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ORIGINAL PAPER

Triangle-Free Subgraphs of Hypergraphs

Jiaxi Nie[1] · Sam Spiro[1] · Jacques Verstraëte[1]

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Abstract

In this paper, we consider an analog of the well-studied extremal problem for triangle-free subgraphs of graphs for uniform hypergraphs. A *loose triangle* is a hypergraph T consisting of three edges e, f and g such that $|e \cap f| = |f \cap g| = |g \cap e| = 1$ and $e \cap f \cap g = \emptyset$. We prove that if H is an n-vertex r-uniform hypergraph with maximum degree $\triangle$, then as $\triangle \to \infty$, the number of edges in a densest T-free subhypergraph of H is at least

$$\frac{e(H)}{\triangle^{\frac{r-2}{r-1}+o(1)}}.$$

For $r = 3$, this is tight up to the $o(1)$ term in the exponent. We also show that if H is a random n-vertex triple system with edge-probability p such that $pn^3 \to \infty$ as $n \to \infty$, then with high probability as $n \to \infty$, the number of edges in a densest T-free subhypergraph is

$$\min\left\{(1-o(1))p\binom{n}{3}, p^{\frac{1}{3}}n^{2-o(1)}\right\}.$$

We use the method of containers together with probabilistic methods and a connection to the extremal problem for arithmetic progressions of length three due to Ruzsa and Szemerédi.

Keywords Loose triangle · Maximum degree · Random hypergraph · Hypergraph container

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Extended author information available on the last page of the article

1 Introduction

The *Turán numbers* for a graph F are the quantities $\mathrm{ex}(n, F)$ denoting the maximum number of edges in an F-free n-vertex graph. The study of Turán numbers is a cornerstone of extremal graph theory, going back to Mantel's Theorem [22] and Turán's Theorem [29]. A more general problem involves studying $\mathrm{ex}(G, F)$, which is the maximum number of edges in an F-free subgraph of a graph G. Some celebrated open problems are instances of this problem, such as the case when G is the n-dimensional hypercube—see Conlon [7] for recent results.

In the case that F is a triangle, $\mathrm{ex}(G, F) \geq \frac{1}{2} e(G)$ for every graph G, which can be seen by taking a maximum cut of G, which is essentially tight. In the case $G = G_{n,p}$, the *Erdős–Rényi random graph*, $\mathrm{ex}(G, F) \sim \frac{1}{2} p \binom{n}{2}$ with high probability provided p is not too small, and furthermore every maximum triangle-free subgraph is bipartite—see di Marco and Kahn [10] and also Kohayakawa, Łuczak and Rödl [21] and di Marco, Hamm and Kahn [9] for related stability results. The study of F-free subgraphs of random graphs when F has chromatic number at least three is undertaken in seminal papers of Friedgut, Rödl and Schacht [16], Conlon and Gowers [8], and Schacht [28].

1.1 Triangle-Free Subgraphs of Hypergraphs

In this paper, we consider a generalization of the problem of determining $\mathrm{ex}(G, F)$ when F is a triangle to uniform hypergraphs. We write *r-graph* instead of r-uniform hypergraph. If G and F are r-graphs, then $\mathrm{ex}(G, F)$ denotes the maximum number of edges in an F-free subgraph of G. A *loose triangle* is a hypergraph T consisting of three edges e, f and g such that $|e \cap f| = |f \cap g| = |g \cap e| = 1$ and $e \cap f \cap g = \emptyset$. We write T^r for the loose r-uniform triangle. The Turán problem for loose triangles in r-graphs was essentially solved by Frankl and Füredi [15], who showed for each $r \geq 3$ that $\mathrm{ex}(n, T^r) = \binom{n-1}{r-1}$ for n is large enough, with equality only for the r-graph S_n^r of all r-sets containing a fixed vertex. We remark that the Turán problem for r-graphs is notoriously difficult in general, and the asymptotic behavior of $\mathrm{ex}(n, K_t^r)$ is a well-known open problem of Erdős [11]—the celebrated Turán conjecture states $\mathrm{ex}(n, K_4^3) \sim \frac{5}{9} \binom{n}{3}$.

The extremal problem for loose triangles is closely connected to the extremal problem for three-term arithmetic progressions in sets of integers. Specifically, Ruzsa and Szemerédi [26] made the connection that if Γ is an abelian group and $A \subseteq \Gamma$ has no three term arithmetic progression, then the tripartite linear triple system $H(A, \Gamma)$ whose parts are equal to Γ and where $(\gamma, \gamma + a, \gamma + 2a)$ is an edge if $a \in A$—in other words, the edges are three-term progressions whose common difference is in A—is triangle-free and has $|A||\Gamma|$ edges. Ruzsa and Szemerédi [26] showed that every n-vertex triangle-free linear triple system has $o(n^2)$ edges, and applying this to $H(A, \Gamma)$ one obtains Roth's Theorem [24] that $|A| = o(|\Gamma|)$. A construction of Behrend [6] gives in $\mathbb{Z}/n\mathbb{Z}$ a set A without three-term progressions of size $n/\exp(O(\sqrt{\log n}))$, and so $H(A, \mathbb{Z}/n\mathbb{Z})$ has $n^{2-o(1)}$ edges in this case. We make use of the following theorem:

Theorem 1.1 (Ruzsa and Szemerédi [26]; Erdős, Frankl, and Rödl [12]) *For all n there exists an n-vertex r-graph which is linear, loose triangle-free, and which has $n^2 e^{-c\sqrt{\log n}}$ edges for some positive constant c.*

This theorem is an important ingredient for our first theorem, giving a general lower bound on the number of edges in a densest triangle-free subgraphs of r-graphs:

Theorem 1.2 *Let $r \geq 3$ and let G be an r-graph with maximum degree $\triangle$. Then as $\triangle \to \infty$,*

$$\mathrm{ex}(G, T^r) \geq \triangle^{-\frac{r-2}{r-1}-o(1)} e(G).$$

If a positive integer t is chosen so that $\binom{t-1}{r-1} < \triangle \leq \binom{t}{r-1}$ and $t|n$, then the n-vertex r-graph G consisting of n/t disjoint copies of a clique K_t^r has maximum degree at most $\triangle$ whereas

$$\mathrm{ex}(G, T^r) = \binom{t-1}{r-1}\frac{n}{t} = \frac{r}{t}e(G) = O(\triangle^{-\frac{1}{r-1}}) \cdot e(G).$$

Here we used the result of Frankl and Füredi [15] that S_t^r is the extremal T^r-free subgraph of K_t^r for t large enough. Therefore for $r = 3$, Theorem 1.2 is sharp up to the $o(1)$ term in the exponent of $\triangle$. For $r \geq 4$, the best construction we have gives the following proposition:

Proposition 1.3 *For $r \geq 4$ there exists an r-graph G with maximum degree $\triangle$ such that as $\triangle \to \infty$,*

$$\mathrm{ex}(G, T^r) = O(\triangle^{-\frac{1}{2}}) \cdot e(G).$$

We leave it as an open problem to determine the smallest c such that $\mathrm{ex}(G, T^r) \geq \triangle^{-c-o(1)} \cdot e(G)$ for every graph G of maximum degree $\triangle$. We conjecture the following for $r = 3$:

Conjecture 1.4 *For $\triangle \geq 1$, there exists a triple system G with maximum degree $\triangle$ such that as $\triangle \to \infty$, every T^3-free subgraph of G has $o(\triangle^{-1/2}) \cdot e(G)$ edges.*

1.2 Triangle-Free Subgraphs of Random Hypergraphs

Our next set of results concern random hosts. To this end, we say that a statement depending on n holds *asymptotically almost surely* (abbreviated a.a.s.) if the probability that it holds tends to 1 as n tends to infinity. Let $G^r_{n,p}$ denote random r-graph where edges of K^r_n are sampled independently with probability p. For the $r = 2$ case we simply write $G_{n,p}$.

A central conjecture of Kohayakawa, Łuczak and Rödl [21] was resolved independently by Conlon and Gowers [8] and by Schacht [28], and determines the asymptotic value of $\mathrm{ex}(G_{n,p}, F)$ whenever F has chromatic number at least three. The situation when F is bipartite is more complicated, partly due to the fact that the order of magnitude of Turán numbers $\mathrm{ex}(n, F)$ is not known in general—see Füredi and Simonovits [17] for a survey of bipartite Turán problems. The case of even cycles was studied by Kohayakawa, Kreuter and Steger [20] and Morris and Saxton [23] and complete bipartite graphs were studied by Morris and Saxton [23] and by Balogh and Samotij [5].

If F consists of two disjoint r-sets, then $\mathrm{ex}(n, F)$ is given by the celebrated Erdős–Ko–Rado Theorem [13], and $\mathrm{ex}(n,F) = \binom{n-1}{r-1}$. A number of researchers studied $\mathrm{ex}(G^r_{n,p}, F)$ in this case [2], with the main question being the smallest value of p such that an extremal F-free subgraph of $G^r_{n,p}$ consists of all r-sets on a vertex of maximum degree—$(1 + o(1))p\binom{n-1}{r-1}$ edges. The same subgraphs are also T^r-free, however the extremal subgraphs in that case are denser and appear to be more difficult to describe. Our second main result is as follows:

Theorem 1.5 *For all $n \geq 2$ and $p = p(n) \leq 1$ with $pn^3 \to \infty$ as $n \to \infty$, there exists a constant $c > 0$ such that asymptotically almost surely*

$$\min\left\{(1 - o(1))p\binom{n}{3}, p^{\frac{1}{3}}n^2 e^{-c\sqrt{\log n}}\right\} \leq \mathrm{ex}(G^3_{n,p}, T^3) \leq \min\left\{(1 + o(1))p\binom{n}{3}, p^{\frac{1}{3}}n^{2+o(1)}\right\},$$

and more accurately, for any constant $\delta > 0$, when $n^{-3/2+\delta} \leq p \leq n^{-\delta}$, we have

$$\mathrm{ex}(G^3_{n,p}, T^3) \leq p^{\frac{1}{3}}n^2(\log n)^c.$$

We believe that perhaps the lower bound is closer to the truth.

Since $G^3_{n,p}$ for $p > n^{-2+o(1)}$ has maximum degree $\triangle \sim p\binom{n-1}{2}$ asymptotically almost surely, Theorem 1.2 only gives $\mathrm{ex}(G^3_{n,p}, T^3) \geq p^{1/2-o(1)}n^2$ a.a.s. The upper bound in Theorem 1.5 employs the method of *containers* developed by Balogh, Morris and Samotij [3] and Saxton and Thomason [27].

We do not have tight bounds for $\mathrm{ex}(G^r_{n,p}, T^r)$ in general for all p and $r \geq 4$. Partial results and conjectures are discussed in the concluding remarks.

1.3 Counting Triangle-Free Hypergraphs

Balogh, Narayanan and Skokan [4] showed that the number of triangle-free n-vertex r-graphs is $2^{\Theta(n^{r-1})}$ using the method of containers. Note that a lower bound follows easily by counting all subgraphs of the r-graph S_n^r on n vertices consisting of all r-sets containing a fixed vertex. In this section, we adapt the method to count triangle-free hypergraphs with a specified number of edges. We let $N(r, m)$ denote the number of T^r-free r-graphs with n vertices and m edges. Analogs of Theorems 1.6 and 5.1 for graphs were proven by Balogh and Samotij [5].

Theorem 1.6 *Let $n \geq 2$, $\epsilon(n)$ be a function such that $\frac{\epsilon(n)\log n}{\log\log n} \to \infty$ as $n \to \infty$. Let $\delta = \delta(n)$ be a function such that $\epsilon(n) < \delta < 1/2 - \epsilon(n)$ and let $m = n^{2-\delta}$. Then*

$$N(3, m) \leq \left(\frac{n^2}{m}\right)^{3m+o(m)}.$$

The upper bound on $\mathrm{ex}(G^3_{n,p}, T^3)$ in Theorem 1.5 will follow from the bound on $N(3, m)$ in Theorem 1.6 by taking $m = p^{1/3-o(1)}n^2$, see details in Sect. 4.

2 Proofs of Theorem 1.2 and Proposition 1.3

For graphs, Foucaud, Krivelevich and Perarnau [14] used certain random homomorphisms to obtain good lower bounds on $\mathrm{ex}(G, F)$. We briefly summarize these ideas. Let $\mathcal{M}(F)$ denote the family of graphs F' such that there exists a graph homomorphism $\phi : V(F) \to V(F')$ and such that ϕ induces a bijection from $E(F)$ to $E(F')$. Let H be an $\mathcal{M}(F)$-free graph with many edges, which we will use as a template for our subgraph of G. Specifically, we take a random mapping $\chi : V(G) \to V(H)$ and then constructs a subgraph $G' \subseteq G$ such that $uv \in E(G')$ if and only if $\chi(u)\chi(v) \in E(H)$ and such that $\chi(u)\chi(v) \neq \chi(u)\chi(w)$ for any other edge $uw \in E(G)$ (that is, we do not keep edges which are incident and map to the same edge). It is then proven in [14] that G' will be F-free because H is $\mathcal{M}(F)$-free, and that in expectation G' will have many edges provided H does.

For general r-graphs, it is not immediately clear how to extend these ideas in such a way that we can both construct a subgraph with many edges and such that the subgraph is F-free. Fortunately for T^r we are able to do this. In particular, for this case it turns out we can avoid a hypergraph analog of the family $\mathcal{M}(F)$ provided our template r-graph is linear. This is where the Ruzsa–Szemeredi construction of Theorem 1.1 plays its crucial role.

***Proof of Theorem* 1.2** Let t be an integer to be determined later. Let χ be a random map from $V(G)$ to $[t]$ and G_t be the r-graph on $[t]$ from Theorem 1.1. For ease of notation define $\chi(e) = \{\chi(v_1), \ldots, \chi(v_r)\}$ when $e = \{v_1, \ldots, v_r\}$. Let G' be the subgraph of G which contains the edge e if and only if

(1) $\chi(e)$ is an edge of G_t, and

(2) $\chi(e') \not\subset \chi(e)$ for any $e' \in E(G)$ with $|e \cap e'| = 1$.

We claim that G' is T^r-free. Indeed, let T be a T^r of G', say with edges e_1, e_2, e_3 and $e_i \cap e_j = \{x_{ij}\}$ for $i \neq j$. Because G_t is linear, if e, e' are (possibly non-distinct) edges of G_t, then $|e \cap e'|$ is either 0, 1, or r. Note that $\chi(e_i), \chi(e_j)$ are edges of G_t by (1). Because $e_i \cap e_j = \{x_{ij}\}$ for $i \neq j$, $\chi(x_{ij}) \in \chi(e_i) \cap \chi(e_j)$, and by (2) the size of this intersection is strictly less than r. Thus $\chi(e_i) \cap \chi(e_j) = \{\chi(x_{ij})\}$. Further, we must have, say, $\chi(x_{ij}) \neq \chi(x_{ik})$ for $k \neq i, j$. This is because (1) guarantees that $\chi(x)$ is a distinct element for each $x \in e_i$, so in particular this holds for $x_{ij}, x_{ik} \in e_i$. In total this implies $\chi(e_1), \chi(e_2), \chi(e_3)$ forms a T^r in G_t, a contradiction.

We wish to compute how large $e(G')$ is in expectation. Fix some $e \in E(G)$. The probability that e satisfies (1) is exactly $e(G_t)r!/t^r$. Let $\{e_1, \ldots, e_d\}$ be the edges in $E(G)$ with $|e_i \cap e| = 1$. Given that e satisfies (1), the probability that $\chi(e_1) \not\subset \chi(e)$ is exactly $1 - (r/t)^{r-1}$. Note that for any $v \notin e \cup e_1$, the event $\chi(v) \in \chi(e)$ is independent of the event $\chi(e_1) \not\subset \chi(e)$, so we have

$$\Pr[\chi(v) \in \chi(e) |\ e \text{ satisfies (1)},\ \chi(e_1) \not\subset \chi(e)] = \frac{r}{t}.$$

On the other hand, if $v \in e_1 \setminus e$, then

$$\Pr[\chi(v) \in \chi(e) |\ e \text{ satisfies (1)},\ \chi(e_1) \not\subset \chi(e)] < \frac{r}{t},$$

as knowing some subset containing $\chi(v)$ is not contained in $\chi(e)$ makes it less likely that $\chi(v) \in \chi(e)$. By applying these observations to each vertex of $e_2 \setminus e$, we conclude that

$$\Pr[\chi(e_2) \not\subset \chi(e) |\ e \text{ satisfies (1)},\ \chi(e_1) \not\subset \chi(e)] \geq 1 - \left(\frac{r}{t}\right)^{r-1}.$$

By repeating this logic for each e_i, and using that $e(G_t) = t^{2-o(1)}$, we conclude that

$$\Pr[e \text{ satisfies (1), (2)}] \geq \frac{e(G_t)r!}{t^r}\left(1 - \left(\frac{r}{t}\right)^{r-1}\right)^{r\triangle} = t^{2-r-o(1)}\left(1 - \left(\frac{r}{t}\right)^{r-1}\right)^{r\triangle}.$$

By taking $t = r(r\triangle)^{1/(r-1)}$ and using that $(1 - x^{-1})^x$ is a decreasing function in x, we conclude by linearity of expectation that

$$\mathbb{E}[e(G')] \geq \triangle^{-1+\frac{1}{r-1}-o(1)} \cdot e(G).$$

In particular, there exists some T^r-free subgraph of G with at least this many edges, giving the desired result. □

We close this section with a proof of Proposition 1.3.

***Proof of Proposition* 1.3** According to Rödl and Thoma [25], there exists an r-graph G with $\Theta(n^3)$ edges such that every three vertices is contained in at most one edge. Let G' be a T^r-free subgraph of G. Define G'' by deleting every edge of G' which contains two vertices that are contained in at most $2r$ edges. Note that

$e(G') - e(G'') \leq 2r\binom{n}{2}$.

Assume G'' contains an edge $e = \{v_1, \ldots, v_r\}$. Because v_1, v_2 are contained in an edge of G'', there exist a set $E_{12} \subseteq E(G')$ of at least $2r + 1$ many edges containing v_1 and v_2. As G contained at most one edge containing v_1, v_2, and v_3, any $e_{12} \neq e$ in E_{12} does not contain v_3. Fix such e_{12}. Because v_2, v_3 are contained in an edge of G'', there exists a set $E_{23} \subseteq E(G')$ of at least $2r + 1 \geq r - 1$ edges containing v_2, v_3. Because G contains at most one edge containing v_2, v_3, u_i for any $u_i \in e_{12} \backslash \{v_2\}$, we conclude that there exists some $e_{23} \in E_{23}$ such that $e_{12} \cap e_{23} = \{v_2\}$. Fix such e_{23}. Because v_1, v_3 are contained in an edge of G'', there exists a set $E_{13} \subseteq E(G')$ of at least $2r + 1 \geq 2r - 3$ edges containing v_1, v_3. Because G contains at most one edge containing v_1, v_3, u_i for any $u_i \in e_{12} \cup e_{23} \backslash \{v_1, v_3\}$, we conclude that there exists some $e_{13} \in E_{13}$ such that $e_{13} \cap e_{12} = \{v_1\}$ and $e_{13} \cap e_{23} = \{v_3\}$. These three edges form a T^r in G', a contradiction. We conclude that G'' contains no edges, and hence $e(G') \leq 2r\binom{n}{2}$ for any T^r-free subgraph G' of G. As G has maximum degree $\triangle = \Theta(n^2)$, we conclude that $\text{ex}(G, T^r) = O(n^2) = O(\triangle^{-1/2}) \cdot e(G)$. □

We note that one can replace the G used in the above proof with an appropriate Steiner system to obtain a regular graph which serves as an upper bound. It has recently been proven by Keevash [19] and Glock, Kühn, Lo, and Osthus [18] that such Steiner systems exist whenever n satisfies certain divisibility conditions and is sufficiently large.

3 Proof of Theorem 1.5: Lower Bound

As noted in the introduction, the bound of Theorem 1.2 is sharp for $r = 3$ by considering the disjoint union of cliques, so we can not improve upon this bound in general. However, we are able to do better when G contains few copies of T^r by using a deletion argument.

Proposition 3.1 *Let $R(G)$ denote the number of copies of T^r in the r-graph G. Then for some positive constant c and any integer $t \geq 1$,*

$$\text{ex}(G, T^r) \geq (e(G)t^{2-r} - R(G)r^{3r}t^{5-3r})e^{-c\sqrt{\log t}}.$$

Proof Let χ be a random map from $V(G)$ to $[t]$ and G_t the r-graph on $[t]$ from Theorem 1.1. For ease of notation, if $e = \{v_1, \ldots, v_r\}$ we define $\chi(e) := \{\chi(v_1), \ldots, \chi(v_r)\}$. Let G' be the subgraph of G which contains the edge e if and only if $\chi(e)$ is an edge of G_t.

We claim that $e_1, e_2, e_3 \in E(G')$ form a T^r in G' if and only if e_1, e_2, e_3 form a T^r in G and $\chi(e_1) = \chi(e_2) = \chi(e_3)$ is an edge of G_t. Indeed, the backwards direction is clear. Assume for contradiction that these edges form a T^r in G' and that $\chi(e_1) \neq \chi(e_2)$. Let x_{ij} for $i \neq j$ be such that $e_i \cap e_j = \{x_{ij}\}$. Because G_t is linear, if

e, e' are (possibly non-distinct) edges of G_t, then $|e \cap e'|$ is either 0, 1, or r. Because each e_i is in $E(G')$, we have $\chi(e_i) \in E(G_t)$ by construction. In particular, as $e_1 \cap e_2 = \{x_{12}\}$ and $\chi(e_1) \neq \chi(e_2)$, we must have $\chi(e_1) \cap \chi(e_2) = \{\chi(x_{12})\}$. As e_3 contains an element in e_1 (namely x_{13}) and an element not in e_1 (namely x_{23}), we must have $\chi(e_1) \cap \chi(e_3) = \{\chi(x_{13})\}$. Similarly we have $\chi(e_2) \cap \chi(e_3) = \{\chi(x_{23})\}$. Because $\chi(e_i)$ is an r-set for each i, we have $\chi(x_{ij}) \neq \chi(x_{ik})$ for $\{i,j,k\} = \{1,2,3\}$. Thus $\chi(e_1), \chi(e_2), \chi(e_3)$ form a T^r in G_t, a contradiction.

The probability that a given $T^r \subseteq G$ maps onto a given $f \in E(G_t)$ is at most $(r/t)^{3r-3}$, since in particular each vertex of T^r must map onto one of the r vertices of f. By the claim above this is the only way that a T^r can appear in G', so by linearity of expectation we find

$$\mathbb{E}[R(G')] \leq \frac{e(G_t) r^{3r-3}}{t^{3r-3}} R(G).$$

Let $G'' \subseteq G'$ be a subgraph obtained by deleting an edge from each T^r of G'. By construction G'' is T^r-free. Since $e(G_t) \geq t^2 e^{-c\sqrt{\log t}}$ for some positive constant c, we conclude by linearity of expectation that

$$\begin{aligned} \mathrm{ex}(G, T^r) &\geq \mathbb{E}[e(G'')] \geq \mathbb{E}[e(G') - R(G')] \\ &= \frac{e(G_t) r!}{t^r} e(G) - \frac{e(G_t) r^{3r-3}}{t^{3r-3}} R(G) \\ &\geq (e(G) t^{2-r} - R(G) r^{3r} t^{5-3r}) e^{-c_2\sqrt{\log t}}. \end{aligned}$$

□

Corollary 3.2 *For any integer* $r \geq 3$, *and function* $p = p(n) \leq 1$ *such that* $p^{2/(2r-3)} n \geq 2$, *we have*

$$\mathbb{E}[\mathrm{ex}(G^r_{n,p}, T^r)] \geq p^{\frac{1}{2r-3}} n^2 e^{-c\sqrt{\log n}},$$

for some constant $c > 0$.

Proof Note for $n \geq 4$ that $\mathbb{E}[e(G^r_{n,p})] = p\binom{n}{r} \geq pn^r r^{-r}$, and that $\mathbb{E}[R(G^r_{n,p})] \leq p^3 n^{3r-3}$. Plugging these into the bound of Proposition 3.1 gives for some positive constant c_1

$$\mathbb{E}[\mathrm{ex}(G^r_{n,p}, T^r)] \geq (pn^r r^{-r} t^{2-r} - p^3 n^{3r-3} r^{3r} t^{5-3r}) e^{-c_1\sqrt{\log t}}.$$

Taking $t = (2r^{4r})^{\frac{1}{2r-3}} p^{2/(2r-3)} n$, we conclude for some positive constant c_2 and sufficiently large n that

$$\mathbb{E}[\mathrm{ex}(G^r_{n,p}, T^r)] \geq p^{\frac{1}{2r-3}} n^2 e^{-c_2\sqrt{\log n}}.$$

□

To get the a.a.s. result of Theorem 1.5, we use Azuma's inequality (see for example in Alon and Spencer [1]) applied to the edge exposure martingale.

Lemma 3.3 *Let f be a function on r-graphs such that* $|f(G) - f(H)| \le 1$ *whenever H is obtained from G by adding or deleting one edge. Then for any* $\lambda > 0$,

$$\Pr\left[|f(G^r_{n,p}) - \mathbb{E}[f(G^r_{n,p})]| > \lambda\sqrt{\binom{n}{r}}\right] < e^{-\frac{\lambda^2}{2}}.$$

***Proof of Theorem 1.5* (*Lower bounds*)** Let $\epsilon(n) = e^{k\sqrt{\log n}}$, where $k > 0$ is some large enough constant. For $p \le n^{-3/2}/\epsilon(n)$, it is not difficult to show that a.a.s. $G^3_{n,p}$ contains $o(pn^3)$ copies of T^3, and by deleting an edge from each of these loose cycles we see that $\mathrm{ex}(G^3_{n,p}, T^3) \ge (1 - o(1)p\binom{n}{3}$ a.a.s..

For $n^{-3/2}/\epsilon(n) \le p \le n^{-3/2}\epsilon(n)$, we do an extra round of random sampling on the edges of $G^r_{n,p}$ and keep each edge with probability $p' := \epsilon(n)^{-2}$. The r-graph we obtained is equivalent to $G^r_{n,pp'}$, with $pp' \le n^{-3/2}/\epsilon(n)$. Thus $\mathrm{ex}(G^3_{n,p}, T^3) \ge (1 - o(1))pp'\binom{n}{3} = (1 - o(1))p\binom{n}{3}/\epsilon(n)^2$ a.a.s.. Using $p \ge n^{-3/2}/\epsilon(n)$, we conclude that $\mathrm{ex}(G^3_{n,p}, T^3) \ge p^{1/3}n^2e^{-3k\sqrt{\log n}}$ a.a.s. in this range.

We now consider $p \ge n^{-3/2}\epsilon(n)$. The bound in expectation follows from Corollary 3.2. To show that this result holds a.a.s., we observe that $f(G) = \mathrm{ex}(G, T^3)$ satisfies the conditions of Lemma 3.3. For ease of notation let $X_{n,p} = \mathrm{ex}(G^3_{n,p}, T^3)$ and let $B_{n,p} = p^{1/3}n^2e^{-c_2\sqrt{\log n}}$ be the lower bound for $\mathbb{E}[X_{n,p}]$ given in Corollary 3.2. Setting $\lambda = \frac{1}{2}B_{n,p}\binom{n}{3}^{-1/2}$ and applying Azuma's inequality, we find

$$\begin{aligned}\Pr\left[X_{n,p} < \frac{1}{2}B_{n,p}\right] &\le \Pr\left[X_{n,p} - \mathbb{E}[X_{n,p}] < -\lambda\binom{n}{3}^{\frac{1}{2}}\right]\\ &\le \Pr\left[\left|X_{n,p} - \mathbb{E}[X_{n,p}]\right| > \lambda\binom{n}{3}^{\frac{1}{2}}\right] \le \exp\left(-\frac{\lambda^2}{2}\right).\end{aligned}$$

Note that for $p \ge n^{-3/2}\epsilon(n)$ we have $\lambda \ge e^{(k/3-c_2)\sqrt{\log n}} \to \infty$ as $n \to \infty$. So we conclude the a.a.s. result. □

4 Containers

The method of containers developed by Balogh, Morris and Samotij [3] and Saxton and Thomason [27] is a powerful technique that has been used to solve a number of combinatorial problems. Roughly, the idea is for a suitable hypergraph H to find a family of sets $\mathcal{C}$ which contain every independent set of H, and in such a way that $|\mathcal{C}|$ is small and each $C \in \mathcal{C}$ contains few edges. For example, by letting H be the 3-uniform hypergraph where each edge is a K_3 in some graph G, we see that independent sets of H correspond to triangle-free subgraphs of G. The existence of containers then allows us to better understand how these subgraphs of G behave.

We proceed with the technical details of this approach. Given an r-graph $H = (V, E)$, let $v(H) = |V|$, $e(H) = |E|$, and let $\mathcal{P}(V)$ be the family of all subsets of V. For a set A of vertices in H, let $d(A)$ be the number of edges in H that contain A. Let $\bar{d}(H)$ be the average vertex degree of H, and let $\triangle_j(H) = \max_{|A|=j} d(A)$. In order to establish our upper bounds, we need to use the following container lemma for hypergraphs:

Lemma 4.1 (Balogh, Morris and Samotij [3]) *Let* $r, b, l \in \mathbb{N}$, $\delta = 2^{-r(r+1)}$, *and* $H = (V, E)$ *an* r*-graph such that*

$$\triangle_j(H) \leq \left(\frac{b}{v(H)}\right)^{j-1} \frac{e(H)}{l},\ j \in \{1, 2, \ldots, r\}.$$

Then there exists a collection $\mathcal{C}$ *of subsets of* V *such that*:

(1) *For every independent set* I *of* H, *there exists* $C \in \mathcal{C}$ *such that* $I \subset C$.
(2) *For every* $C \in \mathcal{C}$, $|C| \leq v(H) - \delta l$.
(3) $|\mathcal{C}| \leq \sum_{s=0}^{(k-1)b} \binom{v(H)}{s}$.

We will use this container lemma to give an upper bound for $N(r, m)$. The idea is to consider the 3-graph H with $V(H) = E(K_n^r)$ and $E(H)$ consisting of T^r in K_n^r. Notice that the container lemma requires upper bounds for the maximum codegrees of the hypergraph. In order to meet this requirement, we will use a balanced-supersaturation lemma for T^r:

Lemma 4.2 (Balogh, Narayanan and Skokan [4]) *For any integer* $r \geq 3$, *there exists* $c = c(r)$ *such that the following hold for all* $n \in \mathbb{N}$. *Given any* r*-graph* G *on* $[n]$ *with* $e(G) = tn^{r-1}$, $t \geq 6(r-1)$, *let* $S = tn^{r-4}$ *if* $r \geq 4$ *and* $S = 1$ *if* $r = 3$. *Then there exists a* 3*-graph* H *on* $E(G)$, *where each edge of* H *is a copy of* T^r *in* G, *such that*:

(1) $\bar{d}(H) \geq c^{-1} t^3 S^2$.
(2) $\triangle_j(H) \leq ct^{5-2j} S^{3-j}$ *for each* $j = 1, 2$.

Using the previous two lemmas, we derive the following container lemma for T^3-free hypergraphs. Similar result for T^r-free hypergraphs can also be obtained using the same idea, and we omit the details.

Lemma 4.3 *Let $c_1 = c(3)$ be the constant obtained in Lemma 4.2 with $r = 3$. For any integer n and positive number t with $\max(12, c_1) \le t \le \binom{n}{3}/n^2$, there exists a collection $\mathcal{C}$ of subgraphs of K_n^3 such that for some constant c_2:*

(1) *For any T^3-free subgraph J of K_n^3, there exists $C \in \mathcal{C}$ such that $J \subset C$.*
(2) $|\mathcal{C}| \le \exp\left(\frac{c_2(\log t)n^2}{\sqrt{t}}\right)$.
(3) *For every $C \in \mathcal{C}$, $e(C) \le tn^2$.*

Proof By Lemma 4.2 with $r = 3$, there exists a positive constant c_1 such that for any 3-graph G on $[n]$ with $e(G) = t_0 n^2$, where $t_0 \ge t$, there exists a 3-graph H on $E(G)$ such that each edge of H is a copy of T^3 in G, such that:

(1) $\bar{d}(H) \ge c_1^{-1} t_0^3$.
(2) $\triangle_j(H) \le c_1 t_0^{5-2j}, j = 1, 2.$ $\triangle_3(H) = 1$.

We can then use Lemma 4.1 on H with $l = t_0 n^2/(3c_1^2)$ and $b = n^2/\sqrt{c_1 t_0}$ to get a collection $\mathcal{C}$ of subgraphs of G such that they contain all T^3-free subgraphs of G, and for each $C \in \mathcal{C}$, $e(C) \le (1-\epsilon)t_0 n^2$ for some constant $\epsilon > 0$. Also, we have

$$|\mathcal{C}| \le \sum_{s=0}^{2b} \binom{t_0 n^2}{s} \le \exp\left(\frac{c_2(\log t_0)n^2}{\sqrt{t_0}}\right)$$

for some constant $c_2 > 0$.

We use the above argument on $G = K_n^3$ to get a family of containers $\mathcal{C}_1$. Notice that the containers of $\mathcal{C}_1$ are also 3-graph on $[n]$, so we can repeat this argument on each $C \in \mathcal{C}_1$ with more than tn^2 edges to get a new collection of containers $\mathcal{C}_2$. We do this repeatedly until all containers have less than tn^2 edges. Since in each step the number of edges will decrease by a constant $(1-\epsilon)$, this process must stop after at most $\log(n/t)/\epsilon$ steps. Let $t_0 = \binom{n}{3}/n^2$, $t_{k+1} = (1-\epsilon)t_k$ for $k \ge 0$, and let M be the largest integer such that $t_M > t$. By definition of t_k, we have $t_{M-i} > (1-\epsilon)^{-i} t$, and hence, there exists a constant $\delta = \delta(\epsilon) > 0$ such that $\frac{\log t_{M-i}}{\sqrt{t_{M-i}}} < (1-\delta)^i \frac{\log t}{\sqrt{t}}$.

Then in the worst case, the number of containers we have in the end is less than

$$\prod_{i=0}^{M} \exp\left(\frac{c_2(\log t_i)n^2}{\sqrt{t_i}}\right) = \exp\left(\sum_{i=0}^{M} \frac{c_2(\log t_i)n^2}{\sqrt{t_i}}\right) < \exp\left(c_2 n^2 \sum_{i=0}^{M} (1-\delta)^i \frac{\log t}{\sqrt{t}}\right) < \exp\left(\frac{c_2(\log t)n^2}{\delta\sqrt{t}}\right).$$

This completes the proof. □

With the lemma above, we are ready to prove Theorem 1.6. The proof of Theorem 5.1 is essentially the same and we omit the details.

***Proof of Theorem* 1.6** Let $\mathcal{C}$ be a collection of containers and c a constant as in Lemma 4.3 with $t = n^{2\delta+\epsilon_1(n)}$, where $\epsilon_1(n) = \frac{2\log\log n}{\log n}$. Since $\epsilon(n) < \delta < 1/2 - \epsilon(n)$, and $\epsilon(n) = \omega(\frac{\log n}{\log\log n})$, t must satisfies the condition in Lemma 4.3. By considering all subgraphs of each $C \in \mathcal{C}$ with $m = n^{2-\delta}$ edges, we conclude that

$$\begin{aligned} N(3,m) &\le \exp\left(\frac{c(\log t)n^2}{\sqrt{t}}\right) \cdot \binom{tn^2}{m} \\ &\le \exp\left(c\log t \cdot \frac{m}{\log n} + (1 + (3\delta + \epsilon_1(n))\log n)m\right) \\ &\le \exp\left(\delta \log n \cdot m\left(3 + (2+o(1))\frac{\log\log n}{\delta \log n}\right)\right) \\ &\le \left(\frac{n^2}{m}\right)^{\left(3+(2+o(1))\frac{\log\log n}{\delta\log n}\right)m}. \end{aligned}$$

Since $\delta > \epsilon(n) = \omega(\frac{\log n}{\log\log n})$, we have

$$N(3,m) = \left(\frac{n^2}{m}\right)^{3m+o(m)}.$$

□

We are now ready to prove the upper bound of Theorem 1.5.

***Proof of Theorem* 1.5 (*Upper bound*)** We will only present the proof of

$$\mathrm{ex}(G^3_{n,p}, T^3) \le \min\left\{(1+o(1))p\binom{n}{3}, p^{\frac{1}{3}}n^{2+o(1)}\right\}$$

for $p \le 1$. The proof of the more accurate upper bound in smaller range is essentially the same, with more careful and explicit computation for the $o(1)$ factor. For $p \le n^{-3/2+o(1)}$, the proof for upper bound is exactly the same as that for lower bound when $p \le n^{-3/2}\epsilon(n)$. We now consider $n^{-3/2+\epsilon(n)} \le p \le n^{-\epsilon(n)}$ for some small function $\epsilon(n) = o(1)$ to be determined. Our goal is to show

$$\Pr[ex(G^3_{n,p}, T^3) \ge m] \to 0, \ \ as\ n \to \infty,$$

for some $m = p^{1/3}n^{2+o(1)}$. Let X_m be the expected number of T^3-free subgraphs in $G^3_{n,p}$ with m edges. By Theorem 1.6, when $n^{3/2+\epsilon_1(n)} \le m \le n^{2-\epsilon_1(n)}$ for some function $\epsilon_1(n) = o(1)$, there exists a function $\epsilon_2(n) = o(1)$ such that the expectation of X_m satisfies

$$\begin{aligned}\mathbb{E}[X_m] &= N(3, m) \cdot p^m \\ &\le \left(\frac{n^2}{m}\right)^{m(3+\epsilon_2(n))} p^m \\ &= \left(\left(\frac{n^2}{m}\right)^{(3+\epsilon_2(n))} p\right)^m.\end{aligned}$$

We can let $m = n^2p^{1/3-\epsilon_3(n)}$ for some small function $\epsilon_3(n) = o(1)$ such that

$$\left(\frac{n^2}{m}\right)^{(3+\epsilon_2(n))} p < 1.$$

Also we can pick some suitable $\epsilon(n)$, so that $n^{3/2+\epsilon_1(n)} \le m \le n^{2-\epsilon_1(n)}$. Thus we have $\mathbb{E}[X_m] \to 0$ as $n \to \infty$. Then by Markov's inequality, we have

$$\Pr[ex(G^3_{n,p}, T^3) \ge m] = \Pr[X_m \ge 1] \le \mathbb{E}[X_m] \to 0, \ \ as\ n \to \infty.$$

So a.a.s. we have

$$ex(G^3_{n,p}, T^3) < m = p^{\frac{1}{3}}n^{2+o(1)}.$$

Finally for $p \ge n^{-o(1)}$, we have $\mathrm{ex}(G^3_{n,p}, T) < \mathrm{ex}(K^3_n, T) = \Theta(n^2) = p^{1/3}n^{2+o(1)}$ a.a.s. □

5 Concluding Remarks

– We are able to generalize Theorem 1.6 to r-graphs as follows:

Theorem 5.1 *Let $r \ge 4$, $n \ge 1$, $0 < \delta < 3/2$ and $m = n^{3-\delta}$. Then*

$$N(r, m) \le \left(\frac{n^{r-1}}{m}\right)^{\left(1+\frac{2\delta}{3r-12+3\delta}\right)m+o(m)}.$$

When $r > 4$, let $m = n^{3+\delta}$ with δ some constant satisfying $0 < \delta < r-4$. Then we have

$$N(r,m) \le \left(\frac{n^{r-1}}{m}\right)^{m+o(m)}.$$

This bound will also leads to an upper bound for $\mathrm{ex}(G^r_{n,p}, T^r)$ when $n^{-r+3/2+o(1)} \le p \le 1$, which is essentially tight for $p = p(n)$ with $n^{-r+4+o(1)} \le p \le 1$. However, there is a gap between the lower bound and upper bound in the range $n^{-r+3/2+o(1)} \le p \le n^{-r+4+o(1)}$.

– Using the same techniques for the $r = 3$ case, we are able to show the following.

Theorem 5.2 *For $r \ge 4$ and $0 \le x \le r$ a constant, let $p = n^{-r+x}$ and define*

$$f_r(x) = \lim_{n\to\infty} \log_n \mathbb{E}[\mathrm{ex}(G^r_{n,p}, T^r)].$$

Then for $0 \le x \le 3/2, f_r(x) = x$; for $4 < x \le r, f_r(x) = x - 1$; and for $3/2 < x \le 4$, we have

$$\max\left\{\frac{x+3r-6}{2r-3}, x-1\right\} \le f_r(x) \le \frac{3x+3}{5}.$$

The bounds for $x \le 3/2$ come from deleting an edge from each triangle in $G^r_{n,p}$. For $x > 3/2$, the upper bound follow from Theorem 5.1, the first lower bound follows from Corollary 3.2, and the second lower bound follows from taking every edge containing a given vertex.

– We believe that the upper bound is perhaps closer to the truth and have the following conjecture.

Conjecture 5.3 *For $r \ge 4$ and $0 \le x \le r$ a constant, let $p = n^{-r+x}$ and $f_r(x)$ as defined in Theorem 5.2. Then for $\frac{3}{2} < x \le 4$,*

$$f_r(x) = \frac{3x+3}{5}$$

– For the deterministic case, we note that one can extend the proof of Theorem 1.2 to other F by defining maps $\chi : V(G) \to V(H)$ for suitable H. In this case a second step must be done to effectively bound $\mathrm{ex}(G, F)$. We plan to do this in a followup paper.

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Declarations

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Authors and Affiliations

Jiaxi Nie[1] · Sam Spiro[1] · Jacques Verstraëte[1]

✉ Jiaxi Nie
jin019@ucsd.edu

Sam Spiro
sspiro@ucsd.edu

Jacques Verstraëte
jverstra@math.ucsd.edu

[1] Department of Mathematics, University of California, San Diego, La Jolla, USA