PAPER

Compressible Navier–Stokes equations with heterogeneous pressure laws

IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **34** (2021) 4115–4162 https://doi.org/10.1088/1361-6544/ac03a1

Compressible Navier–Stokes equations with heterogeneous pressure laws

Didier Bresch[1], **Pierre-Emmanuel Jabin**[2] **and Fei Wang**[3,*]

[1] Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA, 73000 Chambéry, France
[2] Department of Mathematics and Huck Institutes, Pennsylvania State University, State College, PA 16801, United States of America
[3] Department of Mathematics, Shanghai Jiao Tong University, Shanghai, People's Republic of China

E-mail: didier.bresch@univ-smb.fr, pejabin@psu.edu and fwang256@sjtu.edu.cn

Received 20 August 2020, revised 8 April 2021
Accepted for publication 20 May 2021
Published 18 June 2021

Abstract

This paper concerns the existence of global weak solutions *à la Leray* for compressible Navier–Stokes equations with a pressure law which depends on the density and on time and space variables t and x. The assumptions on the pressure contain only locally Lipschitz assumption with respect to the density variable and some hypothesis with respect to the extra time and space variables. It may be seen as a first step to consider heat-conducting Navier–Stokes equations with physical laws such as the truncated virial assumption. The paper focuses on the construction of approximate solutions through a new regularized and fixed point procedure and on the weak stability process taking advantage of the new method introduced by the two first authors with a careful study of an appropriate regularized quantity linked to the pressure.

Keywords: compressible, Navier–Stokes, fluid dynamics, heterogeneous pressure, weak solution

Mathematics Subject Classification numbers: 35Mxx.

1. Introduction and main result

As mentioned in [6], the existence of global weak solutions, in the sense of J Leray [7], to the non-stationary barotropic compressible Navier–Stokes (CNS) system with constant shear and bulk viscosities μ and λ remained a longstanding open problem in space dimension strictly greater than one until the first results by Lions (see [18]) with $P(\rho) = a\rho^{\gamma}$ ($\gamma > 3d/(d+2)$). Many important contributions followed to improve the result including

*Author to whom any correspondence should be addressed.
Recommended by Dr A L Mazzucato.

Feireisl–Novotny–Petzeltova ($\gamma > d/2$, see [14, 19]), Plotnikov–Weigant ($\gamma = d/2$, see [20]), E Feireisl (pressure law $s \mapsto P(s)$ non-monotone on a compact set, see [16]) and more recently Bresch–Jabin (thermodynamically unstable pressure law $s \mapsto P(s)$ or anisotropic viscosities, see [4]).

One of the main issue is that the weak bound of the divergence of the velocity field does not *a priori* rule out singular behaviors by the density which may oscillate, concentrate or even vanish (vacuum state) even if this is not the case initially.

Heat-conducting viscous CNS equations (Navier–Stokes–Fourier) with constant viscosities namely with a pressure law $(\rho, \vartheta) \mapsto P(\rho, \vartheta)$ and an extra equation on the temperature ϑ has been firstly discussed in [18] and solved by E. Feireisl and A. Novotny for specific pressure laws, see [12, 13] which in some sense are monotone with respect to the density after a fixed value. In the present paper, we prepare the resolution of the heat-conducting CNS equations with a truncated virial pressure law

$$P(\rho, \vartheta) = \rho^\gamma + \vartheta \sum_{n=0}^{[\gamma/2]} B_n(\vartheta)\, \rho^n. \tag{1.1}$$

Such pressure law is not monotone with respect to the density after a fixed value and therefore is not thermodynamically stable. This paper concerns the existence of global weak solutions à la Leray for CNS equations with a pressure law which depends on the density and on time and space variables t and x. It may be seen as a first step to consider heat-conducting Navier–Stokes equations with physical laws such as the truncated virial assumption. More precisely, we consider the CNS equations

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \tag{1.2}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda)\nabla \operatorname{div} u + \nabla P = 0 \tag{1.3}$$

with initial condition

$$\rho|_{t=0} = \rho_0 \qquad (\rho u)|_{t=0} = m_0, \tag{1.4}$$

in a periodic box $\Omega = \mathbb{T}^d = [-\pi, \pi]^d$ for $d \geqslant 2$ and μ and λ two constants satisfying the physical constraint $\mu > 0$ and $\lambda + 2\mu/d > 0$. The pressure $P = P(t, x, \rho)$ is a given function depending on the time t, space x, and the density ρ. For simplicity in the redaction we consider in the sequel that the shear viscosity $\mu = 1$ and the bulk viscosity $\lambda = -1$: this does not changed the mathematical proof and result.

For simplicity, we consider the periodic boundary conditions in x, namely $\Omega = \mathbb{T}^d$, even if arguments can be adapted to the whole space case as well. As explained previously, the article should be seen as a first step to solve the truncated virial case where we assume that the temperature $\vartheta(t, x)$ is actually given instead of solving the temperature equation

$$\partial_t(\rho E) + \operatorname{div}_x(\rho E\, u) + \operatorname{div}_x(P(\rho, \vartheta)\, u) = \operatorname{div}_x(\nabla_x u \cdot u) + \operatorname{div}_x(\kappa(\vartheta)\, \nabla \vartheta), \tag{1.5}$$

where $E = |u|^2/2 + e(\rho, \vartheta)$ is the total energy density with $e(\rho, \vartheta)$ is the specific internal energy and initial condition

$$\rho E|_{t=0} = \rho_0 E_0. \tag{1.6}$$

with the virial pressure state law (1.1). The main result presented here will be used in our upcoming article (see [9]) to construct solutions to the full system (1.2)–(1.4), (1.5) and (1.6)

as it provides the starting point for the fixed point procedure that we adopt. If ϑ is given then naturally $P(t,x,\rho)=P(\rho,\vartheta(t,x))$. But there are however several other contexts (for instance in biology) where it is necessary to involve non spatially homogeneous pressure law and for this reason, it is useful to consider more general formulas for P than given by (1.1). Note that as shown in [5, 8], the procedure developed here is also applicable for the compressible Brinkman system (semi-stationary compressible Stokes system) which is standard system that may be seen in porous media and biology.

The construction of appropriate approximate solutions will be a difficulty in our paper. It is based on an original approximate system for which existence of solutions is obtained through a regularization and a fixed point approach. The weak stability property on the sequence of approximate solution is obtained using the new method introduced by the two first authors in [4] and taking care of the regularized term linked to the pressure state law which involves serious difficulties.

We assume **hypothesis on the pressure law** $(t,x,s)\mapsto P(t,x,s)$: some of them are used to ensure the propagation of energy and the others are used to guarantee the propagation of compactness on the density.

More precisely, let us present:

-Assumptions to ensure the *propagation of energy*.

Let $\gamma>3d/(d+2)$:

(P1) There exist $q>2,\ 0\leqslant\bar\gamma\leqslant\gamma/2,$ and a smooth function P_0 such that

$$|P(t,x,s)-P_0(t,x,s)|\leqslant C\,R(t,x)+C\,s^{\bar\gamma}\qquad\text{for } R\in L^q((0,T)\times\mathbb{T}^d). \tag{1.7}$$

(P2) There exist $p<\gamma+\dfrac{2\gamma}{d}-1,\ q>2,\ \Theta_1(t,x)\in L^q((0,T)\times\mathbb{T}^d),$ such that

$$C^{-1}\,s^\gamma-\Theta_1(t,x)\leqslant P_0(t,x,s)\leqslant Cs^p+\Theta_1(t,x). \tag{1.8}$$

(P3) There exist $p<\gamma+\dfrac{2\gamma}{d}-1,$ and $\Theta_2\in L^q([0,\ T]\times\mathbb{T}^d)$ with $q>1$ such that

$$|\partial_t P_0(t,x,s)|\leqslant Cs^p+\Theta_2(t,x). \tag{1.9}$$

(P4) $$|\nabla_x P_0(t,x,s)|\leqslant C\,s^{\gamma/2}+\Theta_3(t,x),\qquad\text{for } \Theta_3\in L^2([0,\ T],\ L^{2d/(d+2)}(\mathbb{T}^d)). \tag{1.10}$$

-Assumptions required for the *propagation of compactness* on the density.

(P5) The pressure P is locally Lipschitz in the sense of that

$$\begin{aligned}|P(t,x,z)-P(t,y,w)|\leqslant{}& Q(t,x,y)+C(z^{\gamma-1}+w^{\gamma-1})\\&+(\widetilde P(t,x)+\widetilde P(t,y))|z-w|,\end{aligned}$$

$$\text{for some } \widetilde P\in L^{s_0}([0,T]\times\mathbb{T}^d)\ \text{ and }\ Q\in L^{s_1}([0,T]\times\mathbb{T}^{2d}) \text{ for some } s_0,s_1>1. \tag{1.11}$$

(P6) The functions $Q,\ \widetilde P$ satisfy that $r_h\to0,$ as $h\to0,$ with

$$\frac{1}{\|\mathcal{K}_h\|_{L^1}}\int_0^T\int_{\mathbb{T}^{2d}}\mathcal{K}_h(x-y)\left(|\widetilde P(t,x)-\widetilde P(t,y)|^{s_0}+|Q(t,x,y)|^{s_1}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t=r_h. \tag{1.12}$$

The total energy of the CNS system. The total energy of the system, which is the sum of the kinetic and the potential energies, reads

$$\mathcal{E}(t,x,\rho,\rho u)=\int_{\mathbb{T}^d}\left(\frac{|\rho u|^2}{2\rho}+\rho e(t,x,\rho)\right)\mathrm{d}x$$

where

$$e(t,x,\rho)=\int_{\rho_{\mathrm{ref}}}^{\rho}\frac{P(t,x,s)}{s^2}\,\mathrm{d}s \tag{1.13}$$

with ρ_{ref} a constant reference density. We also define similarly the reduced total energy $\mathcal{E}_0(t,x,\rho,\rho u)$ which is based on P_0 instead of P, see assumption (1.7). Note that we assume as usually

$$u_0=\frac{m_0}{\rho_0}\ \text{when}\ \rho_0\neq 0\ \text{ and }\ u_0=0\ \text{ elsewhere,} \tag{1.14}$$

$$\frac{|m_0|^2}{\rho_0}=0\ \text{a.e. on}\ \{x\in\Omega:\rho_0(x)=0\}. \tag{1.15}$$

The following is our main result dealing with heterogeneous pressure laws.

Theorem 1.1. *Assume the initial data m_0 and $\rho_0\geqslant 0$ with $\int_{\mathbb{T}^d}\rho_0=M_0>0$ satisfy*

$$\mathcal{E}(\rho_0,m_0)=\int_{\mathbb{T}^d}\left(\frac{|m_0|^2}{2\rho_0}+\rho_0 e(0,x,\rho_0)\right)\mathrm{d}x<\infty.$$

Suppose that the pressure P satisfies (1.7)–(1.12). Assuming for simplicity $\mu=1$ and $\mu+\lambda=0$, then there exists a global weak solution to CNS system (1.2)–(1.4) such that

$$u\in L^2(0,T;H^1(\mathbb{T}^d)),\qquad |m|^2/2\rho\in L^\infty(0,T;L^1(\mathbb{T}^d))$$

$$\rho\in\mathcal{C}([0,T],L^\gamma(\mathbb{T}^d)\,\textit{weak}\,)\cap L^p((0,T)\times\mathbb{T}^d)\quad \textit{where}\quad 0<p<\gamma(d+2)/2-1$$

with the heterogeneous pressure state law P satisfying the energy inequality

$$\begin{aligned}\int_{\mathbb{T}^d}\mathcal{E}_0(\rho,u)\,\mathrm{d}x+\int_0^t\int_{\mathbb{T}^d}|\nabla u(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s&\leqslant\mathcal{E}(\rho_0,u_0)\\&+\int_0^t\int_{\mathbb{T}^d}\mathrm{div}_x\,u(s,x)\,(P(s,x,\rho(s,x))-P_0(s,x,\rho(s,x)))\mathrm{d}s\,\mathrm{d}x\\&+\int_0^t\int_{\mathbb{T}^d}(\rho\,\partial_t e_0(s,x,\rho(s,x))+\rho\,u\cdot\nabla_x e_0(s,x,\rho(s,x)))\,\mathrm{d}x\,\mathrm{d}s\end{aligned}$$

where

$$\mathcal{E}_0(\rho,u)=|\rho u|^2/2\rho+\rho\int_{\rho_{\mathrm{ref}}}^{\rho}\frac{P_0(t,x,s)}{s^2}\,\mathrm{d}s.$$

Remark 1.2. We note that since P_0 is smooth, the reduced internal energy $e_0(t,x,\xi)$ is smooth in each variable. This allows us to give a precise meaning to the terms above

$$\begin{aligned}\partial_t e_0(s,x,\rho(s,x))&=\partial_t e_0(s,x,\xi)|_{\xi=\rho(s,x)},\\\nabla_x e_0(s,x,\rho(s,x))&=\nabla_x e_0(s,x,\xi)|_{\xi=\rho(s,x)}.\end{aligned}$$

Remark 1.3. Note that $u \in L^2(0,T;H^1(\mathbb{T}^d))$ comes from the control of the gradient of the velocity field ∇u in $L^2((0,T)\times\Omega)$ and the control of $|m|^2/\rho$ in $L^1((0,T)\times\mathbb{T}^d)$ using the fact that $\int_\Omega \rho = \int_\Omega \rho_0 = M > 0$. The interested reader is referred to [18].

Remark 1.4. We have assumed in the proof that $\mu = 1$ and $\lambda + \mu = 0$ for simplicity but it is straightforward that it is valuable for $\mu > 0$ and $\lambda + 2\,\mu/d \geqslant 0$ as usually.

2. The approximation systems with a sketch of proof and *a priori* estimates

We present here the approximate system upon which we rely to construct the solution to (1.2)–(1.4) with the pressure law P given by (1.7)–(1.12). As is classical in compressible fluid mechanics, the approximation procedure is performed through several stages, involving different approximate systems.

2.1. The approximate system with artificial and delocalized pressures

One of the main difficulty is to find a proper approximation of the above system so that we may construct a solution of it and prove the compactness of the solutions. We propose to define the approximating system

$$\partial_t \rho_{\varepsilon,\eta} + \operatorname{div}(\rho_{\varepsilon,\eta} u_{\varepsilon,\eta}) = 0 \tag{2.1}$$

$$\partial_t(\rho_{\varepsilon,\eta} u_{\varepsilon,\eta}) + \operatorname{div}(\rho_{\varepsilon,\eta} u_{\varepsilon,\eta}\otimes u_\varepsilon) - \Delta u_{\varepsilon,\eta} + \nabla(P_{\mathrm{art},\eta}(\rho_{\varepsilon,\eta}) + \mathcal{L}_\varepsilon * P) = 0 \tag{2.2}$$

with initial condition

$$\rho_{\varepsilon,\eta}|_{t=0} = \rho_{0,\varepsilon,\eta} \quad \text{and} \quad (\rho_{\varepsilon,\eta} u_{\varepsilon,\eta})|_{t=0} = m_{0,\varepsilon,\eta} \tag{2.3}$$

where an artificial pressure term reads

$$P_{\mathrm{art},\eta}(\rho_{\varepsilon,\eta}) = \eta_1\, \rho_{\varepsilon,\eta}^{\gamma_{\mathrm{art},1}} + \cdots + \eta_m\, \rho_{\varepsilon,\eta}^{\gamma_{\mathrm{art},m}}$$

for some fixed parameters $\gamma_{\mathrm{art}} = \gamma_{\mathrm{art},1} > \gamma_{\mathrm{art},2} > \cdots > \gamma_{\mathrm{art},m}$. The coefficients $\eta_1,\ldots,\eta_m$ will later be let to converge to 0 in that order and the $\gamma_{\mathrm{art},i}$ will be chosen so that

$$\gamma_{\mathrm{art},1} > 2\,\gamma, \qquad \gamma_{\mathrm{art},i+1} + 2\,\frac{\gamma_{\mathrm{art},i+1}}{d} - 1 > \gamma_{\mathrm{art},i}, \qquad \gamma + 2\,\frac{\gamma}{d} - 1 > \gamma_{\mathrm{art},m}.$$

In addition an appropriate regularization of the pressure state law $\mathcal{L}_\varepsilon * (P(t,\cdot,\rho_{\varepsilon,\eta}(t,\cdot))$ has been introduced. More precisely the key step is to construct a suitable mollifying operator $\mathcal{L}_\varepsilon$ defined as follows

$$\mathcal{L}_\varepsilon(x) = \frac{1}{\log 2}\int_\varepsilon^{2\varepsilon} L_{\varepsilon'}(x)\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}.$$

where L_ε is a standard mollifier given by

$$L_\varepsilon(x) = \frac{1}{\varepsilon^d} L\left(\frac{x}{\varepsilon}\right),$$

with L is a non-negative smooth function such that $L \in C_0^\infty(\mathbb{T}^d)$ and $\int_{\mathbb{T}^d} L(x)\,\mathrm{d}x = 1$. Then $L_\varepsilon \to \delta_0$ as $\varepsilon \to 0$, with δ_0 being the Dirac delta function at 0. It is straightforward to check that

$$\int_{\mathbb{T}^d} \mathcal{L}_\varepsilon(x)\,\mathrm{d}x = 1$$

and

$$\mathcal{L}_\varepsilon \to \delta_0, \quad \text{as} \quad \varepsilon \to 0.$$

We observe that we easily have the following global existence result through a fixed point argument that will be presented in the appendix for readers convenience

Theorem 2.1. *Assume that P satisfies (1.7) with $\gamma_{\rm art} > \gamma$ and that the initial data $\rho_{0,\varepsilon,\eta}, u_{0,\varepsilon,\eta}$ satisfy the uniform bound*

$$\sup_{\varepsilon,\eta} \int_{\mathbb{T}^d} \left(\eta_1 \left(\rho_{0,\varepsilon,\eta}(x)\right)^{\gamma_{\rm art},1} + \cdots + \eta_m \left(\rho_{0,\varepsilon,\eta}(x)\right)^{\gamma_{\rm art},m} + \rho_{0,\varepsilon,\eta}(x) \left|u_{0,\varepsilon,\eta}(x)\right|^2 \right) \mathrm{d}x < \infty.$$

There exist $\rho_{\varepsilon,\eta} \in L^\infty([0,\ T],\ L^{\gamma_{\rm art}}(\mathbb{T}^d)) \cap L^p([0,\ T] \times \mathbb{T}^d)$ for any $p < \gamma_{\rm art} + 2\gamma_{\rm art}/d - 1$, $u_{\varepsilon,\eta} \in L^2([0,\ T],\ H^1(\mathbb{T}^d))$ solution to (2.1) and (2.2). Moreover, $\rho_{\varepsilon,\eta}, u_{\varepsilon,\eta}$ satisfy the uniform in ε bounds

$$\sup_{\varepsilon} \sup_{t\in[0,\ T]} \int_{\mathbb{T}^d} \left(\eta_1\, \rho_{\varepsilon,\eta}^{\gamma_{\rm art},1}(t,x) + \cdots + \eta_m\, \rho_{\varepsilon,\eta}^{\gamma_{\rm art},m}(t,x) + \rho_{\varepsilon,\eta}(t,x) \left|u_{\varepsilon,\eta}(t,x)\right|^2 \right) \mathrm{d}x < \infty, \tag{2.4a}$$

$$\sup_{\varepsilon} \int_0^T \int_{\mathbb{T}^d} |\nabla u_{\varepsilon,\eta}|^2 \,\mathrm{d}x\,\mathrm{d}t < \infty, \tag{2.4b}$$

$$\sup_{\varepsilon} \int_0^T \int_{\mathbb{T}^d} \eta_1\, \rho_{\varepsilon,\eta}^p(t,x) \,\mathrm{d}x\,\mathrm{d}t < \infty \quad \textit{for any } p < \gamma_{\rm art} + 2\,\gamma_{\rm art}/d - 1. \tag{2.4c}$$

Finally, we have the explicit energy inequality

$$\begin{aligned} &\int_{\mathbb{T}^d} \left(\eta_1 \frac{\rho_{\varepsilon,\eta}^{\gamma_{\rm art},1}(t,x)}{\gamma_{\rm art},1 - 1} + \cdots + \eta_m \frac{\rho_{\varepsilon,\eta}^{\gamma_{\rm art},m}(t,x)}{\gamma_{\rm art},m - 1} + \rho_{\varepsilon,\eta}(t,x) \left|u_{\varepsilon,\eta}(t,x)\right|^2 \right) \mathrm{d}x \\ &\quad + \int_0^t \int_{\mathbb{T}^d} |\nabla u_{\varepsilon,\eta}(s,x)|^2 \,\mathrm{d}x\,\mathrm{d}s \leqslant \int_0^t \int_{\mathbb{T}^d} \mathrm{div}\, u_{\varepsilon,\eta}\, \mathcal{L}_\varepsilon *_x P \,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_{\mathbb{T}^d} \left(\eta_1 \frac{(\rho_{0,\varepsilon,\eta})^{\gamma_{\rm art},1}(t,x)}{\gamma_{\rm art},1 - 1} + \cdots + \eta_m \frac{(\rho_{0,\varepsilon,\eta})^{\gamma_{\rm art},m}(t,x)}{\gamma_{\rm art},m - 1} \right) \mathrm{d}x. \\ &\quad + \int_{\mathbb{T}^d} \rho_{0,\varepsilon,\eta}(t,x) \left|u_{0,\varepsilon,\eta}(t,x)\right|^2 \mathrm{d}x. \end{aligned} \tag{2.5}$$

The main difficulty and contribution of the present article is the limit passage $\varepsilon \to 0$, with η fixed, given by the following result

Theorem 2.2. *Assume that P satisfies (1.11) and (1.12). Let $\gamma_{\rm art} > \max(2s_0', s_1', 2+d)$, where s_0' and s_1' are the Hölder conjugate exponents of s_0 and s_1 respectively. Suppose that the initial data ρ_ε^0, u_ε^0 of the system (2.1) and (2.2) satisfy that $\rho_{0,\varepsilon,\eta} \to \rho_{0,\eta}$ in $L^{\gamma_{\rm art}}(\mathbb{T}^d)$, $\rho_{0,\varepsilon,\eta} u_{0,\varepsilon,\eta} \to \rho_{0,\eta} u_{0,\eta}$ and $\rho_{0,\varepsilon,\eta}|u_{0,\varepsilon,\eta}|^2 \to |\rho_{0,\eta} u_{0,\eta}|^2$ in $L^1(\mathbb{T}^d)$. Let $(\rho_{\varepsilon,\eta}, u_{\varepsilon;\eta})$ be the corresponding sequence of solutions satisfying the energy estimate (2.4). Then $\rho_{\varepsilon,\eta}$ is compact in $L^p(\mathbb{T}^d)$ for $1 \leqslant p < \gamma_{\rm art}$ as $\varepsilon \to 0$.*

The particular form of the mollifier operator $\mathcal{L}_\varepsilon$ is strongly used for the compactness property on $\{\rho_{\varepsilon,\eta}\}_\varepsilon$ to have enough control of terms involving the pressure terms in the method

introduced by the two first authors in [4]. Using the previous theorem, the limit passage provides a sequence of global weak solutions (ρ_η, u_η) to the following system

$$\partial_t \rho_\eta + \operatorname{div}(\rho_\eta u_\eta) = 0 \tag{2.6}$$

$$\partial_t(\rho_\eta u_\eta) + \operatorname{div}(\rho u_\eta \otimes u_\eta) - \Delta u_\eta + \nabla(P_{\mathrm{art},\eta}(\rho_\eta) + P(t,x,\rho_\eta)) = 0, \tag{2.7}$$

for some large $\gamma_{\mathrm{art}} \geqslant \gamma$ with initial boundary conditions

$$\rho_\eta|_{t=0} = \rho_{0,\eta}, \qquad \rho_\eta u_\eta|_{t=0} = m_{0,\eta}. \tag{2.8}$$

Fortunately once we obtain global weak solutions to (2.6)–(2.8) then passing to the limit as $\eta_1 \to 0$, then $\eta_2 \to 0$ and up to $\eta_m \to 0$, to obtain global weak solutions to (1.2)–(1.4) is in fact a straightforward consequence of [4]. More precisely we have

Theorem 2.3. *Assume that P satisfies (1.7)–(1.12). Consider any sequence $\rho_\eta \in L^\infty([0,\ T],\ L^{\gamma_{\mathrm{art}}}(\mathbb{T}^d))$ with $\gamma_{\mathrm{art},m} < \gamma + 2\gamma/d - 1$, $\gamma_{\mathrm{art},i} < \gamma_{\mathrm{art},i+1} + 2\gamma_{\mathrm{art},i+1}/d - 1$ and $\gamma_{\mathrm{art}} > 2\gamma$, any sequence $u_\eta \in L^2([0,\ T],\ H^1(\mathbb{T}^d))$ of solutions to (2.6) and (2.7) over $[0,T]$. Suppose moreover that $\rho_\eta^0 \to \rho^0$ in $L^\gamma(\mathbb{T}^d)$, $\rho_\eta^0 u_\eta^0 \to \rho^0 u^0$ and $\rho_\eta^0 |u_\eta^0|^2 \to \rho^0 |u^0|^2$ both in $L^1(\mathbb{T}^d)$. Assume finally that $\sup_\eta \sup_{t\in[0,\ T]} \int_{\mathbb{T}^d} \rho_\eta |u_\eta|^2 \,\mathrm{d}x < \infty$. Then ρ_η is compact in $L^1_{t,x}$, u_η is compact in $L^2_{t,x}$ and converge to a global solution to (1.2) and (1.3) with*

$$\begin{aligned}
\int_{\mathbb{T}^d} \mathcal{E}_0(\rho,u)\,\mathrm{d}x + \int_0^t \int_{\mathbb{T}^d} |\nabla u(s,x)|^2 \,\mathrm{d}x\,\mathrm{d}s \leqslant\ & \mathcal{E}(\rho_0,u_0) \\
&+ \int_0^t \int_{\mathbb{T}^d} \operatorname{div}_x u(s,x)\,(P(s,x,\rho(s,x)) - P_0(s,x,\rho(s,x)))\mathrm{d}s\,\mathrm{d}x \\
&+ \int_0^t \int_{\mathbb{T}^d} (\rho(s,x)\,\partial_t e_0(s,x,\rho(s,x)) + \rho(s,x)\,u(s,x)\cdot\nabla_x e_0 \\
&\times (s,x,\rho(s,x)))\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

The proof of theorem 2.3 will be discussed in the appendix of the article for reader's convenience. This will end the proof of the main theorem 1.1.

Important remark. It is important to note that the requirement for having several exponents $\gamma_{\mathrm{art},i}$ in the artificial pressure $P_{\mathrm{art},\eta}$ appears from the constraints in the proofs of theorems 2.1–2.3. To recover the appropriate energy terms in theorem 2.1, we need to treat the actual pressure P as a source term. This is only possible if $\operatorname{div} u\mathcal{L}_\varepsilon \star P$ is integrable uniformly in ε and, as $P \lesssim \rho^\gamma$, it forces that $\gamma_{\mathrm{art}} > 2\gamma$.

On the other hand, assuming that $\gamma_{\mathrm{art},1}, \ldots, \gamma_{\mathrm{art},i-1} = 0$, to pass to the limit in the term $\eta_i \rho^{\gamma_{\mathrm{art},i}}$ as $\eta_i \to 0$ but $\eta_{i+1} > 0$, we again need to have $\rho^{\gamma_{\mathrm{art},i}}$ integrable. From the gain of integrability detailed in the next subsection, this only appears possible if $\gamma_{\mathrm{art},i} < \gamma_{\mathrm{art},i+1} + 2\gamma_{\mathrm{art},i+1}/d - 1$. If we had only one correction in $P_{\mathrm{art},\eta}$, i.e. $m = 1$, then we would actually need both $\gamma_{\mathrm{art}} > 2\gamma$ and $\gamma_{\mathrm{art}} < \gamma + 2\gamma/d - 1$, which is of course not possible if $d \geqslant 2$. The introduction of several exponents $\gamma_{\mathrm{art},i}$ seems to be a fairly straightforward manner of resolving this issue.

2.2. Basic energy estimates

As those are used several times, we collect here the basic energy estimates for the generic system

$$\begin{aligned}&\partial_t\rho + \operatorname{div}(\rho\, u) = 0,\\ &\partial_t(\rho\, u) + \operatorname{div}(\rho\, u\otimes u) - \Delta u + \nabla(P_0(t,x,\rho) + S(t,x)) = 0.\end{aligned} \tag{2.9}$$

We define

$$\begin{aligned}\mathcal{X} = {} & L^1([0,\ T],\ W^{-1,p}(\mathbb{T}^d))\\ & + H^{-1}([0,\ T],\ L^{2pd/(2d+2p-pd)}(\mathbb{T}^d)) \cap W^{-1,\infty}([0,\ T],\ L^{pd/(p+d)}(\mathbb{T}^d)).\end{aligned}$$

There exist a well-known gain in integrability on ρ from the momentum equation. For convenience later, we write it in a slightly more general form.

Lemma 2.4. *Assume that* $\rho \in L^2([0,\ T]\times\mathbb{T}^d) \cap L^\infty([0,\ T],\ L^{\gamma_0}(\mathbb{T}^d))$ *for* $\gamma_0 \geqslant d/2$ *solves* (B.1) *with a velocity field* $u \in L^2([0,\ T],\ H^1(\mathbb{T}^d))$ *and source term* $S \in L^1([0,\ T],\ L^{\gamma_0'}(\mathbb{T}^d)$. *Assume* $\nabla_x(P_0(t,x,\rho(t,x)) + S(t,x)) \in \mathcal{X}$, *then for any* $0 < \theta < \gamma_0/p'$

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}^d} \rho^\theta(s,x)\, P_0(s,x,\rho(s,x))\,\mathrm{d}s\,\mathrm{d}x\\ &\qquad \leqslant C_d\,\|\rho\|^\theta_{L^\infty_t L^{\gamma_0}_x}\,(1+\|u\|_{L^2_t H^1_x})\,\|\nabla_x P_0(\rho)\|_{\mathcal{X}} + C_d\,\|\rho\|^\theta_{L^\infty_t L^{\gamma_0}_x}\,\|S\|_{L^1_t L^{\gamma_0'}_x}.\end{aligned}$$

Proof. We can rewrite the assumption simply as

$$\nabla_x(P_0(t,x,\rho) + S) = \operatorname{div}_x f + \partial_t g,$$

where $f \in L^1([0,\ T],\ L^p(\mathbb{T}^d))$ and $g \in L^2([0,\ T],\ L^{2pd/(2d+2p-pd)}(\mathbb{T}^d))$, with in addition $g \in L^\infty([0,\ T],\ L^{pd/(p+d)}(\mathbb{T}^d))$. For a fixed exponent $\theta > 0$ to be chosen later, we define $c_\theta = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\rho^\theta(t,x)\,\mathrm{d}x$ and $B(t,x) = -\nabla_x\,\Delta_x^{-1}\,(\rho^\theta - c_\theta)$. In the case of a bounded domain with a boundary instead of the torus, one has to be more careful and use the appropriate Bogovski operator (see [12] for example).

The idea is then simply for multiply by B and first notice that

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}^d} B(s,x)\cdot\nabla_x(S + P_0(s,x,\rho))\,\mathrm{d}x\,\mathrm{d}s\\ &\qquad = \int_0^T\int_{\mathbb{T}^d} (\rho^\theta(s,x) - c_\theta)\,(S + P_0(s,x,\rho))\,\mathrm{d}x\,\mathrm{d}s\\ &\qquad \geqslant -C + \int_0^T\int_{\mathbb{T}^d} \rho^\theta(s,x)\,(S + P_0(s,x,\rho(s,x)))\,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

where the constant C depends on ρ, θ, T, S and P_0:

$$C = c_\theta\ (\|S\|_{L^1_{t,x}} + \|P_0(.,.,\rho(.,.))\|_{L^1_{t,x}}) \leqslant c_\theta\,(\|S\|_{L^1_t L^{\gamma_0'}_x} + \|\nabla P_0\|_{\mathcal{X}}).$$

The integral of $\rho^\theta S$ can be bounded immediately to yield the second term in the right-hand side of the lemma,

$$\int_0^T\int_{\mathbb{T}^d} \rho^\theta(s,x)\,S\,\mathrm{d}x\,\mathrm{d}s \leqslant \|\rho^\theta\|_{L^\infty_t L^{\gamma_0}_x}\,\|S\|_{L^\infty_1 L^{\gamma_0'}_x}.$$

On the other hand

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d} B(s,x)\cdot\nabla_x(S+P_0(s,x,\rho))\,\mathrm{d}x\,\mathrm{d}s\\ &= -\int_0^T\int_{\mathbb{T}^d}\nabla_x B(s,x)\,:\,f(s,x)\,\mathrm{d}x\,\mathrm{d}s\\ &\quad -\int_0^T\int_{\mathbb{T}^d}\partial_t B(s,x)\cdot g(s,x)\,\mathrm{d}x\,\mathrm{d}s\\ &\quad +\int_{\mathbb{T}^d}(B(0,x)\cdot g(0,x)-B(0,T)\cdot g(T,x))\,\mathrm{d}x.\end{aligned}$$

By standard Calderon–Zygmund theory, $\|\nabla_x B\|_{L_t^\infty L_x^{\gamma_0/\theta}}\leqslant C_d\,\|\rho\|^\theta_{L_t^\infty L_x^{\gamma_0}}$. Hence the first term in the rhs is directly bounded by

$$\begin{aligned}-\int_0^T\int_{\mathbb{T}^d}\nabla_x B(s,x)\,:\,f(s,x)\,\mathrm{d}x\,\mathrm{d}s &\leqslant \|f\|_{L_t^1 L_x^p}\,\|\nabla B\|_{L_t^\infty L_x^{p'}}\\ &\leqslant C_d\,\|f\|_{L_t^1 L_x^p}\,\|\rho\|^\theta_{L_t^\infty L_x^{\gamma_0}},\end{aligned}$$

since $p'\leqslant\gamma_0/\theta$ as $\theta<\gamma_0/p'$. By Sobolev embedding $\|B\|_{L_t^\infty L_x^q}\leqslant C_d\,\|\rho\|^\theta_{L_t^\infty L_x^{\gamma_0}}$ with $1/q=\theta/\gamma_0-1/d$. Hence we have again that

$$\int_{\mathbb{T}^d}(B(0,x)\cdot g(0,x)-B(0,T)\cdot g(T,x))\,\mathrm{d}x\leqslant C_d\,\|\rho\|^\theta_{L_t^\infty L_x^{\gamma_0}}\,\|g\|_{L_t^\infty L_x^{pd/(p+d)}},$$

since $1-(p+d)/pd=1-1/d-1/p\geqslant\theta/\gamma_0-1/d$ by the same condition on θ. The second term in the rhs is handled by using the continuity equation (B.1) satisfied by ρ. Due to the assumptions that $\rho\in L^2([0,\ T]\times\mathbb{T}^d)$, ρ is a renormalized solution to (B.1) by theorem B.1 and hence we have that

$$\partial_t\rho^\theta+\operatorname{div}(\rho^\theta\,u)=(1-\theta)\,\rho^\theta\operatorname{div}u.$$

We may replace

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d}\partial_t B(s,x)\cdot g(s,x)\,\mathrm{d}x\,\mathrm{d}s=\int_0^T\int_{\mathbb{T}^d}\nabla_x\,\Delta_x^{-1}\,\big((1-\theta)\,\rho^\theta\operatorname{div}u\\ -\operatorname{div}(\rho^\theta\,u)-\widetilde{c}_\theta\big)\cdot g(s,x)\,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

for some time dependent constant $\widetilde{c}_\theta$. Using that $g\in L_t^2 L_x^{2pd/(2d+2p-pd)}$, we bound in a similar manner all the terms and conclude that

$$-\int_0^T\int_{\mathbb{T}^d}\partial_t B(s,x)\cdot g(s,x)\,\mathrm{d}x\,\mathrm{d}s\leqslant C_d\,\|\rho\|^\theta_{L_t^\infty L_x^{\gamma_0}}\,\|u\|_{L_t^2 H_x^1}\,\|g\|_{L_t^2 L_x^{2p/(p+2)}}.$$

□

3. Notations and technical preliminaries

In this section, we give our notations and list technical results with considerations which were mostly developed in [4] and upon which our proof relies.

3.1. Notations

Because we use functions at various points and differences of functions, we introduce specific notations. First, the symbol f^x stands for a function of x, i.e., $f^x = f(x)$. Next, we also denote

$$\delta f(x,\xi) = f(x) - f(x-\xi)$$

and

$$\bar{f}(x,\xi) = f(x) + f(x-\xi).$$

If the argument is not mentioned explicitly then we set $\xi = x - y$, i.e.,

$$\delta f = \delta f(x, x-y) = f(x) - f(y)$$

and

$$\overline{f} = \overline{f}(x, x-y) = f(x) + f(y).$$

We denote the maximum operator by

$$M f(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f(x)| \, \mathrm{d}x.$$

Recall, see that

$$\|M f\|_{L^p} \lesssim \|f\|_{L^p}$$

for $p > 1$ and where the relation $f \lesssim g$ stands for that $f \leqslant Cg$ for some constant $C > 0$. We use bracket to stand for the commutator

$$[f, T]g = f\, T\, g - T(f\, g)$$

where f and g are smooth functions and T is an operator.

3.2. Our compactness criterion

As is classical in compressible fluid mechanics, the main difficulty in obtaining existence is to prove the compactness of a sequence of approximations of the density ρ_ε. As mentioned above, we follow here the general strategy of [4], and we hence rely on the following criterion.

Lemma 3.1. *Let ρ_ε be a family of functions which are bounded in some $L^p([0,T]\times\mathbb{T}^d)$ with $1 \leqslant p < \infty$. Assume that $\mathcal{K}_h$ is a family of positive bounded functions such that*

- $\sup_h \int_{|x|\geqslant\eta} \mathcal{K}_h(x)\,\mathrm{d}x < \infty$ *for any* $\eta > 0$.
- $\|\mathcal{K}_h\|_{L^1} \to \infty$ *as* $h \to 0$.

Assume that for some $q \geqslant 1$

$$\sup_\varepsilon \|\partial_t \rho_\varepsilon\|_{L^q([0,T],W^{-1,1}(\mathbb{T}^d))} < \infty$$

and

$$\lim_{h\to 0} \limsup_\varepsilon \int_0^T \int_{\mathbb{T}^{2d}} \frac{\mathcal{K}_h(x-y)}{\|\mathcal{K}_h\|_{L^1}} |\rho_\varepsilon(x) - \rho_\varepsilon(y)|^p \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s = 0.$$

Then the family of functions ρ_ε is compact in $L^p([0,T]\times\mathbb{T}^d)$. Conversely if ρ_ε is compact in $L^p([0,T]\times\mathbb{T}^d)$, then the above limit is 0.

The construction of a suitable kernel function $\mathcal{K}_h$ for the system that we are considering again follows [4]. We first define a bounded, positive, and symmetric function $\widetilde{K}_h$ such that

$$\widetilde{K}_h(x)=\frac{1}{(h+|x|)^{d+a}},\quad \text{for } |x|\leqslant\frac{1}{4}$$

with some $a>0$ and $\widetilde{K}_h$ independent of h for $|x|\geqslant 1/3$. We will also require that $\widetilde{K}_h\in C^\infty(\mathbb{T}^d\backslash B(0,1/4))$ and that $\operatorname{supp}\widetilde{K}_h\subset B(0,1/2)$. Setting

$$K_h=\frac{\widetilde{K}_h}{\|\widetilde{K}_h\|_{L^1(\mathbb{T}^d)}}$$

we have immediately that

$$\|K_h\|_{L^1(\mathbb{T}^d)}=1$$

and

$$|x||\nabla K_h(x)|\lesssim |K_h(x)|. \tag{3.1}$$

For our compactness argument, we use the operator

$$\mathcal{K}_{h_0}=\int_{h_0}^1 K_h(x)\frac{\mathrm{d}h}{h}. \tag{3.2}$$

Note that

$$\|\mathcal{K}_{h_0}\|_{L^1(\mathbb{T}^d)}=c_0|\log\ h_0|$$

for some positive constant c_0. With the above notation, one of our main steps is to show that

$$\limsup_\varepsilon\int_0^T\int_{\mathbb{T}^{2d}}\mathcal{K}_{h_0}(x-y)|\rho_\varepsilon(x)-\rho_\varepsilon(y)|^p\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\to 0$$

as $h_0\to 0$, from where the compactness of the family ρ_ε follows.

3.3. Technical lemmas

As our main strategy is to control differences $\delta\rho_\varepsilon$, which requires some specific lemmas. One may find proofs for these lemmas in [4]. Our basic way of estimating differences is through

Lemma 3.2. *Let $u\in W^{1,1}$, we have*

$$|u(x)-u(y)|\lesssim (D_{|x-y|}u(x)+D_{|x-y|}u(y))|x-y|,$$

where

$$D_hu(x)=\frac{1}{h}\int_{|z|\leqslant h}\frac{|\nabla u(x+z)|}{|z|^{d-1}}\,\mathrm{d}z.$$

The next lemma provides a bound for the term $D_hu(x)$ in term of the maximal function.

Lemma 3.3. *For any $u \in W^{1,p}$ with $p \geqslant 1$, the following inequality holds*

$$D_h u(x) \lesssim M|\nabla u|(x).$$

Remark 3.4. By the above two lemmas we deduce immediately the classical inequality

$$|u(x) - u(y)| \lesssim (M|\nabla u(x)| + M|\nabla u(y)|)|x - y|. \tag{3.3}$$

In several critical places of the proof, we need to estimate the difference $D_{|z|}u(x) - D_{|z|}u(x - z)$ while relying only on the L^2 regularity of ∇u. Using classical harmonic analysis results, we can get the following.

Lemma 3.5. *Assume that $u \in H^1(\mathbb{T}^d)$. Then for any $1 < p < \infty$, one has*

$$\int_{h_0}^1 \int_{\mathbb{T}^d} K_h(z) \|D_{|z|}u(x) - D_{|z|}u(x - z)\|_{L^p_x} \,\mathrm{d}z \,\frac{\mathrm{d}h}{h} \lesssim \|u\|_{B^1_{p,1}}$$

as a result of which, we further have that

$$\int_{h_0}^1 \int_{\mathbb{T}^d} K_h(z) \|D_{|z|}u(x) - D_{|z|}u(x - z)\|_{L^2_x} \,\mathrm{d}z \,\frac{\mathrm{d}h}{h} \lesssim \|u\|_{H^1} |\log\, h_0|^{1/2}.$$

Moreover, the following estimate holds

$$\int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(z)K_h(\xi) \|D_{|z|}u(x) - D_{|z|}u(x - \xi)\|_{L^2_x} \,\mathrm{d}z \,\mathrm{d}\xi \,\frac{\mathrm{d}h}{h}$$
$$\lesssim \|u\|_{H^1} |\log\, h_0|^{1/2}.$$

In most instances, the above estimate is sufficient. But in several cases, we need the more general version, see [21] for more details.

Lemma 3.6. *Consider a family of kernels $N_r \in W^{s,1}(\mathbb{T}^d)$, where $s > 0$, which satisfy*

- $\sup_{|\xi| \leqslant 1} \sup_r r^{-s} \int_{\mathbb{T}^d} |z|^s |N_r(z) - N_r(z - r\xi)| \,\mathrm{d}z < \infty$,
- $\sup_r (\|N_r\|_{L^1} + r^s \|N_r\|_{W^{s,1}}) < \infty$.

Then the estimate

$$\int_{h_0}^1 \int_{\mathbb{T}^d} K_h(z) \|N_h * u(x) - N_h * u(x - z)\|_{L^p_x} \,\mathrm{d}z \,\frac{\mathrm{d}h}{h} \lesssim \|u\|_{L^p} |\log\, h_0|^{1/2}$$

holds for any $u \in L^p$ with $1 < p \leqslant 2$.

3.4. The choice of the weight function

We now turn to the construction of an appropriate weight function tailored for the proof of theorem 2.2. First we define the function w_ε which satisfies the equations

$$\partial_t w_\varepsilon + u_\varepsilon \cdot \nabla w_\varepsilon = -D_\varepsilon w_\varepsilon \tag{3.4}$$

$$w_\varepsilon(0) = 1 \tag{3.5}$$

where D_ε is given by

$$D_\varepsilon = \lambda(M|\nabla u_\varepsilon| + |\rho_\varepsilon|^\gamma + K_h * (|\mathrm{div}\, u_\varepsilon| + |\mathcal{L}_\varepsilon * P| + |\widetilde{P}^{1+l}_\varepsilon|)). \tag{3.6}$$

Denote

$$w_{\varepsilon,h} = K_h * w_\varepsilon.$$

Then the weight function $W_{\varepsilon,h}$ we use is given by

$$W^{x,y}_{\varepsilon,h} = w_{\varepsilon,h}(x) + w_{\varepsilon,h}(y)$$

which could capture the feature that $W_{\varepsilon,h}$ is big if either one of $w_{\varepsilon,h}(x)$ and $w_{\varepsilon,h}(y)$ is big. Since the function $W^{x,y}_\varepsilon = w_\varepsilon(x) + w_\varepsilon(y)$ satisfies the following equation

$$\partial_t W_\varepsilon + u^x_\varepsilon \cdot \nabla_x W_\varepsilon + u^y_\varepsilon \cdot \nabla_y W_\varepsilon = -(D^x_\varepsilon w_\varepsilon(x) + D^y_\varepsilon w_\varepsilon(y)),$$

it follows that

$$\partial_t W_{\varepsilon,h} + u^x_\varepsilon \cdot \nabla_x W_{\varepsilon,h} + u^y_\varepsilon \cdot \nabla_y W_{\varepsilon,h} = -D^{x,y}_{\varepsilon,h} + \mathrm{Com}^{x,y}_{\varepsilon,h} \tag{3.7}$$

where

$$D^{x,y}_{\varepsilon,h} = K_h * (D_\varepsilon w_\varepsilon)(x) + K_h * (D_\varepsilon w_\varepsilon)(y) \tag{3.8}$$

and

$$\mathrm{Com}^{x,y}_{\varepsilon,h} = [u_\varepsilon \cdot, K_h *] \nabla w_\varepsilon(x) + [u_\varepsilon \cdot, K_h *] \nabla w_\varepsilon(y). \tag{3.9}$$

We conclude the subsection by listing several properties of this weight function without giving a proof (see again [4] for the proof).

Proposition 3.7. *Assume that $(\rho_\varepsilon, u_\varepsilon)$ solves system (2.1) and (2.2) with the bounds (2.4) satisfied. Then there exists a weight function w_ε which satisfies equations (3.4) and (3.5) with D_ε given by (3.6) such that the following hold:*

- *For any t, x, $0 \leqslant w_\varepsilon \leqslant 1$.*
- *If $p \geqslant \gamma + 1$, then we have*

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^d} \rho_\varepsilon(t,x) |\log\, w_\varepsilon(t,x)|\, \mathrm{d}x \leqslant C(1+\lambda). \tag{3.10}$$

- *For $p \geqslant 1 + \gamma$,*

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^d} \rho_\varepsilon(t,x) \mathbf{1}_{K_h * w_\varepsilon \leqslant \eta}\, \mathrm{d}x \leqslant C \frac{1+\lambda}{|\log\, \eta|}. \tag{3.11}$$

- *Setting $\mathcal{D} = |\mathrm{div}\, u_\varepsilon| + |\mathcal{L}_\varepsilon * P| + |\widetilde{P}^{1+l}_\varepsilon|$ for penalization, for $p > \gamma$ we have the following commutator estimate*

$$\int_{h_0}^1 \int_0^t \|K_h * (w_\varepsilon\, K_h * \mathcal{D}) - w_{\varepsilon,h}\, K_h * \mathcal{D}\|_{L^q}\, \mathrm{d}t\, \frac{\mathrm{d}h}{h} \leqslant C |\log\, h_0|^{1/2} \tag{3.12}$$

with $q = \min(2, p/\gamma)$.

4. Proof of theorem 2.2

In this section, we give a proof of the theorem 2.2 using the compactness argument provided in lemma 3.1. Because all coefficients η_i are fixed for this section, we drop the index η in our notations to keep them simple.

In order to carry out our approach, we introduce a smooth function $\chi(\xi) \in C^1(\mathbb{R})$ given by

$$\chi(\xi) = |\xi|^{1+l} \tag{4.1}$$

where $0 < l < 1/2$ is to be specified below. Throughout this section, χ is used as a function of $\delta\rho_\varepsilon(x)$ or $\delta\rho_\varepsilon(x,y)$. We recall that

$$\delta\rho_\varepsilon = \delta\rho_\varepsilon(x, x-y) = \rho_\varepsilon(x) - \rho_\varepsilon(y),$$

together with

$$\bar{f} = \bar{f}(x, x-y) = f(x) + f(y)$$

for a general function f; see subsection 3.1 for the notations convention. To make the presentation compact, we also denote

$$\chi = \chi(\delta\rho_\varepsilon), \qquad \chi(x,y) = \chi(\delta\rho_\varepsilon(x,y)).$$

Notations for χ' are similarly defined. We aim to show

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y)\chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y \to 0 \quad \text{as } h_0 \to 0. \tag{4.2}$$

We follow the general strategy developed in [4] by introducing weights and propagate the following quantity instead:

$$T_{h_0,\varepsilon}(t) = \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y) W_{\varepsilon,h}^{x,y} \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}$$

where as before

$$W_{\varepsilon,h}^{x,y} = w_{\varepsilon,h}^x + w_{\varepsilon,h}^y.$$

Since the weight w_ε satisfies (3.4), the time derivative of $T_{h_0,\varepsilon}$ will exhibit the penalization (3.6) which is necessary to control some of the other terms.

Compared with [4], we however have a different approximation system (2.1) and (2.2). While many terms in the proof are similar, the pressure term is considerably more intricate and its treatment constitutes the main innovation in this paper performed in subsection 4.5.

We calculate the time derivative $T_{h_0,\varepsilon}$ in lemma 4.1 in the next subsection. The proof of statement (4.2) will then follow from the bounds on the various terms in the right-hand side that are divided into several lemmas.

4.1. The time evolution of $T_{h_0,\varepsilon}(t)$

Before stating the first lemma, we recall some notation used in subsection 3.4. The penalization term is defined as

$$D_{\varepsilon,h}^{x,y} = K_h * (D_\varepsilon w_\varepsilon)(x) + K_h * (D_\varepsilon w_\varepsilon)(y)$$

and the commutator term is given as

$$\mathrm{Com}^{x,y}_{\varepsilon,h} = [u_\varepsilon\cdot, K_h*]\nabla w_\varepsilon(x) + [u_\varepsilon\cdot, K_h*]\nabla w_\varepsilon(y).$$

We then have,

Lemma 4.1. *Let ρ_ε and u_ε be a sequence of solutions to the system (2.1) and (2.2) satisfying the bound (2.4) with $\gamma_{\mathrm{art}} \geqslant 3d/(d+2)$. Assume that the pressure P satisfies (1.7), (1.8), and (1.11). Then we have the estimate*

$$T_{h_0,\varepsilon}(t) \lesssim T_{h_0,\varepsilon}(0) + I_1 + I_2 + I_3 + I_4 + I_5, \tag{4.3}$$

where the terms I_1–I_5 are given by

$$I_1 = \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} \delta u_\varepsilon \nabla_x K_h(x-y) W^{x,y}_{\varepsilon,h} \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \tag{4.4}$$

$$I_2 = -\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y) D^{x,y}_{\varepsilon,h} \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \tag{4.5}$$

$$I_3 = \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y) \mathrm{Com}^{x,y}_{\varepsilon,h} \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \tag{4.6}$$

$$I_4 = -\frac{1}{2}\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y) W^{x,y}_{\varepsilon,h} \chi'(\delta\rho_\varepsilon)\overline{\rho}_\varepsilon\,\delta(\mathrm{div}\,u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \tag{4.7}$$

$$I_5 = \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y) W^{x,y}_{\varepsilon,h} \left(\chi(\delta\rho_\varepsilon) - \frac{1}{2}\chi'(\delta\rho_\varepsilon)\delta\rho_\varepsilon\right) \overline{\mathrm{div}_x\,u_\varepsilon}\,\mathrm{d}x\,\mathrm{d}y\frac{\mathrm{d}h}{h}\,\mathrm{d}s. \tag{4.8}$$

For the estimate of the terms I_1, I_2, and I_3 defined in lemma 4.1, we use similar ideas as in [4]. However I_4 and I_5 require a more complex approach. The main difference is that div u_ε *involves pressure terms while we use a delocalized pressure $\mathcal{L}_\varepsilon$ in system (2.1) and (2.2). Unfortunately the estimates in [4] strongly relied on having appropriate pointwise control on the pressure, which is not available here because of the convolution with $\mathcal{L}_\varepsilon$.*

Proof. From (2.1), one gets an equation for $\delta\rho_\varepsilon$

$$\partial_t\delta\rho_\varepsilon + \mathrm{div}_x(\rho_\varepsilon u_\varepsilon)(x) - \mathrm{div}_y(\rho_\varepsilon u_\varepsilon)(y) = 0,$$

which may be rewritten as

$$\begin{aligned}\partial_t\delta\rho_\varepsilon + \mathrm{div}_x(\delta\rho_\varepsilon\, u_\varepsilon(x)) + \mathrm{div}_y(\delta\rho_\varepsilon\, u_\varepsilon(y)) + \rho_\varepsilon(y)\mathrm{div}_x\, u_\varepsilon(x)\\ - \rho_\varepsilon(x)\mathrm{div}_y\, u_\varepsilon(y) = 0.\end{aligned} \tag{4.9}$$

Note that the terms $\rho_\varepsilon(y)\mathrm{div}_x\, u_\varepsilon(x)$ and $\rho_\varepsilon(x)\mathrm{div}_y\, u_\varepsilon(y)$ are well-defined since $\rho_\varepsilon \in L^2$ and $\mathrm{div}_x\, u_\varepsilon \in L^2$. By (2.4), we have $\rho_\varepsilon^{1+l} \in L^2$ for $\gamma_{\mathrm{art}} > 2(1+l)$ and $\nabla_x u_\varepsilon \in L^2$. Hence, by theorem B.1, $\delta\rho_\varepsilon$ is a renormalized solution for the system (4.9). Noticing that

$$\begin{aligned}&-\rho_\varepsilon(y)\mathrm{div}_x\, u_\varepsilon(x) + \rho_\varepsilon(x)\mathrm{div}_y\, u_\varepsilon(y)\\ &\quad= \frac{1}{2}(\delta\rho_\varepsilon\,(\mathrm{div}_x\, u_\varepsilon(x) + \mathrm{div}_y\, u_\varepsilon(y)) - \bar{\rho}_\varepsilon\,(\mathrm{div}_x\, u_\varepsilon(x) - \mathrm{div}_y\, u_\varepsilon(y))),\end{aligned}$$

we arrive at

$$\begin{aligned}\partial_t\chi(\delta\rho_\varepsilon)+\mathrm{div}_x(\chi(\delta\rho_\varepsilon)(x)u_\varepsilon(x))+\mathrm{div}_y(\chi(\delta\rho_\varepsilon)(x)u_\varepsilon(y))\\ =\left(\chi(\delta\rho_\varepsilon)-\frac{1}{2}\chi'(\delta\rho_\varepsilon)\delta\rho_\varepsilon\right)(\mathrm{div}_x\,u_\varepsilon(x)+\mathrm{div}_y\,u_\varepsilon(y))\\ -\frac{1}{2}\chi'(\delta\rho_\varepsilon)\,\bar\rho_\varepsilon\,(\mathrm{div}_x\,u_\varepsilon(x)-\mathrm{div}_y\,u_\varepsilon(y)).\end{aligned}\tag{4.10}$$

From the definition of χ in (4.1), one gets easily

$$\chi(\delta\rho_\varepsilon)+|\chi'(\delta\rho_\varepsilon)|\,\overline{\rho}_\varepsilon\leqslant C\overline{\rho}_\varepsilon^{1+l},$$

which implies that $\chi(\delta\rho_\varepsilon),\chi'(\delta\rho_\varepsilon)\bar\rho\in L^2$. Since $\nabla_x u_\varepsilon\in L^2$, all the terms on the right side of (4.10) make sense. By (3.7), we obtain

$$\begin{aligned}\partial_t(K_h(x-y)W^{x,y}_{\varepsilon,h}\chi(\delta\rho_\varepsilon))&=K_h(x-y)\partial_t W^{x,y}_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)+K_h(x-y)W^{x,y}_{\varepsilon,h}\,\partial_t(\chi(\delta\rho_\varepsilon))\\ &=-K_h(x-y)u_\varepsilon(x)\nabla_x W^{x,y}_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)-K_h(x-y)\\ &\quad\times\ u_\varepsilon(y)\,\nabla_y W^{x,y}_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)-K_h(x-y)D_{\varepsilon,h}\\ &\quad\times\ \chi(\delta\rho_\varepsilon)+K_h(x-y)\mathrm{Com}_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)\\ &\quad+K_h(x-y)W^{x,y}_{\varepsilon,h}(\chi(\delta\rho_\varepsilon)-\chi'(\delta\rho_\varepsilon)\,\delta\rho_\varepsilon)\\ &\quad\times\ (\mathrm{div}_x\,u_\varepsilon(x)+\mathrm{div}_y\,u_\varepsilon(y))+\frac{1}{2}K_h(x-y)\\ &\quad\times\ W^{x,y}_{\varepsilon,h}\,\chi'(\delta\rho_\varepsilon)\,\delta\rho_\varepsilon\,\overline{\mathrm{div}\,u_\varepsilon}-\frac{1}{2}K_h(x-y)\\ &\quad\times\ W^{x,y}_{\varepsilon,h}\,\chi'(\delta\rho_\varepsilon)\,\overline{\rho}_\varepsilon\,\delta\mathrm{div}\,u_\varepsilon-K_h(x-y)\\ &\quad\times\ W^{x,y}_{\varepsilon,h}\,\mathrm{div}_x(\chi(\delta\rho_\varepsilon)\,u_\varepsilon(x))-K_h(x-y)\\ &\quad\times\ W^{x,y}_{\varepsilon,h}\,\mathrm{div}_y(\chi(\delta\rho_\varepsilon)\,u_\varepsilon(y)).\end{aligned}\tag{4.11}$$

The above equation may be justified as the following. First, in order to show $K_h(x-y)u_\varepsilon(x)\nabla_x W^{x,y}_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)\in L^1_{x,y}$, we just need to prove $K_h(x-y)u_\varepsilon(x)\,\chi(\delta\rho_\varepsilon)\in L^1_{x,y}$ since $\nabla_x W^{x,y}_{\varepsilon,h}\in L^\infty$. Recalling $\chi(\delta\rho_\varepsilon)=\chi(\rho_\varepsilon(x)-\rho_\varepsilon(y))$, by a change of variable we get

$$\begin{aligned}\int_{\mathbb{T}^{2d}}K_h(x-y)|u_\varepsilon(x)|\chi\,\mathrm{d}x\,\mathrm{d}y&=\int_{\mathbb{T}^d}K_h(y)\int_{\mathbb{T}^d}|u_\varepsilon(x)|\chi\,(\rho_\varepsilon(x)\\ &\quad-\rho_\varepsilon(x-y))\,\mathrm{d}x\,\mathrm{d}y\\ &\lesssim\int_{\mathbb{T}^d}K_h(y)\,\mathrm{d}y\lesssim 1.\end{aligned}$$

Therefore, the term $K_h(x-y)u_\varepsilon(x)\nabla_x W^{x,y}_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)$ is well-defined. Similar arguments could show that $K_h(x-y)u_\varepsilon(y)\nabla_y W^{x,y}_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)\in L^1_{x,y}$. Second, noting that

$$K_h(x-y)W^{x,y}_{\varepsilon,h}\leqslant 2K_h(x-y),$$

the term $K_h(x-y)\,W^{x,y}_{\varepsilon,h}\,(\chi(\delta\rho_\varepsilon)-\chi'(\delta\rho_\varepsilon)\,\delta\rho_\varepsilon)(\mathrm{div}_x\,u_\varepsilon(x)+\mathrm{div}_y\,u_\varepsilon(y))$, together with $K_h(x-y)W^{x,y}_{\varepsilon,h}\chi'(\delta\rho_\varepsilon)\,\delta\rho_\varepsilon\,\overline{\mathrm{div}\,u_\varepsilon}$, and $K_h(x-y)W^{x,y}_{\varepsilon,h}\,\chi'(\delta\rho_\varepsilon)\,\overline{\rho}_\varepsilon\,\delta\mathrm{div}\,u_\varepsilon$ belong to $L^1_{x,y}$ by similar arguments as for the first term. □

Third, we note that $D_{\varepsilon,h}$ is smooth and belongs to L^∞. Hence, $K_h(x-y)D_{\varepsilon,h}\chi(\delta\rho_\varepsilon)$ makes sense since $\chi(\delta\rho_\varepsilon)\in L^1_x$. One may check easily that $\rho_\varepsilon^{1+l}u_\varepsilon\in L^1$ for $\gamma_{\text{art}}\geqslant 3d/(d+2)$ and thus $K_h(x-y)\text{Com}_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)\in L^1_{x,y}$. Lastly, $\text{div}_x(\chi(\delta\rho_\varepsilon)u_\varepsilon(x))\in W^{-1,r}$ for some $r>1$ and $K_h(x-y)W^{x,y}_{\varepsilon,h}\in W^{1,r'}$ where r' is the Hölder conjugate exponent of r. Therefore, the terms $K_h(x-y)W^{x,y}_{\varepsilon,h}\,\text{div}_x(\chi(\delta\rho_\varepsilon)\,u_\varepsilon(x))$ and $K_h(x-y)W^{x,y}_{\varepsilon,h}\,\text{div}_y(\chi(\delta\rho_\varepsilon)\,u_\varepsilon(y))$ make sense. Using the product rule, we further rewrite (4.11) as

$$\begin{aligned}\partial_t(K_h(x-y)W^{x,y}_{\varepsilon,h}\chi(\delta\rho_\varepsilon)) = &-\text{div}_x\left(u_\varepsilon(x)\,K_h(x-y)\,W^{x,y}_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)\right)\\ &-\text{div}_y\left(u_\varepsilon(y)\,K_h(x-y)\,W^{x,y}_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)\right)\\ &+\delta u_\varepsilon\,\nabla_x K_h(x-y)\,W^{x,y}_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)-K_h(x-y)\\ &\times\,D_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)+K_h(x-y)\text{Com}_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)\\ &+K_h(x-y)W^{x,y}_{\varepsilon,h}(\chi(\delta\rho_\varepsilon)-\chi'(\delta\rho_\varepsilon)\delta\rho_\varepsilon)\\ &\times\,\overline{\text{div}_x\,u_\varepsilon}+\frac{1}{2}K_h(x-y)W^{x,y}_{\varepsilon,h}\chi'(\delta\rho_\varepsilon),\delta\rho_\varepsilon\\ &\times\,\overline{\text{div}\,u_\varepsilon}-\frac{1}{2}K_h(x-y)W^{x,y}_{\varepsilon,h}\chi'(\delta\rho_\varepsilon)\,\overline{\rho}_\varepsilon\,\delta\,\text{div}\,u_\varepsilon,\end{aligned}$$

which can be justified similarly as the equation (4.11). Integrating the time derivative of $T_{h_0,\varepsilon}(t)$ from 0 to t gives (4.3), concluding the proof. □

4.2. A bound for I_1

In this subsection, we estimate the terms I_1 in the following lemma.

Lemma 4.2. *Let I_1 be given by (4.4). Under the assumptions in lemma 4.1, the estimate*

$$I_1\leqslant C|\log\,h_0|^{1/2}+C\lambda^{-1}D_1$$

holds with the penalization D_1 defined by

$$\begin{aligned}D_1=\lambda\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}}K_h(x-y)\,(K_h*\big(\big(M|\nabla u_\varepsilon|\\ +|\rho_\varepsilon|^\gamma\big)\,w_\varepsilon\big)\,(x)\chi(\delta\rho_\varepsilon)\text{d}x\,\text{d}y\,\frac{\text{d}h}{h}\,\text{d}s\end{aligned}\tag{4.12}$$

for $t\leqslant T$, where T can be any positive number and the constant C depends on time T and a priori *bounds on the solution, in particular through $\|u_\varepsilon\|_{L^2_tH^1_x}$ and $\|\rho_\varepsilon\|_{L^\infty_tL^\gamma_x}$.*

Proof. We first recall

$$I_1=\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}}\delta u_\varepsilon\nabla_x K_h(x-y)W^{x,y}_{\varepsilon,h}\chi(\delta\rho_\varepsilon)\,\text{d}x\,\text{d}y\,\frac{\text{d}h}{h}\,\text{d}s.$$

By lemma 3.2, it follows

$$|\delta u_\varepsilon(x)|=|u_\varepsilon(x)-u_\varepsilon(y)|\lesssim|x-y|(D_{|x-y|}u_\varepsilon(x)+D_{|x-y|}u_\varepsilon(y)),$$

with $D_hu_\varepsilon(x)$ given by

$$D_hu_\varepsilon(x)=\frac{1}{h}\int_{|z|\leqslant h}\frac{|\nabla u_\varepsilon(x+z)|}{|z|^{d-1}}\,\text{d}z.$$

Hence, in view of (3.1), we obtain

$$I_1 \lesssim \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y)(D_{|x-y|}u_\varepsilon(x) + D_{|x-y|}u_\varepsilon(y))W^{x,y}_{\varepsilon,h}\chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\frac{\mathrm{d}h}{h}\,\mathrm{d}s$$
$$= 2\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y)(D_{|x-y|}u_\varepsilon(x) + D_{|x-y|}u_\varepsilon(y))w^x_{\varepsilon,h}\chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\frac{\mathrm{d}h}{h}\,\mathrm{d}s$$

where we used symmetry in x and y of the integral bound in the last step. Since we only have

$$\|u_\varepsilon\|_{L^2H^1} \lesssim 1 \quad \text{and} \quad \|\rho\|_{L^{\gamma_{\mathrm{art}}}} \lesssim 1,$$

we can not expect the last integral to be much smaller than

$$\left\| \int_{h_0}^1 K_h \frac{\mathrm{d}h}{h} \right\|_{L^1} = |\log\, h_0|.$$

Instead, we use the penalty defined in (3.6) to absorb the main contribution of I_1 and prove the remainder is of the size of $|\log h_0|^{1/2}$. In order to proceed, we rewrite

$$\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y)(D_{|x-y|}u_\varepsilon(x) + D_{|x-y|}u_\varepsilon(y))w^x_{\varepsilon,h}\chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\frac{\mathrm{d}h}{h}\,\mathrm{d}s$$
$$= \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y)(D_{|x-y|}u_\varepsilon(y) - D_{|x-y|}u_\varepsilon(x))w^x_{\varepsilon,h}\chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s$$
$$+ 2\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y)D_{|x-y|}u_\varepsilon(x)w^x_{\varepsilon,h}\chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s$$
$$= I_{1,1} + I_{1,2}. \tag{4.13}$$

To estimate the term $I_{1,1}$, we change the variable to arrive at

$$I_{1,1} = \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y)(D_{|x-y|}u_\varepsilon(y) - D_{|x-y|}u_\varepsilon(x))w^x_{\varepsilon,h}\rho_\varepsilon^{1+l}(x)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s$$
$$= \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(z)(D_{|z|}u_\varepsilon(x-z) - D_{|z|}u_\varepsilon(x))w^x_{\varepsilon,h}\rho_\varepsilon^{1+l}(x)\,\mathrm{d}x\,\mathrm{d}z\frac{\mathrm{d}h}{h}\,\mathrm{d}s.$$

From proposition 3.7, we know $0 \leqslant w_\varepsilon \leqslant 1$, which implies

$$0 \leqslant w_{\varepsilon,h} \leqslant 1$$

for any $h > 0$ since $\|K_h\|_{L^1} = 1$. By Hölder's inequality, lemma 3.5, we obtain

$$\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(z)(D_{|z|}u_\varepsilon(x-z) - D_{|z|}u_\varepsilon(x))w^x_{\varepsilon,h}\rho_\varepsilon^{1+l}(x)\,\mathrm{d}x\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s$$
$$\lesssim \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^d} K_h(z)\||D_{|z|}u_\varepsilon(x-z) - D_{|z|}u_\varepsilon(x)|\|_{L^2_x}\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s$$
$$\lesssim |\log\, h_0|^{1/2}\|u_\varepsilon\|_{L^2H^1}.$$

While for the second integral $I_{1,2}$, it is not in a form to which we could directly apply lemma 3.5. Instead, we rewrite it as

$$\begin{aligned} I_{1,2} &= 2\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-z)(D_{|x-y|}u_\varepsilon(x)-D_{|x-y|}u_\varepsilon(z))w_\varepsilon^z \\ &\quad\times \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s + \int_0^t\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-z)D_{|x-y|}u_\varepsilon(z) \\ &\quad\times\ w_\varepsilon^z\chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\ &\leqslant \int_0^t\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-z)(D_{|x-y|}u_\varepsilon(x)-D_{|x-y|}u_\varepsilon(z))w_\varepsilon^z \\ &\quad\times \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s + C\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-z) \\ &\quad\times\ M(|\nabla u_\varepsilon|)(z)w_\varepsilon^z\chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \end{aligned}$$

where we used lemma 3.3 in the last step. By lemma 3.5 and the uniform boundedness of ρ_ε in $L^{\gamma_{\mathrm{art}}}$, we further get

$$\begin{aligned} &\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-z)(D_{|x-y|}u_\varepsilon(x)-D_{|x-y|}u_\varepsilon(z))w_\varepsilon^z \\ &\qquad\times \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\ &\lesssim \int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(y)\,K_h(z)\,\|\,|D_{|y|}u_\varepsilon(x-z)-D_{|y|}u_\varepsilon(x)|\,\|_{L^2_x}\,\mathrm{d}y\,\mathrm{d}z\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\ &\lesssim |\log\,h_0|^{1/2}\|u_\varepsilon\|_{L^2H^1}. \end{aligned} \tag{4.14}$$

Collecting the estimates of $I_{1,1}$ with $I_{1,2}$ and applying them to (4.13) gives

$$\begin{aligned} I_1 &\lesssim |\log\,h_0|^{1/2} + \int_0^t\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)\,K_h(x-z)\,M(|\nabla u_\varepsilon|)(z)\,w_\varepsilon^z \\ &\quad\times \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \end{aligned} \tag{4.15}$$

where the last integral could be bounded by $C\lambda^{-1}D_1$ and the proof is completed. □

4.3. An estimate for I_2

We recall

$$\mathcal{D}(x) = |\mathrm{div}\,u_\varepsilon|(x) + |\mathcal{L}_\varepsilon * P|(x) + |\widetilde{P}_\varepsilon|^{1+l}(x)$$

and

$$D^{x,y}_{\varepsilon,h} = K_h * (D_\varepsilon w_\varepsilon)(x) + K_h * (D_\varepsilon w_\varepsilon)(y)$$

where D_ε is defined in (3.6). The estimate for I_2 is provided in the lemma below.

Lemma 4.3. *Let I_2 be as in (4.5). Under the assumptions in lemma 4.1, then we have that*

$$I_2 \leqslant C|\log\, h_0|^\theta - 2D_1 - 2D_2$$

holds for some $1 > \theta > 0$ with the penalization D_1 defined in (4.12) and D_2 given by

$$D_2 = \lambda \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y) K_h * \mathcal{D}(x) w_{\varepsilon,h}(x) \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \tag{4.16}$$

for $t \leqslant T$, where T can be any positive number and the constant C may depend on time T.

Proof. The term I_2 is negative and helps us in controlling other terms. We pull out the penalization terms D_1 with D_2 and the error is bounded by $C|\log h_0|^{1/2}$. To be more specific, we have

$$\begin{aligned}
I_2 &= -\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y) D^{x,y}_{\varepsilon,h} \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\
&= -\lambda \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y)\, \overline{(K_h * ((M|\nabla u_\varepsilon| + |\rho_\varepsilon|^\gamma) w_\varepsilon)}(x) \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \times \frac{\mathrm{d}h}{h}\,\mathrm{d}s - \lambda \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y)\, \overline{(K_h * (K_h * \mathcal{D} w_\varepsilon)}(x)\, \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \times \frac{\mathrm{d}h}{h}\,\mathrm{d}s.
\end{aligned}$$

By the symmetry in x and y of the above expression, we further get

$$\begin{aligned}
I_2 &= -2\lambda \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y)\, (K_h * ((M|\nabla u_\varepsilon| + |\rho_\varepsilon|^\gamma) w_\varepsilon)(x) \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \times \frac{\mathrm{d}h}{h}\,\mathrm{d}s - 2\lambda \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y)\, (K_h * (K_h * \mathcal{D} w_\varepsilon)(x) \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \times \frac{\mathrm{d}h}{h}\,\mathrm{d}s \\
&= -2D_1 + I_{2,1}.
\end{aligned} \tag{4.17}$$

We extract the second penalization D_2 from $I_{2,1}$ as

$$\begin{aligned}
I_{2,1} &= -2\lambda \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y) K_h * \mathcal{D}(x) w_{\varepsilon,h}(x) \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\
&\quad + 2\lambda \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y) \big(K_h * \mathcal{D}(x) w_{\varepsilon,h}(x) \\
&\quad - K_h * (K_h * \mathcal{D} w_\varepsilon)(x)\big) \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s.
\end{aligned}$$

Noting $w_{\varepsilon,h}(x) = K_h * w_\varepsilon(x)$, in view of (3.12), we may bound the last commutator integral in the above equality by

$$C|\log\, h_0|^\theta$$

for some $1 > \theta > 0$. Therefore, we arrive at

$$I_{2,1} \leqslant -2D_2 + C|\log\, h_0|^{\theta}.$$

Hence, from (4.17) we get

$$I_2 \leqslant -2D_1 - 2D_2 + C|\log\, h_0|^{\theta} \tag{4.18}$$

concluding the proof. □

4.4. Treatment of I_3

We bound the term I_3 in this subsection.

Lemma 4.4. *Let I_3 be given by (4.6). Under the assumptions in lemma 4.1, the estimate*

$$I_3 \leqslant C|\log\, h_0|^{1/2} - C\lambda^{-1}D_1$$

holds with the penalization D_1 defined by (4.12) for $t \leqslant T$, where T can be any positive number and the implicit constant may depend on time T.

Proof. In view of (3.9), we may write I_3 as

$$\begin{aligned} I_3 &= \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y)\mathrm{Com}_{\varepsilon,h}^{x,y}\,\chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\ &= \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y)\left([u_\varepsilon\cdot, K_h*]\nabla w_\varepsilon(x) + [u_\varepsilon\cdot, K_h*]\nabla w_\varepsilon(y)\right) \\ &\quad \times \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\ &= 2\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y)[u_\varepsilon\cdot, K_h*]\nabla w_\varepsilon(x)\chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \end{aligned}$$

where we used the symmetry in x and y in the last step. Expanding the commutator and using the identity

$$u_\varepsilon\cdot\nabla w_\varepsilon(x) = \mathrm{div}(u_\varepsilon w_\varepsilon(x)) - \mathrm{div}(u_\varepsilon)w_\varepsilon(x),$$

we arrive at

$$\begin{aligned} I_3 &= 2\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{3d}} K_h(x-y)(u_\varepsilon^x\cdot\nabla K_h(x-z)w_\varepsilon^z - u_\varepsilon^z\cdot\nabla K_h(x-z)w_\varepsilon^z) \\ &\quad \times \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s + 2\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y)K_h*(\mathrm{div}\,u_\varepsilon w_\varepsilon)(x) \\ &\quad \times \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\ &= 2\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{3d}} K_h(x-y)(u_\varepsilon^x - u_\varepsilon^z)\cdot\nabla K_h(x-z)w_\varepsilon^z\chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\ &\quad + 2\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y)K_h*(\mathrm{div}\,u_\varepsilon w_\varepsilon)(x)\chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \end{aligned}$$

where the second integral in the last equality of the above expression is bounded by $C\lambda^{-1}D_1$ since

$$|\operatorname{div} u_\varepsilon| \leqslant |\nabla u_\varepsilon| \leqslant M|\nabla u_\varepsilon|.$$

By lemma 3.2 and the inequality (3.1), the first integral is estimated as

$$\begin{aligned}
&\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{3d}} K_h(x-y)(u_\varepsilon^x - u_\varepsilon^z)\cdot\nabla K_h(x-z) w_\varepsilon^z \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\
&\quad\lesssim \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{3d}} K_h(x-y)(D_{|x-z|}u_\varepsilon(x) + D_{|x-z|}u_\varepsilon(z)) \\
&\qquad\times |(x-z)\cdot\nabla K_h(x-z)| w_\varepsilon^z \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\
&\quad\lesssim \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-z)|D_{|x-z|}u_\varepsilon(x) - D_{|x-z|}u_\varepsilon(z)| w_\varepsilon^z \\
&\qquad\times \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\
&\qquad+ 2\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-z) D_{|x-z|}u_\varepsilon(z) w_\varepsilon^z \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\qquad\times \frac{\mathrm{d}h}{h}\,\mathrm{d}s
\end{aligned} \tag{4.19}$$

where the second integral in the last inequality is bounded by $C\lambda^{-1}D_1$ by lemma 3.3. By the definition of χ in (4.1), we change the variable to get

$$\begin{aligned}
&\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-z)|D_{|x-z|}u_\varepsilon(x) - D_{|x-z|}u_\varepsilon(z)| w_\varepsilon^z \\
&\qquad\times \chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\
&\quad= \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{3d}} K_h(y)K_h(z)|D_{|z|}u_\varepsilon(x) - D_{|z|}u_\varepsilon(x-z)| w_\varepsilon^{x-z} \\
&\qquad\times \chi(\rho_\varepsilon^x - \rho_\varepsilon^{x-z})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\
&\quad\lesssim \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{3d}} K_h(y)K_h(z)|D_{|z|}u_\varepsilon(x) - D_{|z|}u_\varepsilon(x-z)| w_\varepsilon^{x-z} \\
&\qquad\times (\rho_\varepsilon^{1+l}(x) + \rho_\varepsilon^{1+l}(x-z))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s,
\end{aligned}$$

from where by Hölder's inequality and lemma 3.5 we obtain a further bound of the above integral

$$\begin{aligned}
&\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^d} K_h(z)K_h(y)\|D_{|z|}u_\varepsilon(x) - D_{|z|}u_\varepsilon(x-z)\|_{L_x^2}\,\mathrm{d}z\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\
&\quad\lesssim |\log h_0|^{1/2}\|u_\varepsilon\|_{L^2H^1} \lesssim |\log h_0|^{1/2}.
\end{aligned}$$

Collecting the estimates for the two terms in (4.19), we arrive at

$$I_3 \leqslant C|\log\, h_0|^{1/2} + C\lambda^{-1}D_1 \tag{4.20}$$

proving the lemma. □

4.5. Pressure term

In this section, we treat the terms involving the pressure. Actually the pressure term appears in both I_4 and I_5 in slightly different forms. We introduce an abstract function to give the estimate in a more general form and the corresponding bounds in terms I_4 and I_5 follow easily. We define the following integral

$$\begin{aligned} I_P = -\frac{1}{\log\, 2}\int_0^t\int_\varepsilon^{2\varepsilon}\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-\widetilde{y})f(x,y,\widetilde{y})w_{\varepsilon,h}(x) \\ \times (L_{\varepsilon'} * P(x) - L_{\varepsilon'} * P(y))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,\frac{\mathrm{d}h}{h}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s \end{aligned} \tag{4.21}$$

and recall

$$r_h = \frac{1}{\|\mathcal{K}_h\|_{L^1}}\int_0^T\int_{\mathbb{T}^{2d}} \mathcal{K}_h(x-y)\left(|\widetilde{P}(t,x)-\widetilde{P}(t,y)|^{s_0} + |Q(t,x,y)|^{s_1}\right)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

defined in (1.12) with

$$T_{h_0,\varepsilon}(t) = \int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(x-y)W^{x,y}_{\varepsilon,h}\chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}.$$

The estimate of I_P is established in the lemma 4.5 below.

In the estimate of the first three terms I_1, I_2, and I_3, the argument is still true even if we replace the mollifying kernel $\mathcal{L}_\varepsilon$ by L_ε, i.e., we may have an upper bound point-wise in ε. The kernel $\mathcal{L}_\varepsilon$ is only necessary in the treatment of the pressure term. In fact for the pressure term, it is very difficult to obtain an estimate uniform in ε (using the mollifier L_ε) since when ε is relatively big compared to h_0, the error term Diff defined by (4.26) is out of control because $L_\varepsilon * P$ can not approximate P precisely enough. Therefore, instead of consider a L^∞_ε topology, we consider $L^1_\varepsilon(d\varepsilon/\varepsilon)$. In order to treat the term I_P, we need to study two cases separately, i.e., $h \leqslant \varepsilon'$ and $\varepsilon' \leqslant h$. The case $h \leqslant \varepsilon'$ is easy. We bound the term $\delta(L_{\varepsilon'} * P)$ by the Hölder norm of $L_{\varepsilon'}$, which is under our control since ε' is relatively big. For the case $\varepsilon' \leqslant h$, it is much more difficult. Roughly speaking, we use the fact that the smoothing effect of K_h is dominant since the scaling of $L_{\varepsilon'}$ is smaller. Therefore, we treat $L_{\varepsilon'} * P$ as an approximation of P which is bounded by P in any L^p for $p \in [1,\infty]$ such that $P \in L^p$. The main difficulty of executing this idea is that we can not control $L_{\varepsilon'} * P$ directly with our penalization. Instead, we need to consider the quantity $L_{\varepsilon'} * (w^\theta P)$ for some $\theta > 0$ (see (4.25)). Hence, we have to control commutator between the weight function and the convolution with L_ε to close the estimate.

Lemma 4.5. *Let I_P be defined by (4.21) and $(\rho_\varepsilon, u_\varepsilon)$ be a sequence of solutions to the system (2.1) and (2.2) satisfying the bound (2.4) with $\gamma_{\text{art}} \geqslant \max(2s_0', s_1', 3d/(d+2))$ where s_0' and s_1' are the Hölder conjugate exponent of s_0 and s_1 respectively. Assume the pressure P satisfies (1.7), (1.8), (1.11), and (1.12). Let $f(x,y,\widetilde{y})$ be such that*

$$|f(x,x-y,x-\widetilde{y})| \leqslant C(|\chi'(\delta\rho_\varepsilon(x,y))|\overline{\rho_\varepsilon}(x,y) + |\chi'(\delta\rho_\varepsilon(x,\widetilde{y}))|\overline{\rho_\varepsilon}(x,\widetilde{y})). \tag{4.22}$$

We have

$$|I_P| \leqslant C + C\left(\int_{\varepsilon}^{2\varepsilon} r_{\max(h_0,\varepsilon')}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\right)^{\bar\theta} |\log(h_0)|^{\theta} + C\int_0^t T_{h_0,\varepsilon}(s)\mathrm{d}s + C\lambda^{-1}D_2 + \frac{3D_3}{8}$$

with D_2 given by (4.16) and D_3 by

$$D_3 = \eta(1+l)\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(x-y)W^{x,y}_{\varepsilon,h}\chi(\delta\rho_\varepsilon)\overline{\rho^{\gamma}_{\varepsilon}}^{\mathrm{art}}\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s, \tag{4.23}$$

for some $0<\bar\theta$, $0<\theta<1$, and $t\leqslant T$, where T can be any positive number and the implicit constant may depend on time T.

Proof. Here we give a uniform estimate in ε of this term, which may be divided into two cases: $\varepsilon' < h$ and $\varepsilon' \geqslant h$:

$$\begin{aligned}
I_P &= -\frac{1}{\log 2}\int_0^t\int_{\varepsilon}^{2\varepsilon}\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-\widetilde{y})f w_{\varepsilon,h}(x)\\
&\quad\times\delta(L_{\varepsilon'}*P)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,\frac{\mathrm{d}h}{h}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\\
&= -\frac{1}{\log 2}\int_0^t\int_{\varepsilon}^{2\varepsilon}\int_{h_0}^1\int_{\mathbb{T}^{3d}} (\mathbf{1}_{\varepsilon'\geqslant h}+\mathbf{1}_{\varepsilon'<h})K_h(x-y)K_h(x-\widetilde{y})f w_{\varepsilon,h}(x)\\
&\quad\times\delta(L_{\varepsilon'}*P)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,\frac{\mathrm{d}h}{h}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\\
&= I_b + I_s,
\end{aligned}$$

where I_b and I_s are corresponding to the integrals with characteristic functions $\mathbf{1}_{\varepsilon'\geqslant h}$ and $\mathbf{1}_{\varepsilon'<h}$ in them respectively. As we see below, the term I_b is easier to treat since in this case the K_h is the mollifier playing the key role, which is more consistent with the whole compactness argument. While for term I_s, we need to take the advantage of regularity of the weight function to generate an extra small factor $(\varepsilon')^{\theta}$, which help us control the singularity of K_h around the origin. First we rewrite I_b as

$$\begin{aligned}
|I_b| &= \frac{1}{\log 2}\left|\int_0^t\int_{\varepsilon}^{2\varepsilon}\int_{h_0}^1\int_{\mathbb{T}^{3d}} \mathbf{1}_{\varepsilon'\geqslant h}K_h(x-y)K_h(x-\widetilde{y})\,f\,w_{\varepsilon,h}(x)\right.\\
&\quad\left.\times\delta(L_{\varepsilon'}*P)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,\frac{\mathrm{d}h}{h}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\right|\\
&= \frac{1}{\log 2}\left|\int_0^t\int_{\varepsilon}^{2\varepsilon}\int_{h_0}^{\max(h_0,\varepsilon')}\int_{\mathbb{T}^{4d}} K_h(x-y)K_h(x-\widetilde{y})f w_{\varepsilon,h}(x)\right.\\
&\quad\left.\times P(t,z,\rho_\varepsilon(z))\,(L_{\varepsilon'}(x-z)-L_{\varepsilon'}(y-z))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\right|\\
&= \frac{1}{\log 2}\left|\int_0^t\int_{\varepsilon}^{2\varepsilon}\int_{h_0}^{\max(h_0,\varepsilon')}\int_{\mathbb{T}^{4d}} K_h(y)K_h(\widetilde{y})f(x,x-y,x-\widetilde{y})w_{\varepsilon,h}(x)\right.\\
&\quad\left.\times P(t,z,\rho_\varepsilon(z))(L_{\varepsilon'}(x-z)-L_{\varepsilon'}(x-y-z))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\right|.
\end{aligned}$$

Due to the smoothness of $L_{\varepsilon'}$, we have the uniform bound in $x-z$

$$L_{\varepsilon'}(x-z) - L_{\varepsilon'}(x-y-z) \leqslant C\frac{|y|^\theta}{\varepsilon'^\theta}$$

with $1 > \theta > 0$. By (1.7) and (1.8), we get

$$\int_{\mathbb{T}^d} P(t,z,\rho_\varepsilon(z))\,\mathrm{d}z \lesssim \int_{\mathbb{T}^d} R(t,z) + \Theta_1(z) + \rho^p(z)\,\mathrm{d}z \lesssim 1$$

since $\gamma_{\text{art}} \geqslant p$. Therefore, by (4.22), using the uniform integrability of ρ_ε and the fact that

$$\|K_h\|_{L^1} = 1,$$

we arrive at

$$\begin{aligned}|I_b| &\lesssim \int_\varepsilon^{2\varepsilon}\int_{h_0}^{\max(h_0,\varepsilon')}\int_{\mathbb{T}^d} K_h(y)\frac{|y|^\theta}{\varepsilon'^\theta}\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\\ &\lesssim \int_\varepsilon^{2\varepsilon}\int_{h_0}^{\max(h_0,\varepsilon')}\frac{h^\theta}{\varepsilon'^\theta}\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'} \lesssim 1.\end{aligned}$$

Next we treat the difficult term I_s. Denoting $\widetilde{\varepsilon} = \max(h_0,\varepsilon')$, by assumptions (1.11), we obtain

$$\begin{aligned}|I_s| &\leqslant C\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{4d}} K_h(x-y)K_h(x-\widetilde{y})f(x,y,\widetilde{y})w_{\varepsilon,h}(x)L_{\varepsilon'}(z)|\\ &\quad\times \rho_\varepsilon(x-z)-\rho_\varepsilon(y-z)|(\rho_\varepsilon^{\gamma-1}(x-z)+\rho_\varepsilon^{\gamma-1}(y-z))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,\mathrm{d}z\\ &\quad\times \frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s + C\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{4d}} K_h(x-y)K_h(x-\widetilde{y})f(x,y,\widetilde{y})w_{\varepsilon,h}\\ &\quad\times (x)L_{\varepsilon'}(z)\left(Q_\varepsilon^{x-z,y-z} + (\widetilde{P}_\varepsilon^{x-z}+\widetilde{P}_\varepsilon^{y-z})|\rho_\varepsilon(t,x-z)\right.\\ &\quad\left.-\rho_\varepsilon(t,y-z)|\right)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\\ &= I_{s,1}+I_{s,2}+I_{s,3}\end{aligned} \tag{4.24}$$

where $I_{s,1}$ is the first integral with $I_{s,2}$ and $I_{s,3}$ corresponding to the integrals containing $Q_\varepsilon^{x-z,y-z}$ and $(\widetilde{P}_\varepsilon^{x-z}+\widetilde{P}_\varepsilon^{y-z})$ respectively. For the sake of simplicity, we suppress the constant C in $I_{s,1}$, $I_{s,2}$, and $I_{s,3}$. By making constants in the following estimates bigger if necessary, we may recover the bound for I_s. The first integral $I_{s,1}$ is the most difficult one among the three. In order to estimate this term, we need to use the penalization term D_3 as well as the regularity of the weight function $w_{\varepsilon,h}$. To be more specific, we have

$$\begin{aligned}I_{s,1} &= \int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{4d}} K_h(y)K_h(\widetilde{y})f(x,x-y,x-\widetilde{y})w_{\varepsilon,h}(x)L_{\varepsilon'}(z)\\ &\quad\times|\rho_\varepsilon(x-z)-\rho_\varepsilon(x-y-z)|\,\left(\rho_\varepsilon^{\gamma-1}(x-z)\right.\\ &\quad\left.+\rho_\varepsilon^{\gamma-1}(x-y-z)\right)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\\ &= \bar{I}_{s,1}+\text{Diff}\end{aligned}$$

where we denoted using the notation in subsection 3.1

$$\bar{I}_{s,1} = \int_0^t \int_\varepsilon^{2\varepsilon} \int_{\widetilde{\varepsilon}}^1 \int_{\mathbb{T}^{4d}} K_h(y) K_h(\widetilde{y}) f(x, x-y, x-\widetilde{y}) w_{\varepsilon,h}^{1/\gamma_{\rm art}}(x) w_{\varepsilon,h}^{1-1/\gamma_{\rm art}} \\ \times (x-z) L_{\varepsilon'}(z) |\delta\rho_\varepsilon(x-z,y)| \overline{\rho_\varepsilon^{\gamma-1}}(x-z,y)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \frac{\mathrm{d}h}{h} \frac{\mathrm{d}\varepsilon'}{\varepsilon'}\, \mathrm{d}s \tag{4.25}$$

and

$$\mathrm{Diff} = \int_0^t \int_\varepsilon^{2\varepsilon} \int_{\widetilde{\varepsilon}}^1 \int_{\mathbb{T}^{4d}} K_h(y) K_h(\widetilde{y}) f(x, x-y, x-\widetilde{y}) w_{\varepsilon,h}^{1/\gamma_{\rm art}}(x) L_{\varepsilon'}(z)| \\ \times \delta\rho_\varepsilon(x-z,y)| (w_{\varepsilon,h}^{1-1/\gamma_{\rm art}}(x) - w_{\varepsilon,h}^{1-1/\gamma_{\rm art}}(x-z)) \overline{\rho_\varepsilon^{\gamma-1}} \\ \times (x-z,y)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \frac{\mathrm{d}h}{h} \frac{\mathrm{d}\varepsilon'}{\varepsilon'}\, \mathrm{d}s. \tag{4.26}$$

As we see below, the term $\bar{I}_{s,1}$ is the leading order term and Diff is a perturbation of constant size. Using Hölder's inequality, the term $\bar{I}_{s,1}$ is bounded by

$$\begin{aligned} &\int_0^t \int_\varepsilon^{2\varepsilon} \int_{\widetilde{\varepsilon}}^1 \int_{\mathbb{T}^{2d}} K_h(y) K_h(\widetilde{y}) \| L_{\varepsilon'} * (|\delta\rho_\varepsilon(x,y)| \overline{\rho_\varepsilon^{\gamma-1}}(x,y) w_{\varepsilon,h}^{1-1/\gamma_{\rm art}}) \|_{L_x^{\gamma'_{\rm art}}} \\ &\qquad \times \| f(x, x-y, x-\widetilde{y}) w_{\varepsilon,h}^{1/\gamma_{\rm art}} \|_{L_x^{\gamma_{\rm art}}}\, \mathrm{d}y\, \mathrm{d}\widetilde{y}\, \frac{\mathrm{d}h}{h} \frac{\mathrm{d}\varepsilon'}{\varepsilon'}\, \mathrm{d}s \\ &\leqslant C \int_0^t \int_\varepsilon^{2\varepsilon} \int_{\widetilde{\varepsilon}}^1 \int_{\mathbb{T}^{2d}} K_h(y) K_h(\widetilde{y}) \| \delta\rho_\varepsilon(x,y) \overline{\rho_\varepsilon^{\gamma-1}}(x,y) w_{\varepsilon,h}^{1-1/\gamma_{\rm art}} \|_{L_x^{\gamma'_{\rm art}}} \\ &\qquad \times \| (|\chi'(\delta\rho_\varepsilon)|\, \overline{\rho_\varepsilon}(x,y) + |\chi'(\delta\rho_\varepsilon)|\, \overline{\rho_\varepsilon}(x,\widetilde{y})) w_{\varepsilon,h}^{1/\gamma_{\rm art}} \|_{L_x^{\gamma_{\rm art}}}\, \mathrm{d}y\, \mathrm{d}\widetilde{y} \\ &\qquad \times \frac{\mathrm{d}h}{h} \frac{\mathrm{d}\varepsilon'}{\varepsilon'}\, \mathrm{d}s \\ &\leqslant C \int_0^t \int_\varepsilon^{2\varepsilon} \int_{\widetilde{\varepsilon}}^1 \int_{\mathbb{T}^{2d}} K_h(y) K_h(\widetilde{y}) \| (\delta\rho_\varepsilon)^\sigma(x,y) w_{\varepsilon,h}^{1-\gamma/\gamma_{\rm art}} \|_{L_x^{\alpha_1}} \\ &\qquad \times \| (\delta\rho_\varepsilon)^{1-\sigma}(x,y) \overline{\rho_\varepsilon^{\gamma-1}}(x,y) w_{\varepsilon,h}^{(\gamma-1)/\gamma_{\rm art}} \|_{L_x^{\alpha_2}} \| \left(|\chi'(\delta\rho_\varepsilon)|\, \overline{\rho_\varepsilon}(x,y) \right. \\ &\qquad \left. + |\chi'(\delta\rho_\varepsilon)|\, \overline{\rho_\varepsilon}(x,\widetilde{y}) \right) w_{\varepsilon,h}^{1/\gamma_{\rm art}} \|_{L_x^{\gamma_{\rm art}}}\, \mathrm{d}y\, \mathrm{d}\widetilde{y}\, \frac{\mathrm{d}h}{h} \frac{\mathrm{d}\varepsilon'}{\varepsilon'}\, \mathrm{d}s \end{aligned} \tag{4.27}$$

where α_1, α_2, and σ are given by

$$\alpha_1 = \frac{\gamma_{\rm art}}{\gamma_{\rm art} - \gamma} \qquad \alpha_2 = \frac{\gamma_{\rm art}}{\gamma - 1}, \qquad \sigma = 1 - \frac{(\gamma-1)(1+l)}{\gamma_{\rm art}}.$$

We also require

$$l\gamma_{\rm art} = 1 + l.$$

Using Young's inequality, one further gets

$$\begin{aligned}\bar{I}_{s,1} \leqslant & \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(y) K_h(\widetilde{y}) \left(\frac{C}{\eta} \int |\delta\rho_\varepsilon|^{1+l}(x,y) w_{\varepsilon,h} \,\mathrm{d}x \right. \\ & + \frac{\eta}{16} \int |\delta\rho_\varepsilon|^{1+l}(x,y) \overline{\rho_\varepsilon^{\gamma_{\mathrm{art}}}}(x,z) w_{\varepsilon,h} \,\mathrm{d}x + \frac{\eta}{16} \int |\delta\rho_\varepsilon|^{1+l}(x,\widetilde{y}) \overline{\rho_\varepsilon^{\gamma_{\mathrm{art}}}} \\ & \left. \times (x,y) w_{\varepsilon,h} \,\mathrm{d}x \right) \mathrm{d}y \,\mathrm{d}\widetilde{y} \frac{\mathrm{d}h}{h} \,\mathrm{d}s \\ = & \frac{C}{\eta} \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y) |\delta\rho_\varepsilon|^{1+l} w_{\varepsilon,h} \,\mathrm{d}x \,\mathrm{d}y \frac{\mathrm{d}h}{h} \,\mathrm{d}s \\ & + \frac{\eta}{8} \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} K_h(x-y) |\delta\rho_\varepsilon|^{1+l} \overline{\rho_\varepsilon^{\gamma_{\mathrm{art}}}} w_{\varepsilon,h} \,\mathrm{d}x \,\mathrm{d}y \frac{\mathrm{d}h}{h} \,\mathrm{d}s \end{aligned}$$

where we used $\|K_h\|_{L^1} = 1$ and the last integral may be bounded by $D_3/8$. Next we turn to the term Diff. Noting

$$w_{\varepsilon,h}^{1-1/\gamma_{\mathrm{art}}}(x) - w_{\varepsilon,h}^{1-1/\gamma_{\mathrm{art}}}(x-z) \leqslant C \frac{|z|^{1-1/\gamma_{\mathrm{art}}}}{h^{1-1/\gamma_{\mathrm{art}}}},$$

we obtain

$$\begin{aligned}\mathrm{Diff} \leqslant & \int_0^t \int_\varepsilon^{2\varepsilon} \int_{\widetilde{\varepsilon}}^1 \int_{\mathbb{T}^{4d}} K_h(y) K_h(\widetilde{y}) f(x, x-y, x-\widetilde{y}) w_{\varepsilon,h}^{1/\gamma_{\mathrm{art}}}(x) \\ & \times \frac{|z|^{1-1/\gamma_{\mathrm{art}}}}{h^{1-1/\gamma_{\mathrm{art}}}} L_{\varepsilon'}(z) \\ & \times |\delta\rho_\varepsilon(x-z,y)| \overline{\rho_\varepsilon^{\gamma-1}}(x-z,y) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z \,\mathrm{d}\widetilde{y} \frac{\mathrm{d}h}{h} \frac{\mathrm{d}\varepsilon'}{\varepsilon'} \,\mathrm{d}s \\ \leqslant & C \int_\varepsilon^{2\varepsilon} \int_{\widetilde{\varepsilon}}^1 \int_{\mathbb{T}^{3d}} K_h(y) K_h(\widetilde{y}) \frac{|z|^{1-1/\gamma_{\mathrm{art}}}}{h^{1-1/\gamma_{\mathrm{art}}}} L_{\varepsilon'}(z) \| f(x, x-y, x-\widetilde{y}) \\ & \times w_{\varepsilon,h}^{1/\gamma_{\mathrm{art}}} \|_{L_x^{\gamma_{\mathrm{art}}}} \| |\delta\rho_\varepsilon(x,y)| \overline{\rho_\varepsilon^{\gamma-1}}(x,y) \|_{L_x^{\gamma'_{\mathrm{art}}}}, \,\mathrm{d}y \,\mathrm{d}z \,\mathrm{d}\widetilde{y} \frac{\mathrm{d}h}{h} \frac{\mathrm{d}\varepsilon'}{\varepsilon'} \end{aligned} \tag{4.28}$$

from where using (4.22) and Young's inequality, by the uniform integrability of ρ_ε and $\|K_h\|_{L^1} = 1$, we further get

$$\begin{aligned}\mathrm{Diff} \leqslant & C\nu\eta \int_0^t \int_\varepsilon^{2\varepsilon} \int_{\widetilde{\varepsilon}}^1 \int_{\mathbb{T}^d} \frac{|z|^{1-1/\gamma_{\mathrm{art}}}}{h^{1-1/\gamma_{\mathrm{art}}}} L_{\varepsilon'}(z) \,\mathrm{d}z \int_{\mathbb{T}^{2d}} K_h(x-y) |\delta\rho_\varepsilon|^{1+l} \\ & \times \overline{\rho_\varepsilon^{\gamma_{\mathrm{art}}}} w_{\varepsilon,h} \,\mathrm{d}x \,\mathrm{d}y \frac{\mathrm{d}h}{h} \frac{\mathrm{d}\varepsilon'}{\varepsilon'} \,\mathrm{d}s + \frac{C}{\nu} \int_\varepsilon^{2\varepsilon} \int_{\widetilde{\varepsilon}}^1 \int_{\mathbb{T}^d} \frac{|z|^{1-1/\gamma_{\mathrm{art}}}}{h^{1-1/\gamma_{\mathrm{art}}}} L_{\varepsilon'}(z) \,\mathrm{d}z \\ & \times \frac{\mathrm{d}h}{h} \frac{\mathrm{d}\varepsilon'}{\varepsilon'} \end{aligned}$$

for a small parameter $\nu > 0$. For the second integral in the right side of the above inequality, we have

$$\frac{C}{\nu} \int_\varepsilon^{2\varepsilon} \int_{\widetilde{\varepsilon}}^1 \int_{\mathbb{T}^d} \frac{|z|^{1-1/\gamma_{\mathrm{art}}}}{h^{1-1/\gamma_{\mathrm{art}}}} L_{\varepsilon'}(z) \,\mathrm{d}z \frac{\mathrm{d}h}{h} \frac{\mathrm{d}\varepsilon'}{\varepsilon'} \leqslant \frac{C}{\nu} \int_\varepsilon^{2\varepsilon} \int_{\widetilde{\varepsilon}}^1 \frac{(\varepsilon')^{1-1/\gamma_{\mathrm{art}}}}{h^{1-1/\gamma_{\mathrm{art}}}} \frac{\mathrm{d}h}{h} \frac{\mathrm{d}\varepsilon'}{\varepsilon'} \leqslant \frac{C}{\nu}.$$

Using $\varepsilon' \leqslant h$ and choosing ν sufficiently small, we arrive at

$$\begin{aligned}
&C\nu\eta\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^d}\frac{|z|^{1-1/\gamma_{\mathrm{art}}}}{h^{1-1/\gamma_{\mathrm{art}}}}L_{\varepsilon'}(z)\,\mathrm{d}z\int_{\mathbb{T}^{2d}}K_h(x-y)|\delta\rho_\varepsilon|^{1+l}\overline{\rho_\varepsilon^{\gamma_{\mathrm{art}}}}\\
&\qquad\times\ w_{\varepsilon,h}\,\mathrm{d}x\,\mathrm{d}y\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\\
&\leqslant C\nu\eta\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{2d}}\frac{(\varepsilon')^{1-1/\gamma_{\mathrm{art}}}}{h^{1-1/\gamma_{\mathrm{art}}}}K_h(x-y)|\delta\rho_\varepsilon|^{1+l}\overline{\rho_\varepsilon^{\gamma_{\mathrm{art}}}}w_{\varepsilon,h}\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad\times\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\\
&\leqslant\frac{\eta}{16}\int_0^t\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{2d}}K_h(x-y)|\delta\rho_\varepsilon|^{1+l}\overline{\rho_\varepsilon^{\gamma_{\mathrm{art}}}}w_{\varepsilon,h}\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s
\end{aligned}$$

which may be bounded by $D_3/16$. Therefore, we obtain

$$\mathrm{Diff}\leqslant C+\frac{D_3}{16}.$$

Next we turn to the treatment of the term $I_{s,2}$. By changing variables, we rewrite it as

$$\begin{aligned}
I_{s,2}&=\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{4d}}K_h(x-y)K_h(x-\widetilde{y})f(x,y,\widetilde{y})w_{\varepsilon,h}(x)L_{\varepsilon'}(z)\\
&\qquad\times\ Q_{\widetilde{\varepsilon}}^{x-z,y-z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}\widetilde{y}\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\\
&=\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{4d}}K_h(y)K_h(\widetilde{y})f(x,x-y,x-\widetilde{y})w_{\varepsilon,h}(x)L_{\varepsilon'}(z)\\
&\qquad\times\ Q_{\widetilde{\varepsilon}}^{x-z,x-y-z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}\widetilde{y}\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s.
\end{aligned}$$

In view of $w_{\varepsilon,h}(x)\leqslant 1$, we get

$$\begin{aligned}
I_{s,2}&\leqslant\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{3d}}K_h(y)K_h(\widetilde{y})\,|f(x,x-y,x-\widetilde{y})|\,w_{\varepsilon,h}^{1/\gamma_{\mathrm{art}}}(x)L_{\varepsilon'}(z)\\
&\qquad\times\ Q_\varepsilon\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s
\end{aligned}$$

where $Q_\varepsilon=Q_{\widetilde{\varepsilon}}^{x-z,x-y-z}$. Using (4.22), Hölder's inequality, and that $\|L_{\varepsilon'}\|_{L^1}=1$, we arrive at

$$\begin{aligned}
I_{s,2}&\leqslant\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{2d}}K_h(y)K_h(\widetilde{y})\|f(x,x-y,x-\widetilde{y})w_{\varepsilon,h}^{1/\gamma_{\mathrm{art}}}\|_{L_x^{\gamma_{\mathrm{art}}}}\\
&\qquad\times\left\|\int_{\mathbb{T}^d}L_{\varepsilon'}(z)Q_\varepsilon\,\mathrm{d}z\right\|_{L_x^{\gamma'_{\mathrm{art}}}}\mathrm{d}y\,\mathrm{d}\widetilde{y}\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\\
&\leqslant C\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{2d}}K_h(y)K_h(\widetilde{y})\|\left(|\chi'(\delta\rho_\varepsilon)|\,\overline{\rho_\varepsilon}(x,y)+|\chi'(\delta\rho_\varepsilon)|\right.\\
&\qquad\times\left.\overline{\rho_\varepsilon}(x,\widetilde{y})\right)w_{\varepsilon,h}^{1/\gamma_{\mathrm{art}}}\|_{L_x^{\gamma_{\mathrm{art}}}}\left\|Q_\varepsilon^{x,x-y}\right\|_{L_x^{\gamma'_{\mathrm{art}}}}\mathrm{d}y\,\mathrm{d}\widetilde{y}\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s
\end{aligned}$$

where γ'_{art} is as usual in this paper the Hölder conjugate exponent of γ_{art}. By Young's inequality, we further get

$$I_{s,2} \leqslant \frac{\eta}{8}\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(x-y)|\delta\rho_\varepsilon|^{1+l\overline{\rho_\varepsilon^{\gamma_{\text{art}}}}} w_{\varepsilon,h}\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s + \frac{C}{\eta}\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(x-y)|Q_\varepsilon^{x,x-y}|^{\gamma'_{\text{art}}}\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s$$

where the first integral on the right side is bounded by $D_3/8$. Using Hölder's inequalities, the second integral may be estimated as

$$\begin{aligned}\frac{C}{\eta}\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(x-y)|Q_\varepsilon^{x,x-y}|^{\gamma'_{\text{art}}}\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s &\leqslant \frac{C}{\eta}\left(\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(x-y)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\right)^{(s_1-\gamma'_{\text{art}})/s_1}\\ &\quad\times\left(\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(x-y)|Q_\varepsilon^{x,x-y}|^{s_1}\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s\right)^{\gamma'_{\text{art}}/s_1}\end{aligned}$$

with $s_1-\gamma'_{\text{art}}\geqslant 0$ since $\gamma_{\text{art}}\geqslant s_1'$. From (1.12), the above expression may be further bounded by

$$C\left(\int_\varepsilon^{2\varepsilon} r_{\widetilde{\varepsilon}}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\right)^{\gamma'_{\text{art}}/s_1}|\log\,h_0|^{(s_1-\gamma'_{\text{art}})/s_1}.$$

Therefore, we obtain

$$I_{s,2}\leqslant\frac{D_3}{8}+C\left(\int_\varepsilon^{2\varepsilon} r_{\widetilde{\varepsilon}}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\right)^{\gamma'_{\text{art}}/s_1}|\log\,h_0|^{(s_1-\gamma'_{\text{art}})/s_1}.$$

We estimate the term $I_{s,3}$ next and rewrite it as

$$\begin{aligned}I_{s,3} &= \int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{4d}} K_h(x-y)K_h(x-\widetilde{y})f(x,y,\widetilde{y})w_{\varepsilon,h}(x)L_{\varepsilon'}(z)\\ &\quad\times(\widetilde{P}_\varepsilon^{y-z}-\widetilde{P}_\varepsilon^{x-z})|\rho_\varepsilon(t,x-z)-\rho_\varepsilon(t,y-z)|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}\widetilde{y}\,\frac{\mathrm{d}h}{h}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\\ &\quad+2\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{4d}} K_h(x-y)K_h(x-\widetilde{y})f(x,y,\widetilde{y})w_{\varepsilon,h}(x)L_{\varepsilon'}(z)\\ &\quad\times\widetilde{P}_\varepsilon^{x-z}|\rho_\varepsilon(t,x-z)-\rho_\varepsilon(t,y-z)|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}\widetilde{y}\,\frac{\mathrm{d}h}{h}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s.\end{aligned}\tag{4.29}$$

For the first term, we perform a change of variables and use Hölder's inequality to arrive at

$$\begin{aligned}&\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{4d}} K_h(x-y)\,K_h(x-\widetilde{y})f(x,y,\widetilde{y})w_{\varepsilon,h}(x)L_{\varepsilon'}(z)\\ &\qquad\times(\widetilde{P}_\varepsilon^{y-z}-\widetilde{P}_\varepsilon^{x-z})|\rho_\varepsilon(t,x-z)-\rho_\varepsilon(t,y-z)|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}\widetilde{y}\,\frac{\mathrm{d}h}{h}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\end{aligned}$$

$$\leqslant C\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{2d}} K_h(y)\,K_h(\widetilde{y})\,\Big\||\chi'(\delta\rho_\varepsilon)^{x,y}|\,\overline{\rho_\varepsilon}^{x,y}w_{\varepsilon,h}^{1/\gamma_{\rm art}}(x)\Big\|_{L_x^{\gamma_{\rm art}}}\Big\|(\delta\widetilde{P}_\varepsilon^{x,y})|\,\delta\rho_\varepsilon(x,y)|\Big\|_{L_x^{\gamma'_{\rm art}}}$$
$$+\,|\chi'(\delta\rho_\varepsilon)^{x,y}|\,\overline{\rho_\varepsilon}^{x,y}w_{\varepsilon,h}^{1/\gamma_{\rm art}}(x)\Big\|_{L_x^{\gamma_{\rm art}}}\Big\|(\delta\widetilde{P}_\varepsilon^{x,y})|\,\delta\rho_\varepsilon(x,y)|\Big\|_{L_x^{\gamma'_{\rm art}}}$$
$$\times\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,\frac{\mathrm{d}h}{h}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s$$

where we denote $\delta\widetilde{P}_\varepsilon^{x,y}=\widetilde{P}_\varepsilon(x)-\widetilde{P}_\varepsilon(y)$, and we also used the bound $w_{\varepsilon,h}(x)\leqslant 1$ and $\|L_{\varepsilon'}\|_{L^1}=1$ for any $\varepsilon'>0$. Using Young's inequality and Minkowsky's inequality, we get a further bound for the above term

$$\frac{\eta}{16}\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^d}K_h(y)\||\chi'(\delta\rho_\varepsilon)|\,\overline{\rho_\varepsilon}(x,y)w_{\varepsilon,h}(x)\|_{L_x^{\gamma_{\rm art}}}^{\gamma_{\rm art}}\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s$$
$$+\frac{C}{\eta}\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^d}K_h(y)\Big\|(\delta\widetilde{P}_\varepsilon^{x,y})|\delta\rho_\varepsilon(x,y)|\Big\|_{L_x^{\gamma'_{\rm art}}}^{\gamma'_{\rm art}}\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s.$$

The first integral in the above bound is bounded by $D_3/16$. In order to estimate the second integral, we introduce the truncation function

$$\widetilde{\phi}_\varepsilon^M(x,y)=\bar{\phi}(\rho_\varepsilon^x/M)\bar{\phi}(\rho_\varepsilon^y/M)$$

where $\bar{\phi}$ is a smooth function such that

$$\bar{\phi}(s)=\begin{cases}1, & 0\leqslant s\leqslant 1,\\ 0, & s\geqslant 2\\ \in[0,1], & \text{otherwise}\end{cases}. \tag{4.30}$$

Then we have

$$\frac{C}{\eta}\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^d}K_h(y)\Big\|(\delta\widetilde{P}_\varepsilon^{x,y})\,|\delta\rho_\varepsilon(x,y)|\Big\|_{L_x^{\gamma'_{\rm art}}}^{\gamma'_{\rm art}}\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s$$
$$\leqslant C\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{2d}}K_h(y)\,|\delta\widetilde{P}_\varepsilon^{x,y}|^{\gamma'_{\rm art}}\,\widetilde{\phi}_\varepsilon^M(x,x-y)|$$
$$\times\,\delta\rho_\varepsilon(x,y)|^{\gamma'_{\rm art}}\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s+C\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{2d}}K_h(y)$$
$$\times\,|\delta\widetilde{P}_\varepsilon^{x,y}|^{\gamma'_{\rm art}}\,(1-\widetilde{\phi}_\varepsilon^M(x,x-y))\,|\delta\rho_\varepsilon(x,y)|^{\gamma'_{\rm art}}\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s.$$

Applying Hölder's inequality and using (1.12), we bound the truncated term as

$$\frac{C}{\eta}\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{2d}}K_h(y)\,|\delta\widetilde{P}_\varepsilon^{x,y}|^{\gamma'_{\rm art}}\,\phi_\varepsilon^M(x,x-y)\,|\delta\rho_\varepsilon(x,y)|^{\gamma'_{\rm art}}\,\mathrm{d}x\,\mathrm{d}y\frac{\mathrm{d}h}{h}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s$$
$$\leqslant C\,M^{\gamma'_{\rm art}}\left(\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{2d}}K_h(y)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\right)^{1-\gamma'_{\rm art}/s_0}$$
$$\times\left(\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{2d}}K_h(y)\,|\widetilde{P}_\varepsilon^{x-y}-\widetilde{P}_\varepsilon^{x}|^{s_0}\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\right)^{\gamma'_{\rm art}/s_0}$$

$$\leqslant C M^{\gamma'_{\rm art}} |\log\, h_0|^{1-\gamma'_{\rm art}/s_0} \left(\int_\varepsilon^{2\varepsilon} r_{\widetilde{\varepsilon}} \frac{\mathrm{d}\varepsilon'}{\varepsilon'} \right)^{\gamma'_{\rm art}/s_0}.$$

For the remainder term, (i.e., the term involving $1-\phi_\varepsilon^M$), we use the simple relation

$$(\{\rho(x) \geqslant M\} \cap \{\rho(z) \geqslant M\})^c = \{\rho(x) \geqslant M\}^c \cup \{\rho(z) \geqslant M\}^c$$

to obtain

$$\begin{aligned}
&\int_0^t \int_\varepsilon^{2\varepsilon} \int_{\widetilde{\varepsilon}}^1 \int_{\mathbb{T}^{2d}} K_h(y) |\delta \widetilde{P}_\varepsilon^{x,y}|^{\gamma'_{\rm art}} (1 - \widetilde{\phi}_\varepsilon^M(x, x-y)) |\delta \rho_\varepsilon(x,y)|^{\gamma'_{\rm art}} \,\mathrm{d}x\,\mathrm{d}y \frac{\mathrm{d}h}{h} \frac{\mathrm{d}\varepsilon'}{\varepsilon'} \,\mathrm{d}s \\
&\quad \leqslant \int_0^t \int_\varepsilon^{2\varepsilon} \int_{\widetilde{\varepsilon}}^1 \int_{\mathbb{T}^{2d}} K_h(y) |\delta \widetilde{P}_\varepsilon^{x,y}|^{\gamma'_{\rm art}} (\mathbf{1}_{\{\rho^x \geqslant M\}} + \mathbf{1}_{\{\rho^{x-y} \geqslant M\}}) | \\
&\qquad \times \delta \rho_\varepsilon(x,y)|^{\gamma'_{\rm art}} \,\mathrm{d}x\,\mathrm{d}y \,\frac{\mathrm{d}h}{h} \frac{\mathrm{d}\varepsilon'}{\varepsilon'} \,\mathrm{d}s.
\end{aligned}$$

By Hölder's and Young's inequalities, we get

$$\begin{aligned}
&\frac{C}{\eta} \int_0^t \int_\varepsilon^{2\varepsilon} \int_{\widetilde{\varepsilon}}^1 \int_{\mathbb{T}^{2d}} K_h(y) |\delta \widetilde{P}_\varepsilon^{x,y}|^{\gamma'_{\rm art}} (1 - \phi_\varepsilon^M(x, x-y)) |\delta \rho_\varepsilon(x,y)|^{\gamma'_{\rm art}} \,\mathrm{d}x\,\mathrm{d}y \\
&\qquad \times \frac{\mathrm{d}h}{h} \frac{\mathrm{d}\varepsilon'}{\varepsilon'} \,\mathrm{d}s \\
&\quad \lesssim \int_0^t \int_\varepsilon^{2\varepsilon} \int_{\widetilde{\varepsilon}}^1 \int_{\mathbb{T}^{2d}} K_h(y) |\widetilde{P}_\varepsilon^{x-y} - \widetilde{P}_\varepsilon^x|^{s_0} \,\mathrm{d}x\,\mathrm{d}y \,\frac{\mathrm{d}h}{h} \frac{\mathrm{d}\varepsilon'}{\varepsilon'} \,\mathrm{d}s \\
&\qquad + \int_0^t \int_\varepsilon^{2\varepsilon} \int_{\widetilde{\varepsilon}}^1 \int_{\mathbb{T}^{2d}} K_h(y) \mathbf{1}_{\{\rho^x \geqslant M\}} \rho_\varepsilon(t,x)^{s_0 \gamma'_{\rm art}/(s_0 - \gamma'_{\rm art})} \,\mathrm{d}x\,\mathrm{d}y \\
&\qquad \times \frac{\mathrm{d}h}{h} \frac{\mathrm{d}\varepsilon'}{\varepsilon'} \,\mathrm{d}s \\
&\quad \lesssim r_{h_0} + M^{-(\gamma_{\rm art} - s_0 \gamma'_{\rm art}/(s_0 - \gamma'_{\rm art}))} |\log\, h_0|.
\end{aligned}$$

Note that for $\gamma_{\rm art} \geqslant 2s'_0$, one can easily check that $\gamma_{\rm art} - s_0\gamma'_{\rm art}/(s_0 - \gamma'_{\rm art}) > 0$. For the second term in (4.29), we need to use the penalty function defined in (3.6). More specifically, we need to extract an integral involving $K_h * \widetilde{P}$ and estimate the remainder term with a quantity converging to 0. To proceed, we rewrite this integral as

$$\begin{aligned}
&2 \int_0^t \int_\varepsilon^{2\varepsilon} \int_{\widetilde{\varepsilon}}^1 \int_{\mathbb{T}^{3d}} K_h(y) K_h(\widetilde{y}) f(x, x-y, x-\widetilde{y}) w_{\varepsilon,h}^{1/\gamma_{\rm art}}(x) L_{\varepsilon'}(z) \widetilde{P}_\varepsilon^{x-z} \\
&\qquad \times\ w_{\varepsilon,h}^{1/(1+l)}(x-z) |\rho_\varepsilon(t, x-z) - \rho_\varepsilon(t, x-y-z)| \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \,\frac{\mathrm{d}h}{h} \frac{\mathrm{d}\varepsilon'}{\varepsilon'} \,\mathrm{d}s \\
&\qquad + 2 \int_0^t \int_\varepsilon^{2\varepsilon} \int_{\widetilde{\varepsilon}}^1 \int_{\mathbb{T}^{3d}} K_h(y) K_h(\widetilde{y}) f(x, x-y, x-\widetilde{y}) w_{\varepsilon,h}^{1/\gamma_{\rm art}}(x) L_{\varepsilon'}(z) \\
&(w_{\varepsilon,h}^{1/(1+l)}(x) - w_{\varepsilon,h}^{1/(1+l)}(x-z))\, \widetilde{P}_\varepsilon^{x-z} |\rho_\varepsilon(t, x-z) \\
&\qquad - \rho_\varepsilon(t, x-y-z)| \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \,\frac{\mathrm{d}h}{h} \frac{\mathrm{d}\varepsilon'}{\varepsilon'} \,\mathrm{d}s \\
&\qquad = I_G + \mathrm{Diff}_1.
\end{aligned}$$

The treatment of I_G is slightly more difficult. Similar to previous calculations in (4.27), we change variable and use Hölder's inequality to obtain

$$\begin{aligned}|I_G| \leqslant \frac{\eta}{16}\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{2d}} K_h(y)\,\chi(\delta\rho_\varepsilon)^{x,y}\,\overline{\rho_\varepsilon^{\gamma_{\rm art}}}(x,y)w_{\varepsilon,h}(x)\,\mathrm{d}x\,\mathrm{d}y\\ \times\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s + \frac{C}{\eta}\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{2d}} K_h(y)\widetilde{P}_\varepsilon^{1+l}(x)w_{\varepsilon,h}(x)|\\ \times\delta\rho_\varepsilon(x,y)|^{1+l}\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s.\end{aligned}$$

The first term in the above inequality is bounded by $D_3/16$. To estimate the second term, we need to introduce $K_h * G$ to use the penalty function:

$$\begin{aligned}&\frac{C}{\eta}\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{2d}} K_h(y)\widetilde{P}_\varepsilon^{1+l}(x)w_{\varepsilon,h}(x)\,\chi(\delta\rho_\varepsilon(x,y))\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\\ &\quad\leqslant\frac{C}{\eta}\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{3d}} K_h(y)\,K_h(z)\left|\widetilde{P}_\varepsilon(x)-\widetilde{P}_\varepsilon(x-z)\right|^{1+l} w_{\varepsilon,h}(x)\\ &\qquad\times\chi(\delta\rho_\varepsilon(x,y))\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s + \frac{C}{\eta}\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{3d}} K_h(y)\,K_h(z)\\ &\qquad\times|\widetilde{P}_\varepsilon|^{1+l}(x-z)\,w_{\varepsilon,h}(x)\,\chi(\delta\rho_\varepsilon(x,y))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\end{aligned}$$

where the last term may be bounded by $C\lambda^{-1}D_2$ with $C\lambda^{-1}$ being arbitrarily small provided λ is sufficiently large. By Hölder we bound the first term as

$$\begin{aligned}&\frac{C}{\eta}\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{3d}} K_h(y)\,K_h(z)\left|\widetilde{P}_\varepsilon(x)-\widetilde{P}_\varepsilon(x-z)\right|^{1+l} w_{\varepsilon,h}(x)\\ &\qquad\times\chi(\delta\rho_\varepsilon(x,y))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\\ &\quad\leqslant C\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{2d}} K_h(y)K_h(z)\left\|\left|\widetilde{P}_\varepsilon(x)-\widetilde{P}_\varepsilon(x-z)\right|^{1+l}\right\|_{L_x^{s_0/(1+l)}}\\ &\qquad\times\|\chi(\delta\rho_\varepsilon(x,y))\|_{L_x^{s_0/(s_0-(1+l))}}\,\mathrm{d}z\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s.\end{aligned}$$

Note that for $\gamma_{\rm art}\geqslant 2s_0'$ we always have $s_0(1+l)/(s_0-(1+l))\leqslant\gamma_{\rm art}$. Hence, we get

$$\|\chi(\delta\rho_\varepsilon(x,y))\|_{L_x^{s_0/(s_0-(1+l))}} = \|\chi(\rho_\varepsilon(x)-\rho_\varepsilon(x-y))\|_{L_x^{s_0/(s_0-(1+l))}}\leqslant C.$$

Therefore, we have a further bound

$$\begin{aligned}&\frac{C}{\eta}\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{3d}} K_h(y)K_h(z)\left|\widetilde{P}_\varepsilon(x)-\widetilde{P}_\varepsilon(x-z)\right|^{1+l} w_{\varepsilon,h}(x)\\ &\qquad\times\chi(\delta\rho_\varepsilon(x,y))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\\ &\quad\leqslant C\int_0^t\int_\varepsilon^{2\varepsilon}\int_{\widetilde{\varepsilon}}^1\int_{\mathbb{T}^{d}} K_h(z)\left\|\widetilde{P}_\varepsilon(x)-\widetilde{P}_\varepsilon(x-z)\right\|_{L_x^{s_0}}^{1+l}\mathrm{d}z\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\end{aligned}$$

$$\leqslant C\left(\int_{\varepsilon}^{2\varepsilon} r_{\widetilde{\varepsilon}}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\right)^{(1+l)/s_0}|\log\ h_0|^{(s_0-1-l)/s_0}.$$

By Hölder's inequality, the Diff_1 term is estimated similarly to (4.28) as

$$\begin{aligned}\mathrm{Diff}_1 &\leqslant \frac{\eta}{16}\int_0^t\int_{\varepsilon}^{2\varepsilon}\int_{\widetilde{\varepsilon}}^{1}\int_{\mathbb{T}^{2d}} K_h(y)\,\chi(\delta\rho_\varepsilon)^{x,y}\,\overline{\rho_\varepsilon^{\gamma_{\mathrm{art}}}}(x,y)\,w_{\varepsilon,h}(x)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad\times\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s + C\int_0^t\int_{\varepsilon}^{2\varepsilon}\int_{\widetilde{\varepsilon}}^{1}\int_{\mathbb{T}^{2d}} K_h(y)\big\|\rho^{1+l}\big\|_{L_x^{s_0/(s_0-(1+l))}}L_{\varepsilon'}(z)\\ &\quad\times\left(\frac{z}{h}\right)^{1/(1+l)}\Big\||\widetilde{P}_\varepsilon^{x-z}|^{1+l}\Big\|_{L_x^{s_0/(1+l)}}\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\\ &\leqslant \frac{1}{16}D_3 + C\int_0^t\int_{\varepsilon}^{2\varepsilon}\int_{\widetilde{\varepsilon}}^{1}\int_{\mathbb{T}^{2d}} K_h(y)L_{\varepsilon'}(z)\left(\frac{z}{h}\right)^{1/(1+l)}\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\,\mathrm{d}s\\ &\leqslant \frac{1}{16}D_3 + C\end{aligned}$$

provided $\gamma_{\mathrm{art}} > 2s_0'$. Collecting all the estimates of $I_{s,1}$, $I_{s,2}$, with $I_{s,3}$ and optimizing in M concludes the proof. □

4.6. Term I_4

Before giving the bound for the integral terms I_4 and I_5, we introduce the following lemma needed for the treatment of the effective viscous flux $F = \Delta^{-1}\,\mathrm{div}(\partial_t(\rho_\varepsilon u_\varepsilon) + \mathrm{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon))$. We refer the readers to [4] for a proof of this result.

Lemma 4.6. *Let F be the effective viscous flux introduced above. Assume that $(\rho_\varepsilon, u_\varepsilon)$ is a solution of the system (2.6) and (2.7) satisfying the bound (2.4) with $\gamma_{\mathrm{art}} > d/2$. Suppose that $\Phi \in L^\infty([0,T]\times\mathbb{T}^{2d})$ and that*

$$\begin{aligned}C_\Phi :=& \left\|\int_{\mathbb{T}^d} K_h(x-y)\Phi(t,x,y)\,\mathrm{d}y\right\|_{W^{1,1}(0,T;W_x^{-1,1}(\mathbb{T}^d))}\\ &+\left\|\int_{\mathbb{T}^d} K_h(x-y)\Phi(t,x,y)\,\mathrm{d}x\right\|_{W^{1,1}(0,T;W_y^{-1,1}(\mathbb{T}^d))} < \infty,\end{aligned}$$

then there exists $\theta > 0$ such that

$$\begin{aligned}&\int_0^t\int_{\mathbb{T}^{2d}} K_h(x-y)\Phi(t,x,y)(F(t,x)-F(t,y))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\ &\quad\lesssim h^\theta(C_\Phi + \|\Phi\|_{L^\infty((0,T)\times\mathbb{T}2d)})\end{aligned}$$

holds, where the implicit constant in $\lesssim$ is independent of ε.

Next we estimate I_4 in the lemma below. We use θ to denote a parameter between 0 and 1 which may be different from line to line.

Lemma 4.7. *Let I_4 be defined by (4.7). Under the assumptions of lemma 4.5, it follows*

$$I_4 \leqslant C + C\left(\int_{\varepsilon}^{2\varepsilon} r_{\max(h_0,\varepsilon')}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\right)^{\bar\theta}|\log(h_0)|^\theta + C\int_0^t T_{h_0,\varepsilon}(s)\mathrm{d}s - D_1 - D_2 - \frac{7D_3}{8},$$

with D_1, D_2, D_3, and r_h given by (4.12), (4.16), (4.23), and (1.12) respectively. Here $0<\bar{\theta}$, $0<\theta<1$, and $t\leqslant T$, where T can be any positive number and the implicit constant may depend on time T.

Proof. We first recall

$$I_4=-\frac{1}{2}\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}}K_h(x-y)W_{\varepsilon,h}^{x,y}\,\chi'(\delta\rho_\varepsilon)\,\overline{\rho}_\varepsilon\,\delta(\operatorname{div}u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s.$$

We proceed by getting a representation formula for $\operatorname{div}u_\varepsilon$ from (2.2)

$$\operatorname{div}u_\varepsilon=\eta\rho_\varepsilon^{\gamma_{\mathrm{art}}}+\mathcal{L}_\varepsilon*P+F \tag{4.31}$$

where F is the effective viscous flux:

$$F=\Delta^{-1}\operatorname{div}F(\partial_t(\rho_\varepsilon u_\varepsilon)+\operatorname{div}(\rho_\varepsilon u_\varepsilon\otimes u_\varepsilon)).$$

Then the term I_4 may be rewritten as

$$\begin{aligned}I_4&=-\frac{1}{2}\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}}K_h(x-y)W_{\varepsilon,h}^{x,y}\,\chi'(\delta\rho_\varepsilon)\,\overline{\rho_\varepsilon}\,\delta(\eta\rho_\varepsilon^{\gamma_{\mathrm{art}}}+\mathcal{L}_\varepsilon*P+F)\,\mathrm{d}x\,\mathrm{d}y\frac{\mathrm{d}h}{h}\,\mathrm{d}s\\&=I_{4,1}+I_{4,2}+I_{4,3}\end{aligned}$$

with $I_{4,1}, I_{4,2}$, and $I_{4,3}$ being the integrals corresponding to the three terms in the parentheses of the above formula. Noting that

$$\begin{aligned}\eta\,\chi'(\delta\rho_\varepsilon)\,\overline{\rho_\varepsilon}\delta(\rho_\varepsilon^{\gamma_{\mathrm{art}}})&\geqslant\eta\,\chi'(\delta\rho_\varepsilon)\,\overline{\rho_\varepsilon}(\rho_\varepsilon(x)-\rho_\varepsilon(y))(\rho_\varepsilon^{\gamma_{\mathrm{art}}-1}(x)+\rho_\varepsilon^{\gamma_{\mathrm{art}}-1}(y))\\&=\eta(1+l)\,\chi(\delta\rho_\varepsilon)\,\overline{\rho_\varepsilon}(\rho_\varepsilon^{\gamma_{\mathrm{art}}-1}(x)+\rho_\varepsilon^{\gamma_{\mathrm{art}}-1}(y))\\&\geqslant\eta(1+l)\,\chi(\delta\rho_\varepsilon)\,\overline{\rho_\varepsilon^{\gamma_{\mathrm{art}}}}\end{aligned}$$

we arrive at

$$I_{4,1}\leqslant-\eta(1+l)\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}}K_h(x-y)W_{\varepsilon,h}^{x,y}\,\chi(\delta\rho_\varepsilon)\,\overline{\rho_\varepsilon^{\gamma_{\mathrm{art}}}}\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \tag{4.32}$$

which serves as a penalization. To bound the term $I_{4,2}$, we rewrite it as

$$\begin{aligned}I_{4,2}&=-\frac{1}{2}\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}}K_h(x-y)W_{\varepsilon,h}^{x,y}\,\chi'(\delta\rho_\varepsilon)\,\overline{\rho_\varepsilon}\,\delta(\mathcal{L}_\varepsilon*P)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s\\&=-\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{3d}}K_h(x-y)K_h(x-\widetilde{y})w_{\varepsilon,h}^{x}\,\chi'(\delta\rho_\varepsilon)\,\overline{\rho_\varepsilon}\\&\quad\times\delta(\mathcal{L}_\varepsilon*P)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s.\end{aligned}$$

Let $f(x,y,\widetilde{y})=\chi'(\delta\rho(x,x-y))\overline{\rho_\varepsilon}(x,x-y)$, then it is straightforward to check that f satisfies the condition (4.22). Appealing to the lemma 4.5, we arrive at

$$|I_{4,2}|\leqslant C+C\left(\int_\varepsilon^{2\varepsilon}r_{\max(h_0,\varepsilon')}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\right)^{\bar{\theta}}|\log(h_0)|^\theta+C\int_0^t T_{h_0,\varepsilon}(s)\mathrm{d}s+C\lambda^{-1}D_2+\frac{3D_3}{8}.$$

Finally, we deal with the effective viscous flux term $I_{4,3}$, which is rewritten as

$$I_{4,3} = -\frac{1}{2}\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} \phi_\varepsilon^M K_h(x-y) W_{\varepsilon,h}^{x,y} \chi'(\delta\rho_\varepsilon)\,\overline{\rho}_\varepsilon\,\delta F\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s$$
$$-\frac{1}{2}\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} (1-\phi_\varepsilon^M) K_h(x-y) W_{\varepsilon,h}^{x,y}\,\chi'(\delta\rho_\varepsilon)\,\overline{\rho}_\varepsilon\,\delta F\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s. \quad (4.33)$$

For the second integral, we use the uniform integrability of ρ_ε and $\operatorname{div} u_\varepsilon$ to obtain

$$\left|\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} (1-\phi_\varepsilon^M) K_h(x-y) W_{\varepsilon,h}^{x,y}\,\chi'(\delta\rho_\varepsilon)\,\overline{\rho}_\varepsilon\,\delta F\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s\right|$$
$$= \left|\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} (1-\phi_\varepsilon^M) K_h(y) W_{\varepsilon,h}^{x,x-y}\,\chi'(\delta\rho_\varepsilon)\,\overline{\rho}_\varepsilon\,\delta F(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s\right|$$
$$\lesssim \int_0^t \int_{h_0}^1 \int_{\mathbb{T}^d} K_h(y) \left\|(1-\phi_\varepsilon^M)\,\chi'(\delta\rho_\varepsilon)\,\overline{\rho}_\varepsilon(x,y)\right\|_{L_x^{p/(p-\gamma_{\mathrm{art}})}}$$
$$\times \left\|\delta F(x,y)\right\|_{L_x^{p/\gamma_{\mathrm{art}}}}\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s$$
$$\lesssim |\log h_0| M^{-\theta}$$

with some $1 > \theta > 0$ and $p = \gamma_{\mathrm{art}} + 2\gamma_{\mathrm{art}}/d - 1 - 1/\lambda_0$ for a sufficiently large constant λ_0. Note here $(1+l)p/(p-\gamma_{\mathrm{art}}) < \gamma_{\mathrm{art}}$ since we require $\gamma_{\mathrm{art}} > 2 + d$. While for the first integral in (4.33), we need to use lemma 4.6 with

$$\Phi = W_{\varepsilon,h}^{x,y}\,\chi'(\delta\rho_\varepsilon)\,\overline{\rho}_\varepsilon\,\phi_\varepsilon^M.$$

Obviously we have that $\|\Phi\|_{L^\infty} \lesssim M^{1+l}$. In view of the system (2.6) and (2.7), we get an equation for Φ as

$$\partial_t \Phi + \operatorname{div}_x\left(\Phi u_\varepsilon^x\right) + \operatorname{div}_y\left(\Phi u_\varepsilon^y\right) = f_{\varepsilon,1}^{x,y} \operatorname{div}_x u_\varepsilon^x + f_{\varepsilon,2}^{x,y} \operatorname{div}_y u_\varepsilon^y + f_{\varepsilon,3}^{x,y} \frac{1}{\lambda} D_\varepsilon^x$$
$$+ f_{\varepsilon,4}^{x,y} \frac{1}{\lambda} D_\varepsilon^y$$

where D_ε is the penalization introduced in (3.6) and $f_{\varepsilon,i}^{x,y}$ are polynomials of ρ_ε, w_ε, ϕ_ε^M, and derivatives of ϕ_ε^M for $i = 1, 2, 3, 4$. Noting that

$$\|f_{\varepsilon,i}^{x,y}\|_{L^\infty} \lesssim M^{1+l} \quad \text{for } i = 1, 2, 3, 4,$$

it is not difficult to get that

$$C_\Phi \lesssim M^{1+l}$$

where C_Φ is defined in lemma 4.6. Hence lemma 4.6 implies

$$\left|\int_0^t \int_{h_0}^1 \int_{\mathbb{T}^{2d}} \phi_\varepsilon^M K_h(x-y) W_{\varepsilon,h}^{x,y}\,\chi'(\delta\rho_\varepsilon)\,\overline{\rho}_\varepsilon\,\delta F\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s\right| \lesssim M^{1+l}.$$

Optimizing the bound in M gives

$$I_{4,3} \lesssim |\log h_0|^\theta$$

for some $0 < \theta < 1$. The proof is concluded by collecting the estimates for $I_{4,1}$, $I_{4,2}$, and $I_{4,3}$. □

4.7. Term I_5

We give the estimate for I_5 in this subsection.

Lemma 4.8. *Let I_5 be defined by (4.8). Under the assumptions in lemma 4.5, we have*

$$I_5 \leqslant C + C\left(\int_{\varepsilon}^{2\varepsilon} r_{\max(h_0,\varepsilon')}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\right)^{\bar\theta} |\log(h_0)|^{\theta} + C\int_0^t T_{h_0,\varepsilon}(s)\mathrm{d}s + C\lambda^{-1}D_2 + \frac{D_3}{2}$$

with D_2 and D_3 given by (4.16) and (4.23) respectively, for some $0 < \bar\theta$, $0 < \theta < 1$, and $t \leqslant T$, where T can be any positive number and the implicit constant may depend on time T.

Proof. We recall

$$I_5 = \int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(x-y)W_{\varepsilon,h}^{x,y}\left(\chi(\delta\rho_\varepsilon) - \frac{1}{2}\chi'(\delta\rho_\varepsilon)\,\delta\rho_\varepsilon\right) \times \overline{\mathrm{div}_x\, u_\varepsilon}\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s.$$

By the definition of χ in (4.1), the term I_5 may be rewritten as

$$\begin{aligned} I_5 &= \frac{1-l}{2}\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(x-y)W_{\varepsilon,h}^{x,y}\,\chi(\delta\rho_\varepsilon)\,\overline{\mathrm{div}_x\, u_\varepsilon}\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\ &= \frac{1-l}{2}\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(x-y)W_{\varepsilon,h}^{x,y}\,\chi(\delta\rho_\varepsilon)\,\overline{(P_{\mathrm{art},\eta}(\rho_\varepsilon) + \mathcal{L}_\varepsilon * P + F)}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad\times\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\ &= I_{5,1} + I_{5,2} + I_{5,3}. \end{aligned}$$

Note that since $P_{\mathrm{art},\eta}(\rho_\varepsilon) \leqslant C\,\rho_\varepsilon^{\gamma_{\mathrm{art}}}$, the term $I_{5,1}$ may be absorbed by the term $D_3/2$ in (4.23). Next we treat $I_{5,2}$ as

$$\begin{aligned} I_{5,2} &= \frac{1-l}{2}\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(x-y)w_{\varepsilon,h}(x)\chi(\delta\rho_\varepsilon)\,(\mathcal{L}_\varepsilon * P(x) \\ &\quad + \mathcal{L}_\varepsilon * P(y))\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\ &= (1-l)\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(x-y)w_{\varepsilon,h}(x)\chi(\delta\rho_\varepsilon)\mathcal{L}_\varepsilon * P(x)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s \\ &\quad - \frac{1-l}{2}\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(x-y)w_{\varepsilon,h}(x)\chi(\delta\rho_\varepsilon)\,\delta(\mathcal{L}_\varepsilon * P)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s. \end{aligned}$$

Since the second integral in the right side of the last equality is already estimated in I_4, we only need to consider the first integral. We need to use the penalization D_2 defined in (4.16) to control the main contribution of this term. Note

$$\begin{aligned}
&\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(x-y)w_{\varepsilon,h}(x)\chi(\delta\rho_\varepsilon)\mathcal{L}_\varepsilon * P(x)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s\\
&\quad=\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-z)w_{\varepsilon,h}(x)\chi(\delta\rho_\varepsilon)\mathcal{L}_\varepsilon * P(x)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\
&\qquad\times\frac{\mathrm{d}h}{h}\,\mathrm{d}s\\
&\quad=\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-z)w_{\varepsilon,h}(x)\chi(\delta\rho_\varepsilon)\,(\mathcal{L}_\varepsilon * P(x)\\
&\qquad-\mathcal{L}_\varepsilon * P(z))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s+\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-z)\\
&\qquad\times\ w_{\varepsilon,h}(x)\chi(\delta\rho_\varepsilon)\mathcal{L}_\varepsilon * P(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s
\end{aligned}$$

where the last integral is bounded by $C\lambda^{-1}D_2$. We switch variables to rewrite the first integral as

$$\begin{aligned}
&\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-z)w_{\varepsilon,h}(x)\chi(\delta\rho_\varepsilon)\,(\mathcal{L}_\varepsilon * P(x)\\
&\qquad-\mathcal{L}_\varepsilon * P(z))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s\\
&\quad=\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-\widetilde{y})w_{\varepsilon,h}(x)\chi(\delta\rho_\varepsilon(x,x-\widetilde{y}))\,(\mathcal{L}_\varepsilon * P(x)\\
&\qquad-\mathcal{L}_\varepsilon * P(y))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s.
\end{aligned}$$

Let $f(x,y,\widetilde{y})=\chi(\delta\rho_\varepsilon(x,x-\widetilde{y}))$, then it is easy to check that (4.22) holds. Using lemma 4.5, we arrive at

$$\begin{aligned}
&\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-\widetilde{y})w_{\varepsilon,h}(x)\chi(\delta\rho_\varepsilon(x,x-\widetilde{y}))\,(\mathcal{L}_\varepsilon * P(x)\\
&\qquad-\mathcal{L}_\varepsilon * P(y))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s\\
&\leqslant C+C\left(\int_\varepsilon^{2\varepsilon} r_{\max(h_0,\varepsilon')}\frac{\mathrm{d}\varepsilon'}{\varepsilon'}\right)^{\bar\theta}|\log(h_0)|^\theta+C\int_0^t T_{h_0,\varepsilon}(s)\mathrm{d}s\\
&\qquad+C\lambda^{-1}D_2+\frac{3D_3}{8}.
\end{aligned}$$

At last, we treat the effective viscous flux term as

$$I_{5,3}=(1-l)\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(x-y)w^x_{\varepsilon,h}\,\chi\,(\delta\rho_\varepsilon)\,\overline{F}\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s$$

$$= (1-l)\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(x-y)w^x_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)\,(F(y)-F(x))\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s$$
$$+ 2(1-l)\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(x-y)w^x_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)\,F(x)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s.$$

Note that the first integral is already treated in $I_{4,2}$, and we now deal with the second integral as

$$\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{2d}} K_h(x-y)w^x_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)\,F(x)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s$$
$$= \int_0^t\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-z)w^x_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)$$
$$\times\,(F(x)-F(z))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s + \int_0^t\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-z)$$
$$\times\; w^x_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)\,F(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s.$$

For the first integral, by similar argument as in the treatment of $I_{4,3}$, we arrive at

$$\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-z)w^x_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)\,(F(x)-F(z))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s$$
$$\lesssim |\log\,h_0|^\theta$$

for some $0<\theta<1$. While for the second integral, we use the formula (4.31) to obtain

$$\int_0^t\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-z)w^x_{\varepsilon,h}\,\chi(\delta\rho_\varepsilon)\,F(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s$$
$$\leqslant \int_0^t\int_{h_0}^1\int_{\mathbb{T}^{3d}} K_h(x-y)K_h(x-z)w^x_{\varepsilon,h}\chi(\delta\rho_\varepsilon)|\mathrm{div}\,u_\varepsilon|(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}h}{h}\,\mathrm{d}s$$

which is bounded by $C\lambda^{-1}D_2$. Collecting all the estimate and optimizing in M concludes the proof. □

4.8. Compactness argument

Proof of theorem 2.2. Collecting the estimates from lemmas 4.1, 4.2–4.4, 4.7 and 4.8, choosing λ sufficiently large, and dropping the extra penalization D_1, D_2, and D_3, we have

$$T_{h_0,\varepsilon}(t) \lesssim T_{h_0,\varepsilon}(0) + C\int_0^t T_{h_0,\varepsilon}(s)\mathrm{d}s + |\log\,h_0|^\theta$$

for some $0<\theta<1$. A Gronwall inequality implies

$$T_{h_0,\varepsilon}(t) \lesssim \mathrm{e}^{CT}|\log\,h_0|^\theta$$

for $t \leqslant T$. Recalling the definition of $T_{h_0,\varepsilon}$, in order to get the compactness of the solution ρ_ε, we need to get rid of the weight function. Note that

$$\begin{aligned}\int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y)\chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y &= \int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y)\chi(\delta\rho_\varepsilon)\mathbf{1}_{w^x_{\varepsilon,h}\leqslant\eta}\mathbf{1}_{w^y_{\varepsilon,h}\leqslant\eta}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y)\chi(\delta\rho_\varepsilon) \\ &\quad \times (1-\mathbf{1}_{w^x_{\varepsilon,h}\leqslant\eta}\mathbf{1}_{w^y_{\varepsilon,h}\leqslant\eta})\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

where $\eta > 0$ is a big parameter depending on h_0 to be chosen later. For the first integral, in view of (3.11), we have

$$\begin{aligned}&\int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y)\chi(\delta\rho_\varepsilon)\mathbf{1}_{w^x_{\varepsilon,h}\leqslant\eta}\mathbf{1}_{w^y_{\varepsilon,h}\leqslant\eta}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad\lesssim \int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y)\rho_\varepsilon^{1+l}(x)\mathbf{1}_{w^x_{\varepsilon,h}\leqslant\eta}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y)\rho_\varepsilon^{1+l} \\ &\qquad\times (y)\mathbf{1}_{w^y_{\varepsilon,h}\leqslant\eta}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad\lesssim \int_{\mathbb{T}^d} \rho_\varepsilon^{1+l}(x)\mathbf{1}_{w^x_{\varepsilon,h}\leqslant\eta}\,\mathrm{d}x|\log\,h_0| \lesssim \frac{|\log\,h_0|}{|\log\,\eta|^\alpha}\end{aligned}$$

for some $0 < \alpha < 1$. For the second integral, we use $T_{h_0,\varepsilon}$ to get

$$\begin{aligned}\int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y)\chi(\delta\rho_\varepsilon)(1-\mathbf{1}_{w^x_{\varepsilon,h}\leqslant\eta}\mathbf{1}_{w^y_{\varepsilon,h}\leqslant\eta})\,\mathrm{d}x\,\mathrm{d}y &\leqslant \frac{1}{\eta}T_{h_0,\varepsilon}(t) \\ &\lesssim \frac{1}{\eta}|\log\,h_0|^\theta.\end{aligned}$$

By choosing $\eta = |\log h_0|$, we arrive at

$$\int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y)\chi(\delta\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y \lesssim \frac{|\log\,h_0|}{\log\,|\log\,h_0|},$$

which implies the compactness of the solution ρ_ε by lemma 3.1. □

5. Concluding section

In this paper, we prove the existence of global weak solutions à la Leray for CNS equations with a pressure law which depends on the density and on time and space variables t and x. It may be seen as a first step to consider heat-conducting Navier–Stokes equations with physical laws such as the truncated virial assumption. The paper focuses on two main difficulties:

- The construction of approximate solutions through a new regularized and fixed point procedure: to do so, an artificial pressure term based on a hierarchy cascade is introduced in addition to an appropriate regularization of the pressure law to design a candidate for the approximate pressure law.

- The weak stability process taking advantage of the new method introduced by the first two authors with a careful study of the regularized pressure defined in the first step: its treatment constitutes the main innovation in this paper.

Acknowledgments

The first author is partially supported by the SingFlows project, Grant ANR-18-CE40-0027. The second author is partially supported by NSF DMS Grant 161453, 1908739, 2049020 and NSF Grant RNMS (Ki-Net) 1107444.

Appendix A. Proof of theorems 2.1 and 2.3

A.1. Proof of theorem 2.3

The proof is performed by taking several consecutive limits, first $\eta_1 \to 0$, then $\eta_2 \to 0$ till the last limit $\eta_m \to 0$. The generic step is hence, once we already have $\eta_1 = \cdots = \eta_i = 0$, to pass to the limit $\eta_{i+1} \to 0$. For this reason, we introduce the notation $\rho_{\eta,i}, u_{\eta,i}$ which is obtained by taking the first $i-1$ weak limits $\eta_1 \to 0$, $\eta_{i-1} \to 0$. More precisely, after extracting subsequences, we have that $\rho_{\eta,1} = \rho_\eta$, $u_{\eta,1} = u_\eta$ and

$$\rho_{\eta,i+1} = w - \lim_{\eta_i \to 0} \rho_{\eta,i}, \qquad u_{\eta,i+1} = w - \lim_{\eta_i \to 0} u_{\eta,i}.$$

The final solution that we will obtain is simply $\rho = \rho_{\eta,m+1}, u = u_{\eta,m+1}$ which is independent of all η_i. Assuming that $\rho_{\eta,i}$ is a weak solution to the system

$$\begin{aligned} &\partial_t \rho_{\eta_i} + \operatorname{div}(\rho_{\eta_i} u_{\eta_i}) = 0, \\ &\partial_t(\rho_{\eta_i} u_{\eta_i}) + \operatorname{div}(\rho_{\eta_i} u_{\eta_i} \otimes u_{\eta_i}) - \Delta u_{\eta_i} + \nabla \left(\eta_i \rho_{\eta_i}^{\gamma_{\mathrm{art}},i} + \cdots + \eta_m \rho_{\eta_i}^{\gamma_{\mathrm{art}},m} \right. \\ &\qquad \left. + P(t,x,\rho_{\eta_i}) \right) = 0, \end{aligned} \tag{A.1}$$

then we have to show that $\rho_{\eta,i+1}$ solves the same system with $\eta_i = 0$.

Step 1: basic energy inequality for ρ_η, u_η. We observe that ρ_η, u_η solves (A.1) directly from theorem 2.1. However the *a priori* estimates provided by theorem 2.1 are not uniform in η so that our first step consists in deriving such estimate starting from the energy inequality (2.5).

The first point is to pass to the limit as $\varepsilon \to 0$ in (2.5). Of course the left-hand side is convex in $\rho_{\varepsilon,\eta}, u_{\varepsilon,\eta}$ so it handled in the usual manner. We have that $\operatorname{div} u_{\varepsilon,\eta}$ is uniformly bounded in $L^2_{t,x}$ so $\operatorname{div} u_{\varepsilon,\eta} \to \operatorname{div} u_\eta$ in $w - L^2_{t,x}$.

On the other hand by (1.7) and (1.8), we have that $|P(t,x,\rho_{\varepsilon,\eta})| \leqslant R + \Theta_1 + C\,\rho^p_{\varepsilon,\eta}$ with $p \leqslant \gamma + \frac{2\gamma}{d} - 1$ and $R + \Theta_1 \in L^q_{t,x}$ with $q > 2$. By theorem 2.1, we have that $\rho_{\varepsilon,\eta} \in L^{p_{\mathrm{art}}}_{t,x}$ uniformly in ε for any $p_{\mathrm{art}} \leqslant \gamma_{\mathrm{art}} + 2\gamma_{\mathrm{art}}/d - 1$. Observe that $2\,(\gamma + \frac{2\gamma}{d} - 1) < 2\,\gamma + \frac{4\gamma}{d} - 1 \leqslant \gamma_{\mathrm{art}} + 2\,\gamma_{\mathrm{art}}/d - 1$ since $2\gamma \leqslant \gamma_{\mathrm{art}}$. This is the first place where the assumption $2\gamma \leqslant \gamma_{\mathrm{art}}$ is critical.

Hence $P(t,x,\rho_{\varepsilon,\eta})$ is uniformly bounded in ε in $L^q_{t,x}$ for some $q > 2$. By the compactness of $\rho_{\varepsilon,\eta}$ provided by theorem 2.2, we obtain that $P(t,x,\rho_{\varepsilon,\eta}) \to P(t,x,\rho_\eta)$ strongly in $L^2_{t,x}$.

Therefore this provides a solution ρ_η, u_η to the system (2.6) and (2.7) with, *for a fixed* η, the bounds $\rho_\eta \in L^\infty_t L^{\gamma_{\mathrm{art}}}_x$, $\rho_\eta \in L^p_{t,x}$ for any $p \leqslant \gamma_{\mathrm{art}} + 2\gamma_{\mathrm{art}}/d - 1$, $u_\eta \in L^2_t H^1_x$, and the basic

energy inequality

$$\begin{aligned}\int_{\mathbb{T}^d}\left(\eta_1\frac{\rho_\eta^{\gamma_{\rm art},1}(t,x)}{\gamma_{\rm art},1-1}+\cdots+\eta_m\frac{\rho_\eta^{\gamma_{\rm art},m}(t,x)}{\gamma_{\rm art},m-1}+\rho_\eta(t,x)\,|u_\eta(t,x)|^2\right)\mathrm{d}x\\ +\int_0^t\int_{\mathbb{T}^d}|\nabla u_\eta(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s\leqslant\int_0^t\int_{\mathbb{T}^d}\mathrm{div}\,u_\eta\,P\,\mathrm{d}x\,\mathrm{d}s\\ +\int_{\mathbb{T}^d}\left(\eta_1\frac{(\rho_{0,\varepsilon,\eta})^{\gamma_{\rm art},1}(t,x)}{\gamma_{\rm art},1-1}+\cdots+\eta_m\frac{(\rho_{0,\varepsilon,\eta})^{\gamma_{\rm art},m}(t,x)}{\gamma_{\rm art},m-1}\right)\mathrm{d}x.\\ +\int_{\mathbb{T}^d}\rho_{0,\varepsilon,\eta}(t,x)\,|u_{0,\varepsilon,\eta}(t,x)|^2\,\mathrm{d}x\end{aligned}\tag{A.2}$$

Step 2: modified energy inequality. Our next step is to work with (A.2) to obtain a form that is more suitable to the derivation of *a priori* estimates.

We recall that $\mathcal{E}_0=\rho_\eta\,(|u_\eta|^2/2+e_0(\rho_\eta))$ with $e_0(t,x,\rho)=\int_{\rho_{\rm ref}}^{\rho}P_0(t,x,s)/s^2\,\mathrm{d}s$.

We have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}\rho_\eta\,e_0(t,x,\rho_\eta)\,\mathrm{d}x=\int_{\mathbb{T}^d}(\rho_\eta\partial_t e_0(\rho_\eta)+\rho_\eta\,u_\eta\cdot\nabla_x e_0(\rho_\eta))\,\mathrm{d}x\\ +\int_{\mathbb{T}^d}\mathrm{div}\,u_\eta(\rho_\eta\,e_0(\rho_\eta)-\rho_\eta\,\partial_\rho(\rho_\eta e_0(\rho_\eta)))\,\mathrm{d}x.\end{aligned}$$

From the definition of e_0, we get that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}\rho_\eta\,e_0(t,x,\rho_\eta)\,\mathrm{d}x=\int_{\mathbb{T}^d}(\rho_\eta\partial_t e_0(\rho_\eta)+\rho_\eta\,u_\eta\cdot\nabla_x e_0(\rho_\eta))\,\mathrm{d}x\\ -\int_{\mathbb{T}^d}\mathrm{div}\,u_\eta\,P_0(\rho_\eta)\,\mathrm{d}x.\end{aligned}$$

Note that from (1.8)–(1.10), we have that for a fixed η, $P_0\in L^2$ while $\rho_\eta\partial_t e_0(\rho_\eta)\in L^1_{t,x}$ and $\rho_\eta\,\nabla_x e_0(\rho_\eta)\in L^2_t\,L^{2d/(d+2)}_x$ so that $\rho_\eta\,u_\eta\cdot\nabla_x e_0(\rho_\eta)\in L^1_{t,x}$ as well. Therefore all terms make sense and this is again due to the assumption $\gamma_{\rm art}\geqslant 2\gamma$.

Adding this to (A.2) yields the more precise energy inequality

$$\begin{aligned}\int_{\mathbb{T}^d}(\mathcal{E}_0(\rho_\eta,u_\eta)+\eta_1\frac{\rho_\eta^{\gamma_{\rm art},1}}{\gamma_{\rm art},1-1}+\cdots+\eta_m\frac{\rho_\eta^{\gamma_{\rm art},m}}{\gamma_{\rm art},m-1})\,\mathrm{d}x\\ +\int_0^t\int_{\mathbb{T}^d}|\nabla u_\eta(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s\\ \leqslant\int_0^t\int_{\mathbb{T}^d}\mathrm{div}_x\,u_\eta(s,x)\,(P(s,x,\rho_\eta(s,x))-P_0(s,x,\rho_\eta(s,x)))\mathrm{d}s\,\mathrm{d}x\\ +\int_0^t\int_{\mathbb{T}^d}(\rho_\eta\,\partial_t e_0(\rho_\eta)+\rho_\eta\,u_\eta\cdot\nabla_x e_0(\rho_\eta))\,\mathrm{d}x\,\mathrm{d}s\\ +\int_{\mathbb{T}^d}\left(\eta_1\frac{(\rho_{0,\varepsilon,\eta})^{\gamma_{\rm art},1}(t,x)}{\gamma_{\rm art},1-1}+\cdots+\eta_m\frac{(\rho_{0,\varepsilon,\eta})^{\gamma_{\rm art},m}(t,x)}{\gamma_{\rm art},m-1}\right.\\ \left.+\mathcal{E}_0(\rho_{0,\eta},u_{0,\eta})\right)\mathrm{d}x,\end{aligned}\tag{A.3}$$

which we will use to obtain our *a priori* estimates.

Step 3: a priori estimates on ρ_η, u_η. From (A.3), we first observe that from (1.7) since $\bar\gamma \leqslant \gamma/2$,

$$\int_0^t \int_{\mathbb{T}^d} \mathrm{div}_x\, u_\eta(s,x)\,(P(s,x,\rho_\eta(s,x)) - P_0(s,x,\rho_\eta(s,x)))\,\mathrm{d}s\,\mathrm{d}x \leqslant C + \frac{1}{4}\int_0^t\int_{\mathbb{T}^d} |\nabla u_\eta(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\int_{\mathbb{T}^d} |\rho_\eta(s,x)|^\gamma\,\mathrm{d}x\,\mathrm{d}s.$$

Similarly by (1.9)–(1.10), we can bound

$$\int_0^t \int_{\mathbb{T}^d} (\partial_t e_0(\rho_\eta) + \rho_\eta\, u_\eta\cdot\nabla_x e_0(\rho_\eta))\,\mathrm{d}x\,\mathrm{d}s \leqslant C + \frac{1}{4}\int_0^t\int_{\mathbb{T}^d} |\nabla u_\eta(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\int_{\mathbb{T}^d} |\rho_\eta(s,x)|^\gamma\,\mathrm{d}x\,\mathrm{d}s.$$

By (1.8), we hence obtain that

$$\int_{\mathbb{T}^d}\left(\frac{\rho_\eta^\gamma}{C} + \eta_1\frac{\rho_\eta^{\gamma_{\mathrm{art},1}}}{\gamma_{\mathrm{art},1}-1} + \cdots + \eta_m\frac{\rho_\eta^{\gamma_{\mathrm{art},m}}}{\gamma_{\mathrm{art},m}-1} + \rho_\eta\frac{|u_\eta|^2}{2}\right)\mathrm{d}x + \frac{1}{2}\int_0^t\int_{\mathbb{T}^d}|\nabla u_\eta(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C + C\int_0^t\int_{\mathbb{T}^d}|\rho_\eta(s,x)|^\gamma\,\mathrm{d}x\,\mathrm{d}s.$$

By Gronwall's lemma, we deduce the first main estimate on ρ_η and u_η, for some constant C independent of η

$$\begin{aligned}&\int_{\mathbb{T}^d}\left(\frac{\rho_\eta^\gamma}{C} + \eta_1\frac{\rho_\eta^{\gamma_{\mathrm{art},1}}}{\gamma_{\mathrm{art},1}-1} + \cdots + \eta_m\frac{\rho_\eta^{\gamma_{\mathrm{art},m}}}{\gamma_{\mathrm{art},m}-1} + \rho_\eta\frac{|u_\eta|^2}{2}\right)\mathrm{d}x \leqslant C\,\mathrm{e}^{C\,t},\\ &\int_0^t\int_{\mathbb{T}^d}|\nabla u_\eta(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C\,\mathrm{e}^{C\,t}.\end{aligned} \tag{A.4}$$

Those estimates are convex in ρ_η and u_η. Hence by the definition of the $\rho_{\eta,i}, u_{\eta,i}$, we trivially have as well that

$$\begin{aligned}&\int_{\mathbb{T}^d}\left(\frac{\rho_{\eta,i}^\gamma}{C} + \eta_i\frac{\rho_{\eta,i}^{\gamma_{\mathrm{art},i}}}{\gamma_{\mathrm{art},i}-1} + \cdots + \eta_m\frac{\rho_{\eta,i}^{\gamma_{\mathrm{art},m}}}{\gamma_{\mathrm{art},m}-1} + \rho_{\eta,i}\frac{|u_{\eta,i}|^2}{2}\right)\mathrm{d}x \leqslant C\,\mathrm{e}^{C\,t},\\ &\int_0^t\int_{\mathbb{T}^d}|\nabla u_{\eta,i}(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C\,\mathrm{e}^{C\,t}.\end{aligned} \tag{A.5}$$

When considering the limit $\eta_i \to 0$ on $\rho_{\eta,i}, u_{\eta,i}$, we have that $\eta_{i+1},\ldots,\eta_m > 0$. We hence have all the bounds needed to apply lemma 2.4 with $S = P - P_0$, $\gamma_0 = \gamma_{\mathrm{art},i+1}$ and $1/p = 1 + 1/\gamma_{\mathrm{art},i} + 1 - 2/d$ or $\gamma_{\mathrm{art},i} + 1/p' = 2\gamma_{\mathrm{art},i} + 1/d - 1$. This lets us obtain our last *a priori* estimate

$$\sup_{\eta_i}\int_0^T\int_{\mathbb{T}^d}\rho_{\eta,i}^q(t,x)\,\mathrm{d}x\,\mathrm{d}t < \infty, \quad \forall\ q < \gamma_{\mathrm{art},i+1} + 2\,\gamma_{\mathrm{art},i+1}/d - 1. \tag{A.6}$$

Step 4: passing to the limit. Equipped with those bounds, we have the *weakly converging sub-sequences* as $\eta_i \to 0$: $\rho_{\eta,i} \to \rho_{\eta,i+1}$ in $w - L^\infty_t L^{\gamma_{\mathrm{art},i+1}}_x$ and $w - L^q_{t,x}$ for any $q < \gamma_{\mathrm{art},i} + 1 + 2\gamma_{\mathrm{art},i} + 1/d - 1$, and $u_{\eta,i} \to u_{\eta,i+1}$ in $w - L^2_t H^1_x$.

As usual, this is also enough to show the weak limits $\rho_{\eta,i}u_{\eta,i} \to \rho_{\eta,i+1}u_{\eta,i+1}$ and $\rho_{\eta,i}u_{\eta,i} \otimes u_{\eta,i} \to \rho_{\eta,i+1}u_{\eta,i+1} \otimes u_{\eta,i+1}$. Those bounds also provides equi-integrability on $P(t,x,\rho_{\eta,i})$ by the upper bounds following from (1.7) and (1.8). Equi-integrability also holds on

$$\eta_i \frac{\rho^{\gamma_{\mathrm{art},i}}_{\eta,i}}{\gamma_{\mathrm{art},i} - 1} + \cdots + \eta_m \frac{\rho^{\gamma_{\mathrm{art},m}}_{\eta,i}}{\gamma_{\mathrm{art},m} - 1},$$

since $\gamma_{\mathrm{art},i} < \gamma_{\mathrm{art},i} + 1 + 2\gamma_{\mathrm{art},i} + 1/d - 1$ which is the key relation between the coefficients $\gamma_{\mathrm{art},i}$.

The main remaining question is to prove the compactness of $\rho_{\eta,i}$ in $L^1_{t,x}$. This is in general the difficult question for CNS but, fortunately in this case, we may directly apply the result of [4].

Specifically we invoke theorem 5.1, case (ii) in that article (page 613). Our sequence $\rho_{\eta,i}, u_{\eta,i}$ solves the continuity equation (denoted (5.1) in the article). The momentum equation implies that $u_{\eta,i}$ solves equation (5.2) in the article with constant viscosity and $R_k = 0$. Our *a priori* estimates directly ensures the bounds (5.3)–(5.7) that are required by theorem 5.1 in [4]. Finally the assumption on the pressure law for this theorem is identical to our assumptions (1.11) and (1.12).

We hence deduce the compactness of $\rho_{\eta,i}$ and hence the convergence of $P(t,x,\rho_{\eta,i}) + \eta_i\,\rho^{\gamma_{\mathrm{art},i}}_{\eta,i} + \cdots + \eta_m\,\rho^{\gamma_{\mathrm{art},m}}_{\eta,i}$ to $P(t,x,\rho_{\eta,i+1}) + \eta_{i+1}\,\rho^{\gamma_{\mathrm{art},i+1}}_{\eta,i+1} + \cdots + \eta_m\,\rho^{\gamma_{\mathrm{art},m}}_{\eta,i+1}$. This implies that $\rho_{\eta,i+1}, u_{\eta,i+1}$ solves (A.1) with $\eta_i = 0$ and finally that ρ, u is indeed a global solution to the system (1.2) and (1.3) as claimed with the corresponding estimates for $i = m + 1$ following from (A.5) and (A.6). Finally the energy inequality is directly obtained from (A.3) by taking the successive limits.

A.2. Proof of theorem 2.1

We can obtain solutions to (2.1) and (2.2) through a fixed point theorem. Given any $S \in L^2([0,\ T] \times \mathbb{T}^d))$, we define N_S, U_S as a global weak solution to

$$\begin{aligned} &\partial_t N_S + \operatorname{div}(N_S\,U_S) = 0, \qquad N_S(t=0) = \rho^0_\varepsilon,\\ &\partial_t(N_S\,U_S) + \operatorname{div}(N_S\,U_S \otimes U_S) - \Delta U_S + \nabla(P_\eta(N_S) + S) = 0,\\ &U_\rho(t=0) = u^0_\varepsilon. \end{aligned} \tag{A.7}$$

System (A.7) is in fact the classical CNS system with barotropic pressure law $P_\eta(\rho) = \eta_1\rho^{\gamma_{\mathrm{art},1}} + \cdots + \eta_m\rho^{\gamma_{\mathrm{art},m}}$ and a source term. Provided that $\gamma_{\mathrm{art}} + 2\gamma_{\mathrm{art}}/d - 1 > 2$ with $\gamma_{\mathrm{art}} = \gamma_{\mathrm{art},1} = \max_i \gamma_{\mathrm{art},i}$, which we assumed, existence of global solution to this system is guaranteed by [18] and moreover such solutions satisfy the following energy estimate for some constant C

$$\begin{aligned} &\sup_{t\in[0,\ T]} \int_{\mathbb{T}^d} ((N_S(t,x))^{\gamma_{\mathrm{art}}} + N_S\,|U_S|^2)\,\mathrm{d}x + \int_0^T \int_{\mathbb{T}^d} |\nabla U_S|^2\,\mathrm{d}x\\ &\qquad \leqslant C \int_{\mathbb{T}^d} ((\rho^0_\varepsilon(t,x))^{\gamma_{\mathrm{art}}} + \rho^0_\varepsilon\,|u^0_\varepsilon|^2)\,\mathrm{d}x + C\,\|S\|^2_{L^2_{t,x}}. \end{aligned} \tag{A.8}$$

and the following energy inequality

$$\int_{\mathbb{T}^d}\left(\eta_1\frac{N_S^{\gamma_{\rm art},1}(t,x)}{\gamma_{{\rm art},1}-1}+\cdots+\eta_m\frac{N_S^{\gamma_{\rm art},m}(t,x)}{\gamma_{{\rm art},m}-1}+N_S(t,x)\,|U_S(t,x)|^2\right)\mathrm{d}x$$
$$+\int_0^t\int_{\mathbb{T}^d}|\nabla U_S(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s\leqslant\int_0^t\int_{\mathbb{T}^d}\mathrm{div}\,U_S\cdot S\,\mathrm{d}x\,\mathrm{d}s$$
$$+\int_{\mathbb{T}^d}\left(\eta_1\frac{(\rho_{0,\varepsilon,\eta})^{\gamma_{\rm art},1}(t,x)}{\gamma_{{\rm art},1}-1}+\cdots+\eta_m\frac{(\rho_{0,\varepsilon,\eta})^{\gamma_{\rm art},m}(t,x)}{\gamma_{{\rm art},m}-1}\right)\mathrm{d}x$$
$$+\int_{\mathbb{T}^d}\rho_{0,\varepsilon,\eta}(t,x)|u_{0,\varepsilon,\eta}(t,x)|^2\mathrm{d}x. \tag{A.9}$$

We are now using lemma 2.4 with $P_0=P_\eta(N_S)=\eta_1\,N_S^{\gamma_{\rm art},1}+\cdots+\eta_m\,N_s^{\gamma_{\rm art},m}$.

One has that $N_S\in L^\infty_t L^{\gamma_{\rm art}}_x$ solves (B.1) with $u_\varepsilon\in L^2_tH^1_x$. Since $\gamma_{\rm art}>2$ then $S\in L^2_{t,x}\subset L^1_tL^{\gamma'_{\rm art}}_x$ trivially. On the other hand

$$\sup_\varepsilon\|\Delta U_S\|_{L^2_tH^{-1}}<\infty,$$

and using Sobolev embeddings $U_S\in L^2_tL^q_x$ with $1/q=1/2-1/d$ so that

$$\sup_\varepsilon\|N_S\,U_S\otimes U_S\|_{L^1_tL^p_x}<\infty,\qquad \frac{1}{p}=\frac{1}{\gamma_{\rm art}}+\frac{2}{q}=\frac{1}{\gamma_{\rm art}}+1-\frac{2}{d}.$$

Similarly

$$\sup_\varepsilon\|N_S\,U_S\|_{L^2_tL^r_x}<\infty,\qquad \frac{1}{r}=\frac{1}{\gamma_{\rm art}}+\frac{1}{q}=\frac{1}{\gamma_{\rm art}}+\frac{1}{2}-\frac{1}{d},$$

and one notes that $2pd/(2d+2p-pd)=r$ or $1/r=1/p+1/d-1/2$.

Using the bound on the kinetic energy $\int N_S|U_S|^2\,\mathrm{d}x$, we also have that

$$\int_{\mathbb{T}^d}N_S^s\,|U_S|^s\,\mathrm{d}x\leqslant\left(\int_{\mathbb{T}^d}N_S\,|U_S|^2\right)^{s/2}\left(\int_{\mathbb{T}^d}N_S^{s/(2-s)}\right)^{1-s/2}.$$

Note that $s/(2-s)=\gamma_{\rm art}$ iff $s=2\gamma_{\rm art}/(1+\gamma_{\rm art})$, implying that

$$\sup_\varepsilon\|N_S\,U_S\|_{L^\infty_tL^s_x}<\infty,\qquad s=2\gamma_{\rm art}/(1+\gamma_{\rm art}),$$

with in particular $s=2pd/(d+2p)\geqslant pd/(p+d)$.

We hence deduce that for $\theta<\gamma_{\rm art}/p'$ or $\theta<2\gamma_{\rm art}/d-1$

$$\sup_\varepsilon\int_0^T\int_{\mathbb{T}^d}N_S^\theta\,P_0(N_S)\,\mathrm{d}t\,\mathrm{d}x<\infty,$$

or, in other words, lemma 2.4 implies that

$$\int_0^T\int_{\mathbb{T}^d}N_S^q(t,x)\,\mathrm{d}x\,\mathrm{d}t\leqslant C\int_{\mathbb{T}^d}((\rho^0_\varepsilon(t,x))^{\gamma_{\rm art}}+\rho^0_\varepsilon\,|u^0_\varepsilon|^2)\,\mathrm{d}x+C\,\|S\|^2_{L^2_{t,x}}. \tag{A.10}$$

This leads to defining the following operator

$$F:S\to F(S)(t,x)=\mathcal{L}_\varepsilon\star P(N_S).$$

From the definition, we have that

$$\|F(S)\|^2_{L^2_{t,x}} \leqslant C\,\varepsilon^{-d}\,\|P(N_S)\|^2_{L^2_t\,L^1_x} \leqslant C\,\varepsilon^{-d}\,\|R\|^2_{L^1} + C\,\varepsilon^{-d}\,\|N_S^p\|^2_{L^2_t\,L^1_x},$$

for some $p < \gamma + 2\gamma/d - 1$, by using assumptions (1.7) and (1.8) on P. Since $R \in L^2_{t,x}$,, we deduce that

$$\|F(S)\|^2_{L^2_{t,x}} \leqslant C\,\varepsilon^{-d} + C\,\varepsilon^{-d}\,\|N_S\|^{2p}_{L^{2p}_{t,x}}.$$

Finally, $\gamma_{\rm art} + 2\gamma_{\rm art}/d - 1 \geqslant 2\gamma + 4\gamma/d - 1 > 2p$ since $\gamma_{\rm art} \geqslant 2\gamma$, we have by (A.10)

$$\|F(S)\|^2_{L^2_{t,x}} \leqslant C\,\varepsilon^{-d} + C\,\varepsilon^{-d}\,T^4\,\|S\|^{2\,\theta}_{L^2_{t,x}},$$

for some exponent $\theta < 1$ through the uniform in ε bound on

$$\sup_\varepsilon \sup_t \int_{\mathbb{T}^d} ((\rho_{0,\varepsilon}(t,x))^{\gamma_{\rm art}} + \rho_{0,\varepsilon}(t,x)\,|u_{0,\varepsilon}(t,x)|^2)\,{\rm d}x < \infty.$$

As $\theta < 1$, there exists a ball $B \subset L^2_{t,x}$ with large enough radius such that $F(B) \subset B$.

Moreover $F(S) \in L^2_t H^1_x$ for any $S \in B$ thanks to the convolution in x giving compactness in the space variable. To prove the time compactness, one could observe that the argument in [18] or the quantitative estimates from [4] provide full compactness on the density provided that the source term is compact in space (i.e. without time compactness being required).

However, since it is possible to obtain the time compactness in a straightforward manner and for the sake of completeness, we present the argument here. We need to introduce various regularization and truncations. First of all (1.7) implies that $P/(1+s^p)$ is in $L^2_{t,x}$ uniformly in s. Hence we can choose $P_\eta(t,x,s)$ a regularization of P in t and x with for example

$$|\partial_t P_\eta(t,x,s)| + |\nabla_x P_\eta(t,x,s)| + |\partial_s^2 P_\eta(t,x,s)| \leqslant \frac{C}{\eta}\,(1+s^p),$$
$$\|P(.,.,s) - P_\eta(.,.,s)\|_{L^2_{t,x}} \leqslant f(\eta)\,(1+s^p),$$

for some continuous function f with $f(0) = 0$.

By (1.7) again and since (A.8) shows that $N_S \in L^\infty_t L^{\gamma_{\rm art}}_x$ with $\gamma_{\rm art} > \gamma$, we may immediately deduce from the last point that there exists $\widetilde{f}$ continuous with $\widetilde{f}(0) = 0$ such that for any $S \in B$

$$\|P(.,.,N_S) - P_\eta(.,.,N_S)\|_{L^2_{t,x}} \leqslant \widetilde{f}(\eta). \tag{A.11}$$

Now choosing any standard convolution kernel L, we may write

$$\begin{aligned}
\mathcal{L}_\varepsilon \star P_\eta(N_S)(t,x) &= \int_{\mathbb{T}^d} \mathcal{L}_\varepsilon(x-y)\,P_\eta(t,y,N_S(t,y))\,{\rm d}y \\
&= \int_{\mathbb{T}^d} \mathcal{L}_\varepsilon(x-y)\,L_{\sqrt{\eta}}(y-z)\,P_\eta(t,z,N_S(t,y))\,{\rm d}y\,{\rm d}z \\
&\quad + \int_{\mathbb{T}^d} \mathcal{L}_\varepsilon(x-y)\,L_{\sqrt{\eta}}(y-z)\,\big(P_\eta(t,y,N_S(t,y)) \\
&\quad - P_\eta(t,z,N_S(t,y))\big)\,{\rm d}y\,{\rm d}z.
\end{aligned}$$

Therefore

$$\left\| \mathcal{L}_\varepsilon \star P_\eta(N_S) - \int_{\mathbb{T}^d} \mathcal{L}_\varepsilon(x-y)\, L_{\sqrt{\eta}}(y-z)\, P_\eta(t,z,N_S(t,y))\mathrm{d}y\,\mathrm{d}z \right\|_{L^1_{t,x}} \leqslant C\sqrt{\eta}. \tag{A.12}$$

Since N_S solves the continuity equation (B.1) and $U_S \in L^2_t H^1_x$, we have by theorem B.1 that for any fixed z

$$\begin{aligned} \partial_t(P_\eta(t,z,N_S(t,x))) &= \partial_t P_\eta(t,z,N_S(t,x)) + \mathrm{div}_x \left(P_\eta(t,z,N_S(t,x)) \right. \\ &\quad \left. \times\ U_S(t,x)\right) \\ &= (P_\eta(t,z,N_S(t,x)) - N_S\, \partial_s P_\eta(t,z,N_S(t,x))) \\ &\quad \times\ \mathrm{div}\, U_S. \end{aligned}$$

From this, we obtain that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} \mathcal{L}_\varepsilon(x-y)\, L_{\sqrt{\eta}}(y-z)\, P_\eta(t,z,N_S(t,y))\,\mathrm{d}y\,\mathrm{d}z \\ &\quad = \int_{\mathbb{T}^d} \mathcal{L}_\varepsilon(x-y)\, L_{\sqrt{\eta}}(y-z)\, \partial_t P_\eta(t,z,N_S(t,y))\,\mathrm{d}y\,\mathrm{d}z \\ &\qquad + \int_{\mathbb{T}^d} \mathcal{L}_\varepsilon(x-y)\, \nabla_y L_{\sqrt{\eta}}(y-z)\, P_\eta(t,z,N_S(t,y))\, U_S(t,y)\,\mathrm{d}y\,\mathrm{d}z \\ &\qquad + \int_{\mathbb{T}^d} \mathcal{L}_\varepsilon(x-y)\, L_{\sqrt{\eta}}(y-z)\, \left(P_\eta(t,z,N_S(t,x)) \right. \\ &\qquad \left. - N_S\, \partial_s P_\eta(t,z,N_S(t,x))\right)\, \mathrm{div}\, U_S\,\mathrm{d}y\,\mathrm{d}z. \end{aligned}$$

Bounding directly each term, this implies that

$$\left| \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} \mathcal{L}_\varepsilon(x-y)\, L_{\sqrt{\eta}}(y-z)\, P_\eta(t,z,N_S(t,y))\,\mathrm{d}y\,\mathrm{d}z \right| \leqslant C_\varepsilon\, \eta^{-k}, \tag{A.13}$$

for some exponent $k > 0$.

We may now combine (A.11)–(A.13) to obtain the compactness in time of $\mathcal{L}_\varepsilon \star P(N_S)$ and hence the compactness in $L^2_{t,x}$ of $F(B)$. By the Schauder fixed point, F has a fixed point S in $B \subset L^2_{t,x}$. We now simply choose $\rho_\varepsilon = N_S$ and $u_\varepsilon = U_S$ and since N_S, U_S solve (A.7) with $S = F(S) = \mathcal{L}_\varepsilon \star P(N_S) = \mathcal{L}_\varepsilon \star P(\rho_\varepsilon)$, we obtain a solution to (2.1) and (2.2). The energy bound (A.8) provides all uniform in ε bounds on ρ_ε while the energy inequality (A.9) of course leads to the corresponding inequality in the theorem. Estimate (A.10) provides the extra-integrability on $\rho_{\varepsilon,\eta}$.

Appendix B

B.1. Renormalized solutions

We rely on the concept of renormalized solution (see for instance [11, 17]) to justify several *a priori* formal calculations in the article. For this reason, we recall here the main definitions. Given our system, we naturally focus on the conservative transport equation

$$\partial_t \rho + \mathrm{div}(\rho u) = 0. \tag{B.1}$$

Given a weak solution ρ to the above, it is not *a priori* possible to calculate nonlinear functions of ρ which is precisely what we need here. Hence one introduces the notion of renormalized solutions

Definition B.1. A weak solution $\rho \in L^p_{t,x}$ to (B.1) with $u \in L^q_{t,x}$ for $1/p + 1/q = 1$ is a renormalized solution iff for any $\chi \in C^1(\mathbb{R})$ with $|\chi'(s)| \leqslant C(1 + |s|^{p-1})$, one has that

$$\partial_t \chi(\rho) + \operatorname{div}(\chi(\rho)u) = (\chi(\rho) - \rho\chi'(\rho))\operatorname{div} u \tag{B.2}$$

in the sense of distributions.

Renormalized solutions were first introduced in the famous [11] which in particular proved that if u belongs to the right Sobolev space then all weak solutions are renormalized.

Theorem B.1. *Assume that $\rho \in L^p_{t,x}$ is a solution to (B.1) in the sense of distributions. Suppose that $u \in L^q_t W^{1,q}_x$ with $1/p + 1/q = 1$, then ρ is a renormalized solution to (B.1).*

For linear equations, i.e. when u is given in (B.1), then the theory of renormalized solutions immediately provides many key properties such as the compactness for a sequence or the uniqueness of a solution. For example, assume there are two solutions ρ_1 and ρ_2 to (B.1) for the same u. Applying theorem B.1 to the function $\rho = \rho_1 - \rho_2$ with $\chi(x) = |x|$ and integrating in time and space gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} \chi(\rho)\,\mathrm{d}x = 0$$

which immediately implies that $\rho_1 = \rho_2$.

Observe however that in general and unless $\operatorname{div} u \in L^\infty$, it is not possible to have a general existence result for (B.1) for a *given* $u \in L^q_t W^{1,q}_x$. A solution with only $\operatorname{div} u \in L^2$ may for example concentrate, by forming Dirac masses.

Following [11] and the BV extensions in [3] for the kinetic case and the seminal [1] in the general case, the theory of renormalized solutions is now an extensive field for which we refer for example to the reviews [2, 10].

In the context of compressible fluid mechanics, renormalized solutions have been critical to obtaining the compactness of the density since the first breakthrough in [18] and they also form the basis of the extension introduced in [12, 16]. We in particular cite the straightforward compactness result from [11]

Theorem B.2. *Consider a sequence u_n converging strongly to u in $L^1([0,\ T],\ L^q(\mathbb{T}^d))$ s.t. $\operatorname{div} u_n$ converges to $\operatorname{div} u$ in $L^1([0,\ T],\ L^q(\mathbb{T}^d))$ as well. Consider any sequence ρ_n such that ρ_n, u_n satisfies equation (B.1) and ρ_n uniformly bounded in $L^\infty([0,\ T],\ L^p(\mathbb{T}^d))$ with $1/p + 1/q < 1$. Assume finally that $u \in L^1([0,\ T],\ W^{1,p'}(\mathbb{T}^d)$ with $1/p + 1/p' = 1$. Then the sequence ρ_n is compact in $L^1([0,\ T] \times \mathbb{T}^d)$.*

Theorem B.2 can be deduced from theorem B.1. The proof of theorem B.1 itself relies on a so-called qualitative commutator estimate and in several respects, the method introduced in [4] consists in quantifying this commutator estimate.

References

[1] Ambrosio L 2004 Transport equation and Cauchy problem for BV vector fields *Invent. Math.* **158** 227–60

[2] Ambrosio L and Crippa G 2014 Continuity equations and ODE flows with non-smooth velocity *Proc. R. Soc. Edinburgh* A **144** 1191–244

[3] Bouchut F 2001 Renormalized solutions to the Vlasov equation with coefficients of bounded variation *Arch. Ration. Mech. Anal.* **157** 75–90
[4] Bresch D and Jabin P-E 2018 Global existence of weak solutions for compressible Navier–Stokes equations: thermodynamically unstable pressure and anisotropic viscous stress tensor *Ann. Math.* **188** 577–684
[5] Bresch D and Jabin P-E 2018 *Quantitative Regularity Estimates for Compressible Transport Equations* (*New Trends and Results in Mathematical Description of Fluid Flows*) (Boston, MA: Birkhauser) pp 77–113
[6] Bresch D and Jabin P-E 2019 *Viscous Compressible Flows Under Pressure* (*Advances in Mathematical Fluid Mechanics*) ed T Bodnar, G P Galdi and S Necasova (Boston, MA: Birkhauser)
[7] Leray J 1934 Sur le mouvement d'un liquide visqueux emplissant l'espace *Acta Math.* **63** 193–248
[8] Bresch D and Jabin P-E 2017 *Global Existence of Weak Solutions of PDEs for Compressible Media: A Compactness Criterion to Cover New Physical Situations* (*Springer INdAM-Series*) ed F Colombini, D Del Santo and D Lannes special issue, dedicated to G Métivier pp 33–54
[9] Bresch D, Jabin P-E and Wang F 2020 Compressible Navier–Stokes equations with a truncated virial pressure law forthcoming paper
[10] De Lellis C 2007 Notes on hyperbolic systems of conservation laws and transport equations *Handbook of Differential Equations, Evolutionary Equations* vol 3 (Amsterdam: North-Holland)
[11] DiPerna R J and Lions P L 1989 Ordinary differential equations, transport theory and Sobolev spaces *Invent. Math.* **98** 511–47
[12] Feireisl E 2004 *Dynamics of Viscous Compressible Fluids* (*Oxford Lecture Series in Mathematics and its Applications*) vol 26 (Oxford: Oxford University Press)
[13] Feireisl E and Novotny A 2017 *Singular Limits in Thermodynamics of Viscous Fluids* (*Advances in Mathematical Fluid Mechanics*) (Boston, MA: Birkhauser)
[14] Feireisl E, Novotný A and Petzeltová H 2001 On the existence of globally defined weak solutions to the Navier–Stokes equations *J. Math. Fluid Mech.* **3** 358–92
[15] Feireisl E, Karper T and Pokorny M 2016 *Mathematical Theory of Compressible Viscous Fluids: Analysis and Numerics* (Boston, MA: Birkhauser)
[16] Feireisl E 2002 Compressible Navier–Stokes equations with a non-monotone pressure law *J. Differ. Equ.* **184** 97–108
[17] Le Bris C and Lions P-L 2004 Renormalized solutions of some transport equations with partially $W^{1,1}$ velocities and applications *Ann. Mat. Pura Appl.* **183** 97–130
[18] Lions P-L 1998 *Mathematical Topics in Fluid Mechanics. Vol. 2. Compressible Models* (*Oxford Lecture Series in Mathematics and its Applications*) vol 10 (Oxford: Clarendon)
[19] Novotny A and Straskraba I 2004 *Introduction to the Mathematical Theory of Compressible Flow* (*Oxford Lecture Series in Mathematics and its Applications*) (New York: Oxford University Press)
[20] Plotnikov P I and Weigant W 2015 Isothermal Navier–Stokes equations and radon transform *SIAM J. Math. Anal.* **47** 626–53
[21] Stein E M 1995 *Harmonic Analysis* 2nd edn (Princeton, NJ: Princeton University Press)