EPSILON DICHOTOMY FOR LINEAR MODELS

HANG XUE

Contents

1. Introduction

The goal of this paper is to relate linear models on central simple algebras to local root numbers.

1.1. **Main results.** Let E/F be a quadratic extension of local nonarchimedean fields of characteristic zero and $\eta : F^\times/NE^\times \to \{\pm 1\}$ the quadratic character associated to this extension. We fix a nontrivial additive character $\psi : F \to \mathbb{C}^\times$. Let A be a central simple algebra (CSA) over F of dimension $4n^2$ with a fixed embedding $E \to A$ and let B be the centralizer of E in A. Then B is a CSA over E of dimension n^2. Let $G = A^\times$ and $H = B^\times$, both viewed as algebraic groups over F. Let π be an irreducible admissible representation of G. We say that π is H-distinguished if

$$\mathrm{Hom}_H(\pi, \mathbb{C}) \neq 0.$$

Let $G' = \mathrm{GL}_{2n}(F)$ and π' be the Jacquet–Langlands transfer of π to G'. Let $H' = \mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$, embedded in G' as the centralizer of $\mathrm{diag}[1, -1, \cdots, 1, -1]$. We say that π' is H'-distinguished if

$$\mathrm{Hom}_{H'}(\pi', \mathbb{C}) \neq 0.$$

These Hom spaces are all at most one dimensional [AG09, JR96, BM]. Let $\epsilon(\pi') = \epsilon(\pi', \psi)$ be the local root number. It equals ± 1 and is independent of the choice of ψ as π' is self-dual and symplectic.

Date: August 24th 2020.

Theorem 1.1. *Let the notation be as above. If π is H-distinguished then the following two conditions hold.*

(1) *The Langlands parameter of π' takes value in $\mathrm{Sp}(2n, \mathbb{C})$. If π' is generic, this is equivalent to π' being H'-distinguished.*
(2) *$\epsilon(\pi')\epsilon(\pi' \otimes \eta)\eta(-1)^n = (-1)^r$. Here r is the integer so that $A = M_r(C)$ with C a central division algebra, or in other words, r is the split rank of G.*

Conversely, if π' satisfies conditions (1) and (2) above, and assume that either (a) π' is supercuspidal or (b) π is supercuspidal and $G = \mathrm{GL}_n(D)$ where D is a quaternion algebra over F (split or not), then π is H-distinguished.

This is the combination of Theorem 4.1, Theorem 7.3 and Proposition 7.4. When $n = 1$, this recovers the theorem of Saito and Tunnell. In general it confirms a conjecture of Prasad and Takloo-Bighash [PTB11, Conjecture 1] in many cases. The conjecture of Prasad and Takloo-Bighash assumes further that π' is generic, but it is not necessary according to the result of Suzuki [Suz]. We will extend the converse implication to the case π being a discrete series representation in a subsequent paper, at least when $G = \mathrm{GL}_n(D)$. It requires some techniques of a different nature. We however should note that the converse implication, as stated in the theorem, is not expected to hold if π is not a discrete series representation.

The motivation of this paper comes from the conjecture of Sakellaridis and Venkatesh on the canonical factorization of linear periods. Before we attack the global factorization problem, there are many local issues that need to be addressed. This paper deals with the first main local issue: characterizing the existence of linear models using local root numbers.

Our argument is based on relative trace formulae proposed by Guo [Guo96]. It exploits a novel idea of making use an involution on the space of test functions which we would like to call it the involution method. We will outline the argument below. There is a different approach to this type of problems, first used by Waldspurger to prove the local Gross–Prasad conjecture for orthogonal groups. It makes use of the (usual) local trace formula technique and the theory of twisted endoscopy. This seems applicable to our problem at hand, but our approach is much simpler and conceptual. For one implication, one can also prove it using Prasad's global-to-local argument. See Remark 4.2. It however does not seem to give any results in the other direction.

Remark 1.2. In a previous version of this paper that has been circulated for a while, the theorem is proved under the working hypothesis of the full fundamental lemma and Howe's finiteness conjecture for symmetric spaces. It turns out that in the present situation, both can be bypassed. So now the result is unconditional. Also certain germ expansions of the orbital integrals and spherical characters were developed and played a definitive role in the argument in that version. We avoids them in the present version and hence the argument is much simpler.

1.2. **Outline of the argument.** We very briefly outline the argument using the involution method. We will assume π and π' are supercuspidal representations in this subsection for simplicity.

Assume that π is H-distinguished and we fix a nonzero element $l \in \mathrm{Hom}_H(\pi, \mathbb{C})$. Then one attaches to π a distribution on G by

$$J_\pi(f) = \sum_v l(\pi(f)v)\overline{l(v)}, \quad f \in C_c^\infty(G).$$

This distribution is not identically zero. In particular we can pick f to be an essential matrix coefficient of π so that $J_\pi(f) \neq 0$. Here and below by an essential matrix coefficient we mean that f becomes a matrix coefficient of π after integration along the center of G.

It is relatively easy to prove that π' is H'-distinguished. Lapid and Mao [LM17] defined the following H'-invariant linear form on π'. Let P' be the usual mirabolic subgroup of G' and N' the upper triangular unipotent subgroup of G'. Let $\mathcal{W} = \mathcal{W}(\pi', \psi)$ be the Whittaker model of π'. Then for $W \in \mathcal{W}(\pi', \psi)$, put

$$l(W) = \int_{N' \cap H' \backslash P' \cap H'} W(h) \mathrm{d}h.$$

Lapid and Mao showed that this integral is absolutely convergent and defines a nonzero H'-invariant linear form on π'. It is a curious fact that if π' is H'-distinguished then so is $\pi' \otimes \eta$ (this in fact holds for all irreducible generic representations). So we also put

$$l_\eta(W) = \int_{N' \cap H' \backslash P' \cap H'} W(h)\eta(\det h) \mathrm{d}h.$$

We define a distribution on G' by

$$I_{\pi'}(f') = \sum_W l(\pi'(f')W)\overline{l_\eta(W)}, \quad f' \in C_c^\infty(G'),$$

where W runs over an orthonormal basis of $\mathcal{W}(\pi', \psi)$.

Guo proposed a relative trace formula in [Guo96]. There is a notion of smooth matching of orbits and the test functions on G and on G'. We are going to recall it in the main context of the paper. It follows from the work of C. Zhang [Zha15] that the smooth transfer of functions exists. Strictly speaking his result only covers the cases when $H = \mathrm{GL}_n(E)$. In general there is a little bit of extra work on the matching of orbits. Once we have this matching of orbits, the argument of C. Zhang goes through without any difficulty and can be copied vabatim. Now we assume the full fundamental lemma of Guo for the moment to streamline the argument. In the actual proof of Theorem 1.1, we will use a global version of the argument below to bypass the full fundamental lemma. Assuming the full fundamental lemma, we can show that there is a nonzero constant $\kappa(\pi)$ so that

$$I_{\pi'}(f') = \kappa(\pi) J_\pi(f), \tag{1.1}$$

whenever f and f' are smooth transfer of each other.

Here is the most important observation that is the starting point of this series of work. We refer to this as the involution method. There is an involution $f' \mapsto f'^w$ on $C_c^\infty(G')$ given by

$$f'^w(g) = f'(wgw), \quad w \text{ is the longest Weyl group element in } G'.$$

On the one hand, Lapid and Mao [LM17] have shown that

$$l(\pi'(W)) = \epsilon(\pi')l(W), \quad W \in \mathcal{W}(\pi', \psi).$$

It then follows that

$$I_{\pi'}(f'^w) = \epsilon(\pi')\epsilon(\pi' \otimes \eta)\eta(-1)^n I_{\pi'}(f').$$

On the other hand, the property of smooth matching of test functions implies that if f' and f match and f has only elliptic orbital integrals, then f'^w and $(-1)^r f$ also match. We will show that if we take f to be an essential matrix coefficient of π then f has only elliptic orbital integrals. Therefore we conclude that

$$\epsilon(\pi')\epsilon(\pi' \otimes \eta)\eta(-1)^n \kappa(\pi) J_\pi(f) = \epsilon(\pi')\epsilon(\pi' \otimes \eta)\eta(-1)^n I_{\pi'}(f') = I_{\pi'}(f'^w) = (-1)^r \kappa(\pi) J_\pi(f),$$

for all matrix coefficient f. Since we can take some essential matrix coefficient f so that $J_\pi(f) \neq 0$ and $\kappa(\pi) \neq 0$. We conclude that $\epsilon(\pi')\epsilon(\pi' \otimes \eta)\eta(-1)^n = (-1)^r$. This proves one implication of the theorem.

The converse implication is more involved. Assume that π' is H'-distinguished. We first define two "minimal unipotent orbital integrals" on G', which we denote by $O(\zeta_\pm, \eta, f')$. We show by a lengthy computation that if f' is an essential matrix coefficient of π', then

$$O(\zeta_+, \eta, f') = C I_{\pi'}(f'),$$

where C is some nonzero constant. In particular there is an essential matrix coefficient f' such that $O(\zeta_+, \eta, f') \neq 0$. For this choice of f', by the result of Lapid and Mao again, we have

$$O(\zeta_-, \eta, f') = O(\zeta_+, \eta, f'^w) = \epsilon(\pi')\epsilon(\pi' \otimes \eta)\eta(-1)^n O(\zeta_+, \eta, f')$$

and thus

$$O(\zeta_+, \eta, f') + \epsilon(\pi')\epsilon(\pi' \otimes \eta)\eta(-1)^n O(\zeta_-, \eta, f') \neq 0.$$

This, together with the fact that $\epsilon(\pi')\epsilon(\pi' \otimes \eta)\eta(-1)^n = (-1)^r$, imply that

$$O(\zeta_+, \eta, f') + (-1)^r O(\zeta_-, \eta, f') \neq 0.$$

Now using the involution method and the condition $\epsilon(\pi')\epsilon(\pi' \otimes \eta)\eta(-1)^n = (-1)^r$ again, we can show that any regular orbital integral of f' vanishes if this orbit does not match one on G. Thus there is a function $f \in C_c^\infty(G)$ that matches f'. By direct computation we have (up to some nonzero constant we omit)

$$\int_H f(h)\mathrm{d}h = O(\zeta_+, \eta, f') + (-1)^r O(\zeta_-, \eta, f') \neq 0.$$

Using global arguments we can show that there is an H-distinguished irreducible representation τ of G such that either $I_{\tau'}(f') \neq 0$ or $I_{\tau' \otimes \eta}(f') \neq 0$, τ' being the Jacquet–Langlands transfer of τ to G'. But since f' is an essential matrix coefficient of π', we see that π is isomorphic to either τ or $\tau \otimes \eta$. But as τ is H-distinguished, so is $\tau \otimes \eta$. We have thus shown that π is H-distinguished because η is trivial on H. Note that if we had the full fundamental lemma, we should have been able to conclude that π is isomorphic to τ.

In summary, the main theorem relates the epsilon factor and some geometric data. In the argument, we are making use of the involution both on the spectral and geometric side of the relative trace formula. One applies to the spherical characters (spectral) and pulls out the epsilon factor, the other applies to smooth transfer (geometric) and pulls out $(-1)^r$. The interplay between geometric and spectral information is always the theme of the trace formula.

1.3. **Organization of the paper.** The paper is divided into two parts, Section 2 to 4 is the first and Section 5 to 7 is the second. The first part proves one implication of theorem (computing local root number of distinguished representations), while the second part proves the other implication. Each part contains three sections and are ordered as "orbital integrals", "spherical characters", "proof of the main results". The precise content of each section is reflected by the table of contents.

1.4. **Acknowledgement.** I thank Wei Zhang for bringing the problem with linear periods to my attention and Räphael Beuzart-Plessis, Pierre-Henri Chaudouard, Qirui Li, and Miyu Suzuki for many helpful discussions. I thank Ye Tian and Shouwu Zhang for their constant support. I am also grateful to the anonymous referee whose comments improve and clarify several arguments in the paper. This work is partially supported by the NSF grant DMS #1901862.

2. Orbital integrals and smooth transfer

In this section, we assume that E/F is a quadratic field extension of either local or global fields of characteristic zero.

2.1. **The split side.** Let us define a permutation matrix $\sigma \in G'$ by

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & 2n \\ 1 & 3 & \cdots & 2n-1 & 2 & 4 & \cdots & 2n \end{pmatrix}.$$

Recall that H' is the fixed point of the involution

$$\theta(g) = \mathrm{Ad}\,\mathrm{diag}[1, -1, \cdots, 1, -1],$$

on G'. Then $\sigma^{-1} H' \sigma$ embeds in G' as $n \times n$ diagonal blocks. Let

$$S' = \{g^{-1}\theta(g) \mid g \in G'\} \subset G'$$

This is a closed subvariety of G' over F. Elements of S' are all of the form

$$\sigma \begin{pmatrix} A & B \\ C & D \end{pmatrix} \sigma^{-1}, \quad A^2 = 1_n + BC, \quad D^2 = 1_n + CB, \quad AB = BD, \quad DC = CA.$$

The group $H' \times H'$ acts on G' by left and right multiplication and the group H' acts on S' by conjugation. We say that an element in S' (or the orbit it represents) is θ-semisimple or θ-regular if it is so (in the usual sense) in G'. We say that an element $g \in G'$ (or the orbit it represents) is θ-semisimple or θ-regular if its image in S' is so. We say an element in S' is θ-elliptic if it is θ-regular and its stabilizer in H' is an elliptic torus and an element in G' θ-elliptic if its image in S' is so.

In [Guo96], the following results are proved.

Lemma 2.1. *We have the following assertions.*

(1) *An element* $s = \sigma \begin{pmatrix} A & B \\ C & D \end{pmatrix} \sigma^{-1} \in S'$ *is θ-regular if and only if A is regular in $M_n(F)$ in the usual sense and* $\det(A^2 - 1_n) \neq 0$. *It is θ-elliptic if and only if A is elliptic in $M_n(F)$ in the usual sense.*

(2) *Let* $s_i = \sigma \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \sigma^{-1} \in S'$, $i = 1, 2$, *be θ-regular. Then s_1 and s_2 are in the same H'-orbit if and only if A_1 and A_2 are in conjugate in $M_n(F)$.*

(3) *Every θ-regular orbit in G' contains an element of the form*

$$\sigma \begin{pmatrix} 1_n & a \\ 1_n & 1_n \end{pmatrix} \sigma^{-1},$$

and a is regular in $\mathrm{GL}_n(F)$ *in the usual sense and* $\det(a - 1_n) \neq 0$. *Moreover it is θ-elliptic if and only if a is regular and elliptic in* $\mathrm{GL}_n(F)$ *in the usual sense.*

Let $f' \in C_c^\infty(G')$ and $g \in G'$ be a θ-regular element. By the determinant function on H', we mean $\det h = \det h' \det h''$ where $h = (h', h'') \in H'$, $h', h'' \in \mathrm{GL}_n(F)$. If F is a local field, we define the orbital integral

$$O(g, \eta, f') = \int_{(H' \times H')_g \backslash H' \times H'} f'(h_1 g h_2) \eta(\det h_2) \mathrm{d}h_1 \mathrm{d}h_2.$$

If F is a global field, we define the global orbital integral by the same formula, integrating over the adelic points instead. Here are below in this paper, for any group G which acts on some set X we denote by G_x the stabilizer of x in G_x. To see that η is trivial on $(H \times H)_g$ we may assume that $g = \sigma \begin{pmatrix} 1_n & a \\ 1_n & 1_n \end{pmatrix} \sigma^{-1}$ where $a \in \mathrm{GL}_n(F)$ is regular in the usual sense. Then $(H \times H)_g$ consists of elements of the form (h_1, h_2) where $h_1^{-1} = h_2 = \sigma \begin{pmatrix} h' & \\ & h' \end{pmatrix} \sigma^{-1}$ and $h' \in \mathrm{GL}_n(F)$ commutes with a. Then $\eta(\det h_2) = \eta(\det h')^2 = 1$. As the orbit is closed, this integral is absolutely convergent. The orbital integral is thus well-defined.

Let us define a transfer factor for a θ-regular $g \in G'$ by

$$\Omega(g) = \eta(\det B), \quad g^{-1}\theta(g) = \sigma \begin{pmatrix} A & B \\ C & D \end{pmatrix} \sigma^{-1}.$$

Since g is θ-regular we have $A^2 - 1_n = BC$ is nonsingular. Thus the definition make sense. Let w be the longest Weyl group element in G', i.e. the matrix with antidiagonal elements all ones. Note that $w\sigma = \sigma w$. A little computation gives

$$\frac{\Omega(wgw)}{\Omega(g)} = \eta(\det(A^2 - 1_n)). \tag{2.1}$$

For later use we also need to consider the "Lie algebra" of $H'\backslash G'$. Let

$$\mathfrak{s}' = \left\{ \sigma \begin{pmatrix} & X \\ Y & \end{pmatrix} \sigma^{-1} \;\middle|\; X, Y \in M_n(F) \right\}.$$

The group H' acts on $\mathfrak{s}'$ by conjugation. An element in $\sigma \begin{pmatrix} & X \\ Y & \end{pmatrix} \sigma^{-1}$ is θ-regular (resp. θ-elliptic) if XY is invertible and is regular (resp. elliptic) in $\mathrm{GL}_n(F)$ in the usual sense. Let $f' \in C_c^\infty(\mathfrak{s}')$. Then we define the orbital integral

$$O(\gamma, \eta, f') = \int_{H'_\gamma \backslash H'} f'(h^{-1}\gamma h)\eta(\det h) \mathrm{d}h,$$

if γ is θ-regular.

Let $f' \in C_c^\infty(G')$. We define $\widetilde{f'} \in C_c^\infty(S')$ as

$$\widetilde{f'}(g^{-1}\theta(g)) = \int_{H'} f'(hg) \mathrm{d}h, \quad g \in G'.$$

We fix a H'-invariant neighbourhood ω' of $0 \in \mathfrak{s}'$ so that the Calyey transform

$$\mathfrak{c} : \omega' \to S', \quad \xi \mapsto (1+\xi)^{-1}(1-\xi)$$

is defined and is a homeomorphism, and denote by Ω' be its image in G'. We define $f'_\natural \in C_c^\infty(\mathfrak{s}')$ as

$$f'_\natural(\xi) = \begin{cases} \widetilde{f'}(\mathfrak{c}(\xi)), & \xi \in \omega'; \\ 0, & \xi \notin \omega'. \end{cases}$$

Then one checks that

$$O(\gamma, \eta, f'_\natural) = O(g, \eta, f'), \quad \gamma \in \omega', \quad \mathfrak{c}(\gamma) = g^{-1}\theta(g).$$

Indeed one may take $g = 1 + \gamma$.

2.2. **Nonsplit side: orbits.** Let us recall the setting. Let A be a CSA over F of dimension $4n^2$ with a fixed embedding $E \to A$. Let B be the centralizer of E in A, which itself is a CSA of dimension n^2 over E. We fix a $\tau \in F^\times \backslash F^{\times,2}$ so that $E = F[\sqrt{\tau}]$. Then conjugating by $\sqrt{\tau}$ is an involution on A whose set of fixed points is B. We denote this involution by θ. We denote by ν_A and ν_B the reduced norm on A and B respectively.

Put $G = A^\times$, $H = B^\times$, both viewed as algebraic groups over F, and

$$S = H\backslash G = \{g^{-1}\theta(g) \mid g \in G\}, \quad \rho : G \to S, \quad g \mapsto g^{-1}\theta(g),$$

as usual. We also let $\mathfrak{s} = A^{\theta=-1}$ the (-1)-eigenspace of θ, which is isomorphic to the tangent space of S at 1.

The group H acts on S and $\mathfrak{s}$ by conjugation. We say that an element $s \in S$ is θ-semisimple or θ-regular if it is semisimple or regular semisimple in G in the usual sense. We say that an element $X \in \mathfrak{s}$ is θ-semisimple if $X^2 \in B$ is semisimple. We say that $X \in \mathfrak{s}$ is θ-regular if X^2 is invertible and regular in the usual sense. Assume that $X \in \mathfrak{s}$ is invertible in A. Then we claim that the reduced characteristic polynomial of X^2 have coefficients in F, i.e. $\nu_B(\lambda - X^2) \in F[\lambda]$. We note that X and elements in F commute and $X\sqrt{\tau} = -\sqrt{\tau}X$. Thus conjugation by X is an extension of the Galois conjugate of E/F. Therefore

$$\nu_B(\lambda - X^2) = \overline{\nu_B(\lambda - X^{-1}X^2X)} = \overline{\nu_B(\lambda - X^2)}.$$

Thus $\nu_B(\lambda - X^2) \in F[\lambda]$. We say an element $X \in \mathfrak{s}$ is θ-elliptic if X^2 is regular semisimple in B (in the usual sense) and its reduced characteristic polynomial is irreducible over F.

Let us study the θ-regular orbits in $\mathfrak{s}$ under the action of H.

Lemma 2.2. *Assume that $X_1, X_2 \in \mathfrak{s}$ are both θ-regular. Then X_1 and X_2 are conjugate by an element in H if and only if X_1^2 and X_2^2 are conjugate by an element in H.*

Proof. The "only if" direction is clear. Let us show the "if" direction. By replacing X_1 by its H-conjugate, we may assume that we in fact have $X_1^2 = X_2^2 \in B$.

We note that the case where $n = 1$ is clear and can be checked by hand directly. Even though we have assume that E is a field, when $n = 1$, the analogue of this lemma even holds when $E = F \times F$. We will call this the quaternion algebra case. We are going to reduce the general case to this one.

Let us first assume that X_1 hence X_2 are both θ-elliptic. The argument in this paragraph is communicated to me by Qirui Li, and it is implicit in his thesis. Put $L = F[\lambda]/(\nu_B(\lambda - X_1^2))$. This is a degree n *field* extension of F. We embed L in A by sending λ to X_1^2. Let D be the centralizer of L in A. Then D is a quaternion algebra over L. Since $\theta(L) = L$, we have $\theta(D) = D$. We let $D^- = \mathfrak{s} \cap D$. Then $X_1, X_2 \in D^-$. We note that $L \cap E = F$ as elements in E but not in F do not commute with X_1 while elements in L commute with X_1. But $E \subset D$ and hence $K = L \otimes E = E[\lambda]/(\nu_B(\lambda - X_1^2))$ is a quadratic etale algebra over L and $D \cap B = K$. By the quaternion algebra case, we conclude that there is an $h \in K$ so that $X_1 = hX_2h^{-1}$. This proves the lemma when X_1 and X_2 are θ-elliptic.

In general, when X_1 and X_2 are both θ-regular, let us reduce to the θ-elliptic case. First we note that

$$\nu_A(\lambda - X_1) = \nu_B(\lambda^2 - X_1^2).$$

In fact, as reduced characteristic polynomials are invariant under base field extensions we may extend the base field to the algebraic closure of F and the equality is obvious. We also note that if $f(\lambda) \in F[\lambda]$ is irreducible then $f(\lambda^2)$ is either irreducible or decomposes as $f(\lambda^2) = p(\lambda)p(-\lambda)$ where $p \in F[\lambda]$ is irreducible and $p(\lambda) \neq p(-\lambda)$.

The reduced characteristic polynomial (as an element of B) of X_1^2 decomposes

$$\nu_B(\lambda - X_1^2) = \widetilde{p_1}(\lambda) \cdots \widetilde{p_a}(\lambda)\widetilde{q_1}(\lambda) \cdots \widetilde{q_b}(\lambda),$$

where $p_1, \cdots p_a$ and $q_1, \cdots q_b$ are all irreducible polynomials in $F[\lambda]$, and $p_i(\lambda) = \widetilde{p_i}(\lambda^2) \in F[\lambda]$, $i = 1, \cdots, a$, are irreducible while $\widetilde{q_j}(\lambda^2) = q_j(\lambda)q_j(-\lambda)$, $j = 1, \cdots b$. Therefore the reduced characteristic polynomial of X_1 (as an element of A) factors as

$$\nu_A(\lambda - X_1) = p_1(\lambda) \cdots p_a(\lambda) q_1(\lambda) q_1(-\lambda) \cdots q_b(\lambda) q_b(-\lambda).$$

As $X_1^2 = X_2^2$, we conclude that the reduced characteristic polynomial of X_2 (as an element of A) is the same as that of X_1.

Let us put $V = D^r$ as a right D-module and then A is identified with $\mathrm{End}(V)$ where the action of elements in A is from the left. Let us also put $W_i = \mathrm{Ker}\, p_i(X_1) = \mathrm{Ker}\, \widetilde{p_i}(X_1^2)$, $i = 1, \cdots, a$ and $U_j = \mathrm{Ker}\, q_j(X_1)q_j(-X_1) = \mathrm{Ker}\, \widetilde{q_j}(X_1^2)$, $j = 1, \cdots, b$. Then

$$V = W_1 \oplus \cdots W_a \oplus U_1 \oplus \cdots \oplus U_b.$$

By definition, these W_i's and U_j's are X_1 invariant. Since $X_1^2 = X_2^2$, they are also X_2 invariant. Therefore we have

$$X_1, X_2 \in \mathrm{End}_D(W_1) \times \cdots \times \mathrm{End}_D(W_a) \times \mathrm{End}_D(U_1) \times \cdots \times \mathrm{End}_D(U_b).$$

As A contains E, and $X_1^2 = X_2^2$ commute with E, each of these endomorphism spaces contains E. Moreover X_1 and X_2 anti-commutes with E, i.e. $\sqrt{\tau} X_i = -X_i \sqrt{-\tau}$, $i = 1, 2$, in each of these endomorphism spaces. It is also clear from the construction that X_1 and X_2 are θ-elliptic in each of these endomorphism spaces, as $\widetilde{p_i}$'s and $\widetilde{q_j}$'s are irreducible in $F[\lambda]$. Therefore we are reduced to the θ-elliptic case. □

We now turn to the case of the symmetric space S. We may write an element $g \in G$ as $g = g^+ + g^-$ where $g^+ \in B$ and $g^- \in \mathfrak{s}$. Then $g \in S$ if and only if $(g^+)^2 - 1 = (g^-)^2$ and $g^+ g^- = g^- g^+$. We say that $s \in S$ is θ-semisimple or θ-regular if it is so in G in the usual sense. One can prove, in the same way as [Guo96] that $s = s^+ + s^- \in S$ is θ-regular if and only if $\nu_B((s^+)^2 - 1) \neq 0$ and s^+ is regular in B in the usual sense. In particular we have s^- is invertible (in A). By the same argument as in the case of $\mathfrak{s}$, we see that if $s = s^+ + s^-$ is θ-regular, then the reduced characteristic polynomial of s^+ is in $F[\lambda]$. We say that $s = s^+ + s^-$ is θ-elliptic if it is θ-regular and the reduced characteristic polynomial of s^+ is irreducible over F. We say that an element in G is θ-semisimple, or θ-regular, or θ-elliptic if its image in S is so. One also proves, by the same argument as [Guo96] that every θ-regular orbit in G contains an element of the form $g = 1 + a$ where $a \in \mathfrak{s}$ and $\nu_B(a^2 - 1) \neq 0$. Moreover g is θ-elliptic if and only if a is θ-elliptic.

Lemma 2.3. *Suppose that $s_i = s_i^+ + s_i^- \in S$, $i = 1, 2$, are θ-regular. Then s_1 and s_2 are in the same H-orbit if and only if s_1^+ and s_2^+ are conjugate by H in B.*

Proof. Only the "if" direction needs proof. Let us put

$$\alpha_i = s_i^-(1 + s_i^+)^{-1} \in \mathfrak{s}, \quad i = 1, 2.$$

By assumption s_i^+, $i=1,2$, are regular in B. As $\alpha_i^2=(s_i^+-1)(s_i^++1)^{-1}$, $i=1,2$, we see that α_1 and α_2 are θ-regular elements in $\mathfrak{s}$. Then that s_1^+ and s_2^+ are in the same H-orbit implies that so are α_1^2 and α_2^2. By lemma 2.2, this is equivalent to that α_1 and α_2 are in the same H-orbit. So choose $h\in H$ such that $h^{-1}\alpha_1 h=\alpha_2$. Then $h^{-1}\alpha_1^2 h=\alpha_2^2$. Therefore

$$(h^{-1}s_1^+h-1)(h^{-1}s_1^+h+1)^{-1}=(s_2^+-1)(s_2^++1)^{-1}.$$

This implies that $h^{-1}s_1^+h=s_2^+$. Combing this with $h^{-1}\alpha_1 h=\alpha_2$, i.e.

$$h^{-1}s_1^-h(1+h^{-1}s_1^+h)^{-1}=s_2^-(1+s_2^+)^{-1},$$

we get $h^{-1}s_1^-h=s_2^-$ and hence $h^{-1}s_1h=s_2$. □

2.3. **Nonsplit side: orbital integrals.** Assume that F is a local field. Let $f\in C_c^\infty(G)$ and $g\in G$ be θ-semisimple. Define the orbital integral

$$O(g,f)=\iint_{(H\times H)_g\backslash H\times H} f(h_1gh_2)\mathrm{d}h_1\mathrm{d}h_2.$$

This integral is absolutely convergent. If F is a global field, then we define the orbital integral by the same formula, integrating over the adelic points of $H\times H$ instead.

For the rest of this subsection, let us assume that F is local. We prove a formula which is usually referred to as "parabolic descent" of orbital integrals.

Write $G=\mathrm{GL}_r(C)$ where C is a central division algebra. If $\dim C$ is even, then there is an embedding $E\to C$ and we let D be the centralizer of E in C. We have H is isomorphic to $\mathrm{GL}_r(D)$. Let $r=r_1+\cdots+r_s$ be a partition of r and $P_G=M_GN_G$, $P_H=M_HN_H$ the (standard upper triangular) parabolic subgroups of G and H corresponding to this partition of r respectively. We also put C^- be the subspace of C consisting of element c such that $ce=-ec$ for all $e\in E$. If $\dim C$ is odd, then r is even $C_E=C\otimes_F E$ is again a central division algebra and $H\simeq GL_{r/2}(C_E)$. Let $r=r_1+\cdots+r_s$ be a partition of r by even r_i's . and $P_G=M_GN_G$, $P_H=M_HN_H$ the (standard upper triangular) parabolic subgroups of G and H corresponding to this partition of r respectively, i.e. $M_G\simeq \mathrm{GL}_{r_1}(C)\times\cdots \mathrm{GL}_{r_s}(C)$, $M_H\simeq \mathrm{GL}_{r_1/2}(C_E)\times\cdots\times \mathrm{GL}_{r_s/2}(C_E)$.

Let $g\in M_G$ be θ-regular (in G). Note that this implies that it is also θ-regular for the pair (M_G,M_H). Let $f\in C_c^\infty(G)$. By definition

$$O(g,f)=\int_{(H\times H)_g\backslash H\times H} f(h_1gh_2)\mathrm{d}h_1\mathrm{d}h_2.$$

Choose an open compact subgroup K of H in good position with P_H. Put

$$f_K(g)=\int_K f(g^k)\mathrm{d}k,\quad f^{(P_G)}(g)=\delta_{P_G}(g)^{\frac12}\int_{N_G} f_K(gn)\mathrm{d}n.$$

Let $h_1=k_1m_1n_1$ and $h_2=n_2m_2k_2$ be the Iwasawa decompositions of h_1 and h_2 respectively, where $m_1,m_2\in M_H$, $n_1,n_2\in N_H$ and $k_1,k_2\in K$. Then, up to some nonzero constant depending only on the choice of the measures, we have

$$O(g,f)=\int_{(M_H\times M_H)_g\backslash M_H\times M_H}\int_{N_H\times N_H} f_K(m_1n_1gn_2m_2)\delta_P(m_1m_2^{-1})\mathrm{d}n_1\mathrm{d}n_2\mathrm{d}m_1\mathrm{d}m_2. \tag{2.2}$$

Lemma 2.4. *The map*

$$N_H\times N_H\to N_G,\quad (n_1,n_2)\mapsto g^{-1}n_1gn_2$$

is bijective and submersive everywhere.

Proof. Let $\mathfrak{n}_H$ and $\mathfrak{n}_G$ be the Lie algebra of N_H and N_G respectively. The tangent map at the point (n_1,n_2) is given by

$$\mathrm{d}\delta_g|_{(n_1,n_2)}:\mathfrak{n}_H\times\mathfrak{n}_H\to\mathfrak{n}_G,\quad (\xi_1,\xi_2)\mapsto g^{-1}n_1\xi_1gn_2+g^{-1}n_1gn_2\xi_2.$$

As n_1 and n_2 are both unipotent, the determinant of $\mathrm{d}\delta_g$ is independent of n_1 and n_2, and equals $\mathrm{d}\delta_g=\mathrm{d}\delta_g|_{(1,1)}$ which we now compute. First note that if $m_1,m_2\in M_H$, then we have

$$|\det\mathrm{d}\delta_{m_1gm_2}|=\delta_{P_H}(m_1m_2)^{-1}|\det\mathrm{d}\delta_g|.$$

As g is θ-regular, we may assume that $g=1+a$ where $a\in\mathfrak{s}\cap\mathfrak{n}_G$. Then direct computation shows that the determinant of $\mathrm{d}\delta_g$ equals the multiplication of the determinant of

$$\mathfrak{n}_H\to\mathfrak{n}_H,\quad X\mapsto(1-a^2)^{-1}X,$$

and the determinant of

$$\mathfrak{n}_H\to\mathfrak{s}\cap\mathfrak{n}_G,\quad X\mapsto Xa-aX.$$

As g is θ-regular, both determinant are nonzero. This proves that δ is submersive at any θ-regular element g. □

Let us put

$$\Delta(g)=\delta_{P_G}(g)^{-\frac12}|\det\mathrm{d}\delta_g|^{-1},$$

where δ_g is as in the above lemma. As $\delta_{P_H}(m)=\delta_{P_G}(m)^{\frac12}$ if $m\in M_G$, it follows from (the proof of) the above lemma that $\Delta(g)$ is bi-M_H-invariant. Then by making a change of variable $u=g^{-1}n_1gn_2$ in the integral (2.2), we see that

$$\begin{aligned}O(g,f)&=\int_{(M_H\times M_H)_g\backslash(M_H\times M_H)}\int_{N_G}\Delta(g)\delta_{P_G}(g)^{\frac12}f_K(m_1gum_2)\delta_{P_H}(m_1m_2^{-1})\mathrm{d}u\mathrm{d}m_1\mathrm{d}m_2\\&=\int_{(M_H\times M_H)_g\backslash(M_H\times M_H)}\int_{N_H}\Delta(g)\delta_{P_G}(m_1gm_2)^{\frac12}f_K(m_1gm_2u)\mathrm{d}u\mathrm{d}m_1\mathrm{d}m_2\\&=\Delta(g)\int_{(M_H\times M_H)_g\backslash(M_H\times M_H)}f^{(P_G)}(m_1gm_2)\mathrm{d}m_1\mathrm{d}m_2.\end{aligned}\tag{2.3}$$

This last integral is an orbital integral on M_G of the function $f^{(P_G)}$.

To end this subsection, let us explain that if g is θ-regular but not θ-elliptic in G, then there is a proper parabolic subgroup P_G of the above form so that $g\in M_G$. In fact let $s=g^{-1}\theta(g)$ and $s=s^++s^-$. Then s^+ is regular but not elliptic in B. If $\dim C$ is even (resp. odd), we can find a nontrivial partition of $r = r_1 + \cdots + r_s$ (resp. $\frac{r}{2} = \frac{r_1}{2} + \cdots + \frac{r_s}{2}$), and the parabolic subgroups P_G and P_H so that $s^+ \in M_H$. But s^+ is regular and s^- commutes with s^+ we see that $s^- \in \mathfrak{s} \cap \mathfrak{n}_G$. Therefore $s \in M_G$ and $g \in M_G$.

Lemma 2.5. *Assume that $f \in C_c^\infty(G)$ satisfies that $f^{(P)} = 0$ for all parabolic subgroups of G. Then all non-θ-elliptic orbital integrals of f vanish. In particular all non-θ-elliptic orbital integrals of matrix coefficients of a supercuspidal representation vanish.*

Proof. This follows from the parabolic descent of the orbital integrals. □

We end this subsection by defining the orbital integrals on $\mathfrak{s}$. Let $f \in C_c^\infty(\mathfrak{s})$ and $a \in \mathfrak{s}$ that is θ-regular. Then we put

$$O(a, f) = \int_H f(h^{-1}ah)\mathrm{d}h.$$

Let $f \in C_c^\infty(G)$. We define $\widetilde{f} \in C_c^\infty(S)$ as

$$\widetilde{f}(g^{-1}\theta(g)) = \int_H f(hg)\mathrm{d}h, \quad g \in G.$$

We fix a H-invariant neighbourhood ω of $0 \in \mathfrak{s}$ so that the Calyey transform

$$\mathfrak{c} : \omega \to S, \quad \xi \mapsto (1+\xi)^{-1}(1-\xi)$$

is defined and is a homeomorphism, and denote by Ω be its image in G. For any $f \in C_c^\infty(G)$ we define $f_\natural \in C_c^\infty(\mathfrak{s})$ as

$$f_\natural(\xi) = \begin{cases} \widetilde{f}(\mathfrak{c}(\xi)), & \xi \in \omega; \\ 0, & \xi \notin \omega. \end{cases}$$

Then one checks that

$$O(\gamma, \eta, f_\natural) = O(g, \eta, f), \quad \gamma \in \omega, \quad \mathfrak{c}(\gamma) = g^{-1}\theta(g).$$

Indeed one may take $g = 1 + \gamma$.

2.4. **Smooth matching.** We first consider the smooth matching of orbits.

Let $g \in G$ and $g' \in G'$ be θ-regular elements. Let $s = g^{-1}\theta(g) = s^+ + s^-$ and $s' = g'^{-1}\theta(g') = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. We say that g and g' match if the reduced characteristic polynomial of s^+ equals the characteristic polynomial of A.

Lemma 2.6. *The matching of orbits defines an injective map from the θ-regular $H \times H$-orbits in G to the θ-regular $H' \times H'$-orbits in G'.*

Proof. This follows from Lemma 2.1 and Lemma 2.3. □

We next study the matching of θ-elliptic orbits. For this we just need the case F being a local field. The next lemma shows that the θ-elliptic orbits in G' coming from G consist of "half" of the all θ-elliptic orbits in G'.

Lemma 2.7. *Assume that F is a local field. Let r is the split rank of G. The image of the θ-elliptic elements of G in G' consists of elements $g' \in G'$ such that $g'^{-1}\theta(g') = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ with a' being elliptic and $\eta(\det(a'^2 - 1_n)) = (-1)^r$.*

Proof. Let us begin with the following claim.

Claim. Let $f(\lambda) \in F[\lambda]$ be an irreducible polynomial of degree n, and $\eta((-1)^n f(0)) = (-1)^r$ where r is the split rank of G. Then there is a θ-elliptic $X \in \mathfrak{s}$ so that $\nu_B(\lambda - X^2) = f(\lambda)$. Conversely, if $X \in \mathfrak{s}$ is θ-elliptic, then $\eta(\nu_B(X^2)) = (-1)^r$.

Assuming this claim for the moment, the lemma follows directly. Indeed if $g = 1 + a$ where $a \in \mathfrak{s}$ and a^2 is elliptic in B. We have $s = g^{-1}\theta(g)$, $s = s^+ + s^-$. A simple computation gives

$$s^+ = (1 + a^2)(1 - a^2)^{-1}, \quad s^- = -2a(1 - a^2)^{-1}.$$

Then the reduced characteristic polynomial of s^+ is irreducible since that of a^2 is. If g matches some $g' \in G'$, then s^+ and a' has the same characteristic polynomial. In particular a' is elliptic in $\mathrm{GL}_n(F)$ and

$$\eta(\det(a'^2 - 1)) = \eta(\nu_B((s^+)^2 - 1)) = \eta((s^-)^2) = \eta(a^2) = (-1)^r.$$

The last equality follows from the claim. Conversely, assume that a' is elliptic without eigenvalue ± 1 and $\eta(\det(a'^2 - 1)) = (-1)^r$. Consider $(a' - 1)(a' + 1)^{-1}$ which is again elliptic. Then its reduced characteristic polynomial $f(\lambda)$ has the property of being irreducible and $\eta((-1)^n f(0)) = (-1)^r$. By the claim there is an $a \in \mathfrak{s}$ so that $\nu_B(\lambda - a^2) = f(\lambda)$. Let $g = 1 + a$ then we check directly that g matches g'.

We now prove the claim. Let us write $A = M_r(C)$ where C is a central division algebra over F whose invariant is $\frac{h}{m}$ with $rm = 2n$ and $(m, h) = 1$. Note that r being odd will force h to be odd.

Let us prove the the first assertion. The existence of X when A is a quaternion algebra is easy to prove. We are going to reduce the general case to this one. Assume first that f is irreducible in $E[\lambda]$. Let $L = F[\lambda]/(f(\lambda))$ and $K = L \otimes E$. Then K is a field of degree n over E. Therefore there is an embedding $K \to B$. We let u be the image of λ, then $\nu_B(\lambda - u) = f(\lambda)$. Now assume that $f(\lambda)$ is not irreducible over E. Then $f(\lambda) = p(\lambda)\overline{p(\lambda)}$ where $p(\lambda)$ is an irreducible polynomial in $E[\lambda]$. This implies that $n = \deg f$ is even and $f(0)$ is a norm, which in turn implies that r is even. In this case A is isomorphic to $M_2(C')$ where C' is a CSA over F and B is isomorphic to $C' \otimes E$. Therefore there is an embedding $K = L \otimes E \to B$. We let u be the image of λ, then $\nu_B(\lambda - u) = f(\lambda)$.

We will prove that for such a $u \in B$ we can find an $X \in \mathfrak{s}$ so that $u = X^2$. Let D be the centralizer of L in A, then $K = L \otimes E$ is a subfield of D. By the local class field theory, $\mathrm{inv}(D) = hr/2$. In other words, D is a quaternion division algebra if and only if r is odd. By the local class field theory, we have that $\eta_{E/F}(\nu_B(u)) = \eta_{K/L}(u) = (-1)^r$. Then the existence of X is reduced to the case of quaternion algebras, which is clear.

We now prove the second assertion. We just need to reverse the above argument. Suppose that $X \in \mathfrak{s}$ is θ-elliptic. Then as above, consider the reduced characteristic polynomial $f(\lambda)$ of $X^2 \in B$, which is irreducible over E. Then define $L = F[\lambda]/(f(\lambda))$, $K = L \otimes E$ and D just as above. Then $X^2 \in K \subset D$, and by the local class field theory we have $\eta_{E/F}(\nu_B(X^2)) = \eta_{K/L}(X^2) = (-1)^r$. This finishes the proof of the claim and thus the proof of the lemma. ☐

Lemma 2.8. *Suppose that $g \in G$ and $g' \in G'$ match and are θ-elliptic. Then*

$$\frac{\Omega(wg'w)}{\Omega(g')} = (-1)^r,$$

where r is the split rank of G. Conversely if a θ-elliptic element $g' \in G'$ satisfies the above identity, then there is a θ-elliptic $g \in G$ so that g and g' match.

Proof. This follows from Lemma 2.7 and the identity (2.1). ☐

We define the smooth matching of orbital integrals. Assume that F is a local field. We say that a function $f \in C_c^\infty(G)$ and a function $f' \in C_c^\infty(G')$ match if

$$\Omega(g')O(g', \eta, f') = \begin{cases} O(g, f), & \text{for all } \theta\text{-regular matching } g \in G \text{ and } g' \in G'; \\ 0, & g' \text{ does not match any } g \in G. \end{cases}$$

Proposition 2.9. *Given $f \in C_c^\infty(G)$, there is an f' that matches f. Given f' with $O(g', \eta, f') = 0$ for all θ-regular g' not matching any $g \in G$, there is an $f \in C_c^\infty(G)$ that matches f'.*

Proof. If $A = M_{2n}(F)$ or $A = M_n(D)$ where D is a quaternion over F, this is the result of C. Zhang [Zha15]. The general case can be proved by exactly the same method. We omit the details. ☐

Lemma 2.10. *Assume that f and f' match and all orbital integrals but the θ-elliptic ones vanish. Then $(-1)^r f$ and f'^w also match.*

Proof. It follows from a simple change of variables that $O(g', \eta, f'^w) = O(wg'w, \eta, f')$. Thus $O(g', \eta, f'^w) = 0$ if g' is not θ-elliptic or not match any element in G. If g' matches $g \in G$ and they are θ-elliptic, it is straightforward to check that $wg'w$ and g also match. Then we have

$$\Omega(g')O(g', \eta, f'^w) = (-1)^r \Omega(wg'w)O(wg'w, \eta, f') = O(g, f).$$

This proves the lemma. ☐

Let us consider the matching of orbits and orbital integrals on the level of Lie algebra. We will be brief since it is almost identical to the case of symmetric spaces. We just need the case F being a local field.

We say that two θ-regular elements $a \in \mathfrak{s}$ and $\begin{pmatrix} & X \\ Y & \end{pmatrix}$ match if the reduced characteristic polynomial of a^2 and the characteristic polynomial of XY are the same. This sets up an injective map from θ-regular orbits in $\mathfrak{s}$ to the set of θ-regular orbits in $\mathfrak{s}'$. Moreover the image of θ-elliptic orbits in $\mathfrak{s}$ under this map consists of θ-elliptic orbits in $\mathfrak{s}'$ represented by elements $\begin{pmatrix} & X \\ Y & \end{pmatrix}$ with $\eta(\det XY) = (-1)^r$. We define a transfer factor ω on $\mathfrak{s}'$ by

$$\omega\left(\begin{pmatrix} & X \\ Y & \end{pmatrix}\right) = \eta(\det X).$$

If $f \in C_c^\infty(\mathfrak{s})$ and $f' \in C_c^\infty(\mathfrak{s}')$, we say that they match if

$$\omega(\gamma')O(\gamma', \eta, f') = \begin{cases} O(\gamma, f), & \text{for all } \theta\text{-regular matching } \gamma \text{ and } \gamma', \\ 0, & \text{if } \gamma' \text{ does not match any element in } \mathfrak{s}. \end{cases}$$

Then again given $f \in C_c^\infty(\mathfrak{s})$ there is an $f' \in C_c^\infty(\mathfrak{s}')$ that matches it. Conversely if f' satisfies the condition that $O(\gamma', \eta, f') = 0$ for all γ's that do not match any element in $\mathfrak{s}$, then there is an $f \in C_c^\infty(\mathfrak{s})$ that matches f'. The main point to consider the Lie algebra version of matching is the following result [Zha15, Theorem 5.16]. If $f \in C_c^\infty(\mathfrak{s})$, we define its Fourier transform by

$$\widehat{f}(\xi) = \int_{\mathfrak{s}} f(\gamma)\psi(\operatorname{Tr}\xi\gamma)\mathrm{d}\gamma.$$

Similarly we have the Fourier transform of functions on $C_c^\infty(\mathfrak{s}')$.

Proposition 2.11. *There is a constant c_0 that is an eighth root of unity, such that if $f \in C_c^\infty(\mathfrak{s})$ and $f' \in C_c^\infty(\mathfrak{s}')$ match, then so do $c_0\widehat{f}$ and $\widehat{f'}$.*

Though not stated explicitly in the theorem, this constant is computed at the end of [Zha15] when $H = \mathrm{GL}_n(E)$. It is a ratio of Weil indices and hence an eighth root of unity. The general case follows by a similar computation.

Lemma 2.12. *Suppose that $f \in C_c^\infty(G)$ and $f' \in C_c^\infty(G')$ match. Then $f_\natural$ and $\eta(-2)^n f'_\natural$ also match.*

Proof. If $a \in \mathfrak{s}$ and $\xi \in \mathfrak{s}'$ are θ-regular and match, then so are $1 + a \in G$ and $1 + \xi \in G'$. We have

$$O(a, f_\natural) = O(1 + a, f), \quad O(\xi, \eta, f'_\natural) = O(1 + \xi, \eta, f').$$

One also checks that

$$\Omega(1 + \xi) = \eta(-2)^n\omega(\xi).$$

The lemma then follows. □

2.5. **The fundamental lemma.** Assume that E/F is an unramified quadratic extension of local fields. Let $\mathcal{H}$ be the usual Hecke algebra of $\mathrm{GL}_{2n}(F)$, i.e. bi-$\mathrm{GL}_{2n}(\mathfrak{o}_F)$-invariant compactly supported function on $\mathrm{GL}_{2n}(F)$. We consider the case $A = M_{2n}(F)$ and $B = M_n(E)$. Note that in this case $G = G'$. The fundamental lemma is the following conjecture.

Conjecture 2.13. *The functions $f = f' \in \mathcal{H}$ match.*

The main result of [Guo96] is to confirm this conjecture in the case $f = f'$ being the unit element of $\mathcal{H}$.

Proposition 2.14. *The functions $f = f' = \mathbf{1}_{\mathrm{GL}_{2n}(\mathfrak{o}_F)}$ match.*

We expect that Conjecture 2.13 should follows from Proposition 2.14 by a global argument.

2.6. **Categorical quotients.** We briefly explain the orbits and orbital integrals in terms of categorical quotients. Let Q be the n-dimensional affine space over F. Let $q : \mathfrak{s} \to Q$ be the morphism so that for all $\gamma \in \mathfrak{s}$, $q(\gamma)$ is the coefficients of the characteristic polynomial of γ^2. Let $q' : \mathfrak{s}' \to Q$ be the morphism so that for all $\gamma' = \begin{pmatrix} & X \\ Y & \end{pmatrix} \in \mathfrak{s}'$, $q'(\gamma')$ is the coefficients of the characteristic polynomial of XY.

Lemma 2.15. *The morphism $q : \mathfrak{s} \to Q$ and $q' : \mathfrak{s}' \to Q$ are categorical quotients.*

Proof. This is a geometric statement so we may and will base change to the algebraic closure $\overline{F}$ of F. Over the $\overline{F}$, the statement for q and q' are the same. Simple linear algebra yields that q' is surjective, and θ-regular elements form an open subset of $\mathfrak{s}'$ which is the complement of a principal divisor. The lemma then follows from the criterion of Igusa [VP89, Theorem 4.13]. $\square$

The induced map $\mathfrak{s}'(F) \to Q(F)$ is surjective while $\mathfrak{s}(F) \to Q(F)$ is not. Let $Q_{\theta-\mathrm{reg}} \subset Q$ be the open subscheme whose inverse image is the set of θ-regular elements in $\mathfrak{s}'$. Two θ-regular elements $\gamma \in \mathfrak{s}(F)$ and $\gamma' \in \mathfrak{s}'(F)$ match if and only if $q(\gamma) = q(\gamma') \in Q(F)$. Assume that F is a local field. There is a unique measure on $Q_{\theta-\mathrm{reg}}(F)$ so that for all $f \in C_c^\infty(\mathfrak{s})$ and $f' \in C_c^\infty(\mathfrak{s}')$ we have

$$\int_{\mathfrak{s}} f(\gamma)\mathrm{d}\gamma = \int_{Q_{\theta-\mathrm{reg}}(F)} O(\gamma, f)\mathrm{d}q(\gamma)$$

and

$$\int_{\mathfrak{s}'} f'(\gamma')\omega(\gamma')\mathrm{d}\gamma' = \int_{Q_{\theta-\mathrm{reg}}(F)} O(\gamma', \eta, f')\omega(\gamma')\mathrm{d}q'(\gamma').$$

The measure on $Q_{\theta-\mathrm{reg}}(F)$ is given by the Weyl integraion formula, c.f. [RR96, p. 106], see also [Zha15, p. 1829].

3. Spherical characters

For the rest of this paper, E/F is a quadratic extension of nonarchimedean local fields of characteristic zero.

3.1. **The split side.** Let π' be an irreducible generic representation of G'. We say that π' is H'-distinguished if $\mathrm{Hom}_{H'}(\pi', \mathbb{C}) \neq 0$. If π' is H'-distinguished, then it is self-dual and has trivial central character [Mat15a], and moreover $\mathrm{Hom}_{H'}(\pi', \mathbb{C})$ is one dimensional [JR96].

Lemma 3.1. *Suppose that π' is an irreducible generic representation of G'. Then π' is H'-distinguished if and only if $\pi' \otimes \eta$ is H'-distinguished.*

Proof. By [Mat17, Corollary 1.1], π' is H'-distinguished if and only if it admits a Shalika model (see the *loc. cit.* for a explanation of Shalika models). It follows directly from the definition of the Shalika model that π' has a Shalika model if and only if $\pi' \otimes \eta$ has a Shalika model. The lemma then follows. $\square$

Let P' be the usual mirabolic subgroup of G', N' be the upper triangular unipotent subgroup of G'. Recall that we have fixed a nontrivial additive character ψ and it naturally defines a generic character of N' as usual. Let $\mathcal{W} = \mathcal{W}(\pi', \psi)$ the corresponding Whittaker model of π'. Define

$$l(W) = \int_{H' \cap N' \backslash H' \cap P'} W(p) \mathrm{d}p, \quad W \in \mathcal{W}(\pi', \psi).$$

By [LM15b, Proposition 3.2] this integral is absolutely convergent and l defines a nonzero element in $\mathrm{Hom}_{H'}(\pi, \mathbb{C})$. As $\pi' \otimes \eta$ is also H'-distinguished, we put

$$l_\eta(W) = \int_{H' \cap N' \backslash H' \cap P'} W(p) \eta(\det p) \mathrm{d}p, \quad W \in \mathcal{W}(\pi', \psi).$$

This defines a nonzero element in $\mathrm{Hom}_{H'}(\pi' \otimes \eta, \mathbb{C})$.

We denote by $\epsilon(\pi') = \epsilon(\pi', \psi) = \pm 1$ the local root number. The second equality follows from the fact π' is self-dual and is of symplectic type.

Recall that $w \in G'$ is the longest weyl group element.

Proposition 3.2. *We have $l(\pi'(w)W) = \epsilon(\pi') l(W)$ and $l_\eta(\pi'(w)W) = \epsilon(\pi' \otimes \eta)\eta(-1)^n l_\eta(W)$ for any $W \in \mathcal{W}(\pi', \psi)$.*

Proof. The first equality is [LM17, Theorem 3.2]. The second one follows from the first one and the fact that $g \mapsto W(g)\eta(\det g)$ is a Whittaker function for $\pi' \otimes \eta$. $\square$

Define a spherical character as follows. For any $f' \in C_c^\infty(G')$, put

$$I_{\pi'}(f') = \sum_{W \in \mathcal{W}(\pi', \psi)} l(\pi'(f')W) \overline{l_\eta(W)},$$

where the sum runs over an orthonormal basis of $\mathcal{W}(\pi', \psi)$.

Define an involution $f' \mapsto f'^w$ on $C_c^\infty(G')$ as follows by setting $f'^w(g) = f'(wgw)$.

Corollary 3.3. *We have $I_{\pi'}(f'^w) = \epsilon(\pi')\epsilon(\pi' \otimes \eta)\eta(-1)^n I_{\pi'}(f')$ for any $f' \in C_c^\infty(G')$.*

Proof. We have

$$\begin{aligned} I_{\pi'}(f'^w) &= \sum_{W \in \mathcal{W}(\pi, \psi)} l(\pi'(w)\pi'(f')\pi'(w)W) \overline{l_\eta(W)} \\ &= \sum_{W \in \mathcal{W}(\pi', \psi)} l(\pi'(w)\pi'(f')W) \overline{l_\eta(\pi'(w)W)} \\ &= \epsilon(\pi')\epsilon(\pi' \otimes \eta)\eta(-1)^n \sum_{W \in \mathcal{W}(\pi', \psi)} l(\pi'(f')W) \overline{l_\eta(W)}. \end{aligned}$$

This proves the corollary. $\square$

3.2. **The nonsplit side.** Let π be an irreducible admissible unitary representation of G. We say that π is H-distinguished if $\mathrm{Hom}_H(\pi, \mathbb{C}) \neq 0$. In this case, by [BM], π is self-dual and this Hom space is one dimensional. We fix a nonzero element $l \in \mathrm{Hom}_H(\pi, \mathbb{C})$ and define the abstract spherical character attached to π by

$$J_\pi(f) = \sum_v l(\pi(f)v)\overline{l(v)}, \quad f \in C_c^\infty(G), \tag{3.1}$$

where v runs over an orthonormal basis of π. We also note that π is H-distinguished if and only if $\pi \otimes \eta$ is, as η is trivial on H.

3.3. **Global arguments.** We recall the relative trace formulae of Guo and derive some consequences from it.

We assume that $A = M_r(C)$ where C is a central division algebra over F of dimension d^2, with invariant $\frac{c}{d}$. Note that c and d are coprime. Thus r and cr have the same parity.

We first globalize the CSAs.

Lemma 3.4. *We can find the following data.*

1. *A quadratic extension of number fields L/K that splits at all archimedean places, a set of inert finite places S, $|S| = 2n - cr$, and a nonarchimedean inert place v_0 of K such that L_{v_0}/K_{v_0} is isomorphic to E/F. Note that S contains at least two places.*
2. *A CSA $\mathcal{A}$ over K with an embedding $L \to \mathcal{A}$ whose centralizer is $\mathcal{B}$, with the property that $(\mathcal{A}_{v_0}, \mathcal{B}_{v_0})$ is isomorphic to (A, B), the invariant of $\mathcal{A}_v$ is $1/2n$ if $v \in S$, and $\mathcal{A}_v \simeq M_{2n}(K_v)$ for all $v \notin S \cup \{v_0\}$.*

Proof. The existence of L/K is clear. The existence of $\mathcal{A}$ and $\mathcal{B}$ follows from the global class field theory. □

Put $\mathcal{G} = \mathcal{A}^\times$ and $\mathcal{H} = \mathcal{B}^\times$, both viewed as algebraic groups over K. Let $\mathcal{Z}$ be the center of $\mathcal{G}$. We now globalize the representation π.

Lemma 3.5. *With the $\mathcal{A}$ and $\mathcal{B}$ found in the previous lemma, we can find an irreducible cuspidal automorphic representation Π of $\mathcal{G}(\mathbb{A}_K)$, such that the integral*

$$\int_{\mathcal{H}(K)\mathcal{Z}(\mathbb{A}_K)\backslash\mathcal{H}(\mathbb{A}_K)} \varphi(h)\mathrm{d}h$$

is not identically zero, where $\Pi_{v_0} \simeq \pi$, Π_v is the trivial representation of $\mathcal{G}(K_v)$ if $v \in S$, Π_w is a supercuspidal $\mathcal{H}(K_w)$-distinguished representation for some split place w of K, and Π_u is unramified at all other nonarchimdean places u.

Proof. This follows from [PSP08, Theorem 4.1]. All we need to note is that $\mathcal{G}(K_v)$ is compact modulo the center if $v \in S$ and $\mathcal{H}/\mathcal{Z}$ has no rational characters. □

Remark 3.6. The last condition that "Π_u is unramified" will not be used in our argument. It is however needed in order to implement the idea of Prasad, c.f. Remark 4.2. If we drop this last condition, then the lemma also holds if we only assume that π is a discrete series representation, c.f. [SV17, Remark 16.4.1].

Let us now recall the relative trace formula of Guo and Jacquet. This is a slight extension of the relative trace formula presented in [FMW18]. We start from the nonsplit side. Let σ be an irreducible cuspidal automorphic representation of $\mathcal{G}(\mathbb{A}_K)$. We say that σ is globally $\mathcal{H}(\mathbb{A}_K)$-distinguished if there is a $\varphi \in \sigma$ such that

$$P(\varphi) = \int_{\mathcal{H}(K)\mathcal{Z}(\mathbb{A}_K)\backslash\mathcal{H}(\mathbb{A}_K)} \varphi(h)\mathrm{d}h \neq 0.$$

If σ is globally $\mathcal{H}(\mathbb{A}_K)$-distinguished, define a global distribution

$$J_\sigma(\mathbf{f}) = \sum_{\varphi} P(\sigma(\mathbf{f})\varphi)\overline{P(\varphi)}, \quad \mathbf{f} \in C_c^\infty(\mathcal{G}(\mathbb{A}_K)),$$

where φ runs through an orthonormal basis of σ. Then σ is globally $\mathcal{H}(\mathbb{A}_K)$-distinguished if and only if $J_\sigma \neq 0$.

We next consider the split side. We denote by $\mathcal{G}' = \mathrm{GL}_{2n,K}$ and $\mathcal{H}'$ the subgroup of $\mathcal{G}'$ which is the centralizer of $\mathrm{diag}[1, -1, \cdots, 1, -1]$. Let $\mathcal{Z}'$ be the center of $\mathcal{G}'$. Let σ' be an irreducible cuspidal automorphic representation of $\mathcal{G}'(\mathbb{A}_K)$. For $\varphi \in \sigma'$, we define the global linear period

$$P'(\varphi) = \int_{\mathcal{Z}'(\mathbb{A}_K)\mathcal{H}'(K)\backslash\mathcal{H}'(\mathbb{A}_K)} \varphi(h)\mathrm{d}h, \quad P'_\eta(\varphi) = \int_{\mathcal{Z}'(\mathbb{A}_K)\mathcal{H}'(K)\backslash\mathcal{H}'(\mathbb{A}_K)} \varphi(h)\eta(\det h)\mathrm{d}h.$$

We say that σ' is globally $\mathcal{H}'(\mathbb{A}_K)$-distinguished if P' is not identically zero. We define a global spherical character attached to σ' by

$$I_{\sigma'}(\mathbf{f}') = \sum_{\varphi} P'(\sigma'(\mathbf{f}')\varphi)\overline{P'_\eta(\varphi)},$$

where φ runs through an orthonormal basis of σ'. In the Appendix, we will show that $I_{\sigma'}$ factors as a product of certain L-functions and the local spherical characters $I_{\sigma'_v}$. In particular if $\mathbf{f}' = \otimes f'_v$, $I_{\sigma'}(\mathbf{f}') \neq 0$, then $I_{\sigma'_v}(f'_v) \neq 0$.

Let $\mathbf{f} = \otimes f_v \in C_c^\infty(\mathcal{G}(\mathbb{A}_K))$ and $\mathbf{f}' = \otimes f'_v \in C_c^\infty(\mathcal{G}'(\mathbb{A}_K))$, we say that $\mathbf{f}$ and $\mathbf{f}'$ match if for all places v of K, the test functions f_v and f'_v match. Remark that we explained only the matching at nonsplit nonarchimedean places, but at the split places (including the archimedean ones), the matching is trivial: if v is split, then $\mathcal{G}(K_v) = \mathcal{G}'(K_v)$ and we simply take $f_v = f'_v$.

Proposition 3.7. *Let σ be an irreducible cuspidal automorphic representation of $\mathcal{G}(\mathbb{A}_K)$ with the property that $\sigma_{v_0} \simeq \pi$, σ_v is the trivial representation if $v \in S$, and σ_w is an $\mathcal{H}(L_w)$-distinguished supercuspidal representation of $\mathcal{G}(K_w)$ for some split place w. Let σ' be the Jacquet–Langlands transfer of σ to $\mathcal{G}'(\mathbb{A}_K)$. Suppose that $\mathbf{f} \in C_c^\infty(\mathcal{G}(\mathbb{A}_K))$ and $\mathbf{f}' \in C_c^\infty(\mathcal{G}'(\mathbb{A}_K))$ match. Assume that*

$\mathbf{f}_v$ *(and hence $\mathbf{f}'_v$) is supported in the θ-regular locus if $v \in S$ and $\mathbf{f}_w = \mathbf{f}'_w$ is an essential matrix coefficient of σ_w. Then (with the suitable choice of measures) we have*

$$J_\sigma(\mathbf{f}) + J_{\sigma\otimes\eta}(\mathbf{f}) = I_{\sigma'}(\mathbf{f}') + I_{\sigma'\otimes\eta}(\mathbf{f}').$$

Proof. This is proved in [FMW18, Theorem 6.1, identity (6.3)] in a slightly different setting and our case can be handled in exactly the same way. Note that as we take the trivial representations of $\mathcal{G}(K_v)$, which is compact (modulo center), all orbits appearing in the relative trace formula are automatically θ-elliptic. □

Proposition 3.8. *Let π be an irreducible H-distinguished supercuspidal representation and $\pi' = \mathrm{JL}(\pi)$ be its Jacquet–Langlands transfer to G'. If π is H-distinguished, then π' is H'-distinguished.*

Proof. By assumtion, for some $f \in C_c^\infty(G)$, we have $J_\pi(f) \neq 0$. By Lemma 3.5, we can find

- an irreducible cuspidal automorphic representation σ,
- an $\mathbf{f} \in C_c^\infty(\mathcal{G}(\mathbb{A}_K))$ with $\mathbf{f}_{v_0} = f$,

so that they satisfy the conditions in Proposition 3.7 and with $J_\sigma(\mathbf{f}) \neq 0$. By [FMW18, Lemma 4.3], if we suitably modify this $\mathbf{f}$ at some nonsplit place of K, then we can even achieve $J_\sigma(\mathbf{f}) + J_{\sigma\otimes\eta}(\mathbf{f}) \neq 0$. Let $\mathbf{f}' \in C_c^\infty(\mathcal{G}'(\mathbb{A}_K))$ be a test function that matches $\mathbf{f}$. In particular $f' = \mathbf{f}'_{v_0}$ can be taken to be any test function that matches f. Using Proposition 3.7, we conclude that

$$J_\sigma(\mathbf{f}) + J_{\sigma\otimes\eta}(\mathbf{f}) = I_{\sigma'}(\mathbf{f}') + I_{\sigma'\otimes\eta}(\mathbf{f}') \neq 0. \tag{3.2}$$

It follows that either $I_{\sigma'}(\mathbf{f}') \neq 0$ or $I_{\sigma'\otimes\eta}(\mathbf{f}') \neq 0$, i.e. either σ' or $\sigma' \otimes \eta$ is globally $\mathcal{H}'(\mathbb{A}_K)$-distinguished. As π' is a local component of σ', we conclude that either π' or $\pi' \otimes \eta$ is H'-distinguished, but they are equivalent by Lemma 3.1. □

Remark 3.9. As suggested by the anonymous referee, there is a different proof of this proposition without using the relative trace formula. The argument even works without the supercuspidal assumption. We briefly explain the arguments. First by a result of Suzuki, we are reduced to the case π being a disrete series representation. In this case π' is a discrete series representation and by [LM17, Proposition 3.4], π' is H'-distinguished if and only if π' is symplectic in the sense that the image of Weil–Deligne representation attached to π' is contained in $\mathrm{Sp}_{2n}(\mathbb{C})$. As remarked before, by [SV17, Remark 16.4.1] the assertion in Lemma 3.5 still holds for π if we drop the requirement that "Π_u is unramified for all other places". Let Π' be the Jacquet–Langlands transfer of Π to $\mathrm{GL}_{2n}(\mathbb{A}_K)$. It is cuspidal because $\Pi'_w = \Pi_w$ is supercuspidal. Now Π is self-dual as it is globally distinguished by $\mathcal{H}(\mathbb{A}_K)$, then Π' is self-dual by the automorphic Chebotarev density theorem. As Π' is cuspidal, it is either orthogonal (all local components are orthogonal and $L(s, \Pi', \mathrm{Sym}^2)$ has a simple pole at $s = 1$) or symplectic (all local components are symplectic and $L(s, \Pi', \wedge^2)$ has a simple pole at $s = 1$). It cannot be orthogonal because Π'_w is H'-distinguished and is thus symplectic. Therefore Π' is symplectic and so are all its local components. In particular π' is symplectic and hence H'-distinguished.

4. Computing local root numbers

We restate one implication of the main theorem as follows.

Theorem 4.1. *Let π be an irreducible H-distinguished representation of G. Let π' be its Jacquet–Langlands transfer to G'. Then π' is H'-distinguished and*

$$\epsilon(\pi')\epsilon(\pi'\otimes\eta)\eta(-1)^n = (-1)^r$$

where r is the split rank of G.

The proof occupies this section.

4.1. **The supercuspidal case.** We present a proof which is essentially the one explained in the Introduction, but bypasses the full fundamental lemma using a global variant of the argument.

Proof of Theorem 4.1 assuming π is supercuspidal. Assume that π is H-distinguished and supercuspidal. Let us keep the notation from the proof of Proposition 3.8, in particular the identity (3.2). We can find matching test functions $\mathbf{f}\in C_c^\infty(\mathcal{G}(\mathbb{A}_K))$ and $\mathbf{f}'\in C_c^\infty(\mathcal{G}'(\mathbb{A}_K))$ so that

$$J_\sigma(\mathbf{f}) + J_{\sigma\otimes\eta}(\mathbf{f}) = I_{\sigma'}(\mathbf{f}') + I_{\sigma'\otimes\eta}(\mathbf{f}')\neq 0 \tag{4.1}$$

with the property that $f=\mathbf{f}_{v_0}$ is essentially a matrix coefficient of π. The function $f'=\mathbf{f}'_{v_0}$ matches f. By the parabolic descent of orbital integrals, i.e. Lemma 2.5, $O(g,f)=0$ if g is not θ-elliptic in G. Thus by Lemma 2.10, we have that $(-1)^r f$ and f'^w also match. Let $\mathbf{f}'^w$ be the test function obtained by $\mathbf{f}'$ by replacing $f'=\mathbf{f}'_{v_0}$ by f'^w. Then $(-1)^r\mathbf{f}$ and $\mathbf{f}'^w$ match. Now by the factorization of (split) linear periods (see Appendix) and Corollary 3.3, we have

$$I_{\sigma'}(\mathbf{f}'^w) = \epsilon(\pi')\epsilon(\pi'\otimes\eta)\eta(-1)^n I_{\sigma'}(\mathbf{f}'),\quad I_{\sigma'\otimes\eta}(\mathbf{f}'^w) = \epsilon(\pi')\epsilon(\pi'\otimes\eta)\eta(-1)^n I_{\sigma'\otimes\eta}(\mathbf{f}').$$

Therefore

$$(-1)^r(J_\sigma(\mathbf{f})+J_{\sigma\otimes\eta}(\mathbf{f})) = \epsilon(\pi')\epsilon(\pi'\otimes\eta)\eta(-1)^n(I_{\sigma'}(\mathbf{f}')+I_{\sigma'\otimes\eta}(\mathbf{f}')).$$

Comparing this with (4.1) we get

$$\epsilon(\pi')\epsilon(\pi'\otimes\eta)\eta(-1)^n = (-1)^r.$$

This proves the theorem when π is supercuspidal. □

Remark 4.2. There is a different argument in the supercuspidal case following the idea of Prasad which we present here for the comparison of methods. Let us keep the notation from the proof of Theorem 4.1. Recall that π is a local component of a globally distinguished representation σ. By (4.1) we conclude that either $I_{\sigma'}$ or $I_{\sigma'\otimes\eta}$ is not identically zero. By the factorization of split linear periods (see Appendix), this implies that

$$L(\frac{1}{2},\sigma')L(\frac{1}{2},\sigma'\otimes\eta)\neq 0,$$

and hence

$$\prod_v \epsilon(\sigma'_v)\epsilon(\sigma'_v \otimes \eta_v) = 1.$$

Now if $v \notin S \cup \{v_0\}$, then either v splits or σ'_v is unramified, then

$$\epsilon(\sigma'_v)\epsilon(\sigma'_v \otimes \eta_v)\eta_v(-1)^n = 1.$$

If $v \in S$, then σ'_v is the Steinberg representation of $\mathcal{G}'(K_v)$ and it is well-known that

$$\epsilon(\sigma'_v)\epsilon(\sigma'_v \otimes \eta_v)\eta_v(-1)^n = -1.$$

The only place remaining is $v = v_0$ where $\sigma'_{v_0} \simeq \pi'$. Since $|S| = 2n - cr$ which has the same parity with r, it follows that

$$\epsilon(\pi')\epsilon(\pi' \otimes \eta_{v_0})\eta_{v_0}(-1)^n = (-1)^r,$$

This finishes the proof.

4.2. **Reduction to the supercuspidal case.** Let us setup some notation before we proceed. If $G = \mathrm{GL}_r(C)$ where C is a central division algebra of dimension d^2 over F, $r = r_1 + r_2 + \cdots + r_s$, $\rho_1, \cdots \rho_s$ are irreducible representations of $\mathrm{GL}_{r_1}(C), \cdots, \mathrm{GL}_{r_s}(C)$ respectively, we denote by

$$\rho_1 \times \cdots \times \rho_s$$

the full parabolic induced representation (from the usual standard upper triangular parabolic subgroup corresponding to the partition $r = r_1 + \cdots r_s$). We make the convention that all parabolic inductions are normalized. We also denote by ν the absolute value of the reduced norm of any CSA. Suppose that $r = sl$ and ρ is a supercuspidal representation (not necessarily unitary) of $\mathrm{GL}_s(C)$. Assume that $C = F$ first. Then $G = G'$ and

$$\rho \times \rho\nu \times \cdots \times \rho\nu^{l-1}$$

has an unique irreducible quotient which is a discrete series representation of G and any irreducible discrete series representation of G is obtained in this way. In general assume that ρ' is the Jacquet–Langlands transfer of ρ to $\mathrm{GL}_{sd}(F)$ and then it is an irreducible quotient of

$$\tau \times \cdots \times \tau\nu^{q-1}$$

as above. Put $\nu_\rho = \nu^q$. Then the induced representation

$$\rho \times \rho\nu_\rho \cdots \times \rho\nu_\rho^{l-1}$$

has a unique quotient representation which is a discrete series representation. All discrete series representations of G arise in this way. Such a representation, or equivalently the set $\{\rho, \cdots, \rho\nu_\rho^{l-1}\}$, is called a segment. Let

$$\Delta = \{\rho, \cdots, \rho\nu_\rho^{l-1}\}, \quad \Delta' = \{\rho', \cdots, \rho'\nu_{\rho'}^{l'-1}\},$$

be two segments (or equivalently two discrete series representations). We say that Δ and Δ' are linked if $\Delta\cup\Delta'$ is again a segment but neither one is contained in the other (in particular $\nu_\rho=\nu_{\rho'}$). We say that Δ proceeds Δ' if they are linked and $\rho'=\rho\nu_\rho^j$ for some $j>0$.

To each irreducible representation π' of G', there is an associated to Weil–Deligne representation. Let $WD_F=W_F\times \mathrm{SL}_2(\mathbb{C})$ be the Weil–Deligne group, then the local Langlands correspondence gives rise to a representation

$$\phi_{\pi'}: WD_F\to \mathrm{GL}_{2n}(\mathbb{C}).$$

If π' is supercuspidal, then $\phi_{\pi'}$ is an irreducible representation of W_F and is trivial on $\mathrm{SL}_2(\mathbb{C})$. If π' is a segment of the form

$$\{\tau,\cdots,\tau\nu^{l-1}\},$$

then $\phi_{\pi'}=\phi_\tau\boxtimes \mathrm{Sym}^{l-1}$ where ϕ_τ is the irreducible representation of W_F associated to τ and Sym^{l-1} is the (unique) dimension l algebraic representation of $\mathrm{SL}_2(\mathbb{C})$. The local root number of π' is given by

$$\epsilon(\pi')=\epsilon(\phi_{\pi'})=\epsilon(\phi_\tau)^l\det(-\mathrm{Frob}\mid \phi_\tau^{I_F})^{l-1},$$

where I_F stands for the inertia subgroup of W_F, and $\phi_\tau^{I_F}$ stands for the subspace of ϕ_τ on which I_F acts trivially. Let us note that if ϕ_τ is not one-dimensional, then $\phi_\tau^{I_F}=0$. This is because any I_F-fixed subspace will be W_F-stable, and by the irreducibility of ϕ_τ, it should be either 0 or the whole space. But apparently I_F cannot act trivially on ϕ_τ since it is not one dimensional. Therefore it must be zero.

Proof of Theorem 4.1 in general. Theorem 4.1 very quickly reduces to the case of discrete series representations as we have the following classification result of M. Suzuki [Suz]. Assume π is H-distinguished, then Suzuki's result states that π is a quotient of $\Delta_1\times\cdots\times\Delta_s$ where Δ_i's are all irreducible discrete series representations and Δ_i does not proceed Δ_j, and after relabeling Δ_i's, we can find an a so that $s-a$ is even, and $\Delta_1,\cdots,\Delta_a$ are all distinguished and $\Delta_{a+2i-1}\simeq\Delta_{a+2i}^\vee$ for all $i=1,\cdots,(s-a)/2$. Here Δ_i, $1\le i\le a$, is distinguished means the following. The representation Δ_i is a discrete series representation of $\mathrm{GL}_{r_i}(C)$. There is an embedding of $E^\times$ in to $\mathrm{GL}_{r_i}(C)$ whose centralizer is denoted by H_i. We say that Δ_i is distinguished if there is an H_i-invariant linear form on it.

By this classification, if we assume Theorem 4.1 for the discrete series representation, we have

$$\epsilon(\pi)\epsilon(\pi\otimes\eta)\eta(-1)^n=\prod_{i=1}^s\epsilon(\Delta_i)\epsilon(\Delta_i\otimes\eta)\eta(-1)^{r_i}=(-1)^{r_1+\cdots r_a}=(-1)^r.$$

Therefore it is enough to prove Theorem 4.1 when π is a discrete series representation.

Assume that π is a discrete series representation of G and we write it as a segment

$$\{\rho\nu_\rho^{-\frac{l-1}{2}},\cdots,\rho\nu_\rho^{\frac{l-1}{2}}\},$$

where ρ is an irreducible supercuspidal representation of $\mathrm{GL}_s(C)$ and $sdl = 2n$, $sl = r$. The case $s = 1$ and ρ being one-dimensional has been taken care of by [Cho] so for the rest of the proof we assume that ρ is not one dimensional.

Assume first that l is even. Note that this implies that r is even. In this case by [BM, Proposition 5.6] (or rather its proof), ρ is self-dual. It is easy to see that π' is a segment

$$\{\tau\nu^{-\frac{l'-1}{2}}, \cdots, \tau\nu^{\frac{l'-1}{2}}\},$$

where τ is a self-dual supercuspidal representation, and l' is a multiple of l which is again even. Let ϕ_τ be the representation of the Weil group of F associated to τ. As we have assumed that ρ is not one-dimensional, ϕ_τ is not one-dimensional and so $\phi_\tau^{I_F} = 0$. Then easy computation with the root numbers gives

$$\epsilon(\pi')\epsilon(\pi' \otimes \eta) = (\det \phi_\tau(-1))^{\frac{l'}{2}}(\det \phi_{\tau\otimes\eta}(-1))^{\frac{l'}{2}} = \eta(-1)^n.$$

which proves the theorem. Now assume that l is odd. Note that in this case sd is even and hence there is an embedding $E^\times \to \mathrm{GL}_s(C)$. Again by [BM, Proposition 5.6] (or rather its proof), ρ is distinguished, in the sense that there is an H_s-invariant linear form on ρ where H_s is the centralizer of $E^\times$ in $\mathrm{GL}_s(C)$. Let ρ' be the Jacquet–Langlands transfer to $\mathrm{GL}_{sd}(F)$. Then ρ' is a segment

$$\{\tau\nu^{-\frac{a-1}{2}}, \cdots, \tau\nu^{\frac{a-1}{2}}\},$$

and π' is a segment

$$\{\tau\nu^{-\frac{la-1}{2}}, \cdots, \tau\nu^{\frac{la-1}{2}}\}.$$

Let $\phi_{\pi'}$ and $\phi_{\rho'}$ be the Weil–Deligne representation corresponding to π' and ρ' respectively, and ϕ_τ be the irreducible representation of the Weil group associated to τ. Again by our assumption ϕ_τ is not one dimensional and thus $\phi_\tau^{I_F} = 0$. If a is even, then the same computation as in the case l being even gives that both

$$\epsilon(\phi_{\pi'})\epsilon(\phi_{\pi'\otimes\eta})\eta(-1)^n = \epsilon(\phi_{\rho'})\epsilon(\phi_{\rho'\otimes\eta})\eta(-1)^{sd/2} = 1.$$

If a is odd, then a and la are both odd and $\dim \phi_\tau$ is even. We have

$$\epsilon(\phi_{\pi'})\epsilon(\phi_{\pi'\otimes\eta})\eta(-1)^n = \epsilon(\phi_{\rho'})\epsilon(\phi_{\rho'\otimes\eta})\eta(-1)^{sd/2} = \epsilon(\phi_\tau)\epsilon(\phi_{\tau\otimes\eta})\eta(-1)^{\frac{\dim \phi_\tau}{2}}.$$

In any case we have

$$\epsilon(\phi_{\pi'})\epsilon(\phi_{\pi'\otimes\eta})\eta(-1)^n = \epsilon(\phi_{\rho'})\epsilon(\phi_{\rho'\otimes\eta})\eta(-1)^{sd/2}.$$

Thus by the supercuspidal case (ρ is supercuspidal by assumption), we conclude that

$$\epsilon(\phi_{\pi'})\epsilon(\phi_{\pi'\otimes\eta})\eta(-1)^n = (-1)^s = (-1)^r.$$

Here the second equality follows from the fact that s and r have the same parity as $sl = r$ and l is odd. This finishes the proof of Theorem 4.1. □

5. Minimal unipotent orbital integrals

5.1. **Definitions.** Let us consider the following elements in G'

$$\zeta_+ = \sigma \begin{pmatrix} 1_n & 1_n \\ & 1_n \end{pmatrix} \sigma^{-1}, \quad \zeta_- = {}^t\zeta_+.$$

The orbits represented by $\zeta_\pm$ are not θ-semisimple in G' and hence are not closed orbits. The goal of this subsection is to define the "orbital integrals" on them. We just need to treat the case ζ_+.

Let $f' \in C_c^\infty(G')$. Put

$$O(s, \zeta_+, \eta, f')$$
$$= \int_{\mathrm{GL}_n(F)^3} f'\left(\sigma \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix} \begin{pmatrix} 1_n & 1_n \\ & 1_n \end{pmatrix} \begin{pmatrix} h_3 & \\ & 1_n \end{pmatrix} \sigma^{-1}\right) \eta(\det h_3) |\det h_3|^{-s} \mathrm{d}h_1 \mathrm{d}h_2 \mathrm{d}h_3.$$

Lemma 5.1. *The above integral is absolutely convergent when $\Re s >> 0$ and has a meromorphic continuation to the whole complex plane. It is holomorphic at $s = 0$. Moreover its value at zero equals*

$$\eta(-1)^n \gamma \int_{M_n(F)} \mathcal{F}_\psi f'_\natural \left(\sigma \begin{pmatrix} & h \\ 0 & \end{pmatrix} \sigma^{-1}\right) \eta(\det h) \mathrm{d}h.$$

Here the partial Fourier transform is defined by

$$\mathcal{F}_\psi f'_\natural \left(\sigma \begin{pmatrix} & X \\ Y & \end{pmatrix} \sigma^{-1}\right) = \int_{M_n(F)} f'_\natural \left(\sigma \begin{pmatrix} & X' \\ Y & \end{pmatrix} \sigma^{-1}\right) \psi\left(\mathrm{Tr}\, X'X\right) \mathrm{d}X',$$

and $\gamma = \prod_{i=0}^{n-1} \gamma(i, \eta, \psi)$ where $\gamma(s, \eta, \psi)$ is the gamma factor defined by Tate. Note that $\gamma(s, \eta, \psi)$ is holomorphic and nonzero at all $s = i$.

Proof. Simple change of variables gives

$$O(s, \zeta_+, \eta, f') = \int_{\mathrm{GL}_n(F)} f'_\natural \left(\sigma \begin{pmatrix} & h \\ 0 & \end{pmatrix} \sigma^{-1}\right) \eta(\det h) |\det h|^s \mathrm{d}h.$$

Note that we put $f_1(X) = f'_\natural \left(\sigma \begin{pmatrix} & X \\ 0 & \end{pmatrix} \sigma^{-1}\right)$, then $f_1 \in C_c^\infty(M_n(F))$. The the above integral is the zeta integral considered by Godement and Jacquet for the representation $\eta \circ \det$ of $\mathrm{GL}_n(F)$. Thus by [GJ72, Theorem 3.3], we have

$$Z(s, \eta, f_1) = \int_{\mathrm{GL}_n(F)} f_1(h_1) \eta(h_1) |\det h_1|^s \mathrm{d}h_1$$

is absolutely convergent when $\Re s >> 0$ and has a meromorphic continuation to the whole complex plane. Moreover by the remark after [GJ72, Proposition 3.3] we have

$$\prod_{i=1}^{n} \gamma(s - i, \eta, \psi) Z(s, \eta, f_1^\vee) = Z(n - s, \eta, f_1),$$

where $f_1^\vee$ is a Fourier transform defined by

$$f_1^\vee(X) = \int_{M_n(F)} f_1(Y)\psi(-\operatorname{Tr} XY)\mathrm{d}Y,$$

and $\gamma(s,\eta,\psi)$ is the gamma factor defined by Tate. Note that as η is not trivial, $\gamma(s,\eta,\psi)$ is holomorphic and nonzero for all $s \in \mathbb{R}$. Thus

$$Z(0,\eta,f_1) = \prod_{i=1}^{n} \gamma(n-i,\eta,\psi) Z(n,\eta,f_1^\vee).$$

We applying this to the integration $O(s,\zeta_+,\eta,f')$. The left hand side just equals $O(0,\zeta_+,\eta,f')$. The right hand side equals

$$\prod_{i=0}^{n-1} \gamma(i,\eta,\psi) \times \int_{M_n(F)} \int_{\mathrm{GL}_n(F)} f\left(\sigma \begin{pmatrix} & Y \\ 0 & \end{pmatrix} \sigma^{-1}\right) \psi(-\operatorname{Tr} hY)\eta(\det h)|\det h|^n \mathrm{d}h\mathrm{d}Y$$

Note that $|\det h|^n\mathrm{d}h$ equals the additive measure on $M_n(F)$. Making a change of variable $h \mapsto -h$ gives the desired result. □

We define

$$O(\zeta_+,\eta,f') = O(s,\zeta_+,\eta,f')|_{s=0}, \quad O(\zeta_-,\eta,f') = O(\zeta_+,\eta,f'^w).$$

Then it is clear that the distributions $f' \mapsto O(\zeta_\pm,\eta,f')$ are nonzero left H'-invariant and right (H',η)-invariant, and are supported on the closure of the orbits of $\zeta_\pm$.

We also need the counterparts of $O(\zeta_\pm,\eta,f')$ on the Lie algebra $\mathfrak{s}'$. Similar to the above lemma, for $f' \in C_c^\infty(\mathfrak{s}')$, the integral

$$O_+(s,f') = \int_{\mathrm{GL}_n(F)} f'\left(\sigma \begin{pmatrix} & h \\ 0 & \end{pmatrix} \sigma^{-1}\right) \eta(\det h)|\det h|^s \mathrm{d}h,$$

is absolutely convergent when $\Re s >> 0$ and has meromorphic continuation to the whole complex plane. Moreover it is holomorphic at $s=0$ and its value equals

$$\eta(-1)^n \gamma \int_{M_n(F)} \mathcal{F}_\psi f'\left(\sigma \begin{pmatrix} & X \\ 0 & \end{pmatrix} \sigma^{-1}\right) \eta(\det X)\mathrm{d}X.$$

This value is denote by $O_+(f)$. Similarly we have $O_-(f')$ which is by definition $O_-(f') = O_+(f'^w)$. It is easy to see that if $f' \in C_c^\infty(G')$, we have

$$O(\zeta_\pm,\eta,f') = O_\pm(f'_\natural).$$

5.2. **An unfolding identity.** Let $\xi_- \in M_{2n}(F)$ is the matrix with all $(i,i-1)$ entries being 1, $i = 2,3,\cdots,2n$, and all other entries being zero. Note that $\xi_- = \sigma \begin{pmatrix} & \xi_{1,-} \\ 1_n & \end{pmatrix} \sigma^{-1}$ where $\xi_{1,-} \in M_n(F)$ is of the same shape as ξ_-.

Let $f' \in C_c^\infty(G')$ and we define a function on $G' \times G'$ by

$$W_{f'}(g_1,g_2) = \int_{N'} f'(g_1^{-1}ug_2)\overline{\psi(\operatorname{Tr}\xi_- u)}\mathrm{d}u.$$

Proposition 5.2. *Let $f' \in C_c^\infty(G')$ and $W_{f'}$ as above. Then*

$$O(\zeta_+, \eta, f') = \eta(-1)^n \gamma \int_{H' \cap N' \backslash H' \cap P'} \int_{(H' \cap N') \backslash H'} W_{f'}(h_1, h_2) \eta(h_2) \mathrm{d}h_1 \mathrm{d}h_2, \tag{5.1}$$

where γ is as in Lemma 5.1 and the integral on the right hand side is absolutely convergent.

Proof. This is a rather technical computation. We proceed in steps.

Step 1: We check that the right hand side of (5.1) is absolutely convergent.

We are going to make use of the following notation in this step. Let T' be the diagonal torus of G'. Let T_1 be diagonal torus in P', $N_1 = N' \cap H'$ and K_1 be a maximal compact subgroup in $\mathrm{GL}_n(F) \times \mathrm{GL}_{n-1}(F)$. Then we have the Iwasawa decomposition $P' \cap H' = N_1 T_1 K_1$. Similarly let T_2 be the diagonal torus in H', $N_2 = N' \cap H'$ and K_2 be a maximal compact subgroup in H'. Then we have the Iwasawa decomposition $H' = N_2 T_2 K_2$. We let δ be the modulus character of T' with respect to N' and δ_1, δ_2 be the modulus character of T_1 and T_2 with respect to $N_1 \cap M_{P'}$ and N_2 respectively, where $M_{P'} \simeq \mathrm{GL}_n(F) \times \mathrm{GL}_{n-1}(F)$ is the reductive part of P'. If a is a diagonal matrix in G', we let a_i be its i-th diagonal entry. Let us also put $\varsigma(x) = \max\{\log|x|, \log|x|^{-1}\}$ and

$$\varsigma(a) = \max_{1 \leq i \leq n} \varsigma(a_i).$$

Let r and ν be the function on G' defined by $r(g) = |\det g|^{\frac{1}{4}}$ and $\nu(g) = 1 + \|e_{2n} g\|$ where $\|\cdot\|$ stands for the L^∞ norm on F_{2n}. Choose a large integer N which will be determined later, and apply [BP, Lemma 2.4.3] to $r^N \nu f$, we end up with the estimate

$$\begin{aligned} |W_f(a_1 k_1, a_2 k_2)| \ll & |\det a_1^{-1} a_2|^{\frac{1}{4}} (1 + |a_{1,2n-1}|)^{-N} (1 + |a_{2,2n}|)^{-N} \\ & \prod_{i=1}^{2n-2} \left(1 + \left|\frac{a_{1,i}}{a_{1,i+1}}\right|\right)^{-N} \prod_{i=1}^{2n-1} \left(1 + \left|\frac{a_{2,i}}{a_{2,i+1}}\right|\right)^{-N} \delta(a_1 a_2)^{\frac{1}{2}} \varsigma(a_1)^d \varsigma(a_2)^d, \end{aligned}$$

for some integer $d > 0$, where $a_i \in T_i$, $k_i \in K_i$, $i = 1, 2$.

Therefore to prove the convergence of the right hand side of (5.1), we need to prove that

$$\begin{aligned} \int_{T_1} \int_{T_2} & |\det a_1^{-1} a_2|^{\frac{1}{4}} (1 + |a_{1,2n-1}|)^{-N} (1 + |a_{2,2n}|)^{-N} \prod_{i=1}^{2n-2} \left(1 + \left|\frac{a_{1,i}}{a_{1,i+1}}\right|\right)^{-N} \prod_{i=1}^{2n-1} \left(1 + \left|\frac{a_{2,i}}{a_{2,i+1}}\right|\right)^{-N} \\ & \delta(a_1 a_2)^{\frac{1}{2}} \delta_1(a_1)^{-1} \delta_2(a_2)^{-1} \varsigma(a_1)^d \varsigma(a_2)^d \mathrm{d}a_1 \mathrm{d}a_2 \end{aligned}$$

is absolutely convergent for sufficiently large N. Note that

$$\delta(a_1)^{\frac{1}{2}} \delta_1(a_1)^{-1} = |\det a_1|^{\frac{1}{2}}, \quad \delta(a_2)^{\frac{1}{2}} \delta_2(a_2)^{-1} = \prod_{i=1}^{n} \left|\frac{a_{2,2i-1}}{a_{2,2i}}\right|^{\frac{1}{2}}.$$

Thus we need to prove that both integrals

$$\int_{T_1} |\det a_1|^{\frac{1}{4}} (1 + |a_{1,2n-1}|)^{-N} \prod_{i=1}^{2n-2} \left(1 + \left|\frac{a_{1,i}}{a_{1,i+1}}\right|\right)^{-N} \varsigma(a_1)^d \mathrm{d}a_1,$$

and

$$\int_{T_2}\prod_{i=1}^{n}|a_{2,2i-1}|^{\frac{3}{4}}|a_{2,2i}|^{-\frac{1}{4}}(1+|a_{1,2n-1}|)^{-N}\prod_{i=1}^{2n-2}\left(1+\left|\frac{a_{2,i}}{a_{2,i+1}}\right|\right)^{-N}\varsigma(a_2)^d\mathrm{d}a_2$$

are absolutely convergent for sufficiently large N. Both of these are implied by the following claim, which can be proved via a simple change of variable.

Claim. Let $s_1,\cdots,s_n$ be real numbers. Fix $d>0$. If $s_1+\cdots+s_k>0$ for all $k=1,\cdots,n$, then we can find a large N so that the integral

$$\int_{F^\times}(1+|x_n|)^{-N}\prod_{i=1}^{n-1}\left(1+\left|\frac{x_i}{x_{i+1}}\right|\right)^{-N}\prod_{i=1}^{n}|x_i|^{s_i}\varsigma(x_i)^d\mathrm{d}x_1\cdots\mathrm{d}x_n$$

is absolutely convergent.

This proves the absolute convergence of the right hand side of (5.1).

Step 2: We reduce (5.1) to an equality on the Lie algebra.

By lemma 5.1, the left hand side of (5.1) equals

$$\eta(-1)^n\gamma\int_{M_n(F)}\mathcal{F}_\psi f'_\natural\left(\sigma\begin{pmatrix} & h\\ 0 & \end{pmatrix}\sigma^{-1}\right)\eta(\det h)\mathrm{d}h.$$

Let us now compute the right hand side of (5.1). Plugging in the definition of $W_{f'}$, we have

$$\begin{aligned}\text{RHS of (5.1)} &= \int_{H'\cap N'\backslash H'\cap P'}\int_{H'\cap N'\backslash H'}\int_{N'}f'(h_1^{-1}uh_2)\overline{\psi(\mathrm{Tr}\,\xi_-u)}\eta(\det h_2)\mathrm{d}u\mathrm{d}h_1\mathrm{d}h_2\\ &= \int_{H'\cap N'\backslash H'\cap P'}\int_{N'\cap H'\backslash N'}\int_{H'}f'(h_1^{-1}uh_2)\overline{\psi(\mathrm{Tr}\,\xi_-u)}\eta(\det h_2)\mathrm{d}u\mathrm{d}h_1\mathrm{d}h_2\\ &= \int_{H'\cap N'\backslash H'\cap P'}\int_{N'\cap H'\backslash N'}\widetilde{f}'(h_2^{-1}u^{-1}\theta(u)h_2)\overline{\psi(\mathrm{Tr}\,\xi_-u)}\eta(\det h_2)\mathrm{d}u\mathrm{d}h_2.\end{aligned}$$

The second equality is valid as the inner two integrals are absolutely convergent. The third identity is the definition of $\widetilde{f}'$. Note that the map

$$N'\cap H'\backslash N'\to\mathfrak{n}'\cap\mathfrak{s}',\quad u\mapsto\mathfrak{c}^{-1}(u^{-1}\theta(u))$$

is submersive of determinant one everywhere and bijective. Moreover

$$\psi(\mathrm{Tr}\,\xi_-u)=\psi(-\mathrm{Tr}\,\xi_-u'),\quad\text{if } u'\in\mathfrak{n}'\cap\mathfrak{s}',\ \mathfrak{c}(u')=u^{-1}\theta(u).$$

We thus conclude that the right hand side of (5.1) equals

$$\int_{H'\cap N'\backslash H'\cap P'}\int_{\mathfrak{n}'\cap\mathfrak{s}'}f'_\natural(h_2^{-1}uh_2)\psi(\mathrm{Tr}\,\xi_-u)\eta(\det h_2)\mathrm{d}u\mathrm{d}h_2. \tag{5.2}$$

As $\mathcal{F}_\psi$ is a bijection from $C_c^\infty(\mathfrak{s}')$ to itself, to prove Proposition 5.2, it is enough to prove that for any $f_1' \in C_c^\infty(\mathfrak{s}')$, we have

$$\begin{aligned}(5.3)\qquad &\int_{\mathrm{GL}_n(F)} f_1'\left(\sigma\begin{pmatrix} & h\\ 0 & \end{pmatrix}\sigma^{-1}\right)\eta(\det h)\mathrm{d}h\\ =&\int_{H'\cap N'\backslash H'\cap P'}\int_{\mathfrak{n}'\cap\mathfrak{s}'}\mathcal{F}_\psi^{-1}f_1'(h^{-1}uh)\psi(\mathrm{Tr}\,\xi_- u)\eta(\det h)\mathrm{d}u\mathrm{d}h,\end{aligned}$$

where the measure $\mathrm{d}h$ on the left hand side is the additive measure.

Step 3: Computing the right hand side of (5.3) via Fourier inversion formula.

For the rest of the proof, we are going to temporarily use the following notation. We let $G_1 = \mathrm{GL}_n(F)$, B_1 be the upper triangular Borel subgroup, N_1 its unipotent subgroup, $B_{1,-}$ and $N_{1,-}$ be the opposite of B_1 and N_1 respectively. We let $\mathfrak{b}_1$, $\mathfrak{n}_1$, $\mathfrak{b}_{1,-}$, $\mathfrak{n}_{1,-}$ be the Lie algebra of B_1, N_1, $B_{1,-}$ and $N_{1,-}$ respectively. Similarly we put $G_2 = \mathrm{GL}_{n-1}(F)$, and define the corresponding subgroup B_2, N_2, $B_{2,-}$, $N_{2,-}$ and their Lie algebras $\mathfrak{b}_2$, $\mathfrak{n}_2$, $\mathfrak{b}_{2,-}$, $\mathfrak{n}_{2,-}$. The group G_2 is embedded in G_1 via $a \mapsto \begin{pmatrix} a & \\ & 1\end{pmatrix}$ and so are their subgroups.

Claim: For any $f' \in C_c^\infty(\mathfrak{s}')$, any $h = \sigma\begin{pmatrix} h_1 & \\ & h_2\end{pmatrix}\sigma^{-1} \in H'$, we have

$$\begin{aligned}&\int_{\mathfrak{n}'\cap\mathfrak{s}'}\mathcal{F}_\psi^{-1}f'(h^{-1}uh)\psi(\mathrm{Tr}\,\xi_- u)\mathrm{d}u\\ =&|\det h_2^{-1}h_1|^n\int_{\mathfrak{n}_2}\int_{N_1} f'\left(\sigma\begin{pmatrix} & h_2^{-1}u_1h_1\\ h_2^{-1}u_2h_1 & \end{pmatrix}\sigma^{-1}\right)\psi(\mathrm{Tr}\,\xi_{1,-}u_2)\mathrm{d}u_1\mathrm{d}u_2.\end{aligned}$$

The right hand side is absolutely convergent.

In fact, this is an application of the Fourier inversion formula. Explicitly the left hand side of the claim equals

$$\int_{\mathfrak{n}_1}\int_{\mathfrak{b}_1}\int_{M_n(F)} f'\left(\sigma\begin{pmatrix} & X\\ h_2^{-1}u_2h_1 & \end{pmatrix}\sigma^{-1}\right)\psi(-\mathrm{Tr}\,Xh_1^{-1}u_1h_2)\psi(\mathrm{Tr}\,u_1)\psi(\mathrm{Tr}\,\xi_{1,-}u_2)\mathrm{d}X\mathrm{d}u_1\mathrm{d}u_2.$$

The integral is convergent in this order. Make a change of variable $X \mapsto h_2^{-1}Xh_1$. Then the integral above equals

$$\begin{aligned}\int_{\mathfrak{n}_1}\int_{\mathfrak{b}_1}\int_{M_n(F)} & f'\left(\sigma\begin{pmatrix} & h_2^{-1}Xh_1\\ h_2^{-1}u_2h_1 & \end{pmatrix}\sigma^{-1}\right)\\ &\psi(-\mathrm{Tr}\,Xu_1)\psi(\mathrm{Tr}\,u_1)\psi(\mathrm{Tr}\,\xi_{1,-}u_2)|\det h_2^{-1}h_1|^n\mathrm{d}X\mathrm{d}u_1\mathrm{d}u_2.\end{aligned}$$

Applying Fourier inversion formula to the inner two integrals, we obtain that this integral equals

$$\int_{\mathfrak{n}_1}\int_{N_1} f'\left(\sigma\begin{pmatrix} & h_2^{-1}u_3h_1\\ h_2^{-1}u_2h_1 & \end{pmatrix}\sigma^{-1}\right)\psi(\mathrm{Tr}\,\xi_{1,-}u_2)|\det h_2^{-1}h_1|^n\mathrm{d}u_3\mathrm{d}u_2.$$

This proves the claim.

Thus to prove (5.3), we only need to compute

$$\int_{N_1\backslash G_1}\int_{N_2\backslash G_2}\int_{\mathfrak{n}_1}\int_{N_1} f'\left(\sigma\begin{pmatrix} & h_2^{-1}u_3h_1 \\ h_2^{-1}u_2h_1 & \end{pmatrix}\sigma^{-1}\right) \psi(\operatorname{Tr}\xi_{1,-}u_2)|\det h_2^{-1}h_1|^n\eta(\det h_1h_2)\mathrm{d}u_3\mathrm{d}u_2\mathrm{d}h_2\mathrm{d}h_1. \tag{5.4}$$

It is straightforward to see that we can (and will) change the order of the inner two integrals or the out two integrals. Combining the integral against u_3 and h_1 and making a change of variable $h_1 \mapsto h_2h_1$, we have that

$$(5.4) = \int_{N_2\backslash G_2}\int_{G_1}\int_{\mathfrak{n}_1} f'\left(\sigma\begin{pmatrix} & h_1 \\ h_2^{-1}u_2h_2h_1 & \end{pmatrix}\sigma^{-1}\right)\psi(\operatorname{Tr}\xi_{1,-}u_2)|\det h_1|^n\eta(\det h_1)\mathrm{d}u_2\mathrm{d}h_1\mathrm{d}h_2.$$

Step 4. An unfolding argument.

It is clear from Step 3 that to prove Proposition 5.2, it is enough to prove the following claim.

Claim: For any $f \in C_c^\infty(M_n(F))$, we have

$$f(0) = \int_{N_2\backslash G_2}\int_{\mathfrak{n}_1} f(h^{-1}uh)\psi(\operatorname{Tr}\xi_{1,-}u)\mathrm{d}u\mathrm{d}h.$$

To prove the claim, we replace the integration over $N_2\backslash G_2$ with the integration over $B_{2,-}$ and recall that we are using the right invariant Haar measure on $B_{2,-}$. We temporarily introduce the following notation. We let A_i, $i = 0, \cdots, n-1$ be the subgroup of $B_{2,-}$ consisting of elements whose upper left $i \times i$ block is the identity matrix. We let L_i, $i = 0, \cdots, n-1$, be the subspace of $\mathfrak{n}_1$ consisting of matrices whose upper left $(i+1)\times(i+1)$ block is zero. Let us introduce the following auxiliary integral

$$I_i = \int_{A_i}\int_{L_i} f(h_i^{-1}u_ih_i)\psi(\operatorname{Tr}\xi_{1,-}u_i)\mathrm{d}u_i\mathrm{d}h_i.$$

The measure $\mathrm{d}h_i$ is the right invariant measure on A_i. Of course, I_0 is the right hand side of the equality in the claim, while the $I_{n-1} = 0$. We are going to prove that $I_i = I_{i+1}$ for all i and this will prove the claim.

Let $h_i \in A_i$. We write $h_i = ah_{i+1}$ where $h_{i+1} \in A_{i+1}$ and a takes the following form

$$a = \begin{pmatrix} 1_i & & \\ v_i & x_i & \\ & & 1_{n-i-2} \end{pmatrix}, \quad v_i \in F_i,\ x_i \in F^\times.$$

The measure $\mathrm{d}h_i$ decomposes as

$$\mathrm{d}h_i = |x_i|^{-(n-i-2)}\mathrm{d}x_i\mathrm{d}v_i\mathrm{d}h_{i+1},$$

where $\mathrm{d}x_i$ is the multiplicative measure on $F^\times$.

Let $u_i \in L_i$. We write $u_i = c + u_{i+1}$ where $u_{i+1} \in L_{i+1}$ and c takes the following form

$$c = \begin{pmatrix} 0_{i+1} & w_{i+1} & \\ & 0 & \\ & & 0_{n-i-2} \end{pmatrix}, \quad w_{i+1} \in F^{i+1}.$$

Then we have

$$I_i = \int_{A_{i+1}} \int_{F_i} \int_{F^\times} \int_{F^{i+1}} \int_{L_{i+1}} f(h_{i+1}^{-1} a^{-1}(c + u_{i+1}) a h_{i+1}) \psi(\operatorname{Tr} \xi_{1,-}(c + u_{i+1})) \\ |x_i|^{-(n-i-2)} \mathrm{d}u_{i+1} \mathrm{d}c \mathrm{d}x_i \mathrm{d}v_i \mathrm{d}h_{i+1}.$$

Let us make change of variables $c \mapsto aca^{-1}$ and $u_{i+1} \mapsto au_{i+1}a^{-1}$. Let us note that

$$\operatorname{Tr} \xi_{1,-} aca^{-1} = (v_i, x_i) w_{i+1}, \quad \operatorname{Tr} \xi_{1,-} a u_{i+1} a^{-1} = \operatorname{Tr} \xi_{1,-} u_{i+1},$$

and

$$\mathrm{d}aca^{-1} = |x_i| \mathrm{d}c, \quad \mathrm{d}au_{i+1}a^{-1} = |x_i|^{n-i-2} \mathrm{d}u_{i+1}.$$

Then we have

$$I_i = \int_{A_{i+1}} \int_{F_i} \int_{F^\times} \int_{F^{i+1}} \int_{L_{i+1}} f(h_{i+1}^{-1}(c + u_{i+1}) h_{i+1}) \psi((v_i, x_i) w_{i+1}) \psi(\operatorname{Tr} \xi_{1,-} u_{i+1}) \\ |x_i| \mathrm{d}u_{i+1} \mathrm{d}c \mathrm{d}x_i \mathrm{d}v_i \mathrm{d}h_{i+1}.$$

Note that $|x_i|\mathrm{d}x_i$ gives the additive measure on F. We apply the Fourier inversion formula to the integration over w_{i+1} and (v_i, x_i). It follows that

$$I_i = \int_{A_{i+1}} \int_{L_{i+1}} f(h_{i+1}^{-1} u_{i+1} h_{i+1}) \psi(\operatorname{Tr} \xi_{1,-} u_{i+1}) \mathrm{d}u_{i+1} \mathrm{d}h_{i+1}.$$

The right hand side is precisely the definition of I_{i+1}. This proves the claim.

This finishes the proof of Proposition 5.2. □

5.3. **Matching.** We now consider matching of unipotent orbital integrals in G and G'. Let $f \in C_c^\infty(G)$. We put

$$O(1, f) = \int_H f(h) \mathrm{d}h.$$

This is the orbital integral of f on the minimal orbit represented by $1 \in G$. By definition we have $f_\natural(0) = O(1, f)$.

Proposition 5.3. *Suppose that $f \in C_c^\infty(G)$ and $f' \in C_c^\infty(G')$ are matching test functions. Then*

$$O(1, f) = \frac{1}{2} c_0 \eta(-2)^n (O(\zeta_+, \eta, f') + (-1)^r O(\zeta_-, \eta, f')),$$

where c_0 is the constant appearing in Proposition 2.11 (i.e. Fourier transform commutes with matching).

Proof. According to the definition, this identity is equivalent to

$$f_\natural(0) = \frac{1}{2}c_0(O_+(f'_\natural) + (-1)^r O_-(f'_\natural)).$$

As $f_\natural(0)$ and $\eta(-2)^n f'_\natural$ are also matching test functions, it is enough to prove that for all matching $f \in C_c^\infty(\mathfrak{s})$ and $f' \in C_c^\infty(\mathfrak{s}')$, we have

$$f(0) = \frac{1}{2}c_0(O_+(f') + (-1)^r O_-(f')).$$

As Fourier transform is a bijection from $C_c^\infty(\mathfrak{s})$ (resp. $C_c^\infty(\mathfrak{s}')$) to itself, and $c_0\widehat{f}$ and $\widehat{f'}$ also match, it is enough to prove that

$$\widehat{f}(0) = \frac{1}{2}(O_+(\widehat{f'}) + (-1)^r O_-(\widehat{f'})). \tag{5.5}$$

On the left hand side, the distribution $f \mapsto \widehat{f}(0)$ is represented by the constant functions 1 on $\mathfrak{s}$. On the right hand side, $f' \mapsto O_+(\widehat{f'})$ is represented by the function $\begin{pmatrix} & X \\ Y & \end{pmatrix} \mapsto \eta(\det X)$ and $f' \mapsto O_-(\widehat{f'})$ is represented by the function $\begin{pmatrix} & X \\ Y & \end{pmatrix} \mapsto \eta(\det Y)$. It follows that

$$f' \mapsto \frac{1}{2}(O_+(\widehat{f'}) + (-1)^r O_-(\widehat{f'}))$$

is represented by the function $\begin{pmatrix} & X \\ Y & \end{pmatrix} \mapsto \eta(\det X)$ if $\begin{pmatrix} & X \\ Y & \end{pmatrix}$ matches an element in $\mathfrak{s}$. This function is precisely the transfer factor. Recall that we have the categorical quotient $Q = \mathfrak{s}//H \simeq \mathfrak{s}'//H'$ and the quotient morphism $q : \mathfrak{s} \to Q$ and $q' : \mathfrak{s}' \to Q$. Therefore for some compatible choice of the measures (see the discussion at the end of Subsection 2.6), we have

$$\widehat{f}(0) = \int_{\mathfrak{s}} f(a)\mathrm{d}a = \int_{Q_{\theta-\mathrm{reg}}(F)} O(\gamma, f)\mathrm{d}q(\gamma),$$

and

$$\frac{1}{2}(O_+(\widehat{f'}) + (-1)^r O_-(\widehat{f'})) = \int_{\mathfrak{s}'} f'(\gamma')\omega(\gamma')\mathrm{d}\gamma' = \int_{Q_{\theta-\mathrm{reg}}(F)} O(\gamma', \eta, f')\omega(\gamma')\mathrm{d}q'(\gamma').$$

Since f and f' match, we have $O(\gamma', \eta, f')\omega(\gamma') = O(\gamma, f)$ if γ and γ' match and $O(\gamma', \eta, f')\omega(\gamma') = 0$ if γ' does not match any $\gamma \in \mathfrak{s}$. The desired equality (5.5) then follows. This proves the lemma. ☐

6. Minimal unipotent orbital integrals of matrix coefficients

6.1. **A functional equation.** Let Z' be the center of G' and π' be an irreducible generic (unitary) representation of G' and $\mathcal{W} = \mathcal{W}(\pi', \psi)$ be its Whittaker model. Let us recall some work of Matringe [Mat15b]. Let F_n be the n-dimensional vector space over F (row vectors) and $e_n = (0, \cdots, 0, 1) \in F_n$. Let $W \in \mathcal{W}(\pi', \psi)$ and $\phi \in C_c^\infty(F_n)$. Put

$$\Psi(s, t, W, \phi) = \int_{N' \cap H' \backslash H'} W(h)\phi(e_n h_2)|\det h_1|^{s+t+\frac{1}{2}}|\det h_2|^{s-t-\frac{1}{2}}\mathrm{d}h, \quad h = \sigma \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix} \sigma^{-1}.$$

We will only make use of this integral when $t = 0$ or $t = -\frac{1}{2}$. By [Mat15b, Proposition 4.16], for a fixed t, this integral is convergent when the real part of s is large, and it has a meromorphic continuation (in the variable s) to the whole complex plane. Moreover there is a $\gamma^{\mathrm{lin}}(s,\pi,\psi)$ in $\mathbb{C}(q^{\pm s})$ that does not depend on W and ϕ so that

$$\Psi\left(\frac{1}{2}-s,-\frac{1}{2},\widehat{W},\widehat{\phi}\right)=\gamma^{\mathrm{lin}}(s,\pi',\psi)\Psi(s,0,W,\phi). \tag{6.1}$$

where $\widehat{\phi}$ is the Fourier transform of ϕ, and $\widehat{W}(g)=W(w\,{}^tg^{-1})$, w being the longest Weyl element in G'. It is expected that

$$|\gamma^{\mathrm{lin}}(s,\pi',\psi)|=|\gamma(s+\frac{1}{2},\pi',\psi)\gamma(2s,\pi',\wedge^2,\psi)|. \tag{6.2}$$

It is showed in [Mat15b] that the analogous equality (without the absolute value) holds for L-factors, and

$$\frac{\gamma^{\mathrm{lin}}(s,\pi',\psi)L(s+\frac{1}{2},\pi')L(2s,\pi',\wedge^2)}{L(-s+\frac{1}{2},\widetilde{\pi}')L(1-2s,\widetilde{\pi}',\wedge^2)}$$

is a unit in $\mathbb{C}[q^{\pm s}]$. Thus at least the zero and poles of $\gamma^{\mathrm{lin}}(s,\pi',\psi)$ is controlled by the L-functions.

If $W\in\mathcal{W}(\pi',\psi)$, we define the following integral

$$l^{\#}(W)=\int_{Z'(H'\cap N')\backslash H'}W(h)\mathrm{d}h,\quad l^{\#}_{\eta}(W)=\int_{Z'(H'\cap N')\backslash H'}W(h)\eta(\det h)\mathrm{d}h.$$

Lemma 6.1. *Assume that π' (hence $\pi'\otimes\eta$) be an H'-distinguished discrete series representation of G'. Then the defining integrals of $l^{\#}$ and $l^{\#}_{\eta}$ are absolutely convergent and*

$$l(W)=\frac{1}{2n}\frac{\gamma^{\mathrm{lin}}(s,\pi',\psi)}{\gamma(s,\mathbf{1},\psi)}\Big|_{s=0}l^{\#}(W),\quad l_{\eta}(W)=\frac{1}{2n}\frac{\gamma^{\mathrm{lin}}(s,\pi'\otimes\eta,\psi)}{\gamma(s,\mathbf{1},\psi)}\Big|_{s=0}l^{\#}_{\eta}(W).$$

Here $\gamma(s,\mathbf{1},\psi)$ is the gamma factor of the trivial character defined by Tate, which as a simple pole at $s=0$. The ratios of the gamma factors at $s=0$ are holomorphic and nonzero.

Proof. We prove the identity for π'. The one for $\pi'\otimes\eta$ is identical. The lemma is a consequence of the functional equation (6.1). Let us evaluate both sides of (6.1) when $s\to 0+$.

First the right hand side. By [Mat15b, Corollary 4.10], as π' is a discrete series representation, the defining integral of $\Psi(s,0,W,\phi)$ is convergent when $\Re s>0$. Moreover as π' is H'-distinguished, $L(2s,\pi',\wedge^2)$ and hence $\Psi(s,0,W,\phi)$, have a simple pole at $s=0$. We have

$$\Psi(s,0,W,\phi)=\int_{Z'(N'\cap H')\backslash H'}\int_{Z'}W(h)\phi(e_nzh_2)|z|^{2ns}|\det h_1|^{s+\frac{1}{2}}|\det h_2|^{s-\frac{1}{2}}\mathrm{d}z\mathrm{d}h.$$

As s approaches $0+$, the inner integral has a simple zero and its leading term (as a function of s) equals that of $\gamma(2ns,\mathbf{1},\psi)\phi(0)$, which is independent of h_2. Thus

$$\lim_{s\to 0+}\gamma^{\mathrm{lin}}(s,\pi',\psi)\Psi(s,0,W,\phi)=\frac{1}{2n}\phi(0)\frac{\gamma^{\mathrm{lin}}(s,\pi',\psi)}{\gamma(s,\mathbf{1},\psi)}\Big|_{s=0}l^{\#}(W).$$

Now let us evaluate the left hand side of (6.1). We have

$$\Psi\left(\frac{1}{2},-\frac{1}{2},\widehat{W},\widehat{\phi}\right)=\int_{N'\cap H'\backslash H'}\widehat{W}(h)\widehat{\phi}(e_nh_2)\eta(\det h_1h_2)|\det h_2|\mathrm{d}h_1\mathrm{d}h_2.$$

It is not hard to see that the right hand side is absolutely convergent. We decompose

$$h_2=pzu,\quad u=\begin{pmatrix}1_{n-1} & \\ u_{n-1} & 1\end{pmatrix},$$

where p is in the mirabolic subgroup of $\mathrm{GL}_n(F)$, $u_{n-1}\in F_{n-1}$ and z is in the center of $\mathrm{GL}_n(F)$. Then we have

$$\Psi\left(\frac{1}{2},-\frac{1}{2},\widehat{W},\widehat{\phi}\right)=\int_{F_{n-1}}\int_{F^\times}l(\pi(u^+)\widehat{W})\widehat{\phi}(u_{n-1},z)\mathrm{d}u_{n-1}\mathrm{d}z,$$

where $u^+=\sigma\begin{pmatrix}1_n & \\ & u\end{pmatrix}\sigma^{-1}$ and the measures are all additive. As l is H'-invariant, we conclude that

$$\Psi\left(\frac{1}{2},-\frac{1}{2},\widehat{W},\widehat{\phi}\right)=\phi(0)l(\widehat{W})=\phi(0)l(W).$$

The last inequality follows from [LM15b, Proposition 3.9].

The upshot of the above computation is that for all $W\in\mathcal{W}$ and $\phi\in C_c^\infty(F_n)$, we have

$$\frac{1}{2n}\phi(0)\frac{\gamma^{\mathrm{lin}}(s,\pi',\psi)}{\gamma(s,\mathbf{1},\psi)}\Big|_{s=0}l^{\#}(W)=\phi(0)l(W).$$

Of course we can choose ϕ with $\phi(0)\neq 0$. All terms in this identity not involving W are nonzero. We thus have proved the desired identity. □

6.2. **Minimal orbital integrals of matrix coefficients.**

Lemma 6.2. *Let π' be an H'-distinguished supercuspidal representation of G'. Let $f'\in C_c^\infty(G')$ be an essential matrix coefficient of π'. Then*

$$O(\zeta_+,\eta,f')=2n\eta(-1)^n\gamma d(\pi')\frac{\gamma(s,\mathbf{1},\psi)}{\gamma^{\mathrm{lin}}(s,\pi',\psi)}\Big|_{s=0}I_{\pi'}(f'),$$

where $d(\pi')$ stands for the formal degree of π'. In particular there is an essential matrix coefficient of π' so that

$$O(\zeta_+,\eta,f')+\epsilon(\pi')\epsilon(\pi'\otimes\eta)\eta(-1)^nO(\zeta_-,\eta,f')\neq 0.$$

Proof. Suppose that

$$\int_{Z'}f'(zg)\mathrm{d}z=\langle\pi'(g)W_1,W_2\rangle,\quad W_1,W_2\in\mathcal{W}(\pi',\psi).$$

Then by [LM15a, Lemma 4.4], we have

$$\int_{Z'}W_{f'}(zg_1,g_2)\mathrm{d}z=W_1(g_2)\overline{W_2(g_1)}.$$

By Proposition 5.2, we have

$$O(\zeta_+,\eta,f')=\eta(-1)^n\gamma l^{\#}(W_1)\overline{l_\eta(W_2)}.$$

Using Lemma 6.1, we have

$$O(\zeta_+,\eta,f') = 2n\eta(-1)^n\gamma\frac{\gamma(s,\mathbf{1},\psi)}{\gamma^{\mathrm{lin}}(s,\pi',\psi)}\Big|_{s=0} l(W_1)\overline{l_\eta(W_2)}.$$

Note that $l(W_1)\overline{l_\eta(W_2)} = d(\pi')I_{\pi'}(f')$. This proves the first assertion. The last assertion is because

$$O(\zeta_-,\eta,f') = O(\zeta_+,\eta,f'^w) = 2n\eta(-1)^n\gamma d(\pi')\frac{\gamma(s,\mathbf{1},\psi)}{\gamma^{\mathrm{lin}}(s,\pi',\psi)}\Big|_{s=0}\epsilon(\pi')\epsilon(\pi'\otimes\eta)\eta(-1)^n I_{\pi'}(f').$$

Therefore

$$O(\zeta_+,\eta,f') + \epsilon(\pi')\epsilon(\pi'\otimes\eta)\eta(-1)^n O(\zeta_-,\eta,f') = 4n\eta(-1)^n\gamma d(\pi')\frac{\gamma(s,\mathbf{1},\psi)}{\gamma^{\mathrm{lin}}(s,\pi',\psi)}\Big|_{s=0} I_{\pi'}(f') \neq 0.$$

This proves the lemma. □

Remark 6.3. It is well-known that the formal degree $d(\pi')$ equals $\frac{1}{2n}|\gamma(0,\pi',\mathrm{Ad},\psi)|$. Since it is expected that

$$|\gamma^{\mathrm{lin}}(s,\pi',\psi)| = |\gamma(s+\frac{1}{2},\pi',\psi)\gamma(2s,\pi',\wedge^2,\psi)|.$$

The constant on the right hand side of the equality in the lemma should, up to some sign, simplify to $\gamma\cdot|\gamma(0,\pi',\mathrm{Sym}^2,\psi)|$.

7. Distinguished representations

7.1. **Global arguments.** The goal of this subsection is to prove a globalization result. We assume that E/F is local.

Proposition 7.1. *Assume that $f\in C_c^\infty(G)$ has the property that $\int_H f(h)\mathrm{d}h\neq 0$. Then one can find the following data.*

1. *Let L/K be a quadratic extension of global fields which splits at all archimedean places and there is a place v_0 of K so that $L_{v_0}/K_{v_0}\simeq E/F$.*
2. *Let $\mathcal{A}$ be a CSA over K containing L, such that there is a place v_1 of K such that $A\otimes K_{v_1}$ is a central division algebra. Let $\mathcal{B}$ be the centralizer of L in A. Let $\mathcal{G}=\mathcal{A}^\times$, $\mathcal{H}=\mathcal{B}^\times$, both being algebraic groups over K.*
3. *Let v_2 be a split nonarchimedean place of K, π_{v_2} be an $\mathcal{H}(K_{v_2})$-distinguished supercuspidal representations of $\mathcal{G}(K_{v_2})$.*
4. *Let $\mathbf{f}=\otimes\mathbf{f}_w\in C_c^\infty(\mathbb{G}(\mathbb{A}_\mathbb{F}))$ be a test function so that*
   (a) *$\mathbf{f}_{v_0}=f_{v_0}$;*
   (b) *$\mathbf{f}_{v_1}$ becomes a constant after integration over the center of $\mathcal{G}(K_{v_1})$ (note that $\mathcal{G}(K_{v_1})$ is compact modulo its center);*
   (c) *$\mathbf{f}_{v_2}$ is an essential matrix coefficient of π_{v_2} and $\int_{\mathcal{H}(K_{v_2})}\mathbf{f}_{v_2}(h)\mathrm{d}h\neq 0$;*
   (d) *we fix a nonsplit place nonarchimedean place v_3;*
   (e) *for all other places $w\neq v_i$, $i=1,2,3$, we choose an arbitrary test function $\mathbf{f}_w$ in $C_c^\infty(\mathcal{G}(K_w))$ with $\int_{\mathcal{H}(K_w)}\mathbf{f}_w(h)\mathrm{d}h\neq 0$;*

(f) *for the place v_3, we choose a test function $\mathbf{f}_{v_3}$ supported in a small neighbourhood of identity with $\int_{\mathcal{H}(K_{v_3})} \mathbf{f}_{v_3}(h)\mathrm{d}h \neq 0$ so that if $\gamma \in \mathcal{G}(K)$ and $\mathcal{H}(\mathbb{A}_K)\gamma\mathcal{H}(\mathbb{A}_K) \cap \operatorname{supp} \mathbf{f} \neq \emptyset$, then $\gamma \in \mathcal{H}(K)$.*

Proof. We only need to explain how to achieve the choice of $\mathbf{f}_{v_3}$, i.e. the test function in (4)(f). Let $\gamma \in \mathcal{G}(\mathbb{A}_K)$ and $\mathcal{H}(\mathbb{A}_K)\gamma\mathcal{H}(\mathbb{A}_K) \cap \operatorname{supp} \mathbf{f} \neq \emptyset$. For the CSA's $\mathcal{A}$ and $\mathcal{B}$, we have the same discussion as in Section 2. Consider $s = \gamma^{-1}\theta(\gamma)$ and the coefficients of the reduced characteristic polynomial of $(s-1)^2$. We viewed it as an element in $\mathbb{A}_K^n$. Of course $\mathcal{H}(\mathbb{A}_K)\gamma\mathcal{H}(\mathbb{A}_K)$ contains an element in $\mathcal{G}(K)$ if and only if the coefficients of the reduced characteristic polynomial of $(s-1)^2$ lie in K^n. Moreover for any place $w \neq v_3$, these coefficients lie in some fixed compact subset Ω_w of K_w^n, containing zero (because we have assumed that $\int_{\mathcal{H}(K_w)} \mathbf{f}_w(h)\mathrm{d}h \neq 0$). Therefore by the product formula we can choose a sufficiently small neighbourhood Ω_{v_3} of zero in $K_{v_3}^n$ so that

$$K^n \cap \prod \Omega_w = \{0\}.$$

Let U_{v_3} be the inverse image of Ω_{v_3} in $\mathcal{G}(K_{v_3})$, then U_{v_3} contains $\mathcal{H}(K_{v_3})$ by definition. We also note that

$$\left(U_{v_3} \times \prod_{w \neq v_3} \mathcal{H}(K_w)(\operatorname{supp} \mathbf{f}_w)\mathcal{H}(K_w) \right) \cap \mathcal{G}(K) = \mathcal{H}(K).$$

Indeed if γ lies in the left hand side, then the reduced characteristic polynomial of $(s-1)^2 \in \mathcal{B}(K)$ is of the form λ^n. By assumption $\mathcal{A}(K_{v_1})$ is a central division algebra, which means that $s = 1 \in \mathcal{A}(K_{v_1})$, hence in $\mathcal{A}(K)$. This is equivalent to that $\gamma \in \mathcal{H}(K)$. Having all this, we can thus choose an $\mathbf{f}_{v_3}$ supported in U_{v_3}. Then if $\gamma \in \operatorname{supp} \mathbf{f} \cap \mathcal{G}(K)$ then $\gamma \in \mathcal{H}(K)$. □

Corollary 7.2. *Let $f \in C_c^\infty(G)$ and $f' \in C_c^\infty(G')$ be matching test functions. Assume that $\int_H f(h)\mathrm{d}h \neq 0$. Then there is an irreducible H-distinguished representation τ of G, τ' being its Jacquet–Langlands transfer to G', such that either $I_{\tau'}(f') \neq 0$ or $I_{\tau'\otimes\eta}(f') \neq 0$.*

Proof. We plug the test function obtained in Proposition 7.1 in to the relative trace formula. Then we have

$$\sum_{g \in \mathcal{H}(K)\backslash\mathcal{G}(K)/\mathcal{H}(K)} O(g, \mathbf{f}) = \sum_{\sigma} J_\sigma(\mathbf{f}),$$

where σ on the right hand side ranges over all globally $\mathcal{H}(\mathbb{A}_K)$-distinguished automorphic representation of $\mathcal{G}(\mathbb{A}_K)$. Note that the group $\mathcal{G}$ and $\mathcal{H}$ are anisotropic so there is no convergence issue. By the choice of the test function $\mathbf{f}$, the left hand side reduces to only one term, i.e. $g = 1$. Thus the left hand side equals

$$\int_{H(\mathbb{A}_F)} \mathbf{f}(h)\mathrm{d}h \neq 0.$$

Therefore there is at least one σ on the right hand side such that $J_\sigma(\mathbf{f}) + J_{\sigma\otimes\eta}(\mathbf{f}) \neq 0$. We now apply the simple relative trace formula of Guo, c.f. identity (3.2), to conclude that

$$I_{\mathrm{JL}(\sigma)}(\mathbf{f}') + I_{\mathrm{JL}(\sigma)\otimes\eta}(\mathbf{f}') \neq 0.$$

Thus one of the two terms are nonzero. The corollary then follows from the factorization of split linear periods. ☐

7.2. **Distinguished representations.** Now is the time to reap the fruit of our long labor. We consider assumption (a) in Theorem 1.1 first. Let us restate the converse implication of the main theorem as follows.

Theorem 7.3. *Let π' be an irreducible H'-distinguished supercuspidal representation of G' and π its Jacquet–Langlands transfer to G. Assume that*

$$\epsilon(\pi')\epsilon(\pi'\otimes\eta)\eta(-1)^n = (-1)^r,$$

where r stands for the split rank of G, then π is H-distinguished.

Proof. By Lemma 6.2, there is an essential matrix coefficient f' of π' so that

$$O(\zeta_+,\eta,f') + (-1)^r O(\zeta_-,\eta,f') \neq 0. \tag{7.1}$$

By parabolic descent (which we did not make explicit but is very similar to the one on G), $O(g',\eta,f') = 0$ if g' is θ-regular but not θ-elliptic. We now consider the function on the θ-elliptic locus of G' given by

$$g' \mapsto \Omega(g')O(g',\eta,f').$$

This function is bi-$H'(F)$-invariant by definition. We now consider

$$g' \mapsto \Omega(g')O(g',\eta,f'^w).$$

On the one hand, we have

$$\Omega(g')O(g',\eta,f'^w) = \Omega(g')O(wg'w,\eta,f') = \Omega(wg'w)O(g',\eta,f'),$$

since $wg'w$ is in the same $H'(F)\times H'(F)$ double coset as g'. On the other hand we have

$$\Omega(g')O(g',\eta,f'^w) = \epsilon(\pi')\epsilon(\pi'\otimes\eta)\eta(-1)^n\Omega(g')O(g',\eta,f'). \tag{7.2}$$

This can be seen as follows. Suppose that

$$\int_{Z_G(F)} f'(zg')\mathrm{d}z = \langle \pi'(g')W_1, W_2\rangle$$

where W_1, W_2 are in the Whittaker model $\mathcal{W}$ of π'. Let us denote temporarily this f' by f'_{W_1,W_2}. Then the linear form

$$(W_1,W_2) \mapsto \Omega(g')O(g',\eta,f'_{W_1,W_2})$$

defines an element (could be zero) in

$$\mathrm{Hom}_{H'}(\pi',\mathbb{C}) \otimes \mathrm{Hom}_{H'}(\pi'\otimes\eta,\mathbb{C}).$$

Then by the uniqueness of linear periods [JR96], we can find a constant A (could be zero), depending on g and π' but not on W_1 and W_2 so that

$$\Omega(g')O(g',\eta,f') = Al(W_1)\overline{l_\eta(W_2)}.$$

Moreover

$$(f'_{W_1,W_2})^w = f'_{\pi'(w)W_1,\pi'(w)W_2}$$

and hence $\Omega(g')O(g',\eta,f'^w) = Al(\pi'(w)W_1)\overline{l_\eta(\pi'(w)W_2)}$. By Proposition 3.2, we get (7.2). We thus conclude that

$$\Omega(wg'w) = \Omega(g')\epsilon(\pi)\epsilon(\pi\otimes\eta)\eta(-1)^n,$$

if $O(g',\eta,f') \neq 0$. Since $\epsilon(\pi)\epsilon(\pi\otimes\eta)\eta(-1)^n = (-1)^r$, by Lemma 2.8, we conclude that g' matches a θ-elliptic element in G. In conclusion, we have shown that if g' is not θ-elliptic or does not match any element in G, then $O(g',\eta,f') = 0$. By Proposition 2.9, there is an $f \in C_c^\infty(G)$ which matches f'. By Proposition 5.3 and the nonvanishing result (7.1), we conclude that $O(1,f) \neq 0$.

By Corollary 7.2, we can find an H-distinguished representation τ of G so that either $I_{\mathrm{JL}(\tau)}(f') \neq 0$ or $I_{\mathrm{JL}(\tau)\otimes\eta}(f') \neq 0$. However by our very choice, f' is essentially a matrix coefficient of π'. Thus we conclude that $\pi' = \mathrm{JL}(\tau)$ or $\mathrm{JL}(\tau)\otimes\eta$. This implies that either $\pi = \tau$ or $\pi = \tau\otimes\eta$. Note that τ is H-distinguished and thus so is $\tau\otimes\eta$ as η is trivial on H. It follows that π is H-distinguished. □

The following proposition takes care of the final piece of the main theorem, i.e. assumption (b) in Theorem 1.1.

Proposition 7.4. *Assume that $G = \mathrm{GL}_n(D)$ where D is the quaternion division algebra over F. Let π be an irreducible supercuspidal representation of G and π' be its Jacquet–Langlands transfer to G'. Assume that $n > 1$ and $\epsilon(\pi')\epsilon(\pi'\otimes\eta)\eta(-1)^n = (-1)^n$. Then π' is supercuspidal. In particular the converse implication under the assumption (b) holds.*

Proof. Assume that π' is not supercuspidal. Then it is a discrete series representation of G'. Let us keep the notation from Subsection 4. By the classification of discrete series representation of G', π' is a the unique irreducible quotient of

$$\tau\nu^{-\frac{q-1}{2}} \times \cdots \times \tau\nu^{\frac{q-1}{2}}$$

where τ is a supercuspidal representation of $\mathrm{GL}_{2n/q}(F)$. By [DKV84, Theorem B.2.b.1)], $2n$ is the least common multiple of 2 and $2n/q$, $q > 1$. It then follows that $q = 2$ and n is odd. By the calculation in Subsection 4.2 (note that we assume $n > 1$), we have $\epsilon(\pi')\epsilon(\pi'\otimes\eta)\eta(-1)^n = 1 \neq (-1)^n$. This is a contradiction and hence π' is supercuspidal.

The converse implication of Theorem 1.1 under the assumption (b) then follows from Theorem 7.3. □

Appendix A. Factorization of split linear periods

In this appendix we factorize global linear periods into local linear forms. The main idea is already in the work of Bump, Friedberg, Jacquet and Matringe on linear periods. We just need to make the constants explicit.

A.1. **Setup.** We consider a slightly more general setting to allow twists in the linear periods. We also make minor modifications in our notation for convenience. We let F be a global field and $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}^\times$ a nontrivial additive character. We denote by $G_n = \mathrm{GL}_{n,F}$, Z_n be its center, P_n the mirabolic subgroup, N_n the upper triangular unipotent subgroup. We let $\iota : G_n \times G_n \to G_{2n}$ be the embedding so that the image is the centralizer of $\mathrm{diag}[1, -1, \cdots, 1, -1]$.

As we are going to make precise factorization with no ambiguity on the constants, we need to normalize our measures carefully. For the unipotent groups, e.g. N_n, we use the self-dual measures both locally and globally. For G_n, let us choose the following measure. Let v be any place of F. we write $g = (g_{ij}) \in G_n(F_v)$, $g_{ij} \in F_v$, and put

$$\mathrm{d}^*g = (\det g)^{-1} \prod_v \mathrm{d}g_{ij}, \quad \mathrm{d}g = \zeta_{F_v}(1)\zeta_{F_v}(2)\cdots\zeta_{F_v}(n)\mathrm{d}^*g.$$

When ψ_v is unramified, under the measure $\mathrm{d}g$ the volume of $G_n(\mathfrak{o}_{F_v})$ equals 1. If $g = (g_v) \in G_n(\mathbb{A}_F)$, we put

$$\mathrm{d}g = \prod_v \mathrm{d}g_v.$$

A.2. **Integral representation.** Let $\phi \in \mathcal{S}(\mathbb{A}_F^n)$ (row vector), and $e_n = (0, \cdots, 0, 1)$. Put

$$f_\phi(s,h) = |\det h|^s \int_{\mathbb{A}_F^\times} \phi(ae_nh)|a|^{ns}\mathrm{d}a,$$

and for $\Re s >> 0$, define the Eisenstein series

$$E(s,h,\phi) = \sum_{\gamma\in Z_n(F)P_n(F)\backslash G_n(F)} f_\phi(s,\gamma h),$$

By [JS81, Lemma 4.2] the integral is absolutely convergent and has a meromorphic continuation to the whole complex plane, with at most simples poles at $s = 0$ or $s = 1$. The residue at $s = 1$ equals

$$\frac{\mathrm{vol}(F^\times\backslash\mathbb{A}_F^1)}{n}\widehat{\phi}(0),$$

where $\widehat{\phi}$ stands for the Fourier transform

$$\widehat{\phi}(y) = \int_{\mathbb{A}_F^n} \phi(x)\psi({}^t xy)\mathrm{d}x.$$

Let π be an irreducible cuspidal automorphic representation of $G_{2n}(\mathbb{A}_F)$ and $\varphi \in \pi$. Let us now consider the integral

$$\begin{aligned} &I(s,\varphi,\chi,\phi) \\ &= \int_{Z_{2n}(\mathbb{A}_F)(G_n\times G_n)(F)\backslash(G_n\times G_n)(\mathbb{A}_F)} \varphi(\iota(h_1,h_2))E(2s,h_2,\phi)|\det h_1h_2^{-1}|^{s-\frac{1}{2}}\chi(\det h_1h_2^{-1})\mathrm{d}h_1\mathrm{d}h_2. \end{aligned}$$

This integral is absolutely convergent away from the poles of the Eisenstein series and has at most simple poles at $s = 0$ and $s = \frac{1}{2}$. We have

$$\mathrm{Res}_{s=\frac{1}{2}} I(s,\chi,\varphi,\phi) = \frac{1}{2n}\mathrm{vol}(F^\times\backslash\mathbb{A}_F^1)\widehat{\phi}(0)P(\varphi,\chi),$$

where $P(\varphi,\chi)$ is the (split) linear period

$$P(\varphi,\chi)=\int_{Z_{2n}(\mathbb{A}_F)(G_n\times G_n)(F)\backslash(G_n\times G_n)(\mathbb{A}_F)}\varphi(\iota(h_1,h_2))\chi(\det h_1h_2^{-1})\mathrm{d}h_1\mathrm{d}h_2.$$

A.3. **Factorization.** Let $\varphi\in\pi$ and

$$W_\varphi(g)=\int_{N_{2n}(F)\backslash N_{2n}(\mathbb{A}_F)}\varphi(ng)\overline{\psi(n)}\mathrm{d}n$$

be the Whittaker function. Let $\mathcal{W}(\pi_v,\psi_v)$ be the Whittaker model of π_v for each place v. We fix a factorization $W_\varphi=\otimes_v W_v$ into local Whittaker functions, $W_v\in\mathcal{W}(\pi_v,\psi_v)$. We also assume that the Schwartz function ϕ is factorizable, i.e. $\phi=\otimes_v\phi_v$ where $\phi_v\in\mathcal{S}(F_v^n)$. Put

$$\begin{aligned}&I_v(s,W_v,\chi_v,\phi_v)\\&=\int_{(N_n\times N_n)(F_v)\backslash(G_n\times G_n)(F_v)}W_v(\iota(h_1,h_2))\phi(e_nh_2)\chi_v(\det h_1h_2^{-1})|\det h_1|^{s-\frac12}|\det h_2|^{s+\frac12}\mathrm{d}h_1\mathrm{d}h_2.\end{aligned}$$

By [Mat15a], this integral is convergent for $\Re s>0$ and in particular at $s=\frac12$. For all s we can choose data so that this integral does not vanish. For a nonarchimedean place v, if ψ_v is unramified, χ_v is unramified, W_v is $G_{2n}(\mathfrak{o}_{F_v})$-fixed, $W_v(1)=1$, $\phi_v=\mathbf{1}_{\mathfrak{o}_{F_v}^n}$, we have

$$I_v(s,\chi_v,W_v,\phi_v)=L(s,\pi_v\otimes\chi_v)L(2s,\pi_v,\wedge^2).$$

Moreover we have

$$I(s,\varphi,\chi,\phi)=\prod_v I_v(s,W_v,\chi_v,\phi_v),$$

when $\Re s>>0$.

Let us compute $I_v(\frac12,W_v,\chi_v,\phi_v)$. We decompose $h_2=pu$ where $p\in P_n$ and $u\in G_n$ is of the form $\begin{pmatrix}1_{n-1}&\\ *&*\end{pmatrix}$. Thus $\phi(e_nh_2)=\phi(e_nu)$ and the measure $|\det h_2|\mathrm{d}h_2$ decomposes as $\mathrm{d}_Rp\mathrm{d}u$ where d_Rp is the right invariant measure and $\mathrm{d}u$ is the additive measure on F_v^n. Here note that if we write $p=\begin{pmatrix}a&v\\&1\end{pmatrix}$, then $\mathrm{d}_Rp=\mathrm{d}a\mathrm{d}v$. By [Mat15b, Corollary 4.18], a linear form on π_v that is $(\iota(G_n\times P_n),\chi_v^{-1})$-invariant is also $(\iota(G_n\times G_n),\chi_v^{-1})$-invariant. It follows that

$$\begin{aligned}I_v(\frac12,W_v,\chi_v,\phi_v)&=\int_{(N_n\times N_n)(F_v)\backslash(G_n\times G_n)(F_v)}W_v(\iota(h_1,h_2))\phi_v(e_nh_2)\chi(\det h_1h_2^{-1})\mathrm{d}h_1\mathrm{d}h_2\\&=\int_{(N_n\times N_n)(F_v)\backslash(G_n\times P_n)(F_v)}W_v(\iota(h_1,pu))\phi_v(e_nu)\chi(\det h_1p^{-1}u^{-1})\mathrm{d}h_1\mathrm{d}_Rp\mathrm{d}u\\&=\widehat{\phi_v}(0)\int_{(N_n\times N_n)(F_v)\backslash(G_n\times P_n)(F_v)}W_v(\iota(h_1,p))\chi(\det h_1p^{-1})\mathrm{d}h_1\mathrm{d}_Rp.\end{aligned}$$

Let us define the local (split) linear periods by

$$P_v(W_v,\chi_v)=\int_{(N_n\times N_n)(F_v)\backslash(G_n\times P_n)(F_v)}W_v(\iota(h_1,p))\chi(\det h_1p^{-1})\mathrm{d}h_1\mathrm{d}_Rp,$$

and its normalized version

$$P_v^\natural(W_v,\chi_v)=\frac{P_v(W_v,\chi_v)}{L(\frac{1}{2},\pi_v\otimes\chi_v)L(1,\pi_v,\wedge^2)}.$$

Then we have

$$P(\varphi,\chi)=n\,\mathrm{vol}(F^\times\backslash\mathbb{A}_F^1)^{-1}L(\frac{1}{2},\pi\otimes\chi)\,\mathrm{Res}_{s=1}L(s,\pi,\wedge^2)\prod_v P_v^\natural(W_v,\chi_v). \tag{A.1}$$

Note that almost all terms equal one in the product.

A.4. **Spherical characters.** With this we can factor the spherical character defined in the main body of the paper. We define the inner product on π and on $\mathcal{W}(\pi_v,\psi_v)$ are given as follows. On π, we define

$$\langle\varphi,\varphi'\rangle=\int_{Z_{2n}(\mathbb{A}_F)G_{2n}(F)\backslash G_{2n}(\mathbb{A}_F)}\varphi(g)\overline{\varphi'(g)}\mathrm{d}g.$$

On $\mathcal{W}(\pi_v,\psi_v)$ we define

$$\langle W_v,W_v'\rangle=\int_{N_{2n}(F_v)\backslash P_{2n}(F_v)}W_v(p)\overline{W_v'(p)}\mathrm{d}_Rp,\quad \langle W_v,W_v'\rangle^\natural=\frac{\langle W_v,W_v'\rangle}{L(1,\pi_v\times\pi_v^\vee)}.$$

Note that if ψ_v is unramified, W_v is $G_{2n}(\mathfrak{o}_{F_v})$-fixed, $W_v(1)=1$, then

$$\langle W_v,W_v\rangle^\natural=1.$$

Then if W_φ is the Whittaker function attached to φ and $W_\varphi=\otimes_v W_v$, then we have

$$\langle\varphi,\varphi'\rangle=\frac{2n\,\mathrm{Res}_{s=1}L(s,\pi\times\pi^\vee)}{\mathrm{vol}(F^\times\backslash\mathbb{A}_F^1)}\prod_v\langle W_v,W_v'\rangle^\natural. \tag{A.2}$$

Recall that (in the present notation), for $f=\otimes f_v\in C_c^\infty(G_{2n}(\mathbb{A}_F))$, we have defined

$$I_\pi(f)=\sum_\varphi P(\pi(f)\varphi,\mathbf{1})\overline{P(\varphi,\eta)},$$

where φ runs over an orthonormal basis of π. we have also defined its local version

$$I_{\pi_v}(f_v)=\sum_{W_v}\frac{P_v(\pi_v(f_v)W_v,\mathbf{1})\overline{P_v(W_v,\eta_v)}}{\langle W_v,W_v\rangle}.$$

where W_v runs over an orthogonal basis of $\mathcal{W}(\pi_v,\psi_v)$. We also have its normalized version

$$I_{\pi_v}^\natural(f_v)=\sum_{W_v}\frac{P_v^\natural(\pi_v(f_v)W_v,\mathbf{1})\overline{P_v^\natural(W_v,\eta_v)}}{\langle W_v,W_v\rangle^\natural},$$

where W_v runs over an orthogonal basis of $\mathcal{W}(\pi_v,\psi_v)$. If E_v/F_v is unramified, ψ_v is unramified, π_v is an unramified representation, $f_v=\mathbf{1}_{\mathrm{GL}_{2n}(\mathfrak{o}_{F_v})}$, then $I_{\pi_v}^\natural(f_v)=1$.

If π does not admits nonzero linear period, then I_π is identically zero. Otherwise I_π is not identically zero and π is self-dual. It then follows from the factorization of linear periods (A.1) and the factorization of inner product (A.2) that

$$I_\pi(f)=\frac{n}{2}\mathrm{vol}(F^\times\backslash\mathbb{A}_F^1)^{-1}\frac{L(\frac{1}{2},\pi)L(\frac{1}{2},\pi\otimes\eta)\,\mathrm{Res}_{s=1}L(s,\pi,\wedge^2)}{L(1,\pi,\mathrm{Sym}^2)}\prod_v I_{\pi_v}^\natural(f_v).$$

References

[AG09] A. Aizenbud and D. Gourevitch, *Generalized Harish-Chandra descent, Gelfand pairs, and an Archimedean analog of Jacquet-Rallis's theorem*, Duke Math. J. **149** (2009), no. 3, 509–567, DOI 10.1215/00127094-2009-044. With an appendix by the authors and Eitan Sayag. MR2553879 ↑1

[BP] R. Beuzart-Plessis, *Plancherel formula for* $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(E)$ *and applications to the Ichino-Ikeda and formal degree conjectures for unitary groups*, available at `https://arxiv.org/abs/1812.00047`. ↑27

[BM] P. Broussous and N. Matringe, *Multiplicity one for pairs of Prasad–Takloo-Bighash type*. To appear in IMRN. ↑1, 18, 24

[Cho] M. Chommaux, *Distinction of the Steinberg representation and a conjecture of Prasad and Takloo-Bighash*, available at `https://arxiv.org/abs/1806.00362`. ↑24

[DKV84] P. Deligne, D. Kazhdan, and M.-F. Vignéras, *Représentations des algèbres centrales simples p-adiques*, Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 1984, pp. 33–117 (French). MR771672 ↑38

[FMW18] B. Feigon, K. Martin, and D. Whitehouse, *Periods and nonvanishing of central L-values for* $\mathrm{GL}(2n)$, Israel J. Math. **225** (2018), no. 1, 223–266, DOI 10.1007/s11856-018-1657-5. MR3805647 ↑19, 20

[GJ72] R. Godement and H. Jacquet, *Zeta functions of simple algebras*, Lecture Notes in Mathematics, Vol. 260, Springer-Verlag, Berlin-New York, 1972. MR0342495 ↑25

[Guo96] J. Guo, *On a generalization of a result of Waldspurger*, Canad. J. Math. **48** (1996), no. 1, 105–142, DOI 10.4153/CJM-1996-005-3. MR1382478 ↑2, 3, 6, 9, 16

[JR96] H. Jacquet and S. Rallis, *Uniqueness of linear periods*, Compositio Math. **102** (1996), no. 1, 65–123. MR1394521 ↑1, 16, 37

[JS81] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic representations. I*, Amer. J. Math. **103** (1981), no. 3, 499–558, DOI 10.2307/2374103. MR618323 ↑39

[LM15a] E. Lapid and Z. Mao, *A conjecture on Whittaker-Fourier coefficients of cusp forms*, J. Number Theory **146** (2015), 448–505, DOI 10.1016/j.jnt.2013.10.003. MR3267120 ↑34

[LM15b] ———, *Model transition for representations of metaplectic type*, Int. Math. Res. Not. IMRN **19** (2015), 9486–9568, DOI 10.1093/imrn/rnu225. With an appendix by Marko Tadić. MR3431601 ↑17, 34

[LM17] ———, *Whittaker-Fourier coefficients of cusp forms on* $\widetilde{\mathrm{Sp}}_n$*: reduction to a local statement*, Amer. J. Math. **139** (2017), no. 1, 1–55, DOI 10.1353/ajm.2017.0000. MR3619910 ↑3, 4, 17, 20

[Mat15a] N. Matringe, *A specialisation of the Bump-Friedberg L-function*, Canad. Math. Bull. **58** (2015), no. 3, 580–595, DOI 10.4153/CMB-2015-014-1. MR3372874 ↑16, 40

[Mat15b] ———, *On the local Bump-Friedberg L-function*, J. Reine Angew. Math. **709** (2015), 119–170, DOI 10.1515/crelle-2013-0083. MR3430877 ↑32, 33, 40

[Mat17] ———, *Shalika periods and parabolic induction for* $GL(n)$ *over a non-archimedean local field*, Bull. Lond. Math. Soc. **49** (2017), no. 3, 417–427, DOI 10.1112/blms.12020. MR3723627 ↑17

[PSP08] D. Prasad and R. Schulze-Pillot, *Generalised form of a conjecture of Jacquet and a local consequence*, J. Reine Angew. Math. **616** (2008), 219–236, DOI 10.1515/CRELLE.2008.023. MR2369492 ↑18

[PTB11] D. Prasad and R. Takloo-Bighash, *Bessel models for GSp(4)*, J. Reine Angew. Math. **655** (2011), 189–243, DOI 10.1515/CRELLE.2011.045. MR2806111 ↑2

[SV17] Y. Sakellaridis and A. Venkatesh, *Periods and harmonic analysis on spherical varieties*, Astérisque **396** (2017), viii+360 (English, with English and French summaries). MR3764130 ↑19, 20

[Suz] M. Suzuki, *Classification standard modules with linear periods*. preprint. ↑2, 23

[RR96] C. Rader and S. Rallis, *Spherical characters on* $\mathfrak{p}$*-adic symmetric spaces*, Amer. J. Math. **118** (1996), no. 1, 91–178. MR1375304 ↑16

[VP89] È. B. Vinberg and V. L. Popov, *Invariant theory*, Algebraic geometry, 4 (Russian), Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989, pp. 137–314, 315 (Russian). MR1100485 ↑16

[Zha15] C. Zhang, *On the smooth transfer for Guo-Jacquet relative trace formulae*, Compos. Math. **151** (2015), no. 10, 1821–1877. MR3414387 ↑3, 14, 15, 16

Department of Mathematics, The University of Arizona, Tucson, AZ 85721

E-mail address: `xuehang@math.arizona.edu`