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Large Deviations for the Single-Server Queue and the Reneging Paradox

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Abstract. For the $M/M/1+M$ model at the law-of-large-numbers scale, the long-run reneging count per unit time does not depend on the individual (i.e., per customer) reneging rate. This paradoxical statement has a simple proof. Less obvious is a large deviations analogue of this fact, stated as follows: the decay rate of the probability that the long-run reneging count per unit time is atypically large or atypically small does not depend on the individual reneging rate. In this paper, the sample path large deviations principle for the model is proved and the rate function is computed. Next, large time asymptotics for the reneging rate are studied for the case when the arrival rate exceeds the service rate. The key ingredient is a calculus of variations analysis of the variational problem associated with atypical reneging. A characterization of the aforementioned decay rate, given explicitly in terms of the arrival and service rate parameters of the model, is provided yielding a precise mathematical description of this paradoxical behavior.

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Keywords: single-server queue • reneging • sample path large deviations • Laplace principle • Euler–Lagrange equations • the reneging paradox

1. Introduction

Despite vast interest in recent years in queueing models with reneging and their asymptotic analysis (see the survey in Ward [18]), large deviation (LD) treatment of even the simplest queueing model accounting for reneging, namely the $M/M/1+M$, is lacking. This paper addresses two LD aspects of this model: a sample path large deviation principle (LDP) and the decay rate of the probability that the long-run reneging count per unit time is atypically large or small when the arrival rate exceeds the service rate. Theorem 3 gives an explicit formula for the aforementioned decay rate, which shows in particular that the decay rate does not depend on the parameter governing the individual (or per customer) reneging rate. An additional fact that arises from the analysis is that, under the optimal change of measure associated with this atypical behavior, the number of reneging events over a large time interval, normalized by the cumulative time customers spend in the queue (summed over these customers), does not change with respect to (w.r.t.) its law of large numbers (LLN) value. As a result, the LD cost associated with reneging vanishes. These two phenomena, that are related to one another, are called in this paper the reneging paradox at the LD scale.

The model under consideration is of a single-server queue with reneging, in which the interarrival times, service times, and patience are exponentially distributed. With $n \in \mathbb{N}$ as a scaling parameter, the arrival rate in the n th system is given by λn , the service rate by μn , and the per-customer reneging rate by θ . That is, the patience of each customer is an exponential random variable with parameter θ . This scaling, in which arrival and service times are accelerated but patience remains constant, is common in the literature on scaling limits of queueing systems with reneging (as, for example, in Atar et al. [1], Kang and Ramanan [10]). The first main result of this paper is Theorem 1, which gives the sample path LDP for the process consisting of the pair: normalized queue length, normalized reneging count, for which normalization refers to dividing by n . The technique for establishing this is based on describing the state dynamics by Poisson random measures (PRM) and proceeds by proving the

Laplace upper and lower bounds using a general variational representation for expectations of functionals of PRM from Budhiraja et al. [6].

Simple considerations based on the balance equation identify the reneging rate in equilibrium at the LLN scale. That is, the number of arrivals over a time window is given by the number of departures plus the number of reneging customers over this window with a correction term that accounts for changes in the queue length. In steady state, this correction term converges in probability to zero at the scaling limit. Moreover, if $\lambda > \mu$, then in steady state, the queue rarely becomes empty in this asymptotic regime, and consequently, the departure rate is well approximated by $n\mu$, yielding reneging rate $\lambda n - \mu n$ with $o(n)$ correction. For the normalized processes, defined by division by n , this rate is given by $\lambda - \mu$. On the other hand, if $\lambda \leq \mu$, then it is not hard to see that the reneging rate is $O(1)$, and therefore, its normalized version is asymptotic to zero. The lack of dependence of the reneging rate on the parameter θ is also suggested by the time asymptotic behavior of the fluid limit equations given in (4). This gives a paradoxical behavior (that one may call the reneging paradox at the LLN scale) that the overall reneging rate is asymptotically independent of the per-customer reneging rate. It is not hard to establish this property rigorously. A control theoretic version of this phenomenon is observed earlier in Puha and Ward [15], in which a problem of minimizing the long-run average reneging cost of a fluid model is studied. An optimal control policy is calculated, and it is noticed that this policy is independent of the reneging distribution (see Puha and Ward [15, section 6]).

In this work, we establish a similar property at the large deviation scale. Specifically, we are concerned with estimating the probabilities of atypically large or small reneging count over a long time interval. We note that the reneging count is an important performance measure for queuing systems and probability of nontypical reneging counts, just as the buffer overflow probabilities for finite buffer queuing systems are natural to analyze using a large deviation scaling. If $R^n(t)$ denotes the reneging count at time t in the n th system and $\bar{R}^n(t) = n^{-1}R^n(t)$ denotes its normalized version, then by the preceding discussion, $\bar{R}^n(t)$ converges in probability to $\gamma_0 t$ as $n \rightarrow \infty$, where $\gamma_0 = (\lambda - \mu)^+$. Then, the quantities of interest are, for $\gamma > \gamma_0$,

$$\chi^+(\gamma) = \limsup_{t \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{t} \frac{1}{n} \log P\left(\bar{R}^n(t) \geq \gamma t\right), \quad \chi^-(\gamma) = \liminf_{t \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{1}{t} \frac{1}{n} \log P\left(\bar{R}^n(t) > \gamma t\right),$$

and for $\gamma \in (0, \gamma_0)$ (in the case $\gamma_0 > 0$),

$$\chi^+(\gamma) = \limsup_{t \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{t} \frac{1}{n} \log P\left(\bar{R}^n(t) \leq \gamma t\right), \quad \chi^-(\gamma) = \liminf_{t \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{1}{t} \frac{1}{n} \log P\left(\bar{R}^n(t) < \gamma t\right).$$

Under the assumption $\lambda \geq \mu$, Theorem 3 (see also Remark 1) shows that $\chi^+(\gamma) = \chi^-(\gamma)$ and provides a formula for this quantity as an explicit function of λ, μ , and γ . The tools are those of the calculus of variations. That is, the variational formula provided by the LDP for the large n asymptotics is analyzed for each t via the Euler–Lagrange equations. This analysis gives an expression for the minimizing trajectories in this variational problem for fixed t that is sufficiently large, which is explicit, except that it involves one scalar parameter A . This parameter is characterized as the solution of a certain nonlinear equation (see (47) and Lemma 6), which for a fixed t does not admit a simple form solution. We study properties of this parameter as a function of the initial condition and the time horizon as this time horizon approaches infinity. Using these properties, we then analyze the scaled optimal cost in the variational problem as $t \rightarrow \infty$ and obtain a simple form expression for the limit as a function of λ, μ , and γ . Finally, it is argued using several nice properties of the rate function that this limit quantity equals $\chi^+(\gamma)$ and $\chi^-(\gamma)$. In this work, we do not consider the case $\lambda < \mu$. The relevant calculus of variations problem for this case is less tractable, and its study will be taken up elsewhere.

We now present a heuristic to justify the formula for the decay rate as well as the two aspects of the reneging paradox at the LD scale. Consider for specificity, the event that the normalized reneging count over an interval of time T exceeds γT , where $\gamma > \gamma_0$. As is well known, a Poisson process with rate α satisfies a sample path LDP in $\mathcal{D}([0, T] : \mathbb{R}_+)$ with rate function given by $I(\varphi) = \int_0^T \alpha \ell(\dot{\varphi}(s)/\alpha) ds$ for absolutely continuous functions $\varphi : [0, T] \rightarrow \mathbb{R}_+$, and $I(\varphi) = \infty$ all other elements of $\mathcal{D}([0, T] : \mathbb{R}_+)$. Here, $\mathcal{D}([0, T] : \mathbb{R}_+)$ is the space of right-continuous functions with left limits from $[0, T]$ to $\mathbb{R}_+$ equipped with the Skorohod topology, and $\ell(u) = u \log u - u + 1$ for $u > 0$, $\ell(0) = 1$ and $\ell(u) = \infty$ for $u < 0$. The function ℓ attains its minimum value zero uniquely at one. In view of this and using balance equation considerations at equilibrium similar to those described for the LLN analysis, one may conjecture the following. The decay rate of the probability of this event as $n \rightarrow \infty$ is given by

$$-T \inf \{ \lambda \ell(\lambda^*/\lambda) + \mu \ell(\mu^*/\mu) + \theta y \ell(\theta^*/\theta) \}, \quad (1)$$

with an $o(T)$ correction, in which the infimum is over $\lambda^*, \mu^*, \theta^*$, and y in $(0, \infty)$, satisfying

$$\lambda^* = \mu^* + \theta^* y, \quad \theta^* y = \gamma.$$

Moreover, the minimizing $(\lambda^*, \mu^*, \theta^*)$ correspond to the parameters of the optimal change of measure in the LD analysis. To solve this optimization problem, note that the first constraint can be written as $\lambda^* = \mu^* + \gamma$, which decouples the problem into two optimization problems, one associated with (λ^*, μ^*) and another with (θ^*, y) . Clearly, the latter is solved by $\theta^* = \theta$, which makes the last term vanish. This heuristic supports the statement that no change of measure is associated with the reneging process under the optimal change of measure governing the event of interest as well as the fact that the solution is independent of θ . In addition, it can be checked that the solution of (1) is $-TC(\gamma)$, where $C(\gamma)$ is defined in (10). The heuristic is made rigorous in Theorem 3.

Whereas this work focuses on a single-server model, very similar results can be established for the multiserver queue with reneging, such as in a setting in which the number of servers grows linearly with the scaling parameter n and each server operates with standard exponential service time distribution (namely, the M/M/ n +M system). In Section 7, we present and discuss these results; however, the proofs are omitted. The specific scaling used in this setting has been referred to as a *many-server* scaling. This model and scaling were studied for their LLN and central limit theorem asymptotics in Kaspi and Ramanan [12, 13] and Reed [16] far beyond the exponential setting.

Our main motivation for this work stems from an approach to obtain robust probability estimates at the LD scale. According to this approach, a family of probabilistic models is specified in terms of Rényi divergence w.r.t. a single reference model (that is often easier to analyze). The approach then provides a tool by which LD estimates on the reference model can be transformed into LD estimates on the whole family. The model G/G/ n +G is a particularly good test case for this approach because (a) the model is hard and (b) robustness w.r.t. underlying distributions is important in applications in which it is often used, such as in models for call centers. The estimates obtained in this paper for the Markovian model can be used by this approach to obtain LD estimates for suitable families of G/G/ n +G in terms of the results of this paper concerning M/M/ n +M. This progress is reported in Atar et al. [2].

1.1. Notation

Let $\mathcal{D}([0, T] : \mathbb{R}^d)$ (respectively, $\mathcal{C}([0, T] : \mathbb{R}^d)$) denote the space of right-continuous functions with left limits (continuous functions) from $[0, T]$ to $\mathbb{R}^d$, equipped with the usual Skorohod (uniform) topology. Define $|x|_{*,T} \doteq \sup_{0 \leq t \leq T} |x(t)|$ for $x \in \mathcal{D}([0, T] : \mathbb{R}^d)$. For a Polish space $\mathcal{E}$, we denote by $C_b(\mathcal{E})$ the space of real continuous and bounded functions on $\mathcal{E}$ and by $\mathcal{P}(\mathcal{E})$ the space of probability measures on $\mathcal{E}$, which is equipped with the topology of weak convergence. We say a collection of $\mathcal{E}$ -valued random variables is tight if the corresponding family of probability laws of the random variables is relatively compact in the space $\mathcal{P}(\mathcal{E})$. A function $I : \mathcal{E} \rightarrow [0, \infty]$ is called a rate function if, for every $m < \infty$, the set $\{x : I(x) \leq m\}$ is compact. Some authors also refer to such a function as a “good rate function”; however, here we drop the adjective “good.” A tight sequence of $\mathcal{D}([0, T] : \mathbb{R}^d)$ -valued random variables is said to be $\mathcal{C}$ -tight if the limit of every weakly convergent subsequence takes values in $\mathcal{C}([0, T] : \mathbb{R}^d)$ almost surely (a.s.). We use $\kappa, \kappa_1, \kappa_2, \dots$ to denote constants in the proof.

2. Large Deviations for M/M/1 with Reneging

We begin by describing the evolution of the scaled state process. For this, it is convenient to represent the jumps in the system through certain Poisson random measures, which are introduced here.

For a locally compact Polish space $\mathbb{S}$, let $\mathcal{M}_{FC}(\mathbb{S})$ be the space of all measures ν on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ such that $\nu(K) < \infty$ for every compact $K \subset \mathbb{S}$. We equip $\mathcal{M}_{FC}(\mathbb{S})$ with the usual vague topology. This topology can be metrized such that $\mathcal{M}_{FC}(\mathbb{S})$ is a Polish space (see Budhiraja et al. [6] for one convenient metric). A PRM N on a locally compact Polish space $\mathbb{S}$ with mean measure (or intensity measure) $\nu \in \mathcal{M}_{FC}(\mathbb{S})$ is an $\mathcal{M}_{FC}(\mathbb{S})$ valued random variable such that, for each $A \in \mathcal{B}(\mathbb{S})$ with $\nu(A) < \infty$, $N(A)$ is Poisson distributed with mean $\nu(A)$, and for disjoint $A_1, \dots, A_k \in \mathcal{B}(\mathbb{S})$, $N(A_1), \dots, N(A_k)$ are mutually independent random variables (cf. Ikeda and Watanabe [9]).

Fix $T \in (0, \infty)$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which we are given three mutually independent PRM N_1, N_2, N_3 on $[0, T] \times \mathbb{R}_+$, $[0, T] \times \mathbb{R}_+$, and $[0, T] \times \mathbb{R}_+^2$, respectively, with intensities $\lambda ds \times dy$, $\mu ds \times dy$, and $\theta ds \times dy \times dz$, respectively. Let $\mathbb{X}_1 \doteq \mathbb{R}_+$, $\mathbb{X}_2 \doteq \mathbb{R}_+$, $\mathbb{X}_3 \doteq \mathbb{R}_+^2$. Define the filtration

$$\hat{\mathcal{F}}_t \doteq \sigma\{N_i((0, s] \times A_i), s \in [0, t], A_i \in \mathcal{B}(\mathbb{X}_i), i = 1, 2, 3\}, t \geq 0$$

and let $\{\mathcal{F}_t\}$ be the P -augmentation of this filtration. Let $\bar{\mathcal{P}}$ be the $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ -predictable σ -field on $\Omega \times [0, T]$. Denote by $\bar{\mathcal{A}}_1, \bar{\mathcal{A}}_3$ the class of all measurable maps from $(\Omega \times [0, T], \bar{\mathcal{P}})$, $(\Omega \times [0, T] \times \mathbb{R}_+, \bar{\mathcal{P}} \times \mathcal{B}(\mathbb{R}_+))$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, respectively. Let $\bar{\mathcal{A}}_2 \doteq \bar{\mathcal{A}}_1$.

For $\varphi \in \bar{\mathcal{A}}_i$, define counting processes N_1^φ , N_2^φ , and N_3^φ on $[0, T]$, $[0, T]$, and $[0, T] \times \mathbb{R}_+$, respectively, by

$$N_i^\varphi([0, t]) \doteq \int_{[0, t] \times \mathbb{R}_+} 1_{[0, \varphi(s)]}(y) N_i(ds dy), \quad t \in [0, T], i = 1, 2,$$

$$N_i^\varphi([0, t] \times A) \doteq \int_{[0, t] \times A \times \mathbb{R}_+} 1_{[0, \varphi(s, y)]}(z) N_i(ds dy dz), \quad t \in [0, T], A \in \mathcal{B}(\mathbb{R}_+), i = 3.$$

We think of N_i^φ as a controlled random measure with which φ is the control process that produces a thinning of the point process N_i in a random but nonanticipative manner to produce a desired intensity. We write N_i^φ as N_i^m if $\varphi \equiv m$ for some constant $m \in \mathbb{R}_+$. Note that N_1^m, N_2^m, N_3^m are PRM on $[0, T], [0, T], [0, T] \times \mathbb{R}_+$ with intensity $m\lambda ds$, $m\mu ds$, and $m\theta ds \times dy$, respectively.

2.1. State Dynamics Using Poisson Random Measures

The process representing the number of customers in the n th system is referred to as the *number in system* process, denoted by $U^n(t)$. Note that the queue length process and the number of customers in service process (the latter being $\{0, 1\}$ -valued) can be expressed as $(U^n(t) - 1)^+$ and $U^n(t) \wedge 1$, respectively. Denote the total reneging by time t by $V^n(t)$. Let rescaled versions of these processes be defined by

$$X^n(t) = \frac{U^n(t)}{n}, \quad Y^n(t) = \frac{V^n(t)}{n}, \quad t \in [0, T].$$

We take $(X^n(0), Y^n(0)) = (x_n, 0)$ with $x_n \rightarrow x_0$ as $n \rightarrow \infty$. We establish an LDP for (X^n, Y^n) in $\mathcal{D}([0, T] : \mathbb{R}_+^2)$ and then deduce the LDP for $Y^n(T)$ using the contraction principle.

Using the PRMs introduced earlier, the state evolution can be written as

$$X^n(t) = x_n + \frac{1}{n} \int_{[0, t]} N_1^n(ds) - \frac{1}{n} \int_{[0, t]} 1_{\{X^n(s-) \neq 0\}} N_2^n(ds) - \frac{1}{n} \int_{[0, t] \times \mathbb{R}_+} 1_{[0, (X^n(s-) - \frac{1}{n})^+]}(y) N_3^n(ds dy).$$

$$Y^n(t) = \frac{1}{n} \int_{[0, t] \times \mathbb{R}_+} 1_{[0, (X^n(s-) - \frac{1}{n})^+]}(y) N_3^n(ds dy).$$

The role of the indicator of $\{X^n(s-) \neq 0\}$ in the second integral on the right-hand side (RHS) is to express the fact that departures occur only when the number in system is nonzero. Moreover, notice that the expression $(X^n - \frac{1}{n})^+$ gives the normalized queue length. Hence, to model reneging according to exponential clocks for each customer in the queue, the third integral expresses reneging that occurs at a rate proportional to the (normalized) queue length.

Define the map $\Gamma : \mathcal{D}([0, T] : \mathbb{R}) \rightarrow \mathcal{D}([0, T] : \mathbb{R}_+)$ by

$$\Gamma(\psi)(t) \doteq \psi(t) - \inf_{0 \leq s \leq t} [\psi(s) \wedge 0], \quad t \in [0, T], \psi \in \mathcal{D}([0, T] : \mathbb{R}). \quad (2)$$

The map Γ is usually referred to as the one-dimensional Skorohod map (see, e.g., Karatzas and Shreve [11, section 3.6.C]). Note that the evolution of X^n can be written as

$$X^n(t) = x_n + \frac{1}{n} \int_{[0, t]} N_1^n(ds) - \frac{1}{n} \int_{[0, t]} N_2^n(ds) - \frac{1}{n} \int_{[0, t] \times \mathbb{R}_+} 1_{[0, (X^n(s-) - \frac{1}{n})^+]}(y) N_3^n(ds dy) + \eta^n(t),$$

where $\eta^n(t) \doteq \frac{1}{n} \int_{[0, t]} 1_{\{X^n(s-) = 0\}} N_2^n(ds)$ increases only when $X^n(t-) = X^n(t) = 0$. Then using a well-known characterization of the Skorohod map (cf. Dupuis and Ishii [8]) one can write the evolution of X^n as

$$X^n(t) = \Gamma \left(x_n + \frac{1}{n} \int_{[0, \cdot]} N_1^n(ds) - \frac{1}{n} \int_{[0, \cdot]} N_2^n(ds) - \frac{1}{n} \int_{[0, \cdot] \times \mathbb{R}_+} 1_{[0, (X^n(s-) - \frac{1}{n})^+]}(y) N_3^n(ds dy) \right)(t). \quad (3)$$

As $n \rightarrow \infty$, (X^n, Y^n) converges in $\mathcal{D}([0, T] : \mathbb{R}_+^2)$, in probability, to (x, y) given by

$$x(t) = \Gamma \left(x_0 + (\lambda - \mu)t(\cdot) - \theta \int_0^t x(s) ds \right)(t), \quad t \in [0, T],$$

$$y(t) = \theta \int_0^t x(s) ds, \quad t \in [0, T], \quad (4)$$

where ι is the identity map on $[0, T]$. Theorem 1 establishes a large deviation principle for (X^n, Y^n) as $n \rightarrow \infty$. We begin by introducing the associated rate function.

2.2. Rate Function and the Large Deviation Principle

For $(\xi, \zeta) \in \mathcal{C}([0, T] : \mathbb{R}_+^2)$, let $\mathcal{U}(\xi, \zeta)$ be the collection of all $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ such that $\varphi_i : [0, T] \rightarrow \mathbb{R}_+$, $i = 1, 2, 3$ are measurable maps, and the following equations are satisfied for all $t \in [0, T]$:

$$\begin{aligned}\xi(t) &= \Gamma \left(x_0 + \lambda \int_0^t \varphi_1(s) ds - \mu \int_0^t \varphi_2(s) ds - \theta \int_0^t \varphi_3(s) \xi(s) ds \right)(t), \\ \zeta(t) &= \theta \int_0^t \varphi_3(s) \xi(s) ds.\end{aligned}\tag{5}$$

Define

$$\ell(x) \doteq x \log x - x + 1, \quad x \geq 0.\tag{6}$$

For $(\xi, \zeta) \in \mathcal{C}([0, T] : \mathbb{R}_+^2)$, define

$$I_T(\xi, \zeta) \doteq \inf_{\varphi \in \mathcal{U}(\xi, \zeta)} \left\{ \lambda \int_0^T \ell(\varphi_1(s)) ds + \mu \int_0^T \ell(\varphi_2(s)) ds + \theta \int_0^T \xi(s) \ell(\varphi_3(s)) ds \right\}.\tag{7}$$

We set $I_T(\xi, \zeta)$ to be ∞ if $\mathcal{U}(\xi, \zeta)$ is empty or $(\xi, \zeta) \in \mathcal{D}([0, T] : \mathbb{R}_+^2) \setminus \mathcal{C}([0, T] : \mathbb{R}_+^2)$.

Theorem 1. *The pair $\{(X^n, Y^n)\}$ satisfies an LDP on $\mathcal{D}([0, T] : \mathbb{R}_+^2)$ with rate function I_T .*

Using the contraction principle (cf. Dupuis and Ellis [7, theorem 1.3.2]), one has the following result. Let, for $\gamma \in \mathbb{R}_+$, $\mathcal{U}^*(\gamma)$ be the collection of all $(\xi, \zeta) \in \mathcal{C}([0, T] : \mathbb{R}_+^2)$ such that $\zeta(T) = \gamma$. Let $I_T^* : \mathbb{R}_+ \rightarrow [0, \infty]$ be defined as

$$I_T^*(\gamma) \doteq \inf_{(\xi, \zeta) \in \mathcal{U}^*(\gamma)} I_T(\xi, \zeta).\tag{8}$$

Theorem 2. *The random variable $\{Y^n(T)\}$ satisfies an LDP on $\mathbb{R}_+$ with rate function I_T^* .*

There does not appear to be a simple form expression for $I_T^*(\gamma)$ for fixed $T > 0$ and $\gamma \geq 0$, and therefore, we consider asymptotics of $I_T^*(\gamma)$ for large T . For this, we restrict attention to the case $\lambda \geq \mu$. Let

$$\begin{aligned}I_{\gamma, T}^* &\doteq I_T^*(\gamma T) = \inf \{I_T(\xi, \zeta) : \zeta(T) = \gamma T\}, \quad \gamma \geq 0, \\ I_{\gamma, T} &\doteq \inf \{I_T(\xi, \zeta) : \zeta(T) \geq \gamma T\}, \quad \gamma \geq \lambda - \mu, \\ \tilde{I}_{\gamma, T} &\doteq \inf \{I_T(\xi, \zeta) : \zeta(T) \leq \gamma T, 0 \leq \gamma \leq \lambda - \mu\}.\end{aligned}\tag{9}$$

Let

$$C(\gamma) \doteq \lambda(1 - z_\gamma^{-1}) + \mu(1 - z_\gamma) - \gamma \log z_\gamma,\tag{10}$$

where

$$z_\gamma \equiv z(\gamma) \doteq \frac{\sqrt{\gamma^2 + 4\lambda\mu} - \gamma}{2\mu} = \frac{2\lambda}{\sqrt{\gamma^2 + 4\lambda\mu} + \gamma}.\tag{11}$$

Theorem 3. *Suppose that $\lambda \geq \mu$. We have the following asymptotic formulae.*

$$\begin{aligned}\lim_{T \rightarrow \infty} \frac{I_{\gamma, T}^*}{T} &= C(\gamma), \quad \gamma \geq 0, \\ \lim_{T \rightarrow \infty} \frac{I_{\gamma, T}}{T} &= C(\gamma), \quad \gamma \geq \lambda - \mu, \\ \lim_{T \rightarrow \infty} \frac{\tilde{I}_{\gamma, T}}{T} &= C(\gamma), \quad 0 \leq \gamma \leq \lambda - \mu.\end{aligned}$$

Remark 1. From Theorem 3 and the convexity and monotonicity properties of $I_{\gamma, T}^*$ given in Lemma 11, it follows that the quantities $\chi^+(\gamma)$ and $\chi^-(\gamma)$ defined in the introduction equal $C(\gamma)$ for every $\gamma > 0$.

To give an interpretation for taking both n and T to be large parameters and to demonstrate a simple use of our results, consider a service capacity allocation problem. In this system, the arrival rate is large, whereas the reneging rate per customer is order one (and unknown). We take $\lambda = 1$, and $\theta > 0$ so that, in the n th system, the arrival rate is n , and the individual reneging rate is θ . We wish to find the smallest service capacity parameter

$\mu \in (0, 1)$ (corresponding in the n th system to service at rate μn) so that it is guaranteed that the reneging count $R^n(T)$ over a large interval of time $[0, T]$, satisfies $P(R^n(T) > \gamma nT) \leq e^{-cnT}$ for given constants γ and c . Consider, for example, $\gamma = 1$ and $c = 0.1$. Then, we have $z(\gamma) = \frac{\sqrt{1+4\mu}-1}{2\mu} = \frac{2}{\sqrt{1+4\mu}+1}$ and $C(\gamma) = 1 + \mu - \sqrt{1+4\mu} - \log \frac{2}{\sqrt{1+4\mu}+1}$. Setting $C(\gamma) = 0.1$ gives $\mu \approx 0.5723$. Note that the calculation does not depend on θ .

3. Proof of Theorem 1

Let $Z^n = (X^n, Y^n)$ and $\mathcal{E} = \mathcal{D}([0, T] : \mathbb{R}_+^2)$. From the equivalence between a large deviation principle and a Laplace principle (Dupuis and Ellis [7, section 1.2]), it suffices to show that the function I_T has compact sublevel sets, and the following bounds hold for all $h \in C_b(\mathcal{E})$.

Laplace upper bound:

$$\liminf_{n \rightarrow \infty} -\frac{1}{n} \log E \exp\{-nh(Z^n)\} \geq \inf_{\phi \in \mathcal{E}} [I_T(\phi) + h(\phi)]. \quad (12)$$

Laplace lower bound:

$$\limsup_{n \rightarrow \infty} -\frac{1}{n} \log E \exp\{-nh(Z^n)\} \leq \inf_{\phi \in \mathcal{E}} [I_T(\phi) + h(\phi)]. \quad (13)$$

The proof of compactness of level sets is analogous to the proof of the upper bound and is therefore omitted. The Laplace upper bound is proved in Section 3.2, and the lower bound is established in Section 3.3. Q.E.D.

3.1. Variational Representation and Weak Convergence of Controlled Processes

We use the following useful representation formula proved in Budhiraja et al. [4, theorem 2.4, 6] (see also Budhiraja and Dupuis [3, theorem 8.12]). Recall that $\bar{\mathcal{A}}_1, \bar{\mathcal{A}}_2, \bar{\mathcal{A}}_3$ denote the class of all measurable maps from $(\Omega \times [0, T], \bar{\mathcal{P}})$, $(\Omega \times [0, T], \bar{\mathcal{P}})$, $(\Omega \times [0, T] \times \mathbb{R}_+, \bar{\mathcal{P}} \times \mathcal{B}(\mathbb{R}_+))$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, respectively. For each $m \in \mathbb{N}$, let

$$\bar{\mathcal{A}}_{b,k} \doteq \left\{ (\varphi_1, \varphi_2, \varphi_3) : \varphi_i \in \bar{\mathcal{A}}_i \text{ for each } i = 1, 2, 3 \text{ such that for all } (\omega, t, y) \in \Omega \times [0, T] \times \mathbb{R}_+, \right. \\ \left. \varphi_1(\omega, t), \varphi_2(\omega, t), \varphi_3(\omega, t, y) \in [1/k, k], \text{ and } \varphi_3(\omega, t, y) = 1 \text{ when } y > k \right\}$$

and let $\bar{\mathcal{A}}_b \doteq \cup_{k=1}^{\infty} \bar{\mathcal{A}}_{b,k}$. (With a slight abuse of notation, here and in what follows, φ_i denote stochastic processes, unlike in Section 2.2 in which φ_i are used to denote deterministic functions). Denote $\mathcal{M} \doteq [\mathcal{M}_{FC}([0, T])]^2 \times \mathcal{M}_{FC}([0, T] \times \mathbb{R}_+)$. Recall the function ℓ defined in (6). Let, for $m \in \mathbb{N}$, $N^m = (N_1^m, N_2^m, N_3^m)$, and for $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \times_{i=1}^3 \bar{\mathcal{A}}_i$, let $N^\varphi = (N_1^{\varphi_1}, N_2^{\varphi_2}, N_3^{\varphi_3})$. N^m, N^φ are regarded as $\mathcal{M}$ -valued random variables. The theorem represents an expected value in terms of infima over both $\times_{i=1}^3 \bar{\mathcal{A}}_i$ and $\bar{\mathcal{A}}_b$. The latter is sometimes more convenient because, for each fixed control, there are uniform upper and lower (away from zero) bounds.

Theorem 4. Let F be a real valued bounded measurable map on $\mathcal{M}$. Then, for $m > 0$,

$$\begin{aligned} -\log E e^{-F(N^m)} &= \inf_{\varphi = (\varphi_i)_{i=1,2,3} \in \times_{i=1}^3 \bar{\mathcal{A}}_i} E \left[m \int_0^T \left(\lambda \ell(\varphi_1(s)) + \mu \ell(\varphi_2(s)) + \theta \int_{\mathbb{R}_+} \ell(\varphi_3(s, y)) dy \right) ds + F(N^{m\varphi}) \right] \\ &= \inf_{\varphi = (\varphi_i)_{i=1,2,3} \in \bar{\mathcal{A}}_b} E \left[m \int_0^T \left(\lambda \ell(\varphi_1(s)) + \mu \ell(\varphi_2(s)) + \theta \int_{\mathbb{R}_+} \ell(\varphi_3(s, y)) dy \right) ds + F(N^{m\varphi}) \right]. \end{aligned}$$

We note that the representation in Budhiraja et al. [4, 6] and Budhiraja and Dupuis [3] is given for a single PRM with points in $[0, T] \times \mathcal{X}$, where $\mathcal{X}$ is some locally compact space. One can identify the triplet $N^m = (N_1^m, N_2^m, N_3^m)$ with a single PRM with points in the space $[0, T] \times \mathbb{R}_+ \times \{1, 2, 3\}$ and intensity $\text{leb}_T \times \nu$, where, leb_T is the Lebesgue measure on $[0, T]$ and ν is a locally finite measure on $\mathcal{X} = \mathbb{R}_+ \times \{1, 2, 3\}$ defined as

$$\begin{aligned} \nu(\{0\} \times \{1\}) &= m\lambda, \quad \nu((0, \infty) \times \{1\}) = 0, \\ \nu(\{0\} \times \{2\}) &= m\mu, \quad \nu((0, \infty) \times \{2\}) = 0, \\ \nu(A \times \{3\}) &= m\theta \text{leb}_\infty(A), \quad A \in \mathcal{B}(\mathbb{R}_+), \end{aligned}$$

where leb_∞ is the Lebesgue measure on $\mathbb{R}_+$. With this identification, the proof of Theorem 4 is immediate from these references. In particular, the proof of the fact that one can restrict the infimum to the smaller class $\bar{\mathcal{A}}_b$ can be found in Budhiraja and Dupuis [3, theorem 8.12].

Remark 2. We mention difficulties to apply the continuous mapping approach.

i. Let $\mathcal{M}_{FC}^0([0, T])$ [resp. $\mathcal{M}_{FC}^0([0, T] \times \mathbb{R}_+)$] be the subset of $\mathcal{M}_{FC}([0, T])$ [resp. $\mathcal{M}_{FC}([0, T] \times \mathbb{R}_+)$] consisting of all “atomic measures” ν of the form such that there is some finite set $F \subset [0, T]$ with $\nu(F^c) = 0$ [resp. for every $k \in \mathbb{N}$, there is a finite set F_k with $\nu(F_k^c \times [0, k]) = 0$]. Let $\mathcal{M}^0 \doteq [\mathcal{M}_{FC}^0([0, T])]^2 \times \mathcal{M}_{FC}^0([0, T] \times \mathbb{R}_+)$. Then, for any $\nu = (\nu_1, \nu_2, \nu_3) \in \mathcal{M}^0$, there is a unique $z^\nu = (x^\nu, y^\nu) \in \mathcal{E}$ that satisfies

$$\begin{aligned} x^\nu(t) &= x_n + \frac{1}{n} \nu_1[0, t] - \frac{1}{n} \int_{[0, t]} \mathbf{1}_{\{x^\nu(s-) \neq 0\}} \nu_2(ds) - \frac{1}{n} \int_{[0, t] \times \mathbb{R}_+} \mathbf{1}_{[0, (x^\nu(s-) - \frac{1}{n})^+]}(y) \nu_3(ds dy) \\ y^\nu(t) &= \frac{1}{n} \int_{[0, t] \times \mathbb{R}_+} \mathbf{1}_{[0, (x^\nu(s-) - \frac{1}{n})^+]}(y) \nu_3(ds dy). \end{aligned}$$

Define $\Psi^n(\nu) = x^\nu$ for $\nu \in \mathcal{M}^0$. For $\nu \in \mathcal{M} \setminus \mathcal{M}^0$, the map is defined in an arbitrary manner so that Ψ^n is a measurable map from $\mathcal{M}$ to $\mathcal{E}$. We then have that $Z^n = \Psi^n(N^n)$ a.s. for each n . We note that the preceding system of equations may not be well posed for a general $\nu \in \mathcal{M} \setminus \mathcal{M}^0$, and for this reason, there is no natural way to define Ψ^n on all of $\mathcal{M}$ with nice regularity properties (e.g., continuity). In particular, there is no obvious way to implement a contraction principle approach using this representation in order to prove the LDP.

ii. It is also possible to write X in terms of Poisson processes on the line M, N, K (of rates λ, μ, θ , respectively) as

$$X = \Gamma \left(x + M - N - K \left(\int_0^\cdot X(s) ds \right) \right),$$

In which case $X|_{[0, t]}$ is a function of $M|_{[0, t]}$, $N|_{[0, t]}$, and $K|_{[0, \infty)}$. Letting M^n, N^n denote accelerated-by- n versions (recall that reneging rate is not accelerated) and X^n the resulting queue length, specializing to zero initial condition, we have $X^n = \Gamma(M^n - N^n - K(\int_0^\cdot X^n(s) ds))$. We can then write versions that are both accelerated and rescaled by letting $\bar{M}^n = n^{-1}M(\cdot)$, $\bar{N}^n = n^{-1}N(\cdot)$, $\bar{K}^n = n^{-1}K(\cdot)$, $\bar{X}^n = n^{-1}X^n$ and thereby obtain $\bar{X}^n = \Gamma(\bar{M}^n - \bar{N}^n - \bar{K}^n(\int_0^\cdot \bar{X}^n(s) ds))$. From this, we see that $\bar{X}^n|_{[0, t]}$ depends on $\bar{K}^n|_{[0, \infty)}$ and not just $\bar{K}^n|_{[0, t]}$, and therefore, again, it is not clear how one can address the problem via the continuous mapping approach.

Fix $h \in C_b(\mathcal{E})$. Because Z^n can be written as $\Psi^n(N^n)$ for some measurable function Ψ^n from $\mathcal{M}$ to $\mathcal{E}$, we have, from the second equality in Theorem 4, that, with $(m, F) = (n, nh \circ \Psi^n)$,

$$-\frac{1}{n} \log E e^{-nh(Z^n)} = \inf E \left[\int_0^T \left(\lambda \ell(\varphi_1^n(s)) + \mu \ell(\varphi_2^n(s)) + \theta \int_{\mathbb{R}_+} \ell(\varphi_3^n(s, y)) dy \right) ds + h(\bar{Z}^n) \right], \quad (14)$$

where the infimum is taken over all $\varphi^n = (\varphi_i^n)_{i=1,2,3} \in \bar{\mathcal{A}}_b$ and $\bar{Z}^n = (\bar{X}^n, \bar{Y}^n)$ solves

$$\bar{X}^n(t) = x_n + \frac{1}{n} N_1^{n\varphi_1^n}(t) - \frac{1}{n} \int_0^t \mathbf{1}_{\{\bar{X}^n(s-) \neq 0\}} N_2^{n\varphi_2^n}(ds) - \frac{1}{n} \int_{[0, t] \times \mathbb{R}_+} \mathbf{1}_{[0, (\bar{X}^n(s-) - \frac{1}{n})^+]}(y) N_3^{n\varphi_3^n}(ds dy) \quad (15)$$

$$= \Gamma \left(x_n + \frac{1}{n} N_1^{n\varphi_1^n}(\cdot) - \frac{1}{n} N_2^{n\varphi_2^n}(\cdot) - \frac{1}{n} \int_{[0, \cdot] \times \mathbb{R}_+} \mathbf{1}_{[0, (\bar{X}^n(s-) - \frac{1}{n})^+]}(y) N_3^{n\varphi_3^n}(ds dy) \right)(t), \quad (16)$$

$$\bar{Y}^n(t) = \frac{1}{n} \int_{[0, t] \times \mathbb{R}_+} \mathbf{1}_{[0, (\bar{X}^n(s-) - \frac{1}{n})^+]}(y) N_3^{n\varphi_3^n}(ds dy). \quad (17)$$

In the proof of both the upper and lower bounds (following Lemma 3 and Proposition 1), it is sufficient to consider a sequence $\{\varphi^n\} \subset \bar{\mathcal{A}}_b$ that satisfies the following uniform bound for some $M_0 < \infty$:

$$\sup_{n \in \mathbb{N}} \int_0^T \left(\ell(\varphi_1^n(s)) + \ell(\varphi_2^n(s)) + \int_{\mathbb{R}_+} \ell(\varphi_3^n(s, y)) dy \right) ds \leq M_0, \quad \text{a.s. } P. \quad (18)$$

In the rest of this section, we study tightness and convergence properties of controlled processes $\{\bar{Z}^n\}$ that are driven by controls $\{\varphi^n = (\varphi_1^n, \varphi_2^n, \varphi_3^n)\}$ that satisfy the a.s. bound (18).

For $0 \leq K < \infty$, let $\mathcal{S}_K$ be the collection of all triplets $g = (g_1, g_2, g_3)$, where $g_1, g_2 : [0, T] \rightarrow \mathbb{R}_+$, $g_3 : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are measurable maps such that

$$\int_0^T \left(\ell(g_1(s)) + \ell(g_2(s)) + \int_{\mathbb{R}_+} \ell(g_3(s, y)) dy \right) ds \leq K.$$

An element $g = (g_1, g_2, g_3) \in \mathcal{S}_K$ can be identified with elements $v^g \in \mathcal{M}$, where

$$v_i^g(A) \doteq \int_A g_i(s) ds, \quad v_3^g(A \times B) \doteq \int_{A \times B} g_3(s, y) ds dy, \quad i = 1, 2, A \in \mathcal{B}([0, T]), B \in \mathcal{B}(\mathbb{R}_+). \quad (19)$$

With this identification $\mathcal{S}_K$ is a compact metric space (cf. Budhiraja et al. [3, lemma A.1]). Let $\tilde{\mathcal{A}}_{b,K} \doteq \{\varphi \in \tilde{\mathcal{A}}_b : \varphi(\omega) \in \mathcal{S}_K \text{ a.s.}\}$.

For $\varphi^n \in \tilde{\mathcal{A}}_b$, define the compensated processes

$$\begin{aligned} \tilde{N}_1^{n\varphi^n}(ds) &\doteq N_1^{n\varphi^n}(ds) - \lambda n \varphi_1^n(s) ds, \\ \tilde{N}_2^{n\varphi^n}(ds) &\doteq N_2^{n\varphi^n}(ds) - \mu n \varphi_2^n(s) ds, \\ \tilde{N}_3^{n\varphi^n}(ds dy) &\doteq N_3^{n\varphi^n}(ds dy) - \theta n \varphi_3^n(s, y) ds dy. \end{aligned}$$

Then, $\tilde{N}_1^{n\varphi^n}([0, t])$, $\tilde{N}_2^{n\varphi^n}([0, t])$, and $\tilde{N}_3^{n\varphi^n}([0, t] \times A)$ are $\{\mathcal{F}_t\}$ -martingales for $A \in \mathcal{B}(\mathbb{R}_+)$ with $E(v_3^{\varphi^n}([0, T] \times A)) < \infty$.

The following lemma summarizes some elementary properties of ℓ . For part (a), we refer to Budhiraja et al. [5, lemma 3.1], and part (b) is an easy calculation that is omitted.

Lemma 1.

- For each $\beta > 1$, there exists $\gamma(\beta) \in (0, \infty)$ such that $\gamma(\beta) \rightarrow 0$ as $\beta \rightarrow \infty$ and $x \leq \gamma(\beta)\ell(x)$ for $x \geq \beta$.
- For $x \geq 0$, $x \leq \ell(x) + 2$.

Now we have all the ingredients to study tightness and convergence properties of controlled processes $\{\tilde{Z}^n\}$ that are driven by controls $\{\varphi^n = (\varphi_1^n, \varphi_2^n, \varphi_3^n)\}$ that satisfy (18).

Lemma 2. Suppose that, for some $M_0 < \infty$, $\{\varphi^n\} \subset \tilde{\mathcal{A}}_{b,M_0}$.

- The sequence of random variables $\{(\varphi^n, \tilde{Z}^n)\}$ is a tight collection of $\mathcal{M} \times \mathcal{E}$ valued random variables.
- Suppose $(\varphi^n, \tilde{Z}^n)$ converges along a subsequence in distribution to $(\varphi, \tilde{Z})$ given on some probability space $(\Omega^*, \mathcal{F}^*, P^*)$, where $\tilde{Z} = (\tilde{X}, \tilde{Y})$. Then, a.s. P^* , $\tilde{Z} \in \mathcal{C}([0, T] : \mathbb{R}_+^2)$, and

$$\tilde{X}(t) = \Gamma \left(x + \lambda \int_0^t \varphi_1(s) ds - \mu \int_0^t \varphi_2(s) ds - \theta \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{[0, \tilde{X}(s)]}(z) \varphi_3(s, y) ds dy \right) (t), \quad (20)$$

$$\tilde{Y}(t) = \theta \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{[0, \tilde{X}(s)]}(z) \varphi_3(s, y) ds dy, \quad t \in [0, T]. \quad (21)$$

Proof. By assumption, $\{\varphi^n\}$ is a sequence of $\mathcal{S}_{M_0}$ -valued random variables and is, therefore, automatically tight. Next, we argue $\mathcal{C}$ -tightness of $\{\tilde{Z}^n\}$. Let $\tilde{X}^n(t)$ be the process appearing in the argument of Γ in (16), namely

$$\tilde{X}^n(t) \doteq x_n + \frac{1}{n} N_1^{n\varphi^n}(t) - \frac{1}{n} N_2^{n\varphi^n}(t) - \frac{1}{n} \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{[0, (\tilde{X}^n(s) - \frac{1}{n})^+]}(y) N_3^{n\varphi^n}(ds dy). \quad (22)$$

We argue tightness of $\{\tilde{X}^n\}$ and get tightness of $\{\tilde{X}^n\}$ by continuity of the Skorohod map Γ . Write

$$\begin{aligned} \frac{1}{n} N_1^{n\varphi^n}(t) &= \lambda \int_0^t \varphi_1^n(s) ds + \frac{1}{n} \tilde{N}_1^{n\varphi^n}(t), \quad \frac{1}{n} N_2^{n\varphi^n}(t) = \mu \int_0^t \varphi_2^n(s) ds + \frac{1}{n} \tilde{N}_2^{n\varphi^n}(t), \\ \frac{1}{n} \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{[0, (\tilde{X}^n(s) - \frac{1}{n})^+]}(y) N_3^{n\varphi^n}(ds dy) &= \theta \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{[0, (\tilde{X}^n(s) - \frac{1}{n})^+]}(y) \varphi_3^n(s, y) ds dy \\ &\quad + \frac{1}{n} \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{[0, (\tilde{X}^n(s) - \frac{1}{n})^+]}(y) \tilde{N}_3^{n\varphi^n}(ds dy). \end{aligned}$$

From (16), we have

$$\begin{aligned} E|\tilde{X}^n|_{*,t}^2 &\leq \kappa_1 E|\tilde{X}^n|_{*,t}^2 \\ &\leq \kappa_2 + \kappa_2 E \left[\int_0^t \left(\varphi_1^n(s) + \varphi_2^n(s) + \int_{\mathbb{R}_+} \mathbf{1}_{[0, \tilde{X}^n(s)]}(y) \varphi_3^n(s, y) dy \right) ds \right]^2 \\ &\quad + \frac{\kappa_2}{n} E \int_0^t \left(\varphi_1^n(s) + \varphi_2^n(s) + \int_{\mathbb{R}_+} \mathbf{1}_{[0, \tilde{X}^n(s)]}(y) \varphi_3^n(s, y) dy \right) ds \\ &\leq \kappa_2 + \kappa_2 E \left[\int_0^t \left(\ell(\varphi_1^n(s)) + 2 + \ell(\varphi_2^n(s)) + 2 + \int_{\mathbb{R}_+} \mathbf{1}_{[0, \tilde{X}^n(s)]}(y) (\ell(\varphi_3^n(s, y)) + 2) dy \right) ds \right]^2 \\ &\quad + \frac{\kappa_2}{n} E \int_0^t \left(\ell(\varphi_1^n(s)) + 2 + \ell(\varphi_2^n(s)) + 2 + \int_{\mathbb{R}_+} \mathbf{1}_{[0, \tilde{X}^n(s)]}(y) (\ell(\varphi_3^n(s, y)) + 2) dy \right) ds \\ &\leq \kappa_3 + \kappa_3 \int_0^t E|\tilde{X}^n|_{*,s}^2 ds, \end{aligned}$$

where the first inequality uses the explicit expression for the Skorohod map Γ in (2), the second uses Doob's inequality, the third uses Lemma 1(b), and the last uses the fact that (18) is satisfied. It then follows from Gronwall's inequality that

$$\sup_{n \in \mathbb{N}} E|\tilde{X}^n|_{*,T}^2 < \infty, \quad \sup_{n \in \mathbb{N}} E|\tilde{X}^n|_{*,T}^2 < \infty. \quad (23)$$

This gives tightness of $\{\tilde{X}^n(t)\}$ for each fixed $t \in [0, T]$. As for the fluctuation of $\tilde{X}^n$, let $\mathcal{T}^\delta$ be the collection of all $[0, T - \delta]$ -valued stopping times τ for each $\delta \in [0, T]$. In order to argue tightness of $\{\tilde{X}^n\}$, by the Aldous–Kurtz tightness criterion (cf. Kurtz [14, theorem 2.7]), it suffices to show that

$$\lim_{\delta \rightarrow 0} \sup_{n \rightarrow \infty} \sup_{\tau \in \mathcal{T}^\delta} E|\tilde{X}^n(\tau + \delta) - \tilde{X}^n(\tau)| = 0. \quad (24)$$

From (22), it follows that, for every $M \in (0, \infty)$,

$$\begin{aligned} E|\tilde{X}^n(\tau + \delta) - \tilde{X}^n(\tau)| &\leq \kappa_4 E \int_\tau^{\tau+\delta} \left(\varphi_1^n(s) + \varphi_2^n(s) + \int_{\mathbb{R}_+} \mathbf{1}_{[0, \tilde{X}^n(s)]}(y) \varphi_3^n(s, y) dy \right) ds \\ &\leq \kappa_4 E \int_\tau^{\tau+\delta} \left(\varphi_1^n(s) \mathbf{1}_{\{\varphi_1^n(s) > M\}} + \varphi_2^n(s) \mathbf{1}_{\{\varphi_2^n(s) > M\}} + \int_{\mathbb{R}_+} \mathbf{1}_{[0, \tilde{X}^n(s)]}(y) \varphi_3^n(s, y) \mathbf{1}_{\{\varphi_3^n(s, y) > M\}} dy \right) ds \\ &\quad + \kappa_4 E \int_\tau^{\tau+\delta} \left(M + M + \int_{\mathbb{R}_+} \mathbf{1}_{[0, \tilde{X}^n(s)]}(y) M dy \right) ds \\ &\leq \kappa_5 \gamma(M) + \kappa_5 M \delta \end{aligned}$$

where the third inequality follows from Lemma 1(a), (18), and (23). Therefore,

$$\lim_{\delta \rightarrow 0} \sup_{n \rightarrow \infty} \sup_{\tau \in \mathcal{T}^\delta} E|\tilde{X}^n(\tau + \delta) - \tilde{X}^n(\tau)| \leq \kappa_5 \gamma(M).$$

Taking $M \rightarrow \infty$ gives (24). Combining this with (23) and (16), we have tightness of $\{(\tilde{X}^n, \bar{X}^n)\}$. A similar estimate gives tightness of $\{\tilde{Y}^n\}$. The $\mathcal{C}$ -tightness of $\{(\tilde{X}^n, \bar{Z}^n)\}$ is clear as its jump size is $O(\frac{1}{n})$.

Suppose now that $(\varphi^n, \tilde{X}^n, \bar{Z}^n)$ converge in distribution along a subsequence in $\mathcal{M} \times \mathcal{D}([0, T] : \mathbb{R}) \times \mathcal{E}$ to $(\varphi, \tilde{X}, \bar{Z})$ given on some probability space $(\Omega^*, \mathcal{F}^*, P^*)$, where $\bar{Z} = (\bar{X}, \bar{Y})$. Assume without loss of generality that the convergence holds along the whole sequence. From the $\mathcal{C}$ -tightness of $\{(\tilde{X}^n, \bar{Z}^n)\}$, we have $(\tilde{X}, \bar{Z}) \in \mathcal{C}([0, T] : \mathbb{R}) \times \mathcal{C}([0, T] : \mathbb{R}_+^2)$ a.s. P^* . Using Doob's inequality, Lemma 1(b), (18) and (23), we have

$$\begin{aligned} E \left[\left(\frac{1}{n} \tilde{N}_1^{n\varphi_1^n}(\cdot) \right)^2 + \left(\frac{1}{n} \tilde{N}_2^{n\varphi_2^n}(\cdot) \right)^2 + \left(\frac{1}{n} \int_{[0, \cdot] \times \mathbb{R}_+} \mathbf{1}_{[0, (\tilde{X}^n(s) - \frac{1}{n})^+]}(y) \tilde{N}_3^{n\varphi_3^n}(ds dy) \right)^2 \right]_{*,T} \\ \leq \frac{\kappa_6}{n} E \int_0^T \left(\varphi_1^n(s) + \varphi_2^n(s) + \int_{\mathbb{R}_+} \mathbf{1}_{[0, \tilde{X}^n(s)]}(y) \varphi_3^n(s, y) dy \right) ds \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, where, for $f : [0, T] \rightarrow \mathbb{R}$, $|f|_{*,T} \doteq \sup_{0 \leq s \leq T} |f(s)|$. By appealing to the Skorohod representation theorem, we can further assume without loss of generality that the preceding convergence, $(\varphi^n, \tilde{X}^n, \bar{Z}^n) \rightarrow (\varphi, \tilde{X}, \bar{Z})$ and

$(\tilde{X}, \tilde{Z}) \in \mathcal{C}([0, T] : \mathbb{R}) \times \mathcal{C}([0, T] : \mathbb{R}_+^2)$ hold a.s. P^* , namely on some set $G \in \mathcal{F}^*$ such that $P^*(G^c) = 0$. Fix $\omega^* \in G$. The rest of the argument is made for such an ω^* which is suppressed from the notation. In order to prove (20) and (21), it suffices to show

$$\int_0^t \varphi_1^n(s) ds + \int_0^t \varphi_2^n(s) ds \rightarrow \int_0^t \varphi_1(s) ds + \int_0^t \varphi_2(s) ds, \quad (25)$$

$$\int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{[0, (\tilde{X}^n(s-) - \frac{1}{n})^+]}(y) \varphi_3^n(s, y) ds dy \rightarrow \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{[0, \tilde{X}(s)]}(y) \varphi_3(s, y) ds dy, \quad (26)$$

as $n \rightarrow \infty$ for each $t \in [0, T]$. The convergence in (25) is immediate from that of $\varphi^n \rightarrow \varphi \in \mathcal{M}$. Next, note that

$$\begin{aligned} & \left| \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{[0, (\tilde{X}^n(s-) - \frac{1}{n})^+]}(y) \varphi_3^n(s, y) ds dy - \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{[0, \tilde{X}(s)]}(y) \varphi_3(s, y) ds dy \right| \\ & \leq \int_{[0,t] \times \mathbb{R}_+} \left| \mathbf{1}_{[0, (\tilde{X}^n(s-) - \frac{1}{n})^+]}(y) - \mathbf{1}_{[0, \tilde{X}(s)]}(y) \right| \varphi_3^n(s, y) ds dy \\ & \quad + \left| \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{[0, \tilde{X}(s)]}(y) (\varphi_3^n(s, y) - \varphi_3(s, y)) ds dy \right|. \end{aligned}$$

For $M \in (0, \infty)$, it follows from Lemma 1(a) and (18) that

$$\begin{aligned} & \int_{[0,t] \times \mathbb{R}_+} \left| \mathbf{1}_{[0, (\tilde{X}^n(s-) - \frac{1}{n})^+]}(y) - \mathbf{1}_{[0, \tilde{X}(s)]}(y) \right| \varphi_3^n(s, y) ds dy \\ & \leq \int_{[0,t] \times \mathbb{R}_+} \varphi_3^n(s, y) \mathbf{1}_{\{\varphi_3^n(s, y) > M\}} ds dy + \int_{[0,t] \times \mathbb{R}_+} \left| \mathbf{1}_{[0, (\tilde{X}^n(s-) - \frac{1}{n})^+]}(y) - \mathbf{1}_{[0, \tilde{X}(s)]}(y) \right| M ds dy \\ & \leq \gamma(M) M_0 + MT |\tilde{X}^n - \tilde{X}|_{\infty, T} + MT/n, \end{aligned}$$

which converges to zero on sending $n \rightarrow \infty$ and then $M \rightarrow \infty$. Because the Lebesgue measure of $\{(s, y) \in [0, T] \times \mathbb{R}_+ : y = \tilde{X}(s)\}$ is zero, it follows from $\varphi^n \rightarrow \varphi \in \mathcal{M}$ that

$$\left| \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{[0, \tilde{X}(s)]}(y) (\varphi_3^n(s, y) - \varphi_3(s, y)) ds dy \right| \rightarrow 0$$

as $n \rightarrow \infty$. Combining these three displays gives (26). This completes the proof. Q.E.D.

3.2. Proof of Laplace Upper Bound

In this section, we prove (12) for a fixed $h \in C_b(\mathcal{E})$. We begin by giving an alternative representation of I_T that is convenient for the proof of this inequality. For $(\xi, \zeta) \in \mathcal{C}([0, T] : \mathbb{R}_+^2)$, let $\tilde{\mathcal{U}}(\xi, \zeta)$ be the collection of all measurable $\tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3)$ such that $\tilde{\varphi}_i : [0, T] \rightarrow \mathbb{R}_+$, $i = 1, 2$, $\tilde{\varphi}_3 : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and the following equations are satisfied for $t \in [0, T]$:

$$\begin{aligned} \xi(t) &= \Gamma \left(x_0 + \lambda \int_0^t \tilde{\varphi}_1(s) ds - \mu \int_0^t \tilde{\varphi}_2(s) ds - \theta \int_{[0, \cdot] \times \mathbb{R}_+} \tilde{\varphi}_3(s, y) \mathbf{1}_{[0, \xi(s)]}(y) ds dy \right)(t), \\ \zeta(t) &= \theta \int_{[0, t] \times \mathbb{R}_+} \tilde{\varphi}_3(s, y) \mathbf{1}_{[0, \xi(s)]}(y) ds dy. \end{aligned}$$

Define for $(\xi, \zeta) \in \mathcal{C}([0, T] : \mathbb{R}_+^2)$

$$\tilde{I}_T(\xi, \zeta) = \inf_{\tilde{\varphi} \in \tilde{\mathcal{U}}(\xi, \zeta)} \left\{ \lambda \int_0^T \ell(\tilde{\varphi}_1(s)) ds + \mu \int_0^T \ell(\tilde{\varphi}_2(s)) ds + \theta \int_{[0, T] \times \mathbb{R}_+} \ell(\tilde{\varphi}_3(s, y)) ds dy \right\}. \quad (27)$$

We set $\tilde{I}_T(\xi, \zeta) \doteq \infty$ if $\tilde{\mathcal{U}}(\xi, \zeta)$ is empty or $(\xi, \zeta) \in \mathcal{D}([0, T] : \mathbb{R}_+^2) \setminus \mathcal{C}([0, T] : \mathbb{R}_+^2)$.

Remark 3. Note that, if, for $(\xi, \zeta) \in \mathcal{C}([0, T] : \mathbb{R}_+^2)$, $\tilde{I}_T(\xi, \zeta) < \infty$ and $\tilde{\varphi} \in \tilde{\mathcal{U}}(\xi, \zeta)$ is such that its cost given by the right side in (27) is finite, then, from the superlinearity of ℓ , we must have that

$$\int_0^T \tilde{\varphi}_i(s) ds < \infty, i = 1, 2, \text{ and } \int_{[0, T] \times [0, M]} \tilde{\varphi}_3(s, y) ds dy < \infty \text{ for all } M < \infty.$$

This says that ζ is absolutely continuous, and $\xi = \Gamma(\tilde{\xi})$ for some $\tilde{\xi}$, which is absolutely continuous as well. It follows that $t \mapsto \inf_{0 \leq s \leq t} \tilde{\xi}(s) \wedge 0$ is also absolutely continuous. Thus, we have that ζ, ξ , and $\xi - \tilde{\xi}$ are all absolutely continuous. This fact is used in the proof of Proposition 1.

Then, we have the following result.

Lemma 3. For all $T \geq 0$ and $(\xi, \zeta) \in \mathcal{D}([0, T] : \mathbb{R}_+^2)$, $I_T(\xi, \zeta) = \tilde{I}_T(\xi, \zeta)$.

Proof. Consider $T \geq 0$ and, for $(\xi, \zeta) \in \mathcal{C}([0, T] : \mathbb{R}_+^2)$, $\tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3) \in \tilde{\mathcal{U}}(\xi, \zeta)$. Define $\varphi_3 : [0, T] \rightarrow \mathbb{R}_+$ by

$$\varphi_3(s) \doteq \mathbf{1}_{\{\xi(s) \neq 0\}} \frac{1}{\xi(s)} \int_{[0, \xi(s)]} \tilde{\varphi}_3(s, y) dy + \mathbf{1}_{\{\xi(s) = 0\}}, \quad s \in [0, T].$$

Note that

$$\int_{[0, t] \times \mathbb{R}_+} \tilde{\varphi}_3(s, y) \mathbf{1}_{[0, \xi(s)]}(y) ds dy = \int_{[0, t]} \xi(s) \varphi_3(s) ds \quad \text{for all } t \in [0, T],$$

and, by convexity of ℓ ,

$$\begin{aligned} \int_{[0, T] \times \mathbb{R}_+} \ell(\tilde{\varphi}_3(s, y)) ds dy &\geq \int_{[0, T] \times \mathbb{R}_+} \mathbf{1}_{\{\xi(s) \neq 0\}} \mathbf{1}_{[0, \xi(s)]}(y) \ell(\tilde{\varphi}_3(s, y)) ds dy \\ &= \int_{[0, T]} \mathbf{1}_{\{\xi(s) \neq 0\}} \xi(s) \left(\frac{1}{\xi(s)} \int_{[0, \xi(s)]} \ell(\tilde{\varphi}_3(s, y)) dy \right) ds \\ &\geq \int_{[0, T]} \mathbf{1}_{\{\xi(s) \neq 0\}} \xi(s) \ell \left(\frac{1}{\xi(s)} \int_{[0, \xi(s)]} \tilde{\varphi}_3(s, y) dy \right) ds \\ &= \int_{[0, T]} \xi(s) \ell(\varphi_3(s)) ds. \end{aligned}$$

Clearly, $\varphi \doteq (\tilde{\varphi}_1, \tilde{\varphi}_2, \varphi_3) \in \mathcal{U}(\xi, \zeta)$. This shows $I_T(\xi, \zeta) \leq \tilde{I}_T(\xi, \zeta)$. The reverse inequality is immediate on observing that, if $(\varphi_1, \varphi_2, \varphi_3) \in \mathcal{U}(\xi, \zeta)$, then with $\tilde{\varphi}_3(s, y) \doteq \varphi_3(s)$ for $y \in [0, 1]$, $(\varphi_1, \varphi_2, \tilde{\varphi}_3) \in \tilde{\mathcal{U}}(\xi, \zeta)$, and the costs for the two controls are identical. The result follows. Q.E.D.

We now return to the proof of (12). For each n , let $\varphi^n \doteq (\varphi_1^n, \varphi_2^n, \varphi_3^n)$ be $\frac{1}{n}$ -optimal in (14), namely

$$-\frac{1}{n} \log E \exp \{-nh(Z^n)\} \geq E \left[h(\bar{Z}^n) + \int_0^T \left(\lambda \ell(\varphi_1^n(s)) + \mu \ell(\varphi_2^n(s)) + \theta \int_{\mathbb{R}_+} \ell(\varphi_3^n(s, y)) dy \right) ds \right] - \frac{1}{n}, \quad (28)$$

where $\bar{Z}^n$ is as introduced as follows (14). Using the boundedness of h , by a standard localization argument (see Budhiraja et al. [6, proof of theorem 4.2]), we can assume without loss of generality that (18) is satisfied for some $M_0 < \infty$.

We have from Lemma 2(a) that $\{(\varphi^n, \bar{Z}^n)\}$ is tight in $\mathcal{M} \times \mathcal{E}$. Assume without loss of generality that $(\varphi^n, \bar{Z}^n)$ converges along the whole sequence, in distribution, to $(\varphi, \bar{Z}) \in \mathcal{M} \times \mathcal{E}$ given on some probability space $(\Omega^*, \mathcal{F}^*, P^*)$, where $\bar{Z} = (\bar{X}, \bar{Y})$. By Lemma 2(b), we have $\bar{Z} \in \mathcal{C}([0, T] : \mathbb{R}_+^2)$ and $\varphi \in \tilde{\mathcal{U}}(\bar{X}, \bar{Y})$ a.s. P^* . From (28), Fatou's lemma and the lower semicontinuity established in Budhiraja et al. [4, lemma A.1] (see proof of (A.1) therein), it follows that

$$\liminf_{n \rightarrow \infty} -\frac{1}{n} \log E \exp \{-nh(Z^n)\} \geq E \left[h(\bar{Z}) + \int_0^T \left(\lambda \ell(\varphi_1(s)) + \mu \ell(\varphi_2(s)) + \theta \int_{\mathbb{R}_+} \ell(\varphi_3(s, y)) dy \right) ds \right]. \quad (29)$$

Thus, we have that

$$\liminf_{n \rightarrow \infty} -\frac{1}{n} \log E \exp \{-nh(Z^n)\} \geq \inf_{\phi \in \mathcal{E}} E \left[h(\phi) + \tilde{I}_T(\phi) \right] = \inf_{\phi \in \mathcal{E}} E \left[h(\phi) + I_T(\phi) \right],$$

where the last equality is from Lemma 3. The proof of the upper bound is complete. Q.E.D.

3.3. Proof of Laplace Lower Bound

In this section, we prove Inequality (13) for a fixed $h \in C_b(\mathcal{E})$.

We begin by establishing a key uniqueness property.

Proposition 1. Fix $(\xi, \zeta) \in \mathcal{C}([0, T] : \mathbb{R}_+^2)$ with $\tilde{I}_T(\xi, \zeta) < \infty$ and let $\varphi \in \tilde{\mathcal{U}}(\xi, \zeta)$ be such that the cost associated with φ given by the right side of (27) is finite. If $(\tilde{\xi}, \tilde{\zeta}) \in \mathcal{C}([0, T] : \mathbb{R}_+^2)$ is another pair such that $\varphi \in \tilde{\mathcal{U}}(\tilde{\xi}, \tilde{\zeta})$, then $(\xi, \zeta) = (\tilde{\xi}, \tilde{\zeta})$.

We first complete the proof of the lower bound assuming that Proposition 1 holds. Fix $\sigma \in (0, 1)$ and choose $(\xi^*, \zeta^*) \in \mathcal{C}([0, T] : \mathbb{R}_+^2)$ such that

$$h(\xi^*, \zeta^*) + I_T(\xi^*, \zeta^*) \leq \inf_{(\xi, \zeta) \in \mathcal{E}} \{h(\xi, \zeta) + I_T(\xi, \zeta)\} + \sigma.$$

Because $\tilde{I}_T(\xi^*, \zeta^*) = I_T(\xi^*, \zeta^*) < \infty$, we can choose $\varphi^* \in \tilde{\mathcal{U}}(\xi^*, \zeta^*)$ such that

$$\int_0^T \left(\lambda \ell(\varphi_1^*(s)) + \mu \ell(\varphi_2^*(s)) + \theta \int_{\mathbb{R}_+} \ell(\varphi_3^*(s, y)) dy \right) ds \leq I_T(\xi^*, \zeta^*) + \sigma.$$

Define the deterministic controls

$$\varphi_i^n(s) \doteq \frac{1}{n} \mathbf{1}_{\{\varphi_i^*(s) \leq 1/n\}} + \varphi_i^*(s) \mathbf{1}_{\{1/n < \varphi_i^*(s) < n\}} + n \mathbf{1}_{\{\varphi_i^*(s) \geq n\}}$$

for $i = 1, 2$ and $s \in [0, T]$, and

$$\varphi_3^n(s, y) \doteq \frac{1}{n} \mathbf{1}_{\{\varphi_3^*(s, y) \leq 1/n, y \leq n\}} + \varphi_3^*(s, y) \mathbf{1}_{\{1/n < \varphi_3^*(s, y) < n, y \leq n\}} + n \mathbf{1}_{\{\varphi_3^*(s, y) \geq n, y \leq n\}} + \mathbf{1}_{\{y > n\}}$$

for $(s, y) \in [0, T] \times \mathbb{R}_+$. Then, $\varphi^n \doteq (\varphi_1^n, \varphi_2^n, \varphi_3^n) \in \bar{\mathcal{A}}_b$. We use that $\ell(z) \geq 0$ and $\ell(1) = 0$ and also that $\ell(z)$ is increasing for $z > 1$ and decreasing for $z < 1$. Then, from (14) and these properties, we have

$$\begin{aligned} -\frac{1}{n} \log E \exp\{-nh(Z^n)\} &\leq E \left[h(\bar{Z}^n) + \int_0^T \left(\lambda \ell(\varphi_1^n(s)) + \mu \ell(\varphi_2^n(s)) + \theta \int_{\mathbb{R}_+} \ell(\varphi_3^n(s, y)) dy \right) ds \right] \\ &\leq E \left[h(\bar{Z}^n) + \int_0^T \left(\lambda \ell(\varphi_1^*(s)) + \mu \ell(\varphi_2^*(s)) + \theta \int_{\mathbb{R}_+} \ell(\varphi_3^*(s, y)) dy \right) ds \right]. \end{aligned} \quad (30)$$

Note that (18) is satisfied with M_0 replaced by $M \doteq (I_T(\xi^*, \zeta^*) + 1)/\min(\lambda, \mu, \theta)$. It then follows from Lemma 2(a) that $\{(\varphi^n, \bar{Z}^n)\}$ is tight. Clearly, $\varphi^n \rightarrow \varphi^* \in \mathcal{M}$. By Lemma 2(b), if $\bar{Z}^n$ converges along a subsequence to $\bar{Z}$, then $\bar{Z}$ must satisfy (20) and (21) with φ replaced with φ^* , namely $\varphi^* \in \tilde{\mathcal{U}}(\bar{Z})$. From the uniqueness in Proposition 1, it follows that $\bar{Z} = (\xi^*, \zeta^*)$. Finally, from (30), we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} -\frac{1}{n} \log E \exp\{-nh(Z^n)\} &\leq \limsup_{n \rightarrow \infty} E \left[h(\bar{Z}^n) + \int_0^T \left(\lambda \ell(\varphi_1^*(s)) + \mu \ell(\varphi_2^*(s)) + \theta \int_{\mathbb{R}_+} \ell(\varphi_3^*(s, y)) dy \right) ds \right] \\ &= h(\xi^*, \zeta^*) + \int_0^T \left(\lambda \ell(\varphi_1^*(s)) + \mu \ell(\varphi_2^*(s)) + \theta \int_{\mathbb{R}_+} \ell(\varphi_3^*(s, y)) dy \right) ds \\ &\leq h(\xi^*, \zeta^*) + I_T(\xi^*, \zeta^*) + \sigma \\ &\leq \inf_{(\xi, \zeta) \in \mathcal{E}} \{h(\xi, \zeta) + I_T(\xi, \zeta)\} + 2\sigma, \end{aligned}$$

which completes the proof of the lower bound. Q.E.D.

We now return to the proof of Proposition 1.

Proof of Proposition 1. Suppose that $(\tilde{\xi}, \tilde{\zeta})$ is another pair such that $\varphi \in \tilde{\mathcal{U}}(\tilde{\xi}, \tilde{\zeta})$. It suffices to show that $\tilde{\xi} = \xi$. Write $\xi(t) = \psi(t) + \eta(t)$, $\tilde{\xi}(t) = \tilde{\psi}(t) + \tilde{\eta}(t)$, where $\psi, \tilde{\psi}$ are the unconstrained processes defined by

$$\begin{aligned} \psi(t) &= x_0 + \lambda \int_0^t \varphi_1(s) ds - \mu \int_0^t \varphi_2(s) ds - \theta \int_{[0, t] \times \mathbb{R}_+} \varphi_3(s, y) \mathbf{1}_{[0, \xi(s)]}(y) ds dy, \\ \tilde{\psi}(t) &= x_0 + \lambda \int_0^t \varphi_1(s) ds - \mu \int_0^t \varphi_2(s) ds - \theta \int_{[0, t] \times \mathbb{R}_+} \varphi_3(s, y) \mathbf{1}_{[0, \tilde{\xi}(s)]}(y) ds dy, \end{aligned}$$

and $\eta, \tilde{\eta}$ are the reflection terms such that (see, e.g., Karatzas and Shreve [11, section 3.6.C])

$$\begin{aligned} \eta(0) &= 0, \eta(t) \text{ is nondecreasing and } \int_0^T \xi(t) d\eta(t) = 0, \\ \tilde{\eta}(0) &= 0, \tilde{\eta}(t) \text{ is nondecreasing and } \int_0^T \tilde{\xi}(t) d\tilde{\eta}(t) = 0. \end{aligned}$$

Recall from Remark 3 that $\xi, \tilde{\xi}, \psi, \tilde{\psi}, \eta, \tilde{\eta}$ are absolutely continuous. For an absolutely continuous function f , we denote its almost everywhere (a.e.) derivative as f' . Then, for $t \in [0, T]$,

$$\begin{aligned} [\xi(t) - \tilde{\xi}(t)]^2 &= 2 \int_0^t (\xi(s) - \tilde{\xi}(s)) (\xi'(s) - \tilde{\xi}'(s)) ds \\ &= 2 \int_0^t (\xi(s) - \tilde{\xi}(s)) (\psi'(s) - \tilde{\psi}'(s)) ds + 2 \int_0^t (\xi(s) - \tilde{\xi}(s)) d(\eta(s) - \tilde{\eta}(s)). \end{aligned}$$

Note that, for a.e. $s \in [0, T]$, whenever $\xi(s) > \tilde{\xi}(s)$, $\xi(s) = \tilde{\xi}(s)$, or $\xi(s) < \tilde{\xi}(s)$,

$$(\xi(s) - \tilde{\xi}(s)) (\psi'(s) - \tilde{\psi}'(s)) = -\theta (\xi(s) - \tilde{\xi}(s)) \int_{\mathbb{R}_+} \varphi_3(s, y) [\mathbf{1}_{[0, \xi(s)]}(y) - \mathbf{1}_{[0, \tilde{\xi}(s)]}(y)] dy \leq 0.$$

Also, because $\tilde{\eta}$ is nondecreasing,

$$\begin{aligned} \int_0^t \mathbf{1}_{\{\xi(s) > \tilde{\xi}(s)\}} (\xi(s) - \tilde{\xi}(s)) d(\eta(s) - \tilde{\eta}(s)) &\leq \int_0^t \mathbf{1}_{\{\xi(s) > \tilde{\xi}(s)\}} (\xi(s) - \tilde{\xi}(s)) d\eta(s) \\ &\leq \int_0^t \mathbf{1}_{\{\xi(s) > 0\}} \xi(s) d\eta(s) \\ &= 0. \end{aligned}$$

Similarly,

$$\int_0^t \mathbf{1}_{\{\xi(s) < \tilde{\xi}(s)\}} (\xi(s) - \tilde{\xi}(s)) d(\eta(s) - \tilde{\eta}(s)) \leq 0.$$

Thus, $[\xi(t) - \tilde{\xi}(t)]^2 = 0$ for all $t \in [0, T]$. This completes the proof. Q.E.D.

4. Properties of the Rate Function

For the rest of this paper, we assume that $\lambda \geq \mu$. This property is not explicitly noted in the statements of various results.

In this section, we give a different representation for the rate function I_T and establish certain convexity properties of the rate function.

Let $\mathcal{C}_T$ be the collection of functions $(\xi, \zeta) \in \mathcal{C}([0, T] : \mathbb{R}_+^2)$ such that

- i. $\xi(0) = x_0, \zeta(0) = 0$.
- ii. ξ, ζ are absolutely continuous, and $\zeta'(t) = 0$ for a.e. $t \in [0, T]$ such that $\xi(t) = 0$.
- iii. ζ is nondecreasing.

We note that $I_T(\xi, \zeta) = \infty$ if $(\xi, \zeta) \notin \mathcal{C}_T$.

For $(x, p, q) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$, let

$$\begin{aligned} L(x, p, q) &\doteq \lambda \ell \left(\frac{\sqrt{(p+q)^2 + 4\lambda\mu} + (p+q)}{2\lambda} \right) + \mu \ell \left(\frac{\sqrt{(p+q)^2 + 4\lambda\mu} - (p+q)}{2\mu} \right) \\ &\quad + \theta x \ell \left(\frac{q}{\theta x} \right) \cdot \mathbf{1}_{\{x > 0\}} + \infty \cdot \mathbf{1}_{\{x = 0, q > 0\}}. \end{aligned}$$

The following lemma gives an alternative representation for $I_T(\xi, \zeta)$ for $(\xi, \zeta) \in \mathcal{C}_T$.

Lemma 4. For $(\xi, \zeta) \in \mathcal{C}_T$,

$$I_T(\xi, \zeta) = \int_0^T L(\xi(s), \xi'(s), \zeta'(s)) ds. \quad (31)$$

Proof. Fix $(\xi, \zeta) \in \mathcal{C}_T$. Recall

$$I_T(\xi, \zeta) = \inf_{\varphi \in \mathcal{U}(\xi, \zeta)} \left\{ \lambda \int_0^T \ell(\varphi_1(s)) ds + \mu \int_0^T \ell(\varphi_2(s)) ds + \theta \int_0^T \xi(s) \ell(\varphi_3(s)) ds \right\}. \quad (32)$$

We would like to find a $\varphi \in \mathcal{U}(\xi, \zeta)$ for which the preceding infimum is achieved.

Consider $t \in [0, T]$ such that $\xi'(t)$ and $\zeta'(t)$ exist. When $\xi(t) > 0$, from (5) we have

$$\xi'(t) + \zeta'(t) = \lambda \varphi_1(t) - \mu \varphi_2(t), \quad (33)$$

$$\zeta'(t) = \theta \varphi_3(t) \xi(t). \quad (34)$$

It is easy to check that, given these constraints, the $\varphi(t)$ that minimizes

$$\lambda \ell(\varphi_1(t)) + \mu \ell(\varphi_2(t)) + \theta \xi(t) \ell(\varphi_3(t))$$

must satisfy $\varphi_1(t)\varphi_2(t) = 1$, namely

$$\varphi_1(t) = \frac{\sqrt{(\xi'(t) + \zeta'(t))^2 + 4\lambda\mu} + (\xi'(t) + \zeta'(t))}{2\lambda}, \quad (35)$$

$$\varphi_2(t) = \frac{\sqrt{(\xi'(t) + \zeta'(t))^2 + 4\lambda\mu} - (\xi'(t) + \zeta'(t))}{2\mu}, \quad (36)$$

$$\varphi_3(t) = \frac{\zeta'(t)}{\theta \xi(t)}. \quad (37)$$

Because the Lebesgue measure of $\{t : \xi(t) = 0, \xi'(t) \neq 0\} = 0$, we must have that, for a.e. t on the set $\{\xi(t) = 0\}$, $\xi'(t) = 0$, $\zeta'(t) = 0$, and using (5),

$$\lambda \varphi_1(t) - \mu \varphi_2(t) \leq 0.$$

Using Lagrange multipliers, it is easy to check that, under this constraint, the minimizer of

$$\lambda \ell(\varphi_1(t)) + \mu \ell(\varphi_2(t)) + \theta \xi(t) \ell(\varphi_3(t)) = \lambda \ell(\varphi_1(t)) + \mu \ell(\varphi_2(t))$$

must satisfy $\varphi_1(t)\varphi_2(t) = 1$ and $\lambda \varphi_1(t) - \mu \varphi_2(t) = 0$, namely

$$\varphi_1(t) = \sqrt{\frac{\mu}{\lambda}}, \quad \varphi_2(t) = \sqrt{\frac{\lambda}{\mu}}, \quad \varphi_3(t) = 1. \quad (38)$$

Because $\lambda \varphi_1(t) - \mu \varphi_2(t) = 0$, the φ defined by (35)–(38) is in $\mathcal{U}(\xi, \zeta)$, and in fact, (5) holds without the Skorohod map on the right side of the first line. Plugging the minimizer φ into the cost in (32) gives the desired result. Q.E.D.

Remark 4.

i. The proof of Lemma 4 in fact shows that, for $(\xi, \zeta) \in \mathcal{C}_T$ and a.e. $s \in [0, T]$, $L(\xi(s), \xi'(s), \zeta'(s)) = \lambda \ell(\varphi_1(s)) + \mu \ell(\varphi_2(s)) + \theta \xi(s) \ell(\varphi_3(s))$,

where $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ is given by (35)–(38).

ii. The proof also shows that the representation (31) holds for any subinterval of $[0, T]$, namely

$$\inf_{\varphi \in \mathcal{U}(\xi, \zeta)} \left\{ \lambda \int_{t_1}^{t_2} \ell(\varphi_1(s)) ds + \mu \int_{t_1}^{t_2} \ell(\varphi_2(s)) ds + \theta \int_{t_1}^{t_2} \xi(s) \ell(\varphi_3(s)) ds \right\} = \int_{t_1}^{t_2} L(\xi(s), \xi'(s), \zeta'(s)) ds$$

for each $(\xi, \zeta) \in \mathcal{C}_T$ and $0 \leq t_1 < t_2 \leq T$.

For general $(\xi, \zeta) \in \mathcal{C}_T$, let $\mathcal{U}_+(\xi, \zeta)$ be the collection of $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ such that $\varphi_i : [0, T] \rightarrow \mathbb{R}_+$, $i = 1, 2, 3$ are measurable maps and

$$\begin{aligned} \xi(t) &= x_0 + \lambda \int_0^t \varphi_1(s) ds - \mu \int_0^t \varphi_2(s) ds - \theta \int_0^t \varphi_3(s) \xi(s) ds, \\ \zeta(t) &= \theta \int_0^t \varphi_3(s) \xi(s) ds. \end{aligned}$$

As noted in the last line of the proof of Lemma 4, we have, in fact, proved the following result.

Corollary 1. For $(\xi, \zeta) \in \mathcal{C}_T$,

$$I_T(\xi, \zeta) = \inf_{\varphi \in \mathcal{U}_+(\xi, \zeta)} \left\{ \lambda \int_0^T \ell(\varphi_1(s)) ds + \mu \int_0^T \ell(\varphi_2(s)) ds + \theta \int_0^T \xi(s) \ell(\varphi_3(s)) ds \right\}.$$

Moreover, the minimizer φ in the preceding infimum is given by (35)–(37) when $\xi(t) > 0$ and (38) when $\xi(t) = 0$.

The following lemma says that L and $I_T(\xi, \zeta)$ are convex.

Lemma 5. L is a convex function on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$, and I_T is a convex function on $\mathcal{C}_T$.

Proof. In view of Lemma 4, it suffices to show convexity of L . Note that the function

$$f_1(x) \doteq x \log \frac{\sqrt{x^2 + 4\lambda\mu} + x}{2\lambda} - \sqrt{x^2 + 4\lambda\mu}$$

is convex in $x \in \mathbb{R}$ because $f_1''(x) = \frac{1}{\sqrt{x^2 + 4\lambda\mu}} > 0$. Also,

$$\lambda \ell \left(\frac{\sqrt{(p+q)^2 + 4\lambda\mu} + (p+q)}{2\lambda} \right) + \mu \ell \left(\frac{\sqrt{(p+q)^2 + 4\lambda\mu} - (p+q)}{2\mu} \right) = f_1(p+q) + \lambda + \mu,$$

so the map taking (p, q) to the left side of the last display is convex on $\mathbb{R} \times \mathbb{R}_+$.

Next, consider the function $f_2(x, y) \doteq x \ell(\frac{y}{x})$ defined on the convex set $O \doteq \{(x, y) : x > 0, y \geq 0\}$. Note that the Hessian matrix of f_2 is

$$\begin{pmatrix} \frac{y}{x^2} & -\frac{1}{x} \\ -\frac{1}{x} & \frac{1}{y} \end{pmatrix}$$

On $\{(x, y) : x > 0, y > 0\}$, which is positive semidefinite. So f_2 is convex on $\{(x, y) : x > 0, y > 0\}$, and because f_2 is continuous on O , it is also convex on O . Let $\bar{f}_2$ be the extension of f_2 such that $\bar{f}_2(0, 0) = 0$ and $\bar{f}_2(x, y) = \infty$ for $(x, y) \in \mathbb{R}_+^2 \setminus (O \cup \{(0, 0)\})$. Note that, for $(x, y) = (0, 0)$, $(\tilde{x}, \tilde{y}) \in O$, and $r \in (0, 1)$,

$$\bar{f}_2((1-r)x + r\tilde{x}, (1-r)y + r\tilde{y}) = r\tilde{x}\ell(\tilde{y}/\tilde{x}) = (1-r)\bar{f}_2(x, y) + r\bar{f}_2(\tilde{x}, \tilde{y}).$$

Therefore, $\bar{f}_2$ is convex on $\mathbb{R}_+^2$ and so

$$(x, q) \mapsto \theta x \ell \left(\frac{q}{\theta x} \right) \cdot \mathbf{1}_{\{x>0\}} + \infty \cdot \mathbf{1}_{\{x=0, q>0\}} = \bar{f}_2(\theta x, q)$$

is convex on $\mathbb{R}_+^2$. We have, thus, shown the convexity of L on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ and, consequently, that of I_T on $\mathcal{C}_T$. Q.E.D.

5. Construction of Minimizer for $I_{\gamma, T}^*$

Recall that we are assuming $\lambda \geq \mu$. In order to analyze the long time asymptotics of $I_{\gamma, T}^*$, $I_{\gamma, T}$, and $\tilde{I}_{\gamma, T}$ introduced in (9), we first formally calculate a candidate minimizer $(\xi, \zeta) \in \mathcal{C}_T$ for $I_{\gamma, T}^*$, and then use that to prove our main result, Theorem 3.

5.1. A Formal Calculation of a Candidate Minimizer

Consider the Euler–Lagrange equations (Troutman [17, chapter 6]) associated with $I_{\gamma, T}^*$, namely

$$L_1 = \frac{d}{dt} L_2, \quad 0 = \frac{d}{dt} L_3,$$

where L_i denotes the partial derivative of L with respect to the i th variable.

From Lemma 4, for $(\xi, \zeta) \in \mathcal{C}_T$,

$$L(\xi(t), \xi'(t), \zeta'(t)) = \lambda \ell(\varphi_1(t)) + \mu \ell(\varphi_2(t)) + \theta \xi(t) \ell(\varphi_3(t)),$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ is given by (35)–(37) when $\xi(t) \neq 0$ and by (38) when $\xi(t) = 0$.

Using the form of φ_i , one can check that (suppressing in the notation the dependence on t and with $L_i = L_i(\xi, \xi', \zeta')$)

$$\begin{aligned} L_1 &= \theta \ell(\varphi_3) + \theta \xi (\log \varphi_3) \left(-\frac{\zeta'}{\theta \xi^2} \right) = \theta (1 - \varphi_3), \\ L_2 &= \lambda (\log \varphi_1) \frac{\xi' + \zeta'}{\sqrt{(\xi' + \zeta')^2 + 4\lambda\mu}} + \mu (\log \varphi_2) \frac{\xi' + \zeta'}{\sqrt{(\xi' + \zeta')^2 + 4\lambda\mu}} = \frac{1}{2} \log \frac{\varphi_1}{\varphi_2} = \log \varphi_1, \\ L_3 &= \log \varphi_1 + \theta \xi (\log \varphi_3) \frac{1}{\theta \xi} = \log(\varphi_1 \varphi_3). \end{aligned}$$

From $0 = \frac{d}{dt}L_3$, we have that $\log(\varphi_1(t)\varphi_3(t))$ is a constant function of t , and thus, for some $B \in (0, \infty)$,

$$\varphi_1(t) = \frac{B}{\varphi_3(t)}, t \in [0, T].$$

Using this equation and the equality $L_1 = \frac{d}{dt}L_2$, we have $-\frac{\varphi_3'}{\varphi_3} = \theta(1 - \varphi_3)$, whose solution is given by

$$\varphi_3(t) = \frac{1}{1 - Ae^{\theta t}}, \text{ for some } A \in (-\infty, e^{-\theta T}), t \in [0, T].$$

Therefore,

$$\varphi_1(t) = B(1 - Ae^{\theta t}), \quad \varphi_2(t) = \frac{1}{B(1 - Ae^{\theta t})}, \quad \varphi_3(t) = \frac{1}{1 - Ae^{\theta t}}, \quad t \in [0, T]. \quad (39)$$

With A, B chosen to satisfy appropriate boundary conditions, the trajectory (ξ, ζ) is the candidate minimizer in the definition of $I_{\gamma, T}^*$.

Because the terminal value $\xi(T)$ in the optimization problem $I_{\gamma, T}^*$ is unspecified, the boundary conditions are given by the transversality condition (see Troutman [17] and the proof of Lemma 10 for the role played by this condition)

$$L_2|_{t=T} = 0 \quad (40)$$

and the initial and terminal value constraints $\xi(0) = x_0$, $\zeta(0) = 0$ and $\zeta(T) = \gamma T$. These together give the unique (A, B) and the trajectories (ξ, ζ) as explained as follows. In view of Corollary 1, and because (ξ', ζ') satisfy (33) and (34), we have

$$\xi' + \theta\varphi_3\xi = \lambda\varphi_1 - \mu\varphi_2,$$

which gives the solution

$$\frac{(1-A)e^{\theta t}}{1-Ae^{\theta t}}\xi(t) = x_0 + \frac{e^{\theta t}-1}{\theta} \left[\lambda B(1-A) - \frac{\mu}{B(1-Ae^{\theta t})} \right],$$

namely

$$\xi(t) = \frac{e^{-\theta t}-A}{1-A}x_0 + \frac{\lambda B}{\theta}(e^{\theta t}-1)(e^{-\theta t}-A) + \frac{\mu}{\theta B} \frac{e^{-\theta t}-1}{1-A}. \quad (41)$$

Because ζ' satisfies (34), we get

$$\begin{aligned} \zeta(t) &= \lambda Bt - \frac{\mu t}{B} + \frac{\mu}{\theta B} \log \frac{1-Ae^{\theta t}}{1-A} + \frac{1-e^{-\theta t}}{1-A} \left[x_0 - \frac{\lambda B(1-A)}{\theta} + \frac{\mu}{\theta B} \right] \\ &= \frac{\lambda B}{\theta} [\theta t - 1 + e^{-\theta t}] + \frac{\mu}{\theta B} \left[\log \frac{e^{-\theta t}-A}{1-A} - \frac{e^{-\theta t}-A}{1-A} + 1 \right] + \frac{1-e^{-\theta t}}{1-A} x_0. \end{aligned} \quad (42)$$

From (41) and (42), we have

$$\xi(t) + \zeta(t) = x_0 + \frac{\lambda B}{\theta} [\theta t - A(e^{\theta t}-1)] + \frac{\mu}{\theta B} \log \frac{e^{-\theta t}-A}{1-A}. \quad (43)$$

From (40), we have

$$B(1 - Ae^{\theta T}) = 1. \quad (44)$$

Remark 5. Taking $A = 0$ and $B = 1$ in (41) and (42) gives the LLN trajectory

$$\xi_0(t) = e^{-\theta t}x_0 + \frac{\lambda - \mu}{\theta}(1 - e^{-\theta t}), \quad (45)$$

$$\zeta_0(t) = (\lambda - \mu)t + (1 - e^{-\theta t}) \left(x_0 - \frac{\lambda - \mu}{\theta} \right), \quad (46)$$

with $I_T(\xi_0, \zeta_0) = 0$.

Equalities in (41)–(44) determine the minimizer (ξ, ζ) and the associated φ in the definition of $I_{\gamma, T}^*$. Let

$$\Lambda(A, T) \doteq \frac{e^{-\theta T} - A}{1 - A}.$$

We now argue that, for every $\gamma > 0$, there is a unique (A, B) that satisfy (41)–(44) together with the terminal condition $\zeta(T) = \gamma T$. The latter condition, along with (42) and (44), leads to the following equation for A (see Lemma 7):

$$\frac{1}{1 - Ae^{\theta T}} = \frac{1}{2\lambda[\theta T - 1 + e^{-\theta T}]} \left\{ \theta[\gamma T - x_0 + x_0\Lambda(A, T)] + \left(\theta^2[\gamma T - x_0 + x_0\Lambda(A, T)]^2 - 4\lambda[\theta T - 1 + e^{-\theta T}]\mu[\log \Lambda(A, T) - \Lambda(A, T) + 1] \right)^{1/2} \right\}. \quad (47)$$

The following lemma gives existence and uniqueness of solutions to the preceding equation and also gives the asymptotic behavior of the corresponding (A, B) as $T \rightarrow \infty$. The resulting (ξ, ζ) introduced in Construction 5.1 plays a key role in the analysis.

Lemma 6. Suppose $\gamma > 0$. Then, the following hold.

- There exists a unique $A \in (-\infty, e^{-\theta T})$ that satisfies (47).
- The unique $A \equiv A(x_0, T)$ satisfies, uniformly for x_0 in compacts, as $T \rightarrow \infty$, $1 - Ae^{\theta T} \rightarrow \frac{2\lambda}{\gamma + \sqrt{\gamma^2 + 4\lambda\mu}} = z_\gamma$. In particular, $A \rightarrow 0$ and $B \doteq \frac{1}{1 - Ae^{\theta T}} \rightarrow z_\gamma^{-1}$, uniformly for x_0 in compacts, as $T \rightarrow \infty$.

Proof.

- Denote the right side of (47) by $R(A, T)$ and the left side of the same display by $L(A, T)$. Because $\log x - x + 1 \leq 0$ for every $x \geq 0$, $R(A, T)$ is well defined for every $A < e^{-\theta T}$. Let

$$f_1(A) \doteq \frac{1}{\log \frac{1-A}{e^{-\theta T}-A}} \cdot L(A, T), \quad f_2(A) \doteq \frac{1}{\log \frac{1-A}{e^{-\theta T}-A}} \cdot R(A, T).$$

Note that (for fixed T) as $A \downarrow -\infty$,

$$L(A, T) \rightarrow 0, \quad R(A, T) \rightarrow \frac{\theta\gamma T}{\lambda[\theta T - 1 + e^{-\theta T}]} > 0.$$

So $f_1(A) < f_2(A)$ for sufficiently small A . Also note that, as $A \uparrow e^{-\theta T}$,

$$L(A, T) = O\left(\frac{1}{e^{-\theta T} - A}\right), \quad R(A, T) = O\left(\sqrt{\log \frac{1}{e^{-\theta T} - A}}\right). \quad (48)$$

So $f_1(A) > f_2(A)$ for A sufficiently close to $e^{-\theta T}$. Because $f_1(A), f_2(A)$ are continuous in A , there must exist some A such that they are equal, which gives the existence of $A \in (-\infty, e^{-\theta T})$.

For uniqueness, it suffices to verify that

$$A \mapsto f_1(A) \text{ is strictly increasing in } A \in (-\infty, e^{-\theta T}), \quad (49)$$

$$A \mapsto f_2(A) \text{ is strictly decreasing in } A \in (-\infty, e^{-\theta T}). \quad (50)$$

For (49), because $\frac{1-A}{e^{-\theta T}-A} > 1$, it suffices to show that

$$A \mapsto g_1(A) \doteq (e^{-\theta T} - A) \log \frac{1-A}{e^{-\theta T}-A} \text{ is strictly decreasing in } A \in (-\infty, e^{-\theta T}).$$

This follows on observing that

$$g_1'(A) = \log \frac{e^{-\theta T} - A}{1 - A} - \frac{e^{-\theta T} - A}{1 - A} + 1 < 0.$$

This proves (49). For (50), let $z \doteq \frac{e^{-\theta T}-A}{1-A} = \Lambda(A, T)$. Note that $z \in (0, 1)$ and is strictly decreasing in $A \in (-\infty, e^{-\theta T})$. It then suffices to show

$$z \mapsto \frac{\theta[\gamma T - x_0 + zx_0] + \sqrt{\theta^2[\gamma T - x_0 + zx_0]^2 - 4\lambda[\theta T - 1 + e^{-\theta T}]\mu[\log z - z + 1]}}{-\log z}$$

is strictly increasing in $z \in (0, 1)$. Because $\frac{\gamma T - x_0 + zx_0}{-\log z}$ is strictly increasing in $z \in (0, 1)$, and $x + \sqrt{x^2 + C}$ is increasing in $x \in \mathbb{R}$ for any $C \in (0, \infty)$, it suffices to show

$$z \mapsto \frac{-4\lambda[\theta T - 1 + e^{-\theta T}]\mu[\log z - z + 1]}{(\log z)^2} \text{ is increasing in } z \in (0, 1).$$

Because $\theta T - 1 + e^{-\theta T} > 0$, it suffices, in turn, to show

$$z \mapsto \frac{-(\log z - z + 1)}{(\log z)^2} \text{ is increasing in } z \in (0, 1).$$

However, this is easy to verify, proving (50). This completes the proof of (a).

b. Fix a compact $K \subset \mathbb{R}_+$. We first claim that

$$\liminf_{T \rightarrow \infty} \inf_{x_0 \in K} (1 - Ae^{\theta T}) \geq c_0 \text{ for some } c_0 > 0.$$

Indeed, note that, if $\lim_{T \rightarrow \infty} Ae^{\theta T} = 1$ for some sequence $\{x_T\} \subset K$, where $A \equiv A(x_T, T)$, then (as $T \rightarrow \infty$)

$$L(A, T) = O\left(\frac{1}{1 - Ae^{\theta T}}\right) \text{ and } R(A, T) = O\left(\sqrt{\frac{1}{T} \log \frac{1}{e^{-\theta T} - A}}\right),$$

which says that $L(A, T)/R(A, T) \rightarrow \infty$ as $T \rightarrow \infty$ and contradicts the fact that $L(A, T) = R(A, T)$. Therefore, the claim holds. Because $\gamma > 0$ and $A = A(x_T, T)$ is the solution of (47), it follows that

$$\liminf_{T \rightarrow \infty} \inf_{x_0 \in K} \frac{1}{1 - Ae^{\theta T}} \geq \frac{\gamma}{\lambda}.$$

So, for sufficiently large T and uniformly for $x_0 \in K$, we have

$$0 < \frac{1}{2}c_0 \leq 1 - Ae^{\theta T} \leq \frac{2\lambda}{\gamma},$$

and hence, uniformly for $x_0 \in K$,

$$\log \Lambda(A, T) = \log \frac{e^{-\theta T} - A}{1 - A} = -\theta T + \log \frac{1 - Ae^{\theta T}}{1 - A} = -\theta T + O(1).$$

Applying this back to (47) gives

$$\limsup_{T \rightarrow \infty} \sup_{x_0 \in K} \left| \frac{1}{1 - Ae^{\theta T}} - \frac{\gamma + \sqrt{\gamma^2 + 4\lambda\mu}}{2\lambda} \right| = 0.$$

This completes the proof. Q.E.D.

Construction 5.1. Let $A \in (-\infty, e^{-\theta T})$ be the unique solution of (47) and let $B = \frac{1}{1 - Ae^{\theta T}}$. Define $(\bar{\xi}, \bar{\zeta})$ as in (41) and (42) with this choice of (A, B) .

The following lemma shows that $(\bar{\xi}, \bar{\zeta})$ as constructed have the correct terminal value.

Lemma 7. Suppose $\gamma > 0$. The function $\bar{\zeta}$ satisfies $\bar{\zeta}(T) = \gamma T$.

Proof. From (47) and the definition $B = \frac{1}{1 - Ae^{\theta T}}$, we see that B is one solution to the equation

$$\lambda[\theta T - 1 + e^{-\theta T}]B^2 - \theta\left[\gamma T - x_0 + x_0 \frac{e^{-\theta T} - A}{1 - A}\right]B + \mu\left[\log \frac{e^{-\theta T} - A}{1 - A} - \frac{e^{-\theta T} - A}{1 - A} + 1\right] = 0.$$

It then follows from (42) that the function $\bar{\zeta}$ satisfies $\bar{\zeta}(T) = \gamma T$. Q.E.D.

5.2. Properties of the Candidate Minimizer

The pair $(\bar{\xi}, \bar{\zeta})$ introduced in Construction 5.1 is our candidate minimizer for $I_{\gamma, T}^*$. In this section, we study some properties of these trajectories. In particular, we show that $(\bar{\xi}, \bar{\zeta}) \in \mathcal{C}_T$, where $\mathcal{C}_T$ is defined at the beginning of

Section 4, when T is sufficiently large. We occasionally denote $(\bar{\xi}, \bar{\zeta})$ as $(\bar{\xi}_{x_0}, \bar{\zeta}_{x_0})$ in order to emphasize dependence on the initial condition. Note that $(\bar{\xi}, \bar{\zeta})$ also depends on T , but that dependence is suppressed in the notation.

The following lemma says that, with (A, B) as identified in Lemma 6, the state process $\bar{\xi}$ given in Construction 5.1 never goes below zero and actually is away from zero for $t > 0$.

Lemma 8. Suppose $\gamma > 0$. Let $(\bar{\xi}, \bar{\zeta})$ be as given in Construction 5.1. For every compact $K \subset \mathbb{R}_+$, there exists some $T_0 \in (0, \infty)$ such that the following hold for every $T \geq T_0$:

- Suppose $x_0 \in K$. Then, $\bar{\xi}(t) > 0$ for all $t \in (0, T]$ and $(\bar{\xi}, \bar{\zeta}) \in \mathcal{C}_T$.
- Suppose $x_0 \in K \setminus \{0\}$. Then, there exists some $c_0 \in (0, \infty)$ such that $c_0^{-1} \leq \bar{\xi}(t) \leq c_0$, $|\bar{\xi}'(t)| \leq c_0$, and $c_0^{-1} \leq \bar{\zeta}'(t) \leq c_0$ for all $t \in [0, T]$.

Proof.

- From (41) and because $e^{-\theta T} - A > 0$, we see that, for any $t \in (0, T]$,

$$\frac{\bar{\xi}(t)}{1 - e^{-\theta t}} \geq \frac{\lambda B}{\theta} (1 - Ae^{\theta t}) - \frac{\mu}{\theta B} \frac{1}{1 - A}. \quad (51)$$

If $A \equiv A(x_0, T) > 0$, then the right side of (51) is decreasing in t . Therefore,

$$\begin{aligned} \frac{\bar{\xi}(t)}{1 - e^{-\theta t}} &\geq \frac{\lambda B}{\theta} (1 - Ae^{\theta T}) - \frac{\mu}{\theta B} \frac{1}{1 - A} \\ &= \frac{\lambda - \mu}{\theta} + \frac{\mu}{\theta} \frac{1 - Ae^{\theta T}}{1 - A} = \frac{\lambda - \mu}{\theta} + \frac{\mu}{\theta} \frac{A}{1 - A} (e^{\theta T} - 1) > \frac{\lambda - \mu}{\theta} \geq 0, \end{aligned}$$

where the first equality on the second line uses (44) and the last inequality follows because $\lambda \geq \mu$.

Now consider the case $A \leq 0$. From Lemma 6(b) and the fact that $z_\gamma^{-1} > \sqrt{\frac{\mu}{\lambda}}$, we can find some $T_0 < \infty$ such that

$$\inf_{T \geq T_0} \inf_{x_0 \in K} B > \sqrt{\frac{\mu}{\lambda}}. \quad (52)$$

Therefore, for all $T \geq T_0$ and $x_0 \in K$, whenever $A \leq 0$, we have $B(1 - A) > \sqrt{\frac{\mu}{\lambda}}$. Applying this to (51) gives

$$\frac{\bar{\xi}(t)}{1 - e^{-\theta t}} \geq \frac{\lambda B}{\theta} (1 - A) - \frac{\mu}{\theta B} \frac{1}{1 - A} > 0$$

for all $t \in (0, T]$. This proves the first statement in (a). The second statement is immediate from the first because $(\bar{\xi}, \bar{\zeta})$ satisfy (33) and (34).

This completes the proof of part (a).

- From (34) and (39), we have $\bar{\zeta}'(t) = \frac{\theta \bar{\xi}(t)}{1 - Ae^{\theta t}}$.

Because $x_0 > 0$, from part (a), we have that $\inf_{t \in [0, T]} \bar{\xi}(t) > 0$ for all $T \geq T_0$. Now, for every $T \geq T_0$, from (52), and because $-\infty < A < e^{-\theta T}$ and $\sup_{x_0 \in K} |\bar{\xi}|_{*, T} < \infty$, we have that $c_1^{-1} \leq \bar{\zeta}'(t) \leq c_1$ for some $c_1 \in (0, \infty)$. Finally, from (33) and (39), we see that, for every $T \geq T_0$, there exists some $c_2 < \infty$ such that $|\bar{\xi}'(t)| \leq c_2$. This completes the proof. Q.E.D.

5.3. Cost Asymptotics for the Candidate Minimizer

The following lemma calculates $I_T(\bar{\xi}, \bar{\zeta})$ for $(\bar{\xi}, \bar{\zeta})$ introduced in Construction 5.1.

Lemma 9. Suppose $\gamma > 0$. Fix a compact $K \subset \mathbb{R}_+$. Then, with $T_0 \in (0, \infty)$ as in Lemma 8, for all $T \geq T_0$, $x_0 \in K$ and $(\bar{\xi}, \bar{\zeta})$ as introduced in Construction 5.1,

$$\begin{aligned} I_T(\bar{\xi}, \bar{\zeta}) &= \left(\gamma T + \bar{\zeta}(T) - x_0 \right) \log B - \bar{\xi}(T) \log \frac{1}{1 - Ae^{\theta T}} + x_0 \log \frac{1}{1 - A} \\ &\quad + \lambda T + \mu T - \frac{\lambda B}{\theta} [\theta T - A(e^{\theta T} - 1)] + \frac{\mu}{B\theta} \log \frac{e^{-\theta T} - A}{1 - A}. \end{aligned}$$

Furthermore,

$$\lim_{T \rightarrow \infty} \frac{I_T(\bar{\xi}, \bar{\zeta})}{T} = C(\gamma)$$

uniformly for $x_0 \in K$ where $C(\gamma)$ is defined as in (10).

Proof. It follows from Lemma 8(a) that, for all $T \geq T_0$ and $x_0 \in K$, $(\bar{\xi}, \bar{\zeta}) = (\bar{\xi}_{x_0}, \bar{\zeta}_{x_0}) \in \mathcal{C}_T$. In the following sequence of equalities, we use (39) for the second equality, (33) and (34) for the third, the relation $\varphi'_3 = \theta\varphi_3(\varphi_3 - 1)$ given before (39) for the fourth, and (39) once more for the fifth equality:

$$\begin{aligned} I_T(\bar{\xi}, \bar{\zeta}) &= \lambda \int_0^T \ell(\varphi_1(t)) dt + \mu \int_0^T \ell(\varphi_2(t)) dt + \theta \int_0^T \bar{\xi}(t) \ell(\varphi_3(t)) dt \\ &= \int_0^T \left[-\lambda \varphi_1(t) \log \frac{\varphi_3(t)}{B} - \lambda \varphi_1(t) + \lambda + \mu \varphi_2(t) \log \frac{\varphi_3(t)}{B} - \mu \varphi_2(t) + \mu \right. \\ &\quad \left. + \theta \bar{\xi}(t) \varphi_3(t) \log \varphi_3(t) - \theta \bar{\xi}(t) \varphi_3(t) + \theta \bar{\xi}(t) \right] dt \\ &= \int_0^T \left[-\bar{\xi}'(t) \log \varphi_3(t) + (\bar{\xi}'(t) + \bar{\zeta}'(t)) \log B - \lambda \varphi_1(t) + \lambda \right. \\ &\quad \left. - \mu \varphi_2(t) + \mu - \theta \bar{\xi}(t) \varphi_3(t) + \theta \bar{\xi}(t) \right] dt \\ &= -\bar{\xi}(t) \log \varphi_3(t) \Big|_{t=0}^T + (\bar{\xi}(t) + \bar{\zeta}(t)) \log B \Big|_{t=0}^T + \int_0^T [-\lambda \varphi_1(t) + \lambda - \mu \varphi_2(t) + \mu] dt \\ &= (\bar{\xi}(T) + \gamma T - x_0) \log B - \bar{\xi}(T) \log \frac{1}{1 - Ae^{\theta T}} + x_0 \log \frac{1}{1 - A} \\ &\quad + \lambda T + \mu T - \frac{\lambda B}{\theta} [\theta T - A(e^{\theta T} - 1)] + \frac{\mu}{B\theta} \log \frac{e^{-\theta T} - A}{1 - A}. \end{aligned} \quad (53)$$

This proves the first statement in the lemma. For the second statement, we use Lemma 6(b), which says that $1 - Ae^{\theta T} \rightarrow z_\gamma$, $A \rightarrow 0$, and $B \rightarrow z_\gamma^{-1}$ as $T \rightarrow \infty$ uniformly for $x_0 \in K$. From these uniform convergence properties and (41), we have

$$\frac{\bar{\xi}(T)}{T} = \frac{1}{T} \left(\frac{e^{-\theta T} - A}{1 - A} x_0 + \frac{\lambda B}{\theta} (e^{\theta T} - 1)(e^{-\theta T} - A) + \frac{\mu}{\theta B} \frac{e^{-\theta T} - 1}{1 - A} \right) \rightarrow 0,$$

uniformly for $x_0 \in K$ as $T \rightarrow \infty$. Similarly,

$$\begin{aligned} \frac{\lambda B}{\theta T} [\theta T - A(e^{\theta T} - 1)] &= \lambda B - \frac{\lambda B}{\theta T} (Ae^{\theta T} - 1 + 1 - A) \rightarrow \lambda z_\gamma^{-1}, \\ \frac{\mu}{B\theta T} \log \frac{e^{-\theta T} - A}{1 - A} &= \frac{\mu}{B\theta T} \left(\log \frac{1 - Ae^{\theta T}}{1 - A} - \theta T \right) \rightarrow -\mu z_\gamma, \end{aligned}$$

uniformly for $x_0 \in K$ as $T \rightarrow \infty$. Combining these, we have

$$\frac{I_T(\bar{\xi}, \bar{\zeta})}{T} \rightarrow \gamma \log z_\gamma^{-1} + \lambda + \mu - \lambda z_\gamma^{-1} - \mu z_\gamma = C(\gamma),$$

uniformly for $x_0 \in K$ as $T \rightarrow \infty$. Q.E.D.

5.4. Verification of the Minimizer Property When $x_0 > 0$

In this section, we show that, when $x_0 > 0$ and $\gamma > 0$, $(\bar{\xi}, \bar{\zeta})$ defined in Construction 5.1 is the minimizer in the variational problem for $I_{\gamma, T}^*$ when T is sufficiently large. Let

$$\mathcal{J}(x_0, \gamma, T) \doteq \{(\xi, \zeta) \in \mathcal{C}_T : \zeta(T) = \gamma T\}. \quad (54)$$

We frequently suppress x_0 and T from the notation and simply write $\mathcal{J}(\gamma)$ for $\mathcal{J}(x_0, \gamma, T)$.

Lemma 10. Suppose $K \subset \mathbb{R}_+$ is compact and $\gamma > 0$. Let $(\bar{\xi}, \bar{\zeta})$ be as in Construction 5.1. Then, there exists some $T_1 \in (0, \infty)$ such that $I_{\gamma, T}^* = I_T(\bar{\xi}, \bar{\zeta})$ for all $T \geq T_1$ and $x_0 \in K \setminus \{0\}$.

Proof. By Lemma 4,

$$I_{\gamma, T}^* = \inf_{(\xi, \zeta) \in \mathcal{J}(\gamma)} I_T(\xi, \zeta) = \inf_{(\xi, \zeta) \in \mathcal{J}(\gamma)} \int_0^T L(\xi(t), \xi'(t), \zeta'(t)) dt.$$

Thus, in view of Lemmas 7 and 8(a), it suffices to show that, with T_0 as in Lemma 8, there is a $T_1 \geq T_0$ such that, for all $T \geq T_1$ and $x_0 \in K \setminus \{0\}$ $(\bar{\xi}, \bar{\zeta})$ is the minimizer of the function

$$G(\xi, \zeta) \doteq \int_0^T L(\xi(t), \xi'(t), \zeta'(t)) dt, \quad (\xi, \zeta) \in \mathcal{J}(\gamma).$$

We prove this via contradiction.

First, note that $\mathcal{J}(\gamma)$ is a convex subset of $\mathcal{C}_T$ and G is a convex function on $\mathcal{J}(\gamma)$ by Lemma 5. Now, suppose there exists some $(\tilde{\xi}, \tilde{\zeta}) \in \mathcal{J}(\gamma)$ such that $G(\tilde{\xi}, \tilde{\zeta}) < G(\bar{\xi}, \bar{\zeta})$. We show that this leads to a contradiction. From Lemma 9, for $T \geq T_0$, we have $G(\tilde{\xi}, \tilde{\zeta}) < \infty$. For $\varepsilon \in [0, 1]$, consider the family of paths $(\xi^\varepsilon, \zeta^\varepsilon) \doteq (1 - \varepsilon)(\tilde{\xi}, \tilde{\zeta}) + \varepsilon(\bar{\xi}, \bar{\zeta})$. Also consider the convex function g on $[0, 1]$ defined by $g(\varepsilon) \doteq G(\xi^\varepsilon, \zeta^\varepsilon)$ for $\varepsilon \in [0, 1]$. Note that $g(1) = G(\bar{\xi}, \bar{\zeta}) < G(\tilde{\xi}, \tilde{\zeta}) = g(0)$. It follows from the convexity that g is left and right differentiable wherever it is finite. We show that $g'_+(0) = 0$, where $g'_+(\cdot)$ denotes the right derivative of g . The convexity of g then gives the desired contradiction.

Let $\Delta\xi \doteq \bar{\xi} - \tilde{\xi}$ and $\Delta\zeta \doteq \bar{\zeta} - \tilde{\zeta}$. Then,

$$\begin{aligned} g(\varepsilon) &= \int_0^T L(\xi^\varepsilon(t), (\xi^\varepsilon)'(t), (\zeta^\varepsilon)'(t)) dt \\ &= \int_0^T L(\bar{\xi}(t) + \varepsilon\Delta\xi(t), \bar{\xi}'(t) + \varepsilon\Delta\xi'(t), \bar{\zeta}'(t) + \varepsilon\Delta\zeta'(t)) dt. \end{aligned} \quad (55)$$

We claim that we can differentiate with respect to ε under the integral sign for $0 < \varepsilon < \frac{1}{4}$. Suppose for the moment that the claim is true. Then,

$$g'(0+) = \int_0^T \left(L_1(\bar{\xi}(t), \bar{\xi}'(t), \bar{\zeta}'(t))\Delta\xi(t) + L_2(\bar{\xi}(t), \bar{\xi}'(t), \bar{\zeta}'(t))\Delta\xi'(t) + L_3(\bar{\xi}(t), \bar{\xi}'(t), \bar{\zeta}'(t))\Delta\zeta'(t) \right) dt.$$

Recall from Section 5.1 that $(\bar{\xi}, \bar{\zeta})$ is such that $L_1 = \frac{d}{dt}L_2$, $0 = \frac{d}{dt}L_3$, and $L_2|_{t=T} = 0$ (when evaluated at $(\bar{\xi}, \bar{\xi}', \bar{\zeta}, \bar{\zeta}')$). It then follows from integration by parts that

$$g'(0+) = \left(L_2(\bar{\xi}(t), \bar{\xi}'(t), \bar{\zeta}'(t))\Delta\xi(t) + L_3(\bar{\xi}(t), \bar{\xi}'(t), \bar{\zeta}'(t))\Delta\zeta(t) \right) \Big|_{t=0}^{t=T}.$$

Because $\Delta\xi(0) = 0$, $L_2(\bar{\xi}(T), \bar{\xi}'(T), \bar{\zeta}'(T)) = 0$, and $\Delta\zeta(0) = 0 = \Delta\zeta(T)$, we have $g'(0+) = 0$. This gives the desired contradiction and shows that $(\bar{\xi}, \bar{\zeta})$ is the minimizer.

Finally, we prove the claim that, in (55), we can differentiate under the integral sign for $0 < \varepsilon < \frac{1}{4}$. Denote the integrand in (55) by $\tilde{g}(t, \varepsilon)$, namely

$$\tilde{g}(t, \varepsilon) \doteq L(\bar{\xi}(t) + \varepsilon\Delta\xi(t), \bar{\xi}'(t) + \varepsilon\Delta\xi'(t), \bar{\zeta}'(t) + \varepsilon\Delta\zeta'(t)).$$

It suffices to establish an integrable bound on $\frac{\partial \tilde{g}(t, \varepsilon)}{\partial \varepsilon}$ that is uniform for $0 < \varepsilon < \frac{1}{4}$. From the formula for L in Section 4 and recalling that $\xi^\varepsilon > 0$, we have

$$\begin{aligned} \frac{\partial \tilde{g}(t, \varepsilon)}{\partial \varepsilon} &= L_1(\xi^\varepsilon(t), (\xi^\varepsilon)'(t), (\zeta^\varepsilon)'(t))\Delta\xi(t) + L_2(\xi^\varepsilon(t), (\xi^\varepsilon)'(t), (\zeta^\varepsilon)'(t))\Delta\xi'(t) + L_3(\xi^\varepsilon(t), (\xi^\varepsilon)'(t), (\zeta^\varepsilon)'(t))\Delta\zeta'(t) \\ &= \Delta\xi(t) \left(\theta - \frac{(\zeta^\varepsilon)'(t)}{\xi^\varepsilon(t)} \right) + \Delta\zeta'(t) \log \frac{(\zeta^\varepsilon)'(t)}{\theta \xi^\varepsilon(t)} \\ &\quad + (\Delta\xi'(t) + \Delta\zeta'(t)) \log \frac{\sqrt{((\xi^\varepsilon)'(t) + (\zeta^\varepsilon)'(t))^2 + 4\lambda\mu} + (\xi^\varepsilon)'(t) + (\zeta^\varepsilon)'(t)}{2\lambda}. \end{aligned}$$

From Lemma 8(b), we have (by choosing a larger T_0 if needed) $\delta \doteq \inf_{t \in [0, T]} \bar{\xi}(t) > 0$ (we remark that δ may depend on x_0). Then, there exists some $\kappa \in (0, \infty)$ such that, for all $0 < \varepsilon < \frac{1}{4}$ and $t \in [0, T]$,

$$\frac{\delta}{2} < \xi^\varepsilon(t) \leq \kappa. \quad (56)$$

It then follows from continuity of $\tilde{\xi}$ and $\bar{\xi}$ that

$$\left| \Delta\xi(t) \left(\theta - \frac{(\zeta^\varepsilon)'(t)}{\xi^\varepsilon(t)} \right) \right| \leq \kappa_1 \left(1 + \tilde{\zeta}'(t) + \bar{\zeta}'(t) \right). \quad (57)$$

Next, by Lemma 8(b) and (56),

$$\left| \Delta\zeta'(t) \log \frac{(\zeta^\varepsilon)'(t)}{\theta \xi^\varepsilon(t)} \right| \leq \kappa_2 + \tilde{\zeta}'(t) \log \tilde{\zeta}'(t). \quad (58)$$

Because $G(\tilde{\xi}, \tilde{\zeta}) < G(\bar{\xi}, \bar{\zeta}) < \infty$, we must have $\tilde{\zeta}', \tilde{\zeta}'$, and $\tilde{\zeta}'(\cdot) \log \tilde{\zeta}'(\cdot)$ are all in $L^1([0, T])$. Therefore, the right side of the preceding display and the right side of (57) are integrable functions $[0, T]$. Finally, by Lemma 8(b)

$$\left| (\Delta \xi'(t) + \Delta \zeta'(t)) \log \frac{\sqrt{((\xi^\varepsilon)'(t) + (\zeta^\varepsilon)'(t))^2 + 4\lambda\mu + (\xi^\varepsilon)'(t) + (\zeta^\varepsilon)'(t)}}{2\lambda} \right|$$

$$\leq \kappa_3 + \kappa_3 |\tilde{\xi}'(t) + \tilde{\zeta}'(t)| \log \frac{\sqrt{(\tilde{\xi}'(t) + \tilde{\zeta}'(t))^2 + 4\lambda\mu + \tilde{\xi}'(t) + \tilde{\zeta}'(t)}}{2\lambda}.$$

Again, because $G(\tilde{\xi}, \tilde{\zeta}) < G(\bar{\xi}, \bar{\zeta}) < \infty$, we must have that the right side of the preceding display is an integrable function on $[0, T]$. Combining this with the integrability of the right sides of (57) and (58), we have an integrable uniform bound on $\frac{\partial \tilde{g}(t, \varepsilon)}{\partial \varepsilon}$ for $\varepsilon \in (0, \frac{1}{4})$. This completes the proof. Q.E.D.

6. Proof of Theorem 3

In this section, we complete the proof of Theorem 3. Throughout the section, $\lambda \geq \mu$. Although Lemmas 9 and 10 together immediately imply the first statement of Theorem 3, when $x_0 > 0$ and $\gamma > 0$, they cannot be directly used to treat the remaining cases. In order to treat the general case we analyze $\limsup_{T \rightarrow \infty} \frac{I_{\gamma, T}^*}{T}$ for $\gamma \geq 0$ and $x_0 \geq 0$ in Proposition 2 and $\liminf_{T \rightarrow \infty} \frac{I_{\gamma, T}^*}{T}$ in Proposition 3 (for $\gamma > 0$ and $x_0 \geq 0$) and Proposition 4 (for $\gamma = 0$ and $x_0 \geq 0$). These three propositions taken together cover the general case $x_0 \geq 0$ and $\gamma \geq 0$. The proof of Proposition 3 makes crucial use of Lemmas 9 and 10.

We begin by establishing some useful properties of $I_{\gamma, T}^*$.

Lemma 11. *The map $\gamma \mapsto I_{\gamma, T}^*$ is a nonnegative, convex function on $[0, \infty)$, and $I_{\gamma_T^*, T}^* = 0$, where $\gamma_T^* = (\lambda - \mu) + \frac{1 - e^{-\theta T}}{T}(x_0 - \frac{\lambda - \mu}{\theta})$. In particular, the function $\gamma \mapsto I_{\gamma, T}^*$ is decreasing for $\gamma \leq \gamma_T^*$ and increasing otherwise.*

Proof. Clearly, $I_{\gamma, T}^* \geq 0$ for every $\gamma \geq 0$. From Remark 5, we see that the LLN zero-cost trajectory (ξ_0, ζ_0) has total reneging $(\lambda - \mu)T + (1 - e^{-\theta T})(x_0 - \frac{\lambda - \mu}{\theta})$ over $[0, T]$, which proves that $I_{\gamma_T^*, T}^* = 0$. In order to see convexity, fix $\gamma_2 > \gamma_1 \geq 0$ and $\varepsilon \in (0, 1)$. Let $\gamma \doteq \varepsilon\gamma_1 + (1 - \varepsilon)\gamma_2$. For $(\xi_i, \zeta_i) \in \mathcal{J}(\gamma_i)$, $i = 1, 2$, let

$$(\xi, \zeta) \doteq \varepsilon(\xi_1, \zeta_1) + (1 - \varepsilon)(\xi_2, \zeta_2).$$

Then, $(\xi, \zeta) \in \mathcal{J}(\gamma)$, and by Lemma 5,

$$I_{\gamma, T}^* \leq I_T(\xi, \zeta) \leq \varepsilon I_T(\xi_1, \zeta_1) + (1 - \varepsilon) I_T(\xi_2, \zeta_2).$$

Taking infimum over $(\xi_i, \zeta_i) \in \mathcal{J}(\gamma_i)$, $i = 1, 2$, gives

$$I_{\gamma, T}^* \leq \varepsilon I_{\gamma_1, T}^* + (1 - \varepsilon) I_{\gamma_2, T}^*.$$

The final monotonicity statement in the lemma is now immediate from convexity and nonnegativity. Q.E.D.

Note that for $\gamma \geq 0$, z_γ given in (11) is the unique positive solution to

$$\lambda z_\gamma^{-1} - \mu z_\gamma - \gamma = 0. \quad (59)$$

Also, note that the map $\gamma \mapsto C(\gamma)$ is continuous, and $C(\lambda - \mu) = 0$. The following proposition bounds the limit superior of $I_{\gamma, T}^*/T$ by $C(\gamma)$.

Proposition 2. *Fix $\gamma \geq 0$. Then,*

$$\limsup_{T \rightarrow \infty} \frac{I_{\gamma, T}^*}{T} \leq C(\gamma).$$

Proof. First, consider the case that $\gamma > 0$. For fixed T , we make use of the trajectory (ξ, ζ) given by (41) and (42) with $A = 0$ and B chosen such that $\zeta(T) = \gamma T$. This means that

$$B = \frac{\theta[\gamma T + (e^{-\theta T} - 1)x_0] + \sqrt{\theta^2[\gamma T + (e^{-\theta T} - 1)x_0]^2 + 4\lambda\mu[\theta T - 1 + e^{-\theta T}]^2}}{2\lambda[\theta T - 1 + e^{-\theta T}]},$$

$$\xi(t) = e^{-\theta t}x_0 + \frac{\lambda B}{\theta}(1 - e^{-\theta t}) - \frac{\mu}{\theta B}(1 - e^{-\theta t}), \quad 0 \leq t \leq T,$$

$$\zeta(t) = \frac{\lambda B}{\theta}[\theta t - 1 + e^{-\theta t}] + \frac{\mu}{\theta B}[-\theta t - e^{-\theta t} + 1] + (1 - e^{-\theta t})x_0.$$

Because $\gamma > 0$, we have

$$\lim_{T \rightarrow \infty} B = \frac{\gamma + \sqrt{\gamma^2 + 4\lambda\mu}}{2\lambda} = z_\gamma^{-1} > \sqrt{\frac{\mu}{\lambda}}.$$

This implies $B > \sqrt{\frac{\mu}{\lambda}}$, $\xi(t) \geq 0$, and $(\xi, \zeta) \in \mathcal{C}_T$ for sufficiently large T , and

$$\lim_{T \rightarrow \infty} \xi(T) = \frac{\lambda}{z_\gamma \theta} - \frac{z_\gamma \mu}{\theta}.$$

We note that, because (53) is satisfied for any φ satisfying (39), and the corresponding $(\bar{\xi}, \bar{\zeta})$ satisfying (33) and (34), this equality holds with $(\bar{\xi}, \bar{\zeta})$ replaced by (ξ, ζ) .

Using this and because $\zeta(T) = \gamma T$, $A = 0$,

$$\begin{aligned} \frac{I_{\gamma,T}^*}{T} &\leq \frac{\xi(T) + \gamma T - x_0}{T} \log B - \frac{\xi(T)}{T} \log \frac{1}{1 - Ae^{\theta T}} + \frac{x_0}{T} \log \frac{1}{1 - A} \\ &\quad + \lambda + \mu - \frac{\lambda B}{\theta T} [\theta T - A(e^{\theta T} - 1)] + \frac{\mu}{B\theta T} \log \frac{e^{-\theta T} - A}{1 - A} \\ &= \frac{\xi(T) + \gamma T - x_0}{T} \log B + \lambda + \mu - \lambda B - \frac{\mu}{B} \end{aligned}$$

for sufficiently large T , and hence,

$$\limsup_{T \rightarrow \infty} \frac{I_{\gamma,T}^*}{T} \leq \gamma \log z_\gamma^{-1} + \lambda + \mu - \lambda z_\gamma^{-1} - \mu z_\gamma = C(\gamma).$$

Next, consider the case in which $\gamma = 0$. Let $z_{-1} = \frac{2\lambda}{\sqrt{4\lambda\mu+1}}$. Take

$$\begin{aligned} \xi(t) &= x_0 - t, \zeta(t) = 0, \varphi_1(t) = z_{-1}^{-1}, \varphi_2(t) = z_{-1}, \varphi_3(t) = 0, \quad t \in [0, x_0), \\ \xi(t) &= 0, \zeta(t) = 0, \varphi_1(t) = z_0^{-1}, \varphi_2(t) = z_0, \varphi_3(t) = 1, \quad t \in [x_0, T]. \end{aligned}$$

Clearly, $(\xi, \zeta) \in \mathcal{C}_T$. Using (59), we see that $\lambda\varphi_1(t) - \mu\varphi_2(t) = -1 = \xi'(t)$ for $t \in [0, x_0)$ and $\lambda\varphi_1(t) - \mu\varphi_2(t) = 0 = \xi'(t)$ for $t \in [x_0, T]$. Therefore, $(\varphi_1, \varphi_2, \varphi_3) \in \mathcal{U}(\xi, \zeta)$ and

$$\begin{aligned} \limsup_{T \rightarrow \infty} \frac{I_{0,T}^*}{T} &\leq \limsup_{T \rightarrow \infty} \frac{\lambda \int_0^T \ell(\varphi_1(t)) dt + \mu \int_0^T \ell(\varphi_2(t)) dt + \theta \int_0^T \xi(t) \ell(\varphi_3(t)) dt}{T} \\ &= \lambda \ell(z_0^{-1}) + \mu \ell(z_0) = \lambda(1 - z_0^{-1}) + \mu(1 - z_0) = C(0). \end{aligned}$$

This completes the proof. Q.E.D.

The following proposition gives the reverse inequality for limit inferior when $\gamma > 0$.

Proposition 3. Fix $\gamma > 0$. Then,

$$\liminf_{T \rightarrow \infty} \frac{I_{\gamma,T}^*}{T} \geq C(\gamma).$$

Proof. First, consider $x_0 > 0$. Then, it is immediate from Lemmas 9 and 10 that

$$\lim_{T \rightarrow \infty} \frac{I_{\gamma,T}^*}{T} = C(\gamma).$$

Next, consider $x_0 = 0$. There are two cases for γ :

Case 1: $\gamma = \lambda - \mu$. In this case, $C(\gamma) = 0$, and the result clearly holds.

Case 2: $\gamma \neq \lambda - \mu$. Let

$$\mathcal{J}(x, \gamma, T) \doteq \{(\xi, \zeta) \in \mathcal{C}_T : \xi(0) = x, \zeta(T) = \gamma T\}.$$

Fix $\delta \in (0, |\gamma - (\lambda - \mu)| \wedge 1)$ and $T \in (0, \infty)$. There exists $(\xi^T, \zeta^T) \in \mathcal{J}(0, \gamma, T)$ such that $I_{\gamma,T}^* + \delta \geq I_T(\xi^T, \zeta^T)$. Let $(\bar{\xi}, \bar{\zeta})$ be as in Construction 5.1 with $x_0 = 0$. By Lemmas 8(a) and 7, there exists $T_1 \in (0, \infty)$ such that $(\bar{\xi}, \bar{\zeta}) \in \mathcal{J}(0, \gamma, T)$ for $T \geq T_1$. Also note that, by Lemma 9,

$$\lim_{T \rightarrow \infty} \frac{I_T(\bar{\xi}, \bar{\zeta})}{T} = C(\gamma).$$

Consider $\mathcal{J}(0, \gamma, T) \ni (\tilde{\xi}^T, \tilde{\zeta}^T) \doteq \delta(\bar{\xi}, \bar{\zeta}) + (1 - \delta)(\xi^T, \zeta^T)$. By Lemma 5, we have

$$I_T(\tilde{\xi}^T, \tilde{\zeta}^T) \leq \delta I_T(\bar{\xi}, \bar{\zeta}) + (1 - \delta) I_T(\xi^T, \zeta^T) \leq \delta I_T(\bar{\xi}, \bar{\zeta}) + I_T(\xi^T, \zeta^T),$$

and hence,

$$\liminf_{T \rightarrow \infty} \frac{I_{\gamma, T}^*}{T} \geq \liminf_{T \rightarrow \infty} \frac{I_T(\xi^T, \zeta^T)}{T} \geq \liminf_{T \rightarrow \infty} \frac{I_T(\tilde{\xi}^T, \tilde{\zeta}^T)}{T} - \delta C(\gamma). \quad (60)$$

Now, let

$$\sigma_T \doteq \inf \left\{ t : \tilde{\xi}^T(t) \geq 1 \text{ or } \tilde{\zeta}^T(t) \geq 1 \right\} \wedge 1.$$

By restricting the attention to the cost over $[\sigma_T, T]$, we have

$$I_T(\tilde{\xi}^T, \tilde{\zeta}^T) \geq \inf_{(\xi, \zeta) \in \mathcal{J}(\tilde{\xi}^T(\sigma_T), \frac{\gamma T - \tilde{\xi}^T(\sigma_T)}{T - \sigma_T}, T - \sigma_T)} I_{T - \sigma_T}(\xi, \zeta).$$

Let $i = 1$ if $\gamma > \lambda - \mu$ and $i = -1$ if $\gamma < \lambda - \mu$. Noting that $\delta \in (0, |\gamma - (\lambda - \mu)| \wedge 1)$ and $\tilde{\xi}^T(\sigma_T), \tilde{\zeta}^T(\sigma_T), \sigma_T \in [0, 1]$, using the last statement in Lemma 11, we can find $T_2 \in (T_1, \infty)$ such that, for all $T \geq T_2$,

$$\inf_{(\xi, \zeta) \in \mathcal{J}(\tilde{\xi}^T(\sigma_T), \frac{\gamma T - \tilde{\xi}^T(\sigma_T)}{T - \sigma_T}, T - \sigma_T)} I_{T - \sigma_T}(\xi, \zeta) \geq \inf_{(\xi, \zeta) \in \mathcal{J}(\tilde{\xi}^T(\sigma_T), \gamma - i\delta, T - \sigma_T)} I_{T - \sigma_T}(\xi, \zeta).$$

Let $(\bar{\xi}^T, \bar{\zeta}^T) \in \mathcal{J}(\tilde{\xi}^T(\sigma_T), \gamma - i\delta, T - \sigma_T)$ be given by Construction 5.1. Noting that $\sigma_T > 0$, from Lemma 8(a), we have $\bar{\xi}(\sigma_T) > 0$, and hence, $0 < \bar{\xi}^T(\sigma_T) \leq 1$. It then follows from Lemma 10 that

$$\inf_{(\xi, \zeta) \in \mathcal{J}(\tilde{\xi}^T(\sigma_T), \gamma - i\delta, T - \sigma_T)} I_{T - \sigma_T}(\xi, \zeta) = I_{T - \sigma_T}(\bar{\xi}^T, \bar{\zeta}^T)$$

for all $T \geq T_3$ for some $T_3 \in (T_2, \infty)$. From Lemma 9 (applied with $K = [0, 1]$),

$$\lim_{T \rightarrow \infty} \frac{I_{T - \sigma_T}(\bar{\xi}^T, \bar{\zeta}^T)}{T - \sigma_T} = C(\gamma - i\delta).$$

Combining the last four displays, we have

$$\liminf_{T \rightarrow \infty} \frac{I_T(\tilde{\xi}^T, \tilde{\zeta}^T)}{T} \geq \liminf_{T \rightarrow \infty} \frac{I_{T - \sigma_T}(\bar{\xi}^T, \bar{\zeta}^T)}{T} = C(\gamma - i\delta).$$

Combining this with (60), taking $\delta \rightarrow 0$, and using the continuity of $C(\gamma)$ completes the proof. Q.E.D.

The following proposition gives the analogue of Proposition 3 when $\gamma = 0$.

Proposition 4.

$$\liminf_{T \rightarrow \infty} \frac{I_{0, T}^*}{T} \geq C(0).$$

Proof. It follows from Lemma 11 and Proposition 3 that

$$\liminf_{T \rightarrow \infty} \frac{I_{0, T}^*}{T} \geq \liminf_{T \rightarrow \infty} \frac{I_{\varepsilon, T}^*}{T} \geq C(\varepsilon)$$

for each $\varepsilon \in (0, \lambda - \mu)$. The result follows on sending $\varepsilon \rightarrow 0$.

We can now complete the proof of Theorem 3. Q.E.D.

Proof of Theorem 3. The first statement follows from Propositions 2–4. For the second and third statements, note that, by Lemma 11, the following holds for sufficiently large T :

$$\begin{aligned} I_{\gamma, T} &= \inf \{ I_T(\xi, \zeta) : \zeta(T) \geq \gamma T \} = \inf_{\tilde{\gamma} \geq \gamma} I_{\tilde{\gamma}, T}^* = I_{\gamma, T}^*, \quad \gamma > \lambda - \mu, \\ \tilde{I}_{\gamma, T} &\doteq \inf \{ I_T(\xi, \zeta) : \zeta(T) \leq \gamma T \} = \inf_{\tilde{\gamma} \leq \gamma} I_{\tilde{\gamma}, T}^* = I_{\gamma, T}^*, \quad 0 \leq \gamma < \lambda - \mu. \end{aligned}$$

This proves these two statements when $\gamma \neq \lambda - \mu$. The result for the case when $\gamma = \lambda - \mu$ is immediate on observing that

$$0 \leq I_{\gamma,T} \leq I_{\gamma,T}^*, \quad 0 \leq \tilde{I}_{\gamma,T} \leq I_{\gamma,T}^*, \quad \lim_{T \rightarrow \infty} \frac{I_{\gamma,T}^*}{T} = 0 = C(\gamma). \quad \text{Q.E.D.}$$

Remark 6. We now provide a heuristic interpretation of the minimizers $(\bar{\xi}, \bar{\zeta})$ given in Construction 5.1. First, observe that, in the case that there is no reneging (i.e., $\theta = 0$), the law of large numbers limit of the state process X^n is given as the solution of the ordinary differential equation (ODE):

$$\dot{\xi}_0(t) = (\lambda - \mu), \quad \xi_0(0) = x_0.$$

Thus, the limit is simply given as the trajectory with initial value the scaled limit of initial queue length and velocity given as the difference between the inflow and the outflow rates. In the case in which the reneging occurs at rate θ , the LLN limit of X^n is given in (45), which can be rewritten as

$$\dot{\xi}_0(t) = -\theta \xi_0(t) + (\lambda - \mu), \quad \xi_0(0) = x_0.$$

Thus, the only difference is that the velocity is decreased by a factor of $\theta \xi_0(t)$ to account for rate θ reneging. Now fix $\gamma > 0$. Then, the minimizing trajectories over the time interval $[0, T]$ associated with an asymptotic reneging rate of γ are given as in Construction 5.1. Recall from Lemma 6(b) that the parameters A, B (which depend on T) associated with these trajectories satisfy $A \rightarrow 0$, $B \rightarrow z_\gamma^{-1}$, and $1 - Ae^{\theta T} \rightarrow z_\gamma$. The trajectory ξ^* obtained by replacing in the definition of $\bar{\xi}$ these asymptotic values of A and B is given as the solution of the ODE

$$\dot{\xi}^*(t) = -\theta \xi^*(t) + (\lambda z_\gamma^{-1} - \mu z_\gamma), \quad \xi^*(0) = x_0.$$

Thus, formally speaking, in order for the queue to experience a long-term reneging rate of γ , the arrival and service rates behave atypically according to λz_γ^{-1} [resp. μz_γ] instead of their nominal values λ [resp. μ]. Recall that, when $\gamma = \lambda - \mu$, $z_\gamma = 1$ in which case ξ^* corresponds to the LLN limit.

7. Multiserver Queue

The techniques developed in this paper for the analysis of the $M/M/1 + M$ model extend to the multiserver setting, namely to the $M/M/n + M$ model. In this section, we outline this extension without proofs. As in the case with the single-server model, the arrival rate considered is scaled up by n , whereas the individual reneging rate remains fixed. However, the service station now consists of n exponential servers each serving at a fixed rate (this is often referred to in the literature as many-server scaling; see Atar et al. [1] and Kang and Ramanan [10]). Again, let U^n and V^n denote the number in system process and the cumulative reneging count, respectively. Then, the queue length process is now given by $(U^n - n)^+$, and the number of customers in service by $U^n \wedge n$. As before, the rescaled versions are denoted by $X^n = U^n/n$ and $Y^n = V^n/n$.

As with the $M/M/1 + M$ setting, it is convenient to introduce certain PRM. Replace N_2 in Section 2 by a PRM on $[0, T] \times \mathbb{R}_+^2$ with intensity $\mu ds \times dy \times dz$ and denote it once more by N_2 . Let $\mathbb{X}_2 \doteq \mathbb{R}_+^2$. Let $\bar{A}_i$ for $i = 1, 3$ be as before and $\bar{A}_2 = \bar{A}_3$. For $\varphi \in \bar{A}_i$, $i = 1, 3$, N_i^φ are defined as before and let N_2^φ be defined using N_2 and φ as N_3^φ is defined using N_3 and φ . The processes (X^n, Y^n) are given as follows.

$$\begin{aligned} X^n(t) &= x_n + \frac{1}{n} \int_{[0,t]} N_1^n(ds) - \frac{1}{n} \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{[0, X^n(s-) \wedge 1]}(y) N_2^n(ds dy) - \frac{1}{n} \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{[0, (X^n(s-)-1)^+]}(y) N_3^n(ds dy). \\ Y^n(t) &= \frac{1}{n} \int_{[0,t] \times \mathbb{R}_+} \mathbf{1}_{[0, (X^n(s-)-1)^+]}(y) N_3^n(ds dy). \end{aligned}$$

On the RHS of the first equation, the three integrals correspond to the arrival, departure, and reneging processes, respectively, normalized by n . Thinning with an indicator of $[0, X^n(s-) \wedge 1]$ corresponds to the fact that the service rate is proportional to the number of customers in service; indeed, normalizing $U^n \wedge n$ by n gives $X^n \wedge 1$. Similarly, the expression in the third integral accounts for the fact that the total reneging rate is proportional to the queue length, $(U^n - n)^+$, which, after normalization, is given by $(X^n - 1)^+$.

The LLN of (X^n, Y^n) is governed by the following equations:

$$\begin{aligned}x(t) &= x_0 + \lambda t - \mu \int_0^t (x(s) \wedge 1) ds - \theta \int_0^t (x(s) - 1)^+ ds, \quad t \in [0, T], \\y(t) &= \theta \int_0^t (x(s) - 1)^+ ds, \quad t \in [0, T].\end{aligned}\tag{61}$$

We now introduce the rate function associated with LDP for (X^n, Y^n) . For $(\xi, \zeta) \in \mathcal{C}([0, T] : \mathbb{R}_+^2)$, let $\mathcal{U}(\xi, \zeta)$ be the collection of all $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ such that $\varphi_i : [0, T] \rightarrow \mathbb{R}_+$, $i = 1, 2, 3$ are measurable maps and the following equations are satisfied for all $t \in [0, T]$:

$$\begin{aligned}\xi(t) &= x_0 + \lambda \int_0^t \varphi_1(s) ds - \mu \int_0^t (\xi(s) \wedge 1) \varphi_2(s) ds - \theta \int_0^t \varphi_3(s) (\xi(s) - 1)^+ ds, \\ \zeta(t) &= \theta \int_0^t \varphi_3(s) (\xi(s) - 1)^+ ds.\end{aligned}\tag{62}$$

For $(\xi, \zeta) \in \mathcal{C}([0, T] : \mathbb{R}_+^2)$, define

$$I_T(\xi, \zeta) \doteq \inf_{\varphi \in \mathcal{U}(\xi, \zeta)} \left\{ \lambda \int_0^T \ell(\varphi_1(s)) ds + \mu \int_0^T (\xi(s) \wedge 1) \ell(\varphi_2(s)) ds + \theta \int_0^T (\xi(s) - 1)^+ \ell(\varphi_3(s)) ds \right\}.\tag{63}$$

Set $I_T(\xi, \zeta)$ to be ∞ if $\mathcal{U}(\xi, \zeta)$ is empty or $(\xi, \zeta) \in \mathcal{D}([0, T] : \mathbb{R}_+^2) \setminus \mathcal{C}([0, T] : \mathbb{R}_+^2)$. The following result can be established using similar methods as for the proof of Theorem 1.

Theorem 5. *The pair $\{(X^n, Y^n)\}$ satisfies an LDP on $\mathcal{D}([0, T] : \mathbb{R}_+^2)$ with rate function I_T .*

Once more, using contraction principle one obtains an LDP for $\zeta(T)$, namely Theorem 2 holds with I_T^* as in (8) and I_T as in (63). For the analysis of the calculus of variations problem, we assume for simplicity that $\lambda \geq \mu$ and that $x_0 \geq 1$. As in Lemma 4, we can give an alternative representation for the rate function I_T in terms of a local rate function L . In calculating the minimizer for $I_{\gamma, T}^*$, one considers again Euler–Lagrange equations as in Section 5.1. One sees that, in finding the minimizer, one can restrict attention to trajectories ξ that satisfy $\xi(t) \geq 1$ for all t . In fact, the minimizer for $\gamma > 0$ is given by $(\bar{\xi}^*, \bar{\zeta}^*) = (\bar{\xi}_{(x_0-1)} + 1, \bar{\zeta}_{(x_0-1)})$, where $(\bar{\xi}_{(x_0-1)}, \bar{\zeta}_{(x_0-1)})$ is given by Construction 5.1 with x_0 replaced by $x_0 - 1$. Using this, one can show that Theorem 3 holds for the $M/M/n + M$ model with the same decay rate $C(\gamma)$. We omit the details.

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References

- [1] Atar R, Kaspi H, Shimkin N (2014) Fluid limits for many-server systems with reneging under a priority policy. *Math. Oper. Res.* 39(3):672–696.
- [2] Atar R, Budhiraja A, Dupuis P, Wu R (2021) Robust bounds and optimization at the large deviations scale for queueing models via Rényi divergence. *Ann. Appl. Probab.* Forthcoming.
- [3] Budhiraja A, Dupuis P (2019) *Analysis and Approximation of Rare Events: Representations and Weak Convergence Methods*, vol. 94 (Springer, New York).
- [4] Budhiraja A, Chen J, Dupuis P (2013) Large deviations for stochastic partial differential equations driven by a Poisson random measure. *Stochastic Processes Their Appl.* 123(2):523–560.
- [5] Budhiraja A, Dupuis P, Ganguly A (2016) Moderate deviation principles for stochastic differential equations with jumps. *Ann. Probab.* 44(3):1723–1775.
- [6] Budhiraja A, Dupuis P, Maroulas V (2011) Variational representations for continuous time processes. *Annales de l’Institut Henri Poincaré(B). Probabilités et Statistiques* 47(3):725–747.
- [7] Dupuis P, Ellis RS (1997) *A Weak Convergence Approach to the Theory of Large Deviations* (John Wiley & Sons, New York).
- [8] Dupuis P, Ishii H (1991) On Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications. *Stochastics* 35(1):31–62.
- [9] Ikeda N, Watanabe S (1981) *Stochastic Differential Equations and Diffusion Processes*, North-Holland Mathematical Library, vol. 24 (Elsevier, New York).
- [10] Kang W, Ramanan K (2010) Fluid limits of many-server queues with reneging. *Ann. Appl. Probab.* 20(6):2204–2260.
- [11] Karatzas I, Shreve SE (1991) *Brownian Motion and Stochastic Calculus*, Graduate Texts in Mathematics, vol. 113 (Springer, New York).
- [12] Kaspi H, Ramanan K (2011) Law of large numbers limits for many-server queues. *Ann. Appl. Probab.* 21(1):33–114.
- [13] Kaspi H, Ramanan K (2013) SPDE limits of many-server queues. *Ann. Appl. Probab.* 23(1):145–229.
- [14] Kurtz TG (1981) *Approximation of Population Processes*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 36 (SIAM, Philadelphia).

- [15] Puha AL, Ward AR (2019) Scheduling an overloaded multiclass many-server queue with impatient customers. Netessine S, ed. *Operations Research & Management Science in the Age of Analytics* (INFORMS, Catonsville, MD), 189–217.
- [16] Reed J (2009) The $G/GI/N$ queue in the Halfin-Whitt regime. *Ann. Appl. Probab.* 19(6):2211–2269.
- [17] Troutman JL (2012) *Variational Calculus and Optimal Control: Optimization with Elementary Convexity* (Springer, New York).
- [18] Ward AR (2012) Asymptotic analysis of queueing systems with reneging: A survey of results for FIFO, single class models. *Surveys Oper. Res. Management Sci.* 17(1):1–14.