Chiral Central Charge from a Single Bulk Wave Function

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(Received 28 October 2021; revised 8 February 2022; accepted 5 April 2022; published 28 April 2022)

A $(2+1)$-dimensional gapped quantum many-body system can have a topologically protected energy current at its edge. The magnitude of this current is determined entirely by the temperature and the chiral central charge, a quantity associated with the effective field theory of the edge. We derive a formula for the chiral central charge that, akin to the topological entanglement entropy, is completely determined by the many-body ground state wave function in the bulk. According to our formula, nonzero chiral central charge gives rise to a topological obstruction that prevents the ground state wave function from being real valued in any local product basis.

DOI: 10.1103/PhysRevLett.128.176402

Topological order [1,2] is an active topic of investigation today, simultaneously admitting connections to deep mathematics and the promise of intrinsically protected quantum devices. Well-known properties of topologically ordered systems in $(2+1)$ dimensions are anyonic excitations in the bulk [3–5] and robust gapless modes on the edge of the system [6–8], both observed in fractional quantum Hall systems experimentally [9–11].

A remarkable observation is that the features characterizing the low-energy excitations of a system can be extracted from its many-body ground-state wave function. Early examples of this observation include topological entanglement entropy [12,13] and the entanglement spectrum [14]. Later work has gone on to extract other related properties, e.g., topological S and T matrices of the anyonic excitations [15,16].

These properties, however, do not exhaustively constitute the data that define the phase of the underlying system. A missing piece of information is the *chiral central charge* (CCC), denoted as c_- here. A nonzero c_- implies the presence of gapless edge excitations, so a natural place in which the CCC appears is the physical edge of the system. At a temperature T that is low compared with the bulk excitation gap, the energy current I along the edge is [8,17–20]

$$I = \frac{\pi}{12} c_- T^2. \tag{1}$$

A prominent physical system with nonzero CCC is the two-dimensional electron gas in a magnetic field [6,10,11]. Besides admitting the well-known quantized electrical response protected by U(1) charge conservation symmetry, this system admits a quantized *thermal* response [8] as well, which manifests as a unidirectional edge energy current. This current corresponds to a nonzero c_- and exists independently of the U(1) symmetry [17–19]. More generally, the chiral central charge is an integer when the edge admits a chiral Luttinger liquid description [7], but can be a rational number in, for instance, $p+ip$ superconductors and systems with non-Abelian anyons [18,21].

While the CCC describes a physical property of the edge, it is also related to certain properties of the bulk. For instance, the energy current at the edge can be related to a 2-current in the bulk, which can be computed from a microscopic Hamiltonian [20,22]. In effective field theory approaches, CCC appears in the gravitational Chern-Simons term of the bulk action, which is responsible for the framing anomaly of the underlying system [23]. This lets us relate CCC to the topological Berry phase under adiabatic variation of the metric [24] and the Hall viscosity on a sphere [24–27]. Moreover, given a set of ground-state wave functions on a torus, CCC can be computed (up to a fixed integer) using topological S and T matrices [15] or momentum polarization [28]. Alternatively, CCC can be inferred from the entanglement spectrum of the bulk reduced density matrix on a disk [14]. However, a succinct closed-form formula that relates CCC to the ground state entanglement of a single wave function—akin to the topological entanglement entropy [12,13]—has been missing.

In this Letter, we introduce a new formula that reveals a connection between CCC and the entanglement structure

of the bulk. Our formula is based on the modular commutator—a new quantity expressed in terms of reduced density matrices of a many-body wave function. We argue that the CCC can be expressed in terms of the modular commutator for ground states of gapped Hamiltonians, with or without symmetries. We numerically confirm our formula up to a small error attributable to finite-size effects in our companion paper [29].

Summary of results.—For a general tripartite state ρ_{ABC} on a finite dimensional Hilbert space, the *modular commutator* is

$$J(A,B,C)_\rho := i\mathrm{Tr}(\rho_{ABC}[K_{AB},K_{BC}]), \tag{2}$$

where $K_A = -\ln\rho_A$ is the modular Hamiltonian [30,31] associated with the reduced density matrix ρ_A on subsystem A. We can readily see that J is real (because $[K_{AB},K_{BC}]$ is an anti-Hermitian operator) and odd under complex conjugation (with respect to any product-state basis over the local degrees of freedom). The latter operation corresponds to time reversal in our physical context, meaning that it can flip the direction of a system's edge current. Thus, the fact that J is odd under time reversal is a necessary property for it to encode information about the CCC.

Plugging in a many-body ground state $\sigma = |\psi\rangle\langle\psi|$ satisfying the area law with a constant subcorrection term [12,13], we relate the modular commutator [Eq. (2)] to the edge energy current [Eq. (1)], obtaining our main result

$$J(A,B,C)_\sigma = \frac{\pi}{3}c_- \tag{3}$$

for subsystems A, B, and C depicted in Fig. 1. Equation (3) is insensitive to continuous deformations of the subsystems, so long as they remain to partition a disk.

In the rest of this Letter, we derive Eq. (3) using theoretical tools developed in Refs. [32–34]. Using the fact that σ satisfies the area law, we show that the state's modular Hamiltonian $K_{\mathfrak{D}}$ for a disk $\mathfrak{D}$ can be decomposed into a sum of local operators. From this decomposition, we can obtain an expression for the "energy current." The value of the "energy current" is determined by the expectation values of commutators of the local terms of $K_{\mathfrak{D}}$, and each nonzero contribution is shown to be of the form of the modular commutator [Eq. (2)], up to a proportionality constant. By viewing $K_{\mathfrak{D}}$ as a physical Hamiltonian describing the same phase as that of σ, we relate $K_{\mathfrak{D}}$'s "energy current" to the physical current from Eq. (1), yielding Eq. (3).

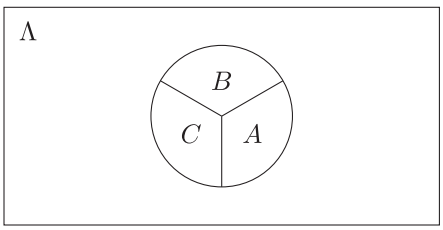

FIG. 1. Partition of a disk-shaped region ABC in the bulk (Λ). Each subsystem is assumed to be sufficiently large compared with the correlation length.

Markov-chain states.—An important fact about the modular commutator is that it vanishes if the underlying state is a quantum Markov chain [35]. Specifically, consider a tripartite state ρ_{XYZ}. This state is a quantum Markov chain if its conditional mutual information—defined as $I(X:Z|Y)_\rho := S(\rho_{XY}) + S(\rho_{YZ}) - S(\rho_Y) - S(\rho_{XYZ})$, where $S(\rho) := -\mathrm{Tr}(\rho\ln\rho)$ is the von Neumann entropy of ρ—is zero. It turns out that

$$I(X:Z|Y)_\rho = 0 \Rightarrow J(X,Y,Z)_\rho = 0, \tag{4}$$

which can be proved via the important relation [35] [This is true for positive definite ρ_{XYZ}. If ρ_{XYZ} has zero eigenvalues, Eq. (5) should be replaced by $K_{XYZ}\rho_{XYZ} = (K_{XY} + K_{YZ} - K_Y)\rho_{XYZ}$, meaning that the same condition holds on the subspace spanned by the eigenstates of the nonzero eigenvalues of ρ_{XYZ}; see the Supplemental Material [36] for details, in particular, on the derivation of Eq. (5) for fermions based on Refs. [35,37–40]]:

$$I(X:Z|Y)_\rho = 0 \Leftrightarrow K_{XYZ} = K_{XY} + K_{YZ} - K_Y. \tag{5}$$

Applying this relation, the modular commutator becomes $J(X,Y,Z) = i\mathrm{Tr}(\rho_{XYZ}[K_{XY}, K_{XYZ} + K_Y])$. The cyclicity of the trace then implies that this expression is zero.

Area law and modular commutator.—Now we shift our focus to ground states of gapped quantum many-body systems in two spatial dimensions. Such states, which we denote as σ, are expected to obey the area law of entanglement entropy [12,13], which means that the following equation holds for any disk-shaped region A:

$$S(\sigma_A) = \alpha|\partial A| - \gamma + \cdots, \tag{6}$$

where $|\partial A|$ is the length of the perimeter of A, α is a nonuniversal constant, and γ is the topological entanglement entropy [12,13]. The remaining $(\ldots)$ term vanishes in the $|A| \to \infty$ limit. While the rigorous proof of the area law [Eq. (6)] is unknown, proofs of analogous statements in one-dimensional gapped systems [41] and two-dimensional locally gapped systems [42] are known.

The area law implies that the modular commutator $J(A,B,C)$ for the partition of the disk in Fig. 1 is invariant under topology-preserving deformation of the regions. Specifically, consider the deformations of subsystems A, B, and C as described in Fig. 2. Deformations away from B leaves the modular commutator invariant for the following reason. Without loss of generality, consider $a \subset \Lambda\backslash(ABC)$ as is shown in the first row of Fig. 2. From $I(a:B|A) = 0$ and Eq. (5), we can obtain $K_{aAB} = K_{AB} + K_{aA} - K_A$;

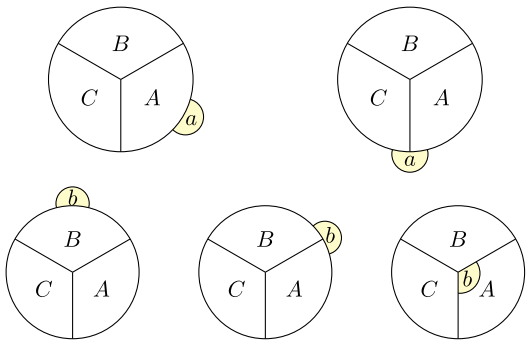

FIG. 2. First row: deformation of A to Aa, where a is a small disk separated from B. Second row: deformations of B to Bb; (for the rightmost figure, $b \subset A$). All cases in the second row can be reduced to the cases in the first row by a change of subsystem.

here the vanishing of conditional mutual information $I(a:B|A) = 0$ follows from Eq. (6). [For our purpose, it will suffice for the size of the region a to be large compared with the correlation length. However, note that $I(a:B|A) = 0$ holds for a of smaller sizes as well, provided that the conditioned region (A) is large. This is a consequence of strong subadditivity, $I(a:B|A) \leq I(aa':B|A)$.] Therefore, $J(Aa, B, C) = J(A, B, C)$. By repeating the same argument, one can freely deform the part of the edges and the triple intersection point that is separated from B.

The deformations of B also leave the modular commutator invariant. To establish this fact, it suffices to consider the deformations depicted in the second row of Fig. 2 and show that $J(A, B, C) = J(A\backslash b, Bb, C)$. While the argument in the previous paragraph does not apply directly, it does apply after we switch the subsystems as follows. Let $\Lambda = ABCD$ and suppose $\sigma = |\psi\rangle\langle\psi|$ is pure. Then it follows from $K_{BC}|\psi\rangle = K_{AD}|\psi\rangle$ that $J(A, B, C) = J(B, A, D)$; similarly, we can put either C or D in the middle entry and have $J(A, B, C) = J(\cdot, C, \cdot) = J(\cdot, D, \cdot)$ with appropriate choices of the first and third entries. Thus, we can always make the middle entry to be away from the place that the deformation occurs. The argument in the previous paragraph now applies. Therefore, given subsystems A, B, and C which are topologically equivalent to Fig. 1, $J(A, B, C)$ is invariant under any smooth deformation of A, B, or C.

Locality of modular Hamiltonian.—We now discuss the "local" structure of the modular Hamiltonian $K_{\mathfrak{D}}$ of a many-body area-law state σ on a disk-shaped region $\mathfrak{D}$. The area law implies that $I(X:Z|Y)_\sigma = 0$ for any "chainlike" region XYZ, i.e., for which X, Y, and Z are simply connected, and X and Z do not share a boundary. Partitioning the disk $\mathfrak{D} = XYZ$ into such a region, Eq. (5) allows us to express the modular Hamiltonian $K_{\mathfrak{D}}$ using terms of smaller support. This process can

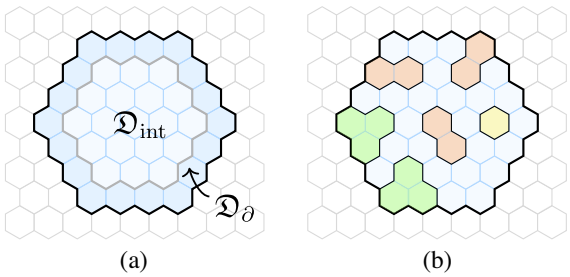

FIG. 3. (a) A disk $\mathfrak{D}$ and its partition into $\mathfrak{D}_{\text{int}}$ and $\mathfrak{D}_\partial$. (b) A disk $\mathfrak{D}$ (blue) and the one-site terms K_v (yellow), two-site terms K_e (orange), and three-site terms K_f (green) in the local decomposition [Eq. (7)] of modular Hamiltonian $K_{\mathfrak{D}}$.

then be applied recursively to the resulting terms $\{K_{XY}, K_{YZ}, K_Y\}$, yielding an evermore local decomposition for $K_{\mathfrak{D}}$.

To take advantage of this process, let us coarse grain the subsystem into a triangular lattice of supersites; see Fig. 3(a). This lattice has *vertices* ⬡ (denoted by hexagonal unit cells at each site in the figure), edges ⬡⬡, as well as three-site combinations such as faces ⬡⬡⬡, ⬡⬡⬡ and chainlike regions ⬡⬡⬡⬡, ⬡⬡⬡⬡, … The above procedure eliminates all terms supported on chainlike regions, leaving the faces $f \in F(\mathfrak{D})$ as the only source of three-site supports for the decomposition. The decomposition also admits single-site terms, which, due to the nature of Eq. (5), only occupy vertices v on the interior $\mathfrak{D}_{\text{int}}$, i.e., the set of all vertices that have no neighbors outside of $\mathfrak{D}$. Similarly, the remaining two-site terms are supported on any edges $e \in E(\mathfrak{D})$ that do not lie exclusively on the boundary, i.e., the set $E(\mathfrak{D})/E(\mathfrak{D}_\partial)$. These terms make up the decomposition

$$K_{\mathfrak{D}} = \sum_{f\in F(\mathfrak{D})} K_f - \sum_{e\in E(\mathfrak{D})/E(\mathfrak{D}_\partial)} K_e + \sum_{v\in\mathfrak{D}_{\text{int}}} K_v, \quad (7)$$

which is local with respect to our triangulation; examples of each term are depicted in Fig. 3(b).

Invariance of $J(A, B, C)$ under smooth deformations implies that commutators of the above terms are 0, $\pm J$, where we have set

$$J := J(u, v, w) \quad \text{on a face of the form} \quad \begin{matrix} & v & \\ w & & u \end{matrix}. \quad (8)$$

This reference value of the modular commutator either stays the same or changes sign if we pick another face or rearrange the order of the sites.

Modular current.—Because $K_{\mathfrak{D}}$ is local, we can define its "energy current," which we refer to as the modular current. We rewrite the modular Hamiltonian [Eq. (7)] in terms of $\tilde{K}_v^{\mathfrak{D}}$, which collects all terms whose support contains site v, multiplied by the appropriate fraction so as to satisfy $K_{\mathfrak{D}} = \sum_{v\in\mathfrak{D}} \tilde{K}_v^{\mathfrak{D}}$:

$$\tilde{K}_v^{\mathfrak{D}} = \begin{cases} \frac{1}{3}\sum_{f:v\in f} K_f - \frac{1}{2}\sum_{e:v\in e} K_e + K_v, & v \in \mathfrak{D}_{\mathrm{int}} \\ \frac{1}{3}\sum_{\substack{f:v\in f,\\ f\in F(\mathfrak{D})}} K_f - \frac{1}{2}\sum_{\substack{e:v\in e,\\ e\in E(\mathfrak{D})\setminus E(\mathfrak{D}_\partial)}} K_e, & v \in \mathfrak{D}_\partial \end{cases}. \tag{9}$$

The modular current from site u to v is then simply

$$f_{uv}^{\mathfrak{D}} := i\langle[\tilde{K}_u^{\mathfrak{D}}, \tilde{K}_v^{\mathfrak{D}}]\rangle, \tag{10}$$

quantifying the noncommutativity of the modular Hamiltonian terms.

The modular current has properties analogous to that of the energy current of some local Hamiltonian at finite temperature [20]. For instance, $f_{uv}^{\mathfrak{D}}$ vanishes for any pair of points $u, v \in \mathfrak{D}$ which are sufficiently far apart. Moreover, the current is conserved:

$$\sum_{v\in\mathfrak{D}} f_{uv}^{\mathfrak{D}} = 0. \tag{11}$$

Lastly, one can explicitly show that the bulk modular current vanishes: $f_{uv}^{\mathfrak{D}} = 0$ for $u, v \in \mathfrak{D}_{\mathrm{int}}$; see the Supplemental Material [36] for details.

Since the modular current vanishes in the bulk of the disk, nontrivial current flows only along the edge. We define the *edge modular current* I_σ to be a sum of $f_{uv}^{\mathfrak{D}}$ over $u \in L$ and $v \in R$, where $L, R \subset \mathfrak{D}$ lie on opposite sides of a cut perpendicular to the boundary. This current is insensitive to the choice of regions L, R, as long as they are sufficiently large. Since there is no current deep enough in the bulk, nonzero contributions only come from sites that are at most two sites away from the edge. Moreover, because the current flows between nearby sites and is conserved, it suffices to consider only a few sites along the edge. For the cut depicted in Fig. 4, nonzero contributions only come from $\{a, b, c\} \subset L$ and $\{x, y, z, w\} \subset R$, yielding

$$I_\sigma = \sum_{u\in L}\sum_{v\in R} f_{uv}^{\mathfrak{D}} = \frac{1}{4}J. \tag{12}$$

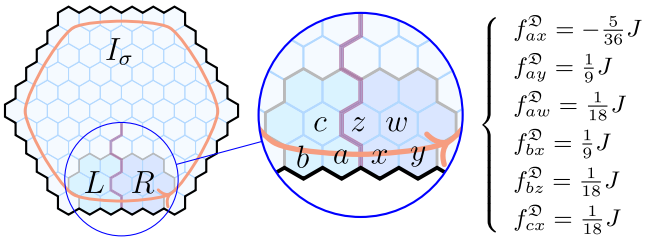

FIG. 4. The calculation of the edge modular current I_σ [Eq. (12)]. The computation result of all the nonvanishing contribution from modular currents passing through a specific cut is summarized (see the Supplemental Material [36] for details).

The choice of regions depends on the decomposition of $K_{\mathfrak{D}}$, and a coarser decomposition requires even smaller regions.

Chiral central charge.—We now relate I_σ to c_- by viewing $K_{\mathfrak{D}}$ as a physical Hamiltonian describing the same phase as that of the state σ. First, $\sigma_{\mathfrak{D}} = e^{-K_{\mathfrak{D}}}$ can be viewed as a thermal state of the local Hamiltonian $\sum_{v\in\mathfrak{D}} \tilde{K}_v^{\mathfrak{D}} = K_{\mathfrak{D}}$ with the temperature $T = 1$. Second, $e^{-\sum_{v\in\mathfrak{D}} \tilde{K}_v^{\mathfrak{D}}} = \sigma_{\mathfrak{D}}$ obeys an entanglement area law in $\mathfrak{D}$, and is indistinguishable from the ground state σ over the same region, an indication of *low* temperature for a system with a bulk energy gap.(Let us remark on a subtlety. Ground states of realistic gapped many-body systems would satisfy $e^{-\sum_{v\in\mathfrak{D}} \tilde{K}_v^{\mathfrak{D}}} = \sigma_{\mathfrak{D}}$ only approximately. In order for our argument on low temperature to work, the size of the hexagon-shaped subsystems partitioning $\mathfrak{D}$ must be sufficiently large compared to the correlation length, yet smaller than the radius of $\mathfrak{D}$. Furthermore, in order to make the error in the local decomposition sufficiently small, the size of the subsystems we use must implicitly depend on the size of $\mathfrak{D}$. Understanding this dependence is left for future study.) Third, the temperature is *high enough* so that the edge correlation length is small compared with the length of the disk. (In other words, the temperature is low as far as the bulk is concerned, and the temperature is high from the perspective of the edge.) This is the right temperature range for which we can apply Eq. (1); see, e.g., Ref. [20], Appendix D.2.

Now, we can make a nontrivial but a reasonable physical assumption; if two local Hamiltonians with a bulk energy gap have the same bulk reduced density matrix, then their energy currents at the boundary at a temperature that is low compared with the bulk gap both obey Eq. (1), independent of other microscopic details. This expression is determined completely by the temperature and the CCC of the edge theory [20] and cannot change unless the bulk undergoes a quantum phase transition. As is observed in the previous paragraph, the "temperature" $T = 1$ lies in the right temperature range. Using Eq. (1) and identifying the energy current to the edge modular current [Eq. (12)], we arrive at our main result:

$$J = \frac{\pi}{3}c_-. \tag{13}$$

Discussion.—In this Letter, we propose a new formula for the chiral central charge (CCC). The formula is based on the modular commutator J, a new quantity that is expressed in terms of reduced density matrices and that is odd under time reversal (i.e., complex conjugation with respect to a local basis). Compared with other similar works in this direction [15,28,43], the main advantage of our formula is that it can be directly computed from a single ground-state wave function. Together with the prior work that established a connection between ground-state entanglement and

the anyonic data [12–15,32,44], our result strengthens the idea that *all* the universal properties of $(2+1)$-dimensional gapped quantum phases may be encoded in a single ground state. Our formula is thus a useful addition to the existing toolkit for diagnosing topological properties of quantum many-body systems.

Aside from drawing a deep connection between ground state entanglement and the CCC, our result [Eq. (13)] also has an intriguing implication for the *intrinsic sign problem* [45]. Specifically, if our formula for the CCC is correct for all gapped systems in $(2+1)$ dimensions, then a system with nonzero CCC cannot have a parent Hamiltonian which is sign-problem free. This is because a sign-problem free Hamiltonian necessarily admits a ground state with nonnegative coefficients in a local product basis, for which the CCC would be zero because J is odd under complex conjugation. Thus, a rigorous proof of our formula will lead to a microscopic proof that such systems have an intrinsic sign problem; no local Hamiltonian free of the sign problem can have ground states with nonzero c_-. This corroborates recent arguments pointing to a similar conclusion [46–48].

A curious fact is that the modular commutator did not possess any ultraviolet-divergent contributions in our calculation. Whether this is a general property of the modular commutator or something specific to gapped phases in $(2+1)$ dimensions is unclear. Perhaps field-theoretic calculations of our result can shed light on this problem.

For future work, it will be interesting to find analogs of J that are suitable for resolving symmetry-protected topological invariants that can be obtained from the ground-state wave function, e.g., Refs. [49–59].

The authors thank Nikita Sopenko, Michael Gullans, Tarun Grover, and Maissam Barkeshli for valuable input and inspiring discussions. I. K. was supported by the Australian Research Council via the Centre of Excellence in Engineered Quantum Systems (EQUS) Project No. CE170100009. B. S. is supported by University of California Laboratory Fees Research Program, Grant No. LFR-20-653926, and the Simons Collaboration on Ultra-Quantum Matter, Grant No. 651440 from the Simons Foundation. K. K. is supported by MEXT Quantum LeapFlagship Program (MEXT Q-LEAP) Grant No. JP-MXS0120319794. Contributions to this work by NIST, an agency of the US government, are not subject to US copyright. Any mention of commercial products does not indicate endorsement by NIST.

Note added.—After completion of our work, we became aware of the independent work of Liu *et al.* [60] and Siva *et al.* [61] which considered a different entanglement measure to extract properties of the ungappable edge degrees of freedom.

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