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On the Bövik-Benveniste methodology and related approaches for modeling thin layers

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This paper reviews several leading approaches for asymptotic modeling of thin layers in elastostatics and wave propagation phenomena. The issues related to applications of the so-called "equivalent" or "effective" boundary conditions and their interpretations are highlighted. Comparative analysis of asymptotic models is performed for two-dimensional elastostatic case using a novel complex variables-based modeling tool. Its implementation allows for straightforward derivations of higher order boundary conditions for problems with layers of arbitrary sufficiently smooth curvatures. Explicit expressions for the conditions up to the third order are provided. All models are tested using available benchmark solutions and the solutions for the limiting cases of the layer parameters.

1. Introduction

Asymptotic-based approaches have long been used for modeling elastostatics and wave propagation problems with thin layers. They eliminate the need for direct simulations of the layers, which could be a challenging task due to significant variations in characteristic scales of the problems. They also represent attractive alternative to the approach in which the layers are modeled by means of various structural elements, e.g. beams, shells, or plates (see, e.g., [1–4]), as the latter require decisions on which type of elements to use, e.g., [5]).

Unlike, phenomenological models (see reviews in e.g., [6–9]) that represent a layer by a surface endowed with its own energetic structure and require supplementary data (e.g., interface constitutive laws, material parameters, etc.), the asymptotic models are derived analytically from the fully resolved problems that includes thin layers.

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Asymptotic procedures for elastic and wave propagation problems were developed independently but more or less simultaneously. However, this happened sometimes without awareness of the developments in respective areas. Thus, one goal of present paper is to highlight the connections between the leading modeling techniques used in both areas. Additionally, there are still a few issues with the interpretation and/or results obtained with some of the procedures. So, the other goal of the paper is to highlight and address those issues.

The analysis of the relevant literature identified the following two groups of modeling techniques used in both areas.

The approaches of the first group use standard asymptotic expansion method in which the dimensionless layer thickness ε is chosen as a small parameter. The reviews of early literature can be found in [10–12]; more recent developments are reported in, e.g., [13–21]. Such an approach produces a set of boundary conditions across a single surface "interface" that eliminates the layer. The resulting boundary conditions are referred to as "equivalent" or "effective" in the literature on wave phenomena and as "jump conditions" in the literature on elastostatic problems, in which the asymptotic models are referred to as "imperfect interface models." An order of boundary conditions or an interface model is defined by the highest power of ε involved in the asymptotic series. The obtained models are for the layers characterized by the parameters that are of order ε^M. Another approach of that group is based on the energy minimization method, see e.g., [17]; there it was demonstrated that the asymptotic expansion- and energy minimization-based methods are consistent and produce the same types of asymptotic models.

Most of the reported boundary/jump conditions obtained with the approaches of the first group are of low orders and only allow to capture the layer behavior up to the so-called membrane type ($M = -1, 0, 1$), or their derivations require separate asymptotic analysis (and result in separate type of boundary conditions) for each value of M, see e.g. [5, 22]. In addition, some procedures involve recursive relations to obtain higher order terms of asymptotic series and, in general, result in implicit forms of jump conditions. More importantly, most of the available models are for problems with planar and very thin layers. We are only aware of a few papers that deal with curved layers, see e.g., [16, 23], where the first order models were obtained. However, it is clear from the plots presented on Figs.2,3 of [16] that the results are only satisfactory for very thin layers and limited cases of material parameters.

The approaches of the second group use the Taylor expansions (in terms of the layer thickness h) for the fields involved; an order of asymptotic model is defined by the truncation order of the expansions. Several types of such approaches could be distinguished depending on the interpretation of the resulting boundary/jump conditions.

One approach was suggested by Hashin in [24, 25] for elastostatic problems. There, the fields inside the layer were expanded in the normal direction about the points located at its inner boundary and the expansions were used to evaluate the fields at the outer boundary. The use of perfect bond conditions at the boundaries allowed to exclude the fields inside the layer. The obtained jumps in fields across the layer were treated as those across the interface (coincided with the inner boundary). It is clear that this approach can only be used for cases of very thin layers.

Different approach was proposed in a series of papers by Rokhlin and his collaborators in the context of wave phenomena, see, [8, 26, 27]. In their papers, the primary and secondary fields on both sides of a layer are arranged in vector forms and connected by the so-called transfer matrix that contains information on the layer properties. This matrix is asymptotically expanded in terms of thickness of the layer and an order of the model is defined by the truncation order of the series. The model resulting from that approach is referred to as the Rokhlin-Wang model. The papers relevant to this approach reported the models of the first and second orders. However, the second order models were obtained not from rigorous asymptotic considerations but approximately.

In the Rokhlin-Wang model, the boundary conditions are formulated as jumps across a layer without any consideration of an interface. Somewhat different approach that results in the same type of boundary conditions was used in [28, 29] for problems of ultrasonic guided wave propagation in thin anisotropic layers. In context of elastostatics, similar type of boundary

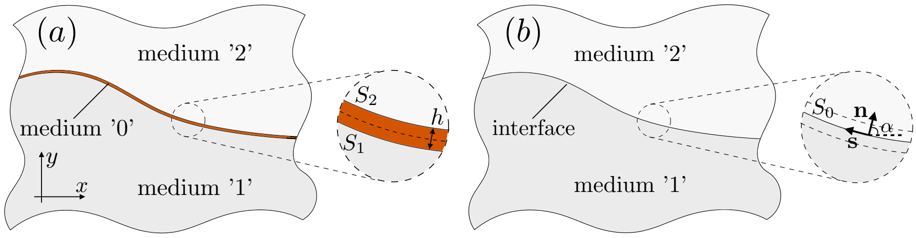

Figure 1: (a) Three-phase configuration problem, (b) two-phase configuration problem

conditions of the first and second orders were introduced in e.g., [30–32], where they were labeled as the jump conditions for "*Model I*." Rokhlin-Wang's model and *Model I* allow for simulations of curved layers and for a relatively wide range of the problem parameters. However, despite some attempts to study the boundary value problems associated with those models, see e.g., [7, 31], their application may potentially be hampered by the lack of rigorous studies.

Yet another approach of that group was proposed by Bövik in [33, 34] for wave scattering and further developed and generalized in the context of elastostatics and conduction phenomena by Benveniste in e.g., [30, 31, 35, 36]. This approach is now referred to as the Bövik-Benveniste methodology. The methodology, that can be used for curved layers, involves a two-step procedure in which the first step coincides with the procedure used for derivations of jump conditions across a layer in *Model I*. At the second step, additional Taylor expansions are used for the jumps of the first steps to obtain explicit expressions for the jumps across an interface (typically a mid-surface of the layer). The resulting model is labeled "*Model II*" in [30–32]. The studies of boundary value problems with the boundary conditions similar to those for *Model II* are reported in e.g., [37–40].

While the Bövik-Benveniste methodology was successfully used in various applications (see, e.g., [41–46]) and incorporated in a few numerical procedures, e.g., [47–51], some controversy and issues related to its implementation remain. For example, in [15] it was suggested that the methodology does not work for the case when all phases in the original problem have the same properties. There, the jumps across an interface (obtained at the second step of the methodology) were confused with the jumps across a layer. The former, as expected vanish, while non-zero jumps across the layer are recovered at the post-processing step of the methodology, in which the problem, of a domain subjected to some load and prescribed vanishing jumps across the interface, is solved analytically and the fields at the traces of the layer boundaries are exactly evaluated. More serious issue was reported in [30], where it was stated that : *"the numerical results for the $O(h^2)$ version ... revealed a serious deficiency consisting in the fact that for very stiff interphases its predictions do not improve over the corresponding results of its $O(h)$ version ... and even fall more distant from the exact solution"*, and concluded that *"the construction of an $O(h^2)$ version ... in elasticity which behaves satisfactorily at all ranges of interphase stiffness remains an open issue."*

Even though the models of the second group are somewhat more versatile, in a sense that they can be used for curved and relatively thick layers, they are still of low (up to the second) order. Until recently, higher-order interface models were proposed for conductivity problems only, e.g., [9, 36]. In this paper, we derive the boundary conditions associated with *Model I* and *Model II* up to the third order for two-dimensional elastostatics problems with layers of arbitrary sufficiently smooth curvatures. To do that we develop a novel complex variables-based approach that results in explicit expressions for the boundary conditions for the two models. We demonstrate that the use of the third order *Model II* takes care of the issue identified in [30]. Using obtained boundary conditions, we perform comparative analysis of *Model I* and *Model II* and demonstrate that they can be used for simulating layers with all ranges of interphase stiffnesses.

2. Problem formulation

Consider two-dimensional, linearly elastic, problem shown in Fig. 1a in which two media '1' and '2' are separated by a thin layer (that constitutes medium '0') with the boundaries S_1, S_2 and constant thickness h. Assume that all media are homogeneous and isotropic and characterized by the shear module $\mu^{(p)}$ and Poisson's ratios $\nu^{(p)}$, where $p=\{0,1,2\}$. The following perfect bond boundary conditions are imposed:

$$\mathbf{u}^{(0)}\Big/_{S_q} = \mathbf{u}^{(q)}\Big/_{S_q}, \qquad \mathbf{T}^{(0)}\Big/_{S_q} = \mathbf{T}^{(q)}\Big/_{S_q}, \qquad q=\{1,2\}, \tag{2.1}$$

where $\mathbf{u}^{(p)}$ and $\mathbf{T}^{(p)} = \boldsymbol{\sigma}^{(p)} \cdot \mathbf{n}_q$ are respectively the displacement and traction vectors in the corresponding medium 'p', $\boldsymbol{\sigma}^{(p)}$ is the stress tensor in the medium 'p', and $\mathbf{n}_q$ is a unit vector normal to the corresponding boundary of the layer S_q.

The idea is behind asymptotic modeling of thin layer consists in simulating the presence of the layer by a set of jump conditions in the displacements and tractions across either the layer (Fig 1a) or the interface S_0 (Fig 1b) (typically a mid-line of the layer).

Below, we will use the following notations for the jumps:

$$[\![(\cdot)]\!] = (\cdot)^{(2)}\Big/_{S_2} - (\cdot)^{(1)}\Big/_{S_1}, \qquad [(\cdot)]_{S_0} = (\cdot)^{(2)}\Big/_{S_0} - (\cdot)^{(1)}\Big/_{S_0}, \tag{2.2}$$

where $[\![(\cdot)]\!]$ and $[(\cdot)]_{S_0}$ identify the jump across the layer and interface, respectively.

3. Modeling tools

Our approach is based on the use of two implementation tools: i) geometrical description in terms of complex variables, ii) fields representations in terms of holomorphic functions.

(a) Geometrical description in terms of complex variables

Assume that S_0, S_1, and S_2 are sufficiently smooth parallel and equidistant curves of arbitrary geometry and identify points $t_p \in S_p$ $(p=0,1,2)$ by complex numbers $t_p = x_p + iy_p$, where x_p and y_p are Cartesian coordinates of the point t_p and $i^2=-1$. We choose the points $t_1 \in S_1$ and $t_2 \in S_2$ as

$$t_q = t_0 + (-1)^q \,\frac{h}{2}\, e^{i\alpha}, \qquad \text{for } q=1,2 \tag{3.1}$$

that is equivalent to the following expression for the points $\mathbf{x}_p = \{x_p, y_p\}$ $(p=0,1,2)$:

$$\mathbf{x}_q = \mathbf{x}_0 + (-1)^q \frac{h}{2}\mathbf{n}, \tag{3.2}$$

where $\mathbf{n}$ is the unit vector normal to S_0 at point $\mathbf{x}_0$ and α is the angle between the x-axis and $\mathbf{n}$ (see Fig. 1).

Parametrizing S_0 by its arc-length s_0, we express the local curvature at s_0, as [52]

$$\varkappa_0(s_0) = \frac{\mathrm{d}\beta}{\mathrm{d}s_0} = \frac{\mathrm{d}\alpha}{\mathrm{d}s_0} = \frac{1}{R_0(s_0)}, \tag{3.3}$$

where $\beta = \alpha + \pi/2$ is the angle between the x-axis and the tangent vector $\mathbf{s}_0$ to the curve, and $R_0(s_0)$ is the radius of curvature. It can be shown that (see [52])

$$\frac{\mathrm{d}t_0}{\mathrm{d}s_0} = \cos(\beta) + i\sin(\beta) = e^{i\beta} = ie^{i\alpha}, \qquad \text{or} \qquad \frac{\mathrm{d}s_0}{\mathrm{d}t_0} = -ie^{-i\alpha}. \tag{3.4}$$

Since t_1 and t_2 are connected to t_0 via Eq. (3.1), the following expression can be obtained:

$$\frac{\mathrm{d}t_q}{\mathrm{d}s_0} = ie^{i\alpha}\, F_q(s_0), \qquad \text{for } q=\{1,2\}, \tag{3.5}$$

where

$$F_q(s_0) = 1 + (-1)^q \, \frac{h}{2} \, \varkappa_0(s_0), \qquad \text{for } q = \{1, 2\}. \tag{3.6}$$

Note, that $F_q(s_0)$ is the real function that represents the metric coefficient of the curvilinear coordinate system (see, Section 3.1 in [22]). Below, for the sake of brevity, the arguments, such as s_0, used to identify geometrical parameters and functions, e.g., $F_q(s_0)$ and $\varkappa_0(s_0)$, are omitted.

Assuming that F_q are nonvanishing functions for any s_0 (in other words, $h\varkappa_0 \neq 2$ for any s_0), we obtain from Eq. (3.5) that

$$\frac{\mathrm{d}s_0}{\mathrm{d}t_q} = -ie^{-i\alpha} F_q^{-1}, \qquad \text{for } q = 1, 2, \tag{3.7}$$

where $F_q^{-1} = 1/F_q$.

(b) Fields representations in terms of holomorphic functions

Introduce the complex displacement and complex traction at the point $z = x + iy$ as

$$u(z) = u_x + iu_y, \qquad \sigma(z) = \sigma_n + i\sigma_s, \tag{3.8}$$

where u_x and u_y are the displacement components in the global Cartesian coordinates (x, y) and σ_n and σ_s are the normal and shear traction components in the local coordinates.

Complex displacement u also can also be formulated in terms of tangential and normal components (u_s and u_n) as

$$u = e^{i\alpha} \left(u_n + iu_s \right). \tag{3.9}$$

Additionally, introduce the resultant force $f(z)$ as

$$f'(z) = \sigma(z), \tag{3.10}$$

where the symbol $(\cdot)'$ identifies the complex derivative $\mathrm{d}(\cdot)/\mathrm{d}z$.

The perfect bond boundary conditions can be reformulated as

$$u^{(0)} \Big/_{S_q} = u^{(q)} \Big/_{S_q}, \qquad \sigma^{(0)} \Big/_{S_q} = \sigma^{(q)} \Big/_{S_q}, \qquad q = \{1, 2\}. \tag{3.11}$$

Sometimes, it might be convenient to reformulate the second condition of Eqs. (3.11) as

$$f^{(0)} \Big/_{S_q} = f^{(q)} \Big/_{S_q}. \tag{3.12}$$

The complex displacements and tractions of two-dimensional elasticity can be represented via holomorphic functions $\varphi(z)$ and $\psi(z)$ (Kolosov-Muskhelishvili potentials), as [53]:

$$\begin{aligned} 2\mu u(z) &= \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \\ \sigma(z) &= \varphi'(z) + \overline{\varphi'(z)} + \frac{\mathrm{d}\overline{z}}{\mathrm{d}z} \left(z\overline{\varphi''(z)} + \overline{\psi'(z)} \right). \end{aligned} \tag{3.13}$$

where $\kappa = 3 - 4\nu$ in plane strain and $\kappa = (3 - \nu)/(1 + \nu)$ in plain stress, a bar over a symbol denotes complex conjugation, and $\mathrm{d}\overline{z}/\mathrm{d}z = -e^{-2i\alpha}$.

The resultant force $f(z)$ can also be represented via the holomorphic functions as

$$f(z) = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}. \tag{3.14}$$

With the use of Eqs. (3.13) and (3.14), the potentials $\varphi(z)$ and $\psi(z)$ can be expressed as

$$\varphi = \frac{1}{\kappa + 1} \left(2\mu u + f \right), \qquad \psi = \frac{1}{\kappa + 1} \left[\kappa\overline{f} - 2\mu\overline{u} - \overline{z} \left(2\mu u' + f' \right) \right]. \tag{3.15}$$

The use of holomorphic functions is advantageous for the following reasons. First, those functions are infinitely differentiable and equal locally to their Taylor series expansions that are essential for proposed asymptotic procedure. Second, the complex derivatives involved in Taylor expansions can be expressed using surface derivatives (discussed below) via Eqs. (3.4), (3.7).

In addition to the notion of holomorphic function, we use that of the derivative of an arbitrary complex-valued function $g_0(t_0)$ defined on curve S_0, as [52]:

$$\left.\frac{\mathrm{d}g_0(t_0)}{\mathrm{d}t_0}\right|_{t_0=a_0} = \lim_{t_0 \to a_0} \frac{g_0(a_0) - g_0(t_0)}{a_0 - t_0}, \tag{3.16}$$

where $t_0,\ a_0 \in S_0$. Below, for the sake of brevity, the arguments of functions are omitted.

By the use of Eq. (3.4), it can be shown that all derivatives of g_0 with respect to t_0 can be expressed via the derivatives with respect to s_0. For instance, the corresponding expressions for the first two derivatives are

$$\begin{aligned} \frac{\mathrm{d}g_0}{\mathrm{d}t_0} &= \frac{\mathrm{d}g_0}{\mathrm{d}s_0}\frac{\mathrm{d}s_0}{\mathrm{d}t_0} = -ie^{-i\alpha} g_{0,s_0}, \\ \frac{\mathrm{d}^2 g_0}{\mathrm{d}t_0^2} &= \frac{\mathrm{d}s_0}{\mathrm{d}t_0}\left(-ie^{-i\alpha} g_{0,s_0}\right)_{,s_0} = e^{-2i\alpha}\left(i\varkappa_0 g_{0,s_0} - g_{0,s_0 s_0}\right), \end{aligned} \tag{3.17}$$

where $(\cdot)_{,s_0}$ and $(\cdot)_{,s_0 s_0}$ refer to the derivatives of corresponding function with respect to s_0.

Using Eq. (3.7), the derivative of complex function g_q defined on S_q, $q = 1, 2$, can be expressed via the derivative with respect s_0 as

$$\frac{\mathrm{d}g_q}{\mathrm{d}t_q} = \frac{\mathrm{d}g_q}{\mathrm{d}s_0}\frac{\mathrm{d}s_0}{\mathrm{d}g_q} = -ie^{-i\alpha} F_q^{-1} g_{q,s_0}, \tag{3.18}$$

which leads to the following interrelations between the derivatives:

$$g_{q,s_q} = F_q^{-1} g_{q,s_0}, \tag{3.19}$$

where s_q is the arc-length of S_q and $(\cdot)_{,s_q}$ identifies the derivative of corresponding function with respect to s_q. The relations obtained here are similar to those presented in Section 3.1 of [22], using the concept of "parallel curvilinear coordinate systems."

4. Modeling approach

In this section, we outline the main steps of the proposed asymptotic procedures for the models of the first order. The detailed derivations are presented in Supplementary material S1.

(a) Jump conditions across the layer (*Model I*)

First, the complex potentials of Eq. (3.15) are expanded in terms of Taylor expansions about the two points $t_q \in S_q$ $(q = 1, 2)$. For example, the expansion for potential $\varphi^{(0)}(z)$ at $z = t_0 \in S_0$ is

$$\varphi^{(0)}(t_0) = \varphi^{(0)}\Big/_{z=t_q} + \sum_{n=1}^{\infty} \frac{(t_0 - t_q)^n}{n!}\left(\frac{\mathrm{d}^n \varphi^{(0)}}{\mathrm{d}z^n}\right)\Big/_{z=t_q}. \tag{4.1}$$

In the following, it is assumed that potential $\varphi^{(p)}$ $(\psi^{(p)})$ for $p = \{0, 1, 2\}$ has a limit value at S_q, indicated as $\varphi_q^{(p)}$ $(\psi_q^{(p)})$, such that each complex derivative of that potential is equal to the derivative of the corresponding order of $\varphi_q^{(p)}$ $(\psi_q^{(p)})$ with respect to t_q. The subscript q that identifies limit value will be used for all fields considered thereafter.

Then, using Eqs. (3.18) and (3.1), expansions (4.1) are truncated up to the first order as

$$\varphi^{(0)}(t_0) = \varphi_q^{(0)} + (-1)^q \frac{ih}{2} F_q^{-1} \varphi_{q,s_0}^{(0)} + O(h^2). \tag{4.2}$$

Subtraction of the value of $\varphi^{(0)}(t_0)$ given by Eq. (4.2) for $q = 2$ from that given by the same equation for $q = 1$ leads to

$$\varphi_2^{(0)} - \varphi_1^{(0)} = -\frac{ih}{2}\sum_{q=1}^{2} F_q^{-1} \varphi_{q,s_0}^{(0)} + O(h^2). \tag{4.3}$$

Assuming that $h\varkappa_0/2<1$, F_q^{-1} could be expanded as

$$F_q^{-1}=\left(1+(-1)^q\frac{h}{2}\varkappa_0\right)^{-1}=1+\sum_{k=1}^{\infty}(-1)^{k(q+1)}\left(\frac{h}{2}\varkappa_0\right)^k. \tag{4.4}$$

Using expansions (4.4) and boundary condition expressed in terms of potentials as well as Eq. (3.1), the first jump condition across the layer can be obtained from (4.3) as

$$\begin{aligned}&\sum_{q=1}^{2}(-1)^q\left\{a^{(q)}\varphi_q^{(q)}-b^{(q)}\left(it_0e^{i\alpha}\overline{\varphi_{q,s_0}^{(q)}}+\overline{\psi_q^{(q)}}\right)\right\}=-\frac{ih}{2}\sum_{q=1}^{2}\left\{a^{(q)}\varphi_{q,s_0}^{(q)}\right.\\&\left.-b^{(q)}\left[e^{i\alpha}\left(e^{i\alpha}-2t_0\varkappa_0\right)\overline{\varphi_{q,s_0}^{(q)}}+it_0e^{i\alpha}\overline{\varphi_{q,s_0s_0}^{(q)}}+\overline{\psi_{q,s_0}^{(q)}}\right]\right\}+O(h^2),\end{aligned} \tag{4.5}$$

where $q=\{1,2\}$ and

$$\begin{aligned}a^{(q)}&=\frac{\mu^{(0)}}{\mu^{(q)}}\kappa^{(q)}+1, \qquad b^{(q)}=\frac{\mu^{(0)}}{\mu^{(q)}}-1,\\c^{(q)}&=\kappa^{(0)}-\kappa^{(q)}\frac{\mu^{(0)}}{\mu^{(q)}}, \qquad d^{(q)}=\kappa^{(0)}+\frac{\mu^{(0)}}{\mu^{(q)}}.\end{aligned} \tag{4.6}$$

Following similar procedure for potential $\psi^{(0)}$, the second jump condition across the layer can be obtained:

$$\begin{aligned}&\sum_{q=1}^{2}(-1)^q\left\{c^{(q)}\overline{\varphi_q^{(q)}}+b^{(q)}\overline{t_0}\left[i\left(t_0\varkappa_0+e^{i\alpha}\right)\overline{\varphi_{q,s_0}^{(q)}}+t_0\overline{\varphi_{q,s_0s_0}^{(q)}}\right]\right.\\&\left.+d^{(q)}\psi_q^{(q)}+i\overline{t_0}e^{-i\alpha}\left[\left(a^{(q)}-d^{(q)}\right)\varphi_{q,s_0}^{(q)}-b^{(q)}\overline{\psi_{q,s_0}^{(q)}}\right]\right\}\\&=-\frac{ih}{2}\sum_{q=1}^{2}\left\{\left(a^{(q)}-d^{(q)}\right)e^{-i\alpha}\left(2e^{-i\alpha}\varphi_{q,s_0}^{(q)}+i\overline{t_0}\varphi_{q,s_0s_0}^{(q)}\right)\right.\\&+\left[c^{(q)}+2b^{(q)}\left(1+\varkappa_0\left(t_0e^{-i\alpha}-\overline{t_0}e^{i\alpha}\right)-t_0\overline{t_0}\left(\varkappa_0^2-i\varkappa_{0,s_0}\right)\right)\right]\overline{\varphi_{q,s_0}^{(q)}}\\&+ib^{(q)}\left[\overline{t_0}e^{i\alpha}+t_0\left(3\overline{t_0}\varkappa_0-2e^{-i\alpha}\right)\right]\overline{\varphi_{q,s_0s_0}^{(q)}}+b^{(q)}t_0\overline{t_0}\overline{\varphi_{q,s_0s_0s_0}^{(q)}}\\&\left.+d^{(q)}\psi_{q,s_0}^{(q)}-b^{(q)}e^{-i\alpha}\left[2e^{-i\alpha}\overline{\psi_{q,s_0}^{(q)}}+i\overline{t_0}\overline{\psi_{q,s_0s_0}^{(q)}}\right]\right\}+O(h^2).\end{aligned} \tag{4.7}$$

Eqs. (4.5) and (4.7) for the jumps across the layer represent the boundary conditions (in terms of potentials) for first order *Model I*. They can be reformulated in terms of jumps in displacement and traction components, see Supplementary material S1 for more detail.

(b) Jump conditions across the interface (*Model II*)

To formulate jumps across the interface, it is assumed that conditions (4.5) and (4.7) obtained for the problem of Fig. 1a are valid for the corresponding fields in the problem of Fig. 1b.

Expanding the fields involved in (4.5) and (4.7) about $z=t_0$ as

$$\begin{aligned}\varphi^{(q)}(t_q)&=\varphi^{(q)}/_{z=t_0}+\sum_{k=1}^{\infty}\frac{(t_q-t_0)^k}{k!}\left(\frac{\mathrm{d}^k\varphi^{(q)}}{\mathrm{d}z^k}\right)\Big/_{z=t_0},\\\psi^{(q)}(t_q)&=\psi^{(q)}/_{z=t_0}+\sum_{k=1}^{\infty}\frac{(t_q-t_0)^k}{k!}\left(\frac{\mathrm{d}^k\varphi^{(q)}}{\mathrm{d}z^k}\right)\Big/_{z=t_0},\end{aligned} \tag{4.8}$$

and truncating those expansions up to the first order with use of Eq. (3.18), we get

$$\begin{aligned}\varphi^{(q)}(t_q) &= \varphi_0^{(q)} - (-1)^q \frac{ih}{2}\varphi_{0,s_0}^{(q)} + O(h^2),\\ \psi^{(q)}(t_q) &= \psi^{(q)}/_{S_0} - (-1)^q \frac{ih}{2}\psi_{0,s_0}^{(q)} + O(h^2).\end{aligned} \tag{4.9}$$

Applying expansions (4.9) to jump conditions (4.5) and (4.7), neglecting all terms that include h^n with $n>1$ (the condition is defined by the order of the interface model), and using straightforward algebra, the following jump conditions across the interface are obtained:

$$\begin{aligned}&\sum_{q=1}^{2}(-1)^q\left\{a^{(q)}\varphi_0^{(q)} - b^{(q)}\left(it_0 e^{i\alpha}\overline{\varphi_{0,s_0}^{(q)}} + \overline{\psi_0^{(q)}}\right)\right\}\\ &= -ih\sum_{q=1}^{2} b^{(q)}\left\{t_0 e^{i\alpha}\left[\varkappa_0\overline{\varphi_{0,s_0}^{(q)}} - i\overline{\varphi_{0,s_0s_0}^{(q)}}\right] - \overline{\psi_{0,s_0}^{(q)}}\right\} + O(h^2),\end{aligned} \tag{4.10}$$

$$\begin{aligned}&\sum_{q=1}^{2}(-1)^q\Big\{c^{(q)}\overline{\varphi_0^{(q)}} + b^{(q)}\overline{t_0}\left[i\left(e^{i\alpha}+t_0\varkappa_0\right)\overline{\varphi_{0,s_0}^{(q)}} + t_0\overline{\varphi_{0,s_0s_0}^{(q)}}\right] + d^{(q)}\psi^{(q)}\\ &+ i\overline{t_0}e^{-i\alpha}\left[\left(a^{(q)}-d^{(q)}\right)\varphi_{0,s_0}^{(q)} - b^{(q)}\overline{\psi_{0,s_0}^{(q)}}\right]\Big\}\\ &= -ih\sum_{q=1}^{2}\Big\{\left[c^{(q)} + b^{(q)}\left(1 - t_0\overline{t_0}\left(\varkappa_0^2 - i\varkappa_{0,s_0}\right) + \varkappa_0\left(t_0 e^{-i\alpha} - \overline{t_0}e^{i\alpha}\right)\right)\right]\overline{\varphi_{0,s_0}^{(q)}}\\ &+ b^{(q)}\left[i\left(\overline{t_0}e^{i\alpha} + t_0\left(2\overline{t_0}\varkappa_0 - e^{-i\alpha}\right)\right)\overline{\varphi_{0,s_0s_0}^{(q)}} + t_0\overline{t_0}\,\overline{\varphi_{0,s_0s_0s_0}^{(q)}}\right]\\ &+ e^{-2i\alpha}\left[\left(a^{(q)}-d^{(q)}\right)\varphi_{0,s_0}^{(q)} - b^{(q)}\left(\overline{\psi_{0,s_0}^{(q)}} + i\overline{t_0}e^{i\alpha}\overline{\psi_{0,s_0s_0}^{(q)}}\right)\right]\Big\} + O(h^2).\end{aligned} \tag{4.11}$$

Eqs. (4.10), (4.11) represent the first order imperfect interface boundary conditions (*Model II*) in terms of potentials. They can be reformulated in terms of jumps in complex displacement and traction across the interface, see Supplementary material S1 for more detail.

5. List of the boundary conditions for the higher order models

In this section, the list of the third order boundary conditions for *Model I* and *Model II* is presented. They are obtained using procedures similar to those of Section 4, but truncating the expansions up to corresponding orders. The conditions for the first (second) order models can be obtained from the listed expressions by omitting all terms that are multiplied by h^2 and h^3 (only h^3).

In the expressions below, the notation $\{...\}/_{S_p}$ $(p=0,1,2)$ is used to identify limit values of elastic fields involved in $\{...\}$. The subscript p used for elastic fields (e.g., $\sigma_0^{(1)}$) that served similar purpose is omitted.

(a) Boundary conditions for *Model I*

The boundary conditions, provided in this subsection in terms of jumps in local components, involve derivatives with respect to the arc-length s_0 of layer's mid-line S_0. They can be reformulated in terms of derivatives with respect to the arc-lengths s_q of layer's boundaries S_q, if desired.

$$
\begin{aligned}
[\![u_n]\!] = {} & \frac{h}{2}\sum_{q=1}^{2}\left\{\frac{a_3^{(0)}}{\mu^{(0)}}\sigma_n^{(q)} + \left(1-4a_1^{(0)}\right)\left(\varkappa_0 u_n^{(q)} + u_{s,s_0}^{(q)}\right)\right\}\Big/_{S_q} + \frac{h^2}{8}\sum_{q=1}^{2}(-1)^q\times \\
& \left\{\frac{1}{\mu^{(0)}}\left(a_3^{(0)}\varkappa_0\sigma_n^{(q)} + 2a_1^{(0)}\sigma_{s,s_0}^{(q)}\right) - 2a_3^{(0)}\varkappa_0^2 u_n^{(q)} + \left(1-4a_1^{(0)}\right)\left(u_{n,s_0s_0}^{(q)} - \varkappa_{0,s_0}u_s^{(q)}\right)\right. \\
& \left. - \left(3-8a_1^{(0)}\right)\varkappa_0 u_{s,s_0}^{(q)}\right\}\Big/_{S_q} + \frac{h^3}{48}\sum_{q=1}^{2}\left\{\frac{1}{\mu^{(0)}}\left[-\left(1-4a_1^{(0)}\right)\sigma_{n,s_0s_0}^{(q)} + \varkappa_{0,s_0}\sigma_s^{(q)}\right.\right. \\
& \left. + 2a_2^{(0)}\varkappa_0\sigma_{s,s_0}^{(q)}\right] + \left[6a_3^{(0)}\varkappa_0^3 - \left(1-8a_1^{(0)}\right)\varkappa_{0,s_0s_0}\right]u_n^{(q)} - \left(5-28a_1^{(q)}\right)\varkappa_{0,s_0}u_{n,s_0}^{(q)} \\
& - 6\left(1-4a_1^{(0)}\right)\varkappa_0 u_{n,s_0s_0}^{(q)} + 4\left(2-7a_1^{(0)}\right)\varkappa_0\varkappa_{0,s_0}u_s^{(q)} + \left(11-28a_1^{(0)}\right)\varkappa_0^2 u_{s,s_0}^{(q)} \\
& \left. - \left(1-8a_1^{(0)}\right)u_{s,s_0s_0s_0}^{(q)}\right\}\Big/_{S_q},
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
[\![u_s]\!] = {} & \frac{h}{2}\sum_{q=1}^{2}\left\{\frac{1}{\mu^{(0)}}\sigma_s^{(q)} + \varkappa_0 u_s^{(q)} - u_{n,s_0}^{(q)}\right\}\Big/_{S_q} + \frac{h^2}{8}\sum_{q=1}^{2}(-1)^q\left\{\frac{1}{\mu^{(0)}}(2a_1^{(0)}\sigma_{n,s_0}^{(q)}\right. \\
& \left. + \varkappa_0\sigma_s^{(q)}) + \left(1+4a_1^{(0)}\right)\left(\varkappa_{0,s_0}u_n^{(q)} + u_{s,s_0s_0}^{(q)}\right) + \left(3+4a_1^{(0)}\right)\varkappa_0 u_{n,s_0}^{(q)} - 2\varkappa_0^2 u_s^{(q)}\right\}\Big/_{S_q} \\
& - \frac{h^3}{48}\sum_{q=1}^{2}\left\{\frac{1}{\mu^{(0)}}\left[a_3^{(0)}\varkappa_{0,s_0}\sigma_n^{(q)} + 2a_2^{(0)}\varkappa_0\sigma_{n,s_0}^{(q)} + \left(1+2a_1^{(0)}\right)\sigma_{s,s_0s_0}^{(q)}\right]\right. \\
& + 8\left(1+2a_1^{(0)}\right)\varkappa_0\varkappa_{0,s_0}u_n^{(q)} + \left(11+8a_1^{(0)}\right)\varkappa_0^2 u_{n,s_0}^{(q)} - \left(1+4a_1^{(0)}\right)u_{n,s_0s_0s_0}^{(q)} - \left[6\varkappa_0^3\right. \\
& \left.\left. - \left(1+4a_1^{(0)}\right)\varkappa_{0,s_0s_0}\right]u_s^{(q)} + \left(5+16a_1^{(0)}\right)\varkappa_{0,s_0}u_{s,s_0}^{(q)} + 6\left(1+2a_1^{(0)}\right)\varkappa_0 u_{s,s_0s_0}^{(q)}\right\}\Big/_{S_q},
\end{aligned}
\tag{5.2}
$$

$$
\begin{aligned}
[\![\sigma_n]\!] = {} & -\frac{h}{2}\sum_{q=1}^{2}\left\{2a_3^{(0)}\varkappa_0\sigma_n^{(q)} + \sigma_{s,s_0}^{(q)} - 8\mu^{(0)}a_1^{(0)}\varkappa_0\left(\varkappa_0 u_n^{(q)} + u_{s,s_0}^{(q)}\right)\right\}\Big/_{S_q} \\
& - \frac{h^2}{8}\sum_{q=1}^{2}(-1)^q\left\{2a_3^{(q)}\varkappa_0^2\sigma_n^{(q)} - \left(1-4a_1^{(0)}\right)\sigma_{n,s_0s_0}^{(q)} + 2\varkappa_{0,s_0}\sigma_s^{(q)} + \left(3+4a_1^{(0)}\right)\varkappa_0\sigma_{s,s_0}^{(q)}\right. \\
& + 8\mu^{(0)}a_1^{(0)}\left[\left(\varkappa_0^3 + \varkappa_{0,s_0s_0}\right)u_n^{(q)} + 2\varkappa_{0,s_0}u_{n,s_0}^{(q)} + \varkappa_0\varkappa_{0,s_0}u_s^{(q)} + 2\varkappa_0^2 u_{s,s_0}^{(q)}\right. \\
& \left.\left. + u_{s,s_0s_0s_0}^{(q)}\right]\right\}\Big/_{S_q} + \frac{h^3}{48}\sum_{q=1}^{2}\left\{2a_3^{(q)}\left(\varkappa_{0,s_0s_0}\sigma_n^{(q)} + 2\varkappa_{0,s_0}\sigma_{n,s_0}^{(q)}\right) - 2\varkappa_0\varkappa_{0,s_0}\sigma_s^{(q)}\right. \\
& + 3\left(1-4a_1^{(0)}\right)\varkappa_0\sigma_{n,s_0s_0}^{(q)} - 2a_3^{(q)}\varkappa_0^2\sigma_{s,s_0}^{(q)} + \left(1+4a_1^{(0)}\right)\sigma_{s,s_0s_0s_0}^{(q)} + 8\mu^{(0)}a_1^{(0)}\times \\
& \left[\left(3\varkappa_0^4 + 4\left(\varkappa_{0,s_0}\right)^2 + 2\varkappa_0\varkappa_{0,s_0s_0}\right)u_n^{(q)} + \varkappa_0\varkappa_{0,s_0}u_{n,s_0}^{(q)} - 4\varkappa_0^2 u_{n,s_0s_0}^{(q)} - u_{n,s_0s_0s_0s_0}^{(q)}\right. \\
& \left.\left. + \left(7\varkappa_0^2\varkappa_{0,s_0} + \varkappa_{0,s_0s_0s_0}\right)u_s^{(q)} + \left(7\varkappa_0^3 + 5\varkappa_{0,s_0s_0}\right)u_{s,s_0}^{(q)} + 7\varkappa_{0,s_0}u_{s,s_0s_0}^{(q)} + \varkappa_0 u_{s,s_0s_0s_0}^{(q)}\right]\right\}\Big/_{S_q},
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
[\![\sigma_s]\!] = \frac{h}{2}\sum_{q=1}^{2}\Big\{ \left(1-4a_1^{(0)}\right)\sigma_{n,s_0}^{(q)} - 2\varkappa_0\sigma_s^{(q)} - 8\mu^{(0)}a_1^{(0)}\Big[\varkappa_{0,s_0}u_n^{(q)} + \varkappa_0 u_{n,s_0}^{(q)} \\
+ u_{s,s_0s_0}^{(q)}\Big]\Big\}\Big/_{S_q} + \frac{h^2}{8}\sum_{q=1}^{2}(-1)^q\Big\{2a_3^{(q)}\varkappa_{0,s_0}\sigma_n^{(q)} + \left(3-8a_1^{(0)}\right)\varkappa_0\sigma_{n,s_0}^{(q)} - 2\varkappa_0^2\sigma_s^{(q)} \\
+ \left(1+4a_1^{(0)}\right)\sigma_{s,s_0s_0}^{(q)} + 8\mu^{(0)}a_1^{(0)}\Big[\varkappa_0\varkappa_{0,s_0}u_n^{(q)} - u_{n,s_0s_0s_0}^{(q)} + \varkappa_{0,s_0s_0}u_s^{(q)} + 3\varkappa_{0,s_0}u_{s,s_0}^{(q)} \\
+ \varkappa_0 u_{s,s_0s_0}^{(q)}\Big]\Big\}\Big/_{S_q} + \frac{h^3}{48}\sum_{q=1}^{2}\Big\{2a_3^{(0)}\varkappa_0\left(\varkappa_{0,s_0}\sigma_n^{(q)} + \varkappa_0\sigma_{n,s_0}^{(q)}\right) - \left(1-8a_1^{(0)}\right)\sigma_{n,s_0s_0s_0}^{(q)} \\
+ 2\varkappa_{0,s_0s_0}\sigma_s^{(q)} + 4a_2^{(0)}\varkappa_{0,s_0}\sigma_{s,s_0}^{(q)} + 3\varkappa_0\sigma_{s,s_0s_0}^{(q)} - 8\mu^{(0)}a_1^{(0)}\Big[\left(5\varkappa_0^2\varkappa_{0,s_0} - 2\varkappa_{0,s_0s_0s_0}\right)u_n^{(q)} \\
+ \left(\varkappa_0^3 - 9\varkappa_{0,s_0s_0}\right)u_{n,s_0}^{(q)} - 13\varkappa_{0,s_0}u_{n,s_0s_0}^{(q)} - 5\varkappa_0 u_{n,s_0s_0s_0}^{(q)} + \left(7\left(\varkappa_{0,s_0}\right)^2 + 6\varkappa_0\varkappa_{0,s_0s_0}\right)u_s^{(q)} \\
+ 17\varkappa_0\varkappa_{0,s_0}u_{s,s_0}^{(q)} + 4\varkappa_0^2 u_{s,s_0s_0}^{(q)} - 2u_{s,s_0s_0s_0s_0}^{(q)}\Big]\Big\}\Big/_{S_q},
\end{aligned}
\tag{5.4}
$$

where the following coefficients are used

$$
\begin{aligned}
&\xi^{(p)} = \frac{1}{\kappa^{(p)}+1}, && \text{for } \ p=\{0,1,2\}, && \\
&a_1^{(0)} = \xi^{(0)}, && a_2^{(0)} = \kappa^{(0)}a_1^{(0)}, && a_3^{(0)} = a_2^{(0)} - a_1^{(0)}, \\
&a_4^{(0)} = a_1^{(0)} + 3a_2^{(0)}, && a_5^{(0)} = 2a_2^{(0)} - a_1^{(0)}, &&
\end{aligned}
\tag{5.5}
$$

and the superscript "0" is used to identify the parameters related to the medium of the layer only.

The boundary conditions of first order, that can be extracted from Eqs. (5.1)-(5.4), are identical to those given for *Model I* in [31] (Eqs. (26)-(29), for the case of constant modulii) and to those in [30] (Eqs. (3.5)-(3.6)), if our conditions are reformulated in terms of the derivatives along S_q.

(b) Boundary conditions for *Model II*

We introduce the following coefficients that are related to the terms multiplied by h:

$$
\begin{aligned}
&A_1^{(q)} = \xi^{(0)} - \xi^{(q)}, && A_2^{(q)} = \mu^{(0)}\xi^{(0)} - \mu^{(q)}\xi^{(q)}, \\
&A_3^{(q)} = \frac{\xi^{(0)}}{\mu^{(0)}} - \frac{\xi^{(q)}}{\mu^{(q)}}, && A_4^{(q)} = \frac{\kappa^{(0)}\xi^{(0)}}{\mu^{(0)}} - \frac{\kappa^{(q)}\xi^{(q)}}{\mu^{(q)}}.
\end{aligned}
\tag{5.6}
$$

Similarly, the coefficients related to the terms multiplied by h^2 are

$$
\begin{aligned}
&B_1^{(q)} = \xi^{(0)}\left(\frac{\mu^{(0)}}{\mu^{(q)}} - 1\right), && B_2^{(q)} = A_1^{(q)} - 8\xi^{(q)} B_1^{(q)},\\
&B_3^{(q)} = \left(3 + \kappa^{(q)}\right) A_1^{(q)} - \kappa^{(q)} B_2^{(q)}, && B_4^{(q)} = B_2^{(q)} + B_3^{(q)} - 2A_1^{(q)},\\
&B_5^{(q)} = A_1^{(q)} + 2B_1^{(q)}, && B_6^{(q)} = -A_2^{(q)} + \mu^{(q)}\left(A_1^{(q)} - B_2^{(q)}\right),\\
&B_7^{(q)} = 2A_2^{(q)} + B_6^{(q)}, && B_8^{(q)} = A_2^{(q)},\\
&B_9^{(q)} = \frac{4\left(\kappa^{(q)} - \kappa^{(0)}\right)}{\mu^{(0)}}\xi^{(q)} B_1^{(q)}, &&\\
&B_{10}^{(q)} = A_3^{(q)} - B_9^{(q)} + \frac{1}{\mu^{(0)}}\left(B_3^{(q)} - 3A_1^{(q)}\right), &&\\
&B_{11}^{(q)} = A_4^{(q)} + \frac{4\left(\kappa^{(q)}\kappa^{(0)} + 1\right)}{\mu^{(0)}}\xi^{(q)} B_1^{(q)}, &&\\
&B_{12}^{(q)} = A_4^{(q)} + B_{11}^{(q)} + \frac{1}{\mu^{(0)}}\left(B_2^{(q)} - A_1^{(q)}\right), &&\\
&B_{13}^{(q)} = 4\xi^{(q)}\left(\frac{\mu^{(q)}}{\mu^{(0)}} - 1\right), && B_{14}^{(q)} = A_1^{(q)} - 2\xi^{(0)} B_{13}^{(q)},\\
&B_{15}^{(q)} = \left(1 - \kappa^{(0)}\right) A_1^{(q)} + \kappa^{(0)} B_{14}^{(q)}, && B_{16}^{(q)} = 2A_1^{(q)} - B_{13}^{(q)};
\end{aligned}
\tag{5.7}
$$

and those related to the terms multiplied by h^3 are

$$
\begin{aligned}
&C_1^{(q)} = \frac{1}{4}\left(B_4^{(q)} - 2A_1^{(q)}\right), && C_2^{(q)} = C_1^{(q)} + A_1^{(q)} - B_2^{(q)},\\
&C_3^{(q)} = -A_1^{(q)} + 12B_1^{(q)}, && C_4^{(q)} = A_1^{(q)} - 2B_1^{(q)},\\
&C_5^{(q)} = \frac{1}{2}\left(7A_1^{(q)} - 3B_2^{(q)}\right), && C_6^{(q)} = B_6^{(q)} + A_2^{(q)},\\
&C_7^{(q)} = 2A_2^{(q)} + 9C_6^{(q)}, && C_8^{(q)} = 2A_2^{(q)} + 3C_6^{(q)},\\
&C_9^{(q)} = 2A_2^{(q)} + C_8^{(q)}, && C_{10}^{(q)} = -\frac{1}{2}\left(B_{10}^{(q)} - A_3^{(q)}\right),\\
&C_{11}^{(q)} = -B_9^{(q)}, && C_{12}^{(q)} = \frac{1}{2}\left(B_{11}^{(q)} - A_4^{(q)}\right),\\
&C_{13}^{(q)} = \frac{1}{2}\left(3B_{11}^{(q)} - 6B_{12}^{(q)} + 11A_4^{(q)}\right), && C_{14}^{(q)} = \frac{1}{2}\left(3B_{12}^{(q)} - 3B_{11}^{(q)} + A_4^{(q)}\right),\\
&C_{15}^{(q)} = 4\kappa^{(0)}\xi^{(0)} B_{13}^{(q)} && C_{16}^{(q)} = \xi^{(0)} B_{13}^{(q)} - \frac{1}{4} C_{15}^{(q)},\\
&C_{17}^{(q)} = -A_1^{(q)} + 9\left(\frac{1}{4} C_{15}^{(q)} + C_{16}^{(q)}\right), && C_{18}^{(q)} = A_1^{(q)} - C_{15}^{(q)} - \frac{7}{2} C_{16}^{(q)},\\
&C_{19}^{(q)} = 2A_1^{(q)} - 3\left(\frac{1}{4} C_{15}^{(q)} + C_{16}^{(q)}\right), &&
\end{aligned}
\tag{5.8}
$$

$$\begin{aligned}
&D_1^{(q)} = 3\left(2C_1^{(q)} - C_2^{(q)}\right) - A_1^{(q)}, && D_2^{(q)} = 3\left(2C_1^{(q)} + C_2^{(q)}\right) + A_1^{(q)},\\
&D_3^{(q)} = 3\left(4C_1^{(q)} + 3C_2^{(q)}\right) + A_1^{(q)}, && D_4^{(q)} = 3\left(4C_1^{(q)} + C_2^{(q)}\right) - A_1^{(q)},\\
&D_5^{(q)} = 3C_2^{(q)} - A_1^{(q)}, && D_6^{(q)} = C_7^{(q)} - \frac{1}{2}C_8^{(q)},\\
&D_7^{(q)} = C_{10}^{(q)} + 3C_{11}^{(q)} - B_{10}^{(q)}, && D_8^{(q)} = A_1^{(q)} + \frac{3}{2}\left(C_{15}^{(q)} + C_{16}^{(q)}\right).
\end{aligned} \tag{5.9}$$

With those notations, the boundary condition in terms of local components of displacement and traction vectors are

$$\begin{aligned}
[u_n]_{S_0} = &-\frac{h}{2}\sum_{q=1}^{2}\left\{\left(A_3^{(q)} - A_4^{(q)}\right)\sigma_n^{(q)} + 4A_1^{(q)}\left(\varkappa_0 u_n^{(q)} + u_{s,s_0}^{(q)}\right)\right\}\Big/_{S_0}\\
&-\frac{h^2}{2}\sum_{q=1}^{2}(-1)^q\Big\{\frac{1}{4}\left(B_{10}^{(q)} - B_{11}^{(q)}\right)\varkappa_0\sigma_n^{(q)} + \frac{1}{4}\left(B_9^{(q)} - B_{12}^{(q)}\right)\sigma_{s,s_0}^{(q)}\\
&+\left(B_{13}^{(q)} + B_{15}^{(q)}\right)\varkappa_0^2 u_n^{(q)} - A_1^{(q)}\left(u_{n,s_0s_0}^{(q)} - \varkappa_{0,s_0}u_s^{(q)}\right)\\
&+\left(A_1^{(q)} + B_{13}^{(q)} + B_{15}^{(q)}\right)\varkappa_0 u_{s,s_0}^{(q)}\Big\}\Big/_{S_0} - \frac{h^3}{24}\sum_{q=1}^{2}\Big\{3\left(C_{10}^{(q)} + C_{12}^{(q)}\right)\varkappa_0^2\sigma_n^{(q)}\\
&-\left(3C_{10}^{(q)} - C_{14}^{(q)}\right)\sigma_{n,s_0s_0}^{(q)} + \frac{1}{2}\left(D_7^{(q)} - C_{13}^{(q)}\right)\varkappa_{0,s_0}\sigma_s^{(q)} + \left(3C_{11}^{(q)} - C_{13}^{(q)}\right)\varkappa_0\sigma_{s,s_0}^{(q)}\\
&-\left[6C_{16}^{(q)}\varkappa_0^3 - \left(3C_{16}^{(q)} - 2C_{19}^{(q)}\right)\varkappa_{0,s_0s_0}\right]u_n^{(q)} + 2\left(3C_{16}^{(q)} - 2C_{19}^{(q)}\right)\varkappa_{0,s_0}u_{n,s_0}^{(q)}\\
&-2\left(A_1^{(q)} + C_{19}^{(q)}\right)\varkappa_0 u_{n,s_0s_0}^{(q)} + \left(2A_1^{(q)} + 3C_{16}^{(q)}\right)\varkappa_0\varkappa_{0,s_0}u_s^{(q)}\\
&+\left(2A_1^{(q)} - 3C_{16}^{(q)}\right)\varkappa_0^2 u_{s,s_0}^{(q)} + \left(3C_{16}^{(q)} - 2C_{19}^{(q)}\right)u_{s,s_0s_0s_0}^{(q)}\Big\}\Big/_{S_0},
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
[u_s]_{S_0} = &\frac{h}{2}\sum_{q=1}^{2}\left(A_3^{(q)} + A_4^{(q)}\right)\sigma_s^{(q)}\Big/_{S_0} - \frac{h^2}{2}\sum_{q=1}^{2}(-1)^q\Big\{\frac{1}{4}\left(B_9^{(q)} + B_{12}^{(q)}\right)\sigma_{n,s_0}^{(q)}\\
&-\frac{1}{4}\left(B_{10}^{(q)} + B_{11}^{(q)}\right)\varkappa_0\sigma_s^{(q)} - \left(A_1^{(q)} - B_{13}^{(q)}\right)\left(\varkappa_{0,s_0}u_n^{(q)} + \varkappa_0 u_{n,s_0}^{(q)} + u_{s,s_0s_0}^{(q)}\right)\Big\}\Big/_{S_0}\\
&-\frac{h^3}{24}\sum_{q=1}^{2}\Big\{\frac{1}{2}\left(C_{13}^{(q)} + D_7^{(q)}\right)\varkappa_{0,s_0}\sigma_n^{(q)} + \left(3C_{11}^{(q)} + C_{13}^{(q)}\right)\varkappa_0\sigma_{n,s_0}^{(q)}\\
&-3\left(C_{10}^{(q)} - C_{12}^{(q)}\right)\varkappa_0^2\sigma_s^{(q)} + \left(3C_{10}^{(q)} + C_{14}^{(q)}\right)\sigma_{s,s_0s_0}^{(q)}\\
&-4\left[7A_1^{(q)} - 3\left(C_{16}^{(q)} + C_{19}^{(q)}\right)\right]\varkappa_0\varkappa_{0,s_0}u_n^{(q)} - \left(18A_1^{(q)} - 9C_{16}^{(q)} - 8C_{19}^{(q)}\right)\varkappa_0^2 u_{n,s_0}^{(q)}\\
&+\left(3C_{16}^{(q)} + 2C_{19}^{(q)}\right)\left(u_{n,s_0s_0s_0}^{(q)} - \varkappa_{0,s_0s_0}u_s^{(q)}\right) - \left(10A_1^{(q)} + 3C_{16}^{(q)}\right)\varkappa_{0,s_0}u_{s,s_0}^{(q)}\\
&-6\left(3A_1^{(q)} - C_{16}^{(q)} - C_{19}^{(q)}\right)\varkappa_0 u_{s,s_0s_0}^{(q)}\Big\}\Big/_{S_0},
\end{aligned} \tag{5.11}$$

$$
\begin{aligned}
&[\sigma_n]_{S_0} = 2h\varkappa_0 \sum_{q=1}^{2} \left\{ A_1^{(q)} \sigma_n^{(q)} + 2A_2^{(q)} \left(\varkappa_0 u_n^{(q)} + u_{s,s_0}^{(q)} \right) \right\} \Big/_{S_0} - h^2 \sum_{q=1}^{2} (-1)^q \times \\
&\left\{ -\frac{1}{4} B_4^{(q)} \varkappa_0^2 \sigma_n^{(q)} + \frac{1}{2} A_1^{(q)} \sigma_{n,s_0s_0}^{(q)} + \frac{1}{4} \left(B_2^{(q)} - B_3^{(q)} \right) \varkappa_0 \sigma_{s,s_0}^{(q)} + \left(B_6^{(q)} \varkappa_0^3 \right.\right. \\
&\left. + B_8^{(q)} \varkappa_{0,s_0s_0} \right) u_n^{(q)} + 2B_8^{(q)} \left(\varkappa_{0,s_0} u_{n,s_0}^{(q)} + \varkappa_0 u_{n,s_0s_0}^{(q)} \right) - B_8^{(q)} \left(\varkappa_0 \varkappa_{0,s_0} u_s^{(q)} \right. \\
&\left.\left. - u_{s,s_0s_0s_0}^{(q)} \right) + \left(B_6^{(q)} - B_8^{(q)} \right) \varkappa_0^2 u_{s,s_0}^{(q)} \right\} \Big/_{S_0} - \frac{h^3}{12} \sum_{q=1}^{2} \left\{ \left[6C_1^{(q)} \varkappa_0^3 + \frac{1}{2} \left(D_2^{(q)} \right.\right.\right. \\
&\left.\left. - D_5^{(q)} \right) \varkappa_{0,s_0s_0} \right] \sigma_n^{(q)} + \frac{1}{2} \left(D_3^{(q)} - 3D_5^{(q)} \right) \varkappa_{0,s_0} \sigma_{n,s_0}^{(q)} + 3 \left(C_2^{(q)} + C_4^{(q)} \right) \varkappa_0 \sigma_{n,s_0s_0}^{(q)} \\
&- \frac{1}{2} \left(D_1^{(q)} - D_4^{(q)} \right) \varkappa_0 \varkappa_{0,s_0} \sigma_s^{(q)} + \left(3C_2^{(q)} + C_3^{(q)} \right) \varkappa_0^2 \sigma_{s,s_0}^{(q)} + \left(3C_1^{(q)} + C_5^{(q)} \right) \sigma_{s,s_0s_0s_0}^{(q)} \\
&- 2 \left[3B_6^{(q)} \left(\varkappa_0^4 + (\varkappa_{0,s_0})^2 \right) + B_8^{(q)} \left(3\varkappa_0^4 + (\varkappa_{0,s_0})^2 - 4\varkappa_0 \varkappa_{0,s_0s_0} \right) \right] u_n^{(q)} \\
&+ 4B_8^{(q)} \left(4\varkappa_0 \varkappa_{0,s_0} u_{n,s_0}^{(q)} + 2\varkappa_0^2 u_{n,s_0s_0}^{(q)} - u_{n,s_0s_0s_0s_0}^{(q)} \right) \\
&+ \left[\left(B_8^{(q)} + 3B_6^{(q)} \right) \varkappa_0^2 \varkappa_{0,s_0} + 4B_8^{(q)} \varkappa_{0,s_0s_0s_0} \right] u_s^{(q)} \\
&- \left[\left(5B_8^{(q)} + 3B_6^{(q)} \right) \varkappa_0^3 - \left(11B_8^{(q)} - 3B_6^{(q)} \right) \varkappa_{0,s_0s_0} \right] u_{s,s_0}^{(q)} \\
&\left. + 2 \left(5B_8^{(q)} - 3B_6^{(q)} \right) \varkappa_{0,s_0} u_{s,s_0s_0}^{(q)} + \left(13B_8^{(q)} + 3B_6^{(q)} \right) \varkappa_0 u_{s,s_0s_0s_0}^{(q)} \right\} \Big/_{S_0},
\end{aligned}
\tag{5.12}
$$

$$
\begin{aligned}
&[\sigma_s]_{S_0} = -2h \sum_{q=1}^{2} \left\{ A_1^{(q)} \sigma_{n,s_0}^{(q)} + 2A_2^{(q)} \left(\varkappa_{0,s_0} u_n^{(q)} + \varkappa_0 u_{n,s_0}^{(q)} + u_{s,s_0s_0}^{(q)} \right) \right\} \Big/_{S_0} \\
&- h^2 \sum_{q=1}^{2} (-1)^q \left\{ \frac{1}{4} B_4^{(q)} \varkappa_{0,s_0} \sigma_n^{(q)} + \frac{1}{4} \left(B_2^{(q)} + B_3^{(q)} \right) \varkappa_0 \sigma_{n,s_0}^{(q)} + \frac{1}{2} \left(A_1^{(q)} + 4B_1^{(q)} \right) \sigma_{s,s_0s_0}^{(q)} \right. \\
&+ \left(B_8^{(q)} - 2B_6^{(q)} \right) \varkappa_0 \varkappa_{0,s_0} u_n^{(q)} + \left(B_8^{(q)} - B_6^{(q)} \right) \varkappa_0^2 u_{n,s_0}^{(q)} - B_8^{(q)} u_{n,s_0s_0s_0}^{(q)} \\
&\left. + B_8^{(q)} \varkappa_{0,s_0s_0} u_s^{(q)} + \left(2B_8^{(q)} - B_6^{(q)} \right) \left(\varkappa_{0,s_0} u_{s,s_0}^{(q)} + \varkappa_0 u_{s,s_0s_0}^{(q)} \right) \right\} \Big/_{S_0} \\
&- \frac{h^3}{12} \sum_{q=1}^{2} \left\{ -\frac{1}{2} \left(D_1^{(q)} + D_4^{(q)} \right) \varkappa_0 \varkappa_{0,s_0} \sigma_n^{(q)} + \left(3C_2^{(q)} - C_3^{(q)} \right) \varkappa_0^2 \sigma_{n,s_0}^{(q)} \right. \\
&+ \left(3C_1^{(q)} - C_5^{(q)} \right) \sigma_{n,s_0s_0s_0}^{(q)} - \frac{1}{2} \left[\left(D_2^{(q)} + D_5^{(q)} \right) \varkappa_{0,s_0s_0} \sigma_s^{(q)} + \left(D_3^{(q)} \right.\right. \\
&\left.\left. + 3D_5^{(q)} \right) \varkappa_{0,s_0} \sigma_{s,s_0}^{(q)} \right] - 3 \left(C_2^{(q)} - C_4^{(q)} \right) \varkappa_0 \sigma_{s,s_0s_0}^{(q)} - 2 \left[\left(5B_8^{(q)} + 3B_6^{(q)} \right) \varkappa_{0,s_0s_0s_0} \right. \\
&\left. - 2 \left(4B_8^{(q)} + 3B_6^{(q)} \right) \varkappa_0^2 \varkappa_{0,s_0} \right] u_n^{(q)} + \left(5B_8^{(q)} + 3B_6^{(q)} \right) \left(\varkappa_0^3 - 6\varkappa_{0,s_0s_0} \right) u_{n,s_0}^{(q)} \\
&- \left(29B_8^{(q)} + 15B_6^{(q)} \right) \varkappa_{0,s_0} u_{n,s_0s_0}^{(q)} - \left(13B_8^{(q)} + 3B_6^{(q)} \right) \varkappa_0 u_{n,s_0s_0s_0}^{(q)} \\
&- \left[\left(B_8^{(q)} + 3B_6^{(q)} \right) (\varkappa_{0,s_0})^2 - 3 \left(B_8^{(q)} - B_6^{(q)} \right) \varkappa_0 \varkappa_{0,s_0s_0} \right] u_s^{(q)} \\
&\left. + 8B_8^{(q)} \varkappa_0 \left(2\varkappa_{0,s_0} u_{s,s_0}^{(q)} + \varkappa_0 u_{s,s_0s_0}^{(q)} \right) - 2 \left(5B_8^{(q)} + 3B_6^{(q)} \right) u_{s,s_0s_0s_0s_0}^{(q)} \right\} \Big/_{S_0}.
\end{aligned}
\tag{5.13}
$$

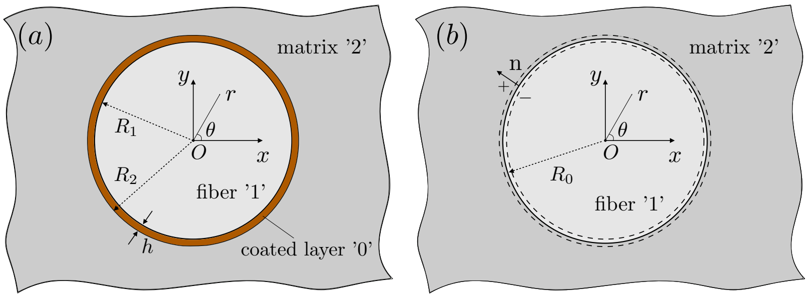

Figure 2: Example problem (a) with perfectly bonded layer, (b) with imperfect interface

The boundary conditions of the first order, that can be extracted from Eqs. (5.10)-(5.13), are identical to those given in [31] for *Model II* (Eqs. (39)-(42), for the case of constant elastic moduli), and to those in [54] (Eqs. (94)-(95) and the plane strain assumption).

6. Example

In this section, *Model I* and *Model II* are tested on the problem of an infinite coated fiber with circular cross-section (Fig. 2a). Simple shear far-field loading ($\sigma_{xx}^{\infty} = -\sigma_{yy}^{\infty} = \sigma_d^{\infty}$, $\sigma_{xy}^{\infty} = 0$) and plane strain settings are imposed and the normal and tangential components of elastic fields are identified by subscripts 'r' and 'θ', respectively. We adopt the following values of governing parameters: $\sigma_d^{\infty}/\mu^{(2)} = 1$, $\mu^{(1)}/\mu^{(2)} = 5$, $\nu^{(p)} = 0.35$ ($p = \{0, 1, 2\}$). Additionally, the layer is assumed to be very thin with $\varepsilon = 0.001$ and very stiff with

$$\frac{\mu^{(0)}}{\mu^{(2)}} = 1.3 \cdot 10^{10} \sim O(\varepsilon^{-3}). \tag{6.1}$$

Condition (6.1) is chosen in order to resolve the issue with stiff layer modeling raised in [30]. We add that this condition identifies the so-called inextensible shell type regime (see, [22]). Its connection to the shell theory was demonstrated in [22]. In [55, 56], it was shown that this regime can simulate the Steigmann–Ogden model of material surface (see [57, 58]), if the residual surface tension involved in the latter model vanishes and the elastic properties and the thickness of the layer are appropriately chosen.

Using appropriately modified (to reflect the problem geometry) boundary conditions of the first, $N = 1$, second, $N = 2$, and third, $N = 3$, orders obtained from (5.1)-(5.4) for *Model I* and from (5.10)-(5.13) for *Model II*, all elastic fields inside the fiber and the matrix are computed. The solutions employed the field representations of [59] that are also used in [30, 60, 61]. Those representations involve unknown coefficients that are found from the linear systems of algebraic equations resulted from the substitution of the representations into prescribed boundary conditions. Similar procedures were implemented in, e.g., [9, 36, 62]. The explicit expressions for the field representations are provided in [62] for two-phase configuration problem (associated with *Model II*). In the current work, we also used those representations to solve the problem associated with *Model I*.

The solutions for the two models are compared with the exact solution of the original problem of Fig. 2a that is available in, e.g., [63, 64]. The normalized jumps in radial and tangential traction components are plotted in Fig. 3 as functions of polar angle $\theta = [0, \pi/2]$.

Both *Model I* and *Model II* of first orders fail to provide accurate approximation for the exact solution. This observation confirms the necessity for using higher order models for accurate simulation of very stiff layers.

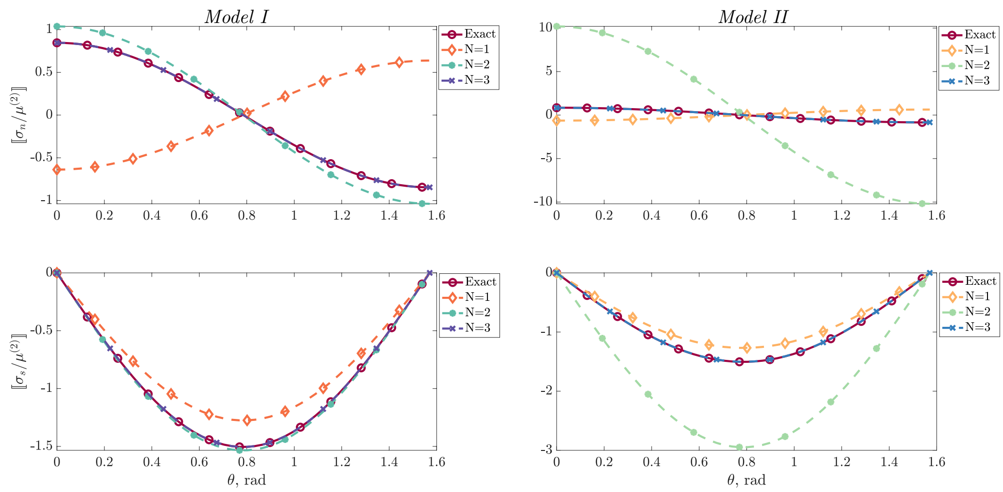

Figure 3: Normalized jumps in traction components computed with *Model I* (left) and *Model II* (right) plotted with respect to θ

Model I of the second order provided visually better results than that of the first order. However, the results obtained by the second order model are still clearly deviate from the exact solutions. The results obtained with *Model II* of the second order are unexpected; they deviate from the exact solution even more than the corresponding results for the first order model. Similar observations were also made by Benveniste in [30].

Model I and *Model II* of third orders provide sufficiently accurate results, thus, demonstrating that such models can accurately simulate the presence of very stiff layers.

Model I and *Model II* were extensively tested, using the same problem of Fig. 2a, for a wide range of governing parameters and results are presented in Supplementary material S2, including parameters that identify the so-called membrane-type interface that is related to the Gurtin-Murdoch model [65, 66], see also [56, 67, 68] and the references therein. Their analysis revealed that *Model I* and *Model II* of all three orders provided accurate results for layers characterized by the parameters associated with the perfect, spring-type, and membrane-type regimes of [22] ($M = -1, 0, 1$). Additionally, for all considered examples, *Model I* led to more accurate results than *Model II*. This could be expected since *Model II* is derived from *Model I* using additional series expansions of the fields involved. However, both third order models provided accurate results for a range of problem parameters that covers all interface regimes of [22]. As also demonstrated in Supplementary material S2, these results can be used to accurately evaluate all elastic fields within the layer.

7. Conclusion

In this paper, we established connections between leading approaches for asymptotic modeling of thin layers in wave phenomena and elastostatics, which should be useful for the researchers working in respective areas. We clarified the misinterpretation of the results obtained with Bövik-Benveniste methodology for the case when all phases of the problem have the same properties and resolved the issue with its implementation for the case of stiff elastic layers. The later became possible by devising a novel modeling approach and deriving the elastostatics models of the third order, which was done here (for the layers of arbitrary sufficiently smooth curvatures) for the first time. Comparative analysis of the results obtained with two models reveals that *Model I* is always more accurate than Model II. However, more rigorous studies of the boundary value problems associated with *Model I* are needed to assure its successful application. From the obtained results (including those presented in Supplementary material S2), it is clear that the use of both third order models allows for accurate evaluation of all governing fields inside the layer. Such data can be used for the numerical evaluation of applicability of various reduced theories for modeling thin layers. The obtained explicit expressions for the boundary conditions contain rich information that can be used for studying the influence of layers curvatures, various asymptotic regimes, etc. Finally we add that it is possible to extend the proposed approach to three-dimensional problems with spherical boundaries by using of either scalar or vectorial spherical harmonics. Such an extension is reported in [69] for conductivity problems. The extension to elasticity problems is work in progress.

Data Accessibility. Electronic supplementary material is available at ...

Authors' Contributions. S.B. and S.M. conceived the project and methodology. S.B. derived the boundary conditions and performed formal analysis and investigation. S.M. supervised the project and performed critical analysis of literature. All authors approved the manuscript to be published.

Competing Interests. The authors declare that they have no competing interests.

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References

1 Abenius E, Edelvik F. 2006 Thin sheet modeling using shell elements in the finite-element time-domain method. *IEEE T Antenn. Prpopag.* **54**, 28–34.
2 Groh R, Tessler A. 2017 Computationally efficient beam elements for accurate stresses in sandwich laminates and laminated composites with delaminations. *Comput. Methods Appl. Mech. Eng.* **320**, 369–395.
3 Tiersten H. 1969 Elastic surface waves guided by thin films. *J. Appl. Phys.* **40**, 770–789.
4 Tiirats T, Chevaugeon N, Moës N, Stolz C, Marouf N, Desdoit E. 2020 How to efficiently apply soft thin coating to existing Finite Element contact model. *Finite Elem. Anal. Des.* **177**, 103420.
5 Kaplunov J, Prikazchikov D, Sultanova L. 2019 Elastic contact of a stiff thin layer and a half-space. *Z. Angew. Math. Phys.* **70**, 1–17.
6 Javili A, Steinmann P, Mosler J. 2017 Micro-to-macro transition accounting for general imperfect interfaces. *Comput. Methods Appl. Mech. Eng.* **317**, 274–317.
7 Martin P. 1992 Boundary integral equations for the scattering of elastic waves by elastic inclusions with thin interface layers. *J. Nondestr. Eval.* **11**, 167–174.
8 Rokhlin S, Wang Y. 1991 Analysis of boundary conditions for elastic wave interaction with an interface between two solids. *J. Acoust. Soc. Am.* **89**, 503–515.
9 Baranova S, Mogilevskaya S, Nguyen T, Schillinger D. 2020 Higher-order imperfect interface modeling via complex variables based asymptotic analysis. *Int. J. Eng. Sci.* **157**, 103399.
10 Klarbring A. 1991 Derivation of a model of adhesively bonded joints by the asymptotic expansion method. *Int. J. Eng. Sci.* **29**, 493–512.
11 Klarbring A, Movchan A. 1998 Asymptotic modelling of adhesive joints. *Mech. Mater.* **28**, 137–145.
12 Mishuris G. 2004 Imperfect transmission conditions for a thin weakly compressible interface. 2D problems. *Arch. Mech.* **56**, 103–115.
13 Schmidt P. 2008 Modelling of adhesively bonded joints by an asymptotic method. *Int. J. Eng. Sci.* **46**, 1291–1324.
14 Lebon F, Rizzoni R. 2011 Asymptotic behavior of a hard thin linear elastic interphase: An energy approach. *Int. J. Solids Struct.* **48**, 441–449.
15 Lebon F, Rizzoni R. 2018 Higher order interfacial effects for elastic waves in one dimensional phononic crystals via the Lagrange-Hamilton's principle. *Eur. J. Mech. A/Solids* **67**, 58–70.
16 Rizzoni R, Lebon F. 2013 Imperfect interfaces as asymptotic models of thin curved elastic adhesive interphases. *Mech. Res. Commun.* **51**, 39–50.

17 Rizzoni R, Dumont S, Lebon F, Sacco E. 2014 Higher order model for soft and hard elastic interfaces. *Int. J. Solids Struct.* **51**, 4137–4148.
18 Serpilli M, Rizzoni R, Lebon F, Dumont S. 2019 An asymptotic derivation of a general imperfect interface law for linear multiphysics composites. *Int. J. Solids Struct.* **180**, 97–107.
19 Serpilli M, Dumont S, Rizzoni R, Lebon F. 2021 Interface Models in Coupled Thermoelasticity. *Technologies* **9**, 17.
20 Dai HH, Kaplunov J, Prikazchikov D. 2010 A long-wave model for the surface elastic wave in a coated half-space. *Proc. R. Soc. Lond. A* **466**, 3097–3116.
21 Kaplunov J, Prikazchikov D, Sultanova L. 2019 On higher order effective boundary conditions for a coated elastic half-space. In *Problems of nonlinear mechanics and physics of materials* Adv. Struct. Mater. pp. 449–462. Springer, Cham.
22 Benveniste Y, Miloh T. 2001 Imperfect soft and stiff interfaces in two-dimensional elasticity. *Mech. Mater.* **33**, 309–323.
23 Serpilli M, Rizzoni R, Dumont S, Lebon F. 2022 Higher order interface conditions for piezoelectric spherical hollow composites: Asymptotic approach and transfer matrix homogenization method. *Compos. Struct.* **279**, 114760.
24 Hashin Z. 2001 Thin interphase/imperfect interface in conduction. *J. Appl. Phys.* **89**, 2261–2267.
25 Hashin Z. 2002 Thin interphase/imperfect interface in elasticity with application to coated fiber composites. *J. Mech. Phys. Solids* **50**, 2509–2537.
26 Huang W, Rokhlin S. 1992 Interface waves along an anisotropic imperfect interface between anisotropic solids. *J. Nondestr. Eval.* **11**, 185–198.
27 Huang W, Rokhlin S, Wang Y. 1997 Analysis of different boundary condition models for study of wave scattering from fiber–matrix interphases. *J. Acoust. Soc. Am.* **101**, 2031–2042.
28 Niklasson AJ, Datta SK, Dunn ML. 2000a On ultrasonic guided waves in a thin anisotropic layer lying between two isotropic layers. *J. Acoust. Soc. Am.* **108**, 2005–2011.
29 Niklasson AJ, Datta SK, Dunn ML. 2000b On approximating guided waves in plates with thin anisotropic coatings by means of effective boundary conditions. *J. Acoust. Soc. Am.* **108**, 924–933.
30 Benveniste Y, Berdichevsky O. 2010 On two models of arbitrarily curved three-dimensional thin interphases in elasticity. *Int. J. Solids Struct.* **47**, 1899–1915.
31 Benveniste Y. 2013a Models of thin interphases with variable moduli in plane-strain elasticity. *Math. Mech. Solids* **18**, 119–134.
32 Benveniste Y. 2013b Models of thin interphases and the effective medium approximation in composite media with curvilinearly anisotropic coated inclusions. *Int. J. Eng. Sci.* **72**, 140–154.
33 Bövik P, Olsson P. 1992 Effective boundary conditions for the scattering of two-dimensional SH waves from a curved thin elastic layer. *Proc. R. Soc. Lond. A* **439**, 257–269.
34 Bövik P. 1994 On the modelling of thin interface layers in elastic and acoustic scattering problems. *Q. J. Mech. Appl. Math.* **47**, 17–42.
35 Benveniste Y. 2006a A general interface model for a three-dimensional curved thin anisotropic interphase between two anisotropic media. *J. Mech. Phys. Solids* **54**, 708–734.
36 Benveniste Y. 2006b An $O(h^N)$ interface model of a three-dimensional curved interphase in conduction phenomena. *Proc. R. Soc. Lond. A* **462**, 1593–1617.
37 Venttsel' AD. 1959 On boundary conditions for multidimensional diffusion processes. *Theory Probab. its Appl.* **4**, 164–177.
38 Goldstein GR. 2006 Derivation and physical interpretation of general boundary conditions. *Adv. Differ. Equ.* **11**, 457–480.
39 Nicaise S, Li H, Mazzucato A. 2017 Regularity and a priori error analysis of a Ventcel problem in polyhedral domains. *Math. Methods Appl. Sci* **40**, 1625–1636.
40 Bonnaillie-Noël V, Dambrine M, Hérau F, Vial G. 2010 On generalized Ventcel's type boundary conditions for Laplace operator in a bounded domain. *SIAM J. Math. Anal.* **42**, 931–945.
41 Zhu QZ, Gu ST, Yvonnet J, Shao JF, He QC. 2011 Three-dimensional numerical modelling by XFEM of spring-layer imperfect curved interfaces with applications to linearly elastic composite materials. *Int. J. Numer. Methods Eng.* **88**, 307–328.

42 Gu S, Monteiro E, He QC. 2011 Coordinate-free derivation and weak formulation of a general imperfect interface model for thermal conduction in composites. *Compos. Sci. Technol.* **71**, 1209–1216.
43 Xu Y, He QC, Gu ST. 2016 Effective elastic moduli of fiber-reinforced composites with interfacial displacement and stress jumps. *Int. J. Solids Struct.* **80**, 146–157.
44 Kushch VI, Mogilevskaya SG. 2021 Anisotropic imperfect interface in elastic particulate composite with initial stress. *Math. Mech. Solids*.
45 Kushch VI. 2021a Elastic equilibrium of spherical particle composites with transversely isotropic interphase and incoherent material interface. *Int. J. Solids Struct.* **232**, 111180.
46 Kushch VI. 2021b Representative unit cell model of elastic spherical particle composite with interphase and/or general imperfect interface. *Mech. Mater.* **158**, 103869.
47 Sussmann C, Givoli D, Benveniste Y. 2011 Combined asymptotic finite-element modeling of thin layers for scalar elliptic problems. *Comput. Methods Appl. Mech. Eng.* **200**, 3255–3269.
48 Han Z, Mogilevskaya SG, Baranova S, Schillinger D. 2021 BEM-based second-order imperfect interface modeling of potential problems with thin layers. *Int. J. Solids Struct.* **230**, 111155.
49 Benvenuti E, Ventura G, Ponara N, Tralli A. 2013 Variationally consistent eXtended FE model for 3D planar and curved imperfect interfaces. *Comput. Methods Appl. Mech. Eng.* **267**, 434–457.
50 Gu ST, Liu JT, He QC. 2014 The strong and weak forms of a general imperfect interface model for linear coupled multifield phenomena. *Int. J. Eng. Sci.* **85**, 31–46.
51 Liu JT, He B, Gu ST, He QC. 2018 Implementation of a physics-based general elastic imperfect interface model in the XFEM and LSM context. *Int. J. Numer. Methods Eng.* **115**, 1499–1525.
52 Muskhelishvili NI. 1968 *Singular integral equations: boundary problems of function theory and their application to mathematical physics, (in Russian)*. Nauka, Moscow 3 edition.
53 Muskhelishvili N. 1963 *Some basic problems of the mathematical theory of elasticity*. Noordhoff, Groningen.
54 Gu S, He QC. 2011 Interfacial discontinuity relations for coupled multifield phenomena and their application to the modeling of thin interphases as imperfect interfaces. *J. Mech. Phys. Solids* **59**, 1413–1426.
55 Zemlyanova AY, Mogilevskaya SG. 2018 Circular inhomogeneity with Steigmann–Ogden interface: Local fields, neutrality, and Maxwell's type approximation formula. *Int. J. Solids Struct.* **135**, 85–98.
56 Mogilevskaya SG, Zemlyanova AY, Kushch VI. 2021 Fiber-and particle-reinforced composite materials with the Gurtin–Murdoch and Steigmann–Ogden surface energy endowed interfaces. *Appl. Mech. Rev.* **73**.
57 Steigmann D, Ogden R. 1997 Plane deformations of elastic solids with intrinsic boundary elasticity. *Proc. R. Soc. A: Math. Phys. Eng. Sci.* **453**, 853–877.
58 Steigmann D, Ogden R. 1999 Elastic surface—substrate interactions. *Proc. R. Soc. A: Math. Phys. Eng. Sci.* **455**, 437–474.
59 Christensen R, Lo K. 1979 Solutions for effective shear properties in three phase sphere and cylinder models. *J. Mech. Phys. Solids* **27**, 315–330.
60 Rubin M, Benveniste Y. 2004 A Cosserat shell model for interphases in elastic media. *J. Mech. Phys. Solids* **52**, 1023–1052.
61 Mogilevskaya SG, Kushch VI, Zemlyanova AY. 2019 Displacements representations for the problems with spherical and circular material surfaces. *Q. J. Mech. Appl. Math.* **72**, 449–471.
62 Baranova S, Mogilevskaya SG. 2022 Case study of the Bövik-Benveniste methodology for imperfect interface modeling of two-dimensional elasticity problems with thin layers. *J. Mech. Mater. Struct.* In press.
63 Mogilevskaya SG, Crouch SL. 2004 A Galerkin boundary integral method for multiple circular elastic inclusions with uniform interphase layers. *Int. J. Solids Struct.* **41**, 1285–1311.
64 Mogilevskaya SG, Zemlyanova AY, Zammarchi M. 2018 On the elastic far-field response of a two-dimensional coated circular inhomogeneity: Analysis and applications. *Int. J. Solids Struct.* **130**, 199–210.
65 Gurtin ME, Murdoch AI. 1975 A continuum theory of elastic material surfaces. *Arch Ration Mech Anal* **57**, 291–323.

66 Gurtin ME, Murdoch AI. 1978 Surface stress in solids. *Int J Solids Struct* **14**, 431–440.
67 Eremeyev VA, Rosi G, Naili S. 2020 Transverse surface waves on a cylindrical surface with coating. *Int. J. Eng. Sci.* **147**, 103188.
68 Gorbushin N, Eremeyev VA, Mishuris G. 2020 On stress singularity near the tip of a crack with surface stresses. *Int. J. Eng. Sci.* **146**, 103183.
69 Kushch VI, Mogilevskaya SG. 2022 Higher order imperfect interface models of conductive spherical interphase. *Math Mech Solids*. Under review.