SIAM J. DISCRETE MATH.
Vol. 36, No. 1, pp. 103–113

RAMSEY NUMBERS FOR NONTRIVIAL BERGE CYCLES*

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Abstract. In this paper, we consider an extension of cycle-complete graph Ramsey numbers to Berge cycles in hypergraphs: for $k \geq 2$, a *nontrivial Berge k-cycle* is a family of sets $e_1, e_2, \ldots, e_k$ such that $e_1 \cap e_2, e_2 \cap e_3, \ldots, e_k \cap e_1$ has a system of distinct representatives and $e_1 \cap e_2 \cap \cdots \cap e_k = \emptyset$. In the case that all the sets e_i have size three, let $\mathcal{B}_k$ denote the family of all nontrivial Berge k-cycles. The *Ramsey numbers* $R(t, \mathcal{B}_k)$ denote the minimum n such that every n-vertex 3-uniform hypergraph contains either a nontrivial Berge k-cycle or an independent set of size t. We prove $R(t, \mathcal{B}_{2k}) \leq t^{1+\frac{1}{2k-1}+\frac{2}{\sqrt{\log t}}}$, and moreover, we show that if a conjecture of Erdős and Simonovits [*Combinatorica*, 2 (1982), pp. 275–288] on girth in graphs is true, then this is tight up to a factor $t^{o(1)}$ as $t \to \infty$.

Key words. Ramsey numbers, Berge cycles, hypergraphs

AMS subject classification. 05

DOI. 10.1137/21M1396770

1. Introduction.

Let $\mathcal{F}$ be a family of r-graphs and $t \geq 1$. The Ramsey numbers $R(t, \mathcal{F})$ denote the minimum n such that every n-vertex r-graph contains either a hypergraph in $\mathcal{F}$ or an independent set of size t. For $k \geq 2$, a *Berge k-cycle* is a family of sets $e_1, e_2, \ldots, e_k$ such that $e_1 \cap e_2, e_2 \cap e_3, \ldots, e_k \cap e_1$ has a system of distinct representatives, and a Berge cycle is *nontrivial* if $e_1 \cap e_2 \cap \cdots \cap e_k = \emptyset$. Let $\mathcal{B}_k^r$ denote the family of nontrivial Berge k-cycles all of whose sets have size r. When $r = 2$, $\mathcal{B}_k^2 = \{C_k\}$, where C_k denotes the graph cycle of length k. In this paper, we let $\mathcal{B}_k = \mathcal{B}_k^3$.

It is a notoriously difficult problem to determine even the order of magnitude of $R(t, C_k)$—the cycle-complete graph Ramsey numbers. Kim [16] proved $R(t, C_3) = \Omega(t^2/\log t)$, which gives the order of magnitude of $R(t, C_3)$ when combined with the results of Ajtai, Komlós, and Szemerédi [2] and Shearer [27]. The current state-of-the-art results on $R(t, C_3)$ are due to Fiz Pontiveros, Griffiths, and Morris [12] and Bohman and Keevash [5], using the random triangle-free process, which determines $R(t, C_3)$ up to a small constant factor:

$$\left(\frac{1}{4} - o(1)\right) \frac{t^2}{\log t} \leq R(t, C_3) \leq (1 + o(1)) \frac{t^2}{\log t}.$$

The case $R(t, C_4)$ is the subject of a notorious conjecture of Erdős [6], where he conjectured that $R(t, C_4) = o(t^{2-\epsilon})$ for some $\epsilon > 0$. The current best upper bound on $R(t, C_{2k})$ is

$$O\left(\left(\frac{t}{\log t}\right)^{k/(k-1)}\right),$$

which come from the work of Caro et al. [8]. For $R(t, C_{2k+1})$, the best upper bound is

*Received by the editors February 3, 2021; accepted for publication (in revised form) September 13, 2021; published electronically January 4, 2022.
https://doi.org/10.1137/21M1396770
Funding: The second author's research was supported by NSF award DMS-1952786.
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$$O\left(\frac{t^{(k+1)/k}}{\log^{1/k} t}\right)$$

due to Sudakov [28]. Recent results using pseudorandom graphs by Mubayi and Verstraëte [23] give the best lower bounds on cycle-complete graph Ramsey numbers:

$$R(C_k, n) = \Omega\left(\frac{t^{(k-1)/(k-2)}}{\log^{2/(k-2)} t}\right).$$

In particular, via random block constructions, they show that

$$R(C_5, t) \geq (1 + o(1))t^{11/8}, \quad R(C_7, t) \geq (1 + o(1))t^{11/9}.$$

For $k \geq 3$, a *loose k-cycle* is a nontrivial Berge k-cycle, denoted C_k^r, with sets $e_1, e_2, \ldots, e_k$ of size r such that $|e_1 \cap e_2| = 1$, $|e_2 \cap e_3| = 1, \ldots, |e_k \cap e_1| = 1$, and for any other pairs of edges e_i,e_j, $e_i \cap e_j = \emptyset$. Ramsey type problems for loose cycles in r-graphs have been studied extensively [4, 9, 10, 13, 14, 15, 16, 17, 18, 21, 23]. For r-uniform hypergraphs with $r \geq 3$, Kostochka, Mubayi, and Verstraëte [17] proved for all $r \geq 3$, there exist constants $a, b > 0$ such that

$$\frac{at^{\frac{3}{2}}}{(\log t)^{\frac{3}{4}}} \leq R(t, C_3^r) \leq bt^{\frac{3}{2}}. \tag{1}$$

The following conjecture was proposed in [17].

Conjecture I. *For $r, k \geq 3$,*

$$R(t, C_k^r) = t^{\frac{k}{k-1}+o(1)}. \tag{2}$$

The conjecture is true for $k = 3$ due to (1). It is shown in [25] that $R(t, C_4^3) \leq t^{4/3+o(1)}$. Méroueh [21] showed $R(t, C_k^3) = O(t^{1+1/\lfloor(k+1)/2\rfloor})$ for $k \geq 3$ and $R(t, C_k^r) = O(t^{1+1/\lfloor k/2\rfloor})$ for $r \geq 4$ and every odd integers $k \geq 5$, improving earlier results of Collier-Cartaino, Graber, and Jiang [9]. Conjecture I motivates our current study of nontrivial Berge k-cycles. In support of the above conjecture, we prove the following result for nontrivial Berge cycles of even length.

Theorem 1. *For $k \geq 3$, and t large enough,*

$$R(t, \mathcal{B}_{2k}) \leq t^{\frac{2k}{2k-1}+\frac{4}{\sqrt{\log t}}}.$$

Erdős and Simonovits [11] conjectured that there exists an n-vertex graph of girth more than $2k$ with $\Theta(n^{1+1/k})$ edges. This notoriously difficult conjecture remains open, except when $k \in \{2, 3, 5\}$, largely due to the existence of generalized polygons [3, 29, 30]. Towards this conjecture, Lazebnik, Ustimenko, and Woldar [19] gave the densest known construction, which has $\Omega(n^{1+2/(3k-2)})$ edges. We prove the following theorem relating this conjecture to lower bounds on Ramsey numbers for nontrivial Berge cycles.

Theorem 2. *Let $k \geq 2$, $r \geq 3$. Suppose there exists an n-vertex graph of girth more than $2k$ with $cn^{1+1/k}$ edges for any integer n large enough and some positive constant c. Then for t large enough and some positive constant $c_{k,r}$ dependent on k and r,*

$$R(t, \mathcal{B}_k^r) \geq c_{k,r}\left(\frac{t}{\log t}\right)^{\frac{k}{k-1}}. \tag{3}$$

This shows that if the Erdős–Simonovits conjecture is true, then Theorem 1 is tight up to a $t^{o(1)}$ factor. Indeed, following the proof of Theorem 2, the known construction of Lazebnik, Ustimenko, and Woldar [19] would give a weaker lower bound of $\Omega((t/\log t)^{(3k-2)/(3k-4)})$.

Let B_k be the family of 3-uniform Berge k-cycles without nontriviality. Random graphs together with the Lovász local lemma give $R(t, B_k) \geq t^{(2k-2)/(2k-3)-o(1)}$; see [1] for similar computation. We prove the following theorem, which gives a substantially better lower bound for B_4 if the Erdős–Simonovits conjecture is true.

THEOREM 3. *Suppose there exists an n-vertex graph of girth more than 8 with $c_1 n^{5/4}$ edges for any integer n large enough and some positive constant c_1. Then for t large enough and some positive constant c_2,*

$$R(t, B_4) \geq \left(\frac{c_2 t}{\sqrt{\log t}}\right)^{16/13}.$$

In fact, this is also a lower bound for $R(t, \{B_2, B_3, B_4\})$. A natural 3-uniform analog of the Erdős–Simovits conjecture is that there exist n-vertex $\{B_2, B_3, \ldots, B_k\}$-free 3-graphs with $n^{1+1/\lfloor k/2\rfloor - o(1)}$ edges. This is true for $k = 3$ due to Ruzsa and Szemeredi [26]. The proof of Theorem 3 makes use of the fact that there exist n-vertex $\{B_2, B_3, B_4\}$-free 3-graphs with $\Omega(n^{3/2})$ edges; that is, the conjecture is true for $k = 4$, which is due to Lazebnik and Verstraëte [20]. More generally, following the proof of Theorem 3, if the 3-uniform analog of the Erdős–Simonovits conjecture is true, then we have $R(t, \{B_2, B_3, \ldots, B_{2k}\}) \geq t^{4k^2/(4k^2-k-1)-o(1)}$ and $R(t, \{B_2, B_3, \ldots, B_{2k+1}\}) \geq t^{(2k+1)2k/(4k^2+k-1)-o(1)}$, which are substantially better than the lower bounds obtained by random graphs.

We prove Theorem 1 in section 5, Theorem 2 in section 2, and Theorem 3 in section 3. Theorem 2 is valid for all values of $k \geq 2$ and $r \geq 3$, while Theorem 1 only works for even values of k and $r = 3$. We believe that Theorem 1 should extend to odd values of k and all $r \geq 3$.

CONJECTURE II. *For all $r, k \geq 3$,*

$$R(t, \mathcal{B}_k^r) \leq t^{\frac{k}{k-1}+o(1)}. \tag{4}$$

Notation and terminology. For a hypergraph H, let $V(H)$ denote the vertex set of H, $v(H) = |V(H)|$, and let $|H|$ be the number of edges in H. If all edges of H have size r, we say H is an *r-uniform hypergraph*, or an *r-graph* for short. For $v \in V(H)$, let $d_H(v) = |\{e \in H : v \in e\}|$ be the *degree of* v in H. We denote the average degree of H by $d(H)$, denote the minimum degree of H by $\delta(H)$, and denote the maximum degree of H by $\Delta(H)$. For $u, v \in V(H)$, let $d_H(u, v) = |\{w : uvw \in H\}|$ denote the *codegree* of the pair $\{u, v\}$. An *independent set* in a hypergraph is a set of vertices containing no edge of the hypergraph. Let $\alpha(H)$ denote the largest size of an independent set in a hypergraph H.

2. Proof of Theorem 2.

We will use the following lemma to get a large bipartite subgraph with large minimum degree and small maximum degree.

LEMMA 4. *Let $k \geq 3$, $c > 0$, and let G be an n-vertex graph of girth more than $2k$ with more than $2cn^{1+1/k}$ edges. Then there exists a bipartite subgraph G' of G such that $\delta(G') \geq cn^{1/k}$, $\Delta(G') \leq n^{1/k}/c^{k-1}$, and $v(G') \geq c^k n$.*

Proof. A maximum cut of G gives a bipartite subgraph with at least $cn^{1+1/k}$ edges. A subgraph G' of this bipartite subgraph of minimum degree at least $cn^{1/k} + 1$

may be obtained by repeatedly removing vertices of degree at most $cn^{1/k}$. Let $\Delta := \Delta(G')$ be the maximum degree of G', and let v be a vertex of maximum degree; then the number of vertices at distance k from v is at least $\Delta c^{k-1} n^{(k-1)/k}$, since G has girth larger than $2k$. In particular, $\Delta c^{k-1} n^{(k-1)/k} \leq n$ and so $\Delta \leq n^{1/k}/c^{k-1}$. The number of vertices in G' is at least $c^k n$, since G' has minimum degree at least $cn^{1/k}+1$ and girth larger than $2k$. ∎

Let $r \geq 2$, a *star* with vertex set V is an r-graph on V consisting of all edges containing a fixed vertex of V; i.e., the edge set of a star is $\{e \subset V : |e| = r, v \in e\}$ for some vertex $v \in V$. Let integers $d \geq m$, and let $S_{d,m}$ be a d-vertex r-graph consisting of m vertex-disjoint stars of size $\lfloor d/m \rfloor$ or $\lceil d/m \rceil$.

LEMMA 5. *Let integer $r \geq 2$, and let integers $d \geq m$. The probability that a uniformly chosen set of s vertices of $S_{d,m}$ is independent is at most*

$$\exp\left(-\frac{m(s-rm)}{2d}\right).$$

Proof. Let the vertex sets of these stars be $V_1, V_2, \ldots, V_m$. The probability that a uniformly chosen set of s_i vertices in V_i is independent in $S_{d,m}$ is at most $1 - s_i/\lceil d/m \rceil \leq 1 - ms_i/2d$ if $s_i \geq r$ and is 1 if $s_i < r$. Hence, this probability is at most $1 - m(s_i - r)/2d$ for $0 \leq s_i \leq d$. Therefore a uniformly chosen set $I \subset S_{d,m}$ of s vertices with $|I \cap V_i| = s_i$ is independent with probability at most

$$\prod_{i=1}^{m}\left(1 - \frac{m(s_i - r)}{2d}\right) \leq \exp\left(-\sum_{i=1}^{m}\frac{m(s_i - r)}{2d}\right) = \exp\left(-\frac{m(s-rm)}{2d}\right).$$ ∎

Now we are ready to prove Theorem 2.

Proof of Theorem 2. It suffices to show that for n large enough, there exists an n-vertex $\mathcal{B}_k^r$-free r-graph with independence number $O(n^{1-\frac{1}{k}} \log n)$. Let G be an n-vertex graph of girth more than $2k$ with $2cn^{1+1/k}$ edges for some positive constant c. By Lemma 4, there exists a bipartite subgraph G' of G with at least $N = c^k n$ vertices, minimum degree at least $cn^{1/k}$, and maximum degree at most $n^{1/k}/c^{k-1}$. Let X, Y be the parts of this bipartite graph where $|Y| \geq |X|$. Let $m = 8 \log n / c^k$. We form an r-graph H with vertex set Y by placing a random copy of $S_{d(x),m}$ on the vertex set $N_{G'}(x)$, the neighborhood of x in G', independently for each $x \in X$. Since G' has girth more than $2k$, it is straightforward to check that H does not contain any nontrivial Berge k-cycles. We now compute the expected number of independent sets of size $t = rmn^{1-1/k}/c^{k+1}$ in H. Clearly, $\log t \geq (1 - 1/k) \log n$. If H has no independent set of size t with positive probability, then since $v(H) \geq N/2$, we find that

$$R(t, \mathcal{B}_k^r) \geq N/2 \geq \frac{c^k}{2}\left(\frac{c^{2k+1} t}{8r \log n}\right)^{\frac{k}{k-1}} \geq c_{k,r}\left(\frac{t}{\log t}\right)^{\frac{k}{k-1}}$$

for some positive constant $c_{k,r}$. This is enough to prove Theorem 2.

For an independent t-set I in H, $I \cap N_{G'}(x)$ is an independent set in $S_{d(x),m}$ for all $x \in X$. Since these events are independent, setting $s(x) = |I \cap N_{G'}(x)|$, and applying Lemma 5 gives

$$\begin{aligned}\mathbb{P}(I \text{ independent in } H) &\leq \prod_{x \in X} \exp\left(-\frac{m(s(x) - rm)}{2d(x)}\right)\\ &= \exp\left(-\sum_{x \in X}\frac{ms(x)}{2d(x)} + \sum_{x \in X}\frac{rm^2}{2d(x)}\right).\end{aligned}$$

For every $x \in X$, $cn^{1/k} \le d(x) \le n^{1/k}/c^{k-1}$, and therefore

$$\mathbb{P}(I \text{ independent in } H) \le \exp\left(-\frac{c^{k-1}m\sum_{x\in X} s(x)}{2n^{1/k}} + \frac{|X|rm^2}{2cn^{1/k}}\right).$$

Now $\sum_{x\in X} s(x)$ is precisely the number of edges of G' between X and I. Since every vertex in I has degree at least $cn^{1/k}$, this number of edges is at least $cn^{1/k}t = rmn/c^k$. Consequently, using $|X| < n/2$,

$$\mathbb{P}(I \text{ independent in } H) \le \exp\left(-\frac{c^k mt}{2} + \frac{c^k mt}{4}\right) = \exp\left(-\frac{c^k mt}{4}\right).$$

The expected number of independent sets of size t is at most

$$\binom{n}{t}\exp\left(-\frac{c^k mt}{4}\right) < \exp\left(t\log n - \frac{c^k mt}{4}\right) = \exp\left(-t\log n\right).$$

This is vanishing as $n \to \infty$, and the proof of Theorem 2 is complete. ∎

3. Proof of Theorem 3. Lazebnik and Verstraëte [20] showed that there exist n-vertex B_4-free 3-graphs with $(1/6 + o(1))n^{3/2}$ triples. More specifically, for n large enough, there exists a linear n-vertex B_4-free 3-graphs J_n with $n^{3/2}/10$ triples and maximum degree at most $n^{1/2}$. We want to find an upper bound for the probability that a random s-set is independent in J_n. We make use of the following lemma, where we make no effort to optimize the constants.

LEMMA 6. *Let n, s be integers such that $s < \sqrt{n}/2$. For n large enough, the probability that a uniformly chosen set of s vertices of J_n is independent is at most*

$$\exp\left(-\frac{s^3 - 216}{80n^{3/2}}\right).$$

When $s \ge \sqrt{n}/2$, the probability is at most $639/640$.

Proof. This is trivial when $s < 6$. When $6 < s < \sqrt{n}/2$, let X be the uniformly chosen s-set. For any edge $e \in E(J_n)$, let A_e be the event that $e \in X$. Then by the inclusion-exclusion principle, for n large enough, the probability that X is not independent is at least

$$\begin{aligned}
&\sum_{e\in E(J_n)} \mathbb{P}(A_e) - \sum_{\{e,f\}\subset E(J_n)} \mathbb{P}(A_e \wedge A_f)\\
&\ge \frac{1}{\binom{n}{s}}\left(\frac{n^{3/2}}{10}\binom{n-3}{s-3} - n\binom{n^{1/2}}{2}\binom{n-5}{s-5} - \binom{n^{3/2}/10}{2}\binom{n-6}{s-6}\right)\\
&\ge \frac{s^3}{40n^{3/2}}\left(1 - \frac{4s^3}{n^{3/2}}\right)\\
&\ge \frac{s^3}{80n^{3/2}}.
\end{aligned}$$

Therefore, for $s > 6$ and n large enough, the probability that X is independent is at most

$$1 - \frac{s^3}{80n^{3/2}} \le \exp\left(-\frac{s^3}{80n^{3/2}}\right) < \exp\left(-\frac{s^3 - 216}{80n^{3/2}}\right).$$

When $s \ge \sqrt{n}/2$, the probability is at most

$$1 - \frac{(\sqrt{n}/2)^3}{80n^{3/2}} = \frac{639}{640}.$$ ∎

Now we are ready to prove Theorem 3.

Proof of Theorem 3. Let G be an n-vertex graph of girth more than 8 with $2c_1n^{5/4}$ edges for some positive constant c_1. By Lemma 4, there exists a bipartite subgraph G' of G with at least $N = c_1^4 n$ vertices, minimum degree at least $c_1 n^{1/4}$, and maximum degree at most $n^{1/4}/c_1^3$. Let X, Y be the parts of this bipartite graph where $|Y| \geq |X|$. We form a 3-graph H with vertex set Y by placing a random copy of $J_{d(x)}$ on the vertex set $N_{G'}(x)$, the neighborhood of x in G, independently for each $x \in X$. Since G has girth more than $2k$, it is straightforward to check that H does not contain any Berge 4-cycles. Let $m = 8c_1^{1/4}\sqrt{\log n}$, and let $t = mn^{13/16}$. Clearly, $\log t > 13 \log n/16$. If H has no independent sets of size t with positive probability, then since $v(H) \geq N/2$, we conclude that

$$R(t, B_4) \geq N/2 \geq \frac{c_1^4}{2}\left(\frac{t}{8c_1^{1/4}\sqrt{\log n}}\right)^{16/13} \geq c_2\left(\frac{t}{\sqrt{\log t}}\right)^{16/13}$$

for some positive constant c_2. This is enough to prove Theorem 3.

Let A be a t-set in Y, and let $X_A = \{x \in X || N_{G'}(x) \cap A| \geq \sqrt{t}/2\}$, $\overline{X}_A = X \backslash A$. We now evaluate the probability that A is independent in H in two cases.

Case 1. When $|X_A| < n^{5/6}$. Since the induced bipartite subgraph of G' on $X_A \cup A$ has girth 8, the number of edges of G' between X_A and A is less than $(n^{5/6})^{5/4} = n^{25/24}$. If A is independent in H, then $N_{G'}(x) \cap A$ is also independent in $J_{d(x)}$ for all $x \in X$. Since these events are independent, setting $s(x) = |N_{G'}(x) \cap A|$ and applying Lemma 6 gives

$$\begin{aligned}\mathbb{P}(A \text{ independent in } H) &\leq \prod_{x\in\overline{X}_A} \exp\left(-\frac{s(x)^3 - 216}{80d(x)^{3/2}}\right)\\ &= \exp\left(-\sum_{x\in\overline{X}_A}\frac{s(x)^3}{80d(x)^{3/2}} + \sum_{x\in\overline{X}_A}\frac{27}{10d(x)^{3/2}}\right).\end{aligned}$$

For every $x \in X$, $c_1n^{1/4} \leq d(x) \leq n^{1/4}/c_1^3$ and hence together with Jenson's inequality we have

$$\begin{aligned}\mathbb{P}(A \text{ independent in } H) &\leq \exp\left(-\frac{c_1^{9/2}\sum_{x\in\overline{X}_A} s(x)^3}{80n^{3/8}} + \frac{27|\overline{X}_A|}{10c_1^{3/2}n^{3/8}}\right)\\ &\leq \exp\left(-\frac{c_1^{9/2}(\sum_{x\in\overline{X}_A} s(x))^3}{80n^{3/8}|\overline{X}_A|^2} + \frac{27|\overline{X}_A|}{10c_1^{3/2}n^{3/8}}\right).\end{aligned}$$

Note that $\sum_{x\in\overline{X}_A} s(x)$ is exactly the number of edges of G' between $\overline{X}_A$ and A, which is at least $tc_1n^{1/4} - n^{25/24} = (1-o(1))c_1mn^{17/16}$. Also note that $|\overline{X}_A| < N/2 = c_1^4 n/2$. Consequently,

$$\begin{aligned}\mathbb{P}(A \text{ independent in } H) &\leq \exp\left(-\frac{(1-o(1))m^3n^{13/16}}{20c_1^{1/2}} + \frac{27c_1^{5/2}n^{5/8}}{20}\right)\\ &< \exp\left(-\frac{m^3n^{13/16}}{32c_1^{1/2}}\right).\end{aligned}$$

Case 2. When $|X_A| \geq n^{5/6}$. Applying Lemma 6 gives

$$\mathbb{P}(A \text{ independent in } H) \leq (639/640)^{|X_A|} \leq \exp(-n^{5/6}/640) < \exp\left(-\frac{m^3 n^{13/16}}{32c_1^{1/2}}\right).$$

In both cases we have $\mathbb{P}(A \text{ independent in } H) < \exp(-\frac{m^3 n^{13/16}}{32c_1^{1/2}})$. Therefore the expected number of independent sets of size t in H is at most

$$\binom{n}{t}\exp\left(-\frac{m^3 n^{13/16}}{32c_1^{1/2}}\right) < \exp\left(mn^{13/16}\log n - \frac{m^3 n^{13/16}}{32c_1^{1/2}}\right) = \exp\left(-mn^{13/16}\log n\right).$$

This is vanishing as $n \to \infty$, which completes the proof of Theorem 3. ∎

4. Degrees, codegrees, and independent sets. We make use of the following elementary lemma, whose proof is a standard probabilistic argument, included for completeness.

LEMMA 7. *Let $d \geq 1$, and let H be a 3-graph of average degree at most d. Then*

$$\alpha(H) \geq \frac{2v(H)}{3d^{\frac{1}{2}}}.$$

Proof. Let X be a subset of $V(H)$ whose elements are chosen independently with probability $p = d^{-1/2}$. We can get an independent set by deleting a vertex for each edge of H contained in X. Then the expected size of such independent set is at least

$$pv(H) - p^3|H| = pv(H) - \frac{p^3 dv(H)}{3} = \frac{2v(H)}{3d^{\frac{1}{2}}}.$$

Hence, there must exist an independent set of size at least the desired lower bound, which completes the proof. ∎

LEMMA 8. *Let H be a 3-graph on n vertices, and $0 < \epsilon < 1/2$. Then there exists an induced subgraph G of H satisfying the following properties:*

1. $v(G) \geq n^{1-\frac{2}{\log_2(\frac{1}{\epsilon})}}$,
2. $\Delta(G) \leq \frac{d(G)}{\epsilon}$.

Proof. Let $H = G^{(0)}$. We do the following for $i \geq 0$. If $\Delta(G^{(i)}) \leq d(G^{(i)})/\epsilon$, we let $G = G^{(i)}$. Otherwise, iteratively delete vertices of $G^{(i)}$ with degree at least $d(G^{(i)})$. Each deleted vertex will result in the loss of at least $d(G^{(i)})$ edges. So we can delete at most

$$\frac{|G^{(i)}|}{d(G^{(i)})} = \frac{v(G^{(i)}) \cdot d(G^{(i)})}{3 \cdot d(G^{(i)})} = \frac{v(G^{(i)})}{3} < \frac{v(G^{(i)})}{2}$$

vertices in this step. Let $G^{(i+1)}$ be the subgraph induced by the remaining vertices. Then we have $v(G^{(i+1)}) > v(G^{(i)})/2$. If $\Delta(G^{(i+1)}) \leq d(G^{(i+1)})/\epsilon$, then we let $G = G^{(i+1)}$. Otherwise, we have

$$d(G^{(i+1)}) \leq \epsilon\Delta(G^{(i+1)}) < \epsilon d(G^{(i)}).$$

Let $K = 2\log_{1/\epsilon} n$. We must obtain an induced subgraph G with $\Delta(G) \leq d(G)/\epsilon$ after at most K repetitions. Otherwise, after K repetitions, since the average degree decreases by at least a factor of ϵ after each repetition, the remaining graph $G^{(K)}$

will have no edge, which satisfies the condition $\Delta(G^{(K)}) \leq d(G^{(K)})/\epsilon$. Suppose after $m \leq K$ repetitions we have the desired induced subgraph G with $\Delta(G) < d(G)/\epsilon$. Since the number of vertices decreases by at most a factor of 2, we also have

$$v(G) > \frac{n}{2^m} \geq n^{1-\frac{2}{\log_2(\frac{1}{\epsilon})}}.$$

This completes the proof. ∎

We use the following slightly weaker version of a lemma due to Méroueh [21]; the lemma is in fact valid for 3-graphs H with no loose k-cycles.

LEMMA 9. *Let H be a $\mathcal{B}_k$-free 3-graph. Then there exists a subgraph H^* of H such that $|H^*| > |H|/(3k^2)$ and each edge of H^* contains a pair of codegree 1.*

Proof. Given a 3-graph G and a pair of vertices x, y, we say that $\{x, y\}$ is *G-light* if $d_G(x, y) < k$. Let $G_1 = H$, and let H_1 consist of all edges of G_1 containing a G_1-light pair, and let $G_2 = G_1 \backslash H_1$. For $i \geq 2$, let H_i consist of all edges of G_i containing a G_i-light pair, and let $G_{i+1} = G_i \backslash H_i$. Suppose for contradiction that G_k is not empty. Let $e_1 = \{v_1, v_2, v_3\}$ be an edge in G_k; then by definition, $\{v_2, v_3\}$ is not a G_{k-1}-light pair, and hence, there exists an edge $e_2 = \{v_2, v_3, v_4\}$ such that $v_4 \neq v_1$. For $2 \leq i \leq k-1$, let $e_i = \{v_i, v_{i+1}, v_{i+2}\}$ be an edge in G_{k+1-i}. By definition, $\{v_{i+1}, v_{i+2}\}$ is not a G_{k-i}-light pair, and hence, there exists an edge $e_{i+1} = \{v_{i+1}, v_{i+2}, v_{i+3}\}$ in G_{k-i} such that v_{i+3} is distinct from all v_j, $1 \leq j \leq i$. Therefore, we have a tight path of length k in $G_1 = H$, that is, a hypergraph consisting of $k+2$ distinct vertices v_i, $1 \leq i \leq k+2$, and k edges $e_i = \{v_i, v_{i+1}, v_{i+2}\}$, $1 \leq i \leq k$. This is also a nontrivial Berge k-cycle. Indeed, when k is even, $\{v_2, v_4, \ldots, v_k, v_{k+1}, v_{k-1}, \ldots, v_3\}$ forms a system of distinct representatives of $\{e_1 \cap e_2, e_2 \cap e_4, e_4 \cap e_6, \ldots, e_{k-2} \cap e_k, e_k \cap e_{k-1}, e_{k-1} \cap e_{k-3}, \ldots, e_3 \cap e_1\}$, and when k is odd, $\{v_2, v_4, \ldots, v_{k+1}, v_k, v_{k-2}, \ldots, v_3\}$ forms a system of distinct representatives of $\{e_1 \cap e_2, e_2 \cap e_4, e_4 \cap e_6, \ldots, e_{k-3} \cap e_{k-1}, e_{k-1} \cap e_k, e_k \cap e_{k-2}, \ldots, e_3 \cap e_1\}$. This results in a contradiction, since H is $\mathcal{B}_k$-free. Therefore, G_k must be empty, and hence H can be partitioned into $k-1$ subgraphs H_i, $1 \leq i \leq k-1$, such that each H_i consists of edges containing a G_i-light pair, which is also H_i-light. Let H' be a subgraph H_i with the most edges; then by the pigeonhole principle,

$$|H'| > \frac{|H|}{k}.$$

Now consider a graph J whose vertex set is the set of 3-edges of H', and two 3-edges of H' form an edge of J if they share an H'-light pair. It is easy to see that J has maximum degree at most $3k-6$. Then we can greedily take an independent set of J of size at least $v(J)/(3k-5)$, and this independent set corresponds to a subgraph H^* of H' such that

$$|H^*| > \frac{|H'|}{3k-5} > \frac{|H|}{3k^2},$$

and each edge of H^* contains a pair of codegree 1. ∎

5. Proof of Theorem 1. A key ingredient of the proof of Theorem 1 is a supersaturation theorem for cycles in graphs: we make use of the following result proved by Simonovits [7] (see Morris and Saxton [22] for stronger supersaturation).

LEMMA 10. *For every $n, k \geq 2$, there exist constants $\gamma, b_0 > 0$, such that for every $b \geq b_0$, any n-vertex graph G with at least $bn^{1+1/k}$ edges contains at least $\gamma b^{2k} n^2$ copies of C_{2k}.*

We next give a simple lemma which says that if a graph has many cycles of length $2k$ containing a fixed edge, then it has many edges.

LEMMA 11. *Let G be a graph containing m cycles of length $2k$, each containing an edge $e \in G$. Then $|G| \geq m^{1/(k-1)}/2$.*

Proof. For each cycle C of length $2k$ containing e, let $M(C)$ be the perfect matching of C containing e. Fixing a matching $M \subset G$ of size k containing e, at most $(k-1)!2^{k-1}$ cycles C have $M(C) = M$. It follows that the number of distinct matchings $M \subset G$ of size k containing e is at least $m/(k-1)!2^{k-1}$, and therefore

$$\binom{|G|-1}{k-1} \geq \frac{m}{(k-1)!2^{k-1}}.$$

We conclude $|G|^{k-1} \geq m/2^{k-1}$, and therefore $|G| \geq m^{1/(k-1)}/2$. ◻

Now we are ready to prove Theorem 1.

Proof of Theorem 1. It suffices to show that for every large enough integer n, an n-vertex $\mathcal{B}_{2k}$-free 3-graph H contains an independent set of size at least $n^{(2k-1)/(2k)-5/(2\sqrt{\log n})}$. By Lemma 8 with $\epsilon = \exp(-\sqrt{\log_2 n})$, we find an induced subgraph H_0 of H with n_0 vertices, average degree d_0, and maximum degree D_0 such that $n_0 \geq n^{1-2/\sqrt{\log_2 n}}$ and $D_0 < d_0/\epsilon$. By Lemma 9, there is a subgraph H_1 of H_0 with at least $|H_0|/(4k^2)$ edges such that each edge of H_1 contains a pair of codegree 1 in H_1. Let $\chi : V(H_1) \to \{1,2,3\}$ be a random 3-coloring, and let H_2 consist of all triples in H_1 such that the pair of vertices of colors 1 and 2 has codegree 1 in H_1 and the last vertex in the triple has color 3. The probability that an edge in H_1 is also an edge in H_2 is at least $1/27$, and therefore the expected number of edges in H_2 is at least $|H_1|/27 \geq |H_0|/(108k^2)$. Fix a coloring so that $|H_2| \geq |H_0|/(108k^2)$. Consider the bipartite graph G comprising all pairs of vertices of colors 1 and 2 contained in an edge of H_2. Thus, $|G| = |H_2|$ and G has average degree $d_G \geq d_0/(108k^2)$. For convenience, let $b > 0$ be defined by $d_G = 2bn_0^{1/k}$ so $|G| = bn_0^{1+1/k}$. By Lemma 10, there exist constants $\gamma, b_0 > 0$ such that if $b > b_0$, then G must contain at least $\gamma b^{2k} n_0^2$ copies of C_{2k}. Notice that we must have $1/\epsilon > b_0$ when n is large enough. The proof is split into two cases.

Case 1. $b \geq 1/\epsilon$. By the pigeonhole principle, there exists an edge e such that the number of C_{2k} containing e in G is at least

$$\frac{2k\gamma b^{2k} n_0^2}{|G|} = 2k\gamma b^{2k-1} n_0^{1-\frac{1}{k}}.$$

Let G' be the union of all $2k$-cycles in G containing e. Then by Lemma 11, for some constant c,

$$|G'| \geq cb^{2+\frac{1}{k-1}} n_0^{\frac{1}{k}} = \frac{1}{2} cb^{1+\frac{1}{k-1}} d_G \geq \frac{1}{216k^2} c\epsilon^{-1-\frac{1}{k-1}} d_0 > D_0$$

provided n is large enough. Let C be a $2k$-cycle in G containing e. Then there exist edges $e_1 \cup \{v_1\}, e_2 \cup \{v_2\}, \ldots, e_{2k} \cup \{v_{2k}\}$ in H_2, where $e_1, e_2, \ldots, e_{2k} \in C$ and $v_1, v_2, \ldots, v_{2k}$ have color 3. Since H_2 is $\mathcal{B}_{2k}$-free, for some vertex z we have $v_1 = v_2 = \cdots = v_{2k} = z$. Since each cycle C in G' contain e, they must have the same z. Now the degree of z in H_2 is at least $|G'| > D_0$, which contradicts the fact that H_0 has maximum degree at most D_0.

Case 2. $b < 1/\epsilon$. In this case, $d_G < 2n_0^{1/k}/\epsilon$, and so $d_0 < (216k^2/\epsilon)n_0^{1/k}$. By Lemma 7 on H_0,

$$\alpha(H) \geq \alpha(H_0) \geq \frac{2n_0}{3d_0^{\frac{1}{2}}} \geq \frac{2}{3}\left(\frac{216k^2}{\epsilon}\right)^{-\frac{1}{2}} n_0^{\frac{2k-1}{2k}} \geq \frac{1}{9\sqrt{6}k} n^{\frac{2k-1}{2k} - \frac{5k-2}{2k\sqrt{\log_2 n}}} > n^{\frac{2k-1}{2k} - \frac{5}{2\sqrt{\log n}}}.$$

Now let $n = t^{\frac{2k}{2k-1} + \frac{4}{\sqrt{\log t}}}$. Clearly, $\log n > \frac{2k}{2k-1}\log t$. Hence, an n-vertex $\mathcal{B}_{2k}$-free 3-graph H contains an independent set of size

$$n^{\frac{2k-1}{2k} - \frac{5}{2\sqrt{\log n}}} = t^{(\frac{2k}{2k-1} + \frac{4}{\sqrt{\log t}})(\frac{2k-1}{2k} - \frac{5}{2\sqrt{\log n}})} > t$$

provided n is large enough. Therefore, we have $R(t, \mathcal{B}_{2k}) < t^{\frac{2k}{2k-1} + \frac{4}{\sqrt{\log t}}}$. ◻

In fact, by more careful computation, we can obtain a slightly better upper bound $R(t, \mathcal{B}_{2k}) < t^{\frac{2k}{2k-1} + \frac{c}{\sqrt{\log t}}}$, where $c > \frac{5k-2}{2k-1} \cdot \sqrt{\frac{(2k)\log 2}{2k-1}}$.

6. Concluding remarks.

- Notice that Theorem 2 is valid for odd values of k; we believe that Theorem 1 should extend to odd values of k. An obstacle to applying the same idea as in the proof for even values of k is that we don't have "good" supersaturation for odd cycles. New ideas may be required to complete the proof for odd values.
- It seems likely that Theorem 1 can be extended to r-uniform hypergraphs with $r \geq 4$; however, when following the proof of Theorem 1, two obstacles arise. The first is that one requires supersaturation for Berge cycles in r-uniform hypergraphs for $r \geq 3$ (in other words, an r-uniform version of Lemma 8). A second obstacle is that an r-uniform analog of Lemma 9 is not straightforward; for instance, if an edge e in an r-graph is contained in m Berge cycles of length $2k$, then the number of edges may be as low as $m^{1/(2k-1)}$: take a graph $2k$-cycle, and replace one edge with the hyperedge e and each other edge with $m^{1/(2k-1)}$ hyperedges. We believe these technical obstacles may be overcome (some of the ideas in the recent paper of Mubayi and Yepremyan [24] may apply).

Acknowledgments. We would like to thank the anonymous referees for their careful reading of the paper and helpful suggestions. In particular, one of the referee's comments on Berge cycles without nontriviality leads to Theorem 3.

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