SOBOLEV ESTIMATES AND DUALITY FOR $\overline{\partial}$ ON DOMAINS IN $\mathbb{CP}^n$

SIQI FU AND MEI-CHI SHAW

Abstract. We study L^2 and Sobolev estimates for solutions of the Cauchy-Riemann equation on pseudoconvex and pseudoconcave domains in $\mathbb{CP}^n$. We also formulate the weak and strong extensions of the $\overline{\partial}$ equation in Sobolev spaces and study their dual problems.

Dedicated to Professor Joseph J. Kohn

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1. Introduction

Since the fundamental work of Kohn ([23, 24]) for the $\overline{\partial}$-Neumann problem on smooth bounded strongly pseudoconvex domains in $\mathbb{C}^n$ and that of Hörmander ([20]) on L^2-estimates of the Cauchy-Riemann operator on bounded pseudoconvex domains in $\mathbb{C}^n$, there has been tremendous progress on L^2-Sobolev theory of the $\overline{\partial}$-operator and the $\overline{\partial}$-Neumann problem for bounded pseudoconvex domains in $\mathbb{C}^n$ (see, for example, monographs [11, 21, 8, 36] for expositions on the subject). One of the most important results is the Sobolev estimates for bounded smooth pseudoconvex domains in $\mathbb{C}^n$ (see [25]). For $s \geq 0$, let $H^{p,q}_{W^s}(\Omega)$ be the Dolbeault cohomology with Sobolev W^s coefficients defined by

$$H^{p,q}_{W^s}(\Omega) = \frac{\{f \in W^s_{p,q}(\Omega) \mid \overline{\partial} f = 0\}}{\{f \in W^s_{p,q}(\Omega) \mid f = \overline{\partial} u, u \in W^s_{p,q-1}(\Omega)\}}.$$

Theorem 1.1 (**Kohn**). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. For every $0 \leq p \leq n$, $0 < q < n$ and $s \geq 0$,*

$$H^{p,q}_{W^s}(\Omega) = 0. \tag{1.1}$$

The theory is less developed for domains in the complex projective space $\mathbb{CP}^n$. The L^2-Sobolev theory of Kohn [25] or Hörmander [20] does not readily generalize to pseudoconvex domains in $\mathbb{CP}^n$, since there is no strictly plurisubharmonic function that can be used as a weight in $\mathbb{CP}^n$. On the other hand, the Fubini-Study metric on $\mathbb{CP}^n$ has a positive holomorphic bisection curvature which can be used to study these problems. In this paper, we discuss some methods and results on L^2 and Sobolev estimates on pseudoconvex and pseudoconcave domains in $\mathbb{CP}^n$.

The plan of the paper is as follows: In Section 2 we discuss the $\overline{\partial}$-Neumann problem for pseudoconvex domains in $\mathbb{CP}^n$. As an application, we give an alternative approach to Hörmander's L^2 existence theorems for bounded domains in $\mathbb{C}^n$ using Bochner-Kodaira-Morrey-Kohn formula and the curvature property of the Fubini-Study metric, but without

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using weights (see Corollary 2.4). In Section 3, we discuss the L^2 theory for $\overline{\partial}$ on Lipschtiz pseudoconvex domains in $\mathbb{CP}^n$. The results in this section is known when the domain has C^2-smooth boundary (see [2] or [4]). There is some necessary modification when generalizing this result to Lipschitz domains. In Section 4, we examine the W^1 estimates on pseudoconcave domains. We show that the range of $\overline{\partial}$ in $W^1(\Omega)$ is closed for all degrees, including the critical case when $q = n-1$. Notice that on a pseudoconcave domain, the cohomology for $q = n-1$ is Hausdorff and infinite dimensional (see [22] and [32, 33]). This can be used to characterize annuli domains in $\mathbb{C}^n$ (see [13]). In Section 5, we discuss the $\overline{\partial}$ operator in the Hilbert space W^s setting and prove duality results. In the L^2 setting, this is done in earlier work (see [20] or [8]). Though Sobolev estimates for $\overline{\partial}$ on pseudoconvex domains in $\mathbb{CP}^n$ remain an open problem, we hope the duality results, Theorems 5.12 and 5.13, will shed some light on this intriguing problem (see Remark at the end of Section 5).

2. Fubini-Study metric and the $\overline{\partial}$ problem in $\mathbb{C}^n$

Let (X, ω) be an n-dimensional Kähler manifold with Kähler form

$$\omega = -\frac{i}{2}\sum_{\alpha,\beta=1}^{n} h_{\alpha\bar\beta}\, dz_\alpha \wedge d\bar z_\beta$$

in local holomorphic coordinates and the associated hermitian metric h. The volume form of X is then given by $dV = \omega^n/n!$. Let ∇ be the Levi-Civita connection for the associated Riemannian metric $g = \text{Re } h$, which is identical to the Chern connection on the holomorphic tangent bundle $T^{1,0}X$ due to the Kähler condition. We will use $|\cdot|_\omega$ and $\langle \cdot, \cdot\rangle_\omega$ to denote respectively the pointwise norm and inner product induced by ω. (Hereafter, we will identify Kähler form with the associated hermitian metric. We might drop the subscript ω when it is clear from the context.) Let

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

be the curvature tensor, extended to be $\mathbb{C}$-linear and to act on tensors of any type. The curvature tensor is then given by

$$R_{\alpha\bar\beta\gamma\bar\delta} = h(R(\frac{\partial}{\partial z_\gamma},\ \frac{\partial}{\partial \bar z_\delta})\frac{\partial}{\partial \bar z_\beta},\ \frac{\partial}{\partial z_\alpha}) \tag{2.1}$$

$$= \frac{\partial^2 h_{\alpha\bar\beta}}{\partial z^\gamma \partial \bar z^\delta} - \sum_{\varepsilon,\tau} h^{\bar\varepsilon\tau} \frac{\partial h_{\alpha\bar\varepsilon}}{\partial z^\gamma}\frac{\partial h_{\tau\bar\beta}}{\partial \bar z^\delta} \tag{2.2}$$

where $h^{\bar\varepsilon\tau}$ denotes the inverse of $h_{\tau\bar\varepsilon}$.

Let $L_1, \ldots, L_n$ be a local orthonormal frame field of type $(1, 0)$ and $\omega^1, \ldots, \omega^n$ be the coframe field. For a (p, q)-form u, we set

$$\langle \Theta u, u\rangle = \sum_{j,k=1}^{n} \langle \bar\omega^j \wedge \big(\bar L_k \lrcorner R(L_j, \bar L_k)u\big),\ u\rangle,$$

where $\lrcorner$ is the usual contraction operator. For a C^2-smooth function φ, we set

$$\langle (\partial\overline{\partial}\varphi)u,\ u\rangle = \sum_{j,k=1}^{n} \partial\overline{\partial}\varphi(L_j, \bar L_k)\langle \bar L_j \lrcorner u, \bar L_k \lrcorner u\rangle.$$

Let Ω be a relatively compact domain in X with C^2-smooth boundary $b\Omega$. Let $\rho(z)$ be the signed distance function from z to $b\Omega$ such that $\rho(z) = -d(z, b\Omega)$ for $z \in \Omega$ and

$\rho(z) = d(z, b\Omega)$ when $z \in X \setminus \Omega$. Let φ be a real-valued C^2 function on $\overline{\Omega}$. Let $L^2_{p,q}(\Omega, e^{-\varphi})$ is the space of (p,q)-forms u on Ω such that

$$\|u\|^2_\varphi = \int_\Omega |u|^2_\omega e^{-\varphi} dV < \infty.$$

We will also use $(\cdot,\ \cdot)_\varphi$ to denote the associated inner product. Let $\overline{\partial}^*_\varphi$ be the adjoint of the maximally defined $\overline{\partial}\colon L^2_{p,q}(\Omega, e^{-\varphi}) \to L^2_{p,q}(\Omega, e^{-\varphi})$. We now recall an integration by parts formula due to Bochner, Kodaira, Morrey, Kohn, and Hörmander that is basic to the study of the complex Laplacian. With the above notations, we can now state the following *Basic Identity* (see [38, 34, 4]).

Theorem 2.1 (**Bochner-Kodaira-Morrey-Kohn-Hörmander**). *Let Ω be a relatively compact domain in a Kähler manifold X with C^2-smooth boundary $b\Omega$. For any $u \in C^1_{p,q}(\overline{\Omega}) \cap \operatorname{dom}(\overline{\partial}^*)$, we have*

$$\|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi = \|\overline{\nabla} u\|^2_\varphi + +(\Theta u,\ u)_\varphi + ((\partial\overline{\partial}\varphi)u,\ u)_\varphi + \int_{b\Omega} \langle (\partial\overline{\partial}\rho)u,\ u\rangle e^{-\varphi} dS \tag{2.3}$$

where dS is the induced surface element on $b\Omega$ and $|\overline{\nabla} u|^2 = \sum_{j=1}^n |\nabla_{\bar{L}_j} u|^2$.

The Kähler form associated with the Fubini-Study metric $g_{\rm FS}$ on the complex projective space $\mathbb{CP}^n$ is given by

$$\omega_{\rm FS} = i\partial\overline{\partial} \log(1 + |z|^2) \tag{2.4}$$

$$= i \sum_{\alpha,\beta=1}^n g_{\alpha\bar{\beta}}(z)\, dz_\alpha \wedge d\bar{z}_\beta \tag{2.5}$$

in local inhomogeneous coordinates, where

$$g_{\alpha\bar{\beta}}(z) = \frac{\partial^2 \log(1 + |z|^2)}{\partial z_\alpha \partial \bar{z}_\beta} = \frac{(1 + |z|^2)\delta_{\alpha\bar{\beta}} - \bar{z}_\alpha z_\beta}{(1 + |z|^2)^2}. \tag{2.6}$$

The volume form is then

$$dV_{\rm FS} = \det(g_{\alpha\bar{\beta}}(z)) dV_{\rm E} = \frac{1}{(1 + |z|^2)^{n+1}} dV_{\rm E} \tag{2.7}$$

where $dV_{\rm E}$ is the Euclidean volume form. The curvature tensor is then given by

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = g_{\alpha\bar{\beta}} g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}} g_{\bar{\beta}\gamma}.$$

It follows that the complex projective space $\mathbb{CP}^n$ with the Fubini-Study metric has constant holomorphic sectional curvature 2 and its holomorphic bisectional curvature is bounded between 1 and 2. Furthermore, we have that if u is a (p,q)-form on $\mathbb{CP}^n$ with $q \geq 1$, then

$$\langle \Theta u, u\rangle = 0, \quad \text{if} \quad p = n; \qquad \langle \Theta u, u\rangle \geq 0, \quad \text{if} \quad p \geq 1; \tag{2.8}$$

and

$$\langle \Theta u, u\rangle = q(2n + 1)|u|^2 \quad \text{if} \quad p = 0. \tag{2.9}$$

For a proof of these results, see [38] or Proposition A.5 in the Appendix in [4].

Proposition 2.2. *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ with C^2 boundary and $1 \leq q \leq n - 1$. Let φ be a plurisubharmonic function on Ω. Then*

$$\|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi \geq q(2n + 1)\|u\|^2_\varphi \tag{2.10}$$

for any $(0,q)$-form $u \in \operatorname{dom}(\overline{\partial}) \cap \operatorname{dom}(\overline{\partial}^*_\varphi)$.

Proof. This is a direct consequence of the curvature property (2.9) and (2.3):

$$\begin{aligned}\|\overline{\partial}u\|_\varphi^2 + \|\overline{\partial}_\varphi^* u\|_\varphi^2 &= \|\overline{\nabla}u\|_\varphi^2 + (\Theta u,\ u)_\varphi + ((\partial\overline{\partial}\varphi)u, u)_\varphi + \int_{b\Omega} \langle(\partial\overline{\partial}\rho)u,\ u\rangle e^{-\varphi} dS \\ &\geq (\Theta u,\ u)_\varphi \geq q(2n+1)\|u\|_\varphi^2.\end{aligned} \tag{2.11}$$

□

Theorem 2.3. *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ such that $\overline{\Omega} \neq \mathbb{CP}^n$ and $1 \leq q \leq n-1$. Let φ be a plurisubharmonic function on Ω. For any $\overline{\partial}$-closed $(0,q)$-form $f \in L^2_{0,q}(\Omega, e^{-\varphi})$, there exists a $(0,q-1)$-form $u \in L^2_{0,q-1}(\Omega, e^{-\varphi})$ such that $\overline{\partial}u = f$ with*

$$\|u\|_\varphi^2 \leq \frac{1}{q(2n+1)} \|f\|_\varphi^2. \tag{2.12}$$

Proof. If Ω has C^2 boundary, estimate (2.12) is then a consequence of (2.11). The general case is then proved by exhausting Ω from inside by pseudoconvex domains with smooth boundaries. □

Corollary 2.4. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with diameter δ, where $\delta = \sup_{z,z' \in \Omega} |z - z'|$. Then for any $f \in L^2_{p,q}(\Omega)$ with $\overline{\partial}f = 0$, there is a $(p, q-1)$-form $u \in L^2_{(p,q-1)}(\Omega)$ such that $\overline{\partial}u = f$ with*

$$\|u\|^2 \leq C_{n,q}\delta^2\|f\| \tag{2.13}$$

where $C_{n,q}$ is a constant depending only on n and q, but is independent of Ω.

Proof. We may assume that $p = 0$. First we assume that Ω has C^2 boundary and its diameter $\delta < 1$. The estimate (2.13) follows from (2.11). The general case can be obtained by exhausting Ω by smooth subdomains with C^2 boundary.

For general bounded domain Ω with radius δ, the estimate (2.13) follows from scaling argument. □

Remark. Corollary 2.4 is a weaker version of the Hörmander's L^2 theory (see [20]), where he proves the L^2 existence with estimate (2.13) with $c_{n,q} = e/q$, which is independent of n.

3. L^2 theory for $\overline{\partial}$ on Lipschitz pseudoconvex domains in $\mathbb{CP}^n$

The following theorem is based on an earlier result of Berndtsson and Charpentier [2, Theorem 2.3] (see also [18, 4]).

Theorem 3.1. *Let (X, ω) be a Kähler manifold of dimension n. Assume that the curvature operator Θ is semi-positive on (p,q)-forms for all $1 \leq q \leq n$. Let Ω be a Stein domain in X. Suppose that there exist a distance function $\rho < 0$ and a constant $\eta > 0$ such that*

$$-i\partial\overline{\partial}(-\rho)^\eta \geq \eta K(-\rho)^\eta\omega$$

on Ω for some constant $K > 0$. Then the $\overline{\partial}$-Neumann Laplacian $\Box$ has a bounded inverse N on $L^2_{p,q}(\Omega)$ and for $u \in Dom(\overline{\partial}) \cap Dom(\overline{\partial}^*)$,

$$\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2 \geq \frac{q\eta K}{4}\|u\|^2. \tag{3.1}$$

Furthermore, the operator N is bounded from $W^s_{p,q}(\Omega) \to W^s_{p,q}(\Omega)$ with

$$\|\overline{\partial}^* N u\|_s^2 \leq C_\eta\|u\|_s^2; \qquad \|\overline{\partial}Nu\|_s^2 \leq C_\eta\|u\|_s^2. \tag{3.2}$$

for any $u \in W^s_{p,q}(\Omega)$ with $0 < s < \eta/2$.

Proof. For any sufficiently small $\varepsilon > 0$, by Richberg's theorem, there exists $\sigma \in C^\infty(\Omega)$ such that

$$-(-\rho)^\eta \le \sigma \le -(1-\varepsilon)(-\rho)^\eta \quad \text{and} \quad -i\partial\overline{\partial}\sigma \ge (1-\varepsilon(-\rho)^\eta)\eta K(-\rho)^\eta\omega.$$

Let $\widetilde{\rho} = -(-\sigma)^{1/\eta}$. Then $\widetilde{\rho} \in C^\infty(\Omega)$, $\widetilde{\rho} < 0$ and

$$-i\partial\overline{\partial}(-\widetilde{\rho})^\eta \ge (1-\varepsilon(-\rho)^\eta)\eta K(-\widetilde{\rho})^\eta\omega \tag{3.3}$$

Let f be a $\overline{\partial}$-close form in $L^2_{p,q}(\Omega)$. Let $\Omega_j \subset\subset \Omega$ be an increasing sequence of smooth bounded pseudoconvex domains whose union is Ω. Let $0 < r < 1$ be a constant to be chosen and let $\varphi = -r\eta\log(-\widetilde{\rho})$. It then follows from (3.3) that

$$i\partial\varphi \wedge \overline{\partial}\varphi \le ri\partial\overline{\partial}\varphi \quad \text{and} \quad i\partial\overline{\partial}\varphi \ge (1-\varepsilon C_j^\eta)r\eta K\omega \tag{3.4}$$

on Ω_j, where $C_j = \max\{-\rho(z) \mid z \in \Omega_j\}$. Let $\alpha \in (0, \eta)$. Applying (2.3) to Ω_j with weight φ, and using the semi-positivity condition on the curvature operator Θ, we then have

$$\|\overline{\partial}u\|^2_{\varphi,\Omega_j} + \|\overline{\partial}^*_\varphi u\|^2_{\varphi,\Omega_j} \ge ((\partial\overline{\partial}\varphi)u,\ u)_\varphi \tag{3.5}$$

for any $u \in C^1_{p,q}(\overline{\Omega_j}) \cap \operatorname{dom}(\overline{\partial}^*_{\Omega_j})$. By Demailly's formulation of Hörmander's L^2-estimates (see [9, Theorem 4.1]), that there exists $u_j \in L^2_{p,q-1}(\Omega, e^{-\varphi})$ such that $\overline{\partial}u_j = f$ and

$$\int_{\Omega_j} |u_j|^2 e^{-\varphi}\, dV \le \int_{\Omega_j} |f|^2_{i\partial\overline{\partial}\varphi} e^{-\varphi}\, dV \tag{3.6}$$

Let u_j be the solution that is orthogonal to $\mathcal{N}(\overline{\partial})$, the nullspace of $\overline{\partial}$, in $L^2(\Omega_j, e^{-\varphi})$. Let $v_j = u_j e^{\varphi}$. Then $v_j \perp \mathcal{N}(\overline{\partial})$ in $L^2(\Omega_j, e^{-2\varphi})$. Applying (3.6) with the weight φ replaced by 2φ, we then have

$$\int_{\Omega_j} |v_j|^2 e^{-2\varphi}\, dV \le \int_{\Omega_j} |\overline{\partial}v_j|^2_{2i\partial\overline{\partial}\varphi} e^{-2\varphi}\, dV$$

Thus

$$\int_{\Omega_j} |u_j|^2\, dV \le \int_{\Omega_j} |\overline{\partial}u_j + \overline{\partial}\varphi \wedge u|^2_{2i\partial\overline{\partial}\varphi}\, dV. \tag{3.7}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
|\overline{\partial}u_j + \overline{\partial}\varphi \wedge u_j|^2_{2i\partial\overline{\partial}\varphi} &\le (1+1/t)|\overline{\partial}u_j|^2_{2i\partial\overline{\partial}\varphi} + (1+t)|\overline{\partial}\varphi \wedge u_j|^2_{2i\partial\overline{\partial}\varphi} \\
&\le (1+1/t)|\overline{\partial}u_j|^2_{2i\partial\overline{\partial}\varphi} + (1+t)|\overline{\partial}\varphi|^2_{2i\partial\overline{\partial}\varphi}|u_j|^2 \\
&\le \frac{1+1/t}{2qr\eta K(1-\varepsilon C_j^\eta)}|\overline{\partial}u_j|^2 + \frac{(1+t)r}{2}|u_j|^2.
\end{aligned}$$

Here in the last inequality, we have used (3.3). It then follows from (3.7) that

$$\int_{\Omega_j} |u_j|^2\, dV \le \frac{1+1/t}{(1-(1+t)r/2)r} \cdot \frac{1}{2q\eta K(1-\varepsilon C_j^\eta)} \int_{\Omega_j} |\overline{\partial}u_j|^2$$

We now take $t = 1$ and $r = 1/2$ to minimize the first factor on the right-hand side of the above inequality. By choosing a sequence of $\varepsilon \to 0$ and letting $j \to \infty$, we then obtain $u \in L^2_{p,q}(\Omega)$ as a weak limit of a subsequence of $\{u_j\}$ such that $\overline{\partial}u = f$ and

$$\int_\Omega |u|^2\, dV \le \frac{4}{q\eta K}\int_\Omega |f|^2\, dV.$$

This yields the basic estimate (3.1). The basic estimate (3.1) implies the existence of the $\overline{\partial}$-Neumann operators for all degrees (see [20] or [8]). The first part of the inequality

(3.2) follows from [2] and the second part follows from [4]. The boundedness of N follows from (3.2) □

Let ω_{FS} be the Kähler form associated with the Fubini-Study metric on $\mathbb{CP}^n$. Let Ω be a (proper) pseudoconvex domain in $\mathbb{CP}^n$ with C^2-smooth boundary. Let $\delta(z) = d(z, b\Omega)$ be the distance, with respect to the Fubini-Study metric, from z to the boundary $b\Omega$. Let $\Omega_\varepsilon = \{z \in \Omega \mid \delta(z) > \varepsilon\}$. It then follows from Takeuchi's theorem [37] that there exists a universal constant $K_0 > 0$ such that

$$i\partial\overline{\partial}(-\log\delta) \geq K_0\omega_{\mathrm{FS}} \tag{3.8}$$

on Ω. In particular, there exists $\epsilon_0 > 0$ such that

$$\partial\overline{\partial}(-\delta)(\zeta, \overline{\zeta}) \geq K_0\epsilon|\zeta|^2_{\omega_{\mathrm{FS}}} \tag{3.9}$$

for all $\zeta \in T^{1,0}_x(b\Omega_\epsilon)$ for $0 \leq \epsilon \leq \epsilon_0$. (See [15, 5] for different proofs of Takeuchi's theorem.) Obsawa and Sibony [30] showed–as a consequence of Takeuchi's theorem–that, there exists $0 < \eta \leq 1$ such that

$$i\partial\overline{\partial}(-\delta^\eta) \geq K\eta\delta^\eta\omega_{\mathrm{FS}} \tag{3.10}$$

on Ω for some constant $K > 0$. (See [5, Proposition 2.3] and [4, Lemma 2.2] for a more streamlined proof of this fact.) Such a constant η is called a Diederich-Fornæss exponent of Ω (see [10]). We refer the reader to [17] for similar results when the boundary is only Lipschitz, and to [14, 1] for relevant results on the Diederich-Fornæss exponent and nonexistence of Levi-flat hypersurfaces in complex manifolds. Combining (3.10) with Theorem 3.1, we then have:

Theorem 3.2. *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ with Lipschitz boundary. Then for $0 \leq p \leq n$ and $1 \leq q < n$, the $\overline{\partial}$-Neumann Laplacian $\Box$ has a bounded inverse N on $L^2_{p,q}(\Omega)$. Furthermore, we have N, $\overline{\partial}^*N$ $\overline{\partial}N$ and the Bergman projection $B = I - \overline{\partial}^*N\overline{\partial}$ are all exact regular on $W^s_{p,q}(\Omega)$ for all $s < \frac{\eta_0}{2}$.*

We remark that if the boundary is smooth, it follows from [26] that there exists $s > 0$ such that Theorem 3.2 holds.

The following proposition is a consequence of the above L^2-theory for $\overline{\partial}$ on $\mathbb{CP}^n$. Its proof follows the same lines of arguments as those in [16, 9, 18, 4] when the boundary is C^2-smooth.

Proposition 3.3. *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ with Lipschitz boundary. Then the L^2 holomorphic $(n,0)$-forms in $L^2_{n,0}(\Omega) \neq \{0\}$. Furthermore, L^2 Holomorphic $(n,0)$-forms separate points.*

Remark. Both Theorem 3.2 and Proposition 3.3 might not hold if we drop the Lipschitz condition. Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$, not necessarily with Lipschitz boundary. Using Theorem 2.3, we still have that $\Box_{0,q}$ has an inverse $N_{0,q}$ where $N_{0,q} : L^2_{0,q}(\Omega) \to L^2_{0,q}(\Omega)$.

When $p > 0$, it is not known if $\Box_{p,q} : L^2_{p,q}(\Omega) \to L^2_{p,q}(\Omega)$ has closed range (see related results in [28] on the Hartogs triangle in $\mathbb{CP}^2$). The reason is that when $p > 0$, the curvature term (2.8) is only nonnegative. Thus p plays a role for domains in $\mathbb{CP}^n$, in contrast to Corollary 2.4 for bounded pseudoconvex domains in $\mathbb{C}^n$.

4. W^1 estimates for $\overline{\partial}$ on pseudoconcave domains

Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ with Lipschitz boundary. Let $\overline{\partial}: L^2_{p,q-1}(\Omega) \to L^2_{p,q}(\Omega)$ be the weak maximal L^2 closure of $\overline{\partial}$ and its Hilbert space adjoint is denoted by $\overline{\partial}^*$. Let

$$\overline{\partial}_c : L^2_{p,q-1}(\Omega) \to L^2_{p,q}(\Omega)$$

be the minimal (strong) closure of $\overline{\partial}$. By this we mean that $f \in \mathrm{Dom}(\overline{\partial}_c)$ if and only if that there exists a sequence of smooth forms f_ν in $C^\infty_{p,n-1}(\Omega)$ compactly supported in Ω such that $f_\nu \to f$ and $\overline{\partial} f_\nu \to \overline{\partial} f$ in L^2. It is easy to see that (see [7])

$$\overline{\partial}_c = - * \overline{\partial}^* *,$$

where $*\colon \Lambda^{p,q} \to \Lambda^{n-p,n-q}$ is the Hodge star operator defined by

$$\langle \phi, \psi \rangle dV = \phi \wedge *\psi.$$

It is well-known that $\overline{\partial}$ has closed range if and only if $\overline{\partial}^*$ has closed range (see [20] or Lemma 4.1.1 in [8]). By using the Hodge star operator, we have that the operator $\overline{\partial}: L^2_{p,q-1}(\Omega) \to L^2_{p,q}(\Omega)$ has closed range if and only if $\overline{\partial}_c : L^2_{n-p,n-q}(\Omega) \to L^2_{n-p,n-q+1}(\Omega)$ has closed range (see [7]).

Lemma 4.1. *Let Ω be a pseudoconvex domain with Lipschitz boundary in $\mathbb{CP}^n$. We have*

$$H^{p,q}_{L^2}(\Omega) \cong H^{n-p,n-q}_{c,L^2}(\Omega) = \{0\}, \qquad q \neq 0.$$

Proof. Using Theorem 3.2, $\overline{\partial}$ has closed range in $L^2_{p,q}(\Omega)$ for all degrees. Thus from the L^2 Serre duality proved in [7], the lemma follows. □

Proposition 4.2. *Let Ω be a pseudoconvex domain with Lipschitz boundary in $\mathbb{CP}^n$, $n \geq 3$. Suppose that $f \in L^2_{p,q}(\Omega)$, where $0 \leq p \leq n$ and $1 \leq q < n$. Assuming that $\overline{\partial} f = 0$ in $\mathbb{CP}^n$ with $f = 0$ outside Ω. Then there exists $u \in L^2_{p,q-1}(\Omega)$ with $u = 0$ outside Ω satisfying $\bar{\partial} u = f$ in the distribution sense in $\mathbb{CP}^n$.*

For $q = n$, if f satisfies the compatibility condition

$$\int_\Omega f \wedge \phi = 0, \qquad \phi \in L^2_{n-p,0}(\Omega) \cap Ker(\overline{\partial}), \tag{4.1}$$

then the same conclusion holds.

Proof. Since the boundary is Lipschtiz, we have that solving $\overline{\partial}_c$ is the same as solving $\overline{\partial}$ with prescribed support in $\overline{\Omega}$ (see Lemma 2.3 in [27]). The proposition then follows from Lemma 4.1. □

Let Ω^+ be the complement of $\overline{\Omega}$ defined by

$$\Omega^+ = \mathbb{CP}^n \setminus \overline{\Omega}.$$

Then the domain Ω^+ is a pseudoconcave domain with Lipschitz boundary. The L^2-theory for $\overline{\partial}$ on Ω^+ is not known in general, unless $\overline{\Omega} \subset \mathbb{C}^n$ (see [31] , [32] or [13]). However, estimates for the $\overline{\partial}$-equation in Sobolev spaces $W^1(\Omega^+)$ can be obtained from the L^2-existence theory of $\overline{\partial}$ in Ω.

Theorem 4.3. *Let Ω be a pseudoconvex domain with Lipschitz boundary $\mathbb{CP}^n$ and let $\Omega^+ = \mathbb{CP}^n \setminus \overline{\Omega}$. For any $\overline{\partial}$-closed $f \in W^1_{p,q}(\Omega^+)$, where $0 \leq p \leq n$, $0 \leq q < n-1$, there exists $F \in L^2_{p,q}(\mathbb{CP}^n)$ with $F|_{\Omega^+} = f$ and $\overline{\partial} F = 0$ in $\mathbb{CP}^n$ in the distribution sense.*

Proof. The theorem is already proved when the boundary is C^2 in [4]. Since Ω has Lipschitz boundary, there exists a bounded extension operator from $W^1(\Omega^+)$ to $W^1(\mathbb{CP}^n)$ (see, e.g., [35]). Let $\tilde f \in W^1_{p,q}(\mathbb{CP}^n)$ be the extension of f so that $\tilde f|_{\Omega^+} = f$ with $\|\tilde f\|_{W^1(\mathbb{CP}^n)} \le C\|f\|_{W^1(\Omega^+)}$. We have $\overline{\partial}\tilde f \in L^2_{p,q+1}(\Omega)$.

From Proposition 4.2, there exists u_c with compact support in $\overline{\Omega}$ such that $\overline{\partial}u_c = \overline{\partial}\tilde f$ in $\mathbb{CP}^n$. Define

$$F = \tilde f - u_c. \tag{4.2}$$

Then $F \in L^2_{p,q}(\mathbb{CP}^n)$ and F is a $\overline{\partial}$-closed extension of f. □

Corollary 4.4. *Let Ω^+ be a pseudoconcave domain in $\mathbb{CP}^n$ with Lipschitz boundary, where $n \ge 2$. Then $W^1_{p,0}(\Omega^+) \cap Ker(\overline{\partial}) = \{0\}$ for every $1 \le p \le n$ and $W^1(\Omega^+) \cap Ker(\overline{\partial}) = \mathbb{C}$.*

Proof. Using Theorem 4.3 for $q = 0$, we have that any holomorphic $(p, 0)$-form on Ω^+ extends to be a holomorphic $(p, 0)$ in $\mathbb{CP}^n$, which are zero (when $p > 0$) or constants (when $p = 0$). □

Corollary 4.5. *Let Ω^+ be a pseudoconcave domain in $\mathbb{CP}^n$ with Lipschitz boundary, where $n \ge 3$. For any $\overline{\partial}$-closed $f \in W^1_{p,q}(\Omega^+)$, where $0 \le p \le n$, $1 \le q < n-1$, $p \ne q$, there exists $u \in W^1_{p,q-1}(\Omega^+)$ with $\overline{\partial}u = f$ in Ω^+.*

Proof. Let $F \in L^2_{p,q}(\mathbb{CP}^n)$ be the $\bar\partial$-closed extension of f from Ω to $\mathbb{CP}^n$. Since $H^{p,q}(\mathbb{CP}^n) = \{0\}$, there exists $u \in L^2_{p,q-1}(\Omega)$ such that $\bar\partial u = F$ on $\mathbb{CP}^n$. By the elliptic theory of the $\bar\partial$-complex on compact complex manifolds, one can choose such a solution $u \in W^1_{p,q-1}(\mathbb{CP}^n)$. □

Next we discuss the situation for the critical degree $q = n - 1$ on Ω^+. For $q = n - 1$, there is an additional compatibility condition for the $\overline{\partial}$-closed extension of $(p, n-1)$-forms from Ω^+ to the whole space $\mathbb{CP}^n$. This case differs from the others since the cohomology group does not vanish in general (see [13]). We first derive the compatibility condition for the extension of $\overline{\partial}$-closed forms when $q = n - 1$.

Lemma 4.6. *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ with Lipschitz boundary and let $\Omega^+ = \mathbb{CP}^n \setminus \overline{\Omega}$. For any $f \in W^1_{p,n-1}(\Omega^+)$ and $\phi \in L^2_{n-p,0}(\Omega) \cap Ker(\overline{\partial})$, the pairing*

$$\int_{b\Omega^+} f \wedge \phi \tag{4.3}$$

is well-defined.

Proof. Since the boundary is Lipschitz, any function in $W^1(\Omega^+)$ has a trace in $W^{\frac12}(b\Omega^+)$. Also Holomorphic L^2 functions or forms have trace in $W^{-\frac12}(b\Omega)$. The pairing (4.3) is well-defined follows from these known facts on Lipschtiz domains. Since we cannot find an exact reference, We will give a proof using the Friedrichs lemma.

Since the boundary is Lipschitz, it follows that smooth forms up to the boundary are dense in the graph norm of $\overline{\partial}$ since the boundary is Lipschitz (see [20] or Lemma 4.3.2 in [8]). For any $\overline{\partial}$-closed (holomorphic) $(n-p, 0)$-form ϕ with $L^2(\Omega)$ coefficients, there exists a sequence $\phi_\nu \in C^\infty_{n-p,0}(\overline{\Omega})$ such that $\phi_\nu \to \phi$ and $\overline{\partial}\phi_\nu \to 0$ in $L^2(\Omega)$.

Let $\tilde f \in W^1_{p,n-1}(\mathbb{CP}^n)$ be a bounded extension of f. We have

$$\int_{b\Omega} f \wedge \phi_\nu = \int_\Omega \overline{\partial}(\tilde f \wedge \phi_\nu) = \int_\Omega \overline{\partial}\tilde f \wedge \phi_\nu \pm \int_\Omega \tilde f \wedge \overline{\partial}\phi_\nu \to \int_\Omega \overline{\partial}\tilde f \wedge \phi. \tag{4.4}$$

Thus the limit on the left-hand-side of (4.4) exists and is independent of the approximating sequence $\{\phi_\nu\}$ that we choose. It is also independent of the extension function $\tilde f$. To see this, let $\tilde f_1 \in W^1_{p,n-1}(\mathbb{CP}^n)$ be another bounded extension of f. We have $\tilde f - \tilde f_1 = f - f = 0$ on $b\Omega$. Thus

$$(4.5) \qquad 0 = \int_{b\Omega} (\tilde f - \tilde f_1) \wedge \phi_\nu = \int_\Omega \overline{\partial}((\tilde f - \tilde f_1) \wedge \phi_\nu) \to \int_\Omega \overline{\partial} \tilde f \wedge \phi - \int_\Omega \overline{\partial} \tilde f_1 \wedge \phi.$$

Hence the pairing

$$\int_{b\Omega} f \wedge \phi = \lim_{\nu\to\infty} \int_{b\Omega} f \wedge \phi_\nu = \int_\Omega \overline{\partial} \tilde f \wedge \phi$$

is well-defined.

The lemma is proved. □

Theorem 4.7. *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ with Lipschitz boundary and let $\Omega^+ = \mathbb{CP}^n \setminus \overline{\Omega}$. For any $\overline{\partial}$-closed $f \in W^1_{p,n-1}(\Omega^+)$, where $0 \le p \le n$ and $p \ne n-1$, the following conditions are equivalent:*

(1) *The restriction of f to $b\Omega^+$ satisfies the compatibility condition*

$$(4.6) \qquad \int_{b\Omega^+} f \wedge \phi = 0, \quad \phi \in L^2_{n-p,0}(\Omega) \cap Ker(\overline{\partial}).$$

(2) *There exists $F \in L^2_{p,n-1}(\mathbb{CP}^n)$ such that $F|_\Omega = f$ in Ω^+ and $\overline{\partial} F = 0$ in $\mathbb{CP}^n$ in the sense of distribution.*

(3) *There exists $u \in W^1_{p,n-2}(\Omega^+)$ satisfying $\overline{\partial} u = f$ in Ω^+.*

Proof. We first prove that (1) implies (2). Suppose that f satisfies the condition (4.6). Using the same notation as in the proof of Theorem 4.3, we first extend f to $\tilde f \in W^1_{p,n-1}(\mathbb{CP}^n)$. Then the form $\overline{\partial} \tilde f$ is in $L^2_{p,n}(\Omega)$. It follows from (4.6) that

$$\int_\Omega \overline{\partial} \tilde f \wedge \phi = \int_{b\Omega} f \wedge \phi = 0$$

for every $\phi \in L^2_{n-p,0}(\Omega) \cap \mathrm{Ker}(\overline{\partial})$. Thus condition (4.1) is satisfied. Using Proposition 4.2 for $q = n$, there exists u_c with compact support in $\overline{\Omega}$ such that $\overline{\partial} u_c = \overline{\partial} \tilde f$ in $\mathbb{CP}^n$. Then $F = \tilde f - u_c$ is an L^2 $\overline{\partial}$-closed extension of f to $\mathbb{CP}^n$. This proves that (1) implies (2).

To show that (2) implies (3), one can solve $F = \overline{\partial} U$ for some $U \in W^1_{p,n-2}(\mathbb{CP}^n)$ since we assume that $p \ne n-1$. Let $u = U$ on Ω, we have $u \in W^1_{p,n-2}(\Omega)$ satisfying $\overline{\partial} u = f$ in Ω. Thus (2) implies (3).

Finally, we prove that (3) implies (1). Suppose that $f = \overline{\partial} u$ with $u \in W^1_{p,n-2}(\Omega^+)$. The trace of u on the boundary has coefficients in $W^{\frac{1}{2}}(b\Omega)$. We have for any $\phi \in L^2_{n-p,0}(\Omega) \cap \mathrm{Ker}(\overline{\partial})$, we claim that

$$(4.7) \qquad \int_{b\Omega^+} f \wedge \phi = \int_{b\Omega^+} \overline{\partial} u \wedge \phi = 0.$$

The integration by parts is justified by an approximation arguments. Since the boundary is Lipschitz, from Friedrichs's Lemma, we can approximate ϕ by smooth forms $\phi_\nu \in C^\infty(\overline{\Omega})$ such that $\phi_\nu \to \phi$ in $L^2_{n-p,0}(\Omega)$ and $\overline{\partial} \phi_\nu \to 0$ in $L^2_{n-p,1}(\Omega)$. Thus we have

$$(4.8) \qquad \int_{b\Omega^+} f \wedge \phi = \lim_{\nu\to\infty} \int_{b\Omega^+} \overline{\partial} u \wedge \phi_\nu = \lim_{\nu\to\infty} (-1)^{p+n-2} \int_{b\Omega^+} u \wedge \overline{\partial} \phi_\nu \to 0.$$

This proves that (3) implies (1). □

Theorem 4.8. *Let $\Omega \subset\subset \mathbb{CP}^n$ be a pseudoconvex domain with Lipschitz boundary and let $\Omega^+ = \mathbb{CP}^n \setminus \overline{\Omega}$. Then $\overline{\partial} : W^1_{p,n-2}(\Omega^+) \to W^1_{p,n-1}(\Omega^+)$ has closed range, where $0 \le p \le n$.*

Proof. Let f be a $\overline{\partial}$-closed $(p, n-1)$-form in $W^1_{p,n-1}(\Omega^+)$. Suppose that f is in the closure of the range of $\overline{\partial} : W^1_{p,n-2}(\Omega^+) \to W^1_{p,n-1}(\Omega^+)$. There exists a sequence $u_\nu \in W^1_{p,n-2}(\Omega^+)$ such that $\overline{\partial} u_\nu \to f$ in $W^1_{p,n-1}(\Omega^+)$. It suffices to show that there exists $u \in W^1_{p,n-2}(\Omega^+)$ such that $\overline{\partial} u = f$.

From Theorem 4.7, it suffices to show that the condition (4.6) is satisfied for every $\phi \in L^2_{n-p,0}(\Omega) \cap \mathrm{Ker}(\overline{\partial})$. This follows from

$$\int_{b\Omega^+} f \wedge \phi = \lim_{\nu\to\infty} \int_{b\Omega^+} \overline{\partial} u_\nu \wedge \phi = \lim_{\nu\to\infty} (-1)^{p+n-2} \int_{b\Omega^+} u_\nu \wedge \overline{\partial}\phi = 0. \tag{4.9}$$

Thus $f = \overline{\partial} u$ for some $u \in W^1_{p,n-2}(\Omega^+)$. Thus the range of $\overline{\partial}$ is closed in $W^1_{p,n-1}(\Omega^+)$. ∎

For $k \ge 0$, we define the Dolbeault cohomology $H^{p,q}_{W^k}(\Omega^+)$ with $W^k(\Omega^+)$-coefficients by

$$H^{p,q}_{W^k}(\Omega^+) = \frac{\{f \in W^k_{p,q}(\Omega^+) \mid \overline{\partial} f = 0\}}{\{f \in W^k_{p,q}(\Omega^+) \mid f = \overline{\partial} u, u \in W^k_{p,q-1}(\Omega^+)\}}.$$

Similarly, $H^{p,0}_{W^k}(\Omega^+)$ is defined to be the space of $(p, 0)$-forms with holomorphic coefficients in $W^k(\Omega)$. When $k = 0$, we use $H^{p,q}_{L^2}(\Omega^+)$ to denote the L^2 Dolbeault cohomology.

Corollary 4.9. *Let Ω^+ be the same as in Theorem 4.8.*
- *If $0 \le q < n-1$ and $p \ne q$, $H^{p,q}_{W^1}(\Omega^+) = 0$.*
- *When $p \ne n-1$, the space $H^{p,n-1}_{W^1}(\Omega^+)$ is Hausdorff and infinite dimensional.*

Proof. When $q \ne n-1$, the corollary follows from Corollary 4.5 for $q > 0$ and Corollary 4.4 when $q = 0$.

When $q = n-1$, the Hausdorff property of $H^{p,n-1}_{W^1}(\Omega^+)$ follows from Theorem 4.8. It remains only to prove that it is infinite dimensional.

Let $f \in W^1_{p,n-1}(\Omega)$ and $\overline{\partial} f = 0$. We define a pairing between $H^{p,n-1}_{W^1}(\Omega)$ and $H^{n-p,0}_{L^2}(\Omega^-)$

$$l : H^{p,n-1}_{W^1}(\Omega^+) \times H^{n-p,0}_{L^2}(\Omega) \tag{4.10}$$

by

$$l([f], h) = \int_{b\Omega^+} f \wedge h. \tag{4.11}$$

It is easy to see that the pairing (4.10) is well-defined. If f satisfies the condition

$$\int_{b\Omega^+} f \wedge \phi = 0, \quad \phi \in L^2_{n-p,0}(\Omega) \cap \mathrm{Ker}(\overline{\partial}),$$

there exists a solution $u \in W^1_{p,n-2}(\Omega)$ satisfying $\overline{\partial} u = f$. This implies that $[f] = 0$.

Thus l is a one-to-one map from $H^{p,n-1}_{W^1}(\Omega^+)$ to $H^{n-p,0}_{L^2}(\Omega)'$, the bounded linear functional on $H^{n-p,0}_{L^2}(\Omega)$. The space $H^{n-p,0}_{L^2}(\Omega)$ is infinite dimensional from Proposition 3.3. We have that $H^{p,n-1}_{W^1}(\Omega^+)$ is infinitely dimensional. Since it is a Hilbert space, it is isomorphic to $H^{n-p,0}_{L^2}(\Omega)$. ∎

Remark. The space of L^2 harmonic forms for the critical degree $q = n-1$ on an annulus between two concentric balls or strongly pseudoconvex domains in $\mathbb{C}^n$ has been computed in [22]. This has been generalized to annulus between two pseudoconvex domains in $\mathbb{C}^n$ in [32, 33]. We also remark that the conditions on the cohomology groups can be used to characterize domains with holes with Lipschtiz boundary in $\mathbb{C}^n$ (see [13]).

When the domain $\Omega \subset \mathbb{CP}^n$ is pseudoconconcave with C^2 boundary, W^1 estimates for $\overline{\partial}$ were obtained earlier in [4] when $q < n-1$ (see also [6]).

5. Extensions of $\overline{\partial}$ in Sobolev spaces and duality

In this section we will formulate the $\overline{\partial}$ operator in Sobolev spaces and study its duality. Let Ω be a bounded domain with Lipschitz boundary in a complex hermitian manifold X. We first extend the $\overline{\partial}$ problem and its dual from $L^2(\Omega)$ to Sobolev spaces $W^s(\Omega)$ for $s > 0$. We restrict ourselves to domains with Lipschitz boundary so that the space $W^s(\Omega)$ can be identified as the restriction from $W^s(X)$ to $W^s(\Omega)$ (see e.g., [35]). When $s = 0$, the L^2 weak and strong extension for $\overline{\partial}$ discussed by way of the Friederichs' lemma is well-known (see Chapter 1 in Hörmander [20]; see also Lemma 4.3 in [8]). The L^2 $\overline{\partial}$-Cauchy problem is used in Section 4. Now we extend $\overline{\partial}$ and the $\overline{\partial}$-Cauchy problem to W^s and its dual spaces. We first remark that the dual space for $L^2(\Omega)$ is itself. The dual space for $W^s(\Omega)$ for $s > 0$ is defined as follows.

Definition 5.1. Let Ω be a bounded Lipschitz domain in a complex hermitian manifold X and let $s > 0$. The dual space of $W^s(\Omega)$, denoted by $W_*^{-s}(\Omega)$, is the space of continuous linear functional f on $W^s(\Omega)$ with the norm

$$\|f\|_{W_*^{-s}(\Omega)} = \sup\{|\langle f, g\rangle|;\ \forall g \in W^s(\Omega), \|g\|_{W^s} \le 1\} < \infty.$$

(Here and in what follows we use the notation $\langle f, g\rangle = f(g)$ for pairing of elements of $W^s(\Omega)$ and $W_*^{-s}(\Omega)$).

Let $W^{-s}(\Omega)$ denote the dual space of $W_0^s(\Omega)$, where $W_0^s(\Omega)$ is the completion in $W^s(\Omega)$ of $\mathcal{D}(\Omega)$, the space of compactly supported smooth functions on Ω. Since

$$\mathcal{D}(\Omega) \hookrightarrow W_0^s(\Omega) \hookrightarrow W^s(\Omega),$$

by taking the transpose, we have

$$\mathcal{E}(\Omega) = C^\infty(\Omega) \hookleftarrow W^{-s}(\Omega) \hookleftarrow W_*^{-s}(\Omega).$$

When $0 \le s \le \frac{1}{2}$, the space $W_0^s(\Omega) = W^s(\Omega)$ (see Theorem 11.1 in [29]). Thus there is no difference between the spaces $W^{-s}(\Omega)$ and $W_*^{-s}(\Omega)$. However, when $s > \frac{1}{2}$, these two spaces are different. We must distinguish between these spaces. Following [3], we use the notation $W_*^{-s}(\Omega)$ to indicate that these spaces demand special attention.

Next we identify the spaces $W_*^{-s}(\Omega)$ as a subspace of distributions in X when the domain Ω is Lipschitz. It is well known that the dual space of $C^\infty(\Omega) = \mathcal{E}(\Omega)$ is the space $\mathcal{E}'(\Omega)$ of distributions with compact support in Ω. The following lemma is also well known (see, *e.g.,* Lemma 2.3 in [27]). We repeat the proof here for the benefit of the reader.

Lemma 5.1. *Let Ω be a relatively compact domain with Lipschitz boundary in a manifold X. The dual space of $C^\infty(\overline{\Omega})$ is the space $\mathcal{E}'_{\overline{\Omega}}(X)$ of distributions on X with support contained in $\overline{\Omega}$.*

Proof. Since the restriction map

$$R \;:\; C^\infty(X) \to C^\infty(\overline{\Omega})$$

is continuous and surjective, the transpose map ${}^tR\colon C^\infty(\overline{\Omega})' \to \mathcal{E}'(X)$ is continuous and injective. Clearly, ${}^tR((C^\infty(\overline{\Omega}))') \subseteq \mathcal{E}'_{\overline{\Omega}}(X)$, the space of distributions on X with support contained in $\overline{\Omega}$. It remains to prove the opposite inclusion. Any distribution $T \in \mathcal{E}'_{\overline{\Omega}}(X)$ defines a continuous linear functional $\widetilde{T}$ on $C^\infty(\overline{\Omega})$ by setting, for $f \in C^\infty(\overline{\Omega})$,

$$\langle \widetilde{T}, f\rangle = \langle T, \widetilde{f}\rangle,$$

where $\widetilde{f}$ is a C^∞-smooth extension of f to X. Since the boundary of Ω is Lipschitz, the space $C_0^\infty(X\setminus\overline{\Omega})$ is dense in the space of C^∞-smooth functions on X with support contained in $X \setminus \Omega$. This implies that $\langle \widetilde{T}, f\rangle$ is independent of the choice of the extension $\widetilde{f}$ of f. For any open set $U \subset \mathbb{C}$, $\widetilde{T}^{-1}(U) = R \circ T^{-1}(U)$. Thus, applying the open mapping theorem on R, we have that $\widetilde{T}$ is continuous linear functional on $C^\infty(\overline{(\Omega)})$. Therefore, the dual space of $C^\infty(\overline{\Omega})$ is the space $\mathcal{E}'_{\overline{\Omega}}(X)$ of distributions on X with support contained in $\overline{\Omega}$. □

Corollary 5.2. *Let Ω be a relatively compact domain with Lipschitz boundary in a hermitian manifold X. We have $W^{-s}_*(\Omega) \subset \mathcal{D}_{\overline{\Omega}}(X)$.*

Proof. Since the space of $C^\infty(\overline{\Omega}) \subset W^s(\Omega)$, we have $W^s(\Omega)' \subset C^\infty(\overline{\Omega})'$. From Lemma 5.1, we have $W^{-s}_*(\Omega) \subset \mathcal{D}_{\overline{\Omega}}(X)$. □

The dual space of $L^2(\Omega)$ is still $L^2(\Omega)$. Any function $f \in L^2(\Omega)$ is an L^2 function on X by extending f as zero outside Ω. When $s > 0$, since $W^s(\Omega) \subsetneq L^2(\Omega)$, the dual space $W^s_*(\Omega)$ of $W^s(\Omega)$ is larger and it contains $L^2(\Omega)$ as a proper subspace. We next identify explicitly the subspace in $\mathcal{D}_{\overline{\Omega}}(X)$ which represents $W^{-s}_*(\Omega)$. For convenience sake, we assume that X is a compact complex hermitian manifold. For $s \in \mathbb{R}$, we define $W^s(X)$ to be the Sobolev spaces of order s. By assuming that X is compact, there is only one way to define these spaces up to equivalent norms. Then $W^s(X)$ and $W^{-s}(X)$ are dual space to each other.

Definition 5.2. Let X be a compact complex hermitian manifold and let Ω be a domain in X with Lipschitz boundary. Let $W^{-s}_{\overline{\Omega}}(X)$ be the subspace of distributions in $W^{-s}(X)$ with support in $\overline{\Omega}$.

Lemma 5.3. *Let Ω be a bounded domain with Lipschitz boundary in a compact complex hermitian manifold X. For any $s \geq 0$, we have $W^{-s}_*(\Omega) = W^{-s}_{\overline{\Omega}}(X)$.*

Proof. Since the boundary of Ω is Lipschitz, every function in $W^s(\Omega)$ can be extended to be a function in $W^s(X)$. Consider the restriction map

$$R : W^s(X) \to W^s(\Omega).$$

It is continuous and surjective. Thus the dual map

$$R' : W^s(\Omega)' = W^{-s}_*(\Omega) \to W^s(X)' = W^{-s}(X)$$

is an injection. Moreover, from Corollary 5.2, the image of $W^s(\Omega)'$ by R' is included in $W^{-s}_{\overline{\Omega}}(X)$. This shows that $W^s(\Omega)' \subset W^{-s}_{\overline{\Omega}}(X)$.

To prove the other direction, we will show that if $f \in W^{-s}_{\overline{\Omega}}(X)$, then $f \in W^s(\Omega)'$. To see this, let ϕ be a function in $W^s(\Omega)$. Since the boundary is Lipschitz, there exists a bounded extension map E (see e.g. [35]) from $W^s(\Omega)$ to $W^s(X)$ with

$$\|E\phi\|_{W^s(X)} \leq C\|\phi\|_{W^s(\Omega)}.$$

We define a linear functional on $W^s(\Omega)$ by $(f,\phi)=(f,E\phi)_X$. Then we have

$$|\langle f,\phi\rangle| = |\langle f, E\phi\rangle_X| \le \|f\|_{W^{-s}(X)}\|E\phi\|_{W^s(X)} \le C\|\phi\|_{W^s(\Omega)}.$$

Thus f is a bounded linear functional on $W^s(\Omega)$. Thus $f\in W_*^{-s}(\Omega)$. This proves the lemma.

□

Lemma 5.4. *Let Ω be a bounded domain with Lipschitz boundary in a compact complex hermitian manifold X. For any $s\ge 0$, the space $C_0^\infty(\Omega)$ is dense in $W_*^{-s}(\Omega)$.*

Proof. When $s=0$, this is true for any domain without the Lipschitz assumption. From Lemma 5.3, every element $f\in W_*^{-s}(\Omega)$ is an element in $W_{\overline{\Omega}}^{-s}(X)$. By a partition of unity, this is a local problem near each $z\in\overline{\Omega}$. Since Ω is Lipschitz, for any point $z\in b\Omega$, there is a neighborhood U of z in X, and for $\varepsilon\ge 0$, a continuous one parameter family t_ε of smooth maps from U into X such that $t_\varepsilon(D\cap U)$ is compactly contained in Ω, and t_ε converges to the identity map on U as $\varepsilon\to 0$. In local coordinates near z, the map t_ε is simply the translation by an amount ε in the inward normal direction. Then we can approximate f locally by $f^{(\varepsilon)}$, where

$$f^{(\varepsilon)} = (t_\varepsilon^{-1})^* f$$

is the pullback of f by the inverse t_ε^{-1} of t_ε. A partition of unity argument now gives a form $f^{(\varepsilon)}\in W_{\overline{\Omega}}^{-s}(X)$ such that $f^{(\varepsilon)}$ is supported inside Ω and as $\varepsilon\to 0$, $f^{(\varepsilon)}\to f$ in $W^{-s}(X)$. The lemma follows from regularization. □

We let $\overline{\partial}$ be the differential operator and ϑ be its formal adjoint.

Definition 5.3. For $s\ge 0$, $0\le p\le n$ and $1\le q\le n$, let

$$\overline{\partial}_s : W^s_{p,q-1}(\Omega)\to W^s_{p,q}(\Omega), \tag{5.1}$$

denote the maximal closed extension of $\overline{\partial}_s$ from the Hilbert space $W^s_{p,q-1}(\Omega)$ to $W^s_{p,q}(\Omega)$. A form $f\in W^s_{p,q-1}(\Omega)\cap \mathrm{Dom}(\overline{\partial}_s)$ if and only if $\overline{\partial}f$ is in $W^s_{p,q}(\Omega)$ in the distribution sense.

Equivalently, let $\mathcal{D}_{p,q}(\Omega)$ denote the smooth (p,q)-forms with compact support in Ω and ϑ denote the formal adjoint of $\overline{\partial}$. A form $f\in\mathrm{Dom}(\overline{\partial}_s)$ if and only if $f\in W^s_{p,q-1}(\Omega)$ and there exists $g\in W^s_{p,q}(\Omega)$ such that

$$\langle\phi,g\rangle = \langle\vartheta\phi,f\rangle, \qquad \text{for every } \phi\in\mathcal{D}_{p,q}(\Omega). \tag{5.2}$$

In this case, we define $\overline{\partial}_s f = g$ in the definition of distribution.

Lemma 5.5. *The $\overline{\partial}_s$ operator is a closed densely defined operator from $W^s_{p,q-1}(\Omega)$ to $W^s_{p,q}(\Omega)$.*

Proof. From the assumption that Ω is Lipschitz, the space $C^\infty_{p,q-1}(\overline{\Omega})\subset W^s_{p,q-1}(\Omega)$ is a dense subspace. Thus the operator $\overline{\partial}_s$ is a densely defined operator since $C^\infty_{p,q-1}(\overline{\Omega})\subset\mathrm{Dom}(\overline{\partial}_s)$, The $\overline{\partial}_s$ operator is a closed operator, i.e, the graph of $\overline{\partial}$ is closed since differentiation in the distributions sense is continuous. □

We have the following lemma which states that the weak equal strong extension for $\overline{\partial}_s$ when the boundary is Lipschitz.

Lemma 5.6. *(**Maximal Weak and Strong extensions for** $\overline{\partial}_s$) Let Ω be a bounded Lipschitz domain in a complex hermitian manifold X. A form $f\in W^s_{p,q-1}(\Omega)\cap Dom(\overline{\partial}_s)$ if* *and only if there exists a sequence $f_\nu \in \mathcal{C}^\infty_{p,q-1}(\overline{\Omega})$ such that $f_\nu \to f$ in $W^s_{p,q-1}(\Omega)$ and $\overline{\partial} f_\nu$ converges to $\overline{\partial} f$ in the $W^s_{p,q}(\Omega)$.*

Proof. Since the boundary is Lipschitz, the lemma follows from the Friedrichs' lemma by regularizing the form f (see [12], [19], [20] or [8]). □

Remark. It is important that the boundary is Lipschitz. If the boundary is not Lipschitz, the lemma might not hold (see the recent paper [28]).

Definition 5.4. The dual $\overline{\partial}'_s$ of $\overline{\partial}_s$ is defined as

$$\overline{\partial}'_s : W^{-s}_{*,p,q}(\Omega) \to W^{-s}_{*,p,q-1}(\Omega); \tag{5.3}$$

An element $f \in \mathrm{Dom}(\overline{\partial}'_s) \cap W^{-s}_{*,p,q}(\Omega)$ if and only if there exists $g \in W^{-s}_{*,p,q-1}(\Omega)$ such that

$$\langle \overline{\partial}_s \phi, f \rangle = \langle \phi, g \rangle, \qquad \phi \in \mathrm{Dom}(\overline{\partial}_s). \tag{5.4}$$

If (5.4) holds, we define $\overline{\partial}'_s f = g$. Let ϑ be the formal adjoint of $\overline{\partial}$. We then have $\vartheta f = g$ in the distribution sense on Ω.

Remark. The pairing in (5.4) is between dual spaces. We must not confuse the dual operator $\overline{\partial}'_s$ with the Hilbert space adjoint $\overline{\partial}^*_s : W^s_{p,q}(\Omega) \to W^s_{p,q-1}(\Omega)$. Only when $s = 0$, we have $\overline{\partial}' = \overline{\partial}^*$ since $W^{-s}_*(\Omega) = W^s(\Omega)$ only when $s = 0$. The importance of taking the dual pairing is that we can use integration by parts when the forms are smooth.

We will show next that $\vartheta f = g$ in the distribution sense in X and generalize the L^2 $\overline{\partial}$-Cauchy problem to the spaces $W^{-s}_*(\Omega)$. From now on, we will fix s, p, q and will sometimes drop the dependence of s, p, q in the function spaces to avoid too many indices when there is no danger of confusion.

Proposition 5.7. *Let Ω be a bounded domain with Lipschitz boundary in a complex hermitian compact manifold X. The form $f \in Dom(\overline{\partial}'_s)$ if and only if $f \in W^{-s}_{\overline{\Omega}}(X)$ and there exists $g \in W^{-s}_{\overline{\Omega}}(X)$ such that $\vartheta f = g$ in the distribution sense in X.*

Proof. From Definition 5.4, $f \in W^{-s}_*(\Omega)$ and there exists $g \in W^{-s}_*(\Omega)$ such that (5.4) holds. Note that $f \in \mathrm{Dom}(\overline{\partial}'_s)$, from Lemma 5.3, both f and g are distributions in X with compact support in $\overline{\Omega}$.

In particular, since $C^\infty_{p,q}(\overline{\Omega}) \subset \mathrm{Dom}(\overline{\partial}_s)$, we have

$$(\overline{\partial}_s \phi, f) = (\overline{\partial} \phi, f) = (\phi, g), \qquad \phi \in \mathcal{C}^\infty_{p,q}(\overline{\Omega}). \tag{5.5}$$

Using Lemma 5.3, both f and g are in $W^{-s}_{\overline{\Omega}}(X)$, i.e., distributions in $W^{-s}(X)$ with compact support in $\overline{\Omega}$. Equality (5.5) implies that

$$(\overline{\partial} \phi, f) = (\phi, g), \qquad \phi \in \mathcal{C}^\infty_{p,q}(X). \tag{5.6}$$

Equality (5.6) means that $\vartheta f = g$ in the distribution sense in X. □

We will also need the following well known result from functional analysis (see, e.g., Section VII.5 in [39], Theorem 1.1.1 in [20], and Proposition 2.5 and Theorem 2.7 in [27]).

Lemma 5.8. *Let X and Y be Hilbert spaces and let T be a densely defined closed operator from X into Y. Let T' be the transpose of T. Then the following statements are equivalent:*

(1) *T has closed range $\mathcal{R}(T)$ in Y.*
(2) *T' has closed range $\mathcal{R}(T')$ in X'.*

(3) $\mathcal{R}(T) = \{g \in Y \mid \langle g, f\rangle = 0$ *for every* $f \in \mathcal{N}(T')\}$.
(4) $\mathcal{R}(T') = \{f \in X' \mid \langle g, f\rangle = 0$ *for every* $g \in \mathcal{N}(T)\}$.
(5) *There is a constant* $C > 0$ *such that*

$$\|f\|_X \leq C\|Tf\|_Y, \qquad f \in \mathrm{dom}\,(T) \cap \mathcal{N}(T)^{\perp} \tag{5.7}$$

(6) *There is a constant* $C > 0$ *such that*

$$\|g\|_{Y'} \leq C\|T'g\|_{X'}, \qquad g \in \mathrm{dom}\,(T') \cap \mathcal{N}(T')^{\perp} \tag{5.8}$$

Proof. The equivalence of (1)-(4) is the Banach closed range theorem (see [39, p. 205]) and holds on Banach spaces. The equivalence between (1) and (5) and between (2) and (6) is a consequence of the closed graph theorem (see [20, Theorem 1.1.1]). □

Applying the above theorem to $\overline{\partial}_s : W^s_{p,q-1}(\Omega) \to W^s_{p,q}(\Omega)$, we have:

Corollary 5.9. *Let* Ω *be a bounded domain with Lipschitz boundary in a complex hermitian compact manifold* X. *Suppose that* $\overline{\partial}_s : W^s_{p,q-1}(\Omega) \to W^s_{p,q}(\Omega)$ *has closed range. Then we have*

(1) $\mathcal{R}(\overline{\partial}_s) = \{g \in W^s_{p,q}(\Omega) \mid \langle g, f\rangle = 0$ *for every* $f \in \mathcal{N}(\overline{\partial}'_s)\}$.
(2) $\mathcal{R}(\overline{\partial}'_s) = \{f \in W^{-s}_{*p,q-1}(\Omega) \mid \langle g, f\rangle = 0$ *for every* $g \in Ker(\overline{\partial}_s)\}$.

Definition 5.5. Let Ω be a bounded Lipschitz domain in a complex hermitian compact manifold X. For $0 \leq p \leq n$, $1 \leq q \leq n$, $s \geq 0$, let

$$\overline{\partial}^s_c : W^{-s}_{\overline{\Omega}}(X) \to W^{-s}_{\overline{\Omega}}(X) \tag{5.9}$$

denote the (weak) closed extension of $\overline{\partial}$ from the Hilbert space $W^{-s}_{\overline{\Omega}}(X)$ to $W^{-s}_{\overline{\Omega}}(X)$. A form $f \in \mathrm{Dom}(\overline{\partial}^s_c)$ if and only if $f \in W^{-s}_{\overline{\Omega}}(X)$ and $\overline{\partial} f \in W^{-s}_{\overline{\Omega}}(X)$.

Let $*$ be the Hodge star operator which maps (p, q)-forms to a $(n-p, n-q)$-forms in X. We have on the compact complex manifold X,

$$\vartheta = - * \overline{\partial} * . \tag{5.10}$$

We also have the following relations between $\overline{\partial}'_s$ and $\overline{\partial}^s_c$.

Lemma 5.10. *Let* Ω *be a bounded Lipschitz domain in a complex hermitian compact manifold* X. *A form* $f \in Dom(\overline{\partial}'_s)$ *if and only if* $*f \in Dom(\overline{\partial}^s_c)$. *We have*

$$\overline{\partial}^s_c = *\overline{\partial}'_s * . \tag{5.11}$$

Proof. When $s = 0$, this is proved in [7]. Let $f \in \mathrm{Dom}(\overline{\partial}'_s)$. From Proposition 5.7, $f \in W^{-s}_{\overline{\Omega}}(X)$ and $\vartheta f = g \in W^{-s}_{\overline{\Omega}}(X)$. Thus $*f$ and $*g$ are forms in $W^{-s}(X)$ with compact support in $\overline{\Omega}$ and $\overline{\partial} * f = *\vartheta f$ in the distribution sense in X. Thus $*f \in \mathrm{Dom}(\overline{\partial}^s_c)$. The other direction is the same. Since on a compact manifold, equation (5.10) holds both for smooth forms and for currents. The equality (5.11) follows.

□

Lemma 5.11. ***(Weak and Strong extensions for*** $\overline{\partial}^s_c$*)* *A form* $f \in Dom(\overline{\partial}^s_c)$ *if and only if* $f \in W^{-s}_{\overline{\Omega}}(X)$ *and there exists* $f_\nu \in \mathcal{D}_{p,q}(\Omega)$ *such that both* $\{f_\nu\}$ *and* $\{\overline{\partial} f_\nu\}$ *converge in* $W^{-s}_{\overline{\Omega}}(X)$ *to* f *and* g *respectively with* $\overline{\partial} f = g$ *in* X.

Proof. When $s = 0$, this is done by Friederichs' Lemma (see [20], [8] or [27]). Assume $f,\ g \in W^{-s}_{\overline{\Omega}}(X)$ such that $\overline{\partial} f = g$ in the distribution sense in X. Both f and g are distribution in X and compactly supported in $\overline{\Omega}$.

We need to construct a sequence f_ν of smooth forms with compact support in Ω which converges in the graph norm of $\overline{\partial}$ in $W^{-s}_{\overline{\Omega}}(X)$ to f. By a partition of unity, this is a again local problem near each $z \in \overline{\Omega}$. Let t_ε and f^ε be the same as before. We approximate f locally by $f^{(\varepsilon)}$, where

$$f^{(\varepsilon)} = (t_\varepsilon^{-1})^* f$$

is the pullback of f by the inverse t_ε^{-1} of t_ε. A partition of unity argument now gives a form $f^{(\varepsilon)} \in W^{-s}_{\overline{\Omega}}(X)$ such that $f^{(\varepsilon)}$ is compactly supported inside Ω and as $\varepsilon \to 0$,

$$f^{(\varepsilon)} \to f \quad \text{in } W^{-s}(X) \quad \overline{\partial} f^{(\varepsilon)} \to \overline{\partial} f \quad \text{in } W^{-s}(X). \tag{5.12}$$

We can apply Friedrich's lemma to regularize the form $f^{(\varepsilon)}$ to construct a sequence of smooth forms f_ν with compact support in Ω with the desired property. ☐

Theorem 5.12. *Let Ω be a bounded Lipschitz domain in a complex hermitian compact manifold X. The following conditions are equivalent:*

1. *The operator $\overline{\partial}_s : W^s_{p,q-1}(\Omega) \to W^s_{p,q}(\Omega)$ has closed range.*
2. *The operator $\overline{\partial}'_s : W^{-s}_{\overline{\Omega},p,q}(X) \to W^{-s}_{\overline{\Omega},p,q-1}(X)$ has closed range.*
3. *The operator $\overline{\partial}^s_c : W^{-s}_{\overline{\Omega},n-p,n-q}(X) \to W^{-s}_{\overline{\Omega},n-p,n-q+1}(X)$ has closed range.*

Proof. Since the boundary is Lipschitz, from Lemma 5.3, the space $W^{-s}_*(\Omega) = W^{-s}_{\overline{\Omega}}(X)$. It then follows from Lemma 5.9 that (1) and (2) are equivalent. That (2) and (3) are equivalent follows from Lemma 5.10. ☐

Theorem 5.13. *Let Ω be a bounded Lipschitz domain in a complex hermitian compact manifold X. Suppose that the range of $\overline{\partial}_s$ is closed in $W^s_{p,q}(\Omega)$. Then $g \in Range(\overline{\partial}^s_c) \cap W^{-s}_{\overline{\Omega},n-p,n-q}(X)$ if and only if*

$$(*g, f) = 0, \qquad \textit{for all } f \in Ker(\overline{\partial}) \cap W^s_{p,q}(\Omega). \tag{5.13}$$

Proof. We first prove the necessity. If g is in the range of $\overline{\partial}^s_c$, there exists $u \in W^{-s}_{\overline{\Omega},n-p,n-q}(X)$ such that $\overline{\partial} u = g$ in X in the distribution sense. Thus there exists a sequence of compactly supported smooth forms $u_\nu \to u$ and $\overline{\partial} u_\nu \to g$. It is easy to see that for every $f \in \mathrm{Ker}(\overline{\partial}_s) \cap W^s_{p,q}(\Omega)$,

$$(*g, f) = (g, *f) = \lim_\nu \int_\Omega \overline{\partial} u_\nu \wedge f = \pm \lim_\nu \int_\Omega u_\nu \wedge \overline{\partial}_s f = 0.$$

Thus (5.13) holds.

On the other hand, if (5.13) holds, we will show that g is in the range of $\overline{\partial}^s_c$, or equivalently, $*g$ is in the range of $\overline{\partial}^*_s$. Let $u \in C^\infty_{p,q-1}(\overline{\Omega})$ and $f = \overline{\partial} u$. Then the linear functional

$$L(\overline{\partial} u) = (*g, u) \le \|g\|_{W^{-s}_*} \|u\|_{W^s} \le \|g\|_{W^{-s}_*} \|\overline{\partial} u\|_{W^s}$$

is a bounded linear functional from $\mathrm{Range}(\overline{\partial}) \subset W^s_{p,q}(\Omega)$ to $\mathbb{C}$. From the Hahn-Banach theorem, L extends to be a bounded linear functional on $W^s_{p,q}(\Omega)$. Thus there exists an $v \in W^{-s}_{*p,q-1}(\Omega)$ such that

$$L(\overline{\partial} u) = (v, \overline{\partial} u) = (*g, u), \qquad u \in C^\infty_{p,q-1}(\overline{\Omega}).$$

This means that $v \in \mathrm{Dom}(\overline{\partial}_s^*)$ and $\overline{\partial}_s^* v = *g$. This proves that $g = \overline{\partial}_c^s * v$ and g is in the range of $\overline{\partial}_c^s$. □

Remark. Not much is known if we consider Ω to be a smooth pseudoconvex domain in $\mathbb{CP}^n$ (or in a complex hermitian manifold) except for small $s < \frac{1}{2}$ (see Theorems 3.1 and 3.2). Suppose that Ω is a pseudoconvex domain in $\mathbb{CP}^n$ with smooth boundary. It is still unknown if

$$\overline{\partial}_s : W^s_{p,q-1}(\Omega) \to W^s_{p,q}(\Omega), \qquad s \in \mathbf{N}$$

has closed range or if (1.1) holds for $s \geq 1$.

However, if we consider the complement $\Omega^+ = \mathbb{CP}^n \setminus \Omega$, then it follows from Corollary 4.9 that

$$\overline{\partial}_1 : W^1_{p,q-1}(\Omega^+) \to W^1_{p,q}(\Omega^+), \qquad s \in \mathbf{N}$$

has closed range for all $1 \leq q \leq n-1$. Here we only need that $b\Omega^+$ to be Lipschitz.

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Department of Mathematical Sciences, Rutgers University-Camden, Camden, NJ 08102
E-mail address: sfu@camden.rutgers.edu

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556
E-mail address: Mei-Chi.Shaw.1@nd.edu