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Quasistationary distributions and ergodic control problems

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Abstract

We introduce and study the basic properties of two ergodic stochastic control problems associated with the quasistationary distribution (QSD) of a diffusion process X relative to a bounded domain. The two problems are in some sense dual, with one defined in terms of the generator associated with X and the other in terms of its adjoint. Besides proving wellposedness of the associated Hamilton–Jacobi–Bellman equations, we describe how they can be used to characterize important properties of the QSD. Of particular note is that the QSD itself can be identified, up to normalization, in terms of the cost potential of the control problem associated with the adjoint.

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1. Introduction

Consider a time homogeneous Markov process $\{X(t), t \geq 0\}$ with state space \mathbb{R}^d , and an open subset $G \subset \mathbb{R}^d$. Suppose the process is started with a distribution γ , and that if we condition on the process remaining in G for any interval $[0, t]$, $t \in (0, \infty)$, then the conditioned distribution of $X(t)$ is also γ . Such a distribution, if it exists, is called a quasistationary distribution (QSD) for X and G . As with ordinary stationary distributions,

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QSDs are of interest in that, under appropriate uniqueness and communication conditions, they can characterize (conditional) behavior over long time intervals when starting with more general initial distributions. However, QSDs are in some ways more complicated than stationary distributions. For example, assuming that the probability to remain in G decays, then characterizing the QSD will also involve characterizing this decay rate $\lambda > 0$. As another example, it turns out that uniqueness issues are in general more subtle for QSDs, in that even for very nice sets and for processes whose communication properties guarantee that there is only one stationary distribution, it can nonetheless happen that there are many quasistationary distributions. For general references we refer to the book [8], the survey paper [16] and the recent work [7].

The purpose of this paper is to identify and study the basic properties of two stochastic control problems that characterize important quantities associated with a QSD when G is a bounded domain. The two problems are in some sense dual, with one defined in terms of the generator associated with X and the other in terms of its adjoint. Both problems identify the rate λ with an optimal average cost per unit time (or ergodic cost). In both of these control problems the admissible controls are absolutely continuous non-anticipative processes under which the state process remains within G at all times a.s. As a consequence the controls are necessarily unbounded and the associated Hamilton–Jacobi–Bellman equations are given in terms of a boundary condition which requires the solution to diverge to ∞ at the boundary of G . We prove the wellposedness of these equations and characterize the solutions as the optimal cost and the cost potential (also referred to as the value function) of the associated ergodic control problems.

Next, we identify the optimally controlled state processes corresponding to the two control problems. The optimally controlled dynamics for the problem phrased in terms of the generator of X has appeared in previous works and, under suitable conditions on the model (see [17] and Remark 2.5), can be identified as the associated Q -process (see [6]) for the underlying diffusion. To the best of our knowledge the optimally controlled diffusion for the ergodic control problem associated with the adjoint of the generator of X has not appeared in previous works on quasistationary distributions and Q -processes, although a direct construction of the semigroup of this Markov process can be found in [11, Section 3]. The two optimally controlled state processes are closely related in that the second process is simply the time reversal of the first when initialized with its unique stationary distribution (see Remark 2.7(3) for a detailed discussion). For the second ergodic control problem, we show that the value function for the control problem identifies the QSD, at least up to normalization.

Two recent papers [1,5] have studied certain stochastic control problems that are closely connected with quasistationary distributions. However, the formulations of the control problems and goals of these papers are different from those in our work. Specifically, the stochastic control problems considered in [1,5] consider controlled state dynamics with extinction states and a cost criteria given in terms of some expectation conditioned on nonextinction. In contrast, in the setting considered in the current work there are no extinction states but rather states that are forbidden for the dynamics which leads to different types of stochastic control problems. Furthermore, our goal here is not to model a controllable system where one is interested in optimizing performance using a stochastic control formulation, but rather to use stochastic control methods as tools for computing quantities associated with quasistationary distributions.

In forthcoming work we will make two interesting uses of the representations. One application is to the study of the non-uniqueness issues mentioned previously, which can occur when the domain G is unbounded. We expect that different QSDs for the same process and

domain can be distinguished according to different classes of controls and costs that are allowed in the ergodic control problem, in a way reminiscent of other well known examples of non-uniqueness in control theory and the calculus of variations [12]. The other application is to numerical approximation. Using the Markov chain approximation method for stochastic control problems [14], we will construct and prove the convergence of schemes for approximating the rate λ and the QSD γ . There has been recent work on numerical approximations for QSD for diffusions in bounded domains [3], however the use of Markov chain approximations for stochastic control problems appears to be novel in the context of numerics for QSD. With the goal of constructing efficient numerical schemes, in the current work we in fact study a collection of ergodic control problems parametrized by a smooth strictly positive function ζ on \mathbb{R}^d . These control problems are associated with the adjoints of the generator of the underlying diffusion on $L^2(\mathbb{R}^d, \zeta(x)dx)$ for the different choices of ζ . It turns out (see Remark 2.7(2)) that although the optimally controlled state process for all of these control problems is the same irrespective of the choice of ζ , the state dynamics and the cost function depend on ζ in an important manner. One expects that the choice of ζ will significantly impact the performance of the numerical schemes based on the corresponding ergodic control problem. These issues will be investigated in a systematic manner in our forthcoming work.

Although in this paper we focus on the case of diffusion process, the analogous results can be formulated for continuous time jump-diffusions. The case of a countable state pure jump Markov process is particularly simple, so long as the state space $\mathcal{X} \subset \mathbb{R}^d$ does not contain accumulation points. The simplicity is due to the fact that in formulating control problems one can simply restrict to feedback controls that do not allow the process to exit $\mathcal{X} \cap G$. The corresponding results will be needed for the numerical approximations mentioned previously, and results analogous to those proved here for diffusions can be obtained with simpler proofs.

The paper is organized as follows. In Section 2 we present our main assumptions and results. In particular, Section 2.1 introduces the two ergodic control problems that are the focus of this work. Theorem 2.4 gives our main results for the ergodic control problem associated with the generator of the diffusion while Theorem 2.6 gives our results for the ergodic control problem for the adjoint of the generator. In Theorem 2.3 we collect some well known results for Dirichlet eigenvalue problems for elliptic operators and their relationship with quasistationary distributions and extinction rates. Proofs of our main results (i.e. Theorems 2.4 and 2.6) are provided in Section 3.

1.1. Notation

The following notation will be used. By a domain G in \mathbb{R}^d we will mean an open connected subset of \mathbb{R}^d . The boundary of such a domain will be denoted by ∂G . Distance between sets A and B in \mathbb{R}^d is defined as $\inf\{\|x - y\| : x \in A, y \in B\}$ and is denoted as $\text{dist}(A, B)$. If A is a singleton $\{x\}$ we write the distance simply as $\text{dist}(x, B)$. For a domain G in \mathbb{R}^d , $\alpha \in (0, 1]$ the Hölder space $C^\alpha(\bar{G})$ consists of continuous real functions u on \bar{G} which satisfy

$$[u]_\alpha \doteq \sup_{x, y \in \bar{G}, x \neq y} \frac{|u(x) - u(y)|}{\|x - y\|^\alpha} < \infty,$$

and for $k \in \mathbb{N}_0$, the space $C^{k, \alpha}(\bar{G})$ consists of real valued functions with continuous partial derivatives up to k th order, and such that all the k th order partial derivatives are in $C^\alpha(\bar{G})$. We denote by $C_0^{k, \alpha}(\bar{G})$ the collection of all $u \in C^{k, \alpha}(\bar{G})$ with $u(x) = 0$ for $x \in \partial G$. $C_0^2(G)$ denotes the collection of all functions in $C^2(G)$ which along with its first two derivatives vanish as x

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approaches ∂G . For $k \in \mathbb{N}$ and $\alpha \in (0, 1]$, we say the ∂G is $C^{k,\alpha}$, if ∂G is a $d - 1$ dimensional $C^{k,\alpha}$ manifold, namely, each point in ∂G has a neighborhood in which ∂G is the graph of a $C^{k,\alpha}$ function of $d - 1$ of the coordinates (cf. [10, Section 6.2]). For twice differentiable $h : \mathbb{R}^d \rightarrow \mathbb{R}$, the gradient $Dh(x)$ is the vector $(h_{x_1}(x), \dots, h_{x_d}(x))'$ and the Hessian $D^2h(x)$ is the $d \times d$ matrix with the (i, j) th entry given as $h_{x_i x_j}(x)$, where $h_{x_i} \doteq \frac{\partial}{\partial x_i} h$ and $h_{x_i x_j} \doteq \frac{\partial^2}{\partial x_i \partial x_j} h$ for $1 \leq i, j \leq d$. Denote by $C_0^2(\mathbb{R}^d)$ the space of all real valued functions on \mathbb{R}^d that are twice continuously differentiable with the function and the derivatives vanishing at ∞ . The trace of a square matrix A will be denoted as $\text{tr}(A)$. For $z \in \mathbb{C}$, $\text{Re}(z)$ is the real part of z . For a Polish space S , denote by $\mathcal{P}(S)$ the space of probability measures on S equipped with the topology of weak convergence. Denote by $\mathcal{C}([0, T] : S)$ (resp., $\mathcal{C}([0, \infty) : S)$) the space of S -valued continuous functions on $[0, T]$ (resp., $[0, \infty)$) equipped with the uniform (resp., locally uniform) topology.

2. Assumptions and main results

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0})$ be a filtered probability space equipped with an m -dimensional standard $\{\mathcal{F}_t\}$ -Brownian motion $\{B(t)\}_{t \geq 0}$. Consider the d -dimensional stochastic differential equation (SDE) given as

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t), \quad X(0) = X_0 \quad (2.1)$$

where X_0 is a \mathcal{F}_0 -measurable random variable and $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ are Lipschitz maps, namely for some $\kappa_1 \in (0, \infty)$

$$\|b(x) - b(y)\| + \|\sigma(x) - \sigma(y)\| \leq \kappa_1 \|x - y\| \quad \text{for all } x, y \in \mathbb{R}^d. \quad (2.2)$$

Under (2.2) there is a unique pathwise solution of the SDE (2.1) for every \mathcal{F}_0 -measurable random variable X_0 . When X_0 has probability distribution $\mu \in \mathcal{P}(\mathbb{R}^d)$, we will write the probability measure P as P_μ . When $\mu = \delta_x$ for some $x \in \mathbb{R}^d$, we will write P_μ simply as P_x . Let G be a bounded domain in \mathbb{R}^d . We are interested in the study of the QSD of the Markov family $(X(\cdot), \{P_x\}_{x \in \mathbb{R}^d})$ on the domain G , which is defined as follows.

Definition 2.1. A probability measure $\gamma \in \mathcal{P}(G)$ is said to be a QSD of the Markov family $(X(\cdot), \{P_x\}_{x \in \mathbb{R}^d})$ on the domain G if

$$P_\gamma(X(t) \in A \mid \tau_\partial > t) = \gamma(A), \quad \text{for every } t \in [0, \infty) \text{ and } A \in \mathcal{B}(G),$$

where $\tau_\partial \doteq \inf\{t \geq 0 : X(t) \in G^c\}$.

Frequently we will refer to γ simply as a QSD for X .

We now introduce additional assumptions on the coefficients and also a condition on the domain. Let $a = \sigma\sigma^T$, where σ^T denotes the transpose of σ .

Assumption 2.2. The boundary ∂G is $C^{2,\alpha}$ for some $\alpha \in (0, 1]$ and the following hold for some open set H that contains \bar{G} .

1. For some $\alpha \in (0, 1]$, $\sigma_{i,j} \in C^{2,\alpha}(H)$, and $b_i \in C^{1,\alpha}(H)$ for all $i = 1, \dots, d$ and $j = 1, \dots, m$.
2. There is a $\kappa_0 \in (0, \infty)$ such that for every $v \in \mathbb{R}^d$

$$\inf_{x \in H} v \cdot a(x)v \geq \kappa_0 \|v\|^2.$$

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Consider the infinitesimal generator \mathcal{L} of the Markov process (2.1) which, when restricted to $\mathcal{C}_0^2(\mathbb{R}^d)$, is given as follows. For $f \in \mathcal{C}_0^2(\mathbb{R}^d)$

$$\mathcal{L}f(x) \doteq b(x) \cdot Df(x) + \frac{1}{2} \text{tr}[a(x)D^2 f(x)], \quad x \in \mathbb{R}^d.$$

The formal adjoint \mathcal{L}^* on $\mathcal{C}_0^2(\mathbb{R}^d)$ of this operator is defined as follows. For $f \in \mathcal{C}_0^2(\mathbb{R}^d)$

$$\mathcal{L}^*f(x) \doteq \beta(x) \cdot Df(x) + \frac{1}{2} \text{tr}[a(x)D^2 f(x)] - c(x)f(x), \quad x \in \mathbb{R}^d,$$

where

$$\beta_i(x) \doteq -b_i(x) + \sum_{j=1}^d (a_{ij}(x))_{x_j}, \quad 1 \leq i \leq d, \quad (2.3)$$

and

$$c(x) \doteq -\frac{1}{2} \sum_{i,j=1}^d (a_{ij}(x))_{x_i x_j} + \sum_{i=1}^d (b_i(x))_{x_i}. \quad (2.4)$$

It will be useful to consider the adjoint of \mathcal{L} with respect to measures other than the Lebesgue measure. Let $\zeta \in \mathcal{C}^2(\mathbb{R}^d) \cap \mathcal{C}^{2,\alpha}(\bar{G})$ be such that $\zeta(x) > 0$ for all $x \in \mathbb{R}^d$. We denote by \mathcal{C}^* the class of all such ζ . Consider the operator \mathcal{L}_ζ^* on $\mathcal{C}_0^2(\mathbb{R}^d)$ defined as follows. For $f \in \mathcal{C}_0^2(\mathbb{R}^d)$,

$$\mathcal{L}_\zeta^*f(x) \doteq \frac{1}{\zeta(x)} \mathcal{L}^*(f\zeta)(x), \quad x \in \mathbb{R}^d. \quad (2.5)$$

Then \mathcal{L}_ζ^* is the adjoint to \mathcal{L} on the space $L^2(\zeta dx)$. Note that when $\zeta \equiv 1$, $\mathcal{L}_\zeta^* = \mathcal{L}^*$. As noted in the introduction, one of the motivations of our work is to develop numerical schemes for computing the QSD γ . Our approach for this is based on Markov chain approximation methods for stochastic control problems (cf. [14]). Theorem 2.6 will show that there is a parametric family of stochastic control problems, indexed by $\zeta \in \mathcal{C}^*$, such that any stochastic control problem in this family can be used to determine the QSD γ and the decay rate λ . However, one expects that the behavior of numerical schemes for these stochastic control problems will depend on the choice of ζ with some choices being numerically advantageous. The function ζ can be viewed as a design variable for the numerical scheme which may be chosen by an algorithm developer suitably based on the problem specific data.

The following result on Dirichlet eigenvalue problems for \mathcal{L} and \mathcal{L}_ζ^* will be used in our work. A $\lambda \in \mathbb{C}$ is a Dirichlet eigenvalue of $-\mathcal{L}$ on the domain G , if for some nonzero $\phi \in \mathcal{C}^2(G) \cap \mathcal{C}(\bar{G})$

$$\begin{aligned} -\mathcal{L}\phi(x) &= \lambda\phi(x), & x \in G, \\ \phi(x) &= 0, & x \in \partial G. \end{aligned}$$

The function ϕ is called a Dirichlet eigenvector of $-\mathcal{L}$ associated with the eigenvalue λ . Dirichlet eigenvalues and eigenvectors of $-\mathcal{L}_\zeta^*$ are defined in a similar manner. Since we will only be concerned with Dirichlet eigenvectors and eigenvalues for $-\mathcal{L}$ and $-\mathcal{L}_\zeta^*$, henceforth we will drop the adjective ‘‘Dirichlet’’.

Parts 1–5 of the following theorem are immediate consequences of well-known results. Part 6 has been shown in [17, Proposition 1.10], under certain technical conditions on the coefficients (see Hypothesis 2 and 3 therein) which, for example, are satisfied when $a^{-1}b = DU$ for some $U \in \mathcal{C}^2(\bar{G})$. We were unable to find a reference that covers part 6 in the generality considered here. The proof of Theorem 2.3 is provided in Section 3.1.

Theorem 2.3. Suppose Assumption 2.2 is satisfied. The following hold.

1. For any $\zeta \in \mathcal{C}^*$, \tilde{v} is an eigenvector of $-\mathcal{L}_\zeta^*$ associated with the eigenvalue z if and only if, $v \doteq \zeta \tilde{v}$ is an eigenvector of $-\mathcal{L}^*$ associated with the eigenvalue z .
2. There exists a $\lambda \in (0, \infty)$ which is an eigenvalue of $-\mathcal{L}$ and $-\mathcal{L}_\zeta^*$ for every $\zeta \in \mathcal{C}^*$, and is such that $\operatorname{Re}(\tilde{\lambda}) \geq \lambda$ for any other eigenvalue $\tilde{\lambda}$ of $-\mathcal{L}$ or $-\mathcal{L}_\zeta^*$, where $\zeta \in \mathcal{C}^*$.
3. There is a unique quasistationary distribution ν for X .
4. There are $\psi, \varphi \in C_0^{2,\alpha}(\bar{G})$ such that $\psi(x) > 0$ and $\varphi(x) > 0$ for all $x \in G$, and ψ [resp., φ] is an eigenvector of $-\mathcal{L}$ [resp., $-\mathcal{L}^*$] associated with the eigenvalue λ . Furthermore, $\int_G \varphi(x) dx = 1$ and $\int_G \psi(x) \nu(dx) = 1$. Any other eigenvector associated with the eigenvalue λ of $-\mathcal{L}$ and $-\mathcal{L}^*$ is a scalar multiple of ψ and φ respectively. For any $\zeta \in \mathcal{C}^*$, $\tilde{\varphi} \doteq \varphi/\zeta$ is an eigenvector of $-\mathcal{L}_\zeta^*$ associated with the eigenvalue λ .
5. The function ψ characterizes the exit rate from G associated with the Markov process (2.1) as follows. For all $x \in G$

$$\psi(x) = \lim_{t \rightarrow \infty} e^{\lambda t} P_x(\tau_\partial > t).$$

6. The unique quasistationary distribution ν of the Markov family $(X(\cdot), \{P_x\}_{x \in \mathbb{R}^d})$ on the domain G can be characterized as $\nu(dx) = \varphi(x) dx$.

2.1. Two ergodic control problems

We now introduce two ergodic control problems, one which is associated with the operator \mathcal{L} and the other associated with \mathcal{L}_ζ^* where $\zeta \in \mathcal{C}^*$.

Consider the d -dimensional SDE on $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0})$

$$dY(t) = b(Y(t))dt + \sigma(Y(t))dB(t) + \sigma(Y(t))u(t)dt, \quad Y(0) = y \in G, \quad (2.6)$$

where $B(t)$ is a m -dimensional \mathcal{F}_t -Brownian motion, $\{u(t), 0 \leq t < \infty\}$ is a \mathbb{R}^m -valued \mathcal{F}_t -progressively measurable process satisfying $\int_0^T \|u(t)\|^2 dt < \infty$ for every $T < \infty$. The process u is regarded as a control process modifying the original dynamics in (2.1). Such a control process is called an *admissible control* (for initial condition $y \in G$) if the following conditions are satisfied.

C.1 There is a unique continuous, \mathbb{R}^d -valued, \mathcal{F}_t -adapted process Y that solves (2.6).

C.2 $P(Y(t) \in G \text{ for all } t \in [0, \infty)) = 1$.

We denote the collection of all admissible controls by $\mathcal{A}(y)$. Although previously the notation P_y and E_y was used for processes X that start from y , with an abuse of notation we will continue to use this notation for probability measures (and expectations) under which $Y(0) = y$. The cost associated with the control problem is defined as follows. For $y \in G$ and $u \in \mathcal{A}(y)$

$$J(y, u) \doteq \limsup_{T \rightarrow \infty} \frac{1}{T} E_y \int_0^T \left[\frac{1}{2} \|u(t)\|^2 \right] dt.$$

The optimal cost, which could in principle depend on the initial condition, is given as follows. For $y \in G$

$$J(y) \doteq \inf_{u \in \mathcal{A}(y)} J(y, u) = \inf_{u \in \mathcal{A}(y)} \limsup_{T \rightarrow \infty} \frac{1}{T} E_y \int_0^T \left[\frac{1}{2} \|u(t)\|^2 \right] dt. \quad (2.7)$$

It will be shown in [Theorem 2.4\(5\)](#) that in fact $J(y)$ is independent of $y \in G$. This is a well known fact for many types of ergodic control problems and the proof here is a variation on classical arguments that accounts for the requirement C.2 of admissible controls and the unboundedness of the control space. The theorem will also give a characterization of J in terms of the following Hamilton–Jacobi–Bellman (HJB) equation:

$$\frac{1}{2} \operatorname{tr}[a(x)D^2\Gamma(x)] + \min_{u \in \mathbb{R}^m} \left[(b(x) + \sigma(x)u) \cdot D\Gamma(x) + \frac{1}{2} \|u\|^2 \right] = \gamma, \quad x \in G, \quad (2.8)$$

$$\Gamma(x) \rightarrow \infty \text{ as } x \rightarrow \partial G.$$

By a solution (Γ, γ) of (2.8) we mean a $\Gamma \in C^2(G)$ and $\gamma \in (0, \infty)$ that satisfy the first line of the equation for each $x \in G$ and such that the divergence statement in the second line holds. In a more classical setup (see, for example, [2]) where either $G = \mathbb{R}^d$ or perhaps a reflecting boundary condition is used, γ identifies the optimal ergodic cost regardless of the starting state, while Γ , which is typically unique only up to an additive constant, gives what is called the *cost potential*. Under suitable conditions, the cost potential generates through the equation an optimal feedback control. The cost potential also indicates the impact of the initial condition over finite but large time intervals, in that heuristically one expects (and can under conditions show rigorously) that if $J_T(x)$ is the minimal cost over $[0, T]$ when starting at x at time $t = 0$, namely,

$$J_T(x) \doteq \inf_{u \in \mathcal{A}(y)} E_x \int_0^T \left[\frac{1}{2} \|u(t)\|^2 \right] dt,$$

then $J_T(x) - J_T(y) \rightarrow \Gamma(x) - \Gamma(y)$ as $T \rightarrow \infty$.

The following theorem relates the ergodic control problem (2.7) with the exit rate of the Markov process X from the domain G .

Theorem 2.4. *Suppose [Assumption 2.2](#) is satisfied. Then the following hold.*

1. Let ψ and λ be as given by [Theorem 2.3](#). Define $\Psi \doteq -\log \psi$. Then the pair (Ψ, λ) solves the HJB Eq. (2.8).
2. Suppose $(\tilde{\Psi}, \tilde{\lambda}) \in C^2(G) \times (0, \infty)$ is a solution of the HJB Eq. (2.8). Then $\tilde{\lambda} = \lambda$ and for some $C \in \mathbb{R}$, $\tilde{\Psi}(x) - \Psi(x) = C$ for all $x \in G$.
3. Consider the control $\{u(t), 0 \leq t < \infty\}$ defined in feedback form by

$$u(t) = -\sigma(Y(t))^T D\Psi(Y(t)), \quad 0 \leq t < \infty, \quad (2.9)$$

where Y is defined by (2.6) with this choice of u , namely

$$dY(t) = b(Y(t))dt + \sigma(Y(t))dB(t) - a(Y(t))D\Psi(Y(t))dt, \quad Y(0) = y \in G. \quad (2.10)$$

Then there is a unique solution to (2.10) which satisfies $P(Y(t) \in G \text{ for all } t \in [0, \infty)) = 1$. In particular, $u \in \mathcal{A}(y)$.

4. The process Y defines a Markov process on G that has a unique invariant probability measure μ given by $\mu(dx) = \varphi(x)\psi(x)dx$, where φ is from [Theorem 2.3](#).
5. $J(y) = \lambda$ for all $y \in G$.
6. The admissible control process defined by (2.9) is optimal, and in fact

$$\lambda = \lim_{T \rightarrow \infty} \frac{1}{T} E_y \int_0^T \frac{1}{2} [D\Psi(Y(t)) \cdot a(Y(t))D\Psi(Y(t))] dt.$$

Furthermore, the eigenvalue λ can be determined from the invariant measure μ by the formula

$$\lambda = \frac{1}{2} \int_G [D\Psi(y) \cdot a(y)D\Psi(y)]\mu(dy). \quad (2.11)$$

Remark 2.5. For fixed $0 \leq t \leq T$, let $\{Y^T(s)\}_{0 \leq s \leq t}$ be a G -valued stochastic process with probability law $Q^{T,t} \in \mathcal{P}(\mathcal{C}([0, t] : G))$ given as

$$Q^{T,t}(A) \doteq P_y(X(\cdot) \in A \mid \tau_\partial > T), \quad A \in \mathcal{B}(\mathcal{C}([0, t] : G)).$$

In [17] it is shown that under certain conditions on the coefficients (see Hypothesis 1 and 2 therein) which, for example, are satisfied when $a^{-1}b = DU$ for some $U \in \mathcal{C}^1(\bar{G})$, as $T \rightarrow \infty$, $Q^{T,t}$ converges in $\mathcal{P}(\mathcal{C}([0, t] : G))$ to some Q^t , for every $t \in (0, \infty)$. This yields a probability measure Q on $\mathcal{C}([0, \infty) : G)$ such that the corresponding induced measure on $\mathcal{C}([0, t] : G)$ is Q^t for every t . Later works [6,11] have proved similar results under more general conditions. The corresponding process whose probability law is Q is sometimes referred to as the Q -process (cf. [6]). The paper [17] shows that the Q -process has the same law as the diffusion Y defined by (2.10) (see also [11] where the measure Q is characterized through the semigroup of the associated Markov process). Thus the process given in (2.10) reduces, under conditions of [17], to the Q -process associated with the diffusion (2.1) and the domain G .

We next introduce an ergodic control problem that can be used to determine the quasistationary distribution of X .

Recall the operator \mathcal{L}_ζ^* from (2.5). Then, for $f \in \mathcal{C}_0^2(\mathbb{R}^d)$

$$\mathcal{L}_\zeta^* f(x) = \tilde{\beta}(x) \cdot Df(x) + \frac{1}{2} \text{tr}[a(x)D^2 f(x)] - \tilde{c}(x)f(x), \quad x \in \mathbb{R}^d, \quad (2.12)$$

where

$$\tilde{\beta}(x) \doteq \beta(x) + \frac{1}{\zeta(x)} a(x)D\zeta(x) \quad (2.13)$$

and

$$\tilde{c}(x) \doteq c(x) - \frac{1}{\zeta(x)} \beta(x) \cdot D\zeta(x) - \frac{1}{2\zeta(x)} \text{tr}[a(x)D^2 \zeta(x)]. \quad (2.14)$$

Consider the d -dimensional SDE on $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0})$ given by

$$dZ(t) = \tilde{\beta}(Z(t))dt + \sigma(Z(t))dB(t) + \sigma(Z(t))u(t)dt, \quad Z(0) = z \in G. \quad (2.15)$$

where $\{u(t), 0 \leq t < \infty\}$ is a \mathbb{R}^m -valued \mathcal{F}_t -progressively measurable process satisfying $\int_0^T \|u(t)\|^2 dt < \infty$ for every $T < \infty$. Such a control process is called an *admissible control* (for initial condition $z \in G$) if the following conditions are satisfied.

C'.1 There is a unique continuous, \mathbb{R}^d -valued, \mathcal{F}_t -adapted process Z that solves (2.15).

C'.2 $P(Z(t) \in G \text{ for all } t \in [0, \infty)) = 1$.

We denote the collection of all admissible controls as $\mathcal{A}^*(z)$. Once again we will write P_z and E_z for probabilities and expectations under which $Z(0) = z$ a.s. The cost associated with the control problem is defined as follows. For $z \in G$ and $u \in \mathcal{A}^*(z)$

$$J^*(z, u) \doteq \limsup_{T \rightarrow \infty} \frac{1}{T} E_z \int_0^T \left[\frac{1}{2} \|u(t)\|^2 + \tilde{c}(Z(t)) \right] dt, \quad (2.16)$$

where \tilde{c} is defined in (2.14). The optimal cost J^* of the control problem is given as follows. For $z \in G$

$$J^*(z) \doteq \inf_{u \in \mathcal{A}^*(z)} J^*(z, u) = \inf_{u \in \mathcal{A}^*(z)} \limsup_{T \rightarrow \infty} \frac{1}{T} E_z \int_0^T \left[\frac{1}{2} \|u(t)\|^2 + \tilde{c}(Z(t)) \right] dt. \quad (2.17)$$

Similar to the case of $J(y)$ in (2.7) and part 5 of Theorem 2.4, in Theorem 2.6(5) it will be shown that the quantity $J^*(z)$ is independent of $z \in G$. Consider now the HJB equation associated with the control problem (2.17):

$$\begin{aligned} \frac{1}{2} \text{tr}[a(x)D^2\Gamma(x)] + \min_{u \in \mathbb{R}^d} \left[(\tilde{\beta}(x) + \sigma(x)u) \cdot D\Gamma(x) + \tilde{c}(x) + \frac{1}{2} \|u\|^2 \right] &= \gamma, \quad x \in G, \\ \Gamma(x) &\rightarrow \infty \text{ as } x \rightarrow \partial G. \end{aligned} \quad (2.18)$$

As before, by a solution (Γ, γ) of (2.18) we mean a $\Gamma \in \mathcal{C}^2(G)$ and $\gamma \in (0, \infty)$ that satisfy the first line of the equation for each $x \in G$ and such that the divergence statement in the second line holds.

Together with Theorem 2.3, the following theorem relates this ergodic control problem with the QSD of the Markov process X . We remind the reader that the function $\tilde{\varphi}$ was introduced in part 4 of Theorem 2.3 and the functions $\tilde{\beta}$, \tilde{c} were defined in (2.13) and (2.14). All three functions depend on our choice of ζ but that dependence has been suppressed in the notation.

Theorem 2.6. *Suppose that Assumption 2.2 is satisfied. Let $\zeta \in \mathcal{C}^*$. Then the following hold.*

1. Let $\tilde{\varphi}$ and λ be as given by Theorem 2.3. Define $\tilde{\Phi} \doteq -\log \tilde{\varphi}$. Then the pair $(\tilde{\Phi}, \lambda)$ solve the HJB Eq. (2.18).
2. Suppose $(\tilde{\Phi}, \bar{\lambda}) \in \mathcal{C}^2(G) \times (0, \infty)$ is a solution of the HJB Eq. (2.8). Then $\bar{\lambda} = \lambda$ and for some $C \in \mathbb{R}$, $\tilde{\Phi}(x) - \bar{\Phi}(x) = C$ for all $x \in G$.
3. Consider the control $\{u(t), 0 \leq t < \infty\}$ defined in feedback form by

$$u(t) = -\sigma(Z(t))^T D\tilde{\Phi}(Z(t)), \quad 0 \leq t < \infty \quad (2.19)$$

where Z is defined by (2.15) with this choice of u , namely

$$dZ(t) = \tilde{\beta}(Z(t))dt + \sigma(Z(t))dB(t) - a(Z(t))D\tilde{\Phi}(Z(t))dt, \quad Z(0) = z \in G. \quad (2.20)$$

Then there is a unique solution to (2.20) which satisfies $P_z(Z(t) \in G \text{ for all } t \in [0, \infty)) = 1$. In particular, $u \in \mathcal{A}^*(z)$.

4. The process Z in (2.20) defines a Markov process on G that has a unique invariant probability measure μ given by $\mu(dx) = \varphi(x)\psi(x)dx$, where φ and ψ are from Theorem 2.3.
5. The optimal cost satisfies $J^*(z) = \lambda$ for all $z \in G$.
6. The admissible control process defined by (2.19) is optimal, and in fact

$$\lambda = \lim_{T \rightarrow \infty} \frac{1}{T} E \int_0^T \left[\frac{1}{2} D\tilde{\Phi}(Z(t)) \cdot a(Z(t))D\tilde{\Phi}(Z(t)) + \tilde{c}(Z(t)) \right] dt.$$

Furthermore, the eigenvalue λ can be determined from the invariant measure μ by the formula

$$\lambda = \int_G \left[\frac{1}{2} D\tilde{\Phi}(y) \cdot a(y)D\tilde{\Phi}(y) + \tilde{c}(y) \right] \mu(dy).$$

Remark 2.7.

1. Although the G -valued diffusion Y in (2.10) has been previously studied in relation to Q -processes (e.g. in [17]), to the best of our knowledge, the diffusion identified in (2.20) has not appeared in previous literature on quasistationary distributions.
2. Note that for every $z \in G$, with $\Phi = -\log \varphi$,

$$\tilde{\beta}(z) - a(z)D\tilde{\Phi}(z) = \beta(z) - a(z)D\Phi(z). \quad (2.21)$$

Thus the equation for the process Z in (2.20) can be written as

$$dZ(t) = \beta(Z(t))dt + \sigma(Z(t))dB(t) - a(Z(t))D\Phi(Z(t))dt, \quad Z(0) = z \in G. \quad (2.22)$$

It follows that the optimally controlled process Z is the same for all $\zeta \in \mathcal{C}^*$. However note that the control problem formulated in (2.16)–(2.17) is different for different choices of ζ . To see this we rewrite the controlled evolution as

$$\begin{aligned} dZ(t) &= \tilde{\beta}(Z(t))dt + \sigma(Z(t))dB(t) + \sigma(Z(t))u(t)dt \\ &= \beta(Z(t))dt + \sigma(Z(t))dB(t) \\ &\quad + \sigma(Z(t)) \left[u(t) + \frac{1}{\zeta(Z(t))} \sigma^T(Z(t))D\zeta(Z(t)) \right] dt \\ &= \beta(Z(t))dt + \sigma(Z(t))dB(t) + \sigma(Z(t))\tilde{u}(t)dt \end{aligned}$$

where

$$\tilde{u}(t) \doteq u(t) + \frac{1}{\zeta(Z(t))} \sigma^T(Z(t))D\zeta(Z(t)).$$

The expectation of the integrand in the cost function can be rewritten as

$$\begin{aligned} E_z \left[\frac{1}{2} \|u(t)\|^2 + \tilde{c}(Z(t)) \right] &= E_z \left[\frac{1}{2} \left\| \tilde{u}(t) - \frac{1}{\zeta(Z(t))} \sigma^T(Z(t))D\zeta(Z(t)) \right\|^2 \right. \\ &\quad \left. + c(Z(t)) - \frac{1}{\zeta(Z(t))} \beta(Z(t)) \cdot D\zeta(Z(t)) \right. \\ &\quad \left. - \frac{1}{2\zeta(Z(t))} \text{tr}[a(Z(t))D^2\zeta(Z(t))] \right]. \end{aligned}$$

With this rewriting, the controlled state dynamics is the same across all $\zeta \in \mathcal{C}^*$, however as seen from the above display, the cost function depends on the first two derivatives of ζ .

3. We note the somewhat surprising fact, previously remarked on in [11], that according to part 4 of Theorem 2.6 and part 4 of Theorem 2.4, the stationary distributions of the two optimally controlled processes, one associated with \mathcal{L} and the other with \mathcal{L}_ζ^* , are in fact the same. Indeed, there is a close connection between the diffusions in (2.10) and (2.20). Specifically, denoting the infinitesimal generators of the diffusions in (2.10) and (2.20) by \mathcal{L}_Y and \mathcal{L}_Z , it can be checked that, for $f \in \mathcal{C}_0^2(G)$,

$$\mathcal{L}_Z(f) = \frac{1}{\eta} \mathcal{L}_Y^*(\eta f), \quad (2.23)$$

where $\eta = \varphi\psi$ is the invariant density of the two diffusions. To see this note that, for $f \in \mathcal{C}_0^2(G)$,

$$\mathcal{L}_Y^*(f) = \hat{\beta} \cdot Df + \frac{1}{2} \text{tr}[aD^2f] - \hat{c}f$$

where

$$\hat{\beta} = \beta + aD\Psi, \quad \hat{c} = c - \sum_{i=1}^d [(aD\Psi)_i]_{x_i}.$$

Thus

$$\begin{aligned} \frac{1}{\eta} \mathcal{L}_Y^*(\eta f) &= \frac{1}{\eta} \left[\hat{\beta} \cdot D(\eta f) + \frac{1}{2} \text{tr}[aD^2(\eta f)] - \hat{c}\eta f \right] \\ &= \hat{\beta} Df + \frac{1}{2} \text{tr}[aD^2 f] + \frac{1}{\eta} D\eta \cdot aDf + \frac{1}{\eta} f \mathcal{L}_Y^*(\eta) \\ &= \beta Df + \frac{1}{2} \text{tr}[aD^2 f] - Df \cdot aD\Phi \\ &= \mathcal{L}_Z(f), \end{aligned}$$

where the third equality follows on recalling that η is the invariant density for Y , and the definitions of $\hat{\beta}$ and η . The final equality follows on recalling that \mathcal{L}_Z is the generator of Z and that the equation for Z can be written as in (2.22).

The equality in (2.23) suggests that the stationary optimal Y is simply the time-reversal of the stationary optimal Z . More precisely, consider the processes Y (resp. Z) solving (2.10) (resp. (2.20)) with $Y(0)$ distributed as μ (resp. $Z(0)$ distributed as μ). Fix $T < \infty$ and for $0 \leq t \leq T$, define $\tilde{Z}(t) \doteq Y(T - t)$. Then the distributions of Z and \tilde{Z} on $\mathcal{C}([0, T] : G)$ are the same. We give the following formal argument, which we believe can be rigorized. For making the argument fully rigorous one needs to argue that $\mathcal{C}_0^2(G)$ is in the domain of the generator of the time reversed process \tilde{Z} and that it is a core for the associated generator. It suffices to check that the generator of Z and \tilde{Z} restricted to $\mathcal{C}_0^2(G)$ are the same. Denote the transition probability semigroups of Y and \tilde{Z} by $\{T_t\}$ and $\{\tilde{T}_t\}$ respectively. Then for $0 \leq s \leq t + s \leq T$ and $f, g \in \mathcal{C}_0^2(G)$

$$E[g(Y(s))f(Y(t+s))] = \int_G g(y)T_t f(y)\eta(y)dy = \int_G f(y)\tilde{T}_t g(y)\eta(y)dy.$$

Subtracting $\int_G f(y)g(y)\eta(y)dy$ from both sides and dividing by t we get

$$\int_G g(y) \frac{[T_t f(y) - f(y)]}{t} \eta(y) dy = \int_G f(y) \frac{[\tilde{T}_t g(y) - g(y)]}{t} \eta(y) dy.$$

Sending $t \rightarrow 0$

$$\int_G g(y) \mathcal{L}_Y f(y) \eta(y) dy = \int_G f(y) \mathcal{L}_{\tilde{Z}} g(y) \eta(y) dy$$

where $\mathcal{L}_{\tilde{Z}}$ is the generator of \tilde{Z} . Noting from (2.23) that

$$\int_G g(y) \mathcal{L}_Y f(y) \eta(y) dy = \int_G \frac{1}{\eta(y)} \mathcal{L}_Y^*[\eta g](y) f(y) \eta(y) dy = \int_G \mathcal{L}_Z g(y) f(y) \eta(y) dy$$

we now see that

$$\int_G \mathcal{L}_Z g(y) f(y) \eta(y) dy = \int_G f(y) \mathcal{L}_{\tilde{Z}} g(y) \eta(y) dy$$

for all $f, g \in \mathcal{C}_0^2(G)$ which shows that the generator of Z and \tilde{Z} restricted to $\mathcal{C}_0^2(G)$ are the same.

In particular when \mathcal{L}_Y is selfadjoint on $L^2(\mu)$, it turns out that the processes Y in (2.10) and Z in (2.20) are the same. As a simple example consider the one dimensional diffusion on the bounded interval $[0, K]$ with absorption at 0 and K , given as

$$X(t) = x + B(t) - at, \quad t \in [0, \infty).$$

In this example, it is easy to check that \mathcal{L}_Y is selfadjoint on $L^2(\mu)$ and both of the optimally controlled processes Y and Z are governed by the stochastic differential equation

$$dU(t) = \frac{\pi}{K} \cot\left(\frac{\pi}{K}U(t)\right) dt + dB(t).$$

3. Proofs

3.1. Proof of Theorem 2.3

Part 1 is immediate from (2.5). From the Krein–Rutman theorem (cf. [9, Theorems 1.1–1.4]) it follows that there is a $\lambda \in (0, \infty)$ which is a simple eigenvalue of both $-\mathcal{L}$ and $-\mathcal{L}^*$ and any other eigenvalue $\tilde{\lambda}$ of these operators satisfies $\operatorname{Re}(\tilde{\lambda}) \geq \lambda$. This together with part 1 proves part 2. For part 3 see [7, Theorem 1.1]. Next denote by ψ and φ the unique eigenvectors of $-\mathcal{L}$ and $-\mathcal{L}^*$, respectively, associated with λ . Then again from the Krein–Rutman theorem (cf. [9, Theorem 1.3]) these functions are strictly positive and (cf. [10, Theorem 6.14]) are in $\mathcal{C}_0^{2,\alpha}(\bar{G})$. We normalize these eigenvectors such that

$$\int_G \varphi(x) dx = 1 \quad \text{and} \quad \int_G \psi(x) \nu(dx) = 1.$$

This together with part 1 completes the proof of part 4.

Part 5 follows from [7, Theorem 1.1] (see also [11, Theorem 3]) on recalling the normalization used to define ψ .

We now prove part 6. For this it suffices to show that for all $f \in \mathcal{C}_0^2(G)$ (recall that $\mathcal{C}_0^2(G)$ denotes the collection of all functions in $\mathcal{C}^2(G)$ which along with its first two derivatives vanish as x approaches ∂G) and $t > 0$, with $\tilde{\nu}(dx) = \varphi(x) dx$.

$$\frac{E_{\tilde{\nu}}[f(X(t))1_{\tau_{\partial} > t}]}{E_{\tilde{\nu}}[1_{\tau_{\partial} > t}]} = \langle f, \tilde{\nu} \rangle, \quad (3.1)$$

where $\langle f, \tilde{\nu} \rangle \doteq \int_G f(x) \tilde{\nu}(dx)$.

To see that it suffices to work with $f \in \mathcal{C}_0^2(G)$, note that

$$P_{\tilde{\nu}}(X(t) \in \partial G, \tau_{\partial} > t) = 0 \quad \text{and} \quad \tilde{\nu}(\partial G) = 0.$$

Using the dominated convergence theorem, given any f that is bounded and measurable we can approximate by functions with the same bound which are nonzero only in compact subsets of G . These in turn can be approximated by elements of $\mathcal{C}_0^2(G)$ using standard methods.

We now show (3.1) for $f \in \mathcal{C}_0^2(G)$. Let $X^\dagger(t)$ be the process defined as $X^\dagger(t) \doteq X(t \wedge \tau_{\partial})$. Then the process restricted to G admits transition subprobability densities $p_t^\dagger(x, y)$, namely for $f \in \mathcal{C}_0^2(G)$,

$$E_x[f(X^\dagger(t))] = \int_G f(y) p_t^\dagger(x, y) dy.$$

Furthermore, $p_t^\dagger(x, y)$ is the Green's function of $\frac{\partial u}{\partial t} = \mathcal{L}u$ on G with a Dirichlet boundary condition on ∂G (see [11, Proposition 1] for these facts). For a function $h(x, y)$ in $\mathcal{C}^2(G \times G)$,

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denote by $\mathcal{L}_x h$ the function obtained by regarding \mathcal{L} as a differential operator in the variable x . Define $\mathcal{L}_y h$ in a similar fashion. Then for all $x, y \in G$ and for all $t > 0$

$$\mathcal{L}_x p_t^\dagger(x, y) = \mathcal{L}_y^* p_t^\dagger(x, y). \quad (3.2)$$

We sketch below the proof of (3.2) which was pointed out to us by Prof. Z.Q. Chen. Let $q_t(x, y)$ denote the Green's function of $\frac{\partial u}{\partial t} = \mathcal{L}^* u$ with a Dirichlet boundary condition on ∂G . For $f \in C_c^\infty(G)$ (the space of functions from $G \rightarrow \mathbb{R}$ which are infinitely differentiable and have a compact support), define

$$\mathcal{P}_t f(x) \doteq \int_G p_t^\dagger(x, y) f(y) dy, \quad \mathcal{P}_t^* f(x) \doteq \int_G q_t(x, y) f(y) dy, \quad x \in G, \quad t > 0.$$

For $\alpha \in (0, \infty)$, define the corresponding resolvents as

$$\mathcal{G}_\alpha f(x) \doteq \int_0^\infty e^{-\alpha t} \mathcal{P}_t f(x) dt, \quad \mathcal{G}_\alpha^* f(x) \doteq \int_0^\infty e^{-\alpha t} \mathcal{P}_t^* f(x) dt, \quad x \in G, \quad f \in C_c^\infty(G).$$

Then by the resolvent identities, for $f \in C_c^\infty(G)$ and $\alpha > 0$

$$(\alpha - \mathcal{L})\mathcal{G}_\alpha f = f, \quad (\alpha - \mathcal{L}^*)\mathcal{G}_\alpha^* f = f.$$

This shows that, for $f, g \in C_c^\infty(G)$

$$\begin{aligned} \int_G \mathcal{G}_\alpha f(x) g(x) dx &= \int_G \mathcal{G}_\alpha f(x) (\alpha - \mathcal{L}^*) \mathcal{G}_\alpha^* g(x) dx = \int_G (\alpha - \mathcal{L}) \mathcal{G}_\alpha f(x) \mathcal{G}_\alpha^* g(x) dx \\ &= \int_G f(x) \mathcal{G}_\alpha^* g(x) dx. \end{aligned}$$

This implies that, for all $\alpha > 0$ and $f, g \in C_c^\infty(G)$

$$\int_{G \times G} \int_0^\infty e^{-\alpha t} p_t^\dagger(x, y) f(y) g(x) dt dy dx = \int_{G \times G} \int_0^\infty e^{-\alpha t} q_t(x, y) f(x) g(y) dt dy dx.$$

Since $f, g \in C_c^\infty(G)$ are arbitrary, this shows that, for all $x, y \in G$ and $\alpha > 0$,

$$\int_0^\infty e^{-\alpha t} p_t^\dagger(x, y) dt = \int_0^\infty e^{-\alpha t} q_t(y, x) dt.$$

The claimed identity in (3.2) now follows by the uniqueness of the Laplace transforms.

Next note that for $f \in C_0^2(G)$

$$\begin{aligned} E_{\tilde{\nu}}[f(X^\dagger(t))] &= E_{\tilde{\nu}}[f(X(t \wedge \tau_\partial))] = \langle f, \tilde{\nu} \rangle + E_{\tilde{\nu}} \left[\int_0^{t \wedge \tau_\partial} \mathcal{L} f(X(s)) ds \right] \\ &= \langle f, \tilde{\nu} \rangle + \int_0^t E_{\tilde{\nu}}[\mathcal{L} f(X^\dagger(s))] ds. \end{aligned}$$

Thus

$$\begin{aligned} E_{\tilde{\nu}}[f(X(t))1_{\tau_\partial > t}] &= E_{\tilde{\nu}}[f(X^\dagger(t))] \\ &= \langle f, \tilde{\nu} \rangle + \int_0^t \int_G \int_G \mathcal{L} f(y) p_s^\dagger(x, y) dy \varphi(x) dx ds \\ &= \langle f, \tilde{\nu} \rangle + \int_0^t \int_G \int_G f(y) \mathcal{L}_y^* p_s^\dagger(x, y) dy \varphi(x) dx ds. \end{aligned}$$

Using (3.2) we have

$$\begin{aligned} E_{\tilde{\nu}}[f(X(t))1_{\tau_{\partial} > t}] &= \langle f, \tilde{\nu} \rangle + \int_0^t \int_G \int_G f(y) \mathcal{L}_x p_s^\dagger(x, y) dy \varphi(x) dx ds \\ &= \langle f, \tilde{\nu} \rangle + \int_0^t \int_G f(y) \int_G p_s^\dagger(x, y) \mathcal{L}^* \varphi(x) dx dy ds \\ &= \langle f, \tilde{\nu} \rangle - \lambda \int_0^t \int_G f(y) \int_G p_s^\dagger(x, y) \varphi(x) dx dy ds \\ &= \langle f, \tilde{\nu} \rangle - \lambda \int_0^t E_{\tilde{\nu}}[f(X(s))1_{\tau_{\partial} > s}] ds. \end{aligned}$$

This shows that

$$E_{\tilde{\nu}}[f(X(t))1_{\tau_{\partial} > t}] = e^{-\lambda t} \langle f, \tilde{\nu} \rangle. \quad (3.3)$$

Now let $g_n \in C_0^2(G)$ be such that $\sup_{n,x} |g_n(x)| < \infty$ and $g_n(x) \rightarrow 1$ for all $x \in G$. Then from (3.3), for all n

$$E_{\tilde{\nu}}[g_n(X(t))1_{\tau_{\partial} > t}] = e^{-\lambda t} \langle g_n, \tilde{\nu} \rangle.$$

Sending $n \rightarrow \infty$, by the dominated convergence theorem

$$P_{\tilde{\nu}}(\tau_{\partial} > t) = E_{\tilde{\nu}}[1_{\tau_{\partial} > t}] = \lim_{n \rightarrow \infty} E_{\tilde{\nu}}[g_n(X(t))1_{\tau_{\partial} > t}] = e^{-\lambda t} \lim_{n \rightarrow \infty} \langle g_n, \tilde{\nu} \rangle = e^{-\lambda t}.$$

Substituting this in (3.3) we obtain (3.1), showing that $\tilde{\nu}$ is a QSD. Recall from part 3 that there is a unique QSD. This completes the proof of 6. \square

3.2. A technical lemma

The following lemma will be used to show that certain stochastic differential equations with drift functions given in terms of a solution of (2.8) or a solution of (2.18) have unique pathwise solutions that remain in the domain G at all times.

Lemma 3.1. *Under Assumption 2.2 the following hold.*

(a) Let $\{Y(t)\}$ solve the SDE

$$dY(t) = b(Y(t))dt + \sigma(Y(t))dB(t) - a(Y(t))D\Gamma(Y(t))dt,$$

where (Γ, γ) is a solution of (2.8). For $\delta \in (0, \infty)$ let

$$\tau_{\delta} \doteq \inf\{t > 0 : \text{dist}(Y(t), \partial G) \leq \delta\}.$$

Then for $y \in G$ and any $T \in (0, \infty)$, $P_y\{\tau_{\delta} \leq T\} \rightarrow 0$ as $\delta \rightarrow 0$.

(b) Let $\{Z(t)\}$ solve the SDE

$$dZ(t) = \tilde{\beta}(Z(t))dt + \sigma(Z(t))dB(t) - a(Z(t))D\Gamma(Z(t))dt,$$

where $\tilde{\beta}$ is as in (2.13) and (Γ, γ) is a solution of (2.18). For $\delta \in (0, \infty)$ define τ_{δ} as in part (a) but with Y replaced with Z . Then for $z \in G$ and any $T \in (0, \infty)$, $P_z\{\tau_{\delta} \leq T\} \rightarrow 0$ as $\delta \rightarrow 0$.

Proof. Consider first part (a). Fix $y \in G$. The argument is based on constructing an appropriate martingale. We use that

$$\Gamma(x_i) \uparrow \infty \text{ whenever } x_i \rightarrow \bar{x} \in \partial G \text{ with } x_i \in G, \quad (3.4)$$

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and that Γ is \mathcal{C}^2 on G . Let

$$\tilde{\mathcal{L}}\Gamma(x) \doteq b(x) \cdot D\Gamma(x) - a(x)D\Gamma(x) \cdot D\Gamma(x) + \frac{1}{2}\text{tr}[a(x)D^2\Gamma(x)], \quad x \in G.$$

Note that, for $x \in G$,

$$\min_{u \in \mathbb{R}^m} \left[(b(x) + \sigma(x)u) \cdot D\Gamma(x) + \frac{1}{2}\|u\|^2 \right] = (b(x) - \frac{1}{2}a(x)D\Gamma(x)) \cdot D\Gamma(x).$$

Using this and since Γ solves (2.8) we get that, for $x \in G$,

$$\begin{aligned} \gamma &= \frac{1}{2}\text{tr}[a(x)D^2\Gamma(x)] + D\Gamma(x) \cdot b(x) - \frac{1}{2}a(x)D\Gamma(x) \cdot D\Gamma(x) \\ &\geq \frac{1}{2}\text{tr}[a(x)D^2\Gamma(x)] + D\Gamma(x) \cdot b(x) - a(x)D\Gamma(x) \cdot D\Gamma(x) = \tilde{\mathcal{L}}\Gamma(x), \end{aligned}$$

where the second line follows on recalling that a is nonnegative definite. Without loss of generality we also assume $\Gamma \geq 0$. Fix $T < \infty$ and suppose that

$$\lim_{\delta \rightarrow 0} P_y\{\tau_\delta \leq T\} \doteq \kappa > 0. \quad (3.5)$$

Note that the above limit exists as the probability on the left side is non-increasing in δ . Choose $M \in (0, \infty)$ such that $\Gamma(y) + \gamma T < \frac{1}{2}\kappa M$. From (3.4) we can find $\delta_0 \in (0, \infty)$ so that

$$\Gamma(x) \geq M \text{ whenever } x \in G, \text{ dist}(x, \partial G) \leq \delta_0.$$

Then, for any $\delta > 0$,

$$E_y[\Gamma(Y(\tau_\delta \wedge T))] - \Gamma(y) = E_y \int_0^{\tau_\delta \wedge T} \tilde{\mathcal{L}}\Gamma(Y(s))ds \leq \gamma T$$

and therefore for any $0 < \delta \leq \delta_0$

$$M P_y(\tau_\delta \leq T) \leq E_y[\Gamma(Y(\tau_\delta))1_{\tau_\delta \leq T}] \leq E_y[\Gamma(Y(\tau_\delta \wedge T))] \leq \Gamma(y) + \gamma T < \frac{1}{2}\kappa M.$$

In particular, $\lim_{\delta \rightarrow 0} P_y\{\tau_\delta \leq T\} \leq \kappa/2$ which contradicts (3.5). The result follows.

The proof of part (b) is similar on using the fact that $\sup_{x \in G} |\tilde{c}(x)| \doteq c_0 < \infty$ and that, if (Γ, γ) is a solution of (2.18), then

$$\tilde{\mathcal{L}}\Gamma(x) \doteq \tilde{\beta}(x) \cdot D\Gamma(x) - a(x)D\Gamma(x) \cdot D\Gamma(x) + \frac{1}{2}\text{tr}[a(x)D^2\Gamma(x)] \leq (\gamma + c_0) \text{ for } x \in G.$$

Indeed, the above estimate gives, for any $\delta > 0$ and $z \in G$,

$$E_z[\Gamma(Z(\tau_\delta \wedge T))] - \Gamma(z) = E_z \int_0^{\tau_\delta \wedge T} \tilde{\mathcal{L}}\Gamma(Z(s))ds \leq (\gamma + c_0)T.$$

The remaining proof is as for part (a). We omit the details. \square

3.3. Proof of Theorem 2.4

Recall that ψ satisfies the equation

$$\begin{aligned} \mathcal{L}\psi(x) &= -\lambda\psi(x), \quad x \in G, \\ \psi(x) &= 0, \quad x \in \partial G. \end{aligned} \quad (3.6)$$

It is easy to verify that $\Psi = -\log \psi$ satisfies

$$D\psi = -\psi D\Psi, \quad \psi_{x_i x_j} = -\psi(\Psi_{x_i x_j} - \Psi_{x_i} \Psi_{x_j}) \quad (3.7)$$

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for all $1 \leq i, j \leq d$. Thus

$$\begin{aligned} \mathcal{L}\psi(x) &= b(x) \cdot D\psi(x) + \frac{1}{2} \text{tr}[a(x)D^2\psi(x)] \\ &= -\psi \left(b(x) \cdot D\Psi(x) + \frac{1}{2} \text{tr}[a(x)D^2\Psi(x)] - \frac{1}{2} D\Psi(x) \cdot a(x)D\Psi(x) \right). \end{aligned}$$

Substituting this in the first line of (3.6) and recalling that $\psi(x) > 0$ for $x \in G$, we have

$$b(x) \cdot D\Psi(x) + \frac{1}{2} \text{tr}[a(x)D^2\Psi(x)] - \frac{1}{2} D\Psi(x) \cdot a(x)D\Psi(x) = \lambda, \quad \text{for all } x \in G. \quad (3.8)$$

Note that, for any $v \in \mathbb{R}^d$

$$\min_{u \in \mathbb{R}^d} \left[(b(x) + \sigma(x)u) \cdot v + \frac{1}{2} \|u\|^2 \right] = b(x) \cdot v - \frac{1}{2} v \cdot a(x)v.$$

Using this in (3.8) we have

$$\frac{1}{2} \text{tr}[a(x)D^2\Psi(x)] + \min_{u \in \mathbb{R}^d} \left[(b(x) + \sigma(x)u) \cdot D\Psi(x) + \frac{1}{2} \|u\|^2 \right] = \lambda, \quad \text{for all } x \in G. \quad (3.9)$$

Also, since $\psi(x) \rightarrow 0$ as $x \rightarrow \partial G$, we have that $\Psi(x) \rightarrow \infty$ as $x \rightarrow \partial G$. Combining these facts we have part 1 of the theorem.

Now consider part 2. Let $(\tilde{\Psi}, \tilde{\lambda}) \in \mathcal{C}^2(G) \times (0, \infty)$ be a solution of the HJB equation in (2.8). Let $\tilde{\psi}(x) \doteq e^{-\tilde{\Psi}(x)}$ for $x \in G$ and $\tilde{\psi}(x) \doteq 0$ for $x \in \partial G$. Then $\tilde{\psi}(x) \in \mathcal{C}^2(G) \cap \mathcal{C}(\bar{G})$ satisfies $\tilde{\psi}(x) > 0$ for all $x \in G$; also it is easily verified that $(\tilde{\psi}, \tilde{\lambda})$ solve (3.6). By the Krein–Rutman theorem once more (cf. [9, Theorem 1.3]) we must have that for some $\kappa_0 \in (0, \infty)$, $\tilde{\psi}(x) = \kappa_0 \psi(x)$ for all $x \in \bar{G}$ and $\lambda = \tilde{\lambda}$. This proves part 2 with $C = -\log \kappa_0$.

Consider now part 3. For $\delta \in (0, \infty)$, let

$$G_\delta = \{x \in G : \text{dist}(x, \partial G) \geq \delta\}.$$

Then, from our assumptions on the coefficients it follows that for every $y \in G_\delta$ there is a continuous $\{\mathcal{F}_t\}$ -adapted process $Y^\delta(t)$ that is unique in the strong sense [13] such that if $\tau_\delta \doteq \inf\{t \geq 0 : Y^\delta(t) \in \partial G_\delta\}$, then for all $0 \leq t < \tau_\delta$,

$$Y^\delta(t) = y + \int_0^t b(Y^\delta(s))ds + \int_0^t \sigma(Y^\delta(s))dB(s) - \int_0^t a(Y^\delta(s))D\Psi(Y^\delta(s))ds.$$

From the uniqueness it follows that for $0 < \delta_1 < \delta_2 < \infty$,

$$Y^{\delta_1}(t) = Y^{\delta_2}(t) \text{ for all } 0 \leq t < \tau_{\delta_2}.$$

Also, note that $G = \cup_{\delta>0} G_\delta$. In order to complete the proof of part 3 it suffices to show that, for every $0 < t < \infty$,

$$P_y(\tau_{\delta_1} < t) \rightarrow 0 \text{ as } \delta_1 \rightarrow 0.$$

However this is an immediate consequence of Lemma 3.1. This proves 3.

From the uniform non-degeneracy condition in Assumption 2.2(2) it follows that Y defined in (2.10) has at most one invariant distribution. In order to show that $\mu(dx) \doteq \varphi(x)\psi(x)dx$ is an invariant measure for Y , it suffices to show that

$$(\mathcal{L}_Y^*[\varphi\psi])(x) = 0 \text{ for all } x \in G,$$

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where \mathcal{L}_Y^* is the formal adjoint of the infinitesimal generator \mathcal{L}_Y of the Markov process Y in (2.10). To simplify notation we suppress the independent variable x . Then for $f \in \mathcal{C}_0^2(G)$

$$\mathcal{L}_Y f = b \cdot Df - Df \cdot aD\Psi + \frac{1}{2}\text{tr}[aD^2 f],$$

and therefore for $h \in \mathcal{C}_0^2(G)$

$$\mathcal{L}_Y^* h = \hat{\beta} \cdot Dh + \frac{1}{2}\text{tr}[aD^2 h] - \hat{c}h,$$

where with β is as defined in (2.3)

$$\hat{\beta} = \beta - \frac{1}{\psi}aD\psi, \quad \hat{\beta}_i = -b_i - \frac{1}{\psi}(aD\psi)_i + \sum_{j=1}^d (a_{ij})_{x_j}, \quad (3.10)$$

and with c is defined in (2.4)

$$\begin{aligned} \hat{c} &= -\frac{1}{2} \sum_{i,j=1}^d (a_{ij})_{x_i x_j} + \sum_{i=1}^d (b_i)_{x_i} - \sum_{i=1}^d \sum_{j=1}^d [a_{ij} \Psi_{x_j}]_{x_i} \\ &= c - \sum_{i=1}^d \sum_{j=1}^d a_{ij} \Psi_{x_i x_j} - \sum_{i=1}^d \sum_{j=1}^d (a_{ij})_{x_i} \Psi_{x_j}. \end{aligned} \quad (3.11)$$

We will use $\Psi_{x_j} = -\psi_{x_j}/\psi$, as well as the identity

$$\varphi\psi \sum_{i,j=1}^d a_{ij} \Psi_{x_i x_j} = -\varphi\text{tr}[aD^2\psi] + \frac{\varphi}{\psi}D\psi \cdot aD\psi,$$

To show $\mathcal{L}_Y^*[\varphi\psi] = 0$, since $\mathcal{L}\psi = -\lambda\psi$ and $\mathcal{L}^*\varphi = -\lambda\varphi$ is suffices to establish

$$\hat{\beta} \cdot D[\varphi\psi] + \frac{1}{2}\text{tr}[aD^2\varphi\psi] - \hat{c}\varphi\psi \stackrel{?}{=} \psi\mathcal{L}^*\varphi - \varphi\mathcal{L}\psi,$$

where ? indicates that the equality is not yet established. Expanding derivatives and using the definitions of \mathcal{L} and \mathcal{L}^* gives the equivalent statement

$$\begin{aligned} &\psi\hat{\beta} \cdot D\varphi + \varphi\hat{\beta} \cdot D\psi + \frac{1}{2}\psi\text{tr}[aD^2\varphi] + \frac{1}{2}\varphi\text{tr}[aD^2\psi] + D\varphi \cdot aD\psi - \hat{c}\varphi\psi \\ &\stackrel{?}{=} \psi\beta \cdot D\varphi + \frac{1}{2}\psi\text{tr}[aD^2\varphi] - c\varphi\psi - \varphi b \cdot D\psi - \frac{1}{2}\varphi\text{tr}[aD^2\psi]. \end{aligned}$$

Cancellation of $\frac{1}{2}\psi\text{tr}[aD^2\varphi]$ and then substitution from (3.10) gives the statement

$$\begin{aligned} &\psi\beta \cdot D\varphi - aD\psi \cdot D\varphi - \varphi b \cdot D\psi - \frac{\varphi}{\psi}aD\psi \cdot D\psi + \varphi \sum_{i,j=1}^d (a_{ij})_{x_j} \psi_{x_i} \\ &\quad + \frac{1}{2}\varphi\text{tr}[aD^2\psi] + D\varphi \cdot aD\psi - \hat{c}\varphi\psi \\ &\stackrel{?}{=} \psi\beta \cdot D\varphi - c\varphi\psi - \varphi b \cdot D\psi - \frac{1}{2}\varphi\text{tr}[aD^2\psi]. \end{aligned}$$

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Next we cancel where obvious, bring the $\frac{1}{2}\varphi\text{tr}[aD^2\psi]$ term to the right hand side, substitute for \hat{c} and use (3.11) to get

$$\begin{aligned} & -\frac{\varphi}{\psi}aD\psi \cdot D\psi + \varphi \sum_{i,j=1}^d (a_{ij})_{x_i} \psi_{x_j} - c\varphi\psi - \varphi\text{tr}[aD^2\psi] + \frac{\varphi}{\psi}D\psi \cdot aD\psi \\ & - \varphi \sum_{i,j=1}^d (a_{ij})_{x_j} \psi_{x_i} \stackrel{?}{=} -c\varphi\psi - \varphi\text{tr}[aD^2\psi]. \end{aligned}$$

Cancellation now gives the desired result. This completes the proof of part 4.

Now let Y be as given in part 3 and for $\delta \in (0, \infty)$ let

$$\tau_\delta \doteq \inf\{t > 0 : \text{dist}(Y(t), \partial G) \leq \delta\}. \quad (3.12)$$

By Itô's formula, and since Ψ satisfies (3.8), for any $t > 0$

$$\begin{aligned} \Psi(Y(t)) - \Psi(Y(0)) &= \int_0^t b(Y(s)) \cdot D\Psi(Y(s))ds + \int_0^t D\Psi(Y(s)) \cdot \sigma(Y(s))dB(s) \\ &\quad - \int_0^t D\Psi(Y(s)) \cdot a(Y(s))D\Psi(Y(s))ds \\ &\quad + \frac{1}{2} \int_0^t \text{tr}[D^2\Psi(Y(s))a(Y(s))]ds \\ &= \int_0^t D\Psi(Y(s)) \cdot \sigma(Y(s))dB(s) \\ &\quad - \frac{1}{2} \int_0^t [D\Psi(Y(s)) \cdot a(Y(s))D\Psi(Y(s))] ds + \lambda t. \end{aligned} \quad (3.13)$$

Applying the above identity with $t = T \wedge \tau_\delta$, taking expectations, and dividing by T throughout

$$\begin{aligned} \frac{1}{2T} E_y \int_0^{T \wedge \tau_\delta} D\Psi(Y(s)) \cdot a(Y(s))D\Psi(Y(s))ds &= \lambda \frac{E_y[T \wedge \tau_\delta]}{T} \\ &\quad + \frac{\Psi(y) - E_y[\Psi(Y(T \wedge \tau_\delta))]}{T} \\ &\leq \lambda + \frac{\Psi(y) + \sup_{y \in G}(\Psi(y))_-}{T}. \end{aligned} \quad (3.14)$$

Sending $\delta \rightarrow 0$, we have by the monotone convergence theorem that

$$\frac{1}{2T} E_y \int_0^T [D\Psi(Y(s)) \cdot a(Y(s))D\Psi(Y(s))] ds \leq \lambda + \frac{\Psi(y) + \sup_{y \in G}(\Psi(y))_-}{T}.$$

Recalling that the process u defined in part 3 is in $\mathcal{A}(y)$, we now have that

$$\begin{aligned} J(y) &\leq \limsup_{T \rightarrow \infty} \frac{1}{2T} E_y \int_0^T \|u(s)\|^2 ds \\ &= \limsup_{T \rightarrow \infty} \frac{1}{2T} E_y \int_0^T [D\Psi(Y(s)) \cdot a(Y(s))D\Psi(Y(s))] ds \leq \lambda. \end{aligned} \quad (3.15)$$

We now consider the reverse direction. Suppose that $u \in \mathcal{A}(y)$ and consider the corresponding state process Y defined according to (2.6). Under Assumption 2.2, it follows from [15] that in terms of the function $\text{dist}(x, G)$ we can define a sequence of bounded domains $\{G_n\}_{n \in \mathbb{N}}$

such that $\text{dist}(G_n^c, G) \rightarrow 0$ as $n \rightarrow \infty$, and for all $n \in \mathbb{N}$, $G_n \supset G_{n+1} \supset G$, $\text{dist}(G_n^c, G) > 0$, G_n is $\mathcal{C}^{2,\alpha}$ for some $\alpha \in (0, 1]$ that is independent of n .

For $n \in \mathbb{N}$, let $\lambda_n \in (0, \infty)$ be the principal eigenvalue of $-\mathcal{L}$ associated with the domain G_n (so that any other eigenvalue $\tilde{\lambda}_n$ of $-\mathcal{L}$ associated with the domain G_n satisfies $\lambda_n \leq \text{Re}(\tilde{\lambda}_n)$). It is known that λ_n is an increasing sequence (cf. [4, Proposition 2.3]) and as $n \rightarrow \infty$ $\lambda_n \rightarrow \lambda$. The latter property follows, for example, from [4, Theorem 1.10].

Let ψ_n be a strictly positive eigenvector associated with the eigenvalue λ_n and let $\Psi_n \doteq -\log \psi_n$. Note that $\Psi_n \in \mathcal{C}^2(G_n)$ and so in particular its restriction to \bar{G} is in $\mathcal{C}^2(\bar{G})$. Applying Itô's formula to $\Psi_n(Y(t))$, and noting that (3.9) is satisfied with (Ψ, λ, G) replaced with (Ψ_n, λ_n, G_n) , we have, for any $T > 0$,

$$\begin{aligned} \Psi_n(Y(T)) - \Psi_n(Y(0)) &= \int_0^T b(Y(s)) \cdot D \Psi_n(Y(s)) ds + \int_0^T D \Psi_n(Y(s)) \cdot \sigma(Y(s)) dB(s) \\ &\quad + \int_0^T D \Psi_n(Y(s)) \cdot \sigma(Y(s)) u(s) ds \\ &\quad + \frac{1}{2} \int_0^T \text{tr}[D^2 \Psi_n(Y(s)) a(Y(s))] ds \\ &\geq \int_0^T D \Psi_n(Y(s)) \cdot \sigma(Y(s)) dB(s) - \frac{1}{2} \int_0^T \|u(s)\|^2 ds + \lambda_n T. \end{aligned}$$

Thus taking expectations, and dividing by T throughout, we have

$$\frac{1}{2T} E_y \int_0^T \|u(s)\|^2 ds \geq \lambda_n + \frac{1}{T} (\Psi_n(y) - E_y[\Psi_n(Y(T))]). \tag{3.16}$$

Finally, sending $T \rightarrow \infty$ and recalling that Ψ_n is bounded on \bar{G} and $Y(T) \in G$ a.s.,

$$J(y) = \inf_{u \in \mathcal{A}(y)} \limsup_{T \rightarrow \infty} \frac{1}{2T} E_y \int_0^T \|u(s)\|^2 ds \geq \lambda_n.$$

Sending $n \rightarrow \infty$ we get that $J(y) \geq \lambda$ which completes the proof of part 5.

Part 6 is now immediate from part 5 and the inequality in (3.15) and (3.16).

Finally, the proof of (2.11) can be completed as in (3.14) and in (3.16) by taking Y to be as given by the equation in (2.10) but with $Y(0)$ distributed according to the stationary distribution μ (and observing that $\int_G |\Psi(y)| \mu(dy) < \infty$). \square

4. Proof of Theorem 2.6

Since many arguments are similar to that in the proof of Theorem 2.4, we will omit some details. Recall from Theorem 2.3 that $(\tilde{\varphi}, \lambda)$ satisfy

$$\begin{aligned} -\mathcal{L}_\zeta^* \tilde{\varphi}(x) &= \lambda \tilde{\varphi}(x), & x \in G \\ \tilde{\varphi}(x) &= 0, & x \in \partial G. \end{aligned} \tag{4.1}$$

Note that, with $\tilde{\Phi} = -\log \tilde{\varphi}$, (3.7) holds with (ψ, Ψ) replaced with $(\tilde{\varphi}, \tilde{\Phi})$. Thus, recalling (2.12),

$$\begin{aligned} \mathcal{L}_\zeta^* \tilde{\varphi}(x) &= \tilde{\beta}(x) \cdot D \tilde{\varphi}(x) + \frac{1}{2} \text{tr}[a(x) D^2 \tilde{\varphi}(x)] - \tilde{c}(x) \tilde{\varphi}(x) \\ &= -\tilde{\varphi}(x) \left(\tilde{\beta}(x) \cdot D \tilde{\Phi} + \frac{1}{2} \text{tr}[a(x) D^2 \tilde{\Phi}(x)] - \frac{1}{2} D \tilde{\Phi}(x) \cdot a(x) D \tilde{\Phi}(x) + \tilde{c}(x) \right). \end{aligned}$$

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Since $\tilde{\varphi}(x) > 0$ for all $x \in G$, we have from (4.1) that

$$\tilde{\beta}(x) \cdot D\tilde{\Phi} + \frac{1}{2}\text{tr}[a(x)D^2\tilde{\Phi}(x)] - \frac{1}{2}D\tilde{\Phi}(x) \cdot a(x)D\tilde{\Phi}(x) + \tilde{c}(x) = \lambda$$

for all $x \in G$ from which part 1 of the theorem follows as in the proof of Theorem 2.4.

Parts 2 and 3 follow exactly as in Theorem 2.4 by using Krein–Rutman theorem (cf. [9, Theorem 1.3]) and Lemma 3.1(b).

Now we consider part 4. It suffices to show that

$$(\mathcal{L}_Z^*[\varphi\psi])(x) = 0 \text{ for all } x \in G,$$

where \mathcal{L}_Z^* is the formal adjoint of the infinitesimal generator \mathcal{L}_Z of the Markov process Z in (2.20). As in the last section, we simplify notation by suppressing the independent variable. For $f \in C_0^2(G)$

$$\mathcal{L}_Z f = \tilde{\beta} \cdot Df - Df \cdot aD\tilde{\Phi} + \frac{1}{2}\text{tr}[aD^2 f] = \beta \cdot Df - Df \cdot aD\tilde{\Phi} + \frac{1}{2}\text{tr}[aD^2 f],$$

where the second equality is from (2.21), and therefore for $h \in C_0^2(G)$

$$\mathcal{L}_Z^* h = \beta^* \cdot Dh + \frac{1}{2}\text{tr}[aD^2 h] - c^* h,$$

where using (2.3)

$$\beta^* = b + aD\tilde{\Phi}$$

and

$$c^* = -c + \frac{1}{\varphi}\text{tr}[aD^2\varphi] - \frac{1}{\varphi^2}D\varphi \cdot aD\varphi + \sum_{ij=1}^d (a_{ij})_{x_i} \frac{\varphi_{x_j}}{\varphi}.$$

Using $D\tilde{\Phi} = -\frac{1}{\varphi}D\varphi$, it follows that

$$\beta^* = b - \frac{1}{\varphi}aD\varphi.$$

To show $\mathcal{L}_Z^*[\varphi\psi] = 0$, since $\mathcal{L}\psi = -\lambda\psi$ and $\mathcal{L}^*\varphi = -\lambda\varphi$ it suffices to show

$$\beta^* \cdot D[\varphi\psi] + \frac{1}{2}\text{tr}[aD^2[\varphi\psi]] - c^*\varphi\psi \stackrel{?}{=} \varphi\mathcal{L}\psi - \psi\mathcal{L}^*\varphi,$$

where again the ? indicates the equality has yet to be established. Substituting for the generators and expanding gives the equivalent statement

$$\begin{aligned} & \psi b \cdot D\varphi + \varphi b \cdot D\psi - \frac{\psi}{\varphi}aD\varphi \cdot D\varphi - aD\varphi \cdot D\psi + \frac{1}{2}\psi\text{tr}[aD^2\varphi] + \frac{1}{2}\varphi\text{tr}[aD^2\psi] \\ & + D\varphi \cdot aD\psi + c\varphi\psi - \psi\text{tr}[D^2\varphi a] + \frac{\psi}{\varphi}D\varphi \cdot aD\varphi - \psi \sum_{ij=1}^d (a_{ij})_{x_i} \varphi_{x_j} \\ & \stackrel{?}{=} \varphi b \cdot D\psi + \frac{1}{2}\varphi\text{tr}[aD^2\psi] + \psi b \cdot D\varphi - \psi \sum_{ij=1}^d (a_{ij})_{x_j} \varphi_{x_i} - \frac{1}{2}\psi\text{tr}[aD^2\varphi] + c\varphi\psi. \end{aligned}$$

Cancellations then give the statement

$$-aD\varphi \cdot D\psi + D\varphi \cdot aD\psi \stackrel{?}{=} 0,$$

and the result follows from the symmetry of a . This completes the proof of 4.

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For proof of 5 we first argue $J^*(z) \leq \lambda$ and then $J^*(z) \geq \lambda$. For the first inequality we introduce τ_δ as in (3.12) with Y replaced with Z and apply Itô's formula to $\tilde{\Phi}(Z(t))$ in a manner similar to (3.13). Then as in (3.14) we see that

$$\begin{aligned} & \frac{1}{T} E_y \int_0^{T \wedge \tau_\delta} \left(\frac{1}{2} D \tilde{\Phi}(Z(s)) \cdot a(Z(s)) D \tilde{\Phi}(Y(s)) + \tilde{c}(Z(s)) \right) ds \\ & \leq \lambda + \frac{\tilde{\Phi}(y) + \sup_{y \in G} (\tilde{\Phi}(y))_-}{T}. \end{aligned}$$

The inequality $J^*(z) \leq \lambda$ follows as in the proof of Theorem 2.4. For the reverse inequality we once more consider domains $G_n \supset G_{n+1} \supset G$ as in Theorem 2.4 and principle eigenvalue–eigenvector pair $(\lambda_n, \tilde{\varphi}_n)$ associated with $-\mathcal{L}_G^*$. We then apply Itô's formula to $\tilde{\Phi}_n(Z(t))$ where $\tilde{\Phi}_n = -\log \tilde{\varphi}_n$ and Z is given by (2.15) with an arbitrary $u \in \mathcal{A}^*(z)$. Then as in the proof of (3.16) we get

$$\frac{1}{T} E_z \int_0^T \left(\frac{1}{2} \|u(s)\|^2 + \tilde{c}(Z(s)) \right) ds \geq \lambda_n + \frac{1}{T} (\tilde{\Phi}_n(y) - E_y[\tilde{\Phi}_n(Y(T))]).$$

The proof of the inequality $J^*(z) \geq \lambda$ is now concluded as in the proof of Theorem 2.4. This completes the proof of part 5. Part 6 is now immediate from part 5. \square

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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