Combinatorics, Probability and Computing (2021), **30**, pp. 344–359
doi:10.1017/S0963548320000279

CAMBRIDGE
UNIVERSITY PRESS

ARTICLE

Independent dominating sets in graphs of girth five

Ararat Harutyunyan[1,*], Paul Horn[2] and Jacques Verstraete[3,†]

[1]LAMSADE, CNRS, Université Paris-Dauphine, PSL Research University, 75016 Paris, France, [2]Department of Mathematics, University of Denver, CO 80210, USA and [3]Department of Mathematics, University of California at San Diego, 9500 Gilman Drive, La Jolla, CA 92093, USA

*Corresponding author. Email: ararat.harutyunyan@dauphine.fr

(Received 3 November 2008; revised 4 May 2020; accepted 2 August 2012; first published online 15 October 2020)

Abstract

Let $\gamma(G)$ and $\gamma_\circ(G)$ denote the sizes of a smallest dominating set and smallest independent dominating set in a graph G, respectively. One of the first results in probabilistic combinatorics is that if G is an n-vertex graph of minimum degree at least d, then

$$\gamma(G) \leqslant \frac{n}{d}(\log d + 1).$$

In this paper the main result is that if G is any n-vertex d-regular graph of girth at least five, then

$$\gamma_\circ(G) \leqslant \frac{n}{d}(\log d + c)$$

for some constant c independent of d. This result is sharp in the sense that as $d \to \infty$, almost all d-regular n-vertex graphs G of girth at least five have

$$\gamma_\circ(G) \sim \frac{n}{d}\log d.$$

Furthermore, if G is a disjoint union of $n/(2d)$ complete bipartite graphs $K_{d,d}$, then $\gamma_\circ(G) = n/2$. We also prove that there are n-vertex graphs G of minimum degree d and whose maximum degree grows not much faster than $d \log d$ such that $\gamma_\circ(G) \sim n/2$ as $d \to \infty$. Therefore both the girth and regularity conditions are required for the main result.

2020 MSC Codes: Primary: 05C35, 05C69, 05C80, 05D40

1. Introduction

Using so-called semirandom methods, many recent results deal with lower bounds on the size of maximum independent sets in d-regular graphs of girth g. The optimal bounds were found by Shearer [15], who showed that the maximum size of an independent set in a d-regular triangle-free graph is asymptotically at least $(n \log d)/d$. Later, Johansson [10] used semirandom methods to show that d-regular triangle-free graphs actually have chromatic number $O(d/\log d)$. Duckworth and Wormald [4] used the differential equations method [17] to determine lower bounds on the size of a maximum independent set in random d-regular graphs for each fixed d. Lauer and Wormald [11] studied the largest independent set in d-regular graphs of large girth. Gamarnik

†Research supported by an Alfred P. Sloan Research Fellowship and NSF grant DMS 0800704.

and Goldberg [7] also study the question of independent sets in d regular graphs of large girth, in particular studying the performance of a randomized greedy algorithm, thus differing somewhat from the semirandom methods used in this work and by others.

Let $\gamma_\circ(G)$ denote the size of a smallest independent dominating set in a graph G. An early result using the probabilistic method is that every n-vertex graph of minimum degree at least d has a dominating set of size at most $(n/d)(1 + \log d)$. This result is due independently to Arnautov [2], Lovász [12] and Payan [14]. In this paper we prove the following theorem.

Theorem 1.1. *There is a constant $c > 0$ such that, for every d-regular n-vertex graph G of girth at least five,*

$$\gamma_\circ(G) \leqslant \frac{n}{d}(\log d + c).$$

The proof of this theorem actually gives a maximal independent set of size roughly $(n/d)(\log d + c)$, which coincides with Shearer's result for triangle-free graphs. However, in our result the girth five requirement is essential, since a graph G consisting of $n/(2d)$ disjoint copies of the complete bipartite graph $K_{d,d}$, when $2d$ divides n, has $\gamma_\circ(G) = n/2$. Alon, Krivelevich and Sudakov [1] extended the theorem of Johansson to graphs with sparse neighbourhoods. It seems likely that Theorem 1.1 can be extended to cases where the number of common vertices of any pair of vertices is much smaller than d.

It is known that as $d, n \to \infty$, with d growing much more slowly than n (say, $d^5 \ll n$), almost all vertices of a random d-regular n-vertex graph lie in no five cycles and every independent dominating set has size asymptotic to $(n \log d)/d$; see Duckworth and Wormald [4] and Zito [18] for a precise study of independent dominating sets in random regular graphs. Theorem 1.1 is also sharp in the following sense.

Theorem 1.2. *For all $m > 1$, there exists $d_0(m)$ such that if $d \geqslant d_0(m)$, then there exists a graph G of minimum degree d, maximum degree at most $\Delta = md$ and girth at least five such that*

$$\gamma_\circ(G) > \left(1 - \frac{4 \log \Delta}{\Delta^{1/2 - 2/(m-1)}}\right) \frac{|V(G)|}{2\Delta^{2/(m-1)}}.$$

For example, if $m/\log d \to \infty$ as $d \to \infty$, this theorem guarantees graphs G of maximum degree md and minimum degree d such that $\gamma_\circ(G) \sim |V(G)|/2$ (again, as $d \to \infty$). It would be interesting for each $m \geqslant 1$ to determine the best possible upper bound on the smallest independent dominating set in an n-vertex graph G of girth five, minimum degree d and maximum degree md. The above theorem does not give any information for $1 < m \leqslant 5$, since the bound in this range is negative, and new ideas seem to be required to find an analogue of Theorem 1.1 for graphs which are not d-regular. We make the following conjecture.

Conjecture 1.3. *For all $\varepsilon > 0$, $m > 1$ there exists $d_0(\varepsilon, m)$ such that if $d \geqslant d_0(\varepsilon, m)$ and G is a graph of girth at least five, minimum degree d and maximum degree at most $\Delta = md$, then*

$$\gamma_\circ(G) \leqslant \frac{|V(G)|}{2\Delta^{(1-\varepsilon)/m}}.$$

1.1 Notation and terminology

If G is a graph, then for a set $S \subset V(G)$ let ∂S denote the set of vertices in $V(G)\backslash S$ which are adjacent to at least one vertex in S. As in the introduction, $\gamma_\circ(G)$ denotes the size of a smallest independent dominating set in G: this is a set $S \subseteq V(G)$ such that no edge of G joins two vertices of S and $S \cup \partial S = V(G)$.

https://doi.org/10.1017/S0963548320000279 Published online by Cambridge University Press

1.2 Organization

The rest of the paper is organized as follows. In Section 2 we define a random process by which an independent dominating set of a d-regular graph of girth five is constructed. The analysis of the process is in Section 3, where we use probabilistic tools (Appendix) to control the degrees of vertices at each stage. The proof of Theorem 1.2 is in Section 4.

2. The process

For an n_0-vertex d_0-regular graph G_0 of girth at least five, a natural way to build an independent dominating set in stages is to select vertices independently and randomly with an appropriate probability. Let S_t be the set of selected vertices at stage t. The set Z_t of selected vertices in the graph G_t which are not adjacent to any other selected vertices are added to the current independent set $Z_0 \cup Z_1 \cup \cdots \cup Z_{t-1}$, and then $Z_t \cup \partial Z_t$ is deleted from G_t to obtain the graph G_{t+1}. The idea is to show that in the remaining graph G_t at each stage t, the degrees of vertices are all roughly the same with positive probability; specifically, the degrees all decrease by a factor roughly $e^{-1/e}$ at each stage with positive probability. To show that this is true requires concentration of degrees of the vertices at each stage. Unfortunately this is not sufficient, since the expected degrees begin to vary substantially if the above process is followed. To fix this problem, we equalize the degrees of the vertices at each stage by putting vertices randomly and independently into an auxiliary set W_t. Another technical consideration is that the random process stops when the degrees of the vertices become too small. We will stop the process at time $T = \lfloor e(\log d_0 - c) \rfloor$ where $c = 2^{100}$.

2.1 Statement of the process

We start with a d_0-regular n_0-vertex graph G_0 of girth at least five. Let $Y_0 = \emptyset$ and $X_0 = V(G_0)$. Having defined graphs G_i, independent sets Z_i and partitions $V(G_i) = X_i \cup Y_i$ for $i < t$, let $d_t = d_0 \prod_{i=1}^{t} \omega_i$ and $n_t = n_0 \prod_{i=1}^{t} \omega_i$, where

$$\sigma_t^2 := 10^5 d_t (\log d_t)^5, \tag{2.1}$$

$$\omega_t := e^{-1/e}\left(1 - \frac{\sigma_{t-1}}{d_{t-1}}\right). \tag{2.2}$$

At stage t, we randomly and independently select vertices from X_{t-1} with probability $1/d_{t-1}$ and let S_t be the set of *selected vertices* of X_{t-1}. Let $Z_t \subseteq S_t$ be the set of selected vertices which have no selected neighbours. Then place vertices $v \in X_{t-1}$ in a set W_t independently with probability $\omega_t(v)$ chosen so that

$$\mathbb{P}(v \notin \partial Z_t \cup W_t) = \mathbb{P}(v \notin \partial Z_t)(1 - \omega_t(v)) = \omega_t. \tag{2.3}$$

The choice of weights $\omega_t(v)$ is made to equalize all the expected degrees of vertices in the graph at stage t, so that they are all roughly d_t. It will be seen that $\mathbb{P}(v \notin \partial Z_t) \geqslant \omega_t$, so that $\omega_t(v)$ is well-defined. Then define

$$X_t := X_{t-1} \backslash (W_t \cup Z_t \cup \partial Z_t), \tag{2.4}$$

$$Y_t := (Y_{t-1} \cup W_t) \setminus \partial Z_t. \tag{2.5}$$

We stop the process when $\log d_{t+1} \leqslant 2^{100}$. Since $d_t \leqslant e^{-t/e} d_0$, this occurs at some time $T \leqslant \lfloor e(\log d_0 - c) \rfloor$, where $c = 2^{100}$. We make no attempt to find the smallest value of c for which our analysis still works.

https://doi.org/10.1017/S0963548320000279 Published online by Cambridge University Press

2.2 Control of degrees and sets

For $t \leqslant T$ and $v \in V(G_{t-1})\backslash Z_t$, let $X_{v,t}$ and $Y_{v,t}$ denote the number of neighbours of v in X_t and Y_t respectively. We shall show that with positive probability, for all $t \leqslant T$ and all $v \in V(G_{t-1})\backslash Z_t$,

$$|X_{v,t} - d_t| \leqslant \sigma_t, \tag{2.6}$$

$$Y_{v,t} \leqslant 100\sigma_t. \tag{2.7}$$

We will use martingales and the Lovász Local Lemma [5] to prove these statements. It will then be shown that, for $t \leqslant T$,

$$|X_t| < n_t + \frac{100\sigma_t n_t}{d_t}, \tag{2.8}$$

$$|Y_t| < \frac{200\sigma_t n_t}{d_t}, \tag{2.9}$$

$$|Z_t| < \frac{n_t}{ed_t} + 200\frac{\sigma_t n_t}{d_t^2}. \tag{2.10}$$

2.3 Proof of Theorem 1.1

The proof of Theorem 1.1 follows from the fact that (2.8)–(2.10) hold for $t \leqslant T$. That is,

$$\begin{aligned}\sum_{t=0}^{T-1}|Z_t| &< \frac{n_0 T}{ed_0} + \frac{200 n_0}{d_0}\sum_{t=0}^{T-1}\frac{\sigma_t}{d_t}\\ &< \frac{n_0 \log d_0}{d_0} - \frac{n_0 c}{d_0} + \frac{200\cdot 10^{5/2} n_0}{d_0}\sum_{t=0}^{T-1}\frac{(\log d_t)^{5/2}}{d_t^{1/2}}\\ &< \frac{n_0\log d_0}{d_0},\end{aligned}$$

where in the last line we have used the facts that $d_{T-1} > c = 2^{100}$ and that d_t grows exponentially with decreasing t to deduce that

$$\sum_{i=0}^{T-1}\frac{(\log d_t)^{5/2}}{d_t^{1/2}} < 1.$$

Let Z be a maximal independent set in $X_T \cup Y_T$. Using (2.8) and (2.9), we have

$$\begin{aligned}|Z| &\leqslant |X_T| + |Y_T|\\ &\leqslant n_T + \frac{300\sigma_T n_T}{d_T}\\ &= \frac{n_T}{d_T}(d_T + 300\sigma_T)\\ &< \frac{2cn_0}{d_0},\end{aligned}$$

where in the last line we used the fact that $d_T + 300\sigma_T < 2d_T < 2c$.

Combining all the bounds, we obtain an independent dominating set $Z_0 \cup Z_1 \cup \cdots \cup Z_{T-1} \cup Z$ of size less than

$$\frac{n_0(\log d_0 + 2c)}{d_0}.$$

This completes the proof of Theorem 1.1 provided we can show that (2.6)–(2.10) hold for $t \leqslant T$.

3. Analysis of degrees

In this section we prove that, for any given vertex v, (2.6) and (2.7) hold with high probability at stage t, assuming they hold for all vertices at stage $t-1$.

Lemma 3.1. *Let $t \leqslant T$ and $v \in V(G_{t-1})$. Suppose (2.6) holds at time $t-1$. Then*

$$\left(1-\frac{\sigma_{t-1}}{d_{t-1}}\right)\cdot\left(\frac{1}{1-1/(d_{t-1})}\right) \leqslant e^{1/e}\mathbb{P}(v \notin \partial Z_t \mid v \notin S_t) \leqslant 1+\frac{\sigma_{t-1}}{d_{t-1}}.$$

Proof. Write $u \leftrightarrow w$ to mean that u and w are adjacent vertices in G_{t-1}. For convenience put $d = d_{t-1}$ and $\sigma = \sigma_{t-1}$. For our coming computations, we recall that since $t \leqslant T$, d and, consequently, σ are large constants. First note that

$$\mathbb{P}(v \notin \partial Z_t \mid v \notin S_t) = \prod_{u \in X_{v,t-1}} \mathbb{P}(u \notin Z_t) = \prod_{u \in X_{v,t-1}} \left(1 - \frac{1}{d}\prod_{\substack{w \in X_{u,t-1}\\ w \neq v}}\left(1-\frac{1}{d}\right)\right).$$

By (2.6), the products have at least $d-\sigma-1$ and at most $d+\sigma$ terms, respectively. Then

$$\begin{aligned}\log \mathbb{P}(v \notin \partial Z_t \mid v \notin S_t) &\leqslant (d-\sigma-1)\cdot\log\left(1-\frac{1}{d}\left(1-\frac{1}{d}\right)^{d+\sigma}\right)\\ &\leqslant -\left(1-\frac{\sigma+1}{d}\right)\cdot\left(1-\frac{1}{d}\right)^{d+\sigma}\end{aligned}$$

using the inequality $\log(1-x) \leqslant -x$ for $x<1$. Also, since

$$\frac{1}{e} \leqslant \left(1-\frac{1}{x}\right)^{x-1} \qquad \text{for } x>1,$$

we obtain

$$\begin{aligned}\log \mathbb{P}(v \notin \partial Z_t \mid v \notin S_t) &\leqslant -\frac{1}{e}\cdot\left(1-\frac{\sigma+1}{d}\right)\cdot\left(1-\frac{1}{d}\right)^{\sigma+1}\\ &\leqslant -\frac{1}{e}\left(1-\frac{\sigma+1}{d}\right)^2\\ &\leqslant -\frac{1}{e}+\log\left(1+\frac{\sigma}{d}\right)\end{aligned}$$

for $t \leqslant T$. This proves the upper bound. Now we prove the lower bound. We will use the inequalities $\log(1-x) \geqslant -x-x^2$, which holds for all $x \in (0,0.5)$, and $1-x<e^{-x}$, which holds for all x. That is,

$$\begin{aligned}\log \mathbb{P}(v \notin \partial Z_t \mid v \notin S_t) &\geqslant (d+\sigma)\cdot\log\left(1-\frac{1}{d}\left(1-\frac{1}{d}\right)^{d-\sigma-1}\right)\\ &\geqslant (d+\sigma)\left(-\frac{1}{d}\left(1-\frac{1}{d}\right)^{d-\sigma-1}-\frac{1}{d^2}\left(1-\frac{1}{d}\right)^{2(d-\sigma-1)}\right)\\ &\geqslant -(d+\sigma)\left(\frac{1}{d}\cdot e^{-1}\cdot\frac{1}{(1-1/d)^{\sigma+1}}+\frac{1}{d^2}\cdot e^{-2}\cdot\frac{1}{(1-1/d)^{2(\sigma+1)}}\right)\\ &\geqslant -(d+\sigma)\left(\frac{1}{d}\cdot e^{-1}\cdot\frac{1}{1-(\sigma+1)/d}+\frac{1}{d^2}\cdot e^{-2}\cdot\frac{1}{1-(2(\sigma+1))/d}\right)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{e}\left(1+\frac{2\sigma+1}{d-\sigma-1}+e^{-1}\left(1+\frac{\sigma}{d}\right)\cdot\frac{1}{d-2\sigma-2}\right)\\
&> -\frac{1}{e}\left(1+\frac{2\sigma+2}{d-2\sigma-2}\right)\\
&> -\frac{1}{e}-\frac{\sigma-1}{d-1}\\
&> -\frac{1}{e}+\log\left(\left(1-\frac{\sigma}{d}\right)\left(\frac{1}{1-1/d}\right)\right). \qquad \square
\end{aligned}
$$

From this, note that

$$
\begin{aligned}
\mathbb{P}(v\notin\partial Z_t) &= \mathbb{P}(v\notin\partial Z_t \mid v\notin S_t)\mathbb{P}(v\notin S_t)+\mathbb{P}(v\notin\partial Z_t \mid v\in S_t)\mathbb{P}(v\in S_t)\\
&\geqslant e^{-1/e}\left(1-\frac{\sigma_{t-1}}{d_{t-1}}\right)\\
&= \omega_t. \qquad (3.1)
\end{aligned}
$$

Therefore $\omega_t(v)$ is well-defined by (2.3).

Similarly,

$$
\mathbb{P}(v\notin\partial Z_t)\leqslant e^{-1/e}\left(1+\frac{\sigma_{t-1}}{d_{t-1}}\right)\left(1-\frac{1}{d_{t-1}}\right)+\frac{1}{d_{t-1}}.
$$

This last bound allows us to give an upper bound on $\omega_t(v)$:

$$
\begin{aligned}
\omega_t(v) &= 1-\frac{\omega_t}{\mathbb{P}(v\notin\partial Z_t)}\\
&\leqslant 1-\left(e^{-1/e}\left(1-\frac{\sigma_{t-1}}{d_{t-1}}\right)\right)\Big/\left(e^{-1/e}\left(1+\frac{\sigma_{t-1}}{d_{t-1}}\right)+\frac{1}{d_{t-1}}\right)\\
&= 1-\frac{e^{-1/e}(d_{t-1}-\sigma_{t-1})}{e^{-1/e}(d_{t-1}+\sigma_{t-1})+1}\\
&= \frac{2e^{-1/e}\sigma_{t-1}+1}{e^{-1/e}(d_{t-1}+\sigma_{t-1})+1}\\
&\leqslant \frac{2\sigma_{t-1}}{d_{t-1}}.
\end{aligned}
$$

Thus

$$
\omega_t(v)\leqslant\frac{2\sigma_{t-1}}{d_{t-1}}. \qquad (3.2)
$$

3.1 Expected degrees

Lemma 3.1 allows us to estimate $\mathbb{E}(X_{v,t})$ and $\mathbb{E}(Y_{v,t})$.

Lemma 3.2. *Let $t\leqslant T$ and $v\in V(G_{t-1})$. Suppose that (2.6) and (2.7) hold at stage $t-1$. Then*

$$
|\mathbb{E}(X_{v,t})-d_t|<0.9\sigma_t, \qquad (3.3)
$$

$$
\mathbb{E}(Y_{v,t})<90\sigma_t. \qquad (3.4)
$$

https://doi.org/10.1017/S0963548320000279 Published online by Cambridge University Press

Proof. By definition,

$$\mathbb{E}(X_{v,t}) = \sum_{u \in X_{v,t-1}} \mathbb{P}(u \notin \partial Z_t \cup W_t) = \omega_t X_{v,t-1}.$$

Using the assumption $|X_{v,t-1} - d_{t-1}| < \sigma_{t-1}$, we easily obtain for $t \leqslant T$

$$|\mathbb{E}(X_{v,t}) - d_t| = |\omega_t X_{v,t-1} - d_t| = \omega_t |X_{v,t-1} - d_{t-1}| < \omega_t \sigma_{t-1} < 0.9\sigma_t.$$

This is enough for (3.3). Next we turn to $\mathbb{E}(Y_{v,t})$. We write $Y_{v,t} = W_{v,t} + U_{v,t}$, where $W_{v,t}$ is the number of neighbours of v in W_t, and $U_{v,t}$ is the number of neighbours of v in $Y_t \backslash W_t$. $W_{v,t}$ reflects the new neighbours of v in Y_t, while the change in $U_{v,t}$ reflects that some neighbours of v in Y_{t-1} are in ∂Z_t. Since

$$0 \leqslant \omega_t(u) \leqslant \frac{2\sigma_{t-1}}{d_{t-1}} \quad \text{for all } u \in V(G_{t-1}),$$

we obtain

$$0 \leqslant \mathbb{E}(W_{v,t}) \leqslant \frac{2\sigma_{t-1} X_{v,t-1}}{d_{t-1}} < 2\sigma_{t-1} + \frac{2\sigma_{t-1}^2}{d_{t-1}}.$$

Then, summing over $u \in Y_{t-1}$ with $u \leftrightarrow v$, by Lemma 3.1 we get

$$\mathbb{E}(U_{v,t}) = \sum_{\substack{u \leftrightarrow v \\ u \in Y_{t-1}}} \mathbb{P}(u \notin \partial Z_t) \leqslant \left(e^{-1/e}\left(1 + \frac{\sigma_{t-1}}{d_{t-1}}\right) + \frac{1}{d_{t-1}} \right) Y_{v,t-1}.$$

Finally, since $Y_{v,t} = U_{v,t} + W_{v,t}$ and $Y_{v,t-1} < 100\sigma_{t-1}$ by assumption,

$$\begin{aligned} \mathbb{E}(Y_{v,t}) &< \left(e^{-1/e}\left(1 + \frac{\sigma_{t-1}}{d_{t-1}}\right) + \frac{1}{d_{t-1}} \right) Y_{v,t-1} + 2\sigma_{t-1} + \frac{2\sigma_{t-1}^2}{d_{t-1}} \\ &< 100\left(\left(1 + \frac{\sigma_{t-1}}{d_{t-1}}\right) e^{-1/e} + \frac{1}{d_{t-1}} \right) \sigma_{t-1} + 2\sigma_{t-1} + \frac{2\sigma_{t-1}^2}{d_{t-1}} \\ &< 90\sigma_t. \end{aligned}$$

These inequalities are contingent on $t \leqslant T$. This completes the proof. □

Remark 3.1. The identities for $\mathbb{E}(X_{v,t})$ and $\mathbb{E}(U_{v,t})$ in the proof of this lemma are crucial. If we did not create the set W_t to equalize expected degrees, then without further analysis we could have vertices v such that $|\mathbb{E}(X_{v,t}) - d_t| > 2e^{-1/(2e)}\sigma_t$, which is problematic since $2e^{-1/(2e)} > 1$. Indeed, in such a case the error terms grow exponentially. This may lead to a situation where, for t large enough (but much smaller than $\lfloor e(\log d_0 - c) \rfloor$ where our process ends), X_t contains much more than $(n_0 \log d_0)/d_0$ vertices of small constant degree. In such a case every maximal independent set in X_t might be much larger than the $(n_0 \log d_0)/d_0$ sized independent dominating set whose existence is posited by Theorem 1.1.

3.2 Concentration of degrees

In this section we show that $X_{v,t}$ is highly concentrated near its expected value, and $Y_{v,t} < 100\sigma_t$ with high probability.

Lemma 3.3. *For $t \leqslant T$ and all $v \in V(G_{t-1}) \backslash Z_t$, if (2.6)–(2.7) hold at stage $t-1$, then*

$$\mathbb{P}(|X_{v,t} - d_t| > \sigma_t) < d_{t-1}^{-9},$$
$$\mathbb{P}(Y_{v,t} > 100\sigma_t) < d_{t-1}^{-9}.$$

The proofs of both inequalities are similar, and are centred around the use of a martingale concentration inequality of Shamir and Spencer [16] (Proposition A.3 in the Appendix). Throughout the proof we fix a v which has neighbourhood $\Gamma(v) = \{u_1, u_2, \ldots, u_k\}$ in X_{t-1}. Let $\Gamma^+(x)$ denote the set of vertices $y \in \Gamma(x)$ at greater distance from v than x. We let χ denote the indicator function and let $\chi(x) := \chi(x \in S_t)$. The event $x \in S_t$ means x is *selected*. And finally, $\mathsf{w}(x)$ will denote the indicator for the event that x was placed in W_t. We say that u_i *survives* if u_i is not in W_t, and for every $x \in \Gamma^+(u_i)$, either x is not selected or x is selected and at least one $y \in \Gamma(x) \setminus \{u_i\}$ is also selected. In terms of characteristic functions, we may write the latter event in terms of x and y as $\chi(x) - \chi(x)\chi(y) = 0$. We let Σ_i be the indicator that u_i survives, so that

$$\Sigma_i = (1 - \mathsf{w}(u_i)) \cdot \prod_{x \in \Gamma^+(u_i)} \left(1 - \prod_{y \in \Gamma(x) \setminus \{u_i\}} (\chi(x) - \chi(x)\chi(y))\right). \tag{3.5}$$

The key to proving Lemma 3.3 is to show that $\alpha_{v,t} = \sum_{i=1}^k \Sigma_i$ is the final state of a martingale α whose difference sequence is very unlikely to be large at any time. Note that $\alpha_{v,t}$ is not the same as $X_{v,t}$, because $\alpha_{v,t}$ ignores the fact when a u_i and a neighbour of u_i are selected. Thus we will further show that $|\alpha_{v,t} - X_{v,t}| \leqslant 10 \log d$ with high probability. Define

$$C_i = \{\mathsf{w}(u_i), \chi(x), \chi(x) \cdot \chi(y) \colon x \in \Gamma^+(u_i), y \in \Gamma^+(x)\}$$

and define the σ-field $F_j = \sigma(C_1 \cup \cdots \cup C_j)$. Then the martingale α is defined by

$$\alpha_j = \sum_{i=1}^k \mathbb{E}(\Sigma_i \mid F_j).$$

Then $\alpha_{v,t} = \alpha_k = \sum_{i=1}^k \Sigma_i$. We note that C_i does not include terms $\chi(x)\chi(y)$ with both x and y at distance two from v, so F_i does not in general determine Σ_i. Nevertheless, as $\chi(x)$ is revealed for all vertices within distance two of v by the last F_k, we have that F_k indeed determines all the Σ_i, $1 \leqslant i \leqslant k$. The central part of the proof of Lemma 3.3 is the following.

Lemma 3.4. *Let $r = (2 \log d)^2$. If $|\alpha_j - \alpha_{j-1}| > r$, then some vertex at distance at most three from v has more than $\log d$ selected neighbours.*

Proof. Throughout the proof, we let

$$p = \frac{1}{d_{t-1}} = \mathbb{E}[\chi(x)]$$

denote the probability that a vertex is selected.

Fix $j \geqslant 1$. We wish to bound

$$\alpha_j - \alpha_{j-1} = \mathbb{E}(\Sigma_j \mid F_j) - \mathbb{E}(\Sigma_j \mid F_{j-1}) + \sum_{i \neq j} (\mathbb{E}(\Sigma_i \mid F_j) - \mathbb{E}(\Sigma_i \mid F_{j-1})).$$

First we refine the filter. Suppose $\Gamma^+(u_j) = \{x_1, x_2, \ldots, x_\ell\}$ and $\Gamma^+(x_i) = \{y_{i1}, y_{i2}, \ldots, y_{im_i}\}$ for $1 \leqslant i \leqslant \ell$. Order the random variables in C_j as follows: first $\mathsf{w}(u_j)$, and then $\chi(x_1)$ and the variables $\chi(x_1)\chi(y_{11}), \chi(x_1)\chi(y_{12}), \ldots, \chi(x_1)\chi(y_{1m_1})$ followed by $\chi(x_2)$ then $\chi(x_2)\chi(y_{21})$, $\chi(x_2)\chi(y_{22}), \ldots, \chi(x_2)\chi(y_{2m_2})$, and so on until $\chi(x_\ell)$ and $\chi(x_\ell)\chi(y_{\ell 1}), \ldots, \chi(x_\ell)\chi(y_{\ell m_\ell})$.

If $s = |C_j|$, consider the σ-fields $G_0, G_1, \ldots, G_s$, where G_m is the σ-field generated by F_{j-1} and the first m random variables in our ordering. Note that $G_0 = F_{j-1}$ and $F_j = G_s$. Then

$$\sum_{i \neq j} \mathbb{E}(\Sigma_i \mid F_j) - \mathbb{E}(\Sigma_i \mid F_{j-1}) = \sum_{m=1}^s \sum_{i \neq j} \mathbb{E}(\Sigma_i \mid G_m) - \mathbb{E}(\Sigma_i \mid G_{m-1}).$$

https://doi.org/10.1017/S0963548320000279 Published online by Cambridge University Press

We wish to bound each $\Delta_{ijm} := \mathbb{E}(\Sigma_i \mid G_m) - \mathbb{E}(\Sigma_i \mid G_{m-1})$ where $i \neq j$. Note that for $m = 1$ we have $\Delta_{ijm} = 0$. Now suppose $m \geqslant 2$. A vertex x is said to be *exposed* at time m if $\mathbb{E}(\chi(x) \mid G_m) \in \{0, 1\}$.

Case 1: We consider first $G_m = \sigma(G_{m-1}, \chi(x))$ where $x \in \Gamma^+(u_j)$. If $\Gamma(x) \cap \Gamma^+(u_i) = \emptyset$, then $\Delta_{ijm} = 0$. Now suppose x^* is a neighbour of x in $\Gamma^+(u_i)$; since G has no cycles of length four, x^* is unique. In that case, we have from (3.5) that

$$|\Delta_{ijm}| \leqslant |\mathbb{E}(\chi(x^*) - \chi(x)\chi(x^*) \mid G_m) - \mathbb{E}(\chi(x^*) - \chi(x)\chi(x^*) \mid G_{m-1})|.$$

If $i < j$, then x^* is already exposed at time $m - 1$, and so $\Delta_{ijm} = 0$ when $i < j$ and $\chi(x^*) = 0$. If $i < j$ and $\chi(x^*) = 1$, then

$$|\Delta_{ijm}| \leqslant \begin{cases} p & \text{if } \chi(x) = 0, \\ 1 & \text{if } \chi(x) = 1. \end{cases}$$

If $i > j$, then x^* is not yet exposed. In that case,

$$|\Delta_{ijm}| \leqslant \begin{cases} p^2 & \text{if } \chi(x) = 0, \\ p & \text{if } \chi(x) = 1. \end{cases}$$

This completes Case 1.

Case 2: The second case is $G_m = \sigma(G_{m-1}, \chi(x)\chi(y))$ where $x \in \Gamma^+(u_j)$ and $y \in \Gamma^+(x)$. First, note that if $\chi(x) = 0$, then $\Delta_{ijm} = 0$, since if x is not selected, then $\chi(x)\chi(y)$ reveals no information about y. This is the key to the proof, and the reason why we use the particular filtration which we use. Suppose $\chi(x) = 1$. If $i < j$, then $\mathbb{E}(\Sigma_i \mid G_m) = \mathbb{E}(\Sigma_i \mid G_{m-1})$. So we may suppose that $i > j$ and $\chi(x) = 1$. Note that the vertex y is adjacent to at most one vertex $x^* \in \Gamma^+(u_i)$, and this vertex is not yet exposed. We get

$$|\Delta_{ijm}| \leqslant \begin{cases} p & \text{if } \chi(y) = 1, \\ p^2 & \text{if } \chi(y) = 0. \end{cases} \tag{3.6}$$

This completes Case 2.

Suppose, for a contradiction, that no vertex within distance three of v has more than $M = \log d$ selected neighbours. Let us count how many times each of $1, p$ and p^2 appear as our best possible bound in our bounds on Δ_{ijm}. Note that in all cases $|\Delta_{ijm}| \leqslant 1$.

We have that $\Delta_{ijm} \leqslant p$ unless $G_m = \sigma(G_{m-1}, \chi(x))$ where $\chi(x) = 1$, and $i < j$. Furthermore, $\Delta_{ijm} = 0$ unless the common neighbour of x and u_i, x^*, has $\chi(x^*) = 1$. Therefore $|\Delta_{ijm}| > p$ at most M^2 times; there are at most M selected neighbours x of u_j such that $\chi(x) = 1$ and each of these has at most M selected neighbours adjacent to some u_i with $i < j$.

The bound $|\Delta_{ijm}| \leqslant p$ is our best bound if $G_m = \sigma(G_{m-1}, \chi(x))$ and either $i < j$ with $\chi(x) = 0$, and the unique common neighbour x^* of x and u_i has $\chi(x^*) = 1$, or $i > j$ and $\chi(x) = 1$. Since the degree of any vertex is less than $2d$ and no vertex has more than M selected neighbours, neither bound is our best more than $2dM$ times. The bound $|\Delta_{ijm}| \leqslant p$ is also the best bound if $G_m = \sigma(G_{m-1}, \chi(x)\chi(y))$ where $\chi(x)\chi(y) = 1$ and $i > j$. Note that there are at most M^2 edges with $\chi(x)\chi(y) = 1$, for each u_i. Again, since the degree of v is at most $2d$, this bound is best possible no more than $2dM^2$ times. In all, p is the best bound for $|\Delta_{ijm}|$ no more than $4dM + 2dM^2$ times.

The bound $|\Delta_{ijm}| \leqslant p^2$ is the best bound if $G_m = \sigma(G_{m-1}, \chi(x))$ with $\chi(x) = 0$ and $i > j$. Using the fact that both v and u_j have maximum degree $2d$, this occurs at most $4d^2$ times. It is also the best bound if $i > j$ and $G_m = \sigma(G_{m-1}, \chi(x)\chi(y))$ with $\chi(x)\chi(y) = 0$ but $\chi(x) = 1$. Since at most M neighbours of u_j are selected and every vertex has degree at most $2d$, there are at most $2dM$ such

https://doi.org/10.1017/S0963548320000279 Published online by Cambridge University Press

edges, and each for at most $2d$ different $|\Delta_{ijm}|$s. In total, p^2 is the best bound for $|\Delta_{ijm}|$ no more than $4d^2 + 4Md^2$ times.

In all other cases, $\Delta_{ijm} = 0$.

In total:

$$\begin{aligned}|\alpha_j - \alpha_{j-1}| &\leqslant 1 + \sum_{i\neq j}\sum_t |\Delta_{ijm}| \\ &\leqslant 1 + M^2 + p(4dM + 2dM^2) + p^2(4d^2 + 4Md^2) \\ &= 1 + M^2 + 4M + 2M^2 + 4 + 4M \\ &= 5 + 8M + 3M^2 \\ &\leqslant r.\end{aligned}$$

Here, the initial one comes from the fact that

$$|\mathbb{E}(\Sigma_j|F_j) - \mathbb{E}(\Sigma_j|F_{j-1})| \leqslant 1.$$

This contradiction completes the proof. □

Proof of Lemma 3.3. Let $(\alpha_j)_{j=0}^k$ be the martingale described above, where $k = |\Gamma(v)|$. Since there are at most $(2d)^3$ vertices at distance at most three from v, we have with $r = (2\log d)^2$

$$\mu := \sum_{j=0}^{k-1} \mathbb{P}(|\alpha_{j+1} - \alpha_j| > r) < |\Gamma(v)|(2d)^3 \cdot \binom{2d}{r}\left(\frac{1}{d}\right)^r < |\Gamma(v)|(2d)^3\left(\frac{2e}{r}\right)^r < d^{-40}.$$

Here $(2d)^3 \cdot \binom{2d}{r}(1/d)^r$ gives an upper bound on the probability that some vertex at distance at most 3 from v has at least r selected neighbours. So $(\alpha_i)_{i=0}^k$ is r-Lipschitz with exceptional probability at most $\mu := d^{-40}$. Note also that, on the event that $v \in V(G_{t-1}) \setminus Z_t$, $|\alpha_k - X_{v,t}|$ is bounded by the number of vertices in $\Gamma(v)$ which are selected. Now, using (2.6) at time $t-1$, it follows by the Chernoff bounds (noting that vertices are in S_t independently) that

$$\mathbb{P}(|\Gamma(v) \cap S_t| > 10\log d) < d^{-10}.$$

Thus we have that

$$\mathbb{P}(|\alpha_k - X_{v,t}| > 10\log d) < d^{-10}.$$

Finally

$$\begin{aligned}|\alpha_0 - \mathbb{E}(X_{v,t})| &\leqslant \mathbb{E}(|\alpha_k - X_{v,t}|) \\ &= \mathbb{E}(|\alpha_k - X_{v,t}| \mid v \notin Z_t)\mathbb{P}(v \notin Z_t) + \mathbb{E}(|\alpha_k - X_{v,t}| \mid v \in Z_t)\mathbb{P}(v \in Z_t) \\ &\leqslant \mathbb{E}|\Gamma(v) \cap S_t| + (2d_{t-1}) \cdot \frac{1}{d_{t-1}} \\ &\leqslant 2 + 2 \\ &\leqslant 0.01\sigma_t,\end{aligned}$$

as $\mathbb{P}(v \in Z_t) \leqslant \mathbb{P}(v \in S_t) = 1/d$.

Let $\lambda := 0.08\sigma_t$. By Proposition A.3 and Lemma 3.2,

$$\begin{aligned}\mathbb{P}(|X_{v,t} - d_t| > \sigma_t) &\leqslant \mathbb{P}(|X_{v,t} - \mathbb{E}X_{v,t}| > 0.1\sigma_t) \\ &\leqslant \mathbb{P}(|X_{v,t} - \alpha_0| > 0.09\sigma_t) \\ &\leqslant \mathbb{P}(|\alpha_k - \alpha_0| > 0.09\sigma_t - 10\log d) + \mathbb{P}(|\alpha_k - X_{v,t}| > 10\log d)\end{aligned}$$

$$\leqslant \mathbb{P}(|\alpha_k - \alpha_0| > \lambda + k^2\mu^{1/2}) + d^{-10}$$
$$\leqslant 2\exp\left(-\frac{\lambda^2}{2kr^2}\right) + 5d^{-10}, \tag{3.7}$$

where we used the fact that $\mu = d^{-40}$ is a bound on the exceptional probability as above, and the fact that k and $|\alpha_k - \alpha_0|$ are both at most $2d$. For $t \leqslant T$ we easily have $\lambda^2 \geqslant 64 \cdot 10d(\log d)^5$ whereas $4dr^2 < 64d(\log d)^4$. Therefore the above probability is less than $2d^{-10} + 5d^{-10} < d^{-9}$ for $t \leqslant T$.

For $Y_{v,t}$, we recall that $Y_{v,t} = U_{v,t} + W_{v,t}$, where $W_{v,t}$ is the number of neighbours of v in W_t and $U_{v,t}$ is the number of neighbours of $v \in Y_t \setminus W_t$. In this case $W_{v,t}$ is bounded by the sum of independent indicators: $W_{v,t} \leqslant \sum_{u \leftrightarrow v} \chi(u)$, where $\chi(u)$ is the indicator random variable of u being selected to be in the set $W_{v,t}$. Then, as seen in the proof of Lemma 3.2,

$$\mathbb{E}(W_{v,t}) \leqslant \sum_{u\leftrightarrow v} \mathbb{E}(\chi(u)) \leqslant 2\sigma_{t-1} + \frac{2\sigma_{t-1}^2}{d_{t-1}}.$$

The Chernoff bounds then imply that

$$\mathbb{P}(W_{v,t} > \mathbb{E}(W_{v,t}) + \sigma_t) \leqslant \exp\left(-\frac{\sigma_t^2}{2(2\sigma_{t-1} + 2\sigma_{t-1}^2/d_{t-1} + \sigma_t/3)}\right) \leqslant d_{t-1}^{-10}.$$

for $t \leqslant T$.

Concentration for $U_{v,t}$ is nearly precisely the same as concentration of $X_{v,t}$ with one slight simplification: in the $X_{v,t}$ case we were required to define random variables Σ_i which were agnostic to the selection of v and its neighbours. This is not necessary here, only the realization that $U_{v,t} = \sum_{u\sim v} \chi(u \notin \partial Z_t)$, where the sum is taken over $u \in Y_{v,t-1}$. For $U_{v,t}$ we use the martingale $(\beta_j)_{j=0}^k$ defined by $\beta_j = \mathbb{E}(U_{v,t} \mid F_j)$ for $j = 1, 2, \ldots, k$ and $\beta_0 = \mathbb{E}(U_{v,t})$, and where the F_j are defined exactly as above (immediately prior to the proof of Lemma 4), only with the $\{u_i\}$ denoting the neighbours of v in Y_{t-1}. Identically as in the proof of Lemma 3.4, with the random variables $\chi(u \notin \partial Z_t)$ taking over the role of Σ_i, $(\beta_j)_{j=0}^k$ is r-Lipschitz with exceptional probability at most μ. Similar to the calculation in (3.7), using Proposition A.3 and Lemma 3.2,

$$\mathbb{P}(|U_{v,t} - \mathbb{E}(U_{v,t})| > 9\sigma_t) = \mathbb{P}(|\beta_k - \beta_0| > 9\sigma_t) \leqslant d_{t-1}^{-10}$$

for $t \leqslant T$. Combining the two bounds, and using the fact that by Lemma 3.2, $\mathbb{E}(Y_{v,t}) < 90\sigma_t$, we see that $\mathbb{P}(Y_{v,t} > 100\sigma_t) < \mathbb{P}(Y_{v,t} - \mathbb{E}[Y_{v,t}] > 10\sigma_t) < d^{-9}$. □

3.3 Lovász Local Lemma

Let $A_{v,t}$ and $B_{v,t}$ be the events that (2.6) and (2.7) do not hold at stage t. We have seen that both these events have probability less than d_{t-1}^{-9} at stage t if they hold at stage $t-1$.

Lemma 3.5. *Suppose $t \leqslant T$ and (2.6)–(2.10) hold at time $t-1$. Then (2.6), (2.7) and (2.10) hold at time t with positive probability.*

Proof. Note that $A_{v,t}$ is mutually independent of any set of events $\{A_{u,t}, B_{u,t} \colon u \in U\}$ if no vertex of U is at distance at most six from v, and similarly for any event $B_{v,t}$. Therefore a dependency graph of these events certainly has maximum degree less than $\Delta = 2^{10}d_{t-1}^6$. By the Lovász Local Lemma with $\delta = 2^{12}d_{t-1}^{-3}$, the probability that no $A_{v,t}$ or $B_{v,t}$ occurs is at least

$$\exp\left(-\frac{8}{d_{t-1}^9} \cdot |V(G_{t-1})|\right).$$

Using the assumption (2.8) and (2.9) at time $t-1$, this product is easily at least $\exp(-n_t/d_t^8)$ if $t \leqslant T$. Now the event that (2.10) does not hold has probability easily less than $\exp(-n_t/d_t^8)$. By (2.8),

$$\mathbb{E}|Z_t| \leqslant \frac{n_t}{ed_t} + 150\frac{\sigma_t n_t}{d_t^2},$$

and concentration follows by considering an ordering $v_1, v_2, \ldots, v_m$ of the vertices of G_{t-1}, and the martingale $(\rho_i)_{i=0}^m$, where $\rho_i = \mathbb{E}(|Z_t| \mid F_i)$ where F_i is the σ-field generated by exposing the first i vertices of G_{t-1}. By (2.6), no vertex of G_{t-1} has degree more than $d_{t-1} + \sigma_{t-1}$, and this is easily less than $2d_t$ for $t \leqslant T$. Then the required bound follows from Hoeffding's inequality (Proposition A.2 in the Appendix) since $(\rho_i)_{i=0}^m$ is $2d_t$-Lipschitz. Therefore, with positive probability, (2.6), (2.7) and (2.10) all hold at time t. □

3.4 Bounds on $|X_t|$ and $|Y_t|$

Lemma 3.5 implies the existence of a choice for G_t (along with X_t, Y_t and Z_t satisfying (2.6), (2.7) and (2.10)). It remains to show that such a choice also satisfies (2.8) and (2.9).

We show that the random variables $|X_t|$ and $|Y_t|$ are deterministically bounded as follows by induction on t.

Lemma 3.6. *Let $t \leqslant T$. Suppose (2.8)–(2.9) hold at stage $t-1$ and (2.6), (2.7) and (2.10) hold at stage t. Then*

$$|X_t| < n_t + \frac{100\sigma_t n_t}{d_t}, \tag{3.8}$$

$$|Y_t| < \frac{200\sigma_t n_t}{d_t}. \tag{3.9}$$

Proof. Observe $|X_0| = n$ and $|Y_0| = 0$, so the inequalities in the lemma hold for $t = 0$. Suppose $t > 0$ and that the inequalities of the lemma hold at stage $t-1$. For inequality (3.8) we count the number of edges between X_{t-1} and X_t. Every $v \in X_{t-1} \backslash Z_t$ has $X_{v,t} \leqslant d_t + \sigma_t$ by (2.6). Similarly, every $v \in X_t$ has $X_{v,t-1} \geqslant d_{t-1} - \sigma_{t-1}$. Therefore

$$(d_{t-1} - \sigma_{t-1})|X_t| \leqslant (d_t + \sigma_t)|X_{t-1}|.$$

For $t \leqslant T$ we have

$$\frac{d_t + \sigma_t}{d_{t-1} - \sigma_{t-1}} = \frac{(d_t + \sigma_t)/d_t}{(d_{t-1} - \sigma_{t-1})/d_t} = e^{-1/e}\left(1 + \frac{\sigma_t}{d_t}\right).$$

Also, a quick computation shows that

$$\frac{\sigma_{t-1}}{d_{t-1}} < \frac{e^{-1/(3e)}\sigma_t}{d_t}.$$

Using (2.8) applied to $|X_{t-1}|$, we obtain

$$\begin{aligned}
|X_t| &\leqslant e^{-1/e}\left(1 + \frac{\sigma_t}{d_t}\right)|X_{t-1}| \\
&< e^{-1/e} n_{t-1}\left(1 + \frac{\sigma_t}{d_t}\right)\left(1 + \frac{100\sigma_{t-1}}{d_{t-1}}\right) \\
&= n_t\left(1 + \frac{\sigma_{t-1}}{d_{t-1} - \sigma_{t-1}}\right)\left(1 + \frac{\sigma_t}{d_t}\right)\left(1 + \frac{100\sigma_{t-1}}{d_{t-1}}\right)
\end{aligned}$$

https://doi.org/10.1017/S0963548320000279 Published online by Cambridge University Press

$$
\begin{aligned}
&< n_t\left(1+\frac{2\sigma_t}{d_t}\right)\left(1+\frac{\sigma_t}{d_t}\right)\left(1+\frac{100e^{-1/(3e)}\sigma_t}{d_t}\right) \\
&< n_t\left(1+\frac{100\sigma_t}{d_t}\right).
\end{aligned}
$$

By (2.7), we have $Y_{v,t} \leqslant 100\sigma_t$ for every $v \in X_t$ and so

$$(d_t - \sigma_t)|Y_t| \leqslant e(X_t, Y_t) \leqslant 100\sigma_t|X_t|.$$

The following calculation gives the required bound on Y_t when $t \leqslant T$:

$$
\begin{aligned}
|Y_t| &\leqslant \frac{100\sigma_t|X_t|}{d_t-\sigma_t} \\
&< \frac{100\sigma_t}{0.8d_t}\left(n_t + \frac{100\sigma_t n_t}{d_t}\right) \\
&= \frac{125\sigma_t n_t}{d_t} + \frac{12500\sigma_t^2 n_t}{d_t^2} \\
&< \frac{200\sigma_t n_t}{d_t}.
\end{aligned}
$$

□

4. Proof of Theorem 1.2

Let $N = \frac{1}{2}n$, and let F be a random k-regular graph on N vertices where $k = (m-1)d$. For convenience, we assume N is even. For $m \leqslant 5$ the bound in the theorem is negative, so we assume $m > 5$. Let G be obtained from F by adding a set I of N independent new vertices, and place an independent random d-regular graph between I and F. We shall show that with positive probability,

$$\gamma_\circ(G) > \left(1 - \frac{4\log\Delta}{\Delta^{1/2-2/(m-1)}}\right)\frac{N}{\Delta^{2/(m-1)}}.$$

It is well known (see *e.g.* Bollobás [3]) that the number of cycles of length at most four in F is asymptotically Poisson. The same calculation shows that the number of cycles of length at most four in G is also asymptotically Poisson as $n \to \infty$ with mean less than $\frac{1}{2}\Delta^4$. Therefore, for large enough N, the probability that G has girth at least five is certainly at least $2\exp(-\Delta^4)$.

Now, from a computation of Frieze and Łuczak (see [6]), it is known that the expected number of independent sets in F of size ℓ is at most

$$2\left(\frac{eN}{\ell}\exp\left(-\frac{k\ell}{2N}\right)\right)^{\ell}.$$

Let $\mathcal{I}_\alpha$ be the expected number of independent sets in F of size $\alpha = (2N\log k)/k$. It follows that

$$\mathcal{I}_\alpha < 2(2\log k)^{-2N\log k/k} < \exp(-N(\log\log k)/k).$$

For large enough N, the probability that every independent set has size at most α is easily at least $1 - \exp(-\Delta^4)$, so we conclude that with probability at least $\exp(-\Delta^4)$, every independent set in F has size at most α and G has girth at least five.

If J is a fixed set of vertices in F, let $\varphi(J)$ denote the number of vertices of I adjacent to no vertex in J. Then

$$\mathbb{E}(\varphi(J)) > |I| \cdot \frac{(N-|J|-d)^d}{N^d} = N\left(1 - \frac{|J|+d}{N}\right)^d.$$

We claim that for $|J| \leqslant \alpha$, this expectation is at least $\beta := N/\Delta^{2/(m-1)}$. Indeed,

$$\begin{aligned}
\left(1-\frac{|J|+d}{N}\right)^d &> \exp\left(-\frac{(|J|+d)d}{N} - d\left(\frac{|J|+d}{N}\right)^2\right) \\
&> \exp\left(-\frac{2d\log k}{k} - \frac{d^2}{N} - d\left(\frac{2\log k}{k} + \frac{d}{N}\right)^2\right) \\
&= ((m-1)d)^{-2/(m-1)} \cdot \exp\left(-\frac{d^2}{N}\right) \cdot \exp\left(-\left(\frac{2\sqrt{d}\log k}{k} + \frac{d\sqrt{d}}{N}\right)^2\right) \\
&= ((m-1)d)^{-2/(m-1)} \cdot \exp\left(-\frac{d^2}{N}\right) \cdot \exp\left(-\left(\frac{2\log((m-1)d)}{(m-1)\sqrt{d}} + \frac{d^{3/2}}{N}\right)^2\right) \\
&> ((m-1)d)^{-2/(m-1)} \cdot \left(\frac{m}{m-1}\right)^{-2/(m-1)} \\
&= \Delta^{-2/(m-1)},
\end{aligned}$$

where in the penultimate step we used the fact that m is fixed and we can take d sufficiently large with respect to m and N sufficiently large with respect to d. Hoeffding's inequality (Proposition A.2 in the Appendix) applied to the 1-Lipschitz vertex exposure martingale, obtained by exposing one by one the vertices of I shows that

$$\mathbb{P}(\varphi(J) < (1-\delta)\beta) < \exp\left(-\frac{1}{2N}\delta^2\beta^2\right).$$

We select the following value of δ, which is less than one if $m > 5$ and $d \geqslant d_0(m)$,

$$\delta = 4\Delta^{2/(m-1)-1/2}\log\Delta.$$

Using this choice of δ, the expected number of sets $J \subseteq F$ of size at most α such that $\varphi(J) < (1-\delta)\beta$ is at most

$$\sum_{j \leqslant \alpha} \binom{N}{j} \cdot \exp\left(-\frac{1}{2N}\delta^2\beta^2\right) < \exp\left(2\alpha\log\Delta - \frac{1}{2N}\delta^2\beta^2\right) < \exp\left(-\frac{1}{8N}\delta^2\beta^2\right).$$

If N is large enough, we conclude that with probability at least $1 - \exp(-\Delta^4)$, every set J of at most α vertices in F has $\varphi(J) \geqslant (1-\delta)\beta$. By the first part of the proof, we conclude that with positive probability, G has girth at least five and every independent set J in F has $\varphi(J) \geqslant (1-\delta)\beta$, and therefore $\gamma_\circ(G) > (1-\delta)\beta$, as required. This completes the proof.

Acknowledgements

We are deeply grateful to the referee for detailed and comprehensive comments and suggested corrections, which tremendously improved the presentation of this paper. We are also grateful to the journal for their understanding of the delay on our part in making these corrections.

References

[1] Alon, N., Krivelevich, M. and Sudakov, B. (1999) Coloring graphs with sparse neighbourhoods. *J. Combin. Theory Ser. B* **77** 73–82.

[2] Arnautov, V. I. (1974) Estimation of the exterior stability number of a graph by means of the minimal degree of the vertices (in Russian). *Prikl. Mat. i Programmirovanie* **11** 3–8.

[3] Bollobás, B. (2001) *Random Graphs*, second edition, Vol. 73 of Cambridge Studies in Advanced Mathematics, Cambridge University Press.
[4] Duckworth, W. and Wormald, N. (2006) On the independent domination number of random regular graphs. *Combin. Probab. Comput.* **15** 513–522.
[5] Erdős, P. and Lovász, L. (1975) Problems and results on 3-chromatic hypergraphs and some related questions. In *Infinite and Finite Sets* (A. Hajnal *et al.*, eds), Vol. 11 of Colloquia Mathematica Societatis János Bolyai, pp. 609–627.
[6] Frieze, A. M. and Łuczak, T. (1992) On the independence and chromatic numbers of random regular graphs. *J. Combin. Theory Ser. B* **54** 123–132.
[7] Gamarnik, D. and Goldberg, D. (2010) Randomized greedy algorithms for independent sets and matchings in regular graphs: exact results and finite girth corrections. *Combin. Probab. Comput.* **19** 61–85.
[8] Godbole, A. and Hitczenko, P. (1998) Beyond the method of bounded differences. In *Microsurveys in Discrete Probability (Princeton, NJ, 1997)* (D. Aldous and J. Propp, eds), pp. 43–58, AMS and DIMACS.
[9] Janson, S., Łuczak, T. and Ruciński, A. (2000) *Random Graphs*, Wiley.
[10] Johansson, A. (1996) Asymptotic choice number for triangle free graphs. DIMACS Technical Report 91-95.
[11] Lauer, J. and Wormald, N. (2007) Large independent sets in regular graphs of large girth. *J. Combin. Theory Ser. B* **97** 999–1009.
[12] Lovász, L. (1975) On the ratio of optimal and integral fractional covers. *Discrete Math.* **13** 383–390.
[13] McDiarmid, C. (1998) Concentration. In *Probabilistic Methods for Algorithmic Discrete Mathematics*, Vol. 16 of Algorithms and Combinatorics, pp. 195–248, Springer.
[14] Payan, C. (1975) Sur le nombre d'absorption d'un graphe simple (in French). *Proc. Colloq. Th. Graphes* (Paris 1974), *C.C.E.R.O* **17** 307–317.
[15] Shearer, J. (1983) A note on the independence number of triangle-free graphs. *Discrete Math.* **46** 83–87.
[16] Shamir, E. and Spencer, J. (1987) Sharp concentration of the chromatic number on random graphs $G_{n,p}$. *Combinatorica* **7** 121–129.
[17] Wormald, N. (1999) The differential equation method for random graph processes and greedy algorithms. In *Lectures on Approximation and Randomized Algorithms* (M. Karoński and H. J. Prömel, eds), pp. 73–155, PWN.
[18] Zito, M. (2001) Greedy algorithms for minimisation problems in random regular graphs. In *Proceedings of the 9th Annual European Symposium on Algorithms*, Vol. 2161 of Lecture Notes in Computer Science, pp. 524–536, Springer.

Appendix A. Concentration inequalities

In this section we present the inequalities which we will need from probability theory involving concentration of measure. All of these inequalities deal with upper bounds for expressions of the form $\mathbb{P}(|X - \mathbb{E}X| > \lambda)$, where X is a random variable and λ is a real number. The most basic inequality of this type for binomial distributions is the Chernoff bound [9].

Proposition A.1 (Chernoff bound). *If a random variable X has binomial distribution with probability p and mean pn, then for any $\varepsilon \in [0, 1]$,*

$$\mathbb{P}(|X - pn| > \varepsilon pn) < 2\exp\left(-\frac{\varepsilon^2 pn}{3}\right).$$

Moreover, the following version also holds, for all $t > 0$,

$$\mathbb{P}(X > \mathbb{E}[X] + t) < \exp\left(-\frac{t^2}{2(np + t/3)}\right).$$

Most of the inequalities we will need concern martingales. One of the most fundamental martingale inequalities is Hoeffding's inequality. Further refinements and generalizations of this inequality may be found in McDiarmid [13].

Proposition A.2 (Hoeffding's inequality). *Let $(\xi_i)_{i=0}^n$ be a martingale with respect to a filtration F_i and with difference sequence $(y_i)_{i=1}^n$, where $-a_i \leqslant y_i \leqslant -a_i + c_i$, and where a_i is a function on (Ω, F_{i-1}) and*

https://doi.org/10.1017/S0963548320000279 Published online by Cambridge University Press

$c_i \in \mathbb{R}$. *Then, for* $t \geqslant 0$ *and* $c := \sum c_i^2$,

$$\mathbb{P}(\xi_n > \mathbb{E}(\xi_n) + \lambda) \leqslant \exp\left(-\frac{2\lambda^2}{c}\right) \quad \text{and} \quad \mathbb{P}(\xi_n < \mathbb{E}(\xi_n) - \lambda) \leqslant \exp\left(-\frac{2\lambda^2}{c}\right).$$

We require the following martingale concentration inequality of Shamir and Spencer [16], which deals with concentration of a martingale which is c-Lipschitz with high probability. An overview of such inequalities is given in [8]. We note that the version in [16] has a larger factor of 8 in the exponential instead of 2, but the version stated here follows from the proof given there.

Proposition A.3. *Suppose* $(\xi_i)_{i=0}^k$ *is a martingale with* ξ_0 *constant satisfying*

$$\sum_{i=0}^{k-1} \mathbb{P}(|\xi_{i+1} - \xi_i| > r) < \mu,$$

$$|\xi_{i+1} - \xi_i| \leqslant k \quad \text{for all } 0 \leqslant i < k.$$

Suppose $k\mu^{1/2} \leqslant r$. *Then*

$$\mathbb{P}(|\xi_k - \xi_0| > \lambda + k^2\mu^{1/2}) < 2\exp\left(\frac{-\lambda^2}{2kr^2}\right) + 2k\mu^{1/2}.$$

A martingale satisfying the hypothesis of Proposition A.3 is called r-Lipschitz with exceptional probability at most μ. A final tool from probability which we require is the Lovász Local Lemma [5].

Proposition A.4 (Lovász Local Lemma). *Let* $A_1, A_2, \ldots, A_n$ *be events in some probability space and suppose that for some set* $J_i \subset [n]$ *of size at most* Δ, A_i *is mutually independent of* $\{A_j : j \notin J_i \cup \{i\}\}$. *Suppose that there is a real* $0 < x < 1$ *such that* $\mathbb{P}(A_i) \leqslant x(1-x)^\Delta$ *for all i. Then*

$$\mathbb{P}\left(\bigcap_{i=1}^n \overline{A_i}\right) \geqslant (1-x)^n.$$

Let $0 < \delta < 1$. *The following corollary is immediate for* $\Delta \geqslant 2$ *by setting* $x = \delta/\Delta$. *If* $\mathbb{P}(A_i) \leqslant \delta/(4\Delta)$ *for all i, then*

$$\mathbb{P}\left(\bigcap_{i=1}^n \overline{A_i}\right) \geqslant e^{-2\delta n/\Delta}.$$

Cite this article: Harutyunyan A, Horn P and Verstraete J (2021). Independent dominating sets in graphs of girth five. *Combinatorics, Probability and Computing* **30**, 344–359. https://doi.org/10.1017/S0963548320000279