* Z *of finite type over* k*, and that* U *is Noetherian and of finite global dimension. Then the completion* $\widehat{Z} \otimes_Z U$ *is also Noetherian and of finite global dimension.*

Proof. Take $\mathsf{U} = \widehat{Z} \otimes_Z U$ the completion. Noetherianity of U follows from the fact that U is finite over the Noetherian algebra $\widehat{Z}$. We have that U is semi-local, and all of the simples are restricted from simples over u. So $\mathsf{U}/\operatorname{Jac}(\mathsf{U}) = \mathsf{u}/\operatorname{Jac}(\mathsf{u})$ is a finite sum of finite-dimensional modules which are annihilated by the augmentation ideal of $\widehat{Z}$. Take $\Lambda = \mathsf{U}/\operatorname{Jac}(\mathsf{U})$ the sum of the simples.

By applying the faithful and exact functor $\widehat{Z} \otimes_Z -$ to any flat resolution of Λ over U we obtain a resolution over U, and so find that $\operatorname{Tor}_*^{\mathsf{U}}(\Lambda, -) = \operatorname{Tor}_*^U(\Lambda, -)$. By our assumption that U is of finite global dimension, we have $\operatorname{Tor}_{>d}^U(\Lambda, -) = 0$ for $d = \operatorname{gldim}(\mathrm{U})$, and so observe $\operatorname{Tor}_{>d}^{\mathsf{U}}(\Lambda, -) = 0$.

By considering minimal resolutions of modules over U, we see that the projective dimension of any finitely generated U-module M is the minimal $d(M)$ so that $\operatorname{Tor}_{>d(M)}^{\mathsf{U}}(\Lambda, M) = 0$. So all finitely generated U-modules are of projective dimension $\leq d$. It follows that U is of global dimension $\leq d$, and in particular of finite global dimension [81, Theorem 4.1.2]. □

3. Basics for deformations of associative algebras

We consider for the moment deformations of associative algebras, without any assumed Hopf structure. Below we explain how such deformations provide specific approaches to cohomology which leverage the deforming algebra as a realization of its corresponding classes in Hochschild cohomology. After providing a relatively general analysis of cohomology, in the present section and Section 6, we give the necessary implications for Hopf algebras in Section 7.

The materials of the present section are basically a review of work of Bezrukavnikov and Ginzburg [21], and provide the technical foundations of our study. We employ standard constructions for dg modules, which can be found throughout the literature (e.g. [58, Chapter III] or [31]).

3.1. **The setup.** In Sections 3–7 we work with a deformation sequence $Z \to Q \to R$ satisfying the following:

- Z is complete local, formally smooth, and essentially of finite type, i.e. isomorphic to $k[\![y_1, \dots, y_n]\!]$ in coordinates.
- Q is finite and flat over Z.

- The reduction $k \otimes_Z Q \cong R$ is finite-dimensional.

The third point here is obviously a consequence of the second. Note that in this situation Q is also Noetherian, since it is finite over the Noetherian subalgebra Z. We always let $1 \in \mathrm{Spf}(Z)$ denote the distinguished (and unique) closed point in $\mathrm{Spf}(Z)$ at which the fiber of Q is identified with R. Here Q and R are not assumed to be Hopf algebras.

Remark 3.1. Our use of the formal spectrum $\mathrm{Spf}(Z)$ [49, Definition 10.1.2] rather than the usual spectrum of Z is simply a matter of preference. In our case $\mathrm{Spf}(Z)$ is just a single point, corresponding to the maximal ideal m_Z, over which Z lives as the global sections. The object $\mathrm{Spf}(Z)$ is also referred to as a formal disk.

Remark 3.2. One can replace the finiteness conditions in the above setup with the assumption that Q is finite over its center. Specifically, one should consider a deformation Q which is finite over a central subalgebra T such that T contains Z, is complete local, and is essentially of finite type over k (cf. Remark 11.4). Of course, in this more general setting one must augment the results of Sections 4–7 in the expected manner. For example, in the statement of Theorem 4.3 one should consider instead finiteness over $T \otimes B_Z$ and $T \otimes A_Z$.

3.2. **Dg algebras, dg modules and semi-projective resolutions.** Consider a dg algebra K over T, where T is a commutative algebra. (Usually for us T will be Z, the parametrizing subalgebra of 3.1.) We consider K-dgmod the category of left dg modules over K, and let $D(K)$ denote the corresponding derived category, $D(K) = (K\text{-dgmod})[\mathrm{quis}^{-1}]$. When T is Noetherian, which is always the case for us, we take

$$D_{coh}(K) = \left\{ \begin{array}{c} \text{The full subcategory of dg modules in } D(K) \\ \text{with finitely generated cohomology over } T \end{array} \right\}$$

When $T = k$ we write $D_{fin}(K)$ for $D_{coh}(K)$, the category of dg modules with finite-dimensional cohomology. Note that when K is a finite-dimensional algebra $D_{fin}(K)$ is just the bounded derived category of finite K-modules.

We say that a dg K-module M is *semi-projective* if it has a dg filtration

$$0 \subset F_0 M \subseteq F_1 M \subseteq \ldots \subseteq M$$

such that each $F_n M / F_{n-1} M$ is a direct summand of a direct sum of shifts of K. (See for example [56, §3.1], where such modules are referred to as having *property (*P*).*) We call a semi-projective dg module *semi-free* if each subquotient $F_n M / F_{n-1} M$ is exactly a sum of shifts of K. For a dg module N, we call a quasi-isomorphism $M \xrightarrow{\sim} N$ from a semi-projective dg module M a *semi-projective resolution* of N. As shown, for example, in [56, Theorem 3.1] or [76, Lemma 20.4], such semi-projective resolutions always exist. Following the usual procedures, we derive tensor products and Hom functors by employing semi-projective resolutions.

For M a dg module (over k) we let M^* denote the corresponding dual dg module $M^* = \mathrm{Hom}_k(M, k)$. Here Hom_k is the standard Hom complex, so that the underlying graded space for M^* is the sum $\oplus_{n \in \mathbb{Z}} (M^{-n})^*$.

3.3. **Deformations, Koszul resolutions, and the Koszul dual pair (A_Z, B_Z).** Consider $Z \to Q \to R$ a sequence as in 3.1. A choice of coordinates for Z is equivalent to a choice of section $\tau : m_Z / m_Z^2 \to m_Z$ from the cotangent space at 1. From τ we can explicitly define the Koszul complex K_Z, which is a dg algebra

resolving k over Z. Specifically, we take K_Z to be the dg Z-algebra $\wedge^*_Z\Omega_Z$ with differential specified on the generating submodule Ω_Z by $d_{y_i} \mapsto y_i$. Here $\{y_i\}_{i=1}^n$ is any choice of basis for the image of τ in m_Z. The quasi-isomorphism $K_Z \xrightarrow{\sim} k$ is given by the augmentation on Z in degree 0.

Remark 3.3. Different choices of sections τ result in resolutions which are isomorphic, as dg Z-algebras, but not literally equal. However, one can check directly that the definition of K_Z is completely independent of the choice of basis $\{y_i\}_{i=1}^n$. We employ the basis $\{y_i\}_i$ only for clarity of presentation.

We change base along the extension $Z \to Q$ to obtain, from K_Z, a complex

$$K_Q := Q \otimes_Z K_Z$$

which provides a resolution of R by a dg algebra which is finite and flat over Z:

$$K_Q \xrightarrow{\sim} Q \otimes_Z k = R.$$

We refer to K_Q as the *Koszul resolution* of R associated to the deformation $Q \to R$.

Lemma 3.4. *The resolution K_Q is finite and free over Q.*

Proof. This follows from the fact that K_Z is finite and free over Z. □

Given a deformation $Q \to R$ as above, we make repeated use of the (graded) commutative dg algebras

$$B_Z := \mathrm{Sym}(\Sigma(m_Z/m_Z^2)) \quad \text{and} \quad A_Z := \mathrm{Sym}(\Sigma^{-2}(m_Z/m_Z^2)^*). \tag{6}$$

Since we are in the dg context B_Z is, as an associative algebra, actually an *exterior* algebra. The two algebras B_Z and A_Z are generated by the shifted cotangent space (in degree -1) and tangent space (in degree 2) of $\mathrm{Spf}(Z)$ at 1, respectively, and can therefore be viewed as functions on corresponding linear dg schemes.

An important point about these two algebras is that they are Koszul dual, in the usual dg sense. In particular, $\mathrm{RHom}_{B_Z}(k,k) \cong A_Z$ and $\mathrm{RHom}_{A_Z}(k,k) \cong B_Z$ (see, for example, [48] [3, Section 3.3]).

3.4. **Koszul resolutions and a map to the center of $D_{fin}(R)$ [21].** We fix a deformation $Q \to R$ as in Section 3.1. Since the Koszul resolution $K_Q \xrightarrow{\sim} R$ is a quasi-isomorphism of dg algebras, reduction and restriction provide mutually inverse equivalences of triangulated categories

$$k \otimes^{\mathrm{L}}_{K_Z} - : D(K_Q) \xrightarrow{\sim} D(R), \quad \mathrm{res} : D(R) \xrightarrow{\sim} D(K_Q). \tag{7}$$

These equivalences restrict to equivalences on the full subcategories of dg modules with coherent (finitely generated) cohomology over Z and k respectively,

$$k \otimes^{\mathrm{L}}_{K_Z} - : D_{coh}(K_Q) \xrightarrow{\sim} D_{fin}(R), \quad \mathrm{res} : D_{fin}(R) \xrightarrow{\sim} D_{coh}(K_Q).$$

Note that $D_{coh}(K_Q)$ and $D_{fin}(R)$ can be identified with the respective derived categories of finitely generated dg modules.

Consider now dg modules M and N over K_Q, and the algebra of extensions

$$\mathrm{Ext}^*_{K_Q}(M,N) = \oplus_{i\in\mathbb{Z}} \mathrm{Hom}^i_{D(K_Q)}(M, \Sigma^i N).$$

Consider also the Koszul dual dg algebras B_Z and A_Z of (6).

Let us explicitly choose coordinates $\tau : m_Z/m_Z^2 \to m_Z$ here, and consider the Koszul complex K_Z defined precisely as in Section 3.3. We enumerate a basis $\{y_i\}_{i=1}^n$ for the image of the section τ, and have the corresponding basis $\{\bar{y}_i\}_{i=1}^n$

of m_Z/m_Z^2. Recall that the algebra B_Z is generated by the cotangent space $T_1^*\operatorname{Spf}(Z) = m_Z/m_Z^2$, so that the basis $\{\bar{y}_i\}_i$ provides a set of generators for this algebra as well.

Consider the product $K_Z \otimes_Z K_Q$, which is a dg algebra over Z. As explained in [21, Lemma 2.4.2], one has a flat extension of dg algebras $j_\tau : B_Z \to K_Z \otimes_Z K_Q$ defined as follows: by definition $K_Z = \wedge_Z^* \Omega_Z$, with differential $d_{y_i} \mapsto y_i$, and we take

$$j_\tau : B_Z \to K_Z \otimes_Z K_Q, \quad j_\tau(\bar{y}_i) = d_{y_i} \otimes 1 - 1 \otimes d_{y_i}. \tag{8}$$

We may view $K_Z \otimes_Z K_Q = K_Q \otimes_Q K_Q$ as a K_Q-bimodule, with commuting B_Z-action provided by the aforementioned algebra map. Flatness of $K_Z \otimes_Z K_Q$ over B_Z implies that the induction functor $(K_Z \otimes_Z K_Q) \otimes_{B_Z} -$ from dg B_Z-modules to dg K_Q-bimodules is exact, and hence is derived simply as

$$(K_Z \otimes_Z K_Q) \otimes^{\mathrm{L}}_{B_Z} - = (K_Z \otimes_Z K_Q) \otimes_{B_Z} - : D(B_Z) \to D(K_Q \otimes K_Q^{op}) \tag{9}$$

One calculates directly $(K_Z \otimes_Z K_Q) \otimes_{B_Z} k = Q \otimes_Z (K_Z \otimes K_Z) \otimes_{B_Z} k = Q \otimes_Z K_Z = K_Q$ (where the middle equality is by [21, Lemma 2.4.2]) to see that the above functor induces a map to Hochschild cohomology

$$i_\tau : A_Z = \operatorname{Ext}^*_{B_Z}(k,k) \to HH^*(K_Q).$$

Remark 3.5. One is free to view the functor $(K_Z \otimes_Z K_Q) \otimes_{B_Z} -$ as having image in the category of dg $K_Z \otimes_Z K_Q^{op}$-modules, or in the category of dg K_Q-bimodules. This depends on whether one views $K_Z \otimes_Z K_Q$ as a Z-central (K_Z, K_Q)-bimodule or a K_Q-bimodule. One advantage of the K_Q-bimodule perspective is that we can then use the quasi-isomorphism $K_Q \to R$ to pass from $D(K_Q \otimes K_Q^{op})$ to $D(R \otimes R^{op})$ and, hence, to identify the Hochschild cohomologies $HH^*(K_Q)$ and $HH^*(R)$ as we do below.

Recall that the center $Z(D_{coh}(K_Q))$ of the derived category $D_{coh}(K_Q)$ is the graded algebra of natural transformations from the identity functor on $D_{coh}(K_Q)$ to its shifts

$$Z(D_{coh}(K_Q)) = \oplus_{n \in \mathbb{Z}} \operatorname{Hom}_{\mathrm{Fun}}(id, \Sigma^n),$$

and recall also the map from Hochschild cohomology to the center. This map takes a class in $HH^*(K_Q)$, which is some map $f : K_Q \to \Sigma^n K_Q$ in the derived category of bimodules, to the transformation $f \otimes_{K_Q} - : id \to \Sigma^n$. So from i_τ we obtain a map to the center $\iota_\tau : A_Z \to Z(D_{coh}(K_Q))$, i.e. an action of A_Z on the derived category of K_Q-modules.

We elaborate on this final point. From the maps $Z(D_{coh}(K_Q)) \to \operatorname{Ext}^*_{K_Q}(M,M)$, $\xi \mapsto \xi_M$, we obtain a natural action of A_Z on arbitrary objects M in the derived category via restricting along ι_τ. Through these actions on objects we obtain, in principle, *two* actions of A_Z on each graded extension group $\operatorname{Ext}^*_{K_Q}(M,N)$, one through M and one through N. Naturality tells us that these two actions agree, and in the case $M = N$ we find that the corresponding algebra map $A_Z \to \operatorname{Ext}^*_{K_Q}(M,M)$ has central image.

Theorem 3.6 ([21])**.** *The algebra map $i_\tau : A_Z \to HH^*(K_Q)$ is independent of the choice of section $\tau : m_Z/m_Z^2 \to m_Z$. In particular, the restriction to the generators $i_\tau : (A_Z)^2 = (m_Z/m_Z^2)^* \to HH^2(K_Q) \cong HH^2(R)$ is the usual deformation map of* [46, 21, 37].

Since the i_τ are independent of the choice of section τ, we can speak of *the* algebra map $A_Z \to HH^*(R)$ determined by the deformation $Q \to R$, parametrized by $\mathrm{Spf}(Z)$, and *the* corresponding action of A_Z on $D_{coh}(K_Q) \cong D_{fin}(R)$.

Definition 3.7. We fix $i_Q : A_Z \to HH^*(R)$ the algebra map associated to a given deformation sequence $Z \to Q \to R$, and view all extensions $\mathrm{Ext}^*_R(M,N)$ as A_Z-modules via i_Q.

4. Deformations and finiteness conditions on cohomology

For this section we fix a deformation $Q \to R$ as in 3.1. We provide a general finite generation result at Theorem 4.3 (see also Lemma 4.2). We then apply Theorem 4.3 to recover, in a uniform manner, finite generation results of [39, 47, 37] in Subsection 4.6. In particular, we find that any integrable Hopf algebra u has finitely generated cohomology.

The key observation of this section is that $\mathrm{RHom}_R(V,W)$ with its A_Z-action as described in 3.6 and $\mathrm{RHom}_Q(V,W)$ with a natural B_Z-action are related by a dg version of Koszul duality for A_Z and B_Z, see Lemma 4.2 as the main technical result. Since Koszul duality preserves finiteness conditions (see [7, 6]), it allows us to translate between finite generation of $\mathrm{Ext}^*_R(V,W)$ as A_Z-module and finite generation of $\mathrm{Ext}^*_Q(V,W)$ as B_Z-module, as done in Theorem 4.3. Finally, for our applications to integrable Hopf algebras, one observes that finite generation of $\mathrm{Ext}^*_Q(V,W)$ as a B_Z-module is equivalent to finite-dimensionality which, in turn, follows from the assumption that Q has final global dimension.

4.1. **Technicalities.** Consider dual bases $\{y^i\}_i$ and $\{y_i\}_i$ for the tangent space and cotangent space of $\mathrm{Spf}(Z)$, and consider $A_Z \otimes B_Z$ as a dg algebra with vanishing differential. We define the functor

$$\begin{aligned} A_Z \otimes^t - : B_Z\text{-dgmod} &\to A_Z \otimes B_Z\text{-dgmod}, \\ A_Z \otimes^t M &:= (A_Z \otimes M, 1 \otimes d_M + \textstyle\sum_i y^i \otimes y_i). \end{aligned} \tag{10}$$

The functor $A_Z \otimes^t -$ respects quasi-isomorphisms on the category B_Z-dgmod$^+$ of bounded below dg B-modules, as it is identified with the functor $\mathrm{Hom}_{B_Z}(F,-)$, where $F = (B_Z \otimes (A_Z)^*, d_F)$ is the Koszul resolution of k over B_Z. So we have the derivation $A_Z \otimes^t - = \mathrm{L}(A_Z \otimes^t -)$ on $D^+(B_Z)$ (see, for example, [63, Section 4], [7]).

Recall that a dg module N over a dg algebra A is called K-flat if the functor $- \otimes_A N$ preserves acyclic objects [75]. For K-flat N the derived and underived tensor products can be identified $- \otimes^{\mathrm{L}}_A N = - \otimes_A N$.

Lemma 4.1. *For any dg B_Z-module M, the twisted product $A_Z \otimes^t M$ is K-flat over A_Z.*

Proof. Consider an acyclic right dg A_Z-module L. We have

$$L \otimes_{A_Z} (A_Z \otimes^t M) = L \otimes^t M,$$

where the latter object is the linear product $L \otimes M$ equipped with the differential $d_L \otimes 1 + 1 \otimes d_M + \sum_i y_i \otimes y^i$. This complex has a bounded filtration provided by the m_{B_Z}-adic filtration on M, and the associated graded complex is the linear product

$$\mathrm{gr}(L \otimes^t M) = L \otimes (\mathrm{gr}\, M).$$

As any complex of k-vector spaces is K-flat over k, we see that the associated graded complex for $L \otimes^t M$ is acyclic. By considering the spectral sequence associated to this filtration, it follows that $L \otimes^t M$ is in fact acyclic. □

We recall also, from the materials of Section 3.4, that $\mathrm{Ext}_R \cong \mathrm{Ext}_{K_Q}$ naturally takes values in the category of dg A_Z-modules. Furthermore, by resolving R-modules by K_Q-modules $M \to V$ which are projective over Q, we see that the functor Ext_Q from R-modules naturally takes values in the category of dg B_Z-modules.

4.2. **Finite generation results.** As we argue below, one can lift the natural actions of A_Z on $\mathrm{Ext}^*_{K_Q}(M,N)$ described in Section 3.4 to the dg level, i.e. to an action of A_Z on $\mathrm{RHom}_{K_Q}(M,N)$. The action of B_Z on $\mathrm{Ext}^*_Q(M,N)$ also lifts to the dg level, at least when N is replaced with an (quasi-isomorphic) dg R-module. After translating from K_Q to R, we establish the following.

Lemma 4.2. (1) *Let V and W be in $D_{fin}(R)$. Then $\mathrm{RHom}_R(V,W)$ has a natural structure of a dg $A_Z \otimes B_Z$-module which lifts the action of A_Z on $\mathrm{Ext}^*_R(V,W)$ of Definition 3.7.*

(2) *There is a natural isomorphism*

$$\mathrm{RHom}_R \cong A_Z \otimes^t \mathrm{RHom}_Q$$

of functors from $D_{fin}(R)^{op} \times D_{fin}(R)$ to $D(A_Z \otimes B_Z)$, where $\otimes^t$ is as defined in (10).

The proof of Lemma 4.2 is given in Section 4.4. In Section 4.5 we prove the following result.

Theorem 4.3 (cf. [8, Theorem 4.2])**.** *Consider a deformation sequence $Z \to Q \to R$ as in Section 3.1. For finite R-modules V and W the following are equivalent:*

(a) *$\mathrm{Ext}^*_Q(V,W)$ is finite over B_Z.*
(b) *$\mathrm{Ext}^*_R(V,W)$ is finite over A_Z.*

Note that finiteness of $\mathrm{Ext}^*_Q(V,W)$ over B_Z is equivalent to boundedness of $\mathrm{Ext}^*_Q(V,W)$, since B_Z is concentrated in finitely many degrees. Theorem 4.3 follows by a basic understanding of the category of dg modules over A_Z (Lemma 4.5) and a direct application of Lemma 4.2.

We apply Theorem 4.3 to recover a number of essential finite generation results for Hopf cohomology, in Section 4.6.

4.3. **Interpretation via noncommutative complete intersections.** One can see Theorem 4.3 as a noncommutative variant of results of Gulliksen and Avramov-Gasharov-Peeva [50, 8] which concern the cohomology of local complete intersections (see also [57]). From the complete intersection perspective [50, 33], A_Z is the "algebra of cohomological operators" deduced from the "noncommutative complete intersection" $Q \to R$ (cf. [57]).

4.4. **Proof of Lemma 4.2.**

Lemma 4.4. *If M is a dg K_Q-module which is semi-projective over Q, then $(K_Z \otimes_Z K_Z) \otimes_{K_Z} M$ is semi-projective over K_Q.*

To be clear, K_Q acts on $(K_Z \otimes_Z K_Z) \otimes_{K_Z} M$ via the K_Z action on the left factor of $K_Z \otimes_Z K_Z$ and the Q-action on M.

Proof. We have by definition a filtration $M = \cup_{i\geq 0} F_i M$ by dg Q-modules so that each quotient $F_iM/F_{i-1}M$ is a summand of a free dg module over Q (recall that we consider Q as a dg algebra concentrated in degree 0). Since K_Z is a bounded complex of flat Z-modules, the functor

$$(K_Z \otimes_Z K_Z) \otimes_{K_Z} - = K_Z \otimes_Z - : D(K_Q) \to D(K_Q)$$

is exact. Hence,

$$K_Z \otimes_Z (F_iM/F_{i-1}M) \cong \frac{K_Z \otimes_Z F_iM}{K_Z \otimes_Z F_{i-1}M}.$$

Since each $F_iM/F_{i-1}M$ is a summand of a free dg Q-module, it follows that $K_Z \otimes_Z (F_iM/F_{i-1}M)$ is a summand of a free dg K_Q-module (that is, direct sum of shifts of K_Q). We consider the filtration $\cup_{i\geq 0} K_Z \otimes_Z F_iM$ on $(K_Z \otimes_Z K_Z) \otimes_{K_Z} M$ to observe a semi-projective structure of this dg module over K_Q. □

We can now prove our essential lemma.

Proof/Construction of Lemma 4.2. Let V and W be finite complexes of R-modules which we consider as dg K_Q-modules via the restriction functor (7). We first construct an action of $A_Z \otimes B_Z$ on $\mathrm{RHom}_{K_Q}(V, W)$ which lifts the action of A_Z on $\mathrm{Ext}_R(V, W) \cong \mathrm{Ext}_{K_Q}(V, W)$ of Definition 3.7 to the dg level.

Choose a resolution $M \to V$ by a dg K_Q-module which is bounded above and projective over Q in each degree. (We may assume the resolution M is bounded when Q is of finite global dimension.) Let $F = B_Z \otimes^t A_Z^*$ be the Koszul resolution of k over B_Z, and consider the induction

$$F^{\mathrm{ind}} = (K_Z \otimes_Z K_Z) \otimes_{B_Z} F$$

along the dg map (8). Since $(K_Z \otimes_Z K_Z) \otimes_{B_Z} k \cong K_Z$ (see [21, Lemma 2.4.2]), and F is the Koszul resolution of k over B_Z, we conclude that

$$F^{\mathrm{ind}} \xrightarrow{\sim} K_Z$$

is a semi-free resolution of K_Z over $K_Z \otimes_Z K_Z$. Therefore, $F^{\mathrm{ind}} \otimes_{K_Z} M$ is a semi-projective resolution of V over K_Q, by Lemma 4.4. We have specifically the quasi-isomorphism

$$F^{\mathrm{ind}} \otimes_{K_Z} M \xrightarrow{\sim} K_Z \otimes_{K_Z} M = M \xrightarrow{\sim} V.$$

Since $F = B_Z \otimes^t A_Z^*$ is a dg (B_Z, A_Z)-bimodule, the induction F^{ind} admits commuting actions of $K_Z \otimes_Z K_Z$ and A_Z, and the product $F^{\mathrm{ind}} \otimes_{K_Z} M$ has commuting actions of K_Z and A_Z. Specifically, we employ the K_Z action on M, and the A_Z-action on F^{ind} to obtain the action of $A_Z \otimes K_Z$ on $F^{\mathrm{ind}} \otimes_{K_Z} M$. The action of $A_Z \otimes K_Z$ on $F^{\mathrm{ind}} \otimes_{K_Z} M$ induces a natural action of $A_Z \otimes B_Z = k \otimes_Z (A_Z \otimes K_Z)$ on the complex

$$\mathrm{RHom}_{K_Q}(V, W) = \mathrm{Hom}_{K_Q}(F^{\mathrm{ind}} \otimes_{K_Z} M, W).$$

A direct comparison verifies that this dg-level action lifts the action of Definition 3.7.

We have the sequence of natural isomorphisms

$$\begin{aligned}
&\mathrm{RHom}_{K_Q}(V, W) = \mathrm{Hom}_{K_Q}(F^{\mathrm{ind}} \otimes_{K_Z} M, W) \\
&\quad = \mathrm{Hom}_{K_Z \otimes_Z K_Z}(F^{\mathrm{ind}}, \mathrm{Hom}_Q(M, W)) && \text{(adjunction)} \\
&\quad = \mathrm{Hom}_{K_Z \otimes_Z K_Z}\big((K_Z \otimes_Z K_Z) \otimes_{B_Z} F, \mathrm{Hom}_Q(M, W)\big) \\
&\quad = \mathrm{Hom}_{B_Z}(F, \mathrm{Hom}_Q(M, W)). && \text{(adjunction)} \\
&\quad = A_Z \otimes^t \mathrm{Hom}_Q(M, W).
\end{aligned}$$

The final expression is $A_Z \otimes^t \mathrm{RHom}_Q(V,W)$, and we obtain the claimed result. □

4.5. **Proof of Theorem 4.3.** We first give a basic lemma on dg modules over regular dg algebras.

Lemma 4.5. *Consider a non-negatively graded dg algebra S with vanishing differential. Suppose also that S is Noetherian and has finite global dimension, as an associative algebra. For any dg S-module M the following are equivalent*

(1) *The cohomology $H^*(M)$ is finitely generated over S.*
(2) *M admits a resolution $M' \to M$ by a finitely generated semi-projective dg S-module M'.*

Proof. The existence of such a resolution implies coherence of cohomology, since S is Noetherian. Suppose now that the cohomology $H^*(M)$ is finitely generated. We prove the existence of the appropriate resolution M by induction on the projective dimension of $H^*(M)$, the projective dimension 0 case being clear. Suppose now that $H^*(M)$ has projective dimension r, and that the desired result holds for bounded below dg modules with cohomology of projective dimension $< r$. Take $P \to H^*(M)$ a surjection from a finitely generated (graded) projective S-module, and $\phi : P \to Z^*(M) \subset M$ any lift of this surjection to the submodule of cocycles. Let C denote the mapping cone of this lift.

By a spectral sequence argument one sees that

$$H^*(C) \cong H^*(\mathrm{cone}(P \to H^*(M))) = \Omega^1 H^*(M),$$

and hence that $H^*(C)$ is finite over S and of projective dimension $r-1$. Let $\psi : M'' \to \Sigma^{-1}C$ be a semi-projective resolution by a finite dg S-module. Then the mapping cone $M' = \mathrm{cone}(M'' \to P)$ of the composite $M'' \to \Sigma^{-1}C \to P$, along with the morphism of dg modules $M' \to M$ induced by the graded S-module maps $\bar{\psi} : M'' \to M$ and $\phi : P \to M$, provides the desired resolution. □

Remark 4.6. The lemma essentially identifies the categories $D_{coh}(S)$ and $\mathrm{thick}_S(S)$ (as defined in, for example, [7]) allowing us to use the conditions interchangeably.

Recall that the symmetric algebra A_Z is of (finite) global dimension $\dim(m_Z/m_Z^2)$. So the above result applies to dg modules over A_Z. We now offer a proof of our theorem.

Proof of Theorem 4.3. Suppose that $\mathrm{RHom}_Q(V,W)$ has finitely generated cohomology over B_Z. Then $\mathrm{RHom}_Q(V,W)$ admits a quasi-isomorphism $N \xrightarrow{\sim} \mathrm{RHom}_Q(V,W)$ from a bounded dg B_Z-module. We have then

$$\mathrm{RHom}_R(V,W) = A_Z \otimes^t \mathrm{RHom}_Q(V,W) \xleftarrow{\sim} A_Z \otimes^t N.$$

The dg module on the right is coherent over A_Z, and thus has coherent cohomology.

Suppose now that $\mathrm{Ext}^*_R(V,W)$ is finite over A_Z. Then $\mathrm{RHom}^*_R(V,W)$ admits a resolution $M \to \mathrm{RHom}_R(V,W)$ by a semi-projective, coherent, dg module over A_Z, by Lemma 4.5. It follows that the cohomology of the fiber

$$k \otimes^{\mathrm{L}}_{A_Z} \mathrm{RHom}_R(V,W) \cong k \otimes_{A_Z} M$$

is finite-dimensional. We have an identification

$$\begin{aligned} k \otimes^{\mathrm{L}}_{A_Z} \mathrm{RHom}_R(V,W) &= k \otimes^{\mathrm{L}}_{A_Z} (A_Z \otimes^t \mathrm{RHom}_Q(V,W)) && \text{(Lemma 4.2)} \\ &= k \otimes_{A_Z} (A_Z \otimes^t \mathrm{RHom}_Q(V,W)) && \text{(Lemma 4.1)} \\ &= \mathrm{RHom}_Q(V,W). \end{aligned}$$

Thus

$$\mathrm{Ext}^*_Q(V,W) \cong H^*(\mathrm{RHom}_Q(V,W)) \cong H^*(k \otimes^{\mathrm{L}}_{A_Z} \mathrm{RHom}_R(V,W))$$

is finite-dimensional, and subsequently finite over B_Z. □

As a corollary to Theorem 4.3 we have the following

Corollary 4.7. *Suppose Q is of finite global dimension, and that $Z \to Q \to R$ is a deformation sequence as in Section 3.1. Then for any finite R-modules V and W, $\mathrm{Ext}^*_R(V,W)$ is a finite A_Z-module.*

4.6. **Implications for Hopf cohomology.** Suppose that u is a finite-dimensional Hopf algebra which admits a smooth integration $\mathsf{U} \to \mathsf{u}$, as in the examples of Section 2. We have the algebra map

$$A_Z \xrightarrow{i_{\mathsf{U}}} HH^*(\mathsf{u}) \xrightarrow{-\otimes^{\mathrm{L}}_{\mathsf{u}} k} \mathrm{Ext}^*_{\mathsf{u}}(k,k). \tag{11}$$

Theorem 4.8. *In the above situation, (11) is a finite algebra map, so that $\mathrm{Ext}^*_{\mathsf{u}}(k,k)$ is a finitely generated algebra. Furthermore, for finite-dimensional u-modules V and W, $\mathrm{Ext}^*_{\mathsf{u}}(V,W)$ is a finite $\mathrm{Ext}^*_{\mathsf{u}}(k,k)$-module via the tensor action $W \otimes - : \mathrm{Ext}^*_{\mathsf{u}}(k,k) \to \mathrm{Ext}^*_{\mathsf{u}}(W,W)$.*

Proof. Corollary 4.7 says directly that $\mathrm{Ext}^*_{\mathsf{u}}(k,k)$ is finite over A_Z in this case. For modules, we have $\mathrm{Ext}^*_{\mathsf{u}}(V,W) = \mathrm{Ext}^*_{\mathsf{u}}(W^* \otimes V, k)$, as an $\mathrm{Ext}^*_{\mathsf{u}}(k,k)$-module. So it suffices to assume $W = k$. But in this case the actions of A_Z on $\mathrm{Ext}^*_{\mathsf{u}}(V,k)$ factors through $\mathrm{Ext}^*_{\mathsf{u}}(k,k)$. So finiteness over A_Z, which holds by Theorem 4.3, implies finiteness over $\mathrm{Ext}^*_{\mathsf{u}}(k,k)$. □

At this point we recover a number of finite generation results, uniformly, via application of Theorem 4.8.

Corollary 4.9. *The following Hopf algebras have finitely generated cohomology, in the strong sense described in the introduction:*

1. *The quantum group $u_q(\mathfrak{g})$* [47, 13] ($\mathrm{char}(k) = 0$).
2. *The quantum Borel $u_q(\mathfrak{b})$* [47, 13] ($\mathrm{char}(k) = 0$).
3. *Restricted enveloping algebras $u^{\mathrm{res}}(\mathfrak{g})$* [39] ($\mathrm{char}(k) < \infty$).
4. *Algebras of functions $\mathscr{O}(\mathcal{G})$ on arbitrary finite group schemes* [80] ($\mathrm{char}(k) < \infty$).
5. *Drinfeld doubles $D(\mathcal{G})$, and relative doubles $D(\mathcal{H}_r, \mathcal{G})$, for $\mathcal{G}$ a height 1 connected group scheme which embeds normally in a smooth algebraic group $\mathcal{H}$* [37] ($\mathrm{char}(k) < \infty$).

Our finite generation results for quantum groups are stronger than those of [47], as we allow q to be of small (odd) order. (Finite generation at arbitrary q was already known, however, and can be found in the text [13].) Our results for doubles are also slightly stronger than those of [37], in the height 1 case, although [37] also addresses doubles of group schemes of height > 1.

5. A meditation on hypersurfaces

We explain the particular implications of the results of Section 4 in the cases in which Q is (homologically) smooth, or of finite global dimension, and subsequently when Q is a "noncommutative hypersurface" in such a smooth algebra.

In this section, and all sections that follow, by a perfect module over a Noetherian algebra we simply mean a finitely generated module which is of finite projective dimension. This is equivalent to perfection of such a module in the corresponding derived category, and we adopt the derived language as it is still meaningful in a dg context.

5.1. **Smooth deformations, finiteness, and naturality.** Fix $Z \to Q \to R$ a deformation sequence as in Section 3.1 with Q *of finite global dimension.* Corollary 4.7 tells us that each $\mathrm{Ext}^*_R(M,N)$ is a finite module over A_Z in this case. So we may view extensions as a bifunctor

$$\mathrm{Ext}^*_R : (R\text{-mod}_{fg})^{op} \times (R\text{-mod}_{fg}) \to A_Z\text{-dgmod}_{fg}.$$

Let $f \in m_Z$ be any function with nonvanishing reduction $\bar{f} \in m_Z/m_Z^2$. We then have the diagram of deformation sequences

$$\begin{array}{ccccc} Z & \twoheadrightarrow & Q & \twoheadrightarrow & R \\ \downarrow{\scriptstyle \pi_Z} & & \downarrow{\scriptstyle \pi_Q} & & \downarrow{\scriptstyle =} \\ Z/(f) & \twoheadrightarrow & Q/(f) & \twoheadrightarrow & R, \end{array}$$

where the $\pi_\#$ are the obvious projections. The projection $Z \to Z/(f)$ also induces an inclusion of tangent spaces $(m_{(Z/f)}/m^2_{(Z/f)})^* \to (m_Z/m_Z^2)^*$ and subsequent algebra inclusion

$$A_{(Z/f)} \to A_Z.$$

We note that the image of this inclusion depends only on the *class* of f in the cotangent space, and not on the choice of f itself. Consider now the possibly non-commuting diagram

$$\begin{array}{ccc} A_{(Z/f)} & \xrightarrow{i_{Q/f}} & HH^*(R) \\ {\scriptstyle \mathrm{incl}}\downarrow & & \downarrow{\scriptstyle =} \\ A_Z & \xrightarrow{i_Q} & HH^*(R). \end{array} \tag{12}$$

Lemma 5.1. *The diagram* (12) *does in fact commute.*

We view the above diagram as a type of naturality property for our setup 3.1.

Proof. Consider any deformation situation $Z' \to Q' \to R'$. The image of a tangent vector $\xi \in (m/m^2)^*$ in Hochschild cohomology is the class of the deformation

$$Q' \otimes_{Z'} k[\epsilon_\xi],$$

where the $k[\epsilon_\xi]$ is $k[\epsilon] = k[\epsilon]/(\epsilon^2)$ equipped with the algebra map $Z' \to k[\epsilon]$ which annihilates m^2 and is defined on the quotient $Z/m^2 \to k[\epsilon]$ by the function $\epsilon \cdot \xi : m/m^2 \to k\epsilon$. Take $Q_f = Q/(f)$ and $Z_f = Z/(f)$. Using this description, one finds that for $\xi \in (m_{Z_f}/m_{Z_f})^* \subset (m_Z/m_Z^2)^*$, i.e. for any function $\xi : m_Z/m_Z^2 \to k$ which vanishes in $\bar{f}$, the projection $Q \to Q_f$ induces an isomorphism of deformations

$$Q \otimes_Z k[\epsilon_\xi] \cong Q_f \otimes_{Z_f} k[\epsilon_\xi].$$

That is to say, the diagram (12) commutes when restricted to the generators, and thus commutes on all of $A_{(Z/f)}$. □

5.2. **Perfection of modules over hypersurfaces.** Commutativity of (12) implies the following.

Corollary 5.2. *Consider $f, g \in m_Z$ elements with equivalent, nonzero, reductions $\bar{f} = \bar{g}$ in m_Z/m_Z^2. Then for finite R-modules V and W,*

$\mathrm{Ext}^*_{Q/(f)}(V,W)$ *vanishes in high degree* $\Leftrightarrow$ $\mathrm{Ext}^*_{Q/(g)}(V,W)$ *vanishes in high degree.*

Proof. By considering Q in Theorem 4.3 to be either of the deformations $Q/(f)$ or $Q/(g)$ of R, the result is a consequence of Theorem 4.3 and Lemma 5.1. □

Locality of Z tells us that all of the finitely generated simples over Q are restricted from simple R-modules along the quotient $Q \to R$. We take $\Lambda = R/\operatorname{Jac}(R) = Q/\operatorname{Jac}(Q)$ to find, for any finitely generated module M over Q,

$$M \text{ is perfect over } Q \;\Leftrightarrow\; \mathrm{Ext}^{\gg 0}_Q(M,\Lambda) = 0.$$

By taking $W = \Lambda$ in Corollary 5.2 we therefore obtain

Corollary 5.3. *Consider $f, g \in m_Z$ elements with equivalent, nonzero, reductions $\bar{f} = \bar{g}$ in m_Z/m_Z^2. Then a finite R-module V is perfect over $Q/(f)$ if and only if V is perfect over $Q/(g)$.*

6. Cohomological supports and hypersurfaces

We show that the cohomological support for R, defined via the action of A_Z on extension $\mathrm{Ext}^*_R(V,W)$, can be identified with a certain hypersurface support. The hypersurface support for a module V is defined by considering (non-)perfection of V over hypersurfaces $Q/(f)$, for varying $f \in m_Z$. (See Definition 6.7 and Corollary 7.2 below.) We then discuss consequences for support in a Hopf theoretic context in Section 7. Throughout $Z \to Q \to R$ is a deformation sequence as in 3.1.

Remark 6.1. The results of this section are noncommutative analogs of results of Avramov-Buchweitz [5].

6.1. **Graded Nakayama.** The following graded version of Nakayama's lemma is standard.

Lemma 6.2. *Let S be a $\mathbb{Z}_{\geq 0}$-graded algebra, and N be a bounded below graded S-module. Then N is 0 if and only if $S_0 \otimes_S N = 0$.*

Note that we have imposed no finiteness conditions on N.

Proof. Suppose that N is nonzero and that the fiber $S_0 \otimes_S N$ vanishes. We have that the minimal degree $N_{\min}$ survives in the fiber $N_{\min} \subset S_0 \otimes_S N$. Hence vanishing of the fiber implies $N_{\min} = 0$, which is nonsense. □

Corollary 6.3. *For S and N as in Lemma 6.2, N is generated by any graded S_0-submodule $N_\omega \subset N$ which surjects onto the fiber $S_0 \otimes_S N$. In particular, if the fiber $S_0 \otimes_S N$ is finite over S_0 then N is finite over S.*

Proof. Take such a lift $N_\omega \subset N$ and consider the graded S-module map $S \otimes_{S_0} N_\omega \to N$ induced by the action of S on N. Let M denote the cokernel, so that we have an exact sequence $S \otimes_{S_0} N_\omega \to N \to M \to 0$. By right exactness of $S_0 \otimes_S -$, we obtain an exact sequence $N_\omega \to S_0 \otimes_S N \to S_0 \otimes_S M \to 0$, from which we conclude that $S_0 \otimes_S M$ vanishes, and hence that M vanishes. So the result holds. □

6.2. **Supports!** In anticipation of the Hopf case, we now impose the following conditions on our deformation $Q \to R$:

(A) Q is of finite global dimension and augmented.
(B) $Q\text{-mod}_{fg}$ comes equipped with a Z-linear endofunctor σ such that, for any finite R-module V and $f \in m_Z$, V is perfect over $Q/(f)$ if and only if $\operatorname{Ext}^{\gg 0}_{Q/(f)}(\sigma V, k) = 0$.

Note that Z-linearity of σ implies that σ restricts to an endofunctor on each full subcategory $Q/IQ\text{-mod}_{fg}$, for any ideal $I \subset Z$, as this category is simply the subcategory of Q-modules for which the action $Z \to \operatorname{End}_Q(M, M)$ vanishes on I.

Example 6.4 (Hopf deformations)**.** Suppose that u is a finite-dimensional Hopf algebra with an integration $\mathsf{U} \to \mathsf{u}$. Then for the sum of the simples $\Lambda = \mathsf{u}/\operatorname{Jac}(\mathsf{u})$ we may take $\sigma = \Lambda \otimes - : \mathsf{U}\text{-mod}_{fg} \to \mathsf{U}\text{-mod}_{fg}$.

In the given context, each $\operatorname{Ext}^*_R(V, W)$ is finite over A_Z, by Corollary 4.7. We fix

$$\mathbb{P} := \mathbb{P}(m_Z/m_Z^2) = \operatorname{Proj}(A_Z).$$

(In a more geometric notation, one could write $\mathbb{P} = \mathbb{P}\big(T_1^* \operatorname{Spf}(Z)\big)$.) For a graded module N over A_Z we let $N^\sim$ denote the associated sheaf on $\mathbb{P}$.

For a deformation $Q \to R$ as above we define two cohomological supports as follows.

Definition 6.5. For V finite over R, define the cohomological support in $\mathbb{P}$ as

$$\operatorname{supp}_{\mathbb{P}}(V) := \operatorname{Supp}_{\mathbb{P}} \operatorname{Ext}^*_R(V, \Lambda)^\sim,$$

where $\Lambda = R/\operatorname{Jac}(R)$ is the sum of the simples for R. We define the cohomological σ-support as

$$\operatorname{supp}^\sigma_{\mathbb{P}}(V) := \operatorname{Supp}_{\mathbb{P}} \operatorname{Ext}^*_R(\sigma V, k)^\sim.$$

Remark 6.6. The reader should note that the supports $\operatorname{supp}_{\mathbb{P}}$ and $\operatorname{supp}^\sigma_{\mathbb{P}}$ depend on the choice of deformation Q for R.

For a closed point $c \in \mathbb{P}$ we let Q_c denote any quotient $Q/(f_c)$ where $f_c \in m_Z$ is any lift of an element in the corresponding line $c \subset m_Z/m_Z^2 - \{0\}$. Recall that the algebra $A_{Z/(f_c)}$ and its corresponding map to Hochschild cohomology are independent of the choice of lift f_c, by Lemma 5.1. We take

$$A_c := A_{(Z/f_cZ)}, \quad \text{for any lift } f_c \text{ of } c \in \mathbb{P}(m_Z/m_Z^2). \tag{13}$$

By Corollary 5.3, for any other choice of lift g_c, and R-module V, perfection of V over $Q/(f_c)$ is equivalent to perfection over $Q/(g_c)$. We may therefore speak unambiguously of perfection of V over Q_c.

Definition 6.7. For V a finite R-module, we define the *hypersurface support* as

$$\operatorname{supp}^{hyp}_{\mathbb{P}}(V) := \{c \in \mathbb{P}(m_Z/m_Z^2) : V \text{ is } \textit{not} \text{ perfect over } Q_c\}^-,$$

where the final bar denotes the closure in $\mathbb{P}$, and c runs over all *closed* points in projective space.

6.3. **Equating cohomological and hypersurface supports.** We maintain the assumptions of the previous subsection. Closed points $c \in \mathbb{P}(m_Z/m_Z^2)$ correspond to graded algebra surjections $\phi_c : A_Z \to k[t]$, where t is given degree 2, and two maps are taken to be equivalent if they differ by a graded automorphism of $k[t]$, i.e. a nonzero scaling of t. Taking the fiber $k[t] \otimes_{A_Z} -$ along any point c is then identified with the reduction

$$k[t] \otimes_{A_Z} - \cong k \otimes_{A_c} -, \tag{14}$$

where $A_c \subset A_Z$ is the subalgebra generated by $\ker(\phi_c|_{A_Z^2})$. One sees directly that the generating subspace $\ker(\phi_c|_{A_Z^2}) \subset (m_Z/m_Z^2)^*$ is the kernel of the associated map $c : (m_Z/m_Z^2)^* \to k$, or rather of the map associated to any choice of representative for $c \in \mathbb{P}(m_Z/m_Z^2)$, so that A_c is precisely the subalgebra of (13).

Lemma 6.8. *Consider a deformation $Q \to R$ as in Section 3.1. Then for any closed point $c \in \mathbb{P}(m_Z/m_Z^2)$, and any finite-dimensional R-modules V and W, the following are equivalent:*

(a) $\mathrm{Ext}^{\gg 0}_{Q_c}(V, W) = 0$.
(b) *The base change $k[t] \otimes_{A_Z} \mathrm{Ext}^*_R(V, W)$ along the map $\phi_c : A_Z \to k[t]$ is finite-dimensional.*
(c) *The base change $k[t, t^{-1}] \otimes_{A_Z} \mathrm{Ext}^*_R(V, W)$ along the localized map $\phi_{c,loc} : A_Z \to k[t, t^{-1}]$ vanishes.*

Proof. Suppose (a) holds. Then Theorem 4.3 implies that $\mathrm{Ext}^*_R(V, W)$ is a finitely generated A_c-module, and hence the fiber

$$k[t] \otimes_{A_Z} \mathrm{Ext}^*_R(V, W) \cong k \otimes_{A_c} \mathrm{Ext}^*_R(V, W)$$

is finite-dimensional. So we see (a) implies (b). Conversely, if the above fiber is finite-dimensional then $\mathrm{Ext}^*_R(V, W)$ is finitely generated over A_c, by Corollary 6.3. Theorem 4.3 then tells us that $\mathrm{Ext}^{\gg 0}_{Q_c}(V, W) = 0$, providing (a).

For the equivalence between (b) and (c), we note that $\mathrm{Ext}^*_R(V, W)$ is finitely-generated over A_Z, by Theorem 4.3, so that $k[t] \otimes_{A_Z} \mathrm{Ext}^*_R(V, W)$ is a finitely-generated $k[t]$-module. We recall that a finitely generated, graded, $k[t]$-module is t-torsion if and only if it is finite-dimensional. So we see that the localization $k[t, t^{-1}] \otimes_{A_Z} \mathrm{Ext}^*_R(V, W)$ vanishes if and only if $k[t] \otimes_{A_Z} \mathrm{Ext}^*_R(V, W)$ is finite-dimensional. □

Theorem 6.9. *Suppose $Q \to R$ is a deformation satisfying* (A) *and* (B) *from Section 6.2. Then for V any finitely generated R-module we have an identification of supports*

$$\mathrm{supp}^{\sigma}_{\mathbb{P}}(V) = \mathrm{supp}^{hyp}_{\mathbb{P}}(V).$$

Proof. It suffices to show that the closed points of $\mathrm{supp}^{\sigma}_{\mathbb{P}}(V)$ are equal to the collection of closed points $c \in \mathbb{P}$ at which V is not perfect over Q_c, in which case taking the closure of this final set recovers $\mathrm{supp}_{\mathbb{P}}(V)$. Equivalently, it suffices to show that the complements of these collections of closed points agree. Recalling our assumption (B), we have

$$V \text{ is not perfect over } Q_c \;\Leftrightarrow\; \mathrm{Ext}^{\gg 0}_{Q_c}(\sigma V, k) = 0.$$

By Lemma 6.8 we therefore find that V is not perfect over Q_c if and only if the base change of $\mathrm{Ext}^*_R(\sigma V, k)$ along the localized map $\phi_c : A_Z \to k[t, t^{-1}]$ vanishes,

$$k[t, t^{-1}] \otimes_{A_Z} \mathrm{Ext}^*_R(\sigma V, k) = 0.$$

This gives the collection of all closed points $c \in \mathbb{P}$ at which V is not perfect as the complement of the support of the coherent $\mathscr{O}_{\mathbb{P}}$-module associated to $\operatorname{Ext}^*_R(\sigma V, k)$, and thus $\operatorname{supp}^{hyp}_{\mathbb{P}}(V) = \operatorname{supp}^{\sigma}_{\mathbb{P}}(V)$. □

As one sees from the proof, the closed points of $\operatorname{supp}^{\sigma}_{\mathbb{P}}(V)$ agree *precisely* with those (closed) points c of $\mathbb{P}$ at which V is non-perfect over the hypersurface Q_c. So we have as a corollary to the proof

Corollary 6.10. *The closed points of* $\operatorname{supp}^{\sigma}_{\mathbb{P}}(V)$ *in* $\mathbb{P}$ *are exactly the set* $\{c \in \mathbb{P} : V$ is *not* perfect over $Q_c\}$. *In particular, this set is already closed (in the topological subspace of k-points) in* $\mathbb{P}$.

The same arguments apply verbatim to the support $\operatorname{supp}_{\mathbb{P}}(V) = \operatorname{Supp}_{\mathbb{P}} \operatorname{Ext}_R(V, \Lambda)^{\sim}$ in $\mathbb{P}$, so that we may identify this cohomological and hypersurface support.

Theorem 6.11 (cf. [5, Theorem 2.5])**.** *For V any finite R-module, there is an identification of supports* $\operatorname{supp}_{\mathbb{P}}(V) = \operatorname{supp}^{hyp}_{\mathbb{P}}(V)$.

Remark 6.12. The cohomological support $\operatorname{supp}_{\mathbb{P}}$ does not play a significant role in our study. However, it is the appropriate object to consider if one is interested in Snashall and Solberg's support theory for associative algebras via Hochschild cohomology [74]. All un-superscripted supports $\operatorname{supp}_?$ which appear in the complement of this section are Hopfy cohomological supports, as defined in Section 7.2 below.

7. Hopfy support and hypersurfaces

We give a precise relation between hypersurface support, defined as in Definition 6.7, and the usual cohomological support for integrable Hopf algebras.

7.1. **Choose a side.** In deciding, in 3.1, that the parametrizing algebra Z for an integration $\mathsf{U} \to \mathsf{u}$ is a right coideal subalgebra in U, we have chosen that $\operatorname{rep}(\mathsf{u})$ acts on the left of the hypersurface categories $\mathsf{U}/(f)\text{-mod}_{fg}$ (see Section 8.1). So the endomorphism $\sigma : \mathsf{U}/(f)\text{-mod}_{fg} \to \mathsf{U}/(f)\text{-mod}_{fg}$ considered in the Hopf situation of Example 6.4 must be tensoring with the simples Λ on the *left*, $\sigma = \Lambda \otimes -$.

To produce consistency with these choices, we consider Hopfy cohomological support according to the algebra maps

$$V \otimes - : \operatorname{Ext}^*_{\mathsf{u}}(k,k) \to \operatorname{Ext}^*_{\mathsf{u}}(V,V) \tag{15}$$

provided by the left action of $\operatorname{rep}(\mathsf{u})$ on itself.

These particular right/left choices do not really matter. What matters, however, is that one does not switch from a "right handed support" to a "left handed support" in a willy-nilly manner. So, we stick to the handedness proposed in Section 3.1, unless explicitly stated otherwise.

7.2. **Hypersurface support and Hopf algebras.** We consider again u a finite-dimensional, integrable, Hopf algebra with chosen integration $\mathsf{U} \to \mathsf{u}$. We observe the algebra map $A_Z \to \operatorname{Ext}^*_{\mathsf{u}}(k,k)$ of equation (11). Fix $\mathsf{Y} := \operatorname{Proj}\left(\operatorname{Ext}^*_{\mathsf{u}}(k,k)\right)_{\mathrm{red}}$ and

$$\kappa : \mathsf{Y} \to \mathbb{P}(m_Z/m_Z^2) \tag{16}$$

the map of varieties dual to (11). When $Z \subset \mathsf{U}$ is specifically *not* a Hopf subalgebra, we may consider the alternate parametrizing subalgebra $Z' = S(Z)$ and corresponding map $\kappa' : \mathsf{Y} \to \mathbb{P}(m_Z/m_Z^2)$.

Recall the standard Hopfy cohomological support

$$\mathrm{supp}_{\mathsf{Y}}(V) := \mathrm{Supp}_{\mathsf{Y}} \mathrm{Ext}^*_{\mathsf{u}}(V,V)^{\sim},$$

where we calculate the sheaf $\mathrm{Ext}^*_{\mathsf{u}}(V,V)^{\sim}$ according to the (graded) action of $\mathrm{Ext}^*_{\mathsf{u}}(k,k)$ on $\mathrm{Ext}^*_{\mathsf{u}}(V,V)$ provided by the algebra map (15).

Theorem 7.1. *Consider* u *a Hopf algebra which admits an integration* $\mathsf{U} \to \mathsf{u}$. *Then the Hopfy cohomological support and hypersurface support satisfy*

$$\kappa\big(\mathrm{supp}_{\mathsf{Y}}(V)\big) = \mathrm{supp}^{hyp}_{\mathbb{P}}(V),$$

where $\kappa : \mathsf{Y} \to \mathbb{P}$ *is as above.*

Proof. For $\sigma = \Lambda \otimes -$ we have $\kappa(\mathrm{supp}_{\mathsf{Y}}(V)) = \mathrm{supp}^{\sigma}_{\mathbb{P}}(V)$. Specifically, the modules $\mathrm{Ext}^*_{\mathsf{u}}(V,V)$ and $\mathrm{Ext}^*_{\mathsf{u}}(\Lambda \otimes V, k)$ have the same support over $\mathrm{Ext}^*_{\mathsf{u}}(k,k)$ [18, §5.7]. By commutative algebra, when we restrict along the (finite) algebra map $A_Z \to \mathrm{Ext}^*_{\mathsf{u}}(k,k)$, the support of the given module over A_Z is the image of the support in $\mathrm{Spec}(\mathrm{Ext}^*_{\mathsf{u}}(k,k))_{\mathrm{red}}$ along the map i^*_{U} to $\mathrm{Spec}(A_Z) = \mathbb{A}(m_Z/m_Z^2)$. These supports are conical subvarieties in the given $\mathbb{G}_m$-equivariant varieties, and we projectivize to obtain the claimed identification of $\kappa(\mathrm{supp}_{\mathsf{Y}}(V))$ with $\mathrm{supp}^{\sigma}_{\mathbb{P}}(V)$. So the result follows by Theorems 6.9 and 6.11. □

If we suppose, furthermore, that $\kappa : \mathsf{Y} \to \mathbb{P}$ is a closed embedding then the Hopfy support can be seen as a support theory valued in closed subvarieties in $\mathbb{P}$.

Corollary 7.2. *If the map* $\kappa : \mathsf{Y} \to \mathbb{P}$ *of* (16) *is a closed embedding, then*

$$\mathrm{supp}_{\mathsf{Y}}(V) = \mathrm{supp}^{hyp}_{\mathbb{P}}(V).$$

In particular, the hypersurface support vanishes on the open complement $\mathbb{P} - \mathsf{Y}$.

7.3. **A weak tensor product property.** We consider again an integrable Hopf algebra u with chosen integration $\mathsf{U} \to \mathsf{u}$. We consider the following additional conditions:

(S1) Z is a Hopf subalgebra in U.
(S2) u is local.
(S3) $\mathrm{rep}(\mathsf{u})$ is braided.

Recall that a braiding on $\mathrm{rep}(\mathsf{u})$ is a choice of natural swap operation $c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$ on products of u-representations. The $c_{V,W}$ here are assumed to be natural in both V and W, so that in total we have a natural isomorphism $c_{-,-}$ between the tensor product on $\mathrm{rep}(\mathsf{u})$ and its opposite, and are also assumed to satisfy the braid relations [53, §1].

Proposition 7.3. *Suppose that the integration* $\mathsf{U} \to \mathsf{u}$ *satisfies any one of the conditions* (S1)–(S3) *above. Then for arbitrary* V *and* W *in* $\mathrm{rep}(\mathsf{u})$, *hypersurface support satisfies the following:*

(1) $\mathrm{supp}^{hyp}_{\mathbb{P}}(V \otimes W) \subset \big(\mathrm{supp}^{hyp}_{\mathbb{P}}(V) \cap \mathrm{supp}^{hyp}_{\mathbb{P}}(W)\big)$.
(2) $\mathrm{supp}^{hyp}_{\mathbb{P}}(V) = \mathrm{supp}^{hyp}_{\mathbb{P}}(V^*)$.

In the proof we employ the notion of a thick subcategory in a triangulated category. The definition of such a subcategory is recalled in Section 9 below.

Proof. In the local case (S2) we have that $V \otimes W$ is in the thick subcategory generated by V and also the thick subcategory generated by W, where here we consider specifically the thick subcategories in $D_{coh}(\mathsf{U}/(f))$ for varying hypersurface. This implies the inclusion (1).

Consider now the braided case (S3). We note that at any function $f \in m_Z$ the category $\mathsf{U}/(f)$-mod$_{fg}$ is a left rep(u)-module category, in the sense that for any V in rep(u), and $\mathsf{U}/(f)$-module M, the action of U on $V \otimes M$ descends to an action of $\mathsf{U}/(f)$. The duality

$$\mathrm{Hom}_{\mathsf{U}/(f)}(V \otimes M, -) \cong \mathrm{Hom}_{\mathsf{U}/(f)}(M, {}^*V \otimes -)$$

implies that the endomorphism $V \otimes - : \mathsf{U}/(f)\text{-mod}_{fg} \to \mathsf{U}/(f)\text{-mod}_{fg}$ associated to any object in rep(u) preserves perfection. Here ${}^*V$ is the linear dual of V with u acting via the inverse antipode S^{-1} (see [36, §2.10]). So we see that

$$\mathrm{supp}_{\mathbb{P}}^{hyp}(V \otimes W) \subset \mathrm{supp}_{\mathbb{P}}^{hyp}(W).$$

Since $V \otimes W \cong W \otimes V$ in this case, we also have an inclusion into $\mathrm{supp}_{\mathbb{P}}^{hyp}(V)$, establishing (1).

Suppose now that Z is a Hopf subalgebra in U, as in (S1). In this case rep(u) acts on both the left *and the right* of the category $\mathsf{U}/(f)$-mod$_{fg}$. So one again employs dualities, as in the braided case, to obtain (1).

Point (1) now implies, for u satisfying any of the (S#), that any object L which is a summand of a product $W \otimes V \otimes W'$, there is an inclusion $\mathrm{supp}_{\mathbb{P}}^{hyp}(L) \subset \mathrm{supp}_{\mathbb{P}}^{hyp}(V)$. In particular, $\mathrm{supp}_{\mathbb{P}}^{hyp}(V^*) \subset \mathrm{supp}_{\mathbb{P}}^{hyp}(V)$, as V^* is a summand of the product $V \otimes V^* \otimes V$, by definition of the dual ${}^*(V^*) \cong V$ [36, Definition 2.10.1]. One obtains the opposite inclusion similarly. So we have (2). □

The analogous equations (1) and (2), for cohomological support $\mathrm{supp}_{\mathsf{Y}}$, are known to hold in both the local and braided cases (S2) & (S3). However, in the case (S1) we obtain something new. We apply Corollary 7.2 to obtain the following from Proposition 7.3.

Corollary 7.4. *Suppose that* u *admits an integration* $\mathsf{U} \to \mathsf{u}$ *for which* Z *is a Hopf subalgebra in* U *and that the map* κ *of* (16) *is a closed embedding. Then cohomological support for* rep(u) *satisfies the following:*

(1) $\mathrm{supp}_{\mathsf{Y}}(V \otimes W) \subset \big(\mathrm{supp}_{\mathsf{Y}}(V) \cap \mathrm{supp}_{\mathsf{Y}}(W)\big)$.
(2) $\mathrm{supp}_{\mathsf{Y}}(V) = \mathrm{supp}_{\mathsf{Y}}(V^*)$.

We note that the relations of Corollary 7.4 are not obvious when rep(u) is not braided, and not even true in general. One can see Example 10.2 below, or [72] for a more extensive exposition.

Lemma 7.5. *If* u *is such that the reduced spectrum* $\mathrm{Spec}(\mathrm{Ext}^*_{\mathsf{u}}(k,k))_{\mathrm{red}}$ *is isomorphic to an affine space* $\mathbb{A}^n_k$*, then the map* $\kappa : \mathsf{Y} \to \mathbb{P}(m_Z/m_Z^2)$ *of* (16) *is a closed embedding.*

Proof. Take $E = \mathrm{Ext}^*_{\mathsf{u}}(k,k)_{\mathrm{red}}$ and take E' the subalgebra generated by E^2. By assumption, E is isomorphic to a polynomial ring. Since E is graded, we may take homogenous generators ξ_i, $\deg(\xi_i) = d_i$, and have $E = k[\xi_1, \ldots, \xi_d]$. We know that the deformation map

$$\mathrm{Sym}(\Sigma^{-2}(m_Z/m_Z^2)^*) = A_Z \to E \tag{17}$$

is finite, by Theorem 4.8. We note that A_Z has image in the subalgebra E' generated in degree 2, so that E must be finite over E', and E' must be finite over the image of A_Z. We have $k\otimes_{A_Z} E' = \mathrm{Sym}(E^2/\mathrm{im}(m_Z/m_Z^2)^*)$ and $k\otimes_{E'} E = k[\xi_j : \deg(\xi_j) > 2]$. Finiteness of these fibers forces $E^2 = \mathrm{im}(m_Z/m_Z^2)^*$, and $E' = E$. Hence we have surjectivity of (17). □

One applies Lemma 7.5 to see that the relations of Corollary 7.4 hold in a number of "unipotent" and "solvable" settings. We just remark on one.

Corollary 7.6. *For $u_q(\mathfrak{b})$ the small quantum Borel, in arbitrary Dynkin type, at q of odd order greater than the associated Coxeter number h, cohomological support satisfies*

$$\mathrm{supp}_{\mathsf{Y}}(V\otimes W) \subset \big(\mathrm{supp}_{\mathsf{Y}}(V)\cap \mathrm{supp}_{\mathsf{Y}}(W)\big)$$

and also $\mathrm{supp}_{\mathsf{Y}}(V) = \mathrm{supp}_{\mathsf{Y}}(V^*)$.

Proof. The distinguished subalgebra Z_0 in $U_q^{DK}(\mathfrak{b})$ is a Hopf subalgebra and, by [47], the algebra of extensions $\mathrm{Ext}^*_{u_q(\mathfrak{b})}(\mathbb{C},\mathbb{C})$ is a polynomial ring in this case. So the result follows by Lemma 7.5 and Corollary 7.4. □

Remark 7.7. For u the restricted enveloping algebra $u^{\mathrm{res}}(L)$ of reductive L in large characteristic, we expect the map κ is simply the embedding of the (twisted) nilpotent cone $\mathcal{N}^{(1)}$ into $L^{(1)}$, or rather the projectivization of this map. Similarly, for the quantum group $u_q(\mathfrak{g})$ at large order q, κ should also be the projectivized embedding of the nilpotent cone into $\mathfrak{g}$. So, in particular, the hypersurface and cohomological supports should agree here. In the finite characteristic setting, one could explicitly prove this result by identifying the Hochschild map employed in [39], [38, Proposition 5.2], with our deformation map of (11).

Remark 7.8. In Section 13 we will be rather precise in our definition of the quantum Borel. In particular, we acknowledge that there are varying choices of Borels with varying groups of grouplikes. In Corollary 7.6 the particular choice of grouplikes for $u_q(\mathfrak{b})$ does not matter.

8. Background: a preamble to examples

We now move on to the second phase in this work. In the following sections we use hypersurface support to prove the tensor product property for cohomological support in a number of "solvable" examples. We explain what precisely counts as a "solvable" Hopf algebra in Section 8.4 below. The proofs of the tensor product property in these varied contexts rely on certain recurring arguments which are both homological and (mildly) tensor categorical in nature. We provide here the backgrounds needed to understand the examples considered in the latter sections of the text.

We recall that a tensor category is a k-linear, abelian, rigid monoidal [36, §2.10] category $\mathscr{C}$ in which all objects are of finite length and Hom sets are finite-dimensional. We also require that the unit $\mathbf{1}\in\mathscr{C}$ is simple. A tensor category $\mathscr{C}$ is *finite* if it has finitely many simples and enough projectives, and $\mathscr{C}$ is *fusion* if it is finite and semisimple. The only tensor categories we are interested are those of the form $\mathrm{rep}(\mathsf{u})$ for a finite-dimensional Hopf algebra u (in which case $\mathrm{rep}(\mathsf{u})$ is finite), so one needn't concern themselves with the intricacies of theory of tensor categories here. We would claim, however, that the language and philosophies from the subject are useful.

We are also interested in *braided* tensor categories, which, just as in the case of $\mathrm{rep}(\mathsf{u})$ discussed in Section 7.3, are tensor categories $\mathscr{C}$ equipped with a natural swap operation $c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$ on objects [53, §1].

8.1. **Module categories.** We refer the reader to [69, Definition 6.2] for a precise definition of a module category. Basically, a (left) module category over a tensor category $\mathscr{C}$ is a k-linear abelian category $\mathscr{M}$ with a biexact action functor $\otimes : \mathscr{C} \times \mathscr{M} \to \mathscr{M}$ which is compatible with the tensor structure on $\mathscr{C}$, in the obvious ways. One has the analogous notion of a bimodule category.

We have already encountered our primary source of examples: Given $\mathrm{rep}(\mathsf{u})$ for a Hopf algebra u, and B a (left) u-comodule algebra, we obtain an action of $\mathrm{rep}(\mathsf{u})$ on B-mod. In particular, for arbitrary V in $\mathrm{rep}(\mathsf{u})$ and M in B-mod, we employing the coaction $B \to \mathsf{u} \otimes B$ to provide a natural B-action on the tensor products $V \otimes B$. In this way, the linear tensor product $\otimes = \otimes_k$ provides our exact action functor $\otimes : \mathrm{rep}(\mathsf{u}) \times B\text{-mod} \to B\text{-mod}$, endowing B-mod with a $\mathrm{rep}(\mathsf{u})$-module category structure. When B is a u-bicomodule, we similarly find that B-mod is a bimodule category over $\mathrm{rep}(\mathsf{u})$.

By considering the adjunction $\mathrm{Hom}_B(V \otimes M, -) = \mathrm{Hom}_B(M, {}^*V \otimes -)$ we see that the action of $\mathrm{rep}(\mathsf{u})$ on such B-mod preserves the subcategory of compact objects, i.e. finitely presented modules. So, when B is Noetherian, we have that $B\text{-mod}_{fg}$ is a $\mathrm{rep}(\mathsf{u})$-module subcategory in B-mod. The analogous claims hold in the bicomodule/$\mathrm{rep}(\mathsf{u})$-bimodule setting. Such duality arguments also show that the action(s) of $\mathrm{rep}(\mathsf{u})$ on B-mod preserves perfect objects.

8.2. **Drinfeld centeralizers.**

Definition 8.1 ([62])**.** Given a tensor subcategory $\mathscr{D}$ in a tensor category $\mathscr{C}$, we define the Drinfeld centralizer $Z^{\mathscr{D}}(\mathscr{C})$ to be the category of pairs (V, γ_V) where V is an object in $\mathscr{C}$ and γ_V is a half braiding against $\mathscr{D}$. Specifically, $\gamma_V : V \otimes - \to - \otimes V$ is a natural isomorphism between the functors $- \otimes V$, $V \otimes - : \mathscr{D} \to \mathscr{C}$ which satisfies the appropriate braid relations.

The Drinfeld center $Z(\mathscr{C})$ of $\mathscr{C}$ is the centralizer of $\mathscr{C}$ against itself, i.e. the category of pairs of an object V with a global half braiding $\gamma_V : V \otimes - \to - \otimes V$.

Both $Z^{\mathscr{D}}(\mathscr{C})$ and $Z(\mathscr{C})$ inherit obvious tensor structures $(V, \gamma_V) \otimes (W, \gamma_W) = (V \otimes W, (\gamma_V \otimes 1)(1 \otimes \gamma_W))$ from $\mathscr{C}$, and we have a surjective tensor functor

$$\mathrm{Forget} : Z^{\mathscr{D}}(\mathscr{C}) \to \mathscr{C} \tag{18}$$

given by forgetting the half braiding [35, Proposition 3.39]. By surjective we mean that all objects of $\mathscr{C}$ are obtained by taking subquotients of objects from $Z^{\mathscr{D}}(\mathscr{C})$.

In the Hopf setting, $Z(\mathrm{rep}(\mathsf{u}))$ is equivalent to the category of modules over the Drinfeld double $D(\mathsf{u}) = \mathsf{u} \bowtie \mathsf{u}^*$ (see e.g. [61, 54]). For $\mathrm{rep}(\Lambda) \to \mathrm{rep}(\mathsf{u})$ the tensor inclusion corresponding to a Hopf quotient $\mathsf{u} \to \Lambda$, the Drinfeld centralizer is identified with representations over the Hopf subalgebra $\mathsf{u} \bowtie \Lambda^*$ in $D(\mathsf{u})$.

We prefer the categorical expression $Z^{\mathscr{D}}(\mathscr{C})$ to the algebraic one as we mean to emphasize a single point here: It is natural to consider, in some specific contexts, objects V in $\mathrm{rep}(\mathsf{u})$ with a uniform means of moving V past a chosen class of other objects in $\mathrm{rep}(\mathsf{u})$, under the tensor action.

8.3. **Bosonization and half-braidings.** Consider u^+ a finite-dimensional Hopf algebra in a braided tensor category $\mathrm{rep}(\Lambda)$ which, as the notation suggests, is the representation category of a quasitriangular Hopf algebra Λ. Then the category $\mathrm{rep}_\Lambda(\mathsf{u}^+)$ of finite-dimensional left u^+-modules in $\mathrm{rep}(\Lambda)$ forms a tensor category for which the composition

$$\mathrm{rep}_\Lambda(\mathsf{u}^+) \overset{forget}{\to} \mathrm{rep}(\Lambda) \to Vect$$

provides a canonical fiber functor. Such modules are simply u^+-modules with a compatible action of Λ, i.e. modules over the smash product $\mathsf{u} = \mathsf{u}^+ \rtimes \Lambda$, and the tensor structure on $\mathrm{rep}_\Lambda(\mathsf{u}^+)$ induces a Hopf structure on u under which the subalgebra u^+ is a left u-comodule algebra. The algebra u, with its given Hopf structure, is the *bosonization* of u^+. We consider the explicit examples of bosonized quantum complete intersections in Section 11 below.

One can endow all objects in $\mathrm{rep}(\Lambda)$ with the trivial u^+-action, via the counit $\mathsf{u}^+ \to \mathbf{1}$, to obtain a tensor embedding

$$\mathrm{rep}(\Lambda) \to \mathrm{rep}_\Lambda(\mathsf{u}^+) = \mathrm{rep}(\mathsf{u}).$$

When u^+ is local, and Λ is semisimple, this map is an equivalence onto the fusion subcategory of semisimple objects in $\mathrm{rep}(\mathsf{u})$. In the statement of the following lemma c denotes the braiding on $\mathrm{rep}(\Lambda)$.

Lemma 8.2. *For u a bosonized Hopf algebra $\mathsf{u} = \mathsf{u}^+ \rtimes \Lambda$ as above, there is a canonical tensor functor $\mathrm{rep}(\mathsf{u}) \to Z^{\mathrm{rep}(\Lambda)}(\mathrm{rep}(\mathsf{u}))$ which is a section of the forgetful functor. Specifically, for V in $\mathrm{rep}(\mathsf{u})$ and L in $\mathrm{rep}(\Lambda)$, the isomorphisms*

$$\gamma_{V,L} : V \otimes L \to L \otimes V$$

given by the braiding $\gamma_{V,L} := c_{V,L}$ on $\mathrm{rep}(\Lambda)$ provide a $\mathrm{rep}(\Lambda)$-centralizing structure on all objects in $\mathrm{rep}(\mathsf{u})$, and the operations $\gamma_{V,L}$ are natural in V and L.

Proof. Both $V \otimes L$ and $L \otimes V$ have well-defined u-actions, and the braiding $\gamma = c$ is a map of Λ-modules by construction. So we need only check u^+-linearity. Since u^+ acts trivially on L, the action of u^+ on $V \otimes L$ is just induced by the action on V, and the action on $L \otimes V$ employs the braiding

$$\mathrm{act}_{L\otimes V} = (1 \otimes \mathrm{act}_V)(c_{\mathsf{u}^+,L} \otimes 1) : \mathsf{u}^+ \otimes L \otimes V \to L \otimes V.$$

Naturality of the braiding, and the braid relation, then imply commutativity of the diagram

$$\begin{array}{ccc} \mathsf{u}^+ \otimes V \otimes L & \xrightarrow{1\otimes c} & \mathsf{u}^+ \otimes L \otimes V \\ \downarrow{\scriptstyle \mathrm{act}_{V\otimes L}} & & \downarrow{\scriptstyle \mathrm{act}_{L\otimes V}} \\ V \otimes L & \xrightarrow{\quad c \quad} & L \otimes V. \end{array}$$

The above diagram implies that $\gamma_{V,L} = c_{V,L}$ is a map of u^+, and hence u, representations. □

8.4. **Geometrically Chevalley algebras.** In the following definition we let $\mathrm{Rep}(\Lambda)$ denote the monoidal category of infinite-dimensional representations for a finite-dimensional Hopf algebra Λ.

Definition 8.3. Call an integrable Hopf algebra u geometrically Chevalley if

(a) u is the bosonization $\mathsf{u} = \mathsf{u}^+ \rtimes \Lambda$ of a local Hopf algebra u^+ in a (semisimple!) braided fusion category $\mathrm{rep}(\Lambda)$.

(b) u^+ admits a deformation sequence $Z \to \mathsf{U}^+ \to \mathsf{u}^+$ via algebras in $\mathrm{Rep}(\Lambda)$ such that
  (b1) U^+ is a local Hopf algebra in $\mathrm{Rep}(\Lambda)$ which is of finite global dimension, as an associative algebra.
  (b2) Z is a central Hopf subalgebra in U^+ which has trivial Λ-action and admits an algebra isomorphism $Z \cong k[\![y_1, \dots, y_n]\!]$.

We call $\mathsf{U} = \mathsf{U}^+ \rtimes \Lambda$ the corresponding Chevalley integration of u, and have the corresponding integration $\mathsf{U} \to \mathsf{u}$ parametrized by the Hopf subalgebra Z in U.

Examples of geometrically Chevalley algebras include bosonized quantum complete intersections, quantum Borels, and also the Drinfeld doubles $D(B_{(1)})$ for Borel subgroups $B \subset \mathbb{G}$ in almost-simple algebraic group $\mathbb{G}$ (in particular characteristics). Restricted enveloping algebras $u^{\mathrm{res}}(\mathfrak{n})$ of nilpotent restricted Lie algebras are also geometrically Chevalley, with $\Lambda = k$. In the case of $D(B_{(1)})$ we take $\Lambda = kT_1$, where $T \subset B$ is the torus, and the braiding on $\mathrm{rep}(\Lambda) = \mathrm{rep}(T_1)$ is the standard (trivial) symmetry. These examples are all considered in detail in Sections 11 and 13 below.

By Lemma 8.2, all objects in $\mathrm{rep}(\mathsf{u})$ centralize the simples $\mathrm{rep}(\Lambda)$ in this case, and we have a canonical section $\mathrm{rep}(\mathsf{u}) \to Z^{\mathrm{rep}(\Lambda)}(\mathrm{rep}(\mathsf{u}))$ for any geometrically Chevalley u.

9. Noncommutative Hopkins' lemma for hypersurfaces

Let $\mathcal{T}$ be a triangulated category. Recall that a thick subcategory in $\mathcal{T}$ is a full triangulated subcategory which is closed under taking summands in $\mathcal{T}$, and the thick subcategory $\langle X_i : i \in I\rangle$ *generated* by a collection of objects $\{X_i\}_{i\in I}$ in $\mathcal{T}$ is the smallest thick subcategory in $\mathcal{T}$ which contains all the X_i. The most basic example of a thick subcategory for us is that of perfect complexes $\mathrm{perf}(Q) = \langle Q\rangle$ in the derived category of modules over a given algebra Q.

Here we suggest an analysis of thick subcategories for the hypersurface categories $D_{coh}(\mathsf{U}/(f))$–or rather, $D_{coh}(\mathsf{U}^+/(f))$ in the geometrically Chevalley instance– which provides a technical foundation for the work that follows.

9.1. **Hopkins' lemma and noncommutative hypersurfaces.** Consider Q a *commutative*, regular, local algebra and $Q/(f)$ a hypersurface algebra. Then a version of Hopkins' lemma [26, Proposition 5.8] implies that, for M non-perfect over $Q/(f)$, the trivial module k is in the thick subcategory $\langle M\rangle$ generated by M in the derived category of $Q/(f)$. It seems clear that this precise result does *not* hold for a noncommutative hypersurface algebra $Q/(f)$, as one needs to properly account for the noncommutativity in this case.

In a geometrically Chevalley setting $\mathsf{U}^+ \to \mathsf{u}^+$, we consider an alternative implication:

$$M \text{ non-perfect in } \mathsf{U}^+/(f)\text{-mod}_{fg} \overset{\text{question}}{\Rightarrow} k \in \langle \lambda \otimes M : \lambda \in \mathrm{Irrep}(\Lambda)\rangle. \tag{19}$$

We act on the left of $\mathsf{U}^+/(f)$-mod via the left u, and hence Λ, comodule structure. More generally, one may replace the λ in $\mathrm{Irrep}(\Lambda)$ with objects in an arbitrary tensor subcategory $\mathscr{D} \subset \mathrm{rep}(\mathsf{u})$. Our main approach to the tensor product property for the examples considered in Section 10–13 below is to show that the implication (19) *does* hold in these cases, and use this implication to deduce the tensor product property for support.

Lemma 9.1. *Consider* u *geometrically Chevalley, with Chevalley integration* $\mathsf{U} \to \mathsf{u}$. *Fix* $\mathscr{D} \subset \mathrm{rep}(\mathsf{u})$ *an arbitrary tensor subcategory, and consider functions* $f \in m_Z$ *with non-trivial reduction to* m_Z/m_Z^2.

Suppose that for any non-perfect, finitely generated, module M *over a hypersurface* $\mathsf{U}^+/(f)$, k *is in the thick subcategory* $\langle \lambda \otimes M : \lambda \in \mathscr{D} \rangle$ *generated by the* $\mathscr{D}$*-orbit of* M *in* $D_{coh}(\mathsf{U}^+/(f))$. *Then the hypersurface support* $\mathrm{supp}_{\mathbb{P}}^{hyp}$ *on* $D_{fin}(\mathsf{u})$ *satisfies the tensor product property*

$$\mathrm{supp}_{\mathbb{P}}^{hyp}(V \otimes W) = \mathrm{supp}_{\mathbb{P}}^{hyp}(V) \cap \mathrm{supp}_{\mathbb{P}}^{hyp}(W).$$

The analogous result holds when u *is local and admits an integration* $\mathsf{U} \to \mathsf{u}$ *by a local Hopf algebra* $(\mathsf{U}^+ =)\mathsf{U}$.

Proof. In what follows, for local u we take $\Lambda = k$, $\mathrm{rep}(\Lambda) = Vect$. In both the geometrically Chevalley and local case we have an inclusion of supports $\mathrm{supp}_{\mathbb{P}}^{hyp}(V \otimes W) \subset \mathrm{supp}_{\mathbb{P}}^{hyp}(V) \cap \mathrm{supp}_{\mathbb{P}}^{hyp}(W)$, by Proposition 7.3. So, we need only establish opposite inclusion.

Since the positive subalgebra u^+ is local, by hypothesis, the inclusion $\mathrm{rep}(\Lambda) \to \mathrm{rep}(\mathsf{u})$ is an isomorphism onto the fusion subcategory of semisimple modules. Furthermore, this fusion subcategory generates $\mathrm{rep}(\mathsf{u})$ under extension. Therefore, for any $\mathscr{D}$ as in the statement we have

$$\langle \lambda \otimes M : \lambda \in \mathscr{D} \rangle \subset \langle \lambda \otimes M : \lambda \in \mathrm{Irrep}(\Lambda) \rangle.$$

So it suffices to prove the result when $\mathscr{D} = \mathrm{rep}(\Lambda)$.

We must prove the following: If $\mathsf{U}/(f)$ is a hypersurface along which W is non-perfect, and the product $V \otimes W$ is perfect, then V must be perfect. (Here f is a function in the parametrizing algebra Z with non-vanishing linear part.) Now, imperfection of a module over $\mathsf{U}/(f) = (\mathsf{U}^+/(f)) \rtimes \Lambda$ is equivalent to imperfection over the subalgebra $\mathsf{U}^+/(f)$. So we are free to replace $\mathsf{U}/(f)$ with this local subalgebra $\mathsf{U}^+/(f)$, and attempt to establish the same claims regarding imperfection.

By locality of $\mathsf{U}^+/(f)$, imperfection of a finitely generated module M over $\mathsf{U}^+/(f)$ is equivalent to nonvanishing of $\mathrm{Ext}^*_{\mathsf{U}^+/(f)}(M, k)$ in arbitrarily high degrees. We recall also that the u-comodule structure on $\mathsf{U}^+/(f)$ implies a (left) action of $\mathrm{rep}(\mathsf{u})$ on $\mathsf{U}^+/(f)$-mod.

Consider such f, V, and W as prescribed. As W is non-perfect along $\mathsf{U}^+/(f)$ we have

$$k \in \langle \lambda \otimes W : \lambda \in \mathrm{Irrep}(\Lambda) \rangle \ \subset D_{coh}(\mathsf{U}^+/(f)),$$

by hypothesis. Applying the endofunctor $V \otimes - : D_{coh}(\mathsf{U}^+/(f)) \to D_{coh}(\mathsf{U}^+/(f))$ and the centrality claim of Lemma 8.2 gives then

$$V \in \langle V \otimes \lambda \otimes W : \lambda \in \mathrm{Irrep}(\Lambda) \rangle = \langle \lambda \otimes V \otimes W : \lambda \in \mathrm{Irrep}(\Lambda) \rangle.$$

Since $V \otimes W$ is perfect, this final thick subcategory is contained in $\mathrm{perf}(\mathsf{U}^+/(f))$, so that V must be perfect. □

Remark 9.2. It suffices to work stably, so that one may replace the thick subcategory generated by the $\lambda \otimes M$ with the thick subcategory generated by the $\lambda \otimes M$ and $\mathsf{U}^+/(f)$ in equations (19) and Lemma 9.1 above.

Remark 9.3. It is possible that (19) holds in general, although we would not claim such a result at this time.

10. The tensor product property for functions on finite group schemes

We consider representations of $\mathscr{O}(\mathcal{G})$, for a finite group scheme $\mathcal{G}$. In this case we have $\mathrm{rep}(\mathscr{O}(\mathcal{G})) = \mathrm{Coh}(\mathcal{G})$, and we employ the latter notation for the sake of expediency. We prove in Theorem 10.5 below that (Hopfy) cohomological support for $\mathrm{Coh}(\mathcal{G})$ has the tensor product property whenever $\mathcal{G}$ is connected. In the case in which $\mathcal{G}$ is *not* connected, the tensor product property for $\mathrm{Coh}(\mathcal{G})$ simply does not hold (see Example 10.2). However, we prove in Corollary 10.8 a variant of the tensor product property for non-connected $\mathcal{G}$ which is, in a global sense, optimal.

Arguably, the case of $\mathrm{Coh}(\mathcal{G})$ is not the most interesting application of the methods developed in this paper. Indeed, many of the issues we've addressed in the present document are well-established in the commutative setting (see Remark 10.7). However, this very natural class of examples provides an excellent case study from which one can observe a number of important phenomenon. We consider more novel situations in Sections 11 and 13 below.

10.1. **Reducing to the connected component I.** Consider $\mathcal{G}$ a finite group scheme, with identity component $\mathcal{G}_o$. We note that in this case the cohomology rings $\mathrm{Ext}^*_{\mathscr{O}(\mathcal{G})}(k,k)$ and $\mathrm{Ext}^*_{\mathscr{O}(\mathcal{G}_o)}(k,k)$ for the algebras of functions agree. So we write unambiguously Y for the reduced projective spectrum of either of these rings. The following lemma can be deduced from work of Plavnik and Witherspoon [72], and we omit the proof.

Lemma 10.1 ([72, Theorem 3.2])**.** *Let $\mathcal{G}$ be a finite group scheme, with identity component $\mathcal{G}_o \subset \mathcal{G}$ and subgroup of closed points $\pi = \pi_0(\mathcal{G})$. The following are equivalent:*

(a) *Cohomological support for* $\mathrm{Coh}(\mathcal{G})$ *has the tensor product property.*
(b) *Cohomological support for the identity component* $\mathrm{Coh}(\mathcal{G}_o)$ *has the tensor product property and the group of closed points $\pi \subset \mathcal{G}$ acts trivially on Y, under the adjoint action.*

It is easy to construct examples of $\mathcal{G} = \mathcal{G}_o \rtimes \pi$ for which π acts non-trivially on cohomology for $\mathcal{G}_o$. Indeed, one deduces from such pairs $(\mathcal{G}_o, \pi)$ examples of tensor categories for which one has no such inclusion

$$\mathrm{supp}_{\mathsf{Y}}(V \otimes W) \nsubseteq \big(\mathrm{supp}_{\mathsf{Y}}(V) \cap \mathrm{supp}_{\mathsf{Y}}(W)\big).$$

One can compare the following example with Corollary 7.4.

Example 10.2 (cf. [17])**.** Take $k = \overline{\mathbb{F}}_p$. Consider $\mathbb{Z}_2$ acting on $\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$ by swapping the factors. Take $\mathcal{G}_o = \mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$ and $\mathcal{G} = \mathcal{G}_o \rtimes \mathbb{Z}_2$. Let $\mathrm{supp}^{\mathcal{G}_o}$ and $\mathrm{supp}^{\mathcal{G}}$ denote the corresponding supports. Since $\mathscr{O}(\mathcal{G}_o)$ is cocommutative, its support $\mathrm{supp}^{\mathcal{G}_o}$ is known to have the tensor product property [44].

Consider V and V' the $\mathscr{O}(\mathcal{G}_o)$ representations given by projecting onto the second and first factor of $\mathscr{O}(\mathbb{G}_{a(1)})$ respectively. Then one has

$$\mathrm{Ext}^*_{\mathscr{O}(\mathcal{G}_o)}(k,k)_{\mathrm{red}} = k[x,y], \quad \mathrm{Ext}^*_{\mathscr{O}(\mathcal{G}_o)}(k,V) = k[x], \quad \mathrm{Ext}^*_{\mathscr{O}(\mathcal{G}_o)}(k,V') = k[y].$$

We also have $V' = \mathrm{Ad}_\sigma(\mathrm{V})$ in $\mathrm{Coh}(\mathcal{G})$, where $\mathbb{Z}/2\mathbb{Z} = \{1, \sigma\}$. Thus

$$\mathrm{supp}^{\mathcal{G}}(V) = pt_x, \ \mathrm{supp}^{\mathcal{G}}(\sigma \oplus k) = \mathbb{P}^1,$$

and

$$\mathrm{supp}^{\mathcal{G}}(V \otimes (\sigma \oplus k)) = \mathrm{supp}^{\mathcal{G}}(V' \oplus V) = pt_x \cup pt_y,$$

where $\mathbb{P}^1$ is identified with the projective spectrum of $k[x,y] = \mathrm{Ext}^*(k,k)_{\mathrm{red}}$ and pt_x and pt_y are the images of the coordinate axes for x and y. In particular, if we take $\mathrm{supp}_{\mathsf{Y}}$ the cohomological support for $\mathscr{O}(\mathcal{G})$, V as above and $W = \sigma \oplus k$, then

$$pt_x \cup pt_y = \mathrm{supp}_{\mathsf{Y}}(V \otimes W) \not\subseteq \big(\mathrm{supp}_{\mathsf{Y}}(V) \cap \mathrm{supp}_{\mathsf{Y}}(W)\big) = pt_x.$$

We note finally that the Hopfy support and hypersurface support agree for this example, by Lemma 10.4, so that the tensor product property for hypersurface support also fails to hold in this case.

One can similarly consider an n-fold product of $\mathbb{G}_{a(1)}$'s equipped with the permutation action of S_n, and observe an explicit obstruction to the tensor product property in these cases as well. A generalization of the above example, where $\mathscr{O}(\mathcal{G})$ is replaced with an arbitrary Hopf algebra, can be found in [72, Proof of Corollary 3.4].

10.2. **Reducing to the connected component II.** We consider a finite group scheme $\mathcal{G}$ with connected component $\mathcal{G}_o$. Suppose that support for $\mathrm{Coh}(\mathcal{G}_o)$ satisfies the tensor product property, which is in fact the case by Theorem 10.5 below.

We have the action of the finite group $\pi = \pi_0(\mathcal{G})$ of closed points, i.e. simple $\mathscr{O}(\mathcal{G})$-representations, on $\mathrm{Coh}(\mathcal{G})$ via conjugation $\mathrm{Ad}_\lambda(\mathrm{V}) = \lambda \otimes \mathrm{V} \otimes \lambda^{-1}$. This action corresponds to the adjoint action of π on the scheme $\mathcal{G}$, in the sense that the tensor adjoint action permutes sheaves on $\mathcal{G}$ via pushforward along the adjoint action automorphisms $\mathrm{Ad}_\lambda : \mathcal{G} \to \mathcal{G}$.

To say that an object V in $\mathrm{Coh}(\mathcal{G})$ is equivariant with respect to this conjugation action of π is to say that there is a compatible collection of natural isomorphisms $V \otimes \lambda \cong \lambda \otimes V$ for all $\lambda \in \pi$. So we observe that a π-equivariant structure on V is equivalent to a $\mathrm{rep}(\pi)$-centralizing structure, i.e. half-braiding, on V.

In the following theorem we consider non-connected $\mathcal{G}$, and let $\mathrm{supp}_{\mathsf{Y}}$ denote the cohomological support for $\mathrm{Coh}(\mathcal{G})$. We take $F : Z^{\mathrm{Coh}(\pi)}(\mathrm{Coh}(\mathcal{G})) \to \mathrm{Coh}(\mathcal{G})$ the forgetful functor.

Lemma 10.3 (cf. [72])**.** *Let $\mathcal{G}$ be a finite group scheme and suppose that support for the identity component $\mathrm{Coh}(\mathcal{G}_o)$ satisfies the tensor product property. Take $\pi = \pi_0(\mathcal{G})$ the group of closed points.*

For arbitrary W in $\mathrm{Coh}(\mathcal{G})$, and V in the Drinfeld centralizer of $\mathrm{Coh}(\pi)$ in $\mathrm{Coh}(\mathcal{G})$, there is an identification of supports

$$\mathrm{supp}_{\mathsf{Y}}(F(V) \otimes W) = \mathrm{supp}_{\mathsf{Y}}(F(V)) \cap \mathrm{supp}_{\mathsf{Y}}(W).$$

Proof. We may write $V = \oplus_\lambda (V_\lambda \otimes \lambda) = \oplus_\lambda (\lambda \otimes {}_\lambda V)$ for representations V_λ and ${}_\lambda V$ supported on $\mathcal{G}_o$, and adopt similar expressions for W. (Here the sums run over the simples $\lambda \in \pi$.) Let $\mathrm{supp}^{\mathcal{G}}$ and $\mathrm{supp}^{\mathcal{G}_o}$ denote the cohomological supports for $\mathrm{Coh}(\mathcal{G})$ and $\mathrm{Coh}(\mathcal{G}_o)$ respectively. We have

$$\mathrm{supp}^{\mathcal{G}}(V) = \cup_\lambda \mathrm{supp}^{\mathcal{G}_o}({}_\lambda V), \quad \mathrm{supp}^{\mathcal{G}}(W) = \cup_\lambda \mathrm{supp}^{\mathcal{G}_o}({}_\lambda W).$$

We are assuming that V centralizes the simples. This implies an identification of the right handed and left handed supports for V,

$$\cup_\lambda \mathrm{supp}^{\mathcal{G}_o}({}_\lambda V) = \mathrm{supp}^{\mathcal{G}}(V) = \cup_\lambda \mathrm{supp}^{\mathcal{G}_o}(V_\lambda) \tag{20}$$

We have that $\operatorname{supp}^{\mathcal{G}}(V \otimes W)$ is the support of cohomology

$$\begin{aligned}\operatorname{Ext}^*_{\operatorname{Coh}\mathcal{G}}(k, \textstyle\bigoplus_\lambda \lambda \otimes (V \otimes W)) &\cong \operatorname{Ext}^*_{\operatorname{Coh}\mathcal{G}}(k, \textstyle\bigoplus_\lambda (V \otimes \lambda \otimes W)) \\ &= \textstyle\bigoplus_{\lambda,\mu} \operatorname{Ext}^*_{\operatorname{Coh}\mathcal{G}_o}(k, V_\mu \otimes {}_{(\mu\lambda)^{-1}}W), \\ &= \textstyle\bigoplus_{\lambda,\mu} \operatorname{Ext}^*_{\operatorname{Coh}\mathcal{G}_o}(k, V_\mu \otimes {}_\lambda W)\end{aligned}$$

where the sum runs over all invertibles $\lambda \in \pi$. We employ the tensor product property for $\mathcal{G}_o$ and equation (20) to obtain

$$\begin{aligned}\operatorname{supp}^{\mathcal{G}}(V \otimes W) &= \textstyle\bigcup_{\lambda,\mu} \operatorname{supp}^{\mathcal{G}_o}(V_\mu \otimes {}_\lambda W) \\ &= \textstyle\bigcup_{\lambda,\mu} \big(\operatorname{supp}^{\mathcal{G}_o}(V_\mu) \cap \operatorname{supp}^{\mathcal{G}_o}({}_\lambda W)\big) \\ &= \big(\textstyle\bigcup_\mu \operatorname{supp}^{\mathcal{G}_o}(V_\mu)\big) \cap \operatorname{supp}^{\mathcal{G}}(W) \ = \operatorname{supp}^{\mathcal{G}}(V) \cap \operatorname{supp}^{\mathcal{G}}(W).\end{aligned}$$

□

10.3. **Support for coherent sheaves on finite group schemes.** Recall that a finite group scheme $\mathcal{G}$ is called infinitesimal if it is connected. Equivalently, $\mathcal{G}$ is infinitesimal if $\mathscr{O}(\mathcal{G})$ is local. Recall also that for any embedding $\mathcal{G} \to \mathcal{H}$ into smooth $\mathcal{H}$, we have the corresponding deformation sequence $\widehat{\mathscr{O}}_{\mathcal{H}/\mathcal{G}} \to \widehat{\mathscr{O}}_{\mathcal{H}} \to \mathscr{O}(\mathcal{G})$ (see Example 2.4). We take in this instance $Z = \widehat{\mathscr{O}}_{\mathcal{H}/\mathcal{G}}$, and as usual $\mathsf{Y} = \operatorname{Proj}(\operatorname{Ext}^*_{\operatorname{Coh}(\mathcal{G})}(k,k))_{\mathrm{red}}$.

Lemma 10.4. *Let $\mathcal{G}$ be an infinitesimal group scheme, with chosen embedding $\mathcal{G} \to \mathcal{H}$ into a smooth algebraic group. Then the associated map $\kappa : \mathsf{Y} \to \mathbb{P}(m_Z/m_Z^2)$ of (16) is a closed embedding.*

Proof. Take $E = \operatorname{Ext}^*_{\operatorname{Coh}(\mathcal{G})}(k,k)_{\mathrm{red}}$. We have an algebra isomorphism $\mathscr{O}(\mathcal{G}) \cong k[x_1, \dots, x_n]/(x_i^{d_i} : 1 \le i \le n)$, abstractly [80], so that E is a graded polynomial ring generated in degree 2, $E = \operatorname{Sym}(E^2)$. Therefore κ is a closed embedding by Lemma 7.5. □

Theorem 10.5. *Let $\mathcal{G}$ be an infinitesimal group scheme. Then for arbitrary V and W in $\operatorname{Coh}(\mathcal{G})$ we have*

$$\operatorname{supp}_{\mathsf{Y}}(V \otimes W) = \operatorname{supp}_{\mathsf{Y}}(V) \cap \operatorname{supp}_{\mathsf{Y}}(W),$$

where $\operatorname{supp}_{\mathsf{Y}}$ denotes the cohomological support.

Proof. Fix an embedding $\mathcal{G} \to \mathcal{H}$ into a smooth algebraic group. By the weak tensor product property (and Lemma 10.4) we have a containment

$$\operatorname{supp}_{\mathsf{Y}}(V \otimes W) \subset \big(\operatorname{supp}_{\mathsf{Y}}(V) \cap \operatorname{supp}_{\mathsf{Y}}(W)\big).$$

We want to show now that if a given closed point in Y is in both $\operatorname{supp}_{\mathsf{Y}}(V)$ and $\operatorname{supp}_{\mathsf{Y}}(W)$, then it is in $\operatorname{supp}_{\mathsf{Y}}(V \otimes W)$. Equivalently, according to Corollary 7.2, we want to show that if $\widehat{\mathscr{O}}_{\mathcal{H}}/(f)$ is a hypersurface along which both V and W are non-perfect, then the product $V \otimes W$ is also non-perfect over $\widehat{\mathscr{O}}_{\mathcal{H}}/(f)$.

Let us fix such a function $f \in \widehat{\mathscr{O}}_{\mathcal{H}/\mathcal{G}}$ and take $\mathscr{O} = \widehat{\mathscr{O}}_{\mathcal{H}}$. It is well-known that imperfection of an object M over the local (commutative) hypersurface $\mathscr{O}/(f)$ implies that the trivial module k is in the thick subcategory $\langle M \rangle$ generated by M, in $D_{coh}(\mathscr{O})$ (see e.g. [78]). By exactness of the operation $V \otimes -$ this implies $V \in \langle V \otimes M \rangle$. Hence, perfection of the product $V \otimes M$ implies

$$V \in \langle V \otimes M \rangle \subset \operatorname{perf}(\mathscr{O}/(f)),$$

so that V is seen to be perfect. Considering the case $M = W$, we observe the desired implication

$$x \in \mathrm{supp}_{\mathsf{Y}}(W) \text{ while } x \notin \mathrm{supp}_{\mathsf{Y}}(V \otimes W) \;\Rightarrow\; x \notin \mathrm{supp}_{\mathsf{Y}}(V),$$

i.e. if W is not perfect over a given hypersurface, while $V \otimes W$ is perfect, then V *must* be perfect over that hypersurface. □

Remark 10.6. In the proof, one can provide a more direct argument to find that $k \in \langle M \rangle$ whenever M is non-perfect over a commutative local hypersurface, using arguments of Carlson-Iyengar [26]. We will follow such a line of reasoning in the proof of Theorem 11.3 below (see also Remark 11.5).

Remark 10.7. Theorem 10.5 can alternatively be proved by employing the classification of thick subcategories in $\mathrm{Coh}(\mathcal{G})$ [26]. One uses this classification to reduce to the case of a product of Carlson modules, then observes the desired result by applying [70, Corollary 4.1].

By applying Lemma 10.3 we obtain a result for general finite group schemes.

Corollary 10.8. *Let $\mathcal{G}$ be an arbitrary finite group scheme, and let π denote the discrete subgroup of closed points in $\mathcal{G}$. For W arbitrary in $\mathrm{Coh}(\mathcal{G})$, and V in the Drinfeld centralizer $Z^{\mathrm{Coh}(\pi)}(\mathrm{Coh}(\mathcal{G}))$, we have an identification*

$$\mathrm{supp}_{\mathsf{Y}}(F(V) \otimes W) = \mathrm{supp}_{\mathsf{Y}}(F(V)) \cap \mathrm{supp}_{\mathsf{Y}}(W), \tag{21}$$

where $\mathrm{supp}_{\mathsf{Y}}$ denotes the cohomological support for $\mathrm{Coh}(\mathcal{G})$ and F is the forgetful functor (18).

We would suggest from examples such as Example 10.2 that the conditions on V in the above result are, in a global sense, optimal. More specifically, the tensor product property simply cannot hold in the non-connected case, and the above corollary gives (arguably) the strongest sense in which the tensor product property does hold for coherent sheaves on arbitrary $\mathcal{G}$.

Remark 10.9. Due to our particular left/right conventions (see Section 7.1), the $\mathrm{rep}(\pi)$-centralizing object V in Corollary 10.8 must appear on the *left*. However, if we take V in the full Drinfeld center $Z(\mathrm{Coh}(\mathcal{G}))$, then the analogous claim (21) is side-independent.

10.4. **A remark on centrality and nonbraided TPP.** We propose an alternate version of the tensor product property which is more robust that the usual version (5), and is satisfied by all examples which currently appear in the literature. Our particular framing here is informed by 4-dimensional topological field theory [25, 24], although we don't elaborate on this point here. From a more pragmatic perspective, we want to provide a version of the tensor product property which *is* satisfied by the examples $\mathrm{Coh}(\mathcal{G})$ considered above.

One can view any nonbraided tensor category $\mathscr{C}$ as a tensor category *over* its Drinfeld center $Z(\mathscr{C})$. We recall that a tensor category $\mathscr{C}$ over a given braided tensor category $\mathscr{Z}$ is a tensor category equipped with a central tensor functor $F : \mathscr{Z} \to \mathscr{C}$. Formally, such central F consists of a choice of tensor functor $F_0 : \mathscr{Z} \to \mathscr{C}$ and a braided lift of this functor to the Drinfeld center $F_1 : \mathscr{Z} \to Z(\mathscr{C})$. Informally, we simply mean that for arbitrary V in $\mathscr{Z}$ and W in $\mathscr{C}$ there is a commutativity relation $F(V) \otimes W \cong W \otimes F(V)$ which is natural in V and W.

Example 10.10. For the quantum Borel $u_q(\mathfrak{b})$, the forgetful functor $\operatorname{rep} u_q(\mathfrak{g}) \to \operatorname{rep} u_q(\mathfrak{b})$ from the associated quantum group admits a central structure $\operatorname{rep} u_q(\mathfrak{g}) \to Z(\operatorname{rep} u_q(\mathfrak{b}))$. This follows from the explicit form of the R-matrix for $u_q(\mathfrak{g})$.

For nonbraided $\mathscr{C}$ we consider again the support $\operatorname{supp}_{\mathsf{Y}}(V) = \operatorname{Supp}_{\mathsf{Y}} \operatorname{Ext}^*_{\mathscr{C}}(V,V)^{\sim}$, but now adopt a tensor relation which expresses a "linearity" property with respect to the action of the Drinfeld center $Z(\mathscr{C})$,

$$\operatorname{supp}_{\mathsf{Y}}(F(V) \otimes W) \overset{\text{question}}{=} \operatorname{supp}_{\mathsf{Y}}(F(V)) \cap \operatorname{supp}_{\mathsf{Y}}(W). \tag{22}$$

Here V is in $Z(\mathscr{C})$, W is in $\mathscr{C}$, and $F : Z(\mathscr{C}) \to \mathscr{C}$ is the forgetful functor. The implicit half-braiding on objects in $Z(\mathscr{C})$ implies that there is no sidedness in the expression (22). One considers the relation (22) as a replacement for the usual tensor product property (5).

More generally, for $F : \mathscr{Z} \to \mathscr{C}$ any central tensor functor we can consider the relation analogous to (22) for $\mathscr{Z}$ acting on $\mathscr{C}$. We leave the proof of the following Lemma to the interested reader.

Lemma 10.11. *The following are equivalent:*

(1) *The relation* (22) *holds for all V in $Z(\mathscr{C})$ and W in $\mathscr{C}$.*
(2) *The analogous relation* (22) *holds for any braided $\mathscr{Z}$ equipped with a central tensor functor $F : \mathscr{Z} \to \mathscr{C}$.*

Furthermore, when $\mathscr{C}$ itself admits a braiding then either of the above two points are equivalent to the usual tensor product property (5) *for $\mathscr{C}$.*

Definition 10.12. We say that support for a tensor category $\mathscr{C}$ satisfies the *centralized tensor product property* if equation (22) holds at all V in $Z(\mathscr{C})$ and W in $\mathscr{C}$.

Corollary 10.8 implies that, for sheaves on non-connected $\mathcal{G}$ for example, the standard tensor product property (5) generally fails while the central version (22) still holds. More explicitly, support for the category $\operatorname{Coh}(\mathcal{G})$ of coherent sheaves on the smash product $\mathcal{G} = (\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}) \rtimes \mathbb{Z}/2\mathbb{Z}$ was shown to *not* have the tensor product property (5), in Example 10.2. But we have shown that support for $\operatorname{Coh}(\mathcal{G})$, for this particular $\mathcal{G}$, does have the central tensor product property outlined above. This alternate tensor product property also holds in all of the (counter)examples considered in [17, 72], where the standard tensor product property was shown to fail.

Furthermore, for pointed Hopf algebras with a "bad" choice of grouplikes the centralized tensor product property (22) is, seemingly, a more appropriate relation to consider, in comparing with the usual version (5). One can see our explicit analysis of quantum complete intersections with various choices of grouplikes given in Theorems 11.3 and 11.6 below.

Remark 10.13. From a more minimalist perspective, one can replace the Drinfeld center $Z(\mathscr{C})$ with the centralizer of the simples in $\mathscr{C}$ in the above presentation. See, for example, [65, §5 & 6]. The main point is that one should employ *some* centralization hypotheses when dealing with support in a non-braided context.

11. Bosonized quantum complete intersections

We consider support for some non-classical analogs of functions on infinitesimal group schemes, over the complex numbers. We focus on a class of examples associated to skew polynomial rings with odd order skewing parameters. We refer to these examples as quantum complete intersections, although the nomenclature *quantum linear spaces* is also common [1, 23].

11.1. **Bosonized quantum complete intersections.** Consider a skew polynomial ring

$$\mathsf{U}^+ = \mathbb{C}\langle\langle x_1, \dots, x_n \rangle\rangle / (x_i x_j - q_{ij} x_j x_i : i \neq j), \quad q_{ij} = q^{a_{ij}},$$

defined by a choice of root of unity q of odd order, and an integer matrix $[a_{ij}]$ which is skew symmetric away from the diagonal $a_{ij} = -a_{ji}$ and of constant value $a_{ii} = 1$ along the diagonal. Fix $l = \mathrm{ord}(q)$.

Take now $G^\vee$ a finite abelian group equipped with a bilinear form $q^{(-,-)} : G^\vee \times G^\vee \to \mathbb{C}^\times$ such that, for distinguished elements $e_1, \dots, e_n$ and $\chi_1, \dots, \chi_n$ in $G^\vee$, we have

$$q^{(e_i, e_j)} = q^{a_{ij}} \quad \text{and} \quad q^{(e_i, \chi_j)} = q^{(e_i, e_j)_{\mathrm{alt}}} \quad \forall \ \ 1 \leq i \leq n. \tag{23}$$

Here $(-,-)_{\mathrm{alt}}$ is the alternating form $(e_i, e_j)_{\mathrm{alt}} = \frac{1}{2}(a_{ij} - a_{ji})$.

One can construct such $G^\vee$ by simply considering a rank $2n$ elementary abelian l-group with free generators $e_1, \dots, e_n, \chi_1, \dots, \chi_n$, and form defined on these generators as above. On the other hand, when the determinant of the matrix $[a_{ij}]$ is invertible mod l, the required elements χ_j already exist in the rank n elementary abelian l-group generated by the e_i, with form defined in accordance with (23).

In any case, we fix $G^\vee$ as above and take $\Lambda = \mathbb{C}G$ the group ring of the character group for $G^\vee$. The form on $G^\vee$ endows the tensor category $\mathrm{rep}(\Lambda)$ with a braided structure. In terms of homogeneous vectors, the braiding on $\mathrm{rep}(\Lambda)$ is given explicitly by

$$c_{V,W} : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto q^{(\deg v, \deg w)} w \otimes v.$$

The algebra U^+ then becomes an algebra in $\mathrm{rep}(\Lambda)$, or rather in its Ind-completion $\mathrm{Rep}(\Lambda)$, by taking each generator $x_i \in \mathsf{U}^+$ to be of $G^\vee$-degree e_i.

The coproduct $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$ on U^+ endows it with the structure of a Hopf algebra in $\mathrm{Rep}(\Lambda)$, and we have the quotient Hopf algebra

$$\mathsf{u}^+ = \mathsf{U}^+ / (x_i^l : 1 \leq i \leq n)$$

in $\mathrm{rep}(\Lambda)$. By applying bosonization (Section 8.3) we obtain Hopf algebras $\mathsf{U} = \mathsf{U}^+ \rtimes \Lambda$ and $\mathsf{u} = \mathsf{u}^+ \rtimes \Lambda$, in $Vect$, and the quotient map $\mathsf{U} \to \mathsf{u}$ deforms u along the trivial commutative Hopf subalgebra $Z = \mathbb{C}[\![x_1^l, \dots, x_n^l]\!]$. (By trivial, we mean that Λ acts trivially on Z.) So u is geometrically Chevalley with Chevalley integration $\mathsf{U} \to \mathsf{u}$. We refer to u constructed in this manner as a bosonized algebra of functions on a quantum complete intersection, or just *a bosonized quantum complete intersection*.

To be clear, as an algebra

$$\mathsf{u} = \left(\mathbb{C}\langle x_1, \dots, x_n \rangle / (x_i x_j - q_{ij} x_j x_i, \ x_i^l)_{i,j}\right) \rtimes \Lambda \tag{24}$$

and the coproduct is given by the formula $\Delta(x_i) = x_i \otimes 1 + K_i \otimes x_i$, where K_i is the character $K_i = q^{(e_i, -)} : G^\vee \to \mathbb{C}^\times$.

Remark 11.1. The name "quantum complete intersection" is somewhat unsatisfactory, as it does not reference the Hopf structure on this algebra. Such algebras are probably more accurately understood as q-analogs of functions on first Frobenius kernels $(\mathbb{G}_a^n)_{(1)}$ in additive group schemes $\mathbb{G}_a^n$ of varying rank.

Remark 11.2. There is some ambiguity in our presentation of the Hopf algebras u, as we only specify the grouplikes within a certain range of parameters. We consider the "standard" choice of grouplikes for such a quantum complete intersection in Section 11.3 below.

Theorem 11.3. *Consider* u *a bosonized quantum complete intersection, with grouplikes* G *as above. Then for arbitrary* V *and* W *in* $\mathrm{rep}(\mathsf{u})$*, we have*

$$\mathrm{supp}_{\mathsf{Y}}(V \otimes W) = \mathrm{supp}_{\mathsf{Y}}(V) \cap \mathrm{supp}_{\mathsf{Y}}(W).$$

We prove Theorem 11.3 in Section 11.2 below. Before giving the proof, let us compare the above result with our findings for functions on non-connected group schemes $\mathscr{O}(\mathcal{G})$, Corollary 10.8. We refer to the situation considered in the present section as the "q-symmetric" situation.

In both the non-connected case of Corollary 10.8 and the q-symmetric case, orbits of objects under the tensor-action of the simples appear in the proof. Furthermore, the centralizing category $Z^{\mathrm{rep}(\Lambda)}(\mathrm{rep}(\mathsf{u}))$ also appears in the proof of Theorem 11.3 (and in the statements of Theorem 11.6 below). There is, however, a fundamental distinction between these two cases. Namely, in the q-symmetric setting, the parametrizing subalgebra $Z \subset \mathsf{U}$ is a Hopf subalgebra in U. This implies that modules over each hypersurfaces $\mathsf{U}/(f)$ have a bimodule structure over u. Such normality does *not* hold in the case of functions on a non-connected group scheme, in general.

Indeed, in the case of non-connected $\mathcal{G}$ the "hard" inclusion

$$\big(\mathrm{supp}_{\mathsf{Y}}(V) \cap \mathrm{supp}_{\mathsf{Y}}(W)\big) \subset \mathrm{supp}_{\mathsf{Y}}(V \otimes W)$$

still holds on $\mathrm{Coh}(\mathcal{G})$. It is the opposite inclusion that fails to hold. The opposite inclusion in the q-symmetric setting is forced by (co)normality of the deformation $\mathsf{U} \to \mathsf{u}$ (Corollary 7.4). In considering Theorem 11.6 below, which makes a claim mirroring almost exactly that of Corollary 10.8, we would still propose that the situations of 11.6 and 10.8 are rather dissimilar. And so, we view the uniform nature of Corollary 10.8 and Theorem 11.6 as a testament to the general framework outlined in Section 10.4.

11.2. **Proof of Theorem 11.3.**

Proof of Theorem 11.3. We first note that $\mathrm{Ext}^*_{\mathsf{u}}(\mathbb{C},\mathbb{C})_{\mathrm{red}}$ is a polynomial ring [20], so that the map $\kappa : \mathsf{Y} \to \mathbb{P}$ is a closed embedding by Lemma 7.5. Hence we have an inclusion of supports $\mathrm{supp}_{\mathsf{Y}}(V \otimes W) \subset \big(\mathrm{supp}_{\mathsf{Y}}(V) \cap \mathrm{supp}_{\mathsf{Y}}(W)\big)$, by Corollary 7.4. We seek the opposite inclusion.

Fix $f \in m_Z$ with non-trivial linear component. According to Lemma 9.1, it suffices to prove that $\mathbb{C}$ is in the thick subcategory $\langle \lambda \otimes M : \lambda \in \mathrm{Irrep}(\Lambda)\rangle$ for finite non-perfect M over $\mathsf{U}^+/(f)$. Fix $R = \mathsf{U}^+/(f)$, $Q = \mathsf{U}^+$, $Z = \mathbb{C}[\![f]\!]$, and consider the corresponding deformation sequence $Z \to Q \to R$. Fix also T to be the central subalgebra $\mathbb{C}[\![x_1^l, \dots, x_n^l]\!]$ in Q.

Take $K_Q = Q \otimes_Z K_Z$ the corresponding dg resolution of R, with K_Z the Koszul resolution of k over Z, as in Section 3.3. We have the reduction $\mathbb{C} \otimes_Q K_Q$, which

is isomorphic to the dual numbers $\mathbb{C}[\varepsilon] = \mathbb{C}[d_f]/(d_f^2)$ with generator in degree -1, and may consider the reduction

$$\mathbb{C} \otimes^{\mathrm{L}}_{Q} - : K_Q\text{-dgmod} \to \mathbb{C}[\varepsilon]\text{-dgmod}.$$

We consider here, specifically, dg K_Q-modules with finite cohomology over T and dg $\mathbb{C}[\varepsilon]$-modules with finite-dimensional cohomology.

By adjunction we see that

$$\mathrm{Ext}^*_{K_Q}(M, \mathbb{C}) = \mathrm{Ext}^*_{\mathbb{C}[\varepsilon]}(\mathbb{C} \otimes^{\mathrm{L}}_{Q} M, \mathbb{C})$$

so that $\mathbb{C} \otimes^{\mathrm{L}}_{Q} M$ is non-perfect over the dual numbers whenever M is non-perfect over K_Q. By a dg version of Hopkins' Theorem [26, Theorem 4.4] for $\mathbb{C}[\varepsilon]$, we have then

$$\mathbb{C} \in \langle \mathbb{C} \otimes^{\mathrm{L}}_{Q} M \rangle \quad \text{whenever} \quad M \text{ is non-perfect over } K_Q, \tag{25}$$

where $\langle L \rangle$ denotes the thick subcategory in $D_{fin}(\mathbb{C}[\varepsilon])$ generated by the given object L. We claim that this implies

$$\mathbb{C} \in \langle \lambda \otimes M : \lambda \in \mathrm{Irrep}(\Lambda) \rangle \subset D_{coh}(K_Q).$$

Let us prove this claim.

We have the q-exterior dg algebra

$$E = \wedge^*_q \mathbb{C}\{d_{x_1}, \dots, d_{x_n}\} = \mathbb{C}\langle d_{x_i} : 1 \leq i \leq n \rangle / (d_{x_i} d_{x_j} + q_{ij} d_{x_j} d_{x_i}, d_{x_i}^2)$$

in the braided category of chain complexes over $\mathrm{rep}(\Lambda)$, where the d_{x_i} are of cohomological degree -1 and respective $G^\vee$-degree e_i. We consider the corresponding Koszul resolution $F = (E \otimes Q, d_F)$ of $\mathbb{C}$ over Q, with explicit differential

$$d_F(d_{x_{i_r}} \dots d_{x_{i_1}} \otimes a) = \sum_j (-1)^{j+1} (\prod_{t<j} q_{i_j i_t}) d_{x_{i_r}} \dots \widehat{d_{x_{i_j}}} \dots d_{x_{i_1}} \otimes x_{i_j} a.$$

This complex is in fact a dg *bimodule* over Q via symmetry of Q. Namely, we have the well-defined left action of Q on F given by the formula

$$b \cdot (v \otimes w) := q^{(\deg(b), \deg(v))_{\mathrm{alt}}} v \otimes bw = q^{(\deg(b), \chi_v)} v \otimes bw, \tag{26}$$

where $(-,-)_{\mathrm{alt}}$ is as in (23) and $\chi_v = \sum_j \chi_{i_j}$ for $v = \prod_j d_{x_{i_j}}$. Hence tensoring by F provides an endofunctor

$$F \otimes^{\mathrm{L}}_{Q} - = F \otimes_Q - : D_{coh}(K_Q) \to D_{coh}(K_Q)$$

which is isomorphic to the endofunctor $\mathbb{C} \otimes^{\mathrm{L}}_{Q} -$, which we interpret as the composition of the reduction to $\mathbb{C}[\varepsilon]$ composed with restriction $D_{fin}(\mathbb{C}[\varepsilon]) \to D_{coh}(K_Q)$.

Let $\mathscr{D} \subset \mathrm{rep}(\Lambda)$ denote the tensor subcategory generated by the simples associated to the characters χ_j. By construction of F and equation (26), we have for any dg K_Q-module M,

$$\begin{aligned} \mathbb{C} \otimes^{\mathrm{L}}_{Q} M \cong F \otimes^{\mathrm{L}}_{Q} M &\in \langle E^i \otimes M : 0 \leq i \leq n \rangle \\ &\subset \langle \chi \otimes M : \chi \text{ a simples in } \mathscr{D} \rangle \\ &\subset \langle \lambda \otimes M : \lambda \text{ a simples in } \mathrm{rep}(\Lambda) \rangle \end{aligned}$$

By (25), imperfection of M over K_Q then implies

$$\mathbb{C} \in \langle \lambda \otimes M : \lambda \in \mathrm{Irrep}(\Lambda) \rangle,$$

as desired. The tensor product property now follows as an application of Lemma 9.1. □

Remark 11.4. The deformation $Q \to R$ employed in the proof does not fit into the setup of Section 3.1, as Q is not finite over the parametrizing algebra $Z = k[\![f]\!]$ in this case. (Although, Q is finite over complete local central subalgebra $T \supset Z$.) However, all we use in this case is the associated Koszul resolution $K_Q \to R$, which is a valid construction for any deformation $Q \to R$ with complete, formally smooth parametrizing algebra Z.

Remark 11.5. By considering Koszul duality $D_{fin}(\mathbb{C}[\varepsilon]) \cong D_{coh}(\mathbb{C}[t])$, $\deg(t) = 2$, and the fact that $\mathbb{C}[t]$ is a PID, one can basically classify thick subcategories in $D_{fin}(\mathbb{C}[\varepsilon])$ by hand. (The classification is rather easy for the stable category $D_{sing}(\mathbb{C}[\varepsilon])$, which is sufficient for our needs.) That is to say, one doesn't actually have to understand the dg version of Hopkins' theorem, in full regalia, in order to prove of Theorem 11.3.

11.3. **Changing grouplikes (standard QCIs).** Consider again the positive algebras of Section 11.1, which we relabel as

$$\mathsf{A}_q^+(P) = \mathbb{C}\langle\langle x_1, \dots, x_n \rangle\rangle / (x_i x_j - q_{ij} x_j x_i)_{i,j}$$

$$\text{and} \quad \mathsf{a}_q^+(P) = \mathbb{C}\langle x_1, \dots, x_n \rangle / (x_i x_j - q_{ij} x_j x_i,\ x_i^l)_{i,j}.$$

Here q is a root of unity of odd order l and $P = [a_{ij}]$ is an integer matrix for which $q_{ij} = q^{a_{ij}}$. As before we assume $a_{ij} = -a_{ji}$ off the diagonal, and all $a_{ii} = 1$. Having fixed such P, we adopt the more efficient notation $A_q^+ = A_q^+(P)$ and $\mathsf{a}_q^+ = \mathsf{a}_q^+(P)$.

Consider G^{st} the rank n elementary abelian l-group with generators K_i, and let G^{st} act on A_q^+ and a_q^+ in the expected way $K_i \cdot x_j = q_{ij} x_j$. Then the smash products

$$\mathsf{A}_q = \mathsf{A}_q^+ \rtimes G^{\mathrm{st}} \quad \text{and} \quad \mathsf{a}_q = \mathsf{a}_q^+ \rtimes G^{\mathrm{st}}$$

become Hopf algebras under the coproduct $\Delta(x_i) = x_i \otimes 1 + K_i \otimes x_i$.

The Hopf algebra A_q integrates a_q, with parametrizing Hopf subalgebra $Z = \mathbb{C}[\![x_1^l, \dots, x_n^l]\!]$, and the positive (coideal) subalgebra A_q^+ deforms a_q^+ along $\mathrm{Spf}(Z)$ as well.

The Hopf algebra(s) a_q constructed as above are what most people would refer to as *the* bosonized quantum complete intersection, or *the* quantum linear space, associated to a root of unity q and matrix $[a_{ij}]$. The algebras u of Section 11.1 and a_q of course differ only in the choice of grouplikes and, by considering the delta characters $\delta_j = q^{(-,e_j)}\chi_j^{-1}$ in $G^\vee$, we see that there are Hopf inclusion

$$\mathrm{incl} : \mathsf{A}_q \to \mathsf{U} \quad \text{and} \quad \mathrm{incl} : \mathsf{a}_q \to \mathsf{u}.$$

These inclusions are the identity on their respective positive parts, and send the K_i in G^{st} to the K_i in G.

We let Λ^{st} and Λ denote the group rings of G^{st} and G respectively, where G is as in Section 11.1.

Theorem 11.6. *Consider $\mathsf{a}_q = \mathsf{a}_q(P)$ the bosonized quantum complete intersection associated to given odd order q and matrix $P = [a_{ij}]$. For V in the Drinfeld centralizer $Z^{\mathrm{rep}(\Lambda^{\mathrm{st}})}(\mathrm{rep}\,\mathsf{a}_q)$, and arbitrary W in $\mathrm{rep}(\mathsf{a}_q)$, we have*

$$\mathrm{supp}_{\mathsf{Y}}(F(V) \otimes W) = \mathrm{supp}_{\mathsf{Y}}(F(V)) \cap \mathrm{supp}_{\mathsf{Y}}(W).$$

As before, $F : Z^{\mathrm{rep}(\Lambda^{\mathrm{st}})}(\mathrm{rep}\,\mathsf{a}_q) \to \mathrm{rep}(\mathsf{a}_q)$ is the forgetful functor.

Proof. We further simplify our notation and take $\mathsf{A} = \mathsf{A}_q(P)$ and $\mathsf{A}^+ = \mathsf{A}_q^+(P)$. The comultiplication on A provides A^+ with a Λ^{st}-comodule algebra structure,

$$\rho : \mathsf{A}^+ \xrightarrow{\Delta} \mathsf{A} \otimes \mathsf{A}^+ \xrightarrow{\pi\otimes 1} \Lambda^{\mathrm{st}} \otimes \mathsf{A}^+.$$

This coaction descends to a coaction on all hypersurface algebras $\mathsf{A}^+/(f)$, for $f \in m_Z$. Via this coaction we get a $\mathrm{rep}(\Lambda^{\mathrm{st}})$-action on the category of modules over any hypersurface algebra $\mathsf{A}^+/(f)$, as well as its corresponding derived category.

Consider $f \in m_Z$ with nonvanishing reduction in m_Z/m_Z^2. We claim that any non-perfect, finite, M over $\mathsf{A}^+/(f)$ is such that

$$\mathbb{C} \in \langle \mu \otimes M : \mu \in \mathrm{Irrep}(\Lambda^{\mathrm{st}})\rangle \subset D_{coh}(\mathsf{A}^+/(f)). \tag{27}$$

It was already shown in the proof of Theorem 11.3 that for such M

$$\mathbb{C} \in \langle \lambda \otimes M : \lambda \in \mathrm{Irrep}(\Lambda)\rangle \subset D_{coh}(\mathsf{U}^+/(f)) = D_{coh}(\mathsf{A}^+/(f)). \tag{28}$$

Now, we have the restriction map $\mathrm{rep}(\Lambda) \to \mathrm{rep}(\Lambda^{\mathrm{st}})$, and claim that for any character $\lambda \in \mathrm{Irrep}(\Lambda)$ with restriction $\mu \in \mathrm{Irrep}(\Lambda^{\mathrm{st}})$, the $\mathsf{A}^+ = \mathsf{U}^+$-modules $\lambda \otimes M$ and $\mu \otimes M$ are literally equal. Indeed, this just follows from the fact that the comultiplication on U sends the positive subalgebra U^+ into the subalgebra $(\mathsf{U}^+ \rtimes G^{\mathrm{st}}) \otimes \mathsf{U}^+$, so that the each endofunctor $\lambda \otimes -$ of U^+-mod depends only on the restriction of λ to G^{st}. (The point here is that the grouplikes K_i live in the subgroup $G^{\mathrm{st}} \subset G$.)

Consider finally a product $V \otimes W$ of a $\mathrm{rep}(\Lambda^{\mathrm{st}})$-centralizing representation V with an arbitrary representation W over a_q. Suppose that W is non-perfect over a given hypersurface $\mathsf{A}/(f)$ while $V \otimes W$ is perfect. One uses (27), and argues as in Lemma 9.1, to see that V must be perfect in this case. Rather, we find that hypersurface support for a_q satisfies the stated tensor product property. Since (reduced) cohomology for a_q is an affine space, cohomological support also satisfies this tensor product property. □

12. (q-)Regular sequences

We consider a particular noncommutativization of the notion of a regular sequence. Such sequences are employed in proofs of the tensor product property for cohomological support, in the examples of Section 13.

12.1. **A choice of ambient braided fusion category $\mathscr{D}$.** Fix $G^\vee$ a finite abelian group with a chosen (exponentiated) form $q^{(-,-)} : G^\vee \times G^\vee \to k^\times$. We are free to assume $\mathrm{ord}(q) | \exp(G^\vee)$ and the form $(-,-)$ can be seen as an additive form which takes values in $\mathbb{Z}/\exp(G^\vee)\mathbb{Z}$. We view $G^\vee$ as the group of characters for its dual group in characteristic 0, or for the dual Hopf algebra in finite characteristic.

We fix

$\mathscr{D}$ = The braided fusion category $Vect_{G^\vee}$, with braiding given by the form $q^{(-,-)}$.

Recall that $Vect_{G^\vee}$ is the fusion category of $G^\vee$-graded vector spaces. The irreducibles in $\mathscr{D}$ are explicitly the 1-dimensional vector spaces supported at a given character. As before, in terms of homogeneous vectors, the braiding on $\mathscr{D}$ is given by

$$c_{V,W} : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto q^{(\deg v, \deg w)} w \otimes v.$$

We impose no symmetry condition on the form $(-,-)$, and we remark that the case $\mathscr{D} = Vect$, $G = \{1\}$, is still of interest.

12.2. **Regular sequences.** Consider A a dg algebra in $\mathscr{D}$, or more precisely the category of complexes over the Ind-category $\operatorname{Ind}\mathscr{D}$ of possibly infinite-dimensional $G^\vee$-graded vector spaces. So, we allow A to be infinite-dimensional, provided it splits into character spaces $A = \oplus_\chi A_\chi$.

We let $D(A)$ denote the derived category of arbitrary dg modules. When A is finite over a central Noetherian subalgebra T of cocycles, one can replace $D(A)$ with $D_{coh}(A)$, the subcategory of dg modules with finitely generated cohomology over T, in all of the results of this section.

Definition 12.1 ([57])**.** A q-regular sequence $x_1, \dots, x_n$ in A is a sequence such that

(a) the x_i are homogenous cocycles, with respect to the character and cohomological gradings, and
(b) for each $1 \leq j \leq n$, x_j is a non-zero divisor in $A/(x_{j+1}, \dots, x_n)$ which is χ_j-central for some character χ_j, in the sense that

$$bx_j = \pm\, q^{(\deg b, \chi_j)} x_j b$$

for all (homogenous) $b \in A/(x_{j+1}, \dots, x_n)$, where $\pm = (-1)^{|x_j||b|}$.

In the case $\mathscr{D} = Vect$, $q = 1$, we refer to such a sequence simply as a regular sequence.

Given such A with fixed q-regular sequence $\{x_1, \dots, x_n\}$ we take $A_j = A/(x_{j+1}, \dots, x_n)$, and have a sequence of projections

$$A_0 \leftarrow A_1 \leftarrow \cdots \leftarrow A_n$$

of dg algebra objects in $\mathscr{D}$.

We consider the categories of dg (bi)modules over the A_j. For any object V in $\mathscr{D}$, and dg A_j-bimodule M, we have the new bimodule $V \otimes M$ with right action induced by that of M and left action given by

$$a \cdot (v \otimes b) = q^{(\deg a, \deg v)} v \otimes ab.$$

Similarly, for any left dg module M we have the new module $V \otimes M$. In this way A_j-dgmod and A_j-dgbimod become left module categories over $\mathscr{D}$. Similarly, $\mathscr{D}$ acts on the derived categories $D(A_j)$.

Lemma 12.2. *For any $r \leq s$, the reduction $A_r \otimes_{A_s} - : A_s\text{-dgmod} \to A_r\text{-dgmod}$ can be given a $\mathscr{D}$-linear structure, so that it is a map of $\mathscr{D}$-module categories, as can be its derived counterpart $A_r \otimes^{\mathrm{L}}_{A_s} - : D(A_s) \to D(A_r)$.*

By a $\mathscr{D}$-linear structure on a functor $F : \mathscr{M} \to \mathscr{N}$ between $\mathscr{D}$-module categories we mean a choice of natural isomorphism of functors $F(V \otimes -) \cong V \otimes F(-)$, for each V in $\mathscr{D}$, which is additionally associative and natural in V. Formally speaking, triangulated categories cannot serve as module categories over $\mathscr{D}$, as they are not abelian. However, by an action of $\mathscr{D}$ on a triangulated category $\mathscr{T}$ here we just mean a choice of associative action map $\otimes : \mathscr{D} \times \mathscr{T} \to \mathscr{T}$ which preserves triangles in $\mathscr{T}$ and sends exact sequences in $\mathscr{D}$ to triangles in $\mathscr{T}$. We would refer to such $\mathscr{T}$ simply as *triangulated* module categories.

Proof of Lemma 12.2. For any V in $\mathscr{D}$ and dg A_s-module M, we have the natural map

$$V \otimes (A_r \otimes M) \to A_r \otimes_{A_s} (V \otimes M), \quad v \otimes (a \otimes m) \mapsto q^{-(\deg a, \deg v)} a \otimes (v \otimes m).$$

Since the quotient $A_s \to A_r$ is a map in $\mathscr{D}$, and thus preserves the character grading, the above natural isomorphism respects the relations of the tensor product over A_s, and hence descends to a well-defined map

$$V \otimes (A_r \otimes_{A_s} M) \xrightarrow{\sim} A_r \otimes_{A_s} (V \otimes M). \tag{29}$$

One similarly constructs the inverse to see that (29) is an isomorphism. This isomorphism is seen to be natural in V and M, and also respects the tensor structure on $\mathscr{D}$, so that we embue the reduction functors $F_{r,s} = A_r \otimes_{A_s} -$ with the structure of a map of $\mathscr{D}$-module categories. One proves the result for the derived categories similarly. □

12.3. **Regular sequences and thick subcategories.** Take A as in Section 12.2, with a fixed q-regular sequence $\{x_1, \dots, x_n\}$. For an object M in A_j-dgmod we let $\langle M \rangle$ denote the thick subcategory in $D(A_j)$ generated by M. We let

$$\mathrm{res}^s_r : D(A_r) \to D(A_s)$$

denote the restriction functor along the projection $A_s \to A_r$, for $r < s$. It is apparent that restriction commutes with the operators $V \otimes -$, for V in $\mathscr{D}$, so that each res^s_r is a map of $\mathscr{D}$-module categories.

Lemma 12.3. *Consider A as in Section 12.2 with fixed q-regular sequence. For arbitrary M in $D(A_r)$, there is a containment*

$$\langle \mathrm{res}^r_{r-1}(A_{r-1} \otimes^{\mathrm{L}}_{A_r} M) \rangle \subset \langle \lambda \otimes M : \lambda \in \mathrm{Irred}(\mathscr{D}) \rangle.$$

of thick subcategories in $D(A_r)$.

Proof. Take $\lambda_r = k1_r$ the irreducible in $\mathscr{D}$ corresponding to the character χ_r, and take m the cohomological degree of x_r. Consider the complex of A_r-bimodules

$$K^r_{r-1} = \left(\Sigma^{m+1}(\lambda_r \otimes A_r) \to A_r, d\right), \quad d(1_r) = x_r.$$

The fact that the differential is A_r-linear on the left follows by the commutativity hypothesis imposed on x_r, by the defintion of a q-regular sequence. Note that K^r_{r-1} is semi-free over A_r on the right, and hence the endomorphism $K^r_{r-1} \otimes_{A_r} -$ on A_r-dgmod preserves quasi-isomorphisms.

Since x_r is a non-zero divisor in A_r, the reduction $A_r \to A_{r-1}$ induces a quasi-isomorphism $K_{r-1} \to A_{r-1}$ of dg A_r-bimodules. It follows that we have a natural isomorphism

$$\mathrm{res}^r_{r-1} \circ (A_{r-1} \otimes^{\mathrm{L}}_{A_r} -) \cong K^r_{r-1} \otimes^{\mathrm{L}}_{A_r} - \cong K^r_{r-1} \otimes_{A_r} -$$

of triangulated endomorphisms of $D(A_r)$. Since

$$K^r_{r-1} \otimes_{A_r} M = \mathrm{cone}(\lambda_r \otimes M \xrightarrow{d} M),$$

we have

$$\mathrm{res}^r_{r-1}(A_{r-1} \otimes^{\mathrm{L}}_{A_r} M) \cong K^r_{r-1} \otimes M \in \langle \lambda \otimes M : \lambda \in \mathrm{Irred}\, \mathscr{D} \rangle.$$

The proposed containment of thick subcategories follows. □

Lemma 12.4. *Consider A as in Section 12.2 with fixed q-regular sequence, and let* res *denote the restriction functor along the projection $A \to A_0$. For arbitrary M in A-dgmod, there is a containment*

$$\langle \mathrm{res}(A_0 \otimes^{\mathrm{L}}_A M) \rangle \subset \langle \lambda \otimes M : \lambda \in \mathrm{Irred}(\mathscr{D}) \rangle.$$

Proof. We claim that there is such a containment

$$\langle \mathrm{res}^s_r(A_r \otimes^{\mathrm{L}}_{A_s} M)\rangle \subset \langle \lambda \otimes M : \lambda \in \mathrm{Irred}(\mathscr{D})\rangle$$

for arbitrary $r < s$ and M over A_s. We proceed by induction on the difference $s-r$. The base case $r = s-1$ is covered by Lemma 12.3. Suppose now that the result holds for $\mathrm{res}^s_r(A_r \otimes^{\mathrm{L}}_{A_s} M)$ and consider the desired containment for $\mathrm{res}^s_{r-1}(A_{r-1} \otimes^{\mathrm{L}}_{A_s} M)$.

Take $N = A_r \otimes^{\mathrm{L}}_{A_s} M$. Then

$$A_{r-1} \otimes^{\mathrm{L}}_{A_s} M \cong A_{r-1} \otimes^{\mathrm{L}}_{A_r} N.$$

and $\mathrm{res}^r_{r-1}(A_{r-1} \otimes^{\mathrm{L}}_{A_s} M) \in \langle \lambda \otimes N : \lambda \in \mathrm{Irred}\, \mathscr{D}\rangle$, by Lemma 12.3. By Lemma 12.2 we have

$$\langle \lambda \otimes N : \lambda \in \mathrm{Irred}\, \mathscr{D}\rangle = \langle A_r \otimes^{\mathrm{L}}_{A_s} (\lambda \otimes M) : \lambda \in \mathrm{Irred}\, \mathscr{D}\rangle$$

so that $\mathrm{res}^r_{r-1}(A_{r-1} \otimes^{\mathrm{L}}_{A_s} M) \in \langle A_r \otimes^{\mathrm{L}}_{A_s} (\lambda \otimes M)\rangle_\lambda$. By restricting further to A_s we find, by our induction hypothesis,

$$\begin{aligned}\mathrm{res}^s_{r-1}(A_{r-1} \otimes^{\mathrm{L}}_{A_s} M) \quad &\in \langle \mathrm{res}^s_r A_r \otimes^{\mathrm{L}}_{A_s} (\lambda \otimes M) : \lambda \in \mathrm{Irred}\, \mathscr{D}\rangle \\ &\subset \langle \mu \otimes \lambda \otimes M : \lambda, \mu \in \mathrm{Irred}\, \mathscr{D}\rangle = \langle \lambda \otimes M : \lambda \in \mathrm{Irred}\, \mathscr{D}\rangle.\end{aligned}$$

So we have the proposed containment. □

12.4. **Regular sequences and deformations.** Consider $Z \to Q \to R$ a deformation sequence. Suppose additionally that all of the given algebras are algebras in $\mathscr{D}$, and that Z is trivial in the sense that Z is supported at the identity in $G^\vee$. Then the Koszul resolution K_Q is a dg algebra in $\mathscr{D}$.

Lemma 12.5. *If $\{x_1, \ldots, x_n\}$ is a q-regular sequence in Q, then $\{x_1, \ldots, x_n\}$ is also a q-regular sequence in the Koszul resolution K_Q of R.*

Proof. Certainly the x_j remain homogenous cocycles in K_Q, and have the same commutativity relations in $K_Q/(x_{j+1}, \ldots, x_n)$, so we need only verify that each x_j is a non-zero divisor in the corresponding quotient. This final property only has to do with the (non-dg) algebra structure of K_Q. If we take $Q_j = Q/(x_{j+1}, \ldots, x_n)$, then as an algebra we have

$$K_Q/(x_{j+1}, \ldots, x_n) \cong Q_j \otimes \wedge^*(m_Z/m_Z^2),$$

and x_j is seen to be a non-zero divisor in this algebra as it is a free module over Q_j. □

13. Restricted enveloping algebras, height 1 doubles, and the quantum Borel

We employ regular sequences, in conjunction with hypersurface support, to obtain the tensor product property for cohomological support in a number of geometrically Chevalley examples (see Section 8.4).

13.1. **A general lemma.** The following general result was essentially argued in the proof of Theorem 11.3, and can be seen as a reduction of the materials of Section 9.1.

Lemma 13.1. *Consider u geometrically Chevalley, with Chevalley integration $\mathsf{U} \to \mathsf{u}$. Fix $\mathscr{D}$ an arbitrary tensor subcategory in $\mathrm{rep}(\mathsf{u})$. For $f \in m_Z$ with non-trivial reduction to m_Z/m_Z^2, let K_f denote the corresponding Koszul resolution of the hypersurface $\mathsf{U}^+/(f)$.*

Suppose that, at arbitrary such $f \in m_Z$, the endomorphism

$$k \otimes^{\mathrm{L}}_{\mathsf{U}^+} - : D(K_f) \to D(K_f)$$

is such that $k \otimes^{\mathrm{L}}_{\mathsf{U}^+} M \in \langle \lambda \otimes M : \lambda \in \mathscr{D} \rangle$, for each M in $D(K_f)$. Then hypersurface support for $D_{fin}(\mathsf{u})$ satisfies the tensor product property

$$\mathrm{supp}^{hyp}_{\mathbb{P}}(V \otimes W) = \mathrm{supp}^{hyp}_{\mathbb{P}}(V) \cap \mathrm{supp}^{hyp}_{\mathbb{P}}(W).$$

The same result holds when u is local and admits a local integration $(\mathsf{U}^+ =)\mathsf{U} \to \mathsf{u}$.

Proof. We claim that the inclusion $k \otimes^{\mathrm{L}}_{\mathsf{U}^+} M \in \langle \lambda \otimes M : \lambda \in \mathscr{D} \rangle$, at arbitrary dg modules M, implies the necessary inclusion $k \in \langle \lambda \otimes M : \lambda \in \mathscr{D} \rangle$ whenever M is non-perfect over the hypersurface $\mathsf{U}^+/(f)$. One then applies Lemma 9.1 to obtain the result.

Via the equivalence $D_{coh}(\mathsf{U}^+/(f)) \to D_{coh}(K_f)$ provided by restriction, it suffices to show that the given inclusion implies $k \in \langle \lambda \otimes M : \lambda \in \mathscr{D} \rangle$ for non-perfect M in $D_{coh}(K_f)$. Take such non-perfect M. As in the proof of Theorem 11.3, one factors the reduction $k \otimes^{\mathrm{L}}_{\mathsf{U}^+} -$ as the composite

$$D_{coh}(K_f) \overset{k \otimes^{\mathrm{L}}_{\mathsf{U}^+} -}{\longrightarrow} D_{fin}(k[\varepsilon]) \overset{\mathrm{res}}{\to} D_{coh}(K_f)$$

and employs the understanding of thick ideals in $D_{fin}(k[\varepsilon])$ from [26, Theorem 4.4] to find that

$$k \in \langle k \otimes^{\mathrm{L}}_{\mathsf{U}^+} M \rangle \ \subset D_{coh}(K_f)$$

at non-perfect M. Our assumption now implies the desired inclusion $k \in \langle \lambda \otimes M : \lambda \in \mathscr{D} \rangle$ at non-perfect M. □

13.2. **Nilpotent restricted Lie algebras over $\overline{\mathbb{F}}_p$.** Consider $\mathfrak{n}$ a nilpotent Lie algebra with a restricted structure, in finite characteristic. Consider any basis $x_1, \dots, x_n$ of $\mathfrak{n}$ which is compatible with the lower central series $\mathfrak{n} \supset \mathfrak{n}^1 \supset \cdots \supset \mathfrak{n}^d = 0$, in the sense that the final m_{d-1} elements form a basis for $\mathfrak{n}^{d-1}$, the following $m_{d-2} + m_{d-1}$ elements form a basis for $\mathfrak{n}^{d-2}$, etc. Then, each element x_r is a central non-zero divisor in $U(\mathfrak{n})_r = U(\mathfrak{n})/(x_{r+1}, \dots, x_n)$, and also in the completion $\widehat{U}(\mathfrak{n})$ along the kernel of the projection $U(\mathfrak{n}) \to u^{\mathrm{res}}(\mathfrak{n})$. Hence the ordered sequence $\{x_1, \dots, x_n\}$ is a regular sequence in $\widehat{U}(\mathfrak{n})$ which generates the kernel of the augmentation $\widehat{U}(\mathfrak{n}) \to k$.

We now recover an essential result of Friedlander and Parshall.

Theorem 13.2 ([41, 77])**.** *For an arbitrary nilpotent restricted Lie algebra $\mathfrak{n}$, the hypersurface support $\mathrm{supp}^{hyp}_{\mathbb{P}}$ on $\mathrm{rep}(u^{\mathrm{res}}(\mathfrak{n}))$ has the tensor product property. Supposing additionally that the reduced spectrum of cohomology for $u^{\mathrm{res}}(\mathfrak{n})$ is an affine space (e.g. supposing the p-th power map on $\mathfrak{n}$ is 0), then the cohomological support for $u^{\mathrm{res}}(\mathfrak{n})$ has the tensor product property*

$$\mathrm{supp}_{\mathsf{Y}}(V \otimes W) = \mathrm{supp}_{\mathsf{Y}}(V) \cap \mathrm{supp}_{\mathsf{Y}}(W).$$

Proof. The fact that the augmentation ideal in $\widehat{U}$ is generated by a regular sequence implies a containment

$$k \otimes^{\mathrm{L}}_{\widehat{U}} M \in \langle M \rangle \ \subset D_{coh}(K_f),$$

at arbitrary dg modules M over the resolution K_f of any hypersurface $\widehat{U}/(f)$, by Lemma 12.4. So Lemma 13.1, considered in the case $\mathscr{D} = Vect$, gives us the tensor product property for hypersurface support. In the case that the spectrum

of cohomology is an affine space, the cohomological support is identified with the hypersurface support by Lemma 10.4. □

13.3. **Solvable height 1 doubles over $\overline{\mathbb{F}}_p$.** Consider U a smooth unipotent group scheme over $k = \overline{\mathbb{F}}_p$. Suppose that U is normal in a larger group B which admits a quasi-logarithm. We recall that a quasi-logarithm is the information of an isomorphism of $\mathfrak{b} = \mathrm{Lie}(B)$-algebras $\widehat{\mathscr{O}}_{B,1} \cong k[\![\mathfrak{b}^*]\!]$, so that we have an isomorphism of algebras (see Example 2.6)

$$\widehat{D}(U_{(1)}) := \widehat{\mathscr{O}}_B \rtimes \widehat{U}(\mathfrak{n}) \cong k[\![\mathfrak{b}^*]\!] \rtimes \widehat{U}(\mathfrak{n}),$$

where $U(\mathfrak{n})$ acts via the coadjoint action of $u^{\mathrm{res}}(\mathfrak{n})$ on the generators $\mathfrak{b}^*$ (see e.g. [55]). We may filter $\mathfrak{b}^*$ by subrepresentations $\mathfrak{b}^* \supset F_{-1}\mathfrak{b}^* \supset \cdots \supset F_{-d}\mathfrak{b}^* = 0$ so that each subquotient is a trivial $\mathfrak{n}$-representation, and consider a corresponding ordered basis $y_1, \ldots, y_n$ for $\mathfrak{b}^*$ with the final m_d elements providing a basis for $F_{-d+1}\mathfrak{b}^*$, the final $m_{d-1}+m_d$ elements forming a basis for $F_{-d+2}\mathfrak{b}^*$, etc. We choose also a basis $x_1, \ldots, x_m$ for $\mathfrak{n}$ as in Section 13.2.

Then the sequence $x_1, \ldots, x_n, y_1, \ldots, y_n$ forms a regular sequence in $\widehat{D}(U_{(1)})$ which generates the kernel of the augmentation $\widehat{D}(U_{(1)}) \to k$. An instance in which the ambient group B admits a quasi-logarithm is the case in which B is a Borel of an almost-simple (smooth) algebraic group $\mathbb{G}$ [37, Corollary 6.4], in very good characteristic for $\mathbb{G}$.

Theorem 13.3. *Let U be a unipotent subgroup in an almost-simple algebraic group $\mathbb{G}$ over $\overline{\mathbb{F}}_p$, which is normalized by a maximal torus. If p is very good for $\mathbb{G}$ then the hypersurface support for the height 1 double $D(U_{(1)})$ satisfies the tensor product property. If $p - 1 > \dim(U)$ then the cohomological support satisfies the tensor product property*

$$\mathrm{supp}_{\mathsf{Y}}(V \otimes W) = \mathrm{supp}_{\mathsf{Y}}(V) \cap \mathrm{supp}_{\mathsf{Y}}(W).$$

Proof. In the case of very good p we have a quasi-logarithm, and hence the augmentation ideal for $\widehat{D}$ is generated by a regular sequence. It follows that for arbitrary M over the Koszul resolution $K_{\widehat{D}}$ we have $k \otimes^{\mathrm{L}}_{\widehat{D}} M \in \langle M \rangle$, by Lemma 12.4. So by Lemma 13.1 the hypersurface support for the double $D(U_{(1)})$ satisfies the tensor product property. When $p-1 > \dim(U)$ the reduced spectrum of cohomology is an affine space [37, Theorem 6.10]. So the hypersurface support and the cohomological support agree, by Lemma 10.4. □

Consider now $B \subset \mathbb{G}$ a Borel subgroup in almost-simple $\mathbb{G}$. We have $D(B_{(1)}) = (\mathscr{O}(B_{(1)}) \rtimes kU_{(1)}) \rtimes k\mathbb{G}^r_{m(1)}$, where U is the unipotent radical in B. We can consider the integration

$$\widehat{D}(B_{(1)}) := \left(\widehat{\mathscr{O}}_B \rtimes \widehat{U}(\mathfrak{n})\right) \rtimes k\mathbb{G}^r_{m(1)}$$

of $D(B_{(1)})$. We note that $\mathrm{rep}(\mathbb{G}^r_{m(1)}) = Vect_{G^\vee}$ as a braided (symmetric) category, for $G^\vee$ a rank r elementary abelian p-group with trivial form. So $D(B_{(1)})$ is geometrically Chevalley and, with $\mathscr{D} = \mathrm{rep}(\mathbb{G}^r_{m(1)})$, fits into the framework of Section 12.2.

Theorem 13.4. *Let B be a Borel subgroup in an almost-simple algebraic group $\mathbb{G}$ over $\overline{\mathbb{F}}_p$. If p is very good for $\mathbb{G}$, then the hypersurface support for the height 1 double $D(B_{(1)})$ satisfies the tensor product property. If $p - 1 > \dim(B)$, then the cohomological support satisfies the tensor product property as well.*

Proof. As remarked above, when p is very good for $\mathbb{G}$, B admits a quasi-logarithm, and when $p-1 > \dim(B)$ the reduced cohomology ring for $D(B_{(1)})$ is a polynomial ring [37, Theorem 6.10]. So, by Lemma 10.4, we need only show that the hypersurface support satisfies the tensor product property. By semisimplicity of $k\mathbb{G}^r_{m(1)}$, a $\widehat{D}$-module is perfect if and only if its restriction to the Hopf subalgebra $\widehat{D}^+ = \widehat{\mathscr{O}} \rtimes \widehat{U}(\mathfrak{n})$ is perfect. One now proceeds as in the proof of Theorem 13.3. □

13.4. **The quantum Borel in small quantum** SL_n**.** Take $q \in \mathbb{C}^\times$ of odd order l. We first consider the simply-connected form of the quantum Borel, then deal with general quantum Borels in type A in Section 13.6.

By the simply-connected form of the quantum Borel $u_q(\mathfrak{b})$ we mean the non-negative subalgebra in the small quantum group $u_q(\mathfrak{sl}_n) = u_q(\mathrm{SL}_n)$. (See [64] and in particular [64, Section 9].) So, $u_q(\mathfrak{b})$ has its usual positive generators E_α, and the grouplikes are given by the character group $G = (P/lQ)^\vee$ of the quotient P/lQ of the weight lattice by the l-th scaling of the root lattice. The standard toral elements K_α are precisely the function $K_\alpha = q^{(\alpha,-)} : P/lQ \to \mathbb{C}^\times$, where $(-,-)$ is the normalized Killing form, $(\alpha,\beta) = d_\alpha\langle\alpha,\beta\rangle$, $d_\alpha = |\alpha|^2/|\text{short root}|^2$. (It happens to be the case that all roots are the same length here, so that all $d_\alpha = 1$.)

Let us consider first the situation in type A_2. For $u_q(\mathfrak{b})$ the quantum Borel in $u_q(\mathfrak{sl}_3)$ we have the standard basis of the positive subalgebras $u_q(\mathfrak{n})$ and $U_q^{DK}(\mathfrak{n})$ provided by ordered monomials in the root vectors

$$E_\alpha,\ E_\beta,\ E_{\alpha+\beta} := E_\alpha E_\beta - q^{-1}E_\beta E_\alpha$$

[59, 29]. One can check by hand, via the Serre relations, that $E_{\alpha+\beta}$ is χ-central in $U_q^{DK}(\mathfrak{n})$, for the character χ of the grouplikes given by $\chi(K_\alpha) = q$, $\chi(K_\beta) = q^{-1}$. In terms of classes $\{\bar{\omega}_\alpha\}_\alpha$ of the fundamental weights $\{\omega_\alpha\}_\alpha$ in P/lQ, $\chi = \bar{\omega}_\alpha - \bar{\omega}_\beta$. So, by considering the monomial basis of $U_q^{DK}(\mathfrak{n})$ in terms of the E_γ, we see that the sequence $\{E_\alpha, E_\beta, E_{\alpha+\beta}\}$ provides a q-regular sequence in $U_q^{DK}(\mathfrak{n})$ which generates the kernel of the augmentation ideal. The root vectors also provide a q-regular sequence for the completion $\widehat{U}_q^{DK}(\mathfrak{n})$ with respect to degree.

The above argument generalizes in type A to provide the following.

Lemma 13.5 ([2])**.** *The augmentation ideal in the completed De Concini-Kac algebra $\widehat{U}_q^{DK}(\mathfrak{n}) \subset \widehat{U}_q^{DK}(\mathfrak{b})$ in type A, at arbitrary odd order q, is generated by a q-regular sequence.*

Proof. We have the standard generators $E_{i,i+1} := E_{\alpha_i}$ of $U_q^{DK}(\mathfrak{n})$ and explicitly define the root vectors $E_{i,j}$, $1 \le i < j \le n$, by

$$E_{i,j} = E_{i,j-1}E_{j-1,j} - q^{(\gamma,\nu)}E_{j-1,j}E_{i,j-1},$$

where $E_{i,j}$ and $E_{i,j-1}$ are of respective degrees γ and ν under the root lattice grading. We order the set of roots $\{E_{i,j} : 1 \le i < j \le n\}$ with respect to the lex(icographic) order, so that $E_{i,j} < E_{i',j'}$ whenever $i < i'$ or $i = i'$ and $j < j'$.

Having established the above explicit expressions, and ordering, we let E_γ denote the unique such vector $E_{i,j}$ of Q-degree γ, for γ a positive root, and adopt the lex ordering on the set $\{E_\gamma : \gamma \in \Phi^+\}$ established above. In [2, Lemmas 6.4, 6.7, 6.8] Andruskiewitsch and Schneider calculate explicitly the q-commutators of the root vectors to find

$$E_\gamma E_\nu - q^{(\gamma,\nu)}E_\gamma E_\nu \in \text{the ideal in } U_q^{DK}(\mathfrak{n}) \text{ generated by } E_\xi \text{ with } ht(\xi) > ht(\gamma),\ ht(\nu),$$

whenever $E_\gamma < E_\nu$ in the lex ordering. (See in particular [2, Equations (6-19) and (6-25)].) By considering the basis of $U_q^{DK}(\mathfrak{n})$ in terms of monomials in the root vectors, it follows that any ordering $\{E_{\gamma_1}, \dots, E_{\gamma_m}\}$ of the root vectors which is compatible with the height, in the sense that $ht(\gamma_i) \leq ht(\gamma_{i+1})$, provides a q-regular sequence in $U_q^{DK}(\mathfrak{n})$ which generates the kernel of the augmentation $U_q^{DK}(\mathfrak{n}) \to \mathbb{C}$. The specific character χ_γ associated to E_γ is the explicit sum of fundamental weights

$$\chi_\gamma = \sum_{\alpha \text{ with } E_\alpha < E_\gamma} (\alpha, \gamma)\bar{\omega}_\alpha - \sum_{\beta \text{ with } E_\gamma < E_\beta} (\beta, \gamma)\bar{\omega}_\beta \ \in G^\vee.$$

By exactness of completion $\mathbb{C}[\![E_\gamma^l : \gamma \in \Phi^+]\!] \otimes_{\mathbb{C}[E_\gamma^l : \gamma \in \Phi^+]} -$, it follows that the ordered set $\{E_{\gamma_1}, \dots, E_{\gamma_n}\}$ also provides a q-regular sequence in the completed algebra $\widehat{U}_q^{DK}(\mathfrak{n})$ which generates the kernel of the augmentation. □

One now applies Lemma 12.4 and Lemma 13.1 to find the following.

Theorem 13.6. *For $u_q(\mathfrak{b})$ the (simply-connected) quantum Borel in type A_n, at arbitrary odd order q, the hypersurface support $\mathrm{supp}_{\mathbb{P}}^{hyp}$ for $\mathrm{rep}(u_q(\mathfrak{b}))$ satisfies the tensor product property*

$$\mathrm{supp}_{\mathbb{P}}^{hyp}(V \otimes W) = \mathrm{supp}_{\mathbb{P}}^{hyp}(V) \cap \mathrm{supp}_{\mathbb{P}}^{hyp}(W).$$

We recall that at q of order greater than the Coxeter number h for $\mathfrak{sl}_{n+1}$, the cohomology ring for $u_q(\mathfrak{b})$ is isomorphic to functions on the nilpotent radical $\mathfrak{n} \cong \mathbb{A}_{\mathbb{C}}^{|\Phi^+|}$ [47]. Indeed, in this case κ is actually an isomorphism from Y onto $\mathbb{P}$. So we obtain the tensor product for cohomological support at such parameters q, by applying Lemma 7.5 and Corollary 7.2.

Corollary 13.7. *Consider $u_q(\mathfrak{b})$ the (simply-connected) quantum Borel in type A_n, at q of odd order $> h$. For arbitrary V and W in $\mathrm{rep}(u_q(\mathfrak{b}))$ we have*

$$\mathrm{supp}_{\mathsf{Y}}(V \otimes W) = \mathrm{supp}_{\mathsf{Y}}(V) \cap \mathrm{supp}_{\mathsf{Y}}(W). \tag{30}$$

We note that the arguments employed in the proof of Theorem 13.6 are specific to type A. In particular, direct calculation indicates that none of the root vectors $\{E_\gamma : \gamma \in \Phi^+\} \subset U_q^{DK}(\mathfrak{b})$, in types D_4 and B_2, are skew central, and so no ordering of the positive roots produces a q-regular sequence. One therefore needs a more robust approach to the quantum Borel in other Dynkin type.

13.5. **Arbitrary Borels in type A.** Consider X any intermediate lattice $Q \subset X \subset P$ between the root and weight lattice in type A, and take X^M the radical of the q-exponentiated Killing form $q^{(-,-)}$ on X. Let $\mathbb{G}$ be the corresponding algebraic group for X, and B be the positive Borel in $\mathbb{G}$.

We have the corresponding small quantum group $u_q(\mathbb{G})$ [64, Section 9] and the quantum Borel $u_q(B)$ in $u_q(\mathbb{G})$, which is explicitly the smash product of the nilpotent subalgebra $u_q(\mathfrak{n})$ with the character group $(X/X^M)^\vee$ of the quotient X/X^M. As stated above, we have taken $u_q(\mathfrak{b})(= u_q(B_{sc}))$ to be the quantum Borel in the simply-connected form $u_q(\mathrm{SL}_n)$.

Theorem 13.8. *Consider $\mathbb{G}$ an arbitrary almost-simple algebraic group in type A, q an odd order root of unity, and $u_q(B)$ the quantum Borel in $u_q(\mathbb{G})$. Then hypersurface support for $u_q(B)$ satisfies the tensor product property. Furthermore,*

when $\mathrm{ord}(q) > h$ *cohomological support for* $u_q(B)$ *also satisfies the tensor product property*

$$\mathrm{supp}_{\mathsf{Y}}(V \otimes W) = \mathrm{supp}_{\mathsf{Y}}(V) \cap \mathrm{supp}_{\mathsf{Y}}(W).$$

Before giving the proof of Theorem 13.8, we provide the following lemma, which reframes our analysis of support for the simply-connected form $u_q(\mathfrak{b})$.

Lemma 13.9. *Consider* $\mathsf{U}_{sc} = \widehat{U}_q^{DK}(\mathfrak{b})$ *the (completed) De Concini-Kac integration of the simply-connected form* $u_q(\mathfrak{b})$ *in type A, and* $f \in m_Z$ *with non-trivial reduction to* m_Z/m_Z^2*. Then for* M *finitely generated and non-perfect over* $\mathsf{U}_{sc}/(f)$ *we have*

$$k \in \langle \lambda \otimes M : \lambda \in \mathrm{Irrep}(G_{sc}) \rangle \ \subset D_{coh}(\mathsf{U}_{sc}/(f)). \tag{31}$$

The point here is that we have replaced the positive subalgebra U^+ with the Hopf algebra U_{sc} in our analysis of support for hypersurfaces (cf. Lemma 9.1).

Proof. Fix f as in the statement of the lemma, and let G_{sc} denote the grouplikes in $u_q(\mathfrak{b})$. Let M' be a finite and non-perfect module over $\mathsf{U}^+/(f)$, which we view also as a dg module over the corresponding Koszul resolution K_f. Consider the reduction $K_f \to k \otimes_{\mathsf{U}^+} K_f = k[\varepsilon]$. The dg version of Hopkins' lemma employed in the proof of Theorem 11.3, [26, Theorem 4.4], therefore implies that

$$k \in \langle k \otimes_{\mathsf{U}^+} M' \rangle \ \subset D_{fin}(k[\varepsilon]).$$

By Lemmas 13.5 and 12.5, the kernel of the reduction $K_f \to k[\varepsilon]$ is generated by a q-regular sequence. So by Lemma 12.4 we have that

$$k \in \langle \lambda \otimes M' : \lambda \in \mathrm{Irrep}(G_{sc}) \rangle \subset D_{coh}(\mathsf{U}^+/(f)).$$

We consider the right adjoint

$$R : \mathsf{U}^+/(f)\text{-mod}_{fg} \to \mathsf{U}_{sc}/(f)\text{-mod}_{fg}$$

to the restriction functor $T : \mathsf{U}_{sc}/(f)\text{-mod}_{fg} \to \mathsf{U}^+/(f)\text{-mod}_{fg}$. Note that both T and R are exact maps of left $\mathrm{rep}(G_{sc})$-module categories [35, §3.3]. When M' is the restriction $M' = TM$ of a $\mathsf{U}_{sc}/(f)$-module M we have

$$R(TM) = \oplus_{\lambda' \in \mathrm{Irrep}(G_{sc})} M \otimes \lambda' \cong \oplus_{\lambda' \in \mathrm{Irrep}(G_{sc})} \lambda' \otimes M,$$

where for the final identity we use the centralizing property of U_{sc}-modules against the irreducibles (Lemma 8.2). So by considering the derived functor $R : D_{coh}(\mathsf{U}^+/(f)) \to D_{coh}(\mathsf{U}_{sc}/(f))$ we find for non-perfect M over $\mathsf{U}_{sc}/(f)$

$$\begin{aligned} \oplus_{\lambda \in \mathrm{Irrep}(G_{sc})} \lambda = R(k) \quad &\in R(\langle \lambda \otimes TM : \lambda \in \mathrm{Irrep}(G_{sc}) \rangle) \\ &\subset \langle R(\lambda \otimes TM) : \lambda \in \mathrm{Irrep}(G_{sc}) \rangle \\ &= \langle \lambda \otimes \lambda' \otimes M : \lambda, \lambda' \in \mathrm{Irrep}(G_{sc}) \rangle \\ &= \langle \lambda \otimes M : \lambda \in \mathrm{Irrep}(G_{sc}) \rangle \ \subset \ D_{coh}(\mathsf{U}_{sc}/(f)). \end{aligned}$$

Since thick subcategories are closed under taking summands we have finally

$$k \in \langle \lambda \otimes M : \lambda \in \mathrm{Irrep}(G_{sc}) \rangle \subset D_{coh}(\mathsf{U}_{sc}/(f))$$

whenever M is non-perfect over $\mathsf{U}_{sc}/(f)$. □

We now prove our theorem.

Proof of Theorem 13.8. Write G_{sc} for the grouplikes in $u_q(\mathfrak{b})$ and G for the grouplikes in $u_q(B)$. Since the q-exponentiated Killing form on $P \times lQ$ is identically 1, we have an inclusion $lQ \subset X^M$. So we observe group maps

$$\begin{array}{ccccccc} & & X/lQ & \xrightarrow{\text{incl}} & P/lQ & \overset{\vee}{\rightsquigarrow} & G_{sc} \xrightarrow{\text{surj}} G' \\ & \overset{\text{surj}}{\swarrow} & & & & & \overset{\text{incl}}{\nwarrow} \\ X/X^M & & & & & & G, \end{array}$$

where G' is the character group of X/lQ. For u' the Hopf algebra $u_q(\mathfrak{n}) \rtimes G'$, we then have a corresponding surjection $u_q(\mathfrak{b}) \to \mathsf{u}'$ and inclusion $u_q(B) \to \mathsf{u}'$.

Note that the Killing form on P/lQ restricts to a form on the intermediate quotient X/lQ, and that the kernel of the quotient $X/lQ \to X/X^M$ is the radical of this form on X/lQ. So each of the Hopf algebras $u_q(\mathfrak{b})$, u', and $u_q(B)$ are constructed via a uniform bosonization process, using the q-exponentiated Killing form. These Hopf algebras also have consistent integrations provided by taking the appropriate smash product with $\mathsf{U}^+ = \widehat{U}_q^{DK}(\mathfrak{n})$. Finally, when $\mathrm{ord}(q) > h$ the spectrum of cohomology for each of these Hopf algebras is an affine space [47], so that the hypersurface and cohomological supports agree in this case. Therefore to prove our result it suffices to show that hypersurface support for $u_q(B)$ splits over tensor products.

Write U_{sc}, U', and U for the integrations of $u_q(\mathfrak{b})$, u', and $u_q(B)$ respectively. Consider $f \in m_Z$ any function with nonzero reduction to m_Z/m_Z^2.

Consider M non-perfect over the algebra $\mathsf{U}'/(f)$, and restrict along the projection $\mathsf{U}_{sc}/(f) \to \mathsf{U}'/(f)$ to consider M as a non-perfect $\mathsf{U}_{sc}/(f)$-module. We have the map

$$L = \mathbb{C}G \otimes_{\mathbb{C}G_{sc}} - : \mathsf{U}_{sc}/(f)\text{-mod}_{fg} \to \mathsf{U}'/(f)\text{-mod}_{fg}$$

which is an exact adjoint to restriction. This functor simply sends a $\mathsf{U}_{sc}/(f)$-module, which is graded by P/lQ, to its $\mathsf{U}'/(f)$-summand which is supported on the subgroup X/lQ. So, in particular, for M restricted from a $\mathsf{U}'/(f)$-module we have

$$L(\lambda \otimes M) = \begin{cases} \lambda \otimes M & \text{if } \lambda \in X/lQ \\ 0 & \text{otherwise.} \end{cases}$$

Applying L to the formula (31) of Lemma 13.9 then gives, for such non-perfect M,

$$k \in \langle \mu \otimes M : \mu \in \mathrm{Irrep}(G') \rangle \subset D_{coh}(\mathsf{U}'/(f)). \tag{32}$$

One therefore argues as in the proof of Lemma 9.1 to find that hypersurface support for the intermediate algebra u' satisfies the tensor product property.

We consider now the inclusion $u_q(B) \to \mathsf{u}'$. One can argue exactly as in the proof of Theorem 11.6 to see that hypersurface support for $u_q(B)$ satisfies the tensor product property. In particular, we observe that the formula (32) implies the analogous formula with U' replaced by the positive subalgebra U^+, and follows precisely the arguments of Theorem 11.6. □

Remark 13.10. Supposing that hypersurface support for the simply-connected form $u_q(\mathfrak{b})$, in arbitrary Dynkin type, satisfies the tensor product property, the arguments of Theorem 13.8 show that supports for all Borels $u_q(B)$ away from the simply-connected form also satisfy the tensor product property.

13.6. **A conjecture in general type.** We conclude with the obvious conjecture.

Conjecture 13.11. *Let $\mathbb{G}$ be an arbitrary almost-simple algebraic group, and $u_q(B)$ be the quantum Borel in $u_q(\mathbb{G})$, at q of arbitrary odd order. Then cohomological support $\mathrm{supp}_{\mathsf{Y}}$ for $\mathrm{rep}(u_q(B))$ satisfies the tensor product property (30).*

In the sequel [65] we expect to show that the conclusions of Theorem 13.8, in type A, hold at q of arbitrary odd order. This informs our omission of a bound on the order of q in the statement of Conjecture 13.11.

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