

STRATIFICATION AND DUALITY FOR UNIPOTENT FINITE SUPERGROUP SCHEMES

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To John Greenlees on his 60th birthday.

ABSTRACT. We survey some methods developed in a series of papers, for classifying localising subcategories of tensor triangulated categories. We illustrate these methods by proving a new theorem, providing such a classification in the case of the stable module category of a unipotent finite supergroup scheme.

1. INTRODUCTION

John Greenlees' influence on mathematics is reflected throughout this volume, and our own work has benefitted enormously from his insights. We should like to mention in particular his collaboration with May [49] and Dwyer [40] on derived completions and local cohomology; on the local cohomology spectral sequence in the context of group cohomology [47] and his subsequent work with Lyubeznik [48]; and on duality in algebra and topology; in particular, his work with Benson [12, 13, 14], with Dwyer and Iyengar [41].

A broad framework for understanding localisation and duality was developed in a series of papers [16]–[27]. Originally geared towards applications to representation theory of finite groups and finite group schemes, the framework has been applied in a number of other areas, such as commutative algebra [34], ring spectra [6], modules over algebras of cochains [15, 17], and equivariant KK-theory [33]. This theory is closely related to that of Balmer [3, 4, 5], but the point of view is different.

The purpose of this paper is to give an outline of the theory, by way of explaining how it applies in the case of the stable module category of a unipotent finite supergroup scheme. In particular, we shall prove the following theorem.

1.1. Theorem. *Let G be a unipotent finite supergroup scheme over a field k of positive characteristic $p \geq 3$. The theory of support gives a one to one correspondence between the localising subcategories of $\mathrm{StMod}(kG)$ and subsets of $\mathrm{Proj} H^{*,*}(G, k)$.*

Here $\mathrm{StMod}(kG)$ denotes the stable module category of kG -modules, $H^{*,*}(G, k)$ the cohomology ring of G , and $\mathrm{Proj} H^{*,*}(G, k)$ its projective spectrum. Supergroup schemes are introduced Section 6 and the result above is proved in Section 10. The statement above speaks of “the” theory of support, but in fact there are at least two, quite distinct, notions of support in this context. One is obtained from specialising the support theory introduced in [16], which is based on cohomology and applies to any compactly generated triangulated category with a ring action; in this case $\mathrm{StMod}(kG)$ with the canonical action of $H^{*,*}(G, k)$. The other notion of support is inspired by the theory of π -points introduced by Friedlander and Pevtsova [45] in the context of finite group schemes. That these two notions of support coincide

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for modules over finite supergroup schemes is one of the crucial steps in proving the theorem above.

Theorem 1.1 was first proved, with no restrictions on the prime p , for finite groups [18], using ideas from commutative algebra, mainly the Bernstein-Gelfand-Gelfand correspondence. A new proof, from a more representation theoretic perspective, was given in [23], exploiting ideas from [45] and a concept of cosupport for representations, which is related to derived completions introduced by Greenlees and May [49]. These ideas also played a pivotal role in our proof of Theorem 1.1 for the case of finite group schemes [24].

Theorem 1.1 is a culmination of our work reported in [26, 27, 28]. We sketch the details of how to put these pieces together in Section 9. This work takes as a starting point the applications of the theory of super polynomial functors to representations and cohomology of finite supergroup schemes, developed in a series of papers by Drupieski and Kujaawa [35, 37, 39]. Their work [36] represents a development parallel to our own. The main difference is that we treat both connected and non-connected cases of unipotent supergroup schemes and are interested in infinite dimensional kG -modules. In [36] one can find many intricate results, examples and calculations for finite dimensional modules over connected group schemes.

Finally, in Section 11 we give a brief overview of the local duality for $\text{StMod}(kG)$, summarised in the following result.

1.2. Theorem. *Let G be a finite supergroup scheme over a field k of positive characteristic $p \geq 3$. For each $\mathfrak{p} \in \text{Proj } H^{*,*}(G, k)$, the corresponding localising subcategory $\Gamma_{\mathfrak{p}} \text{StMod}(kG)$ satisfies local duality.*

Observe that, in contrast with Theorem 1.1, here G is not restricted to be unipotent. For finite groups, such a statement was conjectured by Benson [9] and proved by Benson and Greenlees [14]; see also [10]. In [25] we gave a new proof of their result covering also the case of finite group schemes. It turns out that local duality is a general feature of Gorenstein algebras, and can be viewed as an extension of a duality theorem due to Auslander [1] and Buchweitz [29] generalising Tate duality for finite groups. All this is explained in [22]. One can view the result above as a variation, dealing with graded Gorenstein algebras.

In what follows, we have tried to present the development of the theory from elementary abelian p -groups to finite groups to finite group schemes to finite supergroup schemes. We focus mostly on stratification, and even then cannot hope to present a complete account of the theory in a short survey. Thus the aim is to give just a flavour of numerous techniques and developments that happened in the theory of support varieties in (relatively) recent years many of which were influenced by John Greenlees' work.

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2. LOCAL COHOMOLOGY AND SUPPORT

Let $\mathcal{T}$ be a triangulated category, with shift Σ , admitting all small coproducts, and compactly generated. The examples we have in mind come from algebra, topology and geometry. The *centre* $Z^*(\mathcal{T})$ of $\mathcal{T}$ is the graded ring whose degree n part consists of the natural transformations η from the identity functor to Σ^n satisfying $\eta\Sigma = (-1)^n\Sigma\eta$. These form a graded commutative ring which is usually not Noetherian. We assume that we are given a graded commutative Noetherian ring R together with a homomorphism of rings $R \rightarrow Z^*(\mathcal{T})$. This is called an *action* of R on $\mathcal{T}$. For X, Y in $\mathcal{T}$ set

$$\mathrm{Hom}_{\mathcal{T}}^*(X, Y) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X, \Sigma^n Y);$$

this has a structure of an R -module, compatible with morphisms in $\mathcal{T}$.

A subset V of a topological space X is *specialization closed* if V contains the closure of its points. We write $\mathrm{Spec} R$ for the homogeneous primes ideals of R , with the Zariski topology. For each specialisation closed subset V of $\mathrm{Spec} R$, the subcategory $\mathcal{T}_V$ of V -*torsion* objects in $\mathcal{T}$ is the full subcategory consisting of the objects X such that $\mathrm{Hom}_{\mathcal{T}}^*(C, X)_{\mathfrak{p}} = 0$ for each compact object C in $\mathcal{T}$ and each $\mathfrak{p} \notin V$. For X in $\mathcal{T}$ there is a functorial triangle

$$\Gamma_V X \rightarrow X \rightarrow L_V X$$

where $\Gamma_V X$ is in $\mathcal{T}_V$, and $L_V X$ admits no non-zero maps from any object in $\mathcal{T}_V$.

2.1. Definition. Given a $\mathfrak{p}$ in $\mathrm{Spec} R$ choose specialisation closed subsets V and W with $V \not\subseteq W$ and $V \subseteq W \cup \{\mathfrak{p}\}$, and define

$$\Gamma_{\mathfrak{p}} X := \Gamma_V L_W X = L_W \Gamma_V X.$$

This turns out to be independent of choice of V and W satisfying these conditions, and defines an idempotent functor $\Gamma_{\mathfrak{p}}: \mathcal{T} \rightarrow \mathcal{T}$ which may be thought of as isolating the layer of $\mathcal{T}$ corresponding to the prime $\mathfrak{p}$, and consisting of the objects “supported at $\mathfrak{p}$.” The functor $\Gamma_{\mathfrak{p}}$ has a right adjoint, which we denote $\Lambda^{\mathfrak{p}}$; it has to do with completions along $\mathfrak{p}$, and plays an equally important role in the theory; for details, see [16, 21].

2.2. Definition. The *support* and *cosupport* of an object X in $\mathcal{T}$ are the subsets

$$\begin{aligned} \mathrm{supp}_R(X) &:= \{\mathfrak{p} \in \mathrm{Spec} R \mid \Gamma_{\mathfrak{p}} X \neq 0\} \\ \mathrm{cosupp}_R(X) &:= \{\mathfrak{p} \in \mathrm{Spec} R \mid \Lambda^{\mathfrak{p}} X \neq 0\} \end{aligned}$$

It is not hard to prove that both support and cosupport detect zero objects: an object X in $\mathcal{T}$ is 0 if and only if $\mathrm{supp}_R(X) = \emptyset$, if and only if $\mathrm{cosupp}_R(X) = \emptyset$.

We focus mostly on support, but cosupport resurfaces in Section 10. We write $\mathrm{supp}_R(\mathcal{T})$ for $\bigcup \mathrm{supp}_R(X)$, where X ranges over the objects in $\mathcal{T}$.

Under mild conditions, such as R having finite Krull dimension, the localising subcategory $\mathrm{Loc}_{\mathcal{T}}(X)$ of $\mathcal{T}$ generated by an object X is equal to the localising subcategory $\mathrm{Loc}_{\mathcal{T}}(\{\Gamma_{\mathfrak{p}} X \mid \mathfrak{p} \in \mathrm{supp}_R(\mathcal{T})\})$ generated by the objects $\Gamma_{\mathfrak{p}} X$. If this is the case, we say that the *local-global principle* holds. Then there is a one to one correspondence between localising subcategories of $\mathcal{T}$ and functions assigning to each $\mathfrak{p}$ in $\mathrm{supp}_R(\mathcal{T})$ a localising subcategory of $\Gamma_{\mathfrak{p}} \mathcal{T}$. The function corresponding to a localising subcategory $\mathcal{S}$ sends $\mathfrak{p}$ to $\mathcal{S} \cap \Gamma_{\mathfrak{p}} \mathcal{T}$. This is described in detail in [18].

We say that an action of a graded commutative ring R on $\mathcal{T}$ *stratifies* $\mathcal{T}$ if the local-global principle holds and for each $\mathfrak{p}$ in $\mathrm{supp}_R(\mathcal{T})$, the subcategory $\Gamma_{\mathfrak{p}} \mathcal{T}$ is a minimal with respect to inclusion, among localising subcategories of $\mathcal{T}$. Under these circumstances, there is one to one correspondence between localising subcategories of $\mathcal{T}$ and the subsets of $\mathrm{supp}_R(\mathcal{T})$. The subset corresponding to a localising

subcategory S is its support, $\text{supp}_R S$, by which we mean the set of primes $\mathfrak{p}$ for which $\Gamma_{\mathfrak{p}}$ is not the zero functor on S .

2.3. Example. Let G be a finite group and k a field of characteristic p dividing the order of G . The stable module category $\text{StMod}(kG)$ is a compactly generated triangulated category, with compact objects the finite dimensional kG -modules, and there is an action of the cohomology ring $H^*(G, k)$. In this case, we write $\text{supp}_G(M)$ for $\text{supp}_{H^*(G, k)}(M)$.

The support of $\text{StMod}(kG)$ is $\text{Proj } H^*(G, k)$, which is the set $\text{Spec } H^*(G, k)$ excluding the maximal ideal consisting of all elements of positive degree. When M is finite dimensional, $\text{supp}_G(M)$ is the closed subset of $\text{Proj } H^*(G, k)$ defined by $\text{Ann}_{H^*(G, k)} \text{Ext}^{*,*}(M, M)$.

It is proved in [19] that when G is a p -group this action stratifies $\text{StMod}(kG)$.

Such a stratification does not hold for general finite groups, because the localising subcategories $\Gamma_{\mathfrak{p}} \text{StMod}(kG)$ are *tensor ideal*, namely they are closed under tensor products with all modules, whereas there may be localising subcategories that are not. For a finite p -group, the only simple module is the trivial module k , so the localising subcategory it generates is the whole of $\text{StMod}(kG)$, and hence all localising subcategories are tensor ideal. To address arbitrary finite groups, we must take the tensor structure into account.

Tensor triangulated categories. Suppose now that $\mathcal{T}$ is a compactly generated triangulated category that comes with a symmetric monoidal tensor product $\otimes: \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$, is exact in each variable, preserving small coproducts, and with unit $\mathbb{1}$. In this situation, we say that $\mathcal{T}$ is a *tensor triangulated category*. There is a particularly good kind of action of a graded commutative ring R on such a $\mathcal{T}$; namely, one that factor as $R \rightarrow \text{End}_{\mathcal{T}}^*(\mathbb{1})$ followed by the natural map $\text{End}_{\mathcal{T}}^*(\mathbb{1}) \rightarrow Z^*(\mathcal{T})$ induced by the tensor product. We say that such an action is *canonical*.

Given a canonical action of R on $\mathcal{T}$, the functor $\Gamma_{\mathfrak{p}}$ is naturally isomorphic to $\Gamma_{\mathfrak{p}} \mathbb{1} \otimes -$. The subcategory $\Gamma_{\mathfrak{p}} \mathcal{T}$ is *tensor ideal*, meaning that it is closed under tensoring with arbitrary objects in $\mathcal{T}$. Denote by $\text{Loc}_{\mathcal{T}}^{\otimes}(X)$ the tensor ideal localising subcategory of $\mathcal{T}$ generated by an object X . The tensor version of the local-global principle holds in this situation, and says that $\text{Loc}_{\mathcal{T}}^{\otimes}(X)$ is equal to the tensor ideal localising subcategory $\text{Loc}_{\mathcal{T}}^{\otimes}(\{\Gamma_{\mathfrak{p}} X \mid \mathfrak{p} \in \text{supp}_R(\mathcal{T})\})$. So we get a one to one correspondence between tensor ideal localising subcategories of $\mathcal{T}$ and functions assigning to each $\mathfrak{p}$ in $\text{supp}_R(\mathcal{T})$ a tensor ideal localising subcategory of $\Gamma_{\mathfrak{p}} \mathcal{T}$.

We say that a canonical action of a graded commutative ring R *stratifies* a tensor triangulated category $\mathcal{T}$ if for each $\mathfrak{p} \in \text{supp}_R(\mathcal{T})$, the subcategory $\Gamma_{\mathfrak{p}} \mathcal{T}$ is minimal as a tensor ideal localising subcategory. See also the work of Hovey, Palmieri, and Strickland [50] where such a minimality plays a prominent role.

2.4. Example. Let G be a finite group. The stable module category of kG -modules is a tensor triangulated category, with tensor unit the trivial module k . The Tate cohomology ring $\widehat{H}^*(G, k)$ is none other than the graded ring $\text{End}_{\text{StMod}(kG)}^*(k)$, and so has a canonical action on $\text{StMod}(kG)$. While $\widehat{H}^*(G, k)$ is usually not Noetherian, its subring $H^*(G, k)$ is, which then inherits the canonical action on $\text{StMod}(kG)$.

The following theorem is proved in [19].

2.5. Theorem. *Let G be a finite group and k a field of characteristic p . As a tensor triangulated category, the action of $H^*(G, k)$ stratifies $\text{StMod}(kG)$. In particular, there is a bijection, defined by $\text{supp}_G(-)$, between tensor ideal localising subcategories of $\text{StMod}(kG)$ and subsets of $\text{Proj } H^*(G, k)$. $\square$*

A key step in the proof is a reduction to elementary abelian groups, which depends on the Quillen stratification theorem [53, 54], and work of Chouinard, recalled below.

3. THE RANK VARIETY

For a finite group G , there is another description of support that is much better suited to computation. Chouinard [31] proved that over an arbitrary commutative ring of coefficients k , a kG -module is projective if and only if its restriction to every elementary abelian subgroup E of G is a projective kE -module. If k is a field of characteristic p , we only need to consider elementary abelian p -subgroups. In this case, when k is algebraically closed, Dade [32] proved that a finite dimensional kE -module is projective if and only if its restriction to each cyclic shifted subgroup is projective. The precise statement is as follows. If $E := \langle g_1, \dots, g_r \rangle \cong (\mathbb{Z}/p)^r$, is an elementary abelian p -group, set $X_i = g_i - 1$ in kE for $1 \leq i \leq r$. Then kE is a truncated polynomial ring:

$$kE \cong \frac{k[X_1, \dots, X_r]}{(X_1^p, \dots, X_r^p)}.$$

Let $J(kE)$ be the radical, $(X_1, \dots, X_r)$, of kE . For $\lambda = (\lambda_1, \dots, \lambda_r)$ in $\mathbb{A}^r(k)$ set

$$X_\lambda := \lambda_1 X_1 + \dots + \lambda_r X_r \quad \text{in } kE$$

and $\alpha_\lambda: k[t]/(t^p) \rightarrow kE$ for the homomorphism of k -algebras sending t to X_λ . Given a kE -module M , we write $\alpha_\lambda^*(M)$ for the $k[t]/(t^p)$ -module obtained by restriction along α_λ .

3.1. Theorem (Dade [32]). *Let k be an algebraically closed field of characteristic p , and E an elementary abelian p -group. A finite dimensional kE -module M is projective if and only if $\alpha_\lambda^*(M)$ is projective for all $0 \neq \lambda \in \mathbb{A}^r(k)$. $\square$*

Based on this, Carlson [30] introduced the notion of rank variety.

3.2. Definition. Let k be an algebraically closed field of characteristic p , and E an elementary abelian p -group. The *rank variety* of a finitely generated kE -module M is the subset

$$V_E^r(M) := \{\lambda \in \mathbb{A}^r(k) \mid \alpha_\lambda^*(M) \text{ is not projective}\}$$

of $\mathbb{A}^r(k)$. We write $V_{kE}^r(M)$ if the field of coefficients needs to be specified. Observe that $V_E^r(M)$ contains 0 and is homogenous. It is also closed, for projectivity over $k[t]/(t^p)$ is detected by a rank condition on the operator representing the action of t . This also gives a method for calculating equations defining $V_E^r(M)$. Ostensibly, this depends on the choice of generators for E as an elementary abelian group. However, if $\alpha, \beta: k[t]/(t^p) \rightarrow kE$ are maps such that $\alpha(t)$ and $\beta(t)$ have the same image in $J(kE)/J^2(kE)$ then $\alpha^*(M)$ is projective if and only if $\beta^*(M)$ is projective. So it makes sense to think of the ambient affine space $\mathbb{A}^r(k)$ as identified with $J(kE)/J^2(kE)$. We return to this ambiguity in Section 6 when we discuss π -points.

The cohomology ring of E is well-known and easy to compute:

$$H^*(E, k) = \begin{cases} k[y_1, \dots, y_r] & \text{if } p = 2 \\ k[x_1, \dots, x_r] \otimes \Lambda(y_1, \dots, y_r) & \text{for } p \text{ odd} \end{cases}$$

where y_i is in degree one the x_i is in degree two, and is the Bockstein $\beta(y_i)$ of y_i .

Let $k[Y_1, \dots, Y_r]$ be the coordinate ring of $\mathbb{A}^r(k)$. If $p = 2$, this can be identified with $H^*(E, k)$ in such a way that the Y_i correspond to the y_i . If p is odd, there is a twist: we have to identify x_i with Y_i^p , so that there is a Frobenius twist involved in the identification of $\text{Spec } H^*(E, k)$ with $\text{Spec } k[Y_1, \dots, Y_r]$. Carlson conjectured that this identifies support and rank variety for a finite dimensional kE -module.

3.3. Theorem (Avrunin, Scott [2]). *Let k be an algebraically closed field. Under the usual identification of radical homogeneous ideals in $\text{Spec } H^*(E, k)$ with affine cones in $\mathbb{A}^r(k)$, the radical ideal defining the subset $\text{supp}_E(M)$ corresponds to $V_E^r(M)$ for any finite-dimensional kE -module M .*

Given an extension of fields $k \subseteq K$ and a kE -module M , let M_K be the KE -module $K \otimes_k M$. Benson, Carlson and Rickard [11] extended Dade's theorem to cover infinitely generated modules, as follows:

3.4. Theorem. *A kE -module M is projective if and only if $\alpha_\lambda^*(M_K)$ is projective for all extension fields K of k and all $0 \neq \lambda$ in $\mathbb{A}^r(K)$.*

For $0 \neq \lambda \in \mathbb{A}^r(K)$ let $\mathfrak{p} \subseteq k[Y_1, \dots, Y_r]$ be the ideal consisting of homogeneous polynomials that vanish at λ . This is a prime ideal and λ is a *generic* point for $\mathfrak{p}$. Each $\mathfrak{p}$ in $\text{Proj } k[Y_1, \dots, Y_r]$ occurs this way. If $\lambda \in \mathbb{A}^r(K)$ and $\lambda' \in \mathbb{A}^r(K')$ are generic points for the same prime ideal, then $\alpha_\lambda^*(M_K)$ is projective if and only if $\alpha_{\lambda'}^*(M_{K'})$ is projective. So instead of a rank variety, we assign to M a subset of $\text{Proj } k[Y_1, \dots, Y_r]$.

3.5. Definition. Set $\mathcal{V}_E^r(M)$ to be the set of $\mathfrak{p} \in \text{Proj } k[Y_1, \dots, Y_r]$ such that if $\lambda \in \mathbb{A}^r(K)$ is generic for $\mathfrak{p}$ then $\alpha_\lambda^*(M_K)$ is not projective.

Historically the rank variety was defined as a subset of $\mathbb{A}^r(k)$, but we switch to considering projective varieties since all our constructions are “invariant” under scalar multiplication. It follows from the main theorems of [11] that the subset $\text{supp}_E(M) \subseteq \text{Proj } H^*(E, k)$ corresponds to $\mathcal{V}_E^r(M)$ for *any* kE -module M , once the appropriate identification, involving Frobenius for $p > 2$, of $\text{Proj } H^*(E, k)$ and $\mathbb{P}^{r-1}$ is made.

We shall see in Section 5 that the appropriate generalisation of these concepts to finite group schemes leads to the theory of π -points.

4. FINITE GROUP SCHEMES

In this section, we introduce finite group schemes, following the approach given in [51, Chapter 1]. Throughout k will be a field.

An *affine scheme* over k is a representable functor from the category $\text{CAlg}(k)$ of commutative algebras over k to sets. The representing object of an affine scheme S , denoted $k[S]$, is called its *coordinate ring*. Thus $S: \text{CAlg}(k) \rightarrow \text{Set}$ takes the form $\text{Hom}_{\text{CAlg}(k)}(k[S], -)$.

An *affine group scheme* is a functor $G: \text{CAlg}(k) \rightarrow \text{Grp}$ whose composite with the forgetful functor $\text{Grp} \rightarrow \text{Set}$ is representable. By Yoneda's lemma, the natural transformations given by the group operations give rise to a structure on $k[G]$ of commutative Hopf algebra. This gives a contravariant equivalence of categories from affine group schemes to commutative Hopf algebras, sending G to $k[G]$.

A *finite group scheme* is an affine group scheme G with the property that the coordinate ring $k[G]$ is finite dimensional over k . In this case, we may dualise to get the *group algebra* $kG := \text{Hom}_k(k[G], k)$, which has the structure of cocommutative Hopf algebra. This gives a covariant equivalence of categories from finite group schemes to finite dimensional cocommutative Hopf algebras, sending G to kG .

Finite groups are examples of finite group schemes, but there are many more, including p -restricted Lie algebras, and Frobenius kernels of affine group schemes. See, for example, [23, §1] for examples and an explanation of how finite groups fit into the context.

Friedlander and Suslin [46] proved that for any finite group scheme G , the k -algebra $H^*(G, k)$ is Noetherian; moreover, the $H^*(G, k)$ -module $H^*(G, M)$ is finitely generated for any finite dimensional kG -module M . This opened the door

for the development of support theories for finite group schemes. Another landmark development in this area was the theory of π -points.

5. THE THEORY OF π -POINTS

The theory of π -points for finite group schemes was initiated by Friedlander and Pevtsova [44, 45], and generalises rank varieties discussed in Section 3. This approach does not rely on the choice of generators needed to define cyclic shifted subgroups for elementary abelian p -groups which makes it applicable in a much greater generality.

Let G be a finite group scheme over a field k of positive characteristic p . A π -point α of G is given as follows. We choose an extension field K of k , a unipotent abelian subgroup scheme E of G_K , and a flat map

$$\alpha: K[t]/(t^p) \rightarrow KE \subseteq KG_K.$$

Note that E does not have to come from a subgroup scheme of G by extension of scalars. The flatness condition is equivalent to the statement that the image of t is in $J(KE)$ but not in $J^2(KE)$.

Given such an α consider the composite

$$H^*(G, k) \subseteq K \otimes_k H^*(G, k) \cong H^*(G_K, K) \cong \text{Ext}_{KG_K}^*(K, K) \xrightarrow{\alpha^*} \text{Ext}_{K[t]/(t^p)}^*(K, K).$$

The ring $\text{Ext}_{K[t]/(t^p)}^*(K, K)$ is isomorphic to $K[v]$ with $|v| = 1$ if $p = 2$, and to $K[u, v]/(u^2)$ with $|u| = 1$ and $|v| = 2$ if $p \neq 2$. The nil radical is zero in the first case, and the ideal (u) in the second case. We define $\mathfrak{p}(\alpha)$ to be the inverse image in $H^*(G, k)$ of the nil radical of $\text{Ext}_{K[t]/(t^p)}^*(K, K)$ under the above map. This is a homogeneous prime ideal in $H^*(G, k)$. The following theorem is due to Friedlander and Pevtsova [45].

5.1. Theorem. *If $\alpha: K[t]/(t^p) \rightarrow KG_K$ and $\beta: L[t]/(t^p) \rightarrow LG_L$ are π -points, the following conditions are equivalent.*

- (i) $\mathfrak{p}(\alpha) = \mathfrak{p}(\beta)$;
- (ii) $\alpha^*(M_K)$ is projective if and only if $\beta^*(M_L)$ is projective for a kG -module M ;
- (iii) $\alpha^*(M_K)$ is projective if and only if $\beta^*(M_L)$ is projective for any finite dimensional kG -module M . $\square$

We therefore put an equivalence relation on the set of π -points of G , where $\alpha \sim \beta$ if and only if the equivalent conditions of the theorem hold. The map sending the equivalence class of a π -point α to the prime $\mathfrak{p}(\alpha)$ gives a bijection between the equivalence classes of π -points in G and the set $\text{Proj } H^*(G, k)$.

With this equivalence relation, every π -point is equivalent to one that factors through a subgroup scheme which is not only abelian unipotent, but elementary. Let $\mathbb{G}_a$ be the additive group scheme and for each integer $r \geq 0$, let $\mathbb{G}_{a(r)}$ denote its r th Frobenius kernel; see [51, Chapter 9].

5.2. Definition. A finite group scheme is *elementary* if it is isomorphic to the group scheme $\mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s$ with $r, s \geq 0$.

These group schemes are called “quasi-elementary” by Bendel [7]. For a finite group scheme cohomology is detected, modulo nilpotents, on elementary subgroup schemes over extension fields. This explains their central role in this theory.

5.3. Definition. Let G be a finite group scheme over a field k . The π -support, denoted $\pi\text{-supp}_G(M)$, of a kG -module M is the subset of $\text{Proj } H^*(G, k)$ consisting of primes $\mathfrak{p}(\alpha)$, for $\alpha: K[t]/(t^p) \rightarrow KG_K$ a π -point such that $\alpha^*(M_K)$ is not projective.

The π -support is a “generator-invariant” generalisation of Carlson’s rank variety for any finite group scheme. On the other hand, exactly as for finite groups, one has a canonical action of $H^*(G, k)$ on $\text{StMod}(kG)$ and hence a notion of support for kG -modules; see Section 2, especially 2.3. The following theorem from [45, 24] reconciles these two notions. In doing so, it puts the results for elementary abelian p -groups [2], finite groups [11], restricted Lie algebras [43], and infinitesimal group schemes [8, 52] into a uniform statement.

5.4. Theorem. *One has $\text{supp}_G(M) = \pi\text{-supp}_G(M)$ for any kG -module M .* $\square$

6. FINITE SUPERGROUP SCHEMES

The definition of affine superschemes is parallel to that of affine group schemes. It is obtained replacing the underlying category of vector spaces over k by the category of super vector spaces. A *super vector space* over k is a $\mathbb{Z}/2$ -graded vector space $V = V_0 \oplus V_1$. The tensor product of two such is given by

$$(V \otimes W)_0 = V_0 \otimes W_0 \oplus V_1 \otimes W_1 \quad \text{and} \quad (V \otimes W)_1 = V_0 \otimes W_1 \oplus V_1 \otimes W_0.$$

This tensor product has a symmetric braiding $V \otimes W \cong W \otimes V$ given by sending $v \otimes w$ to $(-1)^{|v||w|} w \otimes v$, where $|v|$ denotes 0 if $v \in V_0$ and 1 if $v \in V_1$. Thus the category $\mathbf{SVec}(k)$ of super vector spaces is a symmetric monoidal abelian category. A *commutative superalgebra* over k is a commutative algebra in this category. Thus it consists of an object A together with a multiplication $A \times A \rightarrow A$ which is commutative with respect to the symmetric braiding, associative, and unital. These form a category $\mathbf{CSAlg}(k)$.

An *affine superscheme* over k is a representable functor from commutative superalgebras over k to sets. The representing object of an affine superscheme S is the coordinate ring $k[S]$. An *affine supergroup scheme* is a functor $G: \mathbf{CSAlg}(k) \rightarrow \mathbf{Grp}$ whose composite with the forgetful functor $\mathbf{Grp} \rightarrow \mathbf{Set}$ is representable. The representing object, the coordinate ring $k[G]$ of G , is a commutative Hopf superalgebra. This way, we obtain a contravariant equivalence of categories from affine supergroup schemes to commutative Hopf superalgebras. A Hopf superalgebra need not be a Hopf algebra, for the diagonal map need not be a map of ungraded algebras.

A *finite supergroup scheme* is an affine supergroup scheme G with the property that the coordinate ring $k[G]$ is finite dimensional over k . In this case, we may dualise to get the *group algebra* $kG = \text{Hom}_k(k[G], k)$, which has the structure of cocommutative Hopf superalgebra. This way, we obtain a covariant equivalence of categories from finite supergroup schemes to finite dimensional cocommutative Hopf superalgebras.

Examples of finite supergroup schemes include finite groups, finite group schemes, exterior algebras, as well as Frobenius kernels of affine supergroup schemes such as the general linear ones $\text{GL}(a|b)$ and the orthosymplectic ones $\text{OSp}(a|2b)$. We write $\mathbb{G}_a^-$ for the finite supergroup scheme whose group algebra is an exterior algebra on one primitive element, in degree 1. This is the simplest example of a finite supergroup scheme which is not a finite group scheme.

A finite supergroup scheme G is *unipotent* if the kernel of the augmentation map $kG \rightarrow k$ is equal to the nil radical. This is equivalent to kG having exactly two simple modules, the trivial module in even degree and the trivial module in odd degree. For example, $\mathbb{G}_a^-$ is unipotent.

The module category for a finite supergroup scheme is an abelian category, with a parity change functor Π . The action of an element $a \in kG$ on $\Pi m \in \Pi M$ is given by $a \cdot \Pi m = (-1)^{|a|} \Pi(am) \in \Pi M$. The stable module category $\text{StMod}(kG)$ is a $\mathbb{Z}/2$ -graded triangulated category. So it comes with an internal shift Π whose

square is naturally isomorphic to the identity, and a cohomological shift Ω^{-1} . The distinguished triangles

$$M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow \Omega^{-1}M_1$$

are those isomorphic to triangles coming from short exact sequences of kG -modules via a pushout diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M_1 & \longrightarrow & I & \longrightarrow & \Omega^{-1}M_1 \longrightarrow 0 \end{array}$$

where I is an injective module into which M_2 embeds.

For a $\mathbb{Z}/2$ -graded triangulated category $\mathcal{T}$, we need a slight modification to the definition of centre, to take account of the grading. We define $Z^{n,j}(\mathcal{T})$ ($n \in \mathbb{Z}$, $j \in \mathbb{Z}/2$) to be the natural transformations η from the identity functor to $\Sigma^n \Pi^j$ satisfying $\eta \Sigma = (-1)^n \Sigma \eta$ and $\eta \Pi = (-1)^j \Pi \eta$. These form a $\mathbb{Z} \times \mathbb{Z}/2$ -graded ring. It is graded commutative, in the sense that if $x \in Z^{m,i}(\mathcal{T})$ and $y \in Z^{n,j}(\mathcal{T})$ then $yx = (-1)^{mn}(-1)^{ij}xy$. The cohomology ring $H^{*,*}(G, k)$ is also $\mathbb{Z} \times \mathbb{Z}/2$ -graded commutative in this sense. For example, $H^{*,*}(\mathbb{G}_a^-, k)$ is a polynomial ring on a single generator in degree $(1, 1)$.

There is a canonical action of $H^{*,*}(G, k)$ on $\text{StMod}(kG)$, so it makes sense to try to stratify $\text{StMod}(kG)$ as a $\mathbb{Z}/2$ -graded tensor triangulated category using this action. The definition of localising subcategory needs to be modified to take account of the extra grading; we only consider localising subcategories closed under the operation Π .

6.1. Definition. We define $\text{Proj } H^{*,*}(G, k)$ to consist of the prime ideals which are homogeneous with respect to both gradings, with the Zariski topology, and as usual we exclude the maximal ideal of elements generated by homogeneous elements whose degree is not $(0, 0)$. Since elements of degree (even, odd) or (odd, even) square to zero, these elements are contained in every prime ideal. Modulo these elements, the $\mathbb{Z}/2$ -grading is just the mod two reduction of the $\mathbb{Z}$ -grading, and so we can ignore it. So we write $H(G, k)$ for the singly graded ring whose degree i component is the degree $(i, 0)$ or $(i, 1)$ component of $H^{*,*}(G, k)$ according as i is even or odd. This is a graded ring which becomes strictly commutative if we reduce modulo nilpotents, and at the level of topological spaces, $\text{Proj } H(G, k)$ may be identified with $\text{Proj } H^{*,*}(G, k)$. Thus for example $H(\mathbb{G}_a^-, k)$ is a polynomial ring on a generator in degree one.

6.2. Definition. For a kG -module M , let $\text{supp}_G(M)$ denote the support defined via the action of $H(G, k)$ on $\text{StMod}(kG)$, as in Section 2.

6.3. Remark. By definition, $\text{supp}_G(M)$ is a subset of $\text{Proj } H(G, k)$. Recall that when M is finite-dimensional, $\text{supp}_G(M)$ is a closed subset defined by the annihilator of $\text{Ext}_{kG}^{*,*}(M, M)$ as a module over $H(G, k)$. Moreover, if G is unipotent, we can take the annihilator of $H^{*,*}(G, M)$.

7. ELEMENTARY SUPERGROUP SCHEMES

The definition of elementary supergroup schemes is more complicated than for finite group schemes, and we have so far only addressed the unipotent case. Drupieski and Kujawa [37, 38, 39] suggest that similar definitions may suffice for arbitrary finite supergroup schemes.

In [26] we construct a family of finite connected unipotent supergroup schemes $E_{m,n}^-$ with $m, n \geq 1$ related to the Witt vectors and declare a supergroup scheme

to be *elementary* if it is isomorphic to a quotient of some $E_{m,n}^- \times (\mathbb{Z}/p)^s$. These are classified; see [26, Remark 8.14] and compare with Definition 5.2.

7.1. Theorem. *Each elementary supergroup scheme is isomorphic to one of:*

- I. $\mathbb{G}_{a(n)} \times (\mathbb{Z}/p)^s$ with $n, s \geq 0$,
- II. $\mathbb{G}_{a(n)} \times \mathbb{G}_a^- \times (\mathbb{Z}/p)^s$ with $n, s \geq 0$,
- III. (i) $E_{m,n}^- \times (\mathbb{Z}/p)^s$ with $m \geq 2, n \geq 1, s \geq 0$,
(ii) $E_{m,n,\mu}^- \times (\mathbb{Z}/p)^s$ with $m, n \geq 1, 0 \neq \mu \in k^\times / (k^\times)^2$ and $s \geq 0$.

Here, $E_{m,n,\mu}^-$ is a quotient of $E_{m+1,n+1}^-$ by a subgroup isomorphic to $\mathbb{G}_{a(1)}$, and only depends on the image of μ in $k^\times / (k^\times)^2$. $\square$

7.2. Definition. The supergroup schemes of type III are said to be *Witt elementary*.

The role played by elementary supergroup schemes is explained by the following analogue from [26] of the theorems of Quillen and Chouinard for finite groups.

7.3. Theorem. *Let G be a unipotent finite supergroup scheme over a field k of characteristic $p \geq 3$. Then the following hold:*

- (i) *An element $x \in H^{*,*}(G, k)$ is nilpotent if and only if for every extension field K of k and every elementary sub-supergroup scheme E of G_K , the restriction of x_K , the image of x in $H^{*,*}(G_K, K)$, to $H^{*,*}(E, K)$ is nilpotent.*
- (ii) *A kG -module M is projective if and only if for every extension field K of k and every elementary sub-supergroup scheme E of G_K , the restriction of M_K to E is projective.* $\square$

Fortunately, if we ignore the comultiplicative structure of these elementary supergroup schemes, the $\mathbb{Z}/2$ -graded algebra structure is easy to describe. The list corresponds to the one from Theorem 7.1.

7.4. Proposition. *If E is an elementary supergroup scheme over k then the algebra kE is isomorphic to one of the following:*

- (i) *a tensor product of copies of $k[s]/(s^p)$,*
- (ii) *a tensor product of copies of $k[s]/(s^p)$ and one copy of $k[\sigma]/(\sigma^2)$,*
- (iii) *a tensor product of copies of $k[s]/(s^p)$ and one copy of $k[s, \sigma]/(s^{p^m}, s^p - \sigma^2)$, where $m \geq 1$,*

with $|s|$ even and $|\sigma|$ odd.

In particular, we have

$$(7.5) \quad kE_{m,n}^- \cong \frac{k[s_1, \dots, s_n, \sigma]}{(s_1^p, \dots, s_{n-1}^p, s_n^{p^m}, s^p - \sigma^2)}$$

with coproduct defined by

$$\begin{aligned} \Delta(s_i) &= S_{i-1}(s_1 \otimes 1, \dots, s_i \otimes 1, 1 \otimes s_1, \dots, 1 \otimes s_i) \\ \Delta(\sigma) &= \sigma \otimes 1 + 1 \otimes \sigma, \end{aligned} \quad (i \geq 1)$$

where the S_i are the maps coming from the comultiplication in $\mathbb{G}_a$.

Although an elementary supergroup scheme is not necessarily commutative as a $\mathbb{Z}/2$ -graded algebra, because of the factors of type 7.4 (iii), it is nonetheless commutative in the ungraded sense.

7.6. Remark. For non-unipotent finite supergroup schemes, these elementary supergroup schemes are definitely not sufficient for detection. Drupieski and Kujawa [37] introduced a slightly more general set of supergroup schemes which they show suffice for $\mathrm{GL}(a|b)_{(r)}$. Following their notation, if f is a p -polynomial, meaning a polynomial of the form $f(t) = \sum_i a_i t^{p^i}$, we shall write $\mathbb{M}_{n,f}$ for the supergroup

scheme defined by replacing $s_n^{p^m}$ by $f(s_n)$ in (7.5), and with the same comultiplication as $E_{m,n}^-$. Similarly, $\mathbb{M}_{n,f,\mu}$ is obtained in the same way from $E_{m,n-1,\mu}^-$. The only unipotent ones among these are our $E_{m,n}^-$ and $E_{m,n,\mu}^-$. These are all quotients of a supergroup scheme $\mathbb{M}_n$. Set

$$k\mathbb{M}_n \cong \frac{k[s_1, \dots, s_{n-1}, \sigma][s_n]}{(s_1^p, \dots, s_{n-1}^p, \sigma^2 - s_n^p)},$$

$$k\mathbb{M}_{n,f,\mu} \cong \frac{k[s_1, \dots, s_n, \sigma]}{(s_1^p, \dots, s_{n-1}^p, f(s_n) + \mu s_1, \sigma^2 - s_n^p)}.$$

Thus $k\mathbb{M}_{n,f,\mu}$ is the group ring over k of the corresponding supergroup scheme. We shall suppress the field from the notation, so that we also regard $\mathbb{M}_n$ as a profinite supergroup scheme over any extension field K of k . Similarly, if f is a p -polynomial with coefficients in K , and $\mu \in K$, we shall write $\mathbb{M}_{n,f,\mu}$. Finally, if $f = t^{p^n}$ and $\mu = 0$, we shall just write $\mathbb{M}_{n,m}$; this is the same as $E_{m,n}^-$.

8. π -POINTS, π -SUPPORTS AND RANK VARIETIES

To generalise the theory of π -points from finite group schemes to finite supergroup schemes, instead of flat maps from $k[t]/(t^p)$, we consider maps of finite flat (or equivalently, projective) dimension from the superalgebra

$$A_k := \frac{k[t, \tau]}{(t^p - \tau^2)} \quad (\tau \text{ odd, } t \text{ even})$$

We aim for theorems about unipotent finite supergroup schemes, but we stay more general for now in the interest of later developments, and also because we shall need to discuss $\mathrm{GL}(a|b)_{(n)}$ as part of the proof. So we shall include the elementary supergroup schemes $\mathbb{M}_{n,f,\mu}$ from Remark 7.6, but note that if G is unipotent then the only ones that occur are the ones listed at the beginning of Section 7.

We view A_k as a cocommutative Hopf superalgebra over k with τ and t primitive:

$$\Delta(\tau) = \tau \otimes 1 + 1 \otimes \tau \quad \text{and} \quad \Delta(t) = t \otimes 1 + 1 \otimes t.$$

This defines a homomorphism of algebras since

$$\begin{aligned} \Delta(\tau^2) &= (\tau \otimes 1 + 1 \otimes \tau)^2 \\ &= \tau^2 \otimes 1 + \tau \otimes \tau - \tau \otimes \tau + 1 \otimes \tau^2 \\ &= t^p \otimes 1 + 1 \otimes t^p \\ &= \Delta(t^p). \end{aligned}$$

The cohomology ring of the k -algebra A_k is easy to compute:

$$\mathrm{Ext}_{A_k}^{*,*}(k, k) = k[\eta] \otimes \Lambda(u)$$

where $|\eta| = (1, 1)$ and $|u| = (1, 0)$. In particular, modulo its radical it is a domain.

8.1. Definition. A π -point α of a finite supergroup scheme G is given as follows. We choose an extension field K of k , an elementary sub-supergroup scheme E of G_K , and a map of superalgebras, but not necessarily respecting the coproduct,

$$\alpha: A_K \rightarrow KE \subseteq KG_K$$

of finite flat dimension.

We put an equivalence relation on π -points, analogous to the one in Section 5.

8.2. Definition. We say that π -points $\alpha: A_K \rightarrow KG_K$ and $\beta: A_L \rightarrow LG_L$ are equivalent if, for all finite dimensional kG -modules M , the module $\alpha^*(M_K)$ has finite flat dimension if and only if $\beta^*(M_L)$ has finite flat dimension. We write $\Pi(G)$ for the set of equivalence classes of π -points of G .

8.3. Definition. We say a π -point α is *K-rational* if it is defined over the field K , so that it is a map $\alpha: A_K \rightarrow KG_K$.

If E is an elementary sub-supergroup scheme of G_K , and $\alpha: A_K \rightarrow KE$ is a π -point, then the radical of the kernel of restriction

$$H^{*,*}(G, k) \subseteq H^{*,*}(G_K, K) \rightarrow H^{*,*}(E, K) \rightarrow \text{Ext}_{A_K}^{*,*}(K, K)$$

is a prime ideal $\mathfrak{p}(\alpha)$, for the target, modulo its radical, is a domain.

8.4. Lemma. *If α and β are equivalent π -points, then $\mathfrak{p}(\alpha) = \mathfrak{p}(\beta)$.*

Proof. A proof of this result is given in Proposition 6.8 of [27] for elementary supergroup schemes; it involves Carlson's L_ζ modules. The same argument applies without any change to the general case. $\square$

8.5. Definition. By the preceding lemma, one gets a well-defined map

$$\Phi_G: \Pi(G) \longrightarrow \text{Proj } H^{*,*}(G, k)$$

where α is sent to $\mathfrak{p}(\alpha)$. By [27, Theorem 6.9], see also [27, §8], it is bijective when G is an elementary supergroup scheme. See Section 9 for the general case.

8.6. Definition. Let G be a finite *unipotent* supergroup scheme over a field k of characteristic $p \geq 3$. The π -support of a kG -module M , denoted $\pi\text{-supp}_G(M)$, is the subset of $\Pi(G)$ consisting of equivalence classes of π -points $\alpha: A_K \rightarrow KG$ such that $\alpha^*(M_K)$ has infinite flat dimension. This is well-defined by design.

If E is an elementary supergroup scheme, then by Proposition 7.4 the group algebra kE is isomorphic to one of the following algebras:

- I. $kE \cong k[s_1, \dots, s_n]/(s_1^p, \dots, s_n^p)$,
- II. $kE \cong k[s_1, \dots, s_n, \sigma]/(s_1^p, \dots, s_n^p, \sigma^2)$,
- III. $kE \cong k[s_1, \dots, s_n, \sigma]/(s_1^p, \dots, s_{n-1}^p, s_n^{p^m}, s_n^p - \sigma^2)$ for some $m \geq 2, n \geq 1$.

In each case, a set of representatives for equivalence classes of π -points of E can be prescribed by choosing an extension field K of k and a point $0 \neq \lambda = (\lambda_1, \dots, \lambda_{n+1})$ in $\mathbb{A}^{n+1}(K)$. The π -point

$$\alpha_\lambda: A_K \rightarrow KE_K$$

is as follows: In case I, which is covered by Carlson's theory of rank varieties, set

$$\alpha_\lambda(t) = \lambda_1 s_1 + \dots + \lambda_n s_n \quad \text{and} \quad \alpha_\lambda(\tau) = 0.$$

In the second case, set

$$\alpha_\lambda(t) = \lambda_1 s_1 + \dots + \lambda_n s_n \quad \text{and} \quad \alpha_\lambda(\tau) = \lambda_{n+1} \sigma.$$

See [27, §8]. In the last case, the π -point is defined by

$$(8.7) \quad \begin{aligned} \alpha_\lambda(t) &= \lambda_1 s_1 + \dots + \lambda_{n-1} s_{n-1} + \lambda_{n-1} s_{n-1} + \lambda_n s_n^{p^{m-1}} + \lambda_{n+1}^2 s_n, \\ \alpha_\lambda(\tau) &= \lambda_{n+1}^p \sigma. \end{aligned}$$

Note that

$$(\alpha_\lambda(t))^p = \lambda_{n+1}^{2p} s_n^p = \lambda_{n+1}^{2p} \sigma^2 = (\alpha_\lambda(\tau))^2$$

in KE_K , so this does indeed define a homomorphism of algebras.

The result below, which reproduces [27, Theorem 4.12], extends Theorem 3.4.

8.8. Theorem. *Let E be an elementary supergroup scheme defined over a field k . An E -module M is projective if and only if $\alpha_\lambda^*(M_K)$ has finite flat dimension for every extension field K of k and every $0 \neq \lambda \in \mathbb{A}^{n+1}(K)$.* $\square$

Combining with Theorem 7.3 yields that π -points detect projectivity of modules.

8.9. Theorem. *Let G be a unipotent finite supergroup scheme over a field k of characteristic $p \geq 3$. A kG -module M is projective if and only if $\pi\text{-supp}_G(M) = \emptyset$.*

Proof. If M is projective, K is an extension field of k , and E is a sub-supergroup scheme of G_K , then $\text{res}_{G_K, E}(M_K)$ is projective. So if $\alpha: A_K \rightarrow E \rightarrow G_K$ is a π -point then $\alpha^*(M_K)$ has finite flat dimension. Thus $\pi\text{-supp}_G(M)$ is empty.

Conversely, if $\pi\text{-supp}_G(M) = \emptyset$ and E is a sub-supergroup scheme of G_K for some extension field K , then $\text{res}_{G_K, E}(M_K)$ has empty π -support. By Theorem 8.8, $\text{res}_{G_K, E}(M_K)$ is projective for every such K and E . It follows from Theorem 7.3 that M is a projective kG -module. $\square$

A crucial property of π -support is the tensor product formula.

8.10. Theorem. *Let G be a unipotent finite supergroup scheme over a field k of characteristic $p \geq 3$, and let M and N be kG -modules. As subsets of $\Pi(G)$ we have*

$$\pi\text{-supp}_G(M \otimes_k N) = \pi\text{-supp}_G(M) \cap \pi\text{-supp}_G(N).$$

Proof. For any elementary sub-supergroup scheme E of G_K , the restriction functor

$$\text{res}_{G_K, E}: \text{Mod } G_K \rightarrow \text{Mod } E$$

commutes with tensor product. Hence, it suffices to prove the formula for elementary supergroup schemes. This is done in [27, Theorem 7.6]. $\square$

For finite-dimensional modules over elementary supergroups π -support has a “rank variety” interpretation, analogous to Carlson’s original construction recalled in Definition 3.5.

Let $k[Y_1, \dots, Y_{n+1}]$ be the coordinate ring of $\mathbb{A}^{n+1}(k)$. Since the map

$$\Phi_E: P(E) \rightarrow \text{Proj } H^{*,*}(E, k) \cong \mathbb{P}^n(k)$$

is a homeomorphism, for any $\mathfrak{p} \in \text{Proj } k[Y_1, \dots, Y_{n+1}]$ there is a “generic” point $\lambda \in \mathbb{A}^{n+1}(K)$ such that $\Phi_E(a_\lambda) = \mathfrak{p}$ where a_λ is the π -point defined in (8.7).

8.11. Definition. Let k be a field of characteristic $p \geq 3$, and let E be a Witt elementary supergroup scheme over k . If M is an E -module, we define $\mathcal{V}_E^r(M)$ to be the set of homogeneous primes $\mathfrak{p} \in \text{Proj } k[Y_1, \dots, Y_{n+1}]$ such that if $\lambda \in \mathbb{A}^{n+1}(K)$ is generic for $\mathfrak{p}$ then $\alpha_\lambda^*(M_K)$ has infinite flat dimension as a KA_K -module.

It follows from [27] that Φ_E takes the π -support of M bijectively to $\mathcal{V}_E^r(M)$.

Here is a free resolution of the trivial A_K -module K :

$$\cdots \rightarrow \Pi A_K \oplus A_K \xrightarrow{\begin{pmatrix} \tau & t \\ -t^{p-1} & -\tau \end{pmatrix}} A_K \oplus \Pi A_K \xrightarrow{\begin{pmatrix} \tau & t \\ -t^{p-1} & -\tau \end{pmatrix}} \Pi A_K \oplus A_K \xrightarrow{(\tau, t)} A_K \rightarrow 0$$

This is periodic from degree one onwards, with period one. So for a π -point $\alpha: A_K \rightarrow KE_K$ and a E -module M , it follows that $\alpha^*(M_K)$ has finite projective dimension as a A_K -module if and only if $\text{Ext}_{A_K}^1(K, \alpha^*(M_K)) = 0$. Taking homomorphisms from the above resolution to $\alpha^*(M_K)$, this is true if and only if

$$\Pi M_K \oplus M_K \xrightarrow{\begin{pmatrix} \alpha_\lambda(\tau) & \alpha_\lambda(t) \\ -\alpha_\lambda^{p-1}(t) & -\alpha_\lambda(\tau) \end{pmatrix}} M_K \oplus \Pi M_K \xrightarrow{\begin{pmatrix} \alpha_\lambda(\tau) & \alpha_\lambda(t) \\ -\alpha_\lambda^{p-1}(t) & -\alpha_\lambda(\tau) \end{pmatrix}} \Pi M_K \oplus M_K$$

is exact. In particular, if M is a finitely generated E -module, of dimension d , we can think of the action of

$$(8.12) \quad \begin{pmatrix} \alpha_\lambda(\tau) & \alpha_\lambda(t) \\ -\alpha_\lambda^{p-1}(t) & -\alpha_\lambda(\tau) \end{pmatrix}$$

on a direct sum of two copies of M_K as a $2d \times 2d$ matrix whose square is zero. Exactness is therefore just the condition that this matrix has rank equal to d , which is the largest possible for a square zero matrix of this size. This condition fails if and only if every $d \times d$ minor of the matrix vanishes. In particular, this is a set of homogeneous polynomial conditions, and therefore defines a closed homogeneous subvariety of $\mathbb{A}^{n+1}(K)$. So the following is an analogue of Carlson’s rank variety.

8.13. Definition. If k is algebraically closed, and M is a finitely generated E -module of dimension d , we define the *rank variety* $V_E^r(M)$ to be $\{0\}$ together with the set of $\lambda \in \mathbb{A}^{n+1}(k)$ such that $\alpha_\lambda^*(M)$ has infinite flat dimension. Equivalently, $V_E^r(M)$ is the set of points λ where the $2d \times 2d$ matrix representing the action of (8.12) on M has rank strictly less than d .

If we extend the field from k to K , the polynomial equations defining the variety $V_{E_K}^r(M_K)$ are exactly the same as those defining $V_E^r(M)$. It follows that a prime $\mathfrak{p}$ is in $V_E^r(M)$ if and only if a generic point for $\mathfrak{p}$ in a suitable $\mathbb{A}^{n+1}(K)$ is contained in the zero set of these polynomials. This happens if and only if $\mathfrak{p}$ is contained in the radical ideal defining $V_E^r(M)$.

8.14. Theorem. For M a finite dimensional kG -module, $V_E^r(M)$ is the Zariski closed subset of $\text{Proj } k[Y_1, \dots, Y_{n+1}]$ defined by the rank variety $V_E^r(M)$.

On the other hand, it can be shown that for infinitely generated E -modules, every subset of $\text{Proj } k[Y_1, \dots, Y_{n+1}]$ occurs as $V_E^r(M)$ for suitably constructed M .

9. Φ_G IS A HOMEOMORPHISM

In this section we show that the map $\Phi_G: \Pi(G) \rightarrow \text{Proj } H^{*,*}(G, k)$, introduced in Definition 8.6, is bijective for a unipotent finite supergroup scheme G . Surjectivity will follow from Theorem 7.3 whereas for injectivity we have to recall some recent results of Drupieski and Kujawa. We shall use supergroup schemes $\mathbb{M}_n$ and $\mathbb{M}_{n;f,\mu}$ introduced in Remark 7.6.

Let $\text{Hom}_{\text{sgs}/k}(\mathbb{M}_n, G)$ be a functor from commutative k -superalgebras to sets defined as

$$\text{Hom}_{\text{sgs}/k}(\mathbb{M}_n, G)(R) = \text{Hom}_{\text{sgs}/R}(\mathbb{M}_{n,R}, G_R).$$

9.1. Definition. It is shown in [37, Theorem 3.3.6] that if G is any affine supergroup scheme of finite type then $\text{Hom}_{\text{sgs}/k}(\mathbb{M}_n, G)$ has the structure of a connected affine superscheme of finite type over k , which we denote $\mathbf{V}_n(G)$. Similarly, $\text{Hom}_{\text{sgs}/k}(\mathbb{M}_{n;f,\mu}, G)$ is an affine sub-superscheme of $\mathbf{V}_n(G)$, which we denote $\mathbf{V}_{n;f,\mu}(G)$. In the case $f = t^{p^m}$ and $\mu = 0$, we shall write $\mathbf{V}_{n;m}(G)$.

The K -points of $\mathbf{V}_n(G)$ are maps $\mathbb{M}_{n,K} \rightarrow G_K$ as above. In particular, we can identify $\mathbf{V}_n(G)$ with $\mathbf{V}_n(G_{(n)})$.

In §6.2 of [37], the authors construct a map of affine superschemes

$$\Psi_G: \mathbf{V}_n(G) \rightarrow \text{Spec } H^{*,*}(G, k)$$

coming from a map of coordinate rings

$$\psi_G: H^{*,*}(G, k) \rightarrow k[\mathbf{V}_n(G)].$$

Let $\text{GL}(a|b)$ be the general linear supergroup; see, for example, [35]. For a connected finite supergroup scheme $\text{GL}(a|b)_{(n)}$ which is the Frobenius kernel of $\text{GL}(a|b)$, we have the following. There is a map constructed in §6 of [37]

$$\bar{\phi}: k[\mathbf{V}_n(\text{GL}(a|b))]_{\text{ev}} = k[\mathbf{V}_n(\text{GL}(a|b)_{(n)})]_{\text{ev}} \rightarrow H^{*,0}(\text{GL}(a|b)_{(n)})$$

with the property that the composite

$$k[\mathbf{V}_n(\text{GL}(a|b))]_{\text{ev}} \xrightarrow{\bar{\phi}} H(\text{GL}(a|b)_{(n)}) \xrightarrow{\psi_{\text{GL}(a|b)_{(n)}}} k[\mathbf{V}_{n;f,\mu}(\text{GL}(a|b))]_{\text{ev}}$$

is the Frobenius morphism F^n followed by the quotient map

$$k[\mathbf{V}_n(\text{GL}(a|b))]_{\text{ev}} \rightarrow k[\mathbf{V}_{n;f,\mu}(\text{GL}(a|b))]_{\text{ev}}.$$

This has the following consequence, as pointed out in Theorem 6.2.3 of [37].

9.2. Theorem. *The image of map*

$$\psi_{\mathrm{GL}(a|b)_{(n)}} : H^{*,*}(\mathrm{GL}(a|b)_{(n)}, k) \rightarrow k[\mathbf{V}_{n;f,\mu}(\mathrm{GL}(a|b)_{(n)})]$$

contains the p^n th power of every element of $k[\mathbf{V}_{n;f,\mu}(\mathrm{GL}(a|b)_{(n)})]$. The map

$$\Psi_{\mathrm{GL}(a|b)_{(n)}} : \mathbf{V}_n(\mathrm{GL}(a|b)_{(n)}) \rightarrow \mathrm{Spec} H^{*,*}(\mathrm{GL}(a|b)_{(n)}, k)$$

is injective on K -points for all extension fields K of k . $\square$

The following is the analogu of [8, Theorem 5.2]; see also [44, Proposition 3.8].

9.3. Theorem. *Let G be connected finite supergroup scheme of height at most n . Then for m large enough, the image of the map*

$$\psi_G : H^{*,*}(G, k) \rightarrow k[\mathbf{V}_{n;m}(G)]$$

contains the p^n th power of every element of $k[\mathbf{V}_{n;m}(G)]$. Consequently, the map $\Psi_G : \mathbf{V}_{n;m}(G) \rightarrow \mathrm{Spec} H^{,*}(G, k)$ is injective.*

Proof. Choose an embedding $G \rightarrow \mathrm{GL}(a|b)_{(r)}$ for suitable a, b and r . Then we have the following diagram

$$\begin{array}{ccc} H^{*,*}(\mathrm{GL}(a|b)_{(n)}, k) & \xrightarrow{\psi_{\mathrm{GL}(a|b)_{(n)}}} & k[\mathbf{V}_{n;m}(\mathrm{GL}(a|b)_{(n)})] \\ \downarrow & & \downarrow \\ H^{*,*}(G, k) & \xrightarrow{\psi_G} & k[\mathbf{V}_{n;m}(G)] \end{array}$$

The upper horizontal map surjects onto p^n th powers by Theorem 9.2, and the right hand vertical map is surjective since $\mathbf{V}_{n;m}(G) \rightarrow \mathbf{V}_{n;m}(\mathrm{GL}(a|b)_{(n)})$ is a closed embedding by [37, Theorem 3.3.6]. It follows that the lower horizontal map surjects onto p^n th powers. $\square$

For the algebra $A = k[t, \tau]/(t^p - \tau^2)$, we have a map

$$(9.4) \quad e : A \rightarrow k\mathbb{M}_n \rightarrow k\mathbb{M}_{n;m}$$

given by $e(\tau) = \sigma$, $e(t) = s_n$. Let $\mathbb{M}_n \rightarrow G_K$ be a homomorphism of supergroup schemes. Since G is unipotent, the group algebra KG_K is finite dimensional and local. Hence, any element in the augmentation ideal is nilpotent. Therefore, the homomorphism factors through some $\mathbb{M}_{n;m}$. We compose to get a π -point of G :

$$e_G : A_K \rightarrow K\mathbb{M}_n \rightarrow K\mathbb{M}_{n;m} \rightarrow KG_K.$$

Thus we have maps

$$\mathbb{P} \mathrm{Hom}_{\mathrm{sgs}/k}(\mathbb{M}_{n;m}, G) \hookrightarrow \mathbb{P} \mathrm{Hom}_{\mathrm{sgs}/k}(\mathbb{M}_n, G) \rightarrow \Pi(G) \xrightarrow{\Phi_G} \mathrm{Proj} H^{*,*}(G, k),$$

where we denote by $\mathbb{P}X$ the projectivization of a conical affine scheme X . It follows from [37, §6.2] that the composition $\mathbb{P} \mathrm{Hom}_{\mathrm{sgs}}(\mathbb{M}_{n;m}, G) \rightarrow \mathrm{Proj} H^{*,*}(G, k)$ is induced by the map Ψ_G .

9.5. Proposition. *The distinguished π -point $e : A \rightarrow k\mathbb{M}_{n;m}$ induces a bijection*

$$e^* : \mathrm{Proj} k[\mathbf{V}_{n;m}(\mathbb{M}_{n;m})] \rightarrow \Pi(\mathbb{M}_{n;m}).$$

Proof. This follows from Lemma 3.3.2 in [37]. $\square$

9.6. Remark. Proposition 9.5 is effectively saying that any equivalence class of π -points of $\mathbb{M}_{n;m}$ has a unique (up to a scalar) representative of the form $\phi \circ e$ where ϕ is an endomorphism of the supergroup scheme $\mathbb{M}_{n;m}$.

This observation almost immediately extends to any finite connected unipotent supergroup scheme G .

9.7. Corollary. *For any finite connected unipotent supergroup scheme G of height n , the π -point $e_G: A \rightarrow kG$ induces a surjection*

$$e_G^*: \text{Proj } k[\mathbf{V}_n(G)] \rightarrow \Pi(G).$$

Proof. Let $\alpha: A_K \rightarrow KG_K$ be a π -point. Then α factors through an elementary supergroup $i: E \hookrightarrow G_K$. We consider the case $E = E_{m,n}^-$, $m \geq 1$, the other three cases are similar.

We factor $\alpha = \iota \circ \alpha': A_K \rightarrow KE_{m,n}^- \rightarrow KG_K$. By Proposition 9.5 there exists an endomorphism $\phi: E_{m,n}^- \rightarrow E_{m,n}^-$ such that α' is equivalent to $\phi \circ e: A_K \rightarrow E_{m,n}^-$ as a π -point of $E_{m,n}^-$. Hence, α is equivalent to $i \circ \phi \circ e = e_G^*(i \circ \phi)$ as a π -point of G . Hence e_G^* is surjective. $\square$

9.8. Corollary. *For G a finite connected unipotent supergroup scheme, the map $\Phi_G: \Pi(G) \rightarrow \text{Proj } H^{*,*}(G, k)$ is injective.*

Proof. Let G be of height n and choose large enough m such that any map $\mathbb{M}_n \rightarrow G$ factors through $\mathbb{M}_{n:m}$. Then the composition

$$\Psi_G: \text{Proj } k[\mathbf{V}_{n:m}(G)] \xrightarrow{e_G^*} \Pi(G) \xrightarrow{\Phi_G} \text{Proj } H^{*,*}(G, k)$$

is bijective by Theorem 9.3 and the first map is surjective by Corollary 9.7. Hence, Φ_G is injective. $\square$

To prove the main result, Theorem 9.14, of this section we wish to extend Corollary 9.8 to all—not necessarily connected—unipotent finite supergroup schemes. The strategy for the reduction from the general to the connected case closely follows [44]. We outline the key steps of the argument citing the proofs from [44] where they apply verbatim.

Let E be an elementary abelian p -group, which we can view as a supergroup scheme concentrated in even degree. Denote by $\sigma_E \in H^*(E, k)$ the cohomology class defined as

$$\sigma_E = \prod_{0 \neq \xi \in H^1(E, \mathbb{F}_p)} \beta(\xi),$$

where β denotes the Bockstein homomorphism. The key property of σ_E is that as a function on $\text{Proj } H^*(E, k)$ it vanishes on any subvariety $\text{Proj } H^*(E', k) \subset \text{Proj } H^*(E, k)$ for a proper subgroup E' of E .

For G a finite supergroup scheme with a group of connected components $\pi_0(G) = E$ an elementary abelian p -group, we have a natural projection $\phi: G \rightarrow E$. We set

$$\sigma_G = \phi^*(\sigma_E) \in H^{*,0}(G, k).$$

The proof of the following result relies on the construction of the *Evens norm* [42] and goes exactly as in [44, 4.4].

9.9. Proposition. *Let $G = G^0 \rtimes \pi$ be a finite supergroup scheme with group of connected components π . Let E be an elementary abelian p -subgroup of π . Then we have an injective map*

$$\text{Spec}(H^{*,*}(G^0 \rtimes E, k)[\sigma_{G^0 \rtimes E}^{-1}]) / N_\pi(E) \rightarrow \text{Spec } H^{*,*}(G, k)$$

induced by the embedding of supergroups $G^0 \rtimes E \hookrightarrow G$. $\square$

9.10. Proposition. [44, 4.5] *Let $G = G^0 \rtimes \pi$ and let $(G^0)^\pi$ be the sub-supergroup scheme of invariants of the connected component G^0 under the action of π . Then the image of the map in cohomology induced by the natural embedding of group schemes $(G^0)^\pi \times \pi \hookrightarrow G^0 \rtimes \pi = G$,*

$$(9.11) \quad H^{*,*}(G, k) \rightarrow H^{*,*}((G^0)^\pi \times \pi, k),$$

contains the $p^{|\pi|}$ th power of every element. $\square$

9.12. Corollary. *Let $G = G^0 \rtimes E$ be a finite supergroup scheme with group of connected components E . The natural map*

$$\mathrm{Proj} H^{*,*}((G^0)^\pi \times E, k) \rightarrow \mathrm{Proj} H^{*,*}(G, k)$$

is an embedding of topological spaces. $\square$

The final preparatory result that we need underlines the connection between rank and support varieties. Let $\alpha: A \rightarrow kG$ be a k -rational π -point and $\mathfrak{p}(\alpha)$ be the corresponding homogeneous prime ideal in $H(G, k)$. Since α is defined over k , $\mathfrak{p}(\alpha)$ defines a closed point on $\mathrm{Proj} H(G, k)$ and we can localise $H^{*,*}(G, M)$ at $\mathfrak{p}(\alpha)$.

9.13. Lemma. *Let G be a unipotent finite supergroup scheme defined over an algebraically closed field k , and M a finite-dimensional kG -module. If α is a k -rational π -point, then $\alpha^*(M)$ has finite flat dimension if and only if $H^{*,*}(G, M)_{\mathfrak{p}(\alpha)} = 0$.*

Proof. This is similar to the second half of the proof of [44, Theorem 4.8]. $\square$

9.14. Theorem. *If G is a unipotent finite supergroup scheme then the map*

$$\Phi_G: \Pi(G) \rightarrow \mathrm{Proj} H^{*,*}(G, k)$$

is bijective. If k is algebraically closed then for any finite-dimensional kG -module M , the map Φ_G takes equivalence classes of k -rational π -points in $\pi\text{-supp}_G(M)$ bijectively onto the closed points of $\mathrm{supp}_G(M)$.

Proof. Surjectivity follows from Theorem 7.3, together with [27, Theorem 6.9] that shows that Φ_G is bijective for G elementary.

We have thus to show that if $\Phi_G(\alpha) = \Phi_G(\beta)$ for π -points α, β , then α is equivalent to β . We first consider the case when $\alpha: A \rightarrow kG$ and $\beta: A \rightarrow kG$ are both defined over the ground field k which we assume to be algebraically closed.

In this case the proof is essentially the same as the proof of Theorem 4.6 of [44] and proceeds in a series of reductions. We sketch the main steps here and refer the reader to [44] for details.

Since k is perfect, we can write $G = G^0 \rtimes \pi$ where G^0 is connected and π is a finite group (the group of connected components of G). By definition, α factors through some elementary supergroup $E_\alpha = E_\alpha^0 \times \pi_0(E_\alpha)$ and similarly β factors through $E_\beta = E_\beta^0 \times \pi_0(E_\beta)$. By choosing representatives α, β within their equivalence classes in such a way that $\pi_0(E_\alpha)$ and $\pi_0(E_\beta)$ are minimal, we can apply Proposition 9.9 to show that $\pi_0(E_\beta)$ is conjugate to $\pi_0(E_\alpha)$ by an element in $N_\pi(\pi_0(E_\alpha))$. Since conjugation preserves equivalence of π -points, we can now assume that both π -points factor through $G^0 \rtimes \pi_0(E_\alpha)$. Hence, we reduce to the case when $G = G^0 \rtimes E$ with the group of connected components elementary abelian.

Since α, β factor through elementary sub-supergroup schemes of $G = G^0 \rtimes E$, they both must factor through $(G^0)^E \times E$. Corollary 9.12 now implies that we can assume that $G = (G^0)^E \times E$ which completes the second reduction step. Finally, since the group algebra of $(G^0)^E \times E$ is isomorphic to a group algebra of a connected unipotent finite supergroup scheme, the statement follows from Corollary 9.8.

The statement about Φ_G identifying equivalence classes of π -points defined over k and closed points of $\mathrm{supp}_G(M)$ is easily seen to be equivalent to Lemma 9.13.

We finally prove injectivity of Φ_G for any π -points.

Let α, β be π -points such that $\mathfrak{p}(\alpha) = \mathfrak{p}(\beta)$. Since extending scalars does not affect $\mathfrak{p}(\alpha)$, we can assume that $\alpha, \beta: A_K \rightarrow KG_K$ are defined over the same field K which is algebraically closed. Suppose α and β are not equivalent. Then there exists a finite dimensional kG -module M such that $\alpha^*(M)$ has infinite flat dimension but $\beta^*(M)$ does not, or vice versa. Let $\mathfrak{p}(\alpha, K)$ be the radical of the kernel of the map $\alpha^*: H(G_K, K) \rightarrow H(A_K, K)$ which is a point in

$\text{Proj } H(G_K, K)$ lying over $\mathfrak{p}(\alpha)$, and similarly for $\mathfrak{p}(\beta, K)$. Since Φ_{G_K} takes π -points defined over K to the K -closed points of $\text{supp}_{G_K}(M_K)$, we conclude that $\mathfrak{p}(\alpha, K) \in \text{supp}_{G_K}(M_K)$. Hence, $\text{Ann}_{H(G_K, K)} H(G_K, M_K) \subset \mathfrak{p}(\alpha, K)$ (see Remark 6.3); and similarly, $\text{Ann}_{H(G_K, K)} H(G_K, M_K) \not\subset \mathfrak{p}(\beta, K)$. But for an ideal $\mathfrak{J}$ of $H(G, k)$ we have $\mathfrak{J} \subset \mathfrak{p}(\alpha) = \mathfrak{p}(\alpha, K) \cap H(G, k)$ if and only if $\mathfrak{J} \subset \mathfrak{p}_{\alpha, K}$. We conclude that $\mathfrak{p}(\alpha) \neq \mathfrak{p}(\beta)$ which is a contradiction. Hence, Φ_G is injective. $\square$

9.15. *Remark.* We can endow $\Pi(G)$ with the structure of the topological space by choosing $\pi\text{-supp}_G(M)$ to be a closed set exactly when M is a finite-dimensional kG -modules. With this topology Φ_G becomes a homeomorphism.

10. STRATIFICATION

Knowing that Φ_G is bijective we can apply the techniques developed for finite group schemes to identify $\pi\text{-supp}$ and supp for *all* kG -modules M , thereby proving the analogue of the result of Avrunin and Scott 3.3 for unipotent supergroup schemes. The arguments for the results stated in this section are similar to the ones for finite group schemes which appear in [24, 45], so we give only a brief outline of how they go, referring the reader to those papers for details.

As always k will be a field of characteristic $p \geq 3$ and G a unipotent finite supergroup scheme over k . We first compute π -supports of the local cohomology modules introduced in § 2. From now on we identify $\text{Proj } H^{*,*}(G, k)$ with $\Pi(G)$, using Theorem 9.14.

10.1. Proposition. *Let G be a unipotent finite supergroup scheme. If V is a specialisation closed subset of $\text{Proj } H^{*,*}(G, k)$ then*

- (i) $\pi\text{-supp}_G(\Gamma_V(k)) = V$,
- (ii) $\pi\text{-supp}_G(L_V(k))$ is the complement of V , and
- (iii) $\pi\text{-supp}_G(\Gamma_{\mathfrak{p}}(k)) = \{\mathfrak{p}\}$.

Proof. The proof of this is the same as in Proposition 6.6 of [45]. For the last statement, we choose specialisation closed subsets V and W as in Definition 2.1, and use the tensor product formula given in Theorem 8.10. $\square$

Given Theorems 7.3 and 8.10, and Proposition 10.1, one can argue as in the proof of [24, Theorem 6.1] to get establish the result below.

10.2. Theorem. *Let G be a unipotent finite supergroup scheme. For any kG -module M one has $\text{supp}_G(M) = \pi\text{-supp}_G(M)$.* $\square$

At this point cosupport can no longer be ignored: We write $\text{cosupp}_G(M)$ for the cosupport of a kG -module M , defined as in 2.2, using the action of $H^{*,*}(G, k)$ on $\text{StMod}(kG)$. The π -cosupport of M consists of equivalence classes of π -points $\alpha: A_K \rightarrow KG_K$ such that $\alpha^*(\text{Hom}_k(K, M))$ has infinite flat dimension; compare with Definition 8.6. It has to be checked that this is well defined; see [27, Theorem 4.12] for the case of elementary supergroup schemes. Then, using [26, Theorem 11.3] one can verify that π -cosupport, like π -support, detects projectivity.

10.3. Theorem. *Let G be a unipotent finite supergroup scheme. A kG -module M is projective if and only if $\pi\text{-cosupp}_G(M) = \emptyset$.* $\square$

Here is the analogue of the tensor-product formula 8.10 for π -support, and proved in the same way: reduce to the super elementary case and apply [27, Theorem 7.6].

10.4. Theorem. *Let G be a unipotent finite supergroup scheme over a field k of characteristic $p \geq 3$, and let M and N be kG -modules. As subsets of $\Pi(G)$ we have*

$$\text{cosupp}_G \text{Hom}_k(M, N) = \pi\text{-supp}_G(M) \cap \pi\text{-cosupp}_G(N). \quad \square$$

It is now a simple matter to establish that $\text{StMod}(kG)$ is stratified by the action of $H^{*,*}(G, k)$. As explained in Section 2 this yields the classification of localising subcategories of $\text{StMod}(kG)$, namely, Theorem 1.1.

10.5. Theorem. *Let G be a unipotent finite supergroup scheme over k . For point $\mathfrak{p}$ in $\text{Proj } H^{*,*}(G, k)$, the localising subcategory $\Gamma_{\mathfrak{p}} \text{StMod}(kG)$ of $\text{StMod}(kG)$ is minimal. In particular, there is a bijection, defined by $\pi\text{-supp}_G(-)$, between localising subcategories of $\text{StMod}(kG)$ and subsets of $\text{Proj } H^{*,*}(G, k)$.*

Proof. Since G is unipotent every localising subcategory of $\text{StMod}(kG)$ is tensor ideal. Thus minimality of $\Gamma_{\mathfrak{p}} \text{StMod}(kG)$ is tantamount to the statement that for any non-projective modules M, N in this subcategory, the kG -module $\text{Hom}_k(M, N)$ is not projective; see [18, Lemma 3.9].

We can verify this in two ways: Using Proposition 10.1, and Theorems 10.3 and 10.4, one can mimic the proof of [23, Theorem 6.1] to prove that

$$\pi\text{-cosupp}_G(M) = \text{cosupp}_G(M) \quad \text{for any } kG\text{-module } M.$$

Fix kG -modules M and N with $\text{supp}_G(M) = \{\mathfrak{p}\} = \text{supp}_G(N)$. Then $\mathfrak{p}$ is also in $\pi\text{-supp}_G(M)$, by Theorem 10.2. Moreover, $\mathfrak{p}$ is in the cosupport of N because $\text{supp}_G(N)$ and $\text{cosupp}_G(N)$ have the same maximal elements [20, Theorem 4.13], and hence also in $\pi\text{-cosupp}_G(N)$. Then using Theorem 10.4 one gets

$$\pi\text{-cosupp}_G \text{Hom}_k(M, N) = \pi\text{-supp}_G(M) \cap \pi\text{-cosupp}_G(N) = \{\mathfrak{p}\}$$

Thus $\text{Hom}_k(M, N)$ is not projective, as desired.

Here is another way, which circumvents the cosupport detection theorem: Let $\mathfrak{m}$ be a closed point of $\text{Proj } H^{*,*}(G, k)$. We argue as in [24, Proposition 6.3]: Choose a finite field extension K of k and a π -point $\alpha: A_K \rightarrow KG$ representing $\mathfrak{m}$. For any kG -module M there is an isomorphism of G_K -modules

$$\text{Hom}_k(K, k) \otimes_k M \cong \text{Hom}_k(K, M)$$

Since the module on the left is a direct sum of copies of M_K , we deduce that $\mathfrak{m}$ is in $\text{supp}_G(M)$ if and only if it is in $\text{cosupp}_G(M)$; this is the crucial observation. For then given non-projective M, N in $\Gamma_{\mathfrak{m}} \text{StMod}(kG)$ it allows us to conclude that $\mathfrak{m}$ is also in $\text{cosupp}_G(N)$, and hence Theorem 10.4 yields

$$\pi\text{-cosupp}_G(\text{Hom}_k(M, N)) = \{\mathfrak{m}\}.$$

Applying the observation once again, we conclude that $\mathfrak{m}$ is in the π -support of $\text{Hom}_k(M, N)$, so it is not projective, by Theorem 8.9.

This settles the desired minimality at closed points in $\text{Proj } H^{*,*}(G, k)$. A technique of reduction to closed points from [24, §8], see also [25, §3], allows us to treat general prime ideals. Note that the extra grading on $H^{*,*}(G, k)$ introduces no new complications because its projective spectrum is the same as that of the singly graded ring $H(G, k)$ discussed in Definition 6.1.

This finishes the proof of the theorem. $\square$

As noted above, one can prove Theorem 10.5 without first establishing that $\pi\text{-cosupp}_G(M) = \text{cosupp}_G(M)$ for a kG -module M . In fact one can use deduce this equality, and hence that π -cosupport detects projectivity, from the theorem above. This is what is done in [24, Part IV], where the reader can find further applications of the theorem.

11. LOCAL DUALITY

Let now G be any (not necessarily unipotent) finite supergroup scheme over a field k of positive characteristic $p \geq 3$. The group algebra kG is Frobenius; thus there is an isomorphism of kG -modules

$$\mathrm{Hom}_k(kG, k) \cong kG \otimes_k \delta_G$$

where δ_G is a one-dimensional kG -module and a super analogue of the modular function for finite group schemes; see [28, §4]. In particular, it has a parity, denoted ϵ_G , that records its internal degree. The isomorphism above leads to a duality on the stable category of G , namely for all M, N in $\mathrm{stmod}(kG)$, there is a natural isomorphism

$$(11.1) \quad \mathrm{Hom}_k(\mathrm{Hom}_{kG}(M, N), k) \cong \mathrm{Hom}_{kG}(N, \Omega(M \otimes_k \delta_G))$$

The isomorphism above can be deduced from Auslander's defect formula; see [25, §4] where this is done for finite group schemes. When G is a finite group, the isomorphism above is nothing but classical Tate duality.

Building on the Tate duality theorem, one can mimic the arguments in [25], or better yet [22] that treats general finite dimensional Gorenstein algebras, to get:

11.2. Theorem. *Set $H(G) := H^{*,*}(G, k)$ and fix $\mathfrak{p}$ in $\mathrm{Proj} H(G)$. The kG -module $\Gamma_{\mathfrak{p}}(\delta_G)$ is a dualising object in $\Gamma_{\mathfrak{p}} \mathrm{StMod}(kG)$, in that, for any kG -module M there is a natural isomorphism*

$$\widehat{\mathrm{Ext}}_{kG}^{i,*}(M, \Gamma_{\mathfrak{p}}(\delta_G)) \cong \mathrm{Hom}_{H(G)}(H^{*-d-i,*+\epsilon_G}(G, M), I(\mathfrak{p}))$$

where $d = \dim H(G)/\mathfrak{p}$ and $I(\mathfrak{p})$ is the injective hull of the residue field at $\mathfrak{p}$. $\square$

From this one gets, for example, a local cohomology spectral sequence:

$$H_{\mathfrak{p}}^{s,t,j} H^{*,*}(G, M)_{\mathfrak{p}} \Rightarrow H_{-s-t-d,j+\epsilon_G}(G, M \otimes \delta_G).$$

discovered by Greenlees in the context of finite groups [47, 48].

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