Linear Quadratic Stochastic Optimal Control Problems with Operator Coefficients: Open-Loop Solutions*

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Abstract: Linear quadratic optimal control problem is considered for linear stochastic differential equations with the coefficients being bounded operators on the spaces of square integrable random variables. The main motivation of our study is linear quadratic optimal control problems for mean-field stochastic differential equations. Open-loop solvability of the problem is investigated. The well-posedness of a relevant system of linear coupled forward-backward stochastic differential equations with operator coefficients is established, which leads to the existence of open-loop optimal control. Finally, as an application of our main results, a general mean-field linear quadratic control problem in the open-loop case is solved.

Keywords: linear stochastic differential equation with operator coefficients, open-loop solvability, forward-backward stochastic differential equations, mean-field linear quadratic control problem.

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a standard one-dimensional Brownian motion $\{W(t), t \geqslant 0\}$ is defined such that $\mathbb{F} = \{\mathcal{F}_t\}_{t\geqslant 0}$ is the natural filtration of $W(\cdot)$ augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Consider the following controlled linear (forward) stochastic differential equation (FSDE, for short) on $[t, T]$:

$$
\begin{cases} dX(s) = \big[\mathcal{A}(s)X(s) + \mathcal{B}(s)u(s) + b(s)\big]ds + \big[\mathcal{C}(s)X(s) + \mathcal{D}(s)u(s) + \sigma(s)\big]dW(s), \quad s \in [t, T], \\ X(t) = x. \end{cases} \tag{1.1}
$$

In the above, $X(\cdot)$ is called the *state process* taking values in the n-dimensional Euclidean space $\mathbb{R}^n$; $u(\cdot)$ is called the *control process* taking values in $\mathbb{R}^m$; (t, x) is called an *initial pair* with $t \in [0, T)$ and square integrable $\mathbb{R}^n$-valued random variable x; $b(\cdot)$ and $\sigma(\cdot)$ are called *non-homogeneous terms.* To explain the coefficients of the system, we first recall the following spaces: For any $t \in [0, T]$,

$$
L^2_{\mathbb{F}}(t, T; \mathbb{R}^n) = \Big\{\varphi : [t, T] \times \Omega \to \mathbb{R}^n \ \big|\ \varphi(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable, } \mathbb{E}\int_t^T |\varphi(s)|^2 ds < \infty\Big\},
$$
$$
L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n) = \big\{\xi : \Omega \to \mathbb{R}^n \ \big|\ \xi \text{ is } \mathcal{F}_t\text{-measurable and } \mathbb{E}|\xi|^2 < \infty\ \big\}, \qquad L^2(\Omega; \mathbb{R}^n) = L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n).
$$

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For any Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, we let $\mathscr{L}(\mathbb{X};\mathbb{Y})$ be the set of all linear bounded operators from $\mathbb{X}$ to $\mathbb{Y}$, and denote $\mathscr{L}(\mathbb{X};\mathbb{X}) = \mathscr{L}(\mathbb{X})$. Also, when $\mathbb{X}$ is a Hilbert space, we let $\mathscr{S}(\mathbb{X})$ be the set of all bounded self-adjoint operators on $\mathbb{X}$. In the state equation (1.1), we assume that

$$\mathcal{A}(s), \mathcal{C}(s) \in \mathscr{L}\Big(L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n)\Big), \quad \mathcal{B}(s), \mathcal{D}(s) \in \mathscr{L}\Big(L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m); L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n)\Big), \qquad \forall s \in [0,T], \tag{1.2}$$

with certain additional conditions. In what follows, the set of all initial pairs is denoted by

$$\mathscr{D} = \Big\{(t,x) \mid t \in [0,T],\ x \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)\Big\},$$

and the set of all *admissible controls* is denoted by $\mathcal{U}[t,T] = L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$.

One can show that under certain conditions, for any initial pair $(t,x) \in \mathscr{D}$ and control $u(\cdot) \in \mathcal{U}[t,T]$, the state equation (1.1) admits a unique strong solution $X(\cdot) \equiv X(\cdot\,; t,x,u(\cdot))$. The performance of the control process is measured by the following cost functional:

$$\begin{aligned} J(t,x;u(\cdot)) = \mathbb{E}\Big[\langle \mathcal{G}X(T), X(T)\rangle + 2\langle g, X(T)\rangle + \int_t^T \Big(\langle \mathcal{Q}(s)X(s), X(s)\rangle + 2\langle \mathcal{S}(s)X(s), u(s)\rangle \\ + \langle \mathcal{R}(s)u(s), u(s)\rangle + 2\langle q(s), X(s)\rangle + 2\langle \rho(s), u(s)\rangle\Big) ds\Big], \end{aligned} \tag{1.3}$$

where $g \in L^2(\Omega;\mathbb{R}^n)$, $q(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, $\rho(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$, and

$$\begin{aligned} &\mathcal{G} \in \mathscr{S}\Big(L^2(\Omega;\mathbb{R}^n)\Big), \qquad \mathcal{Q}(s) \in \mathscr{S}\Big(L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n)\Big), \\ &\mathcal{S}(s) \in \mathscr{L}\Big(L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n); L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m)\Big), \qquad \mathcal{R}(s) \in \mathscr{S}\Big(L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m)\Big), \qquad \forall s \in [0,T], \end{aligned} \tag{1.4}$$

with certain additional conditions.

Our optimal control problem can be stated as follows.

Problem (OLQ). For given $(t,x) \in \mathscr{D}$, find a $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ such that

$$J(t,x;\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x;u(\cdot)). \tag{1.5}$$

The above Problem (OLQ) clearly includes the classical stochastic linear quadratic (LQ, for short) optimal control for which all the coefficients and quadratic weighting operators in the cost functional are matrix-valued processes ([34, 29, 30]). On the other hand, by allowing the coefficients of the state equation and the quadratic weighting operators in the cost functional to be linear bounded operators between Hilbert spaces of square integrable random variables, our problem will cover stochastic LQ optimal control problem for mean-field FSDEs with cost functionals also involving mean-field terms (which is referred to as MF-LQ problems). In [33], for a simple MF-LQ problem (with deterministic coefficients), under proper conditions, optimal control is obtained via the solution to a system of Riccati equations. See [11, 17] for some follow-up works.

For classical LQ optimal control problem and two-person differential games (with deterministic coefficients), in [26] (see also, [27, 28]) open-loop and closed-loop solvabilities/saddle points were introduced, and the following interesting equivalent relations were established for LQ optimal control problems with deterministic coefficients: The open-loop solvability of the LQ problem is equivalent to the solvability of a forward-backward stochastic differential equation (FBSDE, for short), and the closed-loop solvability of the LQ problem is equivalent to the solvability of the corresponding Riccati equation. For two-person differential games, similar results are also valid. In the current paper, we focus on the open-loop solvability of our Problem (OLQ). The studies of closed-loop case and differential game problems will be carried out in our future publications. For the solvability of FBSDEs or Riccati equations arising in the classical LQ stochastic optimal control problems and stochastic differential game problems, one is referred to [22, 8, 36, 38]. We may regard the current work as a continuation of [33, 11, 17] and [26, 27, 28].

Due to the appearance of the operator coefficients in the state equation and the cost functional, some new methods and techniques need to be developed. Actually, our results on the FSDEs and BSDEs with operator coefficients are of independent interests themselves. Based on these preparations, as expected, we will establish the equivalence between the open-loop solvability of Problem (OLQ) and the well-posedness of a coupled FBSDE with operator coefficients. Also, the well-posedness of the revelent FBSDE with operator coefficients is established under some conditions so that Problem (OLQ) is solved. Aa an application of our general abstract results, we present the solution to the mean-field LQ problem (which is a major motivation of the current work). An explicit open-loop optimal control for the MF-LQ control problem is derived.

The rest of this paper is organized as follows. Some motivations of Problem (OLQ) are carefully presented in Section 2. We also develop some general results for FSDEs and BSDEs with operator coefficients. Section 3 is concerned with Problem (OLQ). Open-loop optimal controls are characterized, and the solvability of the revelent coupled FBSDEs with operator coefficients is established by method of continuation. In Section 4, an MF-LQ optimal control problem is worked out.

2 Preliminaries

2.1 Motivations

In this subsection, we look at some motivations of our Problem (OLQ). First of all, for the state equation (1.1), let us look at some special cases.

- The classical linear SDE:

$$\begin{cases} \mathcal{A}(s)\xi = A(s)\xi, \quad \mathcal{C}(s) = C(s)\xi, \qquad \forall \xi \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n), \\ \mathcal{B}(s)\eta = B(s)\eta, \quad \mathcal{D}(s) = D(s)\eta, \qquad \forall \eta \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m), \end{cases}$$

with $A(\cdot), C(\cdot), B(\cdot), D(\cdot)$ being some matrix-valued processes.

- The case of simple mean-field SDE (MF-SDE):

$$\begin{cases} \mathcal{A}(s)\xi = A(s)\xi + \bar{A}(s)\mathbb{E}[\widetilde{A}(s)\xi], \quad \mathcal{C}(s)\xi = C(s)\xi + \bar{C}(s)\mathbb{E}[\widetilde{C}(s)\xi], \quad \forall \xi \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n), \\ \mathcal{B}(s)\eta = B(s)\eta + \bar{B}(s)\mathbb{E}[\widetilde{B}(s)\eta], \quad \mathcal{D}(s)\eta = D(s)\eta + \bar{D}(s)\mathbb{E}[\widetilde{D}(s)\eta], \quad \forall \eta \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m), \end{cases}$$

for some matrix-valued processes $A(\cdot), \bar{A}(\cdot), \widetilde{A}(\cdot)$, etc. In the case that all the coefficients are deterministic (as in [33, 11, 17]), the above will be reduced to the following simpler form:

$$\begin{cases} \mathcal{A}(s)\xi = A(s)\xi + \bar{A}(s)\mathbb{E}[\xi], \quad \mathcal{C}(s)\xi = C(s)\xi + \bar{C}(s)\mathbb{E}[\xi], \quad \forall \xi \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n), \\ \mathcal{B}(s)\eta = B(s)\eta + \bar{B}(s)\mathbb{E}[\eta], \quad \mathcal{D}(s)\eta = D(s)\eta + \bar{D}(s)\mathbb{E}[\eta], \quad \forall \eta \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m), \end{cases}$$

for some matrix-valued deterministic functions $A(\cdot), \bar{A}(\cdot)$, etc.

- The extended MF-SDE:

$$\begin{cases} \mathcal{A}(s)\xi = A(s)\xi + \int_{\mathbb{R}} \bar{A}_\lambda(s)\mathbb{E}\big[\widetilde{A}_\lambda(s)\xi\big]\mu(d\lambda), \\ \mathcal{C}(s)\xi = C(s)\xi + \int_{\mathbb{R}} \bar{C}_\lambda(s)\mathbb{E}\big[\widetilde{C}_\lambda(s)\xi\big]\mu(d\lambda), \end{cases} \quad \forall \xi \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n),$$
$$\begin{cases} \mathcal{B}(s)\eta = B(s)\eta + \int_{\mathbb{R}} \bar{B}_\lambda(s)\mathbb{E}\big[\widetilde{B}_\lambda(s)\eta\big]\mu(d\lambda), \\ \mathcal{D}(s)\eta = D(s)\eta + \int_{\mathbb{R}} \bar{D}_\lambda(s)\mathbb{E}\big[\widetilde{D}_\lambda(s)\eta\big]\mu(d\lambda), \end{cases} \quad \forall \eta \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m), \qquad (2.1)$$

with $A(\cdot), C(\cdot), B(\cdot), D(\cdot)$ being matrix-valued processes and $\bar{A}_\lambda(\cdot), \widetilde{A}_\lambda(\cdot)$, etc. being families of matrix-valued processes parameterized by $\lambda \in \mathbb{R}$, and $\mu(\cdot)$ being a Borel measure on $\mathbb{R}$. A special case of the above is the following (with $\mu(\cdot)$ supported at $\{1, 2, 3, \cdots\}$):

$$
\left\{
\begin{aligned}
&\mathcal{A}(s)\xi = A(s)\xi + \sum_{k\geqslant 1} \bar{A}_k(s)\mathbb{E}\big[\widetilde{A}_k(s)\xi\big] \equiv A(s)\xi + \bar{\mathbf{A}}(s)^\top \mathbb{E}\big[\widetilde{\mathbf{A}}(s)\xi\big], && \\
&\mathcal{C}(s)\xi = C(s)\xi + \sum_{k\geqslant 1} \bar{C}_k(s)\mathbb{E}\big[\widetilde{C}_k(s)\xi\big] \equiv C(s)\xi + \bar{\mathbf{C}}(s)^\top \mathbb{E}\big[\widetilde{\mathbf{C}}(s)\xi\big], && \forall \xi \in L^2_{\mathcal{F}_s}(\Omega; \mathbb{R}^n), \\
&\mathcal{B}(s)\eta = B(s)\eta + \sum_{k\geqslant 1} \bar{B}_k(s)\mathbb{E}\big[\widetilde{B}_k(s)\eta\big] \equiv B(s)\eta + \bar{\mathbf{B}}(s)^\top \mathbb{E}\big[\widetilde{\mathbf{B}}(s)\eta\big], && \\
&\mathcal{D}(s)\eta = D(s)\eta + \sum_{k\geqslant 1} \bar{D}_k(s)\mathbb{E}\big[\widetilde{D}_k(s)\eta\big] \equiv D(s)\eta + \bar{\mathbf{D}}(s)^\top \mathbb{E}\big[\widetilde{\mathbf{D}}(s)\eta\big], && \forall \eta \in L^2_{\mathcal{F}_s}(\Omega; \mathbb{R}^m),
\end{aligned}
\right.
\tag{2.2}
$$

with

$$
\bar{\mathbf{A}}(s)^\top = (\bar{A}_1(s), \bar{A}_2(s), \cdots), \quad \widetilde{\mathbf{A}}(s)^\top = (\widetilde{A}_1(s)^\top, \widetilde{A}_2(s)^\top, \cdots),
$$

etc., for some matrix-valued processes $\bar{A}_k(\cdot), \widetilde{A}_k(\cdot)$, etc.

From the above, we see that by allowing $\mathcal{A}(\cdot), \mathcal{C}(\cdot), \mathcal{B}(\cdot), \mathcal{D}(\cdot)$ to be operator-valued processes (not just matrix-valued processes), our state equation can cover a very big class of stochastic linear systems.

Next, we look at the cost functional. To get some feeling about the operators in the cost functional, let us look at the case of (2.2). Let

$$
\widetilde{\mathbf{X}}(T) = \begin{pmatrix} X(T) \\ \mathbb{E}[\widetilde{\mathbf{G}}X(T)] \end{pmatrix}, \qquad \mathbf{X}(s) = \begin{pmatrix} X(s) \\ \mathbb{E}[\widetilde{\mathbf{Q}}(s)X(s)] \end{pmatrix}, \qquad \mathbf{u}(s) = \begin{pmatrix} u(s) \\ \mathbb{E}[\widetilde{\mathbf{R}}(s)u(s)] \end{pmatrix},
$$

where

$$
\widetilde{\mathbf{G}} = \begin{pmatrix} \widetilde{G}_1 \\ \widetilde{G}_2 \\ \vdots \end{pmatrix}, \qquad \widetilde{\mathbf{Q}}(s) = \begin{pmatrix} \widetilde{Q}_1(s) \\ \widetilde{Q}_2(s) \\ \vdots \end{pmatrix}, \qquad \widetilde{\mathbf{R}}(s) = \begin{pmatrix} \widetilde{R}_1(s) \\ \widetilde{R}_2(s) \\ \vdots \end{pmatrix}.
$$

Then the quadratic term in the terminal cost could look like the following:

$$
\begin{aligned}
&\mathbb{E}\langle \mathbf{G}\widetilde{\mathbf{X}}(T), \widetilde{\mathbf{X}}(T)\rangle = \mathbb{E}\left\langle \begin{pmatrix} G & \bar{\mathbf{G}}^\top \\ \bar{\mathbf{G}} & \widehat{\mathbf{G}} \end{pmatrix} \begin{pmatrix} X(T) \\ \mathbb{E}[\widetilde{\mathbf{G}}X(T)] \end{pmatrix}, \begin{pmatrix} X(T) \\ \mathbb{E}[\widetilde{\mathbf{G}}X(T)] \end{pmatrix} \right\rangle \\
&= \mathbb{E}\Big[\langle GX(T), X(T)\rangle + 2\langle \bar{\mathbf{G}}X(T), \mathbb{E}[\widetilde{\mathbf{G}}X(T)]\rangle + \langle \widehat{\mathbf{G}}\mathbb{E}[\widetilde{\mathbf{G}}X(T)], \mathbb{E}[\widetilde{\mathbf{G}}X(T)]\rangle\Big] \\
&= \mathbb{E}\langle GX(T) + \widetilde{\mathbf{G}}^\top \mathbb{E}[\bar{\mathbf{G}}X(T)] + \bar{\mathbf{G}}^\top \mathbb{E}[\widetilde{\mathbf{G}}X(T)] + \widetilde{\mathbf{G}}^\top \mathbb{E}[\widehat{\mathbf{G}}]\mathbb{E}[\widetilde{\mathbf{G}}X(T)], X(T)\rangle \\
&\equiv \mathbb{E}\langle \mathcal{G}X(T), X(T)\rangle,
\end{aligned}
$$

with

$$
G^\top = G, \qquad \bar{\mathbf{G}} = \begin{pmatrix} G_{10} \\ G_{20} \\ \vdots \end{pmatrix}, \qquad \widehat{\mathbf{G}} = \begin{pmatrix} G_{11} & G_{12} & \cdots \\ G_{21} & G_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \qquad \widehat{\mathbf{G}}^\top = \widehat{\mathbf{G}}.
$$

The quadratic terms in the integrand of the running cost should look like the following:

$$
\begin{aligned}
&\mathbb{E}\langle \mathbf{Q}(s)\mathbf{X}(s), \mathbf{X}(s)\rangle = \mathbb{E}\left\langle \begin{pmatrix} Q(s) & \bar{\mathbf{Q}}(s)^\top \\ \bar{\mathbf{Q}}(s) & \widehat{\mathbf{Q}}(s) \end{pmatrix} \begin{pmatrix} X(s) \\ \mathbb{E}[\widetilde{\mathbf{Q}}(s)X(s)] \end{pmatrix}, \begin{pmatrix} X(s) \\ \mathbb{E}[\widetilde{\mathbf{Q}}(s)X(s)] \end{pmatrix} \right\rangle \\
&= \mathbb{E}\Big[\langle Q(s)X(s), X(s)\rangle + 2\langle \bar{\mathbf{Q}}(s)X(s), \mathbb{E}[\widetilde{\mathbf{Q}}(s)X(s)]\rangle + \langle \widehat{\mathbf{Q}}(s)\mathbb{E}[\widetilde{\mathbf{Q}}(s)X(s)], \mathbb{E}[\widetilde{\mathbf{Q}}(s)X(s)]\rangle\Big] \\
&= \mathbb{E}\langle Q(s)X(s) + \widetilde{\mathbf{Q}}(s)^\top \mathbb{E}[\bar{\mathbf{Q}}(s)X(s)] + \bar{\mathbf{Q}}(s)^\top \mathbb{E}[\widetilde{\mathbf{Q}}(s)X(s)] + \widetilde{\mathbf{Q}}(s)^\top \mathbb{E}[\widehat{\mathbf{Q}}(s)]\mathbb{E}[\widetilde{\mathbf{Q}}(s)X(s)], X(s)\rangle \\
&\equiv \mathbb{E}\langle \mathcal{Q}(s)X(s), X(s)\rangle,
\end{aligned}
$$

with

$$Q(s)^\top = Q(s), \quad \bar{\mathbf{Q}}(s) = \begin{pmatrix} Q_{10}(s) \\ Q_{20}(s) \\ \vdots \end{pmatrix}, \quad \widehat{\mathbf{Q}}(s) = \begin{pmatrix} Q_{11}(s) & Q_{12}(s) & \cdots \\ Q_{21}(s) & Q_{22}(s) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad \widehat{\mathbf{Q}}(s)^\top = \widehat{\mathbf{Q}}(s),$$

and

$$\begin{aligned}
&\mathbb{E}\langle \mathbf{R}(s)\mathbf{u}(s), \mathbf{u}(s)\rangle = \mathbb{E}\left\langle \begin{pmatrix} R(s) & \bar{\mathbf{R}}(s)^\top \\ \bar{\mathbf{R}}(s) & \widehat{\mathbf{R}}(s) \end{pmatrix} \begin{pmatrix} u(s) \\ \mathbb{E}[\widetilde{\mathbf{R}}(s)u(s)] \end{pmatrix}, \begin{pmatrix} u(s) \\ \mathbb{E}[\widetilde{\mathbf{R}}(s)u(s)] \end{pmatrix} \right\rangle \\
&= \mathbb{E}\Big[\langle R(s)u(s), u(s)\rangle + 2\langle \bar{\mathbf{R}}(s)u(s), \mathbb{E}[\widetilde{\mathbf{R}}(s)u(s)]\rangle + \langle \widehat{\mathbf{R}}(s)\mathbb{E}[\widetilde{\mathbf{R}}(s)u(s)], \mathbb{E}[\widetilde{\mathbf{R}}(s)u(s)]\rangle\Big] \\
&= \mathbb{E}\langle R(s)u(s) + \widetilde{\mathbf{R}}(s)^\top \mathbb{E}[\bar{\mathbf{R}}(s)u(s)] + \bar{\mathbf{R}}(s)^\top \mathbb{E}[\widetilde{\mathbf{R}}(s)u(s)] + \widetilde{\mathbf{R}}(s)^\top \mathbb{E}[\widehat{\mathbf{R}}(s)]\mathbb{E}[\widetilde{\mathbf{R}}(s)u(s)], u(s)\rangle \\
&\equiv \mathbb{E}\langle \mathcal{R}(s)u(s), u(s)\rangle,
\end{aligned}$$

with

$$R(s)^\top = R(s), \quad \bar{\mathbf{R}}(s) = \begin{pmatrix} R_{10}(s) \\ R_{20}(s) \\ \vdots \end{pmatrix}, \quad \widehat{\mathbf{R}}(s) = \begin{pmatrix} R_{11}(s) & R_{12}(s) & \cdots \\ R_{21}(s) & R_{22}(s) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad \widehat{\mathbf{R}}(s)^\top = \widehat{\mathbf{R}}(s).$$

Also,

$$\begin{aligned}
&\mathbb{E}\langle \mathbf{S}(s)\mathbf{X}(s), \mathbf{u}(s)\rangle = \mathbb{E}\left\langle \begin{pmatrix} S(s) & \widetilde{\mathbf{S}}(s) \\ \bar{\mathbf{S}}(s) & \widehat{\mathbf{S}}(s) \end{pmatrix} \begin{pmatrix} X(s) \\ \mathbb{E}[\widetilde{\mathbf{Q}}(s)X(s)] \end{pmatrix}, \begin{pmatrix} u(s) \\ \mathbb{E}[\widetilde{\mathbf{R}}(s)u(s)] \end{pmatrix} \right\rangle \\
&= \mathbb{E}\Big[\langle S(s)X(s), u(s)\rangle + \langle \bar{\mathbf{S}}(s)X(s), \mathbb{E}[\widetilde{\mathbf{R}}(s)u(s)]\rangle \\
&\qquad + \langle \widetilde{\mathbf{S}}(s)\mathbb{E}[\widetilde{\mathbf{Q}}(s)X(s)], u(s)\rangle + \langle \widehat{\mathbf{S}}(s)\mathbb{E}[\widetilde{\mathbf{Q}}(s)X(s)], \mathbb{E}[\widetilde{\mathbf{R}}(s)u(s)]\rangle\Big] \\
&= \mathbb{E}\langle S(s)X(s) + \widetilde{\mathbf{R}}(s)^\top \mathbb{E}[\bar{\mathbf{S}}(s)X(s)] + \widetilde{\mathbf{S}}(s)\mathbb{E}[\widetilde{\mathbf{Q}}(s)X(s)] + \widetilde{\mathbf{R}}(s)^\top \mathbb{E}[\widehat{\mathbf{S}}(s)]\mathbb{E}[\widetilde{\mathbf{Q}}(s)X(s)], u(s)\rangle \\
&\equiv \mathbb{E}\langle \mathcal{S}(s)X(s), u(s)\rangle,
\end{aligned}$$

with

$$\bar{\mathbf{S}}(s) = \begin{pmatrix} S_{10}(s) \\ S_{20}(s) \\ \vdots \end{pmatrix}, \qquad \widetilde{\mathbf{S}}(s) = \begin{pmatrix} S_{01}(s) \\ S_{02}(s) \\ \vdots \end{pmatrix}^\top, \qquad \widehat{\mathbf{S}}(s) = \begin{pmatrix} S_{11}(s) & S_{12}(s) & \cdots \\ S_{21}(s) & S_{22}(s) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

The linear terms in the cost functional could look like the following:

$$\begin{aligned}
&\mathbb{E}\langle \mathbf{g}, \widetilde{\mathbf{X}}(T)\rangle = \mathbb{E}\left\langle \begin{pmatrix} g_0 \\ \bar{\mathbf{g}} \end{pmatrix}, \begin{pmatrix} X(T) \\ \mathbb{E}[\widetilde{\mathbf{G}}X(T)] \end{pmatrix} \right\rangle = \mathbb{E}\langle g_0 + \widetilde{\mathbf{G}}^\top \mathbb{E}[\bar{\mathbf{g}}], X(T)\rangle \equiv \mathbb{E}\langle g, X(T)\rangle, \\
&\mathbb{E}\langle \mathbf{q}(s), \mathbf{X}(s)\rangle = \mathbb{E}\left\langle \begin{pmatrix} q_0(s) \\ \bar{\mathbf{q}}(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ \mathbb{E}[\widetilde{\mathbf{Q}}(s)X(s)] \end{pmatrix} \right\rangle = \mathbb{E}\langle q_0(s) + \widetilde{\mathbf{Q}}(s)^\top \mathbb{E}[\bar{\mathbf{q}}(s)], X(s)\rangle \equiv \mathbb{E}\langle q(s), X(s)\rangle, \\
&\mathbb{E}\langle \boldsymbol{\rho}(s), \mathbf{u}(s)\rangle = \mathbb{E}\left\langle \begin{pmatrix} \rho_0(s) \\ \bar{\boldsymbol{\rho}}(s) \end{pmatrix}, \begin{pmatrix} u(s) \\ \mathbb{E}[\widetilde{\mathbf{R}}(s)u(s)] \end{pmatrix} \right\rangle = \mathbb{E}\langle \rho_0(s) + \widetilde{\mathbf{R}}(s)^\top \mathbb{E}[\bar{\boldsymbol{\rho}}(s)], u(s)\rangle \equiv \mathbb{E}\langle \rho(s), u(s)\rangle,
\end{aligned}$$

with $\widetilde{\mathbf{G}}$, $\widetilde{\mathbf{Q}}(\cdot)$ and $\widetilde{\mathbf{R}}(\cdot)$ as above, and

$$\bar{\mathbf{g}} = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \end{pmatrix}, \qquad \bar{\mathbf{q}}(s) = \begin{pmatrix} q_1(s) \\ q_2(s) \\ \vdots \end{pmatrix}, \qquad \bar{\boldsymbol{\rho}}(s) = \begin{pmatrix} \rho_1(s) \\ \rho_2(s) \\ \vdots \end{pmatrix}.$$

Hence, in the above case, we have

$$
(2.3)\quad \begin{cases}
\mathcal{G}\xi = G\xi + \widetilde{\mathbf{G}}^\top \mathbb{E}[\bar{\mathbf{G}}\xi] + \bar{\mathbf{G}}^\top \mathbb{E}[\widetilde{\mathbf{G}}\xi] + \widetilde{\mathbf{G}}^\top \mathbb{E}[\widehat{\mathbf{G}}]\mathbb{E}[\widetilde{\mathbf{G}}\xi], \quad \xi \in L^2(\Omega;\mathbb{R}^n),\\
\mathcal{Q}(s)\xi = Q(s)\xi + \widetilde{\mathbf{Q}}(s)^\top \mathbb{E}[\bar{\mathbf{Q}}(s)\xi] + \bar{\mathbf{Q}}(s)^\top \mathbb{E}[\widetilde{\mathbf{Q}}(s)\xi] + \widetilde{\mathbf{Q}}(s)^\top \mathbb{E}[\widehat{\mathbf{Q}}(s)]\mathbb{E}[\widetilde{\mathbf{Q}}(s)\xi], \quad \xi \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n),\\
\mathcal{S}(s)\xi = S(s)\xi + \widetilde{\mathbf{R}}(s)^\top \mathbb{E}[\bar{\mathbf{S}}(s)\xi] + \widetilde{\mathbf{S}}(s)\mathbb{E}[\widetilde{\mathbf{Q}}(s)\xi] + \widetilde{\mathbf{R}}(s)^\top \mathbb{E}[\widehat{\mathbf{S}}(s)]\mathbb{E}[\widetilde{\mathbf{Q}}(s)\xi], \quad \xi \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n),\\
\mathcal{R}(s)\eta = R(s)\eta + \widetilde{\mathbf{R}}(s)^\top \mathbb{E}[\bar{\mathbf{R}}(s)\eta] + \bar{\mathbf{R}}(s)^\top \mathbb{E}[\widetilde{\mathbf{R}}(s)\eta] + \widetilde{\mathbf{R}}(s)^\top \mathbb{E}[\widehat{\mathbf{R}}(s)]\mathbb{E}[\widetilde{\mathbf{R}}(s)\eta], \quad \eta \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m),\\
g = g_0 + \widetilde{\mathbf{G}}^\top \mathbb{E}[\bar{\mathbf{g}}], \quad q(s) = q_0(s) + \widetilde{\mathbf{Q}}(s)^\top \mathbb{E}[\bar{\mathbf{q}}(s)], \quad \rho(s) = \rho_0(s) + \widetilde{\mathbf{R}}(s)^\top \mathbb{E}[\bar{\boldsymbol{\rho}}(s)].
\end{cases}
$$

When all the weighting functions in the cost functional are deterministic, the above will be reduced to the following: (see [33])

$$
\begin{cases}
\mathcal{G}\xi = G\xi + \bar{G}\mathbb{E}[\xi], \quad \xi \in L^2(\Omega;\mathbb{R}^n),\\
\mathcal{Q}(s)\xi = Q(s)\xi + \bar{Q}(s)\mathbb{E}[\xi], \quad \mathcal{S}(s)\xi = S(s)\xi + \bar{S}(s)\mathbb{E}[\xi], \qquad \xi \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n),\\
\mathcal{R}(s)\eta = R(s)\eta + \bar{R}(s)\mathbb{E}[\eta], \quad \eta \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m).
\end{cases}
$$

The above suggests that if the coefficients of the state equation are given by (2.1), the corresponding operators in the cost functional could look like the following:

$$
(2.4)\quad \begin{cases}
\mathcal{G}\xi = G\xi + \displaystyle\int \Big(\widetilde{G}_\lambda^\top \mathbb{E}[\bar{G}_\lambda\xi] + \bar{G}_\lambda^\top \mathbb{E}[\widetilde{G}_\lambda\xi]\Big)\mu(d\lambda) + \iint \widetilde{G}_\lambda^\top \widehat{G}_{\lambda\nu}\mathbb{E}[\widetilde{G}_\nu\xi]\mu(d\lambda)\mu(d\nu), \quad \xi \in L^2(\Omega;\mathbb{R}^n),\\
\mathcal{Q}(s)\xi = Q(s)\xi + \displaystyle\int \Big(\widetilde{Q}_\lambda(s)^\top \mathbb{E}[\bar{Q}_\lambda(s)\xi] + \bar{Q}_\lambda(s)^\top \mathbb{E}[\widetilde{Q}_\lambda(s)\xi]\Big)\mu(d\lambda)\\
\qquad\qquad + \displaystyle\iint \widetilde{Q}_\lambda(s)^\top \widehat{Q}_{\lambda\nu}(s)\mathbb{E}[\widetilde{Q}_\nu(s)\xi]\mu(d\lambda)\mu(d\nu), \qquad \xi \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n),\\
\mathcal{S}(s)\xi = S(s)\xi + \displaystyle\int \Big(\widetilde{R}_\lambda(s)^\top \mathbb{E}[\bar{S}_\lambda(s)\xi] + \widetilde{S}_\lambda(s)\mathbb{E}[\widetilde{Q}(s)\xi]\Big)\mu(d\lambda)\\
\qquad\qquad + \displaystyle\iint \widetilde{R}_\lambda(s)^\top \widehat{S}_{\lambda\nu}(s)\mathbb{E}[\widetilde{Q}_\nu(s)\xi]\mu(d\lambda)\mu(d\nu), \qquad \xi \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n),\\
\mathcal{R}(s)\eta = R(s)\eta + \displaystyle\int \Big(\widetilde{R}_\lambda(s)^\top \mathbb{E}[\bar{R}_\lambda(s)\eta] + \bar{R}_\lambda(s)^\top \mathbb{E}[\widetilde{R}_\lambda(s)\eta]\Big)\mu(d\lambda)\\
\qquad\qquad + \displaystyle\iint \widetilde{R}_\lambda(s)^\top \widehat{R}_{\lambda\nu}(s)\mathbb{E}[\widetilde{R}_\nu(s)\eta]\mu(d\lambda)\mu(d\nu), \qquad \eta \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m).
\end{cases}
$$

In the above, $\widehat{G}_{\lambda\nu}, \widehat{Q}_{\lambda\nu}(\cdot), \widehat{S}_{\lambda\nu}(\cdot), \widehat{R}_{\lambda\nu}(\cdot)$ are deterministic, and

$$
(2.5)\qquad \widehat{G}_{\lambda\nu}^\top = \widehat{G}_{\nu\lambda}, \qquad \widehat{Q}_{\lambda\nu}(s)^\top = \widehat{Q}_{\nu\lambda}(s), \qquad \widehat{R}_{\lambda\nu}(s)^\top = \widehat{R}_{\nu\lambda}(s), \qquad \forall \lambda,\nu \in \mathbb{R},\ s \in [0,T].
$$

The above shows that our framework can cover many problems involving mean fields.

2.2 The state equation and the cost functional

We return to our state equation (1.1) and cost functional (1.3). Recall the spaces

$$
L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n) \subseteq L^2(\Omega;\mathbb{R}^n) \equiv L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n), \qquad s \in [0,T],
$$

each of which is a Hilbert space with the norm defined by the following:

$$
\|\xi\|_2 = \Big(\mathbb{E}|\xi|^2\Big)^{\frac{1}{2}}, \qquad \forall \xi \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n).
$$

Next, we introduce the following space:

$$
L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n)) = \Big\{\varphi : [t,T]\times\Omega \to \mathbb{R}^n \ \Big|\ \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted}, s \mapsto \varphi(s,\omega) \text{ is continuous almost surely}, \mathbb{E}\Big[\sup_{s\in[t,T]} |\varphi(s)|^2\Big] < \infty\Big\}.
$$

Now, we introduce the following definition.

Definition 2.1. An operator-valued process $\mathcal{B}:[0,T]\to\mathscr{L}\Big(L^2(\Omega;\mathbb{R}^m);L^2(\Omega;\mathbb{R}^n)\Big)$ is said to be *strongly* $\mathbb{F}$*-progressively measurable* if

$$\mathcal{B}(s)\in\mathscr{L}\Big(L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m);L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n)\Big),\qquad\forall s\in[0,T],$$

and for any $\eta(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$, $\mathcal{B}(\cdot)\eta(\cdot)$ is $\mathbb{F}$-progressively measurable. The set of all strongly $\mathbb{F}$-progressively measurable operator-valued processes in $\mathscr{L}\Big(L^2(\Omega;\mathbb{R}^m);L^2(\Omega;\mathbb{R}^n)\Big)$ is denoted by $\mathscr{L}_{\mathbb{F}}\Big(L^2(\Omega;\mathbb{R}^m);L^2(\Omega;\mathbb{R}^n)\Big)$, and denote

$$\mathscr{L}_{\mathbb{F}}\Big(L^2(\Omega;\mathbb{R}^n);L^2(\Omega;\mathbb{R}^n)\Big)=\mathscr{L}_{\mathbb{F}}\Big(L^2(\Omega;\mathbb{R}^n)\Big).$$

Further, for any $p\in[1,\infty]$, we denote

$$\mathscr{L}^p_{\mathbb{F}}\Big(L^2(\Omega;\mathbb{R}^m);L^2(\Omega;\mathbb{R}^n)\Big)=\Big\{\mathcal{B}(\cdot)\in\mathscr{L}_{\mathbb{F}}\Big(L^2(\Omega;\mathbb{R}^m);L^2(\Omega;\mathbb{R}^n)\Big)\mid\|\mathcal{B}(\cdot)\|\in L^p(0,T;\mathbb{R})\Big\},$$

where $\|\mathcal{B}(\cdot)\|$ stands for the norm of the operator $\mathcal{B}(\cdot)$, and

$$\mathscr{L}^p_{\mathbb{F}}\Big(L^2(\Omega;\mathbb{R}^n);L^2(\Omega;\mathbb{R}^n)\Big)=\mathscr{L}^p_{\mathbb{F}}\Big(L^2(\Omega;\mathbb{R}^n)\Big).$$

Also, we let

$$\mathscr{S}^p_{\mathbb{F}}\Big(L^2(\Omega;\mathbb{R}^n)\Big)=\Big\{\mathcal{B}(\cdot)\in\mathscr{L}^p_{\mathbb{F}}\Big(L^2(\Omega;\mathbb{R}^n)\Big)\mid\mathcal{B}(s)\in\mathscr{S}\Big(L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n)\Big),\ \forall\ s\in[0,T]\Big\}.$$

Strong measurability for operator-valued functions can be found in [35]. Our operator-valued processes have an additional feature of $\mathbb{F}$-adaptiveness. Therefore, the above definition is necessary.

Now, we introduce the following linear FSDE with operator-valued coefficients:

$$\left\{\begin{array}{l}dX(s)=[\mathcal{A}(s)X(s)+\varphi(s)]ds+[\mathcal{C}(s)X(s)+\psi(s)]dW(s),\quad s\in[t,T],\\X(t)=x.\end{array}\right.\tag{2.6}$$

We have the following well-posedness of FSDE (2.6).

Proposition 2.2. *Let $\mathcal{A}(\cdot),\mathcal{C}(\cdot)\in\mathscr{L}^2_{\mathbb{F}}\Big(L^2(\Omega;\mathbb{R}^n)\Big)$ and $\varphi(\cdot),\psi(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$. Then for any $(t,x)\in\mathscr{D}$, there exists a unique solution $X(\cdot)\equiv X(\cdot\,;t,x)\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))$ to (2.6) and the following estimate holds:*

$$\mathbb{E}\Big[\sup_{s\in[t,T]}|X(s)|^2\Big]\leqslant K\mathbb{E}\Big[|x|^2+\int_t^T\Big(|\varphi(r)|^2+|\psi(r)|^2\Big)dr\Big],\tag{2.7}$$

for some constant $K>0$ depending on $\|\mathcal{A}(\cdot)\|_{L^2(t,T;\mathbb{R})}$ and $\|\mathcal{C}(\cdot)\|_{L^2(t,T;\mathbb{R})}$.

Proof. Let $(t,x)\in\mathscr{D}$ be fixed. For any $\bar X(\cdot)\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))$, we define the process $X(\cdot)$ by

$$X(s)=x+\int_t^s\Big(\mathcal{A}(r)\bar X(r)+\varphi(r)\Big)dr+\int_t^s\Big(\mathcal{C}(r)\bar X(r)+\psi(r)\Big)dW(r),\quad s\in[t,T].$$

By the classical theory of SDEs, we see that $\bar X(\cdot)\mapsto X(\cdot)$ is a map from $L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))$ into itself. For any $t\leqslant t_1<t_2\leqslant T$ and any $s\in[t_1,t_2]$,

$$|X(s)|^2\leqslant3\Big\{|X(t_1)|^2+\Big(\int_{t_1}^s\big|\mathcal{A}(r)\bar X(r)+\varphi(r)\big|dr\Big)^2+\Big|\int_{t_1}^s\big(\mathcal{C}(r)\bar X(r)+\psi(r)\big)dW(r)\Big|^2\Big\}.$$

By the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{\tau\in[t_1,s]}|X(\tau)|^2\Big]&\leqslant3\mathbb{E}\Big\{|X(t_1)|^2+(s-t_1)\int_{t_1}^s\Big|\mathcal{A}(r)\bar X(r)+\varphi(r)\Big|^2dr+c_2\int_{t_1}^s\Big|\mathcal{C}(r)\bar X(r)+\psi(r)\Big|^2dr\Big\}\\&\leqslant3\mathbb{E}\Big\{|X(t_1)|^2+2(s-t_1)\int_{t_1}^s\Big(|\mathcal{A}(r)\bar X(r)|^2+|\varphi(r)|^2\Big)dr+2c_2\int_{t_1}^s\Big(|\mathcal{C}(r)\bar X(r)|^2+|\psi(r)|^2\Big)dr\Big\},\end{aligned}$$

where $c_2 > 0$ is the constant in the Burkholder–Davis–Gundy inequality.

Let $K = 3\max\{1, 2T, 2c_2\}$. We have

$$\mathbb{E}\Big[\sup_{\tau\in[t_1,s]} |X(\tau)|^2\Big] \leqslant K\Big\{\mathbb{E}\big[|X(t_1)|^2\big] + \mathbb{E}\int_{t_1}^{s} \Big(|\varphi(r)|^2 + |\psi(r)|^2\Big)dr + \int_{t_1}^{s} \Big(\|\mathcal{A}(r)\|^2 + \|\mathcal{C}(r)\|^2\Big)\mathbb{E}\big[|\bar{X}(r)|^2\big]dr\Big\}. \tag{2.8}$$

Thus

$$\mathbb{E}\Big[\sup_{\tau\in[t_1,s]} |X(\tau)|^2\Big] \leqslant K\Big\{\mathbb{E}\big[|X(t_1)|^2\big] + \mathbb{E}\int_{t_1}^{s} \Big(|\varphi(r)|^2 + |\psi(r)|^2\Big)dr\Big\} + K\int_{t_1}^{s} \Big(\|\mathcal{A}(r)\|^2 + \|\mathcal{C}(r)\|^2\Big)dr\ \mathbb{E}\Big[\sup_{\tau\in[t_1,s]} |\bar{X}(\tau)|^2\Big].$$

Since for $\delta > 0$ small enough, we have

$$K\int_{t}^{t+\delta} \Big(\|\mathcal{A}(r)\|^2 + \|\mathcal{C}(r)\|^2\Big)dr < 1,$$

the map $\bar{X}(\cdot) \mapsto X(\cdot)$ is a contraction from $L^2_{\mathbb{F}}(\Omega; C([t, t+\delta]; \mathbb{R}^n))$ into itself. Therefore, it admits a unique fixed point which is a solution to the state equation on $[t, t+\delta]$. Repeating the same argument, we can obtain the unique existence of the solution $X(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n))$ to the state equation.

Moreover, for the solution $X(\cdot)$, from (2.8), we have

$$\mathbb{E}\Big[\sup_{\tau\in[t,s]} |X(\tau)|^2\Big] \leqslant K\mathbb{E}\Big\{|x|^2 + \int_{t}^{s} \Big(|\varphi(r)|^2 + |\psi(r)|^2\Big)dr\Big\} + K\int_{t}^{s} \Big(\|\mathcal{A}(r)\|^2 + \|\mathcal{C}(r)\|^2\Big)\mathbb{E}\Big[\sup_{\tau\in[t,r]} |X(\tau)|^2\Big]dr.$$

Then (2.7) is derived by Gronwall's inequality. $\square$

For the coefficients of state equation (1.1), we introduce the following hypothesis.

(H1) The coefficients of the state equation satisfy:

$$\begin{aligned}
&\mathcal{A}(\cdot) \in \mathscr{L}^2_{\mathbb{F}}\Big(L^2(\Omega; \mathbb{R}^n)\Big), \quad \mathcal{C}(\cdot) \in \mathscr{L}^\infty_{\mathbb{F}}\Big(L^2(\Omega; \mathbb{R}^n)\Big),\\
&\mathcal{B}(\cdot), \mathcal{D}(\cdot) \in \mathscr{L}^\infty_{\mathbb{F}}\Big(L^2(\Omega; \mathbb{R}^m); L^2(\Omega; \mathbb{R}^n)\Big), \quad b(\cdot), \sigma(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n).
\end{aligned}$$

Clearly, under (H1), Proposition 2.2 leads to the well-posedness of state equation (1.1). Now, for the cost functional, we introduce the following hypothesis.

(H2) The operator $\mathcal{G} \in \mathscr{S}\Big(L^2(\Omega; \mathbb{R}^n)\Big)$ and the operator-valued processes

$$\mathcal{Q}(\cdot) \in \mathscr{S}^1_{\mathbb{F}}\Big(L^2(\Omega; \mathbb{R}^n)\Big), \quad \mathcal{S}(\cdot) \in \mathscr{L}^2_{\mathbb{F}}\Big(L^2(\Omega; \mathbb{R}^n); L^2(\Omega; \mathbb{R}^m)\Big), \quad \mathcal{R}(\cdot) \in \mathscr{S}^\infty_{\mathbb{F}}\Big(L^2(\Omega; \mathbb{R}^m)\Big).$$

Also,

$$g \in L^2(\Omega; \mathbb{R}^n), \quad q(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n), \quad \rho(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^m).$$

We have the following result.

Proposition 2.3. *Let* (H1)–(H2) *hold. Then for any* $(t, x) \in \mathscr{D}$ *and any* $u(\cdot) \in \mathcal{U}[t, T]$*, the cost functional* $J(t, x; u(\cdot))$ *is well-defined.*

Proof. First of all, Proposition 2.2 implies that the state equation (1.1) admits a unique state process $X(\cdot)\equiv X(\cdot;t,x,u(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))$. Let us observe the following estimates:

$$
\begin{aligned}
&\mathbb{E}|\langle \mathcal{G}X(T),X(T)\rangle|\leqslant \|\mathcal{G}\|\mathbb{E}|X(T)|^2,\quad \mathbb{E}|\langle g,X(T)\rangle|\leqslant (\mathbb{E}|g|^2)^{\frac12}(\mathbb{E}|X(T)|^2)^{\frac12};\\
&\int_t^T\mathbb{E}|\langle \mathcal{R}(s)u(s),u(s)\rangle|ds\leqslant \int_t^T\|\mathcal{R}(s)\|\mathbb{E}|u(s)|^2ds\leqslant\Big(\sup_{s\in[t,T]}\|\mathcal{R}(s)\|\Big)\Big(\mathbb{E}\int_t^T|u(s)|^2ds\Big);\\
&\int_t^T\mathbb{E}|\langle \mathcal{S}(s)X(s),u(s)\rangle|ds\leqslant \Big(\int_t^T\mathbb{E}|\mathcal{S}(s)X(s)|^2ds\Big)^{\frac12}\Big(\mathbb{E}\int_t^T|u(s)|^2ds\Big)^{\frac12};\\
&\qquad\qquad\qquad\qquad\qquad\qquad\leqslant\Big(\int_t^T\|\mathcal{S}(s)\|^2ds\Big)^{\frac12}\Big(\mathbb{E}\Big[\sup_{s\in[t,T]}|X(s)|^2\Big]\Big)^{\frac12}\Big(\mathbb{E}\int_t^T|u(s)|^2ds\Big)^{\frac12};\\
&\int_t^T\mathbb{E}|\langle \mathcal{Q}(s)X(s),X(s)\rangle|ds\leqslant \int_t^T\|\mathcal{Q}(s)\|\mathbb{E}|X(s)|^2ds\leqslant\Big(\int_t^T\|\mathcal{Q}(s)\|ds\Big)\mathbb{E}\Big[\sup_{s\in[t,T]}|X(s)|^2\Big];\\
&\int_t^T\mathbb{E}|\langle q(s),X(s)\rangle|ds\leqslant (T-t)^{\frac12}\Big(\mathbb{E}\int_t^T|q(s)|^2ds\Big)^{\frac12}\Big(\mathbb{E}\Big[\sup_{s\in[t,T]}|X(s)|^2\Big]\Big)^{\frac12};\\
&\int_t^T\mathbb{E}|\langle \rho(s),u(s)\rangle|ds\leqslant \Big(\mathbb{E}\int_t^T|\rho(s)|^2ds\Big)^{\frac12}\Big(\mathbb{E}\int_t^T|u(s)|^2ds\Big)^{\frac12}.
\end{aligned}
$$

This implies that the cost functional $J(t,x;u(\cdot))$ is well-defined. $\square$

Next we look at the quadratic form of the cost functional (1.3), from which we will get some abstract results for Problem (OLQ).

It is clear that for given $t\in[0,T)$, $(x,u(\cdot))\mapsto X(\cdot\,;t,x,u(\cdot))$ is affine. Therefore, we may write

$$X(\cdot;t,x,u(\cdot))=[F_1(t)u(\cdot)](\cdot)+[F_0(t)x](\cdot)+f_0(t,\cdot),$$

where

$$F_1(t)\in\mathscr{L}\Big(\mathcal{U}[t,T];L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)\Big),\quad F_0(t)\in\mathscr{L}\Big(L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n);L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)\Big),\quad f_0(t,\cdot)\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^n).$$

Let

$$\widehat F_1(t)u(\cdot)=[F_1(t)u(\cdot)](T),\quad \widehat F_0(t)x=[F_0(t)x](T),\quad \widehat f_0(t)=f_0(t,T).$$

Consequently,

$$
\begin{aligned}
&J(t,x;u(\cdot))\\
&=\mathbb{E}\Big[\big\langle\mathcal{G}\big\{\widehat F_1(t)u(\cdot)+\widehat F_0(t)x+\widehat f_0(t)\big\},\widehat F_1(t)u(\cdot)+\widehat F_0(t)x+\widehat f_0(t)\big\rangle+2\big\langle g,\widehat F_1(t)u(\cdot)+\widehat F_0(t)x+\widehat f_0(t)\big\rangle\\
&\quad+\int_t^T\Big(\big\langle\mathcal{Q}(s)\big\{[F_1(t)u(\cdot)](s)+[F_0(t)x](s)+f_0(t,s)\big\},[F_1(t)u(\cdot)](s)+[F_0(t)x](s)+f_0(t,s)\big\rangle\\
&\qquad\qquad+2\langle\mathcal{S}(s)\big\{[F_1(t)u(\cdot)](s)+[F_0(t)x](s)+f_0(t,s)\big\},u(s)\rangle+\langle\mathcal{R}(s)u(s),u(s)\rangle\\
&\qquad\qquad+2\big\langle q(s),[F_1(t)u(\cdot)](s)+[F_0(t)x](s)+f_0(t,s)\big\rangle+2\big\langle\rho(s),u(s)\big\rangle\Big)ds\Big]\\
&\equiv\big\langle\mathcal{G}[\widehat F_1u+\widehat F_0x+\widehat f_0],\widehat F_1u+\widehat F_0x+\widehat f_0\big\rangle+2\big\langle g,\widehat F_1u+\widehat F_0x+\widehat f_0\big\rangle\\
&\quad+\big\langle\mathcal{Q}(F_1u+F_0x+f_0),F_1u+F_0x+f_0\big\rangle+2\big\langle\mathcal{S}(F_1u+F_0x+f_0),u\big\rangle+\big\langle\mathcal{R}u,u\big\rangle\\
&\quad+2\big\langle q,F_1u+F_0x+f_0\big\rangle+2\big\langle\rho,u\big\rangle\\
&=\big\langle\Phi_2u(\cdot),u(\cdot)\big\rangle+2\big\langle u(\cdot),\varphi_1\big\rangle+\varphi_0,
\end{aligned}\tag{2.9}
$$

with

$$
\begin{cases}
\Phi_2=\widehat F_1^*\mathcal{G}\widehat F_1+F_1^*\mathcal{Q}F_1+\mathcal{S}F_1+F_1^*\mathcal{S}^*+\mathcal{R},\\
\varphi_1=\widehat F_1^*\Big[\mathcal{G}(\widehat F_0x+\widehat f_0)+g\Big]+F_1^*\Big[\mathcal{Q}(F_0x+f_0)+q\Big]+\mathcal{S}(F_0x+f_0)+\rho,\\
\varphi_0=\langle\mathcal{G}(\widehat F_0x+\widehat f_0),\widehat F_0x+\widehat f_0\rangle+2\langle g,\widehat F_0x+\widehat f_0\rangle+\langle\mathcal{Q}(F_0x+f_0),F_0x+f_0\rangle+2\langle q,F_0x+f_0\rangle.
\end{cases}
$$

Clearly,

$$\Phi_2 \in \mathscr{S}\Big(L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)\Big), \qquad \varphi_1 \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m), \qquad \varphi_0 \in \mathbb{R}.$$

Therefore, according to [21], we have the following result.

Proposition 2.4. *For any $(t,x) \in \mathscr{D}$, the map $u(\cdot) \mapsto J(t,x;u(\cdot))$ admits a minimum in $\mathcal{U}[t,T]$ if and only if*

$$\Phi_2 \geqslant 0, \qquad \varphi_1 \in \mathscr{R}\big(\Phi_2\big) \equiv \text{ the range of } \Phi_2. \tag{2.10}$$

In particular, if the following holds:

$$\Phi_2 \geqslant \delta I, \tag{2.11}$$

for some $\delta > 0$, then (2.10) holds and the map $u(\cdot) \mapsto J(t,x;u(\cdot))$ admits a unique minimum given by the following:

$$\bar{u}(\cdot) = -\Phi_2^{-1}\varphi_1(\cdot). \tag{2.12}$$

It is clear that if

$$\mathcal{G} \geqslant 0, \quad \mathcal{R}(\cdot) \geqslant \delta I, \quad \mathcal{Q}(\cdot) - \mathcal{S}(\cdot)^*\mathcal{R}(\cdot)^{-1}\mathcal{S}(\cdot) \geqslant 0, \tag{2.13}$$

then (2.11) holds. Thus, under (H1)–(H2) and (2.11), Problem (OLQ) admits a unique open-loop optimal control.

If we denote

$$J(t,x;u(\cdot)) = J(t,x;u(\cdot);b(\cdot),\sigma(\cdot),g,q(\cdot),\rho(\cdot)),$$

indicating the dependence on the nonhomogeneous terms $b(\cdot), \sigma(\cdot)$ in the state equation and the linear weighting coefficients $g, q(\cdot), \rho(\cdot)$, then we define

$$J^0(t;u(\cdot)) = J(t,0;u(\cdot);0,0,0,0,0) = \langle \Phi_2 u(\cdot), u(\cdot)\rangle, \qquad \forall u(\cdot) \in \mathcal{U}[t,T]. \tag{2.14}$$

Hence, we see that the following is true.

Proposition 2.5. *Let (H1)–(H2) hold. Then the following are equivalent:*

(i) *$u(\cdot) \mapsto J(t,x;u(\cdot))$ is convex;*

(ii) *$\Phi_2 \geqslant 0$;*

(iii) *$J^0(t;u(\cdot)) \geqslant 0$ for all $u(\cdot) \in \mathcal{U}[t,T]$.*

2.3 BSDEs with operator coefficients

In this subsection, we consider the following BSDE with operator coefficients:

$$\begin{cases} dY(s) = -\Big(\mathcal{A}(s)^*Y(s) + \mathcal{C}(s)^*Z(s) + \varphi(s)\Big)ds + Z(s)dW(s), \quad s \in [t,T], \\ Y(T) = \xi \in L^2(\Omega;\mathbb{R}^n). \end{cases} \tag{2.15}$$

We have the following well-posedness and regularity result for the above BSDE.

Proposition 2.6. *Suppose (H1) holds and $\varphi(\cdot) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$. Then (2.15) admits a unique adapted solution $(Y(\cdot),Z(\cdot)) \in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$, and the following estimate holds:*

$$\mathbb{E}\Big[\sup_{s\in[t,T]}|Y(s)|^2 + \int_t^T |Z(s)|^2 ds\Big] \leqslant K\mathbb{E}\Big[|\xi|^2 + \int_t^T |\varphi(s)|^2 ds\Big], \tag{2.16}$$

where $K > 0$ is a constant.

Proof. Step 1. For any $\beta \geqslant 0$, denote $\mathcal{M}_\beta[t,T]$ the space $L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$, endowed with the following norm

$$\|(Y(\cdot), Z(\cdot))\|_{\mathcal{M}_\beta[t,T]} = \left\{\mathbb{E}\Big[\sup_{s\in[t,T]} |Y(s)|^2 e^{\beta^2 h(s)}\Big] + \mathbb{E}\int_t^T |Z(s)|^2 e^{\beta^2 h(s)} ds\right\}^{\frac{1}{2}},$$

where

$$h(\cdot) := \int_t^{\cdot} \big[\|\mathcal{A}(r)\|^2 + \|\mathcal{C}(r)\|^2 + 2\big] dr.$$

It is clear that the norms $\|\cdot\|_{\mathcal{M}_\beta[t,T]}$ with different $\beta \geqslant 0$ are all equivalent. We shall work with $\mathcal{M}_\beta[t,T]$ and specify β later. For any $(y(\cdot), z(\cdot)) \in \mathcal{M}_\beta[t,T]$, a map $\mathscr{T} : \mathcal{M}_\beta[t,T] \to \mathcal{M}_\beta[t,T]$ is defined by

$$\mathscr{T}(y(\cdot), z(\cdot)) = (Y(\cdot), Z(\cdot)),$$

where $(Y(\cdot), Z(\cdot))$ is the adapted solution to the following BSDE:

$$Y(s) = \xi + \int_s^T \Big(\mathcal{A}(r)^* y(r) + \mathcal{C}(r)^* z(r) + \varphi(r)\Big) dr - \int_s^T Z(r) dW(r), \quad s \in [t,T].$$

The wellposedness of the above BSDE from the classical theory of BSDEs implies that the map $\mathscr{T}$ is well-defined. For any $s \in [t,T]$, applying Itô's formula to $|Y(r)|^2 e^{\beta^2 h(r)}$ on the interval $[s,T]$, we have

$$\begin{aligned} & |Y(s)|^2 e^{\beta^2 h(s)} + \int_s^T \Big[\beta^2 h'(r) |Y(r)|^2 + |Z(r)|^2\Big] e^{\beta^2 h(r)} dr \\ & = |\xi|^2 e^{\beta^2 h(T)} + 2\int_s^T \Big\langle Y(r), \mathcal{A}(r)^* y(r) + \mathcal{C}(r)^* z(r) + \varphi(r)\Big\rangle e^{\beta^2 h(r)} dr \\ & \quad -2\int_s^T \langle Y(r),\ Z(r)\rangle e^{\beta^2 h(r)} dW(r). \end{aligned} \tag{2.17}$$

Then, by letting $\beta > 0$, we get

$$\begin{aligned} & \mathbb{E}\bigg\{|Y(s)|^2 e^{\beta^2 h(s)} + \int_s^T \Big[\beta^2 h'(r)|Y(r)|^2 + |Z(r)|^2\Big] e^{\beta^2 h(r)} dr\bigg\} \\ & = \mathbb{E}\bigg\{|\xi|^2 e^{\beta^2 h(T)} + 2\int_s^T \Big\langle Y(r),\ \mathcal{A}(r)^* y(r) + \mathcal{C}(r)^* z(r) + \varphi(r)\Big\rangle e^{\beta^2 h(r)} dr\bigg\} \\ & \leqslant \mathbb{E}\Big[|\xi|^2 e^{\beta^2 h(T)}\Big] + \int_s^T \bigg\{\beta^2\Big(\|\mathcal{A}(r)\|^2 + \|\mathcal{C}(r)\|^2 + 1\Big)\mathbb{E}|Y(r)|^2 + \frac{1}{\beta^2}\mathbb{E}\Big[|y(r)|^2 + |z(r)|^2 + |\varphi(r)|^2\Big]\bigg\} e^{\beta^2 h(r)} dr. \end{aligned}$$

Therefore,

$$\begin{aligned} & \mathbb{E}\bigg\{|Y(s)|^2 e^{\beta^2 h(s)} + \int_s^T \Big[\beta^2 |Y(r)|^2 + |Z(r)|^2\Big] e^{\beta^2 h(r)} dr\bigg\} \\ & \leqslant \mathbb{E}\bigg\{|\xi|^2 e^{\beta^2 h(T)} + \frac{1}{\beta^2}\int_s^T \Big[|y(r)|^2 + |z(r)|^2 + |\varphi(r)|^2\Big] e^{\beta^2 h(r)} dr\bigg\}. \end{aligned} \tag{2.18}$$

Next, for any $(y_i(\cdot), z_i(\cdot)) \in \mathcal{M}_\beta[t,T]$, $i = 1,2$, let $\mathscr{T}(y_i(\cdot), z_i(\cdot)) = (Y_i(\cdot), Z_i(\cdot)),\ i = 1,2$ and

$$\hat{y}(\cdot) = y_1(\cdot) - y_2(\cdot),\ \hat{z}(\cdot) = z_1(\cdot) - z_2(\cdot),\ \hat{Y}(\cdot) = Y_1(\cdot) - Y_2(\cdot),\ \hat{Z}(\cdot) = Z_1(\cdot) - Z_2(\cdot).$$

Similar to (2.18), we have

$$\begin{aligned} & \mathbb{E}\bigg\{|\hat{Y}(s)|^2 e^{\beta^2 h(s)} + \int_s^T \Big[\beta^2 |\hat{Y}(r)|^2 + |\hat{Z}(r)|^2\Big] e^{\beta^2 h(r)} dr\bigg\} \\ & \leqslant \frac{1}{\beta^2}\mathbb{E}\int_t^T \Big[|\hat{y}(r)|^2 + |\hat{z}(r)|^2\Big] e^{\beta^2 h(r)} dr \leqslant \frac{1}{\beta^2}(T - t + 1)\|(\hat{y}(\cdot), \hat{z}(\cdot))\|^2_{\mathcal{M}_\beta[t,T]}. \end{aligned}$$

Consequently,

$$\mathbb{E}\int_t^T |\hat{Z}(r)|^2 e^{\beta^2 h(r)}dr \leqslant \frac{1}{\beta^2}(T-t+1)\|(\hat{y}(\cdot),\hat{z}(\cdot))\|^2_{\mathcal{M}_\beta[t,T]}, \tag{2.19}$$

$$\sup_{s\in[t,T]}\mathbb{E}\Big[|\hat{Y}(s)|^2 e^{\beta^2 h(s)}\Big] \leqslant \frac{1}{\beta^2}(T-t+1)\|(\hat{y}(\cdot),\hat{z}(\cdot))\|^2_{\mathcal{M}_\beta[t,T]}. \tag{2.20}$$

Similar to (2.17), we have

$$\begin{aligned}|\hat{Y}(s)|^2 e^{\beta^2 h(s)} &\leqslant 2\int_s^T \Big\langle \hat{Y}(r),\ \mathcal{A}(r)^*\hat{y}(r)+\mathcal{C}(r)^*\hat{z}(r)\Big\rangle e^{\beta^2 h(r)}dr - 2\int_s^T \langle \hat{Y}(r),\ \hat{Z}(r)\rangle e^{\beta^2 h(r)}dW(r)\\ &\leqslant 2\int_t^T |\hat{Y}(r)||\mathcal{A}(r)^*\hat{y}(r)+\mathcal{C}(r)^*\hat{z}(r)|e^{\beta^2 h(r)}dr + 2\Big|\int_s^T \langle \hat{Y}(r),\ \hat{Z}(r)\rangle e^{\beta^2 h(r)}dW(r)\Big|.\end{aligned}$$

Then

$$\begin{aligned}&\mathbb{E}\Big[\sup_{s\in[t,T]}|\hat{Y}(s)|^2 e^{\beta^2 h(s)}\Big]\\ &\leqslant 2\mathbb{E}\int_t^T |\hat{Y}(r)||\mathcal{A}(r)^*\hat{y}(r)+\mathcal{C}(r)^*\hat{z}(r)|e^{\beta^2 h(r)}dr + 2\mathbb{E}\Big[\sup_{s\in[t,T]}\Big|\int_s^T \langle \hat{Y}(r),\ \hat{Z}(r)\rangle e^{\beta^2 h(r)}dW(r)\Big|\Big]\\ &\equiv \mathrm{I}+\mathrm{II}.\end{aligned}$$

We first estimate I as follows:

$$\begin{aligned}\mathrm{I} &\leqslant \mathbb{E}\int_t^T \Big(\beta|\hat{Y}(r)|^2 + \frac{1}{\beta}|\mathcal{A}(r)^*\hat{y}(r)+\mathcal{C}(r)^*\hat{z}(r)|^2\Big)e^{\beta^2 h(r)}dr\\ &\leqslant \beta(T-t)\Big\{\sup_{s\in[t,T]}\mathbb{E}\Big[|\hat{Y}(s)|^2 e^{\beta^2 h(s)}\Big]\Big\} + \frac{2}{\beta}\int_t^T \Big[\|\mathcal{A}(r)^*\|^2\mathbb{E}|\hat{y}(r)|^2 + \|\mathcal{C}(r)^*\|^2\mathbb{E}|\hat{z}(r)|^2\Big]e^{\beta^2 h(r)}dr\\ &\leqslant \frac{1}{\beta}(T-t+1)^2\|(\hat{y}(\cdot),\hat{z}(\cdot))\|^2_{\mathcal{M}_\beta[t,T]}\\ &\quad + \frac{2}{\beta}\mathbb{E}\Big[\sup_{s\in[t,T]}|\hat{y}(s)|^2 e^{\beta^2 h(s)}\Big]\Big(\int_t^T \|\mathcal{A}(r)\|^2 dr\Big) + \frac{2}{\beta}\Big(\sup_{s\in[t,T]}\|\mathcal{C}(s)\|^2\Big)\mathbb{E}\int_t^T |\hat{z}(r)|^2 e^{\beta^2 h(r)}dr\\ &\leqslant \frac{1}{\beta}\Big\{(T-t+1)^2 + 2\Big(\int_t^T \|\mathcal{A}(r)\|^2 dr\Big) + 2\Big(\sup_{s\in[t,T]}\|\mathcal{C}(s)\|^2\Big)\Big\}\|(\hat{y}(\cdot),\hat{z}(\cdot))\|^2_{\mathcal{M}_\beta[t,T]}.\end{aligned}$$

On the other hand, by the Burkholder–Davis–Gundy inequality,

$$\begin{aligned}\mathrm{II} &\leqslant 2c_2\mathbb{E}\Big(\int_t^T |\hat{Y}(r)|^2|\hat{Z}(r)|^2 e^{2\beta^2 h(r)}dr\Big)^{\frac{1}{2}}\\ &\leqslant 2c_2\mathbb{E}\Big\{\Big(\sup_{s\in[t,T]}|\hat{Y}(s)|^2 e^{\beta^2 h(s)}\Big)^{\frac{1}{2}}\Big(\int_t^T |\hat{Z}(r)|^2 e^{\beta^2 h(r)}dr\Big)^{\frac{1}{2}}\Big\}\\ &\leqslant \frac{1}{2}\mathbb{E}\Big[\sup_{s\in[t,T]}|\hat{Y}(s)|^2 e^{\beta^2 h(s)}\Big] + 2c_2^2\mathbb{E}\int_t^T |\hat{Z}(r)|^2 e^{\beta^2 h(r)}dr,\end{aligned} \tag{2.21}$$

where $c_2>0$ is the constant in the Burkholder-Davis-Gundy inequality. By (2.19),

$$\mathrm{II} \leqslant \frac{1}{2}\mathbb{E}\Big[\sup_{s\in[t,T]}|\hat{Y}(s)|^2 e^{\beta^2 h(s)}\Big] + \frac{1}{\beta^2}2c_2^2(T-t+1)\|(\hat{y}(\cdot),\hat{z}(\cdot))\|^2_{\mathcal{M}_\beta[t,T]}.$$

Hence,

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{s\in[t,T]}|\hat{Y}(s)|^2e^{\beta^2h(s)}\Big]\\
&\leqslant \bigg\{\frac{2}{\beta}\bigg[(T-t+1)^2+2\bigg(\int_t^T\|\mathcal{A}(r)\|^2dr\bigg)+2\bigg(\sup_{s\in[t,T]}\|\mathcal{C}(s)\|^2\bigg)\bigg]+\frac{4}{\beta^2}c_2^2(T-t+1)\bigg\}\|(\hat{y}(\cdot),\hat{z}(\cdot))\|^2_{\mathcal{M}_\beta[t,T]}.
\end{aligned}
$$

Consequently, by selecting $\beta\geqslant 1$, we have

$$
\|(\hat{Y}(\cdot),\hat{Z}(\cdot))\|_{\mathcal{M}_\beta[t,T]}\leqslant\frac{K}{\beta}\|(\hat{y}(\cdot),\hat{z}(\cdot))\|^2_{\mathcal{M}_\beta[t,T]},
$$

where

$$
K=2(T-t+1)^2+4\bigg(\int_t^T\|\mathcal{A}(r)\|^2dr\bigg)+4\bigg(\sup_{s\in[t,T]}\|\mathcal{C}(s)\|^2\bigg)+(T-t+1)(1+4c_2^2).
$$

By taking $\beta=2K$, which is bigger than 4 obviously, we get the mapping $\mathscr{T}$ to be contractive on $\mathcal{M}_\beta[t,T]$. From the contraction mapping theorem, we know there exists a unique adapted solution $(Y(\cdot),Z(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n)\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ to BSDE (2.15).

Step 2. For the solution $(Y(\cdot),Z(\cdot))$, from (2.18), we have

$$
\begin{aligned}
&\mathbb{E}\bigg\{|Y(s)|^2e^{\beta^2h(s)}+\int_s^T\bigg[\Big(\beta^2-\frac{1}{\beta^2}\Big)|Y(r)|^2+\Big(1-\frac{1}{\beta^2}\Big)|Z(r)|^2\bigg]e^{\beta^2h(r)}dr\bigg\}\\
&\leqslant\mathbb{E}\bigg\{|\xi|^2e^{\beta^2h(T)}+\frac{1}{\beta^2}\int_s^T|\varphi(r)|^2e^{\beta^2h(r)}dr\bigg\}.
\end{aligned}
$$

Consequently,

$$
\mathbb{E}\int_t^T|Z(r)|^2e^{\beta^2h(r)}dr\leqslant\frac{\beta^2}{\beta^2-1}\mathbb{E}\bigg\{|\xi|^2e^{\beta^2h(T)}+\int_t^T|\varphi(r)|^2e^{\beta^2h(r)}dr\bigg\},\tag{2.22}
$$

$$
\sup_{s\in[t,T]}\mathbb{E}\Big[|Y(s)|^2e^{\beta^2h(s)}\Big]\leqslant\mathbb{E}\bigg\{|\xi|^2e^{\beta^2h(T)}+\int_t^T|\varphi(r)|^2e^{\beta^2h(r)}dr\bigg\}.\tag{2.23}
$$

Coming back to (2.17), we have

$$
\begin{aligned}
|Y(s)|^2e^{\beta^2h(s)}\leqslant|\xi|^2e^{\beta^2h(T)}&+2\int_s^T|Y(r)||A(r)^*y(r)+\mathcal{C}(r)^*z(r)+\varphi(r)|e^{\beta^2h(r)}dr\\
&+2\sup_{s\in[t,T]}\bigg|\int_s^T\langle Y(r),\ Z(r)\rangle e^{\beta^2h(r)}dW(r)\bigg|.
\end{aligned}
$$

By virtue of the technique of (2.21),

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{s\in[t,T]}|Y(s)|^2e^{\beta^2h(s)}\Big]\leqslant 2\mathbb{E}\Big[|\xi|^2e^{\beta^2h(T)}\Big]&+6\mathbb{E}\int_t^T|\varphi(r)|^2e^{\beta^2h(r)}dr\\
&+\bigg(\int_t^T\Big[2+6\|\mathcal{A}(r)\|^2\Big]dr\bigg)\bigg\{\sup_{s\in[t,T]}\mathbb{E}\Big[|Y(s)|^2e^{\beta^2h(s)}\Big]\bigg\}\\
&+\Big(6\sup_{s\in[t,T]}\|\mathcal{C}(r)\|^2+4c_2^2\Big)\mathbb{E}\int_t^T|Z(r)|^2e^{\beta^2h(r)}dr.
\end{aligned}
$$

Combining the above inequality with (2.22) and (2.23), and thanks to $1\leqslant e^{\beta^2h(s)}\leqslant e^{\beta^2h(T)}$, $s\in[t,T]$, we obtain the desired estimate (2.16). The proof is completed. $\square$

3 Open-Loop Optimal Controls and FBSDEs

3.1 Solvability of Problem (OLQ)

This subsection is devoted to Problem (OLQ). We focus on the open-loop case for the optimal control. First the definition of the open-loop optimal control is introduced as follows.

Definition 3.1. A control process $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ is called an *open-loop optimal control* of Problem (OLQ) at $(t,x) \in \mathscr{D}$ if

$$J(t,x;\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x;u(\cdot)). \tag{3.1}$$

If $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ exists satisfying (3.1), we say that Problem (OLQ) is *open-loop solvable* at $(t,x) \in \mathscr{D}$. And $\bar{X}(\cdot) \equiv X^{\bar{u}(\cdot)}(\cdot)$ is called the optimal state process.

In this subsection, we will derive the sufficient and necessary conditions for open-loop optimal controls for Problem (OLQ). The main result can be stated as follows.

Theorem 3.2. *Let* (H1)–(H2) *hold. Given $(t,x) \in \mathscr{D}$. Then $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ is an open-loop optimal control of Problem (OLQ) at $(t,x) \in \mathscr{D}$ with $\bar{X}(\cdot)$ being the corresponding open-loop optimal state process if and only if $u(\cdot) \mapsto J(t,x;u(\cdot))$ is convex and the following FBSDE with operator coefficients:*

$$\begin{cases} d\bar{X}(s) = \Big(\mathcal{A}(s)\bar{X}(s) + \mathcal{B}(s)\bar{u}(s) + b(s)\Big)ds + \Big(\mathcal{C}(s)\bar{X}(s) + \mathcal{D}(s)\bar{u}(s) + \sigma(s)\Big)dW(s), \\ d\bar{Y}(s) = -\Big(\mathcal{A}(s)^*\bar{Y}(s) + \mathcal{C}(s)^*\bar{Z}(s) + \mathcal{Q}(s)\bar{X}(s) + \mathcal{S}(s)^*\bar{u}(s) + q(s)\Big)ds + \bar{Z}(s)dW(s), \\ \bar{X}(t) = x, \qquad \bar{Y}(T) = \mathcal{G}\bar{X}(T) + g, \end{cases} \quad s \in [t,T], \tag{3.2}$$

admits an adapted solution $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot))$ such that the following constraint holds:

$$\mathcal{R}(s)\bar{u}(s) + \mathcal{B}(s)^*\bar{Y}(s) + \mathcal{D}(s)^*\bar{Z}(s) + \mathcal{S}(s)\bar{X}(s) + \rho(s) = 0, \qquad s \in [t,T], \text{ a.s.} \tag{3.3}$$

Proof. For $(t,x) \in \mathscr{D}$ and $u(\cdot), \bar{u}(\cdot) \in \mathcal{U}[t,T]$, let $X(\cdot) = X(\cdot\,;t,x,u(\cdot))$ and $\bar{X}(\cdot) = X(\cdot\,;t,x,\bar{u}(\cdot))$ be the state process (1.1) corresponding to $u(\cdot)$ and $\bar{u}(\cdot)$, respectively. Denote

$$\hat{X}(\cdot) = X(\cdot) - \bar{X}(\cdot), \qquad \hat{u}(\cdot) = u(\cdot) - \bar{u}(\cdot).$$

Then $\hat{X}(\cdot)$ satisfies the following FSDE:

$$\begin{cases} d\hat{X}(s) = \Big(\mathcal{A}(s)\hat{X}(s) + \mathcal{B}(s)\hat{u}(s)\Big)ds + \Big(\mathcal{C}(s)\hat{X}(s) + \mathcal{D}(s)\hat{u}(s)\Big)dW(s), \qquad s \in [t,T], \\ \hat{X}(t) = 0. \end{cases}$$

Now, let $(\bar{Y}(\cdot), \bar{Z}(\cdot))$ be the adapted solution to the BSDE in (3.2). Then we have the following duality:

$$\begin{aligned} &\mathbb{E}\langle \mathcal{G}\bar{X}(T) + g, \hat{X}(T)\rangle = \mathbb{E}\langle \hat{X}(T), \bar{Y}(T)\rangle \\ &= \mathbb{E}\int_t^T \Big(\langle \mathcal{A}(s)\hat{X}(s) + \mathcal{B}(s)\hat{u}(s), \bar{Y}(s)\rangle + \langle \mathcal{C}(s)\hat{X}(s) + \mathcal{D}(s)\hat{u}(s), \bar{Z}(s)\rangle \\ &\qquad -\langle \hat{X}(s), \mathcal{A}(s)^*\bar{Y}(s) + \mathcal{C}(s)^*\bar{Z}(s) + \mathcal{Q}(s)\bar{X}(s) + \mathcal{S}(s)^*\bar{u}(s) + q(s)\rangle\Big)ds \\ &= \mathbb{E}\int_t^T \Big(-\langle \hat{X}(s), \mathcal{Q}(s)\bar{X}(s) + \mathcal{S}(s)^*\bar{u}(s) + q(s)\rangle + \langle \hat{u}(s), \mathcal{B}(s)^*\bar{Y}(s) + \mathcal{D}(s)^*\bar{Z}(s)\rangle\Big)ds. \end{aligned}$$

Hence,

$$
\begin{aligned}
J(t,x;u(\cdot)) - J(t,x;\bar u(\cdot)) &= \mathbb{E}\Big\{\langle \mathcal{G}[X(T)+\bar X(T)], \hat X(T)\rangle + 2\langle g, \hat X(T)\rangle \\
&\quad + \int_t^T \Big(\langle \mathcal{Q}(s)[X(s)+\bar X(s)], \hat X(s)\rangle + \langle \mathcal{S}(s)[X(s)+\bar X(s)], \hat u(s)\rangle + \langle \mathcal{S}(s)\hat X(s), u(s)+\bar u(s)\rangle \\
&\qquad + \langle \mathcal{R}(s)[u(s)+\bar u(s)], \hat u(s)\rangle + 2\langle q(s), \hat X(s)\rangle + 2\langle \rho(s), \hat u(s)\rangle\Big) ds\Big\} \\
&= \mathbb{E}\Big\{\langle \mathcal{G}\hat X(T), \hat X(T)\rangle + 2\langle \mathcal{G}\bar X(T) + g, \hat X(T)\rangle \\
&\quad + \int_t^T \Big(\langle \mathcal{Q}(s)\hat X(s), \hat X(s)\rangle + 2\langle \mathcal{S}(s)\hat X(s), \hat u(s)\rangle + \langle \mathcal{R}(s)\hat u(s), \hat u(s)\rangle \\
&\qquad + 2\Big[\langle \mathcal{Q}(s)\bar X(s), \hat X(s)\rangle + \langle \mathcal{S}(s)\bar X(s), \hat u(s)\rangle + \langle \mathcal{S}(s)\hat X(s), \bar u(s)\rangle \\
&\qquad + \langle \mathcal{R}(s)\bar u(s), \hat u(s)\rangle + \langle q(s), \hat X(s)\rangle + \langle \rho(s), \hat u(s)\rangle\Big]\Big) ds\Big\} \\
&= \mathbb{E}\Big\{\langle \mathcal{G}\hat X(T), \hat X(T)\rangle + \int_t^T \Big(\langle \mathcal{Q}(s)\hat X(s), \hat X(s)\rangle + 2\langle \mathcal{S}(s)\hat X(s), \hat u(s)\rangle + \langle \mathcal{R}(s)\hat u(s), \hat u(s)\rangle\Big) ds \\
&\qquad + 2\Big[\langle \mathcal{G}\bar X(T) + g, \hat X(T)\rangle + \int_t^T \Big(\langle \mathcal{Q}(s)\bar X(s), \hat X(s)\rangle + \langle \mathcal{S}(s)\bar X(s), \hat u(s)\rangle \\
&\qquad + \langle \mathcal{S}(s)\hat X(s), \bar u(s)\rangle + \langle \mathcal{R}(s)\bar u(s), \hat u(s)\rangle + \langle q(s), \hat X(s)\rangle + \langle \rho(s), \hat u(s)\rangle\Big) ds\Big]\Big\} \\
&= J^0(t;\hat u(\cdot)) + 2\mathbb{E}\int_t^T \Big\langle \mathcal{R}(s)\bar u(s) + \mathcal{B}(s)^*\bar Y(s) + \mathcal{D}(s)^*\bar Z(s) + \mathcal{S}(s)\bar X(s) + \rho(s), \hat u(s)\Big\rangle ds.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
J(t,x;\bar u(\cdot)+\alpha u(\cdot)) - J(t,x;\bar u(\cdot)) &= \alpha^2 J^0(t;u(\cdot)-\bar u(\cdot)) \\
&\quad + 2\alpha\mathbb{E}\int_t^T \Big\langle \mathcal{R}(s)\bar u(s) + \mathcal{B}(s)^*\bar Y(s) + \mathcal{D}(s)^*\bar Z(s) + \mathcal{S}(s)\bar X(s) + \rho(s), u(s)-\bar u(s)\Big\rangle ds.
\end{aligned}
$$

Thus, if constraint (3.3) holds and $u(\cdot) \mapsto J(t,x;u(\cdot))$ is convex (which is equivalent to $J^0(t;u(\cdot)-\bar u(\cdot)) \geqslant 0$, see Proposition 2.5), we have the optimality of $\bar u(\cdot)$. Conversely, if $\bar u(\cdot) \in \mathcal{U}[t,T]$ is a minimum of $u(\cdot) \mapsto J(t,x;u(\cdot))$, then by letting $\alpha \to \infty$, we see that the constraint (3.3) holds, and $u(\cdot) \mapsto J(t,x;u(\cdot))$ is convex. $\square$

We note that (3.2)–(3.3) is a coupled linear FBSDE with operator coefficients. The above theorem tells us that the open-loop solvability of Problem (OLQ) is equivalent to the solvability of an FBSDE with operator coefficients. A similar result for LQ problem with constant coefficients were established in [27]. The proof presented above is similar to that found in [27], with some simplifications.

3.2 Well-posedness of FBSDEs with operator coefficients

We now look at the solvability of FBSDE (3.2)–(3.3). To abbreviate the notations, we drop the bars in $\bar X, \bar Y, \bar Z, \bar u$ of (3.2) and (3.3), that is, we consider the following:

$$
\begin{cases}
dX(s) = \Big(\mathcal{A}(s)X(s) + \mathcal{B}(s)u(s) + b(s)\Big)ds + \Big(\mathcal{C}(s)X(s) + \mathcal{D}(s)u(s) + \sigma(s)\Big)dW(s), \\
dY(s) = -\Big(\mathcal{A}(s)^*Y(s) + \mathcal{C}(s)^*Z(s) + \mathcal{Q}(s)X(s) + \mathcal{S}(s)^*u(s) + q(s)\Big)ds + Z(s)dW(s), \quad s \in [t,T], \\
X(t) = x, \qquad Y(T) = \mathcal{G}X(T) + g, \\
\mathcal{R}(s)u(s) + \mathcal{B}(s)^*Y(s) + \mathcal{D}(s)^*Z(s) + \mathcal{S}(s)X(s) + \rho(s) = 0.
\end{cases}
\tag{3.4}
$$

For the well-posedness of the above equation, we need the following hypothesis.

(H3) (i) Operator processes $\mathcal{C}(\cdot),\mathcal{D}(\cdot),\mathcal{Q}(\cdot),\mathcal{S}(\cdot),\mathcal{R}(\cdot)$ are compatible in the following sense:

$$\|\mathcal{C}-\mathcal{D}\mathcal{R}^{-1}\mathcal{S}\|,\|\mathcal{S}^*\mathcal{R}^{-1}\|\in L^\infty(t,T;\mathbb{R}),\qquad \|\mathcal{Q}-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{S}\|\in L^2(t,T;\mathbb{R});$$

(ii) For some $\delta>0$,

$$\mathcal{G}\geqslant 0,\quad \mathcal{Q}(\cdot)-\mathcal{S}(\cdot)^*\mathcal{R}(\cdot)^{-1}\mathcal{S}(\cdot)\geqslant 0,\quad \mathcal{R}(\cdot)\geqslant\delta I.$$

Let us make some remarks on (H3). First of all, $\mathcal{R}(\cdot)\geqslant\delta I$ means that

$$\mathbb{E}\langle\mathcal{R}(s)u,u\rangle\geqslant\delta\mathbb{E}|u|^2,\ \forall u\in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m),\ \text{a.e.},\ s\in[t,T].$$

Similarly for $\mathcal{G}\geqslant 0$ and $\mathcal{Q}(\cdot)-\mathcal{S}(\cdot)^*\mathcal{R}(\cdot)^{-1}\mathcal{S}(\cdot)\geqslant 0$.

Next, (H3)–(ii) ensures the existence of the inverse of operator $\mathcal{R}$. Moreover, we have

$$\mathbb{E}\langle\mathcal{R}(s)^{-1}u,u\rangle\geqslant\frac{\delta}{\|\mathcal{R}(s)\|^2}\mathbb{E}|u|^2,\ \text{for any } u\in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m),\ \text{a.e.},\ s\in[t,T].$$

On the other hand, by virtue of Schur's Lemma in operator form, (H3)–(ii) is equivalent to the following:

$$\mathcal{G}\geqslant 0,\qquad \begin{pmatrix}\mathcal{Q}(\cdot) & \mathcal{S}(\cdot)^*\\ \mathcal{S}(\cdot) & \mathcal{R}(\cdot)\end{pmatrix}\geqslant 0,\quad \mathcal{R}(\cdot)\geqslant\delta I.$$

Now, under (H3), $u(\cdot)$ can be expressed explicitly as

$$u(s)=-\mathcal{R}(s)^{-1}\Big(\mathcal{B}^*(s)Y(s)+\mathcal{D}^*(s)Z(s)+\mathcal{S}(s)X(s)+\rho(s)\Big),\quad s\in[t,T]. \tag{3.5}$$

By substituting it into the first two equations of (3.4), the system becomes a coupled FBSDEs with operator coefficients as follows (the dependence on time s is omitted):

$$\begin{cases} dX=\Big((\mathcal{A}-\mathcal{B}\mathcal{R}^{-1}\mathcal{S})X-\mathcal{B}\mathcal{R}^{-1}(\mathcal{B}^*Y+\mathcal{D}^*Z)-\mathcal{B}\mathcal{R}^{-1}\rho+b\Big)ds\\ \qquad\quad+\Big((\mathcal{C}-\mathcal{D}\mathcal{R}^{-1}\mathcal{S})X-\mathcal{D}\mathcal{R}^{-1}(\mathcal{B}^*Y+\mathcal{D}^*Z)-\mathcal{D}\mathcal{R}^{-1}\rho+\sigma\Big)dW(s),\\ dY=-\Big((\mathcal{A}^*-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{B}^*)Y+(\mathcal{C}^*-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{D}^*)Z+(\mathcal{Q}-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{S})X \qquad s\in[t,T],\\ \qquad\quad-\mathcal{S}^*\mathcal{R}^{-1}\rho+q\Big)ds+ZdW(s),\\ X(t)=x,\qquad Y(T)=\mathcal{G}X(T)+g. \end{cases} \tag{3.6}$$

The following is the main result of this subsection.

Theorem 3.3. *Let* (H1)–(H3) *hold. Then there exists a unique adapted solution* $(X(\cdot),Y(\cdot),Z(\cdot))\in[L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))]^2\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ *to coupled FBSDE* (3.6) *with operator coefficients.*

Before presenting a proof of Theorem 3.3, we introduce the following auxiliary FBSDE with operator coefficients parameterized by $\alpha\in[0,1]$:

$$\begin{cases} dX^\alpha=\Big(\alpha(\mathcal{A}-\mathcal{B}\mathcal{R}^{-1}\mathcal{S})X^\alpha-\mathcal{B}\mathcal{R}^{-1}(\mathcal{B}^*Y^\alpha+\mathcal{D}^*Z^\alpha)+\varphi\Big)ds\\ \qquad\quad+\Big(\alpha(\mathcal{C}-\mathcal{D}\mathcal{R}^{-1}\mathcal{S})X^\alpha-\mathcal{D}\mathcal{R}^{-1}(\mathcal{B}^*Y^\alpha+\mathcal{D}^*Z^\alpha)+\psi\Big)dW(s),\\ dY^\alpha=-\Big(\alpha(\mathcal{A}^*-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{B}^*)Y^\alpha+\alpha(\mathcal{C}^*-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{D}^*)Z^\alpha \qquad s\in[t,T],\\ \qquad\qquad+\alpha(\mathcal{Q}-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{S})X^\alpha+\gamma\Big)ds+Z^\alpha dW(s),\\ X^\alpha(t)=\xi,\qquad Y^\alpha(T)=\alpha\mathcal{G}X^\alpha(T)+\eta, \end{cases} \tag{3.7}$$

where

$$(\xi,\varphi,\psi,\gamma,\eta)\in\mathbb{M}[t,T]\equiv L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)\times[L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)]^3\times L^2(\Omega;\mathbb{R}^n).$$

The first preparation is to get an a priori estimate of the adapt solution to the parameterized equation (3.7).

Proposition 3.4. *Let* (H1)–(H3) *hold. Let* (X_1, Y_1, Z_1) *and* (X_2, Y_2, Z_2) *be the solutions to* (3.7) *corresponding to the different* $(\xi_1, \varphi_1, \psi_1, \gamma_1, \eta_1)$, $(\xi_2, \varphi_2, \psi_2, \gamma_2, \eta_2) \in \mathbb{M}[t, T]$, *respectively. Then, for any* $\alpha \in [0, 1]$, *the following estimate holds:*

$$\begin{aligned}&\mathbb{E}\Big\{\sup_{s\in[t,T]}|X_1(s)-X_2(s)|^2+\sup_{s\in[t,T]}|Y_1(s)-Y_2(s)|^2+\int_s^T|Z_1(s)-Z_2(s)|^2ds\Big\}\\&\leqslant K\mathbb{E}\Big\{|\xi_1-\xi_2|^2+|\eta_1-\eta_2|^2+\int_t^T\Big(|\varphi_1-\varphi_2|^2+|\psi_1-\psi_2|^2+|\gamma_1-\gamma_2|^2\Big)dr\Big\},\end{aligned}\tag{3.8}$$

where $K > 0$ *is a constant.*

Proof. For convenience, we denote $(\hat{X}, \hat{Y}, \hat{Z}) := (X_1 - X_2, Y_1 - Y_2, Z_1 - Z_2)$, which satisfies

$$\begin{cases}d\hat{X}=\Big(\alpha(\mathcal{A}-\mathcal{B}\mathcal{R}^{-1}\mathcal{S})\hat{X}-\mathcal{B}\mathcal{R}^{-1}(\mathcal{B}^*\hat{Y}+\mathcal{D}^*\hat{Z})+\hat{\varphi}\Big)ds\\ \qquad\quad+\Big(\alpha(\mathcal{C}-\mathcal{D}\mathcal{R}^{-1}\mathcal{S})\hat{X}-\mathcal{D}\mathcal{R}^{-1}(\mathcal{B}^*\hat{Y}+\mathcal{D}^*\hat{Z})+\hat{\psi}\Big)dW(s),\quad s\in[t,T],\\ d\hat{Y}=-\Big(\alpha(\mathcal{A}^*-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{B}^*)\hat{Y}+\alpha(\mathcal{C}^*-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{D}^*)\hat{Z}+\alpha(\mathcal{Q}-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{S})\hat{X}+\hat{\gamma}\Big)ds\\ \qquad\quad+\hat{Z}dW(s),\quad s\in[t,T],\\ \hat{X}(t)=\hat{\xi},\qquad \hat{Y}(T)=\alpha\mathcal{G}\hat{X}(T)+\hat{\eta},\end{cases}\tag{3.9}$$

with $(\hat{\xi}, \hat{\varphi}, \hat{\psi}, \hat{\gamma}, \hat{\eta}) = (\xi_1 - \xi_2, \varphi_1 - \varphi_2, \psi_1 - \psi_2, \gamma_1 - \gamma_2, \eta_1 - \eta_2)$.

For the forward equation, by applying Proposition 2.2, we have

$$\begin{aligned}&\mathbb{E}\Big[\sup_{s\in[t,T]}|\hat{X}(s)|^2\Big]\\&\leqslant K\mathbb{E}\Big\{|\hat{\xi}|^2+\int_t^T\Big[\Big|-\mathcal{B}\mathcal{R}^{-1}(\mathcal{B}^*\hat{Y}+\mathcal{D}^*\hat{Z})+\hat{\varphi}\Big|^2+\Big|-\mathcal{D}\mathcal{R}^{-1}(\mathcal{B}^*\hat{Y}+\mathcal{D}^*\hat{Z})+\hat{\psi}\Big|^2\Big]dr\Big\}\\&\leqslant K\mathbb{E}\Big\{|\hat{\xi}|^2+\int_t^T\Big[|\hat{\varphi}|^2+|\hat{\psi}|^2+|\mathcal{B}^*\hat{Y}+\mathcal{D}^*\hat{Z}|^2\Big]dr\Big\}.\end{aligned}$$

Similarly, for the backward equation, thanks to $\|\mathcal{C} - \mathcal{D}\mathcal{R}^{-1}\mathcal{S}\| \in L^\infty(0, T; \mathbb{R})$ and $\|\mathcal{Q} - \mathcal{S}^*\mathcal{R}^{-1}\mathcal{S}\| \in L^2(0, T; \mathbb{R})$, Proposition 2.6 leads to

$$\begin{aligned}&\mathbb{E}\Big[\sup_{s\in[t,T]}|\hat{Y}(s)|^2+\int_t^T|\hat{Z}(s)|^2ds\Big]\\&\leqslant K\mathbb{E}\Big\{|\alpha\mathcal{G}\hat{X}(T)+\hat{\eta}|^2+\int_t^T\Big|\alpha(\mathcal{Q}-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{S})\hat{X}+\hat{\gamma}\Big|^2ds\Big\}\\&\leqslant K\mathbb{E}\Big\{|\hat{\eta}|^2+\int_t^T|\hat{\gamma}|^2dr+\Big(\|\mathcal{G}\|^2+\int_t^T\|\mathcal{Q}-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{S}\|^2dr\Big)\sup_{s\in[t,T]}|\hat{X}(s)|^2\Big\}\\&\leqslant K\mathbb{E}\Big\{|\hat{\eta}|^2+\int_t^T|\hat{\gamma}|^2dr+\sup_{s\in[t,T]}|\hat{X}(s)|^2\Big\}.\end{aligned}$$

Then, we get

$$\begin{aligned}&\mathbb{E}\Big[\sup_{s\in[t,T]}|\hat{X}(s)|^2+\sup_{s\in[t,T]}|\hat{Y}(s)|^2+\int_t^T|\hat{Z}(s)|^2ds\Big]\\&\leqslant K\mathbb{E}\Big\{|\hat{\xi}|^2+|\hat{\eta}|^2+\int_t^T\Big[|\hat{\varphi}|^2+|\hat{\psi}|^2+|\hat{\gamma}|^2+|\mathcal{B}^*\hat{Y}+\mathcal{D}^*\hat{Z}|^2\Big]dr\Big\}.\end{aligned}\tag{3.10}$$

On the other hand, by applying Itô's formula to $\langle \hat{X}, \hat{Y}\rangle$, we have

$$
\begin{aligned}
&\mathbb{E}\int_t^T \langle \mathcal{R}^{-1}(\mathcal{B}^*\hat{Y}+\mathcal{D}^*\hat{Z}), \mathcal{B}^*\hat{Y}+\mathcal{D}^*\hat{Z}\rangle dr \\
&= -\alpha\mathbb{E}\Big\{\langle \mathcal{G}\hat{X}(T), \hat{X}(T)\rangle + \int_t^T \langle (\mathcal{Q}-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{S})\hat{X}, \hat{X}\rangle dr\Big\} \\
&\quad +\mathbb{E}\Big\{\langle \hat{\xi}, \hat{Y}(t)\rangle - \langle \hat{\eta}, \hat{X}(T)\rangle + \int_t^T \Big[\langle \hat{\varphi}, \hat{Y}\rangle + \langle \hat{\psi}, \hat{Z}\rangle + \langle \hat{\gamma}, \hat{X}\rangle\Big] dr\Big\}.
\end{aligned}
\tag{3.11}
$$

By Assumption (H3)–(ii),

$$
\mathbb{E}\int_t^T |\mathcal{B}^*\hat{Y}+\mathcal{D}^*\hat{Z}|^2 ds \leqslant \frac{\|\mathcal{R}\|^2}{\delta}\mathbb{E}\Big\{\langle \hat{\xi}, \hat{Y}(t)\rangle - \langle \hat{\eta}, \hat{X}(T)\rangle + \int_t^T \Big[\langle \hat{\varphi}, \hat{Y}\rangle + \langle \hat{\psi}, \hat{Z}\rangle + \langle \hat{\gamma}, \hat{X}\rangle\Big] dr\Big\}.
$$

Then, for any $\varepsilon > 0$, we have

$$
\begin{aligned}
\mathbb{E}\int_t^T |\mathcal{B}^*\hat{Y}+\mathcal{D}^*\hat{Z}|^2 ds &\leqslant \varepsilon\mathbb{E}\Big[\sup_{s\in[t,T]}|\hat{X}(s)|^2 + \sup_{s\in[t,T]}|\hat{Y}(s)|^2 + \int_t^T |\hat{Z}(s)|^2 ds\Big] \\
&\quad + \frac{K}{\varepsilon}\mathbb{E}\Big\{|\hat{\xi}|^2 + |\hat{\eta}|^2 + \int_t^T \Big[|\hat{\varphi}|^2 + |\hat{\psi}|^2 + |\hat{\gamma}|^2\Big] dr\Big\}.
\end{aligned}
\tag{3.12}
$$

Selecting $\varepsilon = 1/(2K)$, we get the desired result (3.8) from (3.10) and (3.12).

□

Remark 3.5. *As we known, for the coupled FBSDEs, the main role of the monotonicity condition is to uncouple the interdependence of the forward $X(\cdot)$ and backward $(Y(\cdot), Z(\cdot))$ when studying the estimates of $(X(\cdot), Y(\cdot), Z(\cdot))$.*

Note that, for the linear system, our condition (H3)-(ii): $\mathcal{G} \geqslant 0,\ \mathcal{Q}(\cdot) - \mathcal{S}(\cdot)^*\mathcal{R}(\cdot)^{-1}\mathcal{S}(\cdot) \geqslant 0,\ \mathcal{R}(\cdot) \geqslant \delta I$ which acts as the classical monotonicity condition, has been used in (3.11).

Next, we give the continuation lemma.

Lemma 3.6. *Let* (H1)–(H3) *hold. Then there exists a constant $\ell_0 > 0$ such that, if for some $\alpha_0 \in [0, 1)$, for any $(\xi, \varphi, \psi, \gamma, \eta) \in \mathbb{M}[t, T]$, FBSDE* (3.7) *has a unique solution, then for $\alpha = \alpha_0 + \ell$ with $\ell \in [0, \ell_0]$, $\alpha_0 + \ell \leqslant 1$,* (3.7) *is also uniquely solvable.*

Proof. Let ℓ_0 be undetermined. Let $\ell \in [0, \ell_0]$, we focus on the following FBSDE:

$$
\begin{cases}
dX = \Big(\alpha_0(\mathcal{A}-\mathcal{B}\mathcal{R}^{-1}\mathcal{S})X - \mathcal{B}\mathcal{R}^{-1}(\mathcal{B}^*Y+\mathcal{D}^*Z) + \ell(\mathcal{A}-\mathcal{B}\mathcal{R}^{-1}\mathcal{S})\mathcal{X} + \varphi\Big)ds \\
\qquad +\Big(\alpha_0(\mathcal{C}-\mathcal{D}\mathcal{R}^{-1}\mathcal{S})X - \mathcal{D}\mathcal{R}^{-1}(\mathcal{B}^*Y+\mathcal{D}^*Z) + \ell(\mathcal{C}-\mathcal{D}\mathcal{R}^{-1}\mathcal{S})\mathcal{X} + \psi\Big)dW(s), \\
dY = -\Big(\alpha_0(\mathcal{A}^*-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{B}^*)Y + \alpha_0(\mathcal{C}^*-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{D}^*)Z + \alpha_0(\mathcal{Q}-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{S})X \\
\qquad +\ell(\mathcal{A}^*-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{B}^*)\mathcal{Y} + \ell(\mathcal{C}^*-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{D}^*)\mathcal{Z} + \ell(\mathcal{Q}-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{S})\mathcal{X} + \gamma\Big)ds \\
\qquad +ZdW(s), \\
X(t) = \xi, \qquad Y(T) = \alpha_0\mathcal{G}X(T) + \ell\mathcal{G}\mathcal{X}(T) + \eta,
\end{cases}
\quad s \in [t, T]
\tag{3.13}
$$

where $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in [L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n))]^2 \times L^2_{\mathbb{F}}(t, T; \mathbb{R}^n)$ is arbitrarily chosen.

Our assumption ensures the solvability of the above equation, so that a mapping $\mathcal{L}$ from $[L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n))]^2 \times L^2_{\mathbb{F}}(t, T; \mathbb{R}^n)$ into itself defined as

$$
(X(\cdot), Y(\cdot), Z(\cdot)) = \mathcal{L}_{\alpha_0+\ell}(\mathcal{X}(\cdot), \mathcal{Y}(\cdot), \mathcal{Z}(\cdot))
$$

makes sense. For another given $(\bar{\mathcal{X}}(\cdot),\bar{\mathcal{Y}}(\cdot),\bar{\mathcal{Z}}(\cdot))\in[L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))]^2\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$, let

$$(\bar{X}(\cdot),\bar{Y}(\cdot),\bar{Z}(\cdot))=\mathcal{L}_{\alpha_0+\ell}(\bar{\mathcal{X}}(\cdot),\bar{\mathcal{Y}}(\cdot),\bar{\mathcal{Z}}(\cdot)).$$

Denote

$$\hat{X}(\cdot)=X(\cdot)-\bar{X}(\cdot),\quad \hat{Y}(\cdot)=Y(\cdot)-\bar{Y}(\cdot),\quad \hat{Z}(\cdot)=Z(\cdot)-\bar{Z}(\cdot).$$

Proposition 3.4 leads to

$$\begin{aligned}
&\mathbb{E}\Big\{\sup_{s\in[t,T]}|\hat{X}(s)|^2+\sup_{s\in[t,T]}|\hat{Y}(s)|^2+\int_t^T|\hat{Z}(s)|^2ds\Big\}\\
&\leqslant K|\ell|^2\mathbb{E}\Big\{|\mathcal{G}\hat{X}(T)|^2+\int_t^T\Big[|(\mathcal{A}-\mathcal{B}\mathcal{R}^{-1}\mathcal{S})\hat{\mathcal{X}}|^2+|(\mathcal{C}-\mathcal{D}\mathcal{R}^{-1}\mathcal{S})\hat{\mathcal{X}}|^2\\
&\qquad +|(\mathcal{A}^*-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{B}^*)\hat{\mathcal{Y}}+(\mathcal{C}^*-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{D}^*)\hat{\mathcal{Z}}+(\mathcal{Q}-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{S})\hat{\mathcal{X}}|^2\Big]dr\Big\}\\
&\leqslant K|\ell|^2\mathbb{E}\Big\{\sup_{s\in[t,T]}|\hat{\mathcal{X}}(s)|^2+\sup_{s\in[t,T]}|\hat{\mathcal{Y}}(s)|^2+\int_t^T|\hat{\mathcal{Z}}(s)|^2ds\Big\},
\end{aligned}\tag{3.14}$$

where $K>0$ is a constant independent of α_0 and ℓ. Therefore, we can choose $\ell_0>0$, such that $K\ell_0^2\leqslant 1/4$. Then, for any $\ell\in[0,\ell_0]$, $\mathcal{L}_{\alpha_0+\ell}(\cdot)$ is a contraction map. Consequently, there exists a unique fixed point for the mapping $\mathcal{L}_{\alpha_0+\ell}$ which is just the unique solution to FBSDE (3.7) with $\alpha=\alpha_0+\ell$. □

Now we present a proof of Theorem 3.3.

Proof of Theorem 3.3. When $\alpha=0$, (3.7) becomes

$$\begin{cases}
dX^0=\Big(-\mathcal{B}\mathcal{R}^{-1}(\mathcal{B}^*Y^0+\mathcal{D}^*Z^0)+\varphi\Big)ds+\Big(-\mathcal{D}\mathcal{R}^{-1}(\mathcal{B}^*Y^0+\mathcal{D}^*Z^0)+\psi\Big)dW(s),\\
dY^0=-\gamma ds+Z^0dW(s), \\
X^0(t)=\xi,\qquad Y^0(T)=\eta,
\end{cases}\quad s\in[t,T].$$

In this case, the solvability of (3.7) is obvious, which is due to the part (Y^0,Z^0) of backward equation can be solved firstly; then substituting it into the forward equation, we solve X^0.

By Lemma 3.6, for any $(\xi,\varphi,\psi,\gamma,\eta)\in\mathbb{M}[t,T]$, and any $\alpha\in[0,1]$, (3.7) is uniquely solvable. Particularly, when $\alpha=1$, (3.7) with

$$\xi=x,\quad \varphi=-\mathcal{B}\mathcal{R}^{-1}\rho+b,\quad \psi=-\mathcal{D}\mathcal{R}^{-1}\rho+\sigma,\quad \gamma=-\mathcal{S}\mathcal{R}^{-1}\rho+q,\quad \eta=g$$

becomes (3.4) which is also uniquely solvable.

□

Remark 3.7. In our setting, the norms of coefficients $\|\mathcal{A}-\mathcal{B}\mathcal{R}^{-1}\mathcal{S}\|$ and $\|\mathcal{Q}-\mathcal{S}^*\mathcal{R}^{-1}\mathcal{S}\|$ are only required to belong to $L^2(t,T;\mathbb{R})$. In other words, these two norms are not necessary to be bounded.

Corollary 3.8. *Under* (H1)–(H3), *Problem (OLQ) admits a unique open-loop optimal control given by* (3.5), *where* $(X(\cdot),Y(\cdot),Z(\cdot))$ *is the unique solution to FBSDE* (3.6).

Proof. Let us denote by $J^0(t;u(\cdot))$ the cost functional (1.3) when $x,b(\cdot),\sigma(\cdot),g,q(\cdot),\rho(\cdot)$ are all 0. Obviously, Assumption (H3)–(ii) implies $J^0(t;u(\cdot))\geqslant 0$ for all $u(\cdot)\in\mathcal{U}[t,T]$. By Proposition 2.5, we know $u(\cdot)\mapsto J(t,x;u(\cdot))$ is convex. Moreover, by Theorem 3.3 and the expression (3.5), there exists a unique $(X(\cdot),Y(\cdot),Z(\cdot),u(\cdot))$ satisfying (3.2) and (3.3). Thanks to Theorem 3.2, we obtain the result. □

4 Mean-field LQ control problem

We have mentioned that the major motivation of this work comes from the study of stochastic LQ problem of mean-field FSDE with cost functional also involving mean-field terms. In this section, we will apply the results obtained in the previous section to the mean-field case.

We shall use the mean-field setting (2.2) and (2.3) given in Subsection 2.1. That is, the involved operator coefficients of Problem (OLQ) are, $\forall s \in [t,T]$,

$$
\begin{cases}
\mathcal{A}(s)\xi = A(s)\xi + \bar{\mathbf{A}}(s)^\top \mathbb{E}\big[\widetilde{\mathbf{A}}(s)\xi\big], \\
\mathcal{C}(s)\xi = C(s)\xi + \bar{\mathbf{C}}(s)^\top \mathbb{E}\big[\widetilde{\mathbf{C}}(s)\xi\big],
\end{cases} \quad \forall \xi \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n),
$$
$$
\begin{cases}
\mathcal{B}(s)\eta = B(s)\eta + \bar{\mathbf{B}}(s)^\top \mathbb{E}\big[\widetilde{\mathbf{B}}(s)\eta\big], \\
\mathcal{D}(s)\eta = D(s)\eta + \bar{\mathbf{D}}(s)^\top \mathbb{E}\big[\widetilde{\mathbf{D}}(s)\eta\big],
\end{cases} \quad \forall \eta \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m), \tag{4.1}
$$

and

$$
\begin{cases}
\mathcal{G}\xi = G\xi + \widetilde{\mathbf{G}}^\top \mathbb{E}[\bar{\mathbf{G}}\xi] + \bar{\mathbf{G}}^\top \mathbb{E}[\widetilde{\mathbf{G}}\xi] + \widetilde{\mathbf{G}}^\top \mathbb{E}[\widehat{\mathbf{G}}]\mathbb{E}[\widetilde{\mathbf{G}}\xi], \quad \xi \in L^2(\Omega;\mathbb{R}^n), \\
\mathcal{Q}(s)\xi = Q(s)\xi + \widetilde{\mathbf{Q}}(s)^\top \mathbb{E}[\bar{\mathbf{Q}}(s)\xi] + \bar{\mathbf{Q}}(s)^\top \mathbb{E}[\widetilde{\mathbf{Q}}(s)\xi] + \widetilde{\mathbf{Q}}(s)^\top \mathbb{E}[\widehat{\mathbf{Q}}(s)]\mathbb{E}[\widetilde{\mathbf{Q}}(s)\xi], \quad \xi \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n), \\
\mathcal{S}(s)\xi = S(s)\xi + \widetilde{\mathbf{R}}(s)^\top \mathbb{E}[\bar{\mathbf{S}}(s)\xi] + \widetilde{\mathbf{S}}(s)\mathbb{E}[\widetilde{\mathbf{Q}}(s)\xi] + \widetilde{\mathbf{R}}(s)^\top \mathbb{E}[\widehat{\mathbf{S}}(s)]\mathbb{E}[\widetilde{\mathbf{Q}}(s)\xi], \quad \xi \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n), \\
\mathcal{R}(s)\eta = R(s)\eta + \widetilde{\mathbf{R}}(s)^\top \mathbb{E}[\bar{\mathbf{R}}(s)\eta] + \bar{\mathbf{R}}(s)^\top \mathbb{E}[\widetilde{\mathbf{R}}(s)\eta] + \widetilde{\mathbf{R}}(s)^\top \mathbb{E}[\widehat{\mathbf{R}}(s)]\mathbb{E}[\widetilde{\mathbf{R}}(s)\eta], \quad \eta \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m), \\
g = g_0 + \widetilde{\mathbf{G}}^\top \mathbb{E}[\bar{\mathbf{g}}], \quad q(s) = q_0(s) + \widetilde{\mathbf{Q}}(s)^\top \mathbb{E}[\bar{\mathbf{q}}(s)], \quad \rho(s) = \rho_0(s) + \widetilde{\mathbf{R}}(s)^\top \mathbb{E}[\bar{\boldsymbol{\rho}}(s)].
\end{cases} \tag{4.2}
$$

For any $(t,x) \in \mathscr{D}$, our state equation is given by

$$
\begin{cases}
dX(s) = \Big(A(s)X(s) + \bar{\mathbf{A}}(s)^\top \mathbb{E}[\widetilde{\mathbf{A}}(s)X(s)] + B(s)u(s) + \bar{\mathbf{B}}(s)^\top \mathbb{E}[\widetilde{\mathbf{B}}(s)u(s)] + b(s)\Big)ds \\
\quad + \Big(C(s)X(s) + \bar{\mathbf{C}}(s)^\top \mathbb{E}[\widetilde{\mathbf{C}}(s)X(s)] + D(s)u(s) + \bar{\mathbf{D}}(s)^\top \mathbb{E}[\widetilde{\mathbf{D}}(s)u(s)] + \sigma(s)\Big)dW(s), \ \ s \in [t,T], \\
X(t) = x,
\end{cases} \tag{4.3}
$$

and the quadratic cost functional is

$$
\begin{aligned}
J(t,x;u(\cdot)) = \mathbb{E}\Big\{ & \langle GX(T),X(T)\rangle + 2\langle \bar{\mathbf{G}}X(T), \mathbb{E}[\widetilde{\mathbf{G}}X(T)]\rangle + \langle \widehat{\mathbf{G}}\mathbb{E}[\widetilde{\mathbf{G}}X(T)], \mathbb{E}[\widetilde{\mathbf{G}}X(T)]\rangle \\
& + 2\langle g_0, X(T)\rangle + 2\langle \bar{\mathbf{g}}, \mathbb{E}[\widetilde{\mathbf{G}}X(T)]\rangle + \int_t^T \Big[\langle QX,X\rangle + 2\langle \bar{\mathbf{Q}}X, \mathbb{E}[\widetilde{\mathbf{Q}}X]\rangle + \langle \widehat{\mathbf{Q}}\mathbb{E}[\widetilde{\mathbf{Q}}X], \mathbb{E}[\widetilde{\mathbf{Q}}X]\rangle \\
& + 2\langle SX,u\rangle + 2\langle \bar{\mathbf{S}}X, \mathbb{E}[\widetilde{\mathbf{R}}u]\rangle + 2\langle \widetilde{\mathbf{S}}\mathbb{E}[\widetilde{\mathbf{Q}}X], u\rangle + \langle \widehat{\mathbf{S}}\mathbb{E}[\widetilde{\mathbf{Q}}X], \mathbb{E}[\widetilde{\mathbf{R}}u]\rangle + \langle Ru,u\rangle \\
& + 2\langle \bar{\mathbf{R}}u, \mathbb{E}[\widetilde{\mathbf{R}}u]\rangle + \langle \widehat{\mathbf{R}}\mathbb{E}[\widetilde{\mathbf{R}}u], \mathbb{E}[\widetilde{\mathbf{R}}u]\rangle + 2\langle q_0, X\rangle + 2\langle \bar{\mathbf{q}}, \mathbb{E}[\widetilde{\mathbf{Q}}X]\rangle + 2\langle \rho_0, u\rangle + 2\langle \bar{\boldsymbol{\rho}}, \mathbb{E}[\widetilde{\mathbf{R}}u]\rangle\Big] ds\Big\}.
\end{aligned} \tag{4.4}
$$

The argument s is suppressed in the above functional. Here,

$$
\begin{cases}
A(\cdot), \bar{\mathbf{A}}(\cdot), \widetilde{\mathbf{A}}(\cdot), B(\cdot), \bar{\mathbf{B}}(\cdot), \widetilde{\mathbf{B}}(\cdot), C(\cdot), \bar{\mathbf{C}}(\cdot), \widetilde{\mathbf{C}}(\cdot), D(\cdot), \bar{\mathbf{D}}(\cdot), \widetilde{\mathbf{D}}(\cdot), \\
Q(\cdot), \bar{\mathbf{Q}}(\cdot), \widetilde{\mathbf{Q}}(\cdot), \widehat{\mathbf{Q}}(\cdot), S(\cdot), \bar{\mathbf{S}}(\cdot), \widetilde{\mathbf{S}}(\cdot), \widehat{\mathbf{S}}(\cdot), R(\cdot), \bar{\mathbf{R}}(\cdot), \widetilde{\mathbf{R}}(\cdot), \widehat{\mathbf{R}}(\cdot)
\end{cases}
$$

are given bounded $\mathbb{F}$-progressively measurable matrix-valued processes with appropriate dimensions. G, $\bar{\mathbf{G}}$, $\widetilde{\mathbf{G}}$, $\widehat{\mathbf{G}}$ are given bounded $\mathcal{F}_T$-measurable matrix-valued random variables with appropriate dimensions. Moreover, $Q(\cdot)$, $\widehat{\mathbf{Q}}(\cdot)$, $R(\cdot)$, $\widehat{\mathbf{R}}(\cdot)$, G, $\widehat{\mathbf{G}}$ are symmetric. $b(\cdot)$, $\sigma(\cdot)$, $q_0(\cdot)$, $\bar{\mathbf{q}}(\cdot)$, $\rho_0(\cdot)$, $\bar{\boldsymbol{\rho}}(\cdot)$ are given square integrable $\mathbb{F}$-progressively measurable processes with appropriate dimensions, and g_0, $\bar{\mathbf{g}}$ are given square integrable $\mathcal{F}_T$-measurable random variables with appropriate dimensions.

Remark 4.1. *Note that all the coefficients in bold are all infinite dimensions as shown in Subsection 2.1. Taking a bounded process* $\bar{\mathbf{A}}(\cdot)^\top = (\bar{A}_1(\cdot), \bar{A}_2(\cdot), \cdots)$ *as an example,* $\bar{\mathbf{A}}(\cdot)$ *is bounded means*

$$\|\bar{\mathbf{A}}(\cdot)^\top\| = \operatorname*{esssup}_{(s,\omega)} \sup_i |\bar{A}_i(s)| < \infty.$$

As the square-integrable property, take $\bar{\mathbf{q}}(\cdot) = \begin{pmatrix} q_1(\cdot) \\ q_2(\cdot) \\ \vdots \end{pmatrix}$ *as an example.* $\bar{\mathbf{q}}(\cdot)$ *is square-integrable means*

$$\|\bar{\mathbf{q}}(\cdot)\| = \mathbb{E}\int_0^T \sum_{i=1}^{\infty} |q_i(s)|^2 ds < \infty.$$

There are similar explanations for the above used another infinite-dimensional bounded and square-integral processes.

According to the classical well-posedness of MF-SDE (refer to [2]), for any $u(\cdot) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$, there exists a unique adapted solution $X^u(\cdot)$ satisfying (4.3), then the cost functional $J(t,x;u(\cdot))$ is well-defined.

Further, the cost functional can be rewritten as follows:

$$\begin{aligned} J(t,x;u(\cdot)) = \mathbb{E}\Big\{ &\langle \mathbf{G}\widetilde{\mathbf{X}}(T), \widetilde{\mathbf{X}}(T)\rangle + 2\langle \mathbf{g}, \widetilde{\mathbf{X}}(T)\rangle + \int_t^T \Big[\langle \mathbf{Q}(s)\mathbf{X}(s), \mathbf{X}(s)\rangle \\ &+2\langle \mathbf{S}(s)\mathbf{X}(s), \mathbf{u}(s)\rangle + \langle \mathbf{R}(s)\mathbf{u}(s), \mathbf{u}(s)\rangle + 2\langle \mathbf{q}(s), \mathbf{X}(s)\rangle + 2\langle \boldsymbol{\rho}(s), \mathbf{u}(s)\rangle\Big] ds\Big\}, \end{aligned} \tag{4.5}$$

where, for any $s \in [t,T]$, the above used notations (introduced in Subsection 2.1) are repeated again as

$$\widetilde{\mathbf{X}}(T) = \begin{pmatrix} X(T) \\ \mathbb{E}[\widetilde{\mathbf{G}}X(T)] \end{pmatrix}, \qquad \mathbf{X}(s) = \begin{pmatrix} X(s) \\ \mathbb{E}[\widetilde{\mathbf{Q}}(s)X(s)] \end{pmatrix}, \qquad \mathbf{u}(s) = \begin{pmatrix} u(s) \\ \mathbb{E}[\widetilde{\mathbf{R}}(s)u(s)] \end{pmatrix},$$

$$\mathbf{G} = \begin{pmatrix} G & \bar{\mathbf{G}}^\top \\ \bar{\mathbf{G}} & \widehat{\mathbf{G}} \end{pmatrix}, \quad \mathbf{Q}(s) = \begin{pmatrix} Q(s) & \bar{\mathbf{Q}}(s)^\top \\ \bar{\mathbf{Q}}(s) & \widehat{\mathbf{Q}}(s) \end{pmatrix}, \quad \mathbf{S}(s) = \begin{pmatrix} S(s) & \widetilde{\mathbf{S}}(s) \\ \bar{\mathbf{S}}(s) & \widehat{\mathbf{S}}(s) \end{pmatrix},$$

$$\mathbf{R}(s) = \begin{pmatrix} R(s) & \bar{\mathbf{R}}(s)^\top \\ \bar{\mathbf{R}}(s) & \widehat{\mathbf{R}}(s) \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} g_0 \\ \bar{\mathbf{g}} \end{pmatrix}, \quad \mathbf{q}(s) = \begin{pmatrix} q_0(s) \\ \bar{\mathbf{q}}(s) \end{pmatrix}, \quad \boldsymbol{\rho}(s) = \begin{pmatrix} \rho_0(s) \\ \bar{\boldsymbol{\rho}}(s) \end{pmatrix},$$

where $\mathbf{G}$, $\mathbf{Q}$, $\mathbf{S}$, $\mathbf{R}$ satisfy

$$\mathbf{G} \geqslant 0, \quad \begin{pmatrix} \mathbf{Q}(\cdot) & \mathbf{S}(\cdot)^\top \\ \mathbf{S}(\cdot) & \mathbf{R}(\cdot) \end{pmatrix} \geqslant 0, \quad \mathbf{R} \geqslant \delta \begin{pmatrix} I & O \\ O & O \end{pmatrix}. \tag{4.6}$$

with O being the matrices zero with different appropriate dimensions.

Now, we propose the MF-LQ stochastic optimal control problem as follows:

Problem (MF-LQ). Find an admissible control $\bar{u}(\cdot) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$ such that

$$J(t,x;\bar{u}(\cdot)) = \inf_{u(\cdot)\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)} J(t,x;u(\cdot)). \tag{4.7}$$

In the above $\bar{u}(\cdot)$ is called an open-loop optimal control of Problem (MF-LQ), and the corresponding optimal state trajectory $X^{\bar{u}}(\cdot)$ is denoted by $\bar{X}(\cdot)$.

The following result characterizes the optimal control $\bar{u}(\cdot)$ of Problem (MF-LQ).

Theorem 4.2. *For any given $(t,x)\in\mathscr{D}$, $\bar u(\cdot)\in\mathcal{U}[t,T]=L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$ is an open-loop optimal control of Problem (MF-LQ) at (t,x) with $\bar X(\cdot)$ being the corresponding open-loop optimal state process, if and only if $u(\cdot)\mapsto J(t,x;u(\cdot))$ is convex and $(\bar X(\cdot),\bar Y(\cdot),\bar Z(\cdot),\bar u(\cdot))\in[L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))]^2\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)\times\mathcal{U}[t,T]$ solves the following system (the argument s is suppressed):*

$$
\left\{
\begin{aligned}
&R\bar u+2\bar{\mathbf{R}}^\top\mathbb{E}[\widetilde{\mathbf{R}}\bar u]+\widetilde{\mathbf{R}}^\top\mathbb{E}[\widehat{\mathbf{R}}]\mathbb{E}[\widetilde{\mathbf{R}}\bar u]+B^\top\bar Y+\widetilde{\mathbf{B}}^\top\mathbb{E}[\bar{\mathbf{B}}\bar Y]+D^\top\bar Z+\widetilde{\mathbf{D}}^\top\mathbb{E}[\bar{\mathbf{D}}\bar Z]\\
&\qquad+S\bar X+\widetilde{\mathbf{R}}^\top\mathbb{E}[\bar{\mathbf{S}}\bar X]+\widetilde{S}\mathbb{E}[\widetilde{\mathbf{Q}}\bar X]+\widetilde{\mathbf{R}}^\top\mathbb{E}[\widehat{\mathbf{S}}]\mathbb{E}[\widetilde{\mathbf{Q}}\bar X]+\rho=0,\quad s\in[t,T],\\
&d\bar X=\Big(A\bar X+\bar{\mathbf{A}}^\top\mathbb{E}[\widetilde{\mathbf{A}}\bar X]+B\bar u+\bar{\mathbf{B}}^\top\mathbb{E}[\widetilde{\mathbf{B}}\bar u]+b\Big)ds\\
&\qquad+\Big(C\bar X+\bar{\mathbf{C}}^\top\mathbb{E}[\widetilde{\mathbf{C}}\bar X]+D\bar u+\bar{\mathbf{D}}^\top\mathbb{E}[\widetilde{\mathbf{D}}\bar u]+\sigma\Big)dW,\quad s\in[t,T],\\
&d\bar Y=-\Big(A^\top\bar Y+\widetilde{\mathbf{A}}^\top\mathbb{E}[\bar{\mathbf{A}}\bar Y]+C^\top\bar Z+\widetilde{\mathbf{C}}^\top\mathbb{E}[\bar{\mathbf{C}}\bar Z]+Q\bar X+\widetilde{\mathbf{Q}}^\top\mathbb{E}[\bar{\mathbf{Q}}\bar X]+\bar{\mathbf{Q}}^\top\mathbb{E}[\widetilde{\mathbf{Q}}\bar X]\\
&\qquad+\widetilde{\mathbf{Q}}^\top\mathbb{E}[\widehat{\mathbf{Q}}]\mathbb{E}[\widetilde{\mathbf{Q}}\bar X]+S^\top\bar u+\bar{\mathbf{S}}^\top\mathbb{E}[\widetilde{\mathbf{R}}\bar u]+\widetilde{\mathbf{Q}}^\top\mathbb{E}[\widetilde{\mathbf{S}}^\top\bar u]+\widetilde{\mathbf{Q}}^\top\mathbb{E}[\widehat{\mathbf{S}}]\mathbb{E}[\widetilde{\mathbf{R}}\bar u]+q\Big)ds\\
&\qquad+\bar Z dW,\quad s\in[t,T],\\
&\bar X(t)=x,\\
&\bar Y(T)=G\bar X(T)+\widetilde{\mathbf{G}}^\top\mathbb{E}[\bar{\mathbf{G}}\bar X(T)]+\bar{\mathbf{G}}^\top\mathbb{E}[\widetilde{\mathbf{G}}\bar X(T)]+\widetilde{\mathbf{G}}^\top\mathbb{E}[\widehat{\mathbf{G}}]\mathbb{E}[\widetilde{\mathbf{G}}\bar X(T)]+g_0+\widetilde{\mathbf{G}}^\top\mathbb{E}[\bar{\mathbf{g}}].
\end{aligned}
\right.
\tag{4.8}
$$

Moreover, if (4.6) holds true, then the above system (4.8) admits a unique solution, and Problem (MF-LQ) admits a unique optimal open-loop control.

Proof. According to the results obtained in Section 3 (Theorem 3.2, Theorem 3.3 and Corollary 3.8), we only need to verify the operators in (4.1) and (4.2) satisfy Assumptions (H1), (H2) and (H3).

It is easy to check that Assumptions (H1), (H2) and (H3)-(i) are satisfied.

Next we will show (4.6) implies Assumption (H3)-(ii). Firstly, for any $\xi\in L^2(\Omega;\mathbb{R}^n)$,

$$
\begin{aligned}
&\mathbb{E}\langle\mathcal{G}\xi,\xi\rangle=\mathbb{E}\langle G\xi+\widetilde{\mathbf{G}}^\top\mathbb{E}[\bar{\mathbf{G}}\xi]+\bar{G}^\top\mathbb{E}[\widetilde{\mathbf{G}}\xi]+\widetilde{\mathbf{G}}^\top\mathbb{E}[\widehat{G}]\mathbb{E}[\widetilde{\mathbf{G}}\xi],\xi\rangle\\
&=\mathbb{E}\langle G\xi,\xi\rangle+2\mathbb{E}\langle\bar{\mathbf{G}}\xi,\mathbb{E}[\widetilde{\mathbf{G}}\xi]\rangle+\mathbb{E}\langle\widehat{\mathbf{G}}\mathbb{E}[\widetilde{\mathbf{G}}\xi],\mathbb{E}[\widetilde{\mathbf{G}}\xi]\rangle\\
&=\mathbb{E}\langle\mathbf{G}\widetilde{\boldsymbol{\xi}},\widetilde{\boldsymbol{\xi}}\rangle\geqslant 0,
\end{aligned}
$$

where $\widetilde{\boldsymbol{\xi}}^\top\equiv(\xi^\top,(\mathbb{E}[\widetilde{\mathbf{G}}\xi])^\top)$.

Secondly, for any $s\in[t,T]$, any $\xi\in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n)$, any $\eta\in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m)$, we have (the argument s is suppressed for simplicity):

$$
\begin{aligned}
&\mathbb{E}\left\langle\begin{pmatrix}\mathcal{Q}&\mathcal{S}^*\\ \mathcal{S}&\mathcal{R}\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix},\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right\rangle=\mathbb{E}\langle\mathcal{Q}\xi,\xi\rangle+2\mathbb{E}\langle\mathcal{S}\xi,\eta\rangle+\mathbb{E}\langle\mathcal{R}\eta,\eta\rangle\\
&=\Big\{\mathbb{E}\langle Q\xi,\xi\rangle+2\mathbb{E}\langle\bar{\mathbf{Q}}\xi,\mathbb{E}[\widetilde{\mathbf{Q}}\xi]\rangle+\mathbb{E}\langle\widehat{\mathbf{Q}}\mathbb{E}[\widetilde{\mathbf{Q}}\xi],\mathbb{E}[\widetilde{\mathbf{Q}}\xi]\rangle\Big\}\\
&\quad+2\Big\{\mathbb{E}\langle S\xi,\eta\rangle+\mathbb{E}\langle\bar{\mathbf{S}}\xi,\mathbb{E}[\widetilde{\mathbf{R}}\eta]\rangle+\mathbb{E}\langle\widetilde{\mathbf{S}}\mathbb{E}[\widetilde{\mathbf{Q}}\xi],\eta\rangle+\mathbb{E}\langle\widehat{\mathbf{S}}\mathbb{E}[\widetilde{\mathbf{Q}}\xi],\mathbb{E}[\widetilde{\mathbf{R}}\eta]\rangle\Big\}\\
&\quad+\Big\{\mathbb{E}\langle R\eta,\eta\rangle+2\mathbb{E}\langle\bar{\mathbf{R}}\eta,\mathbb{E}[\widetilde{\mathbf{R}}\eta]\rangle+\mathbb{E}\langle\widehat{\mathbf{R}}\mathbb{E}[\widetilde{\mathbf{R}}\eta],\mathbb{E}[\widetilde{\mathbf{R}}\eta]\rangle\Big\}\\
&=\mathbb{E}\langle\mathbf{Q}\boldsymbol{\xi},\boldsymbol{\xi}\rangle+2\mathbb{E}\langle\mathbf{S}\boldsymbol{\xi},\boldsymbol{\eta}\rangle+\mathbb{E}\langle\mathbf{R}\boldsymbol{\eta},\boldsymbol{\eta}\rangle\\
&=\mathbb{E}\left\langle\begin{pmatrix}\mathbf{Q}&\mathbf{S}^\top\\ \mathbf{S}&\mathbf{R}\end{pmatrix}\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{pmatrix},\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{pmatrix}\right\rangle\geqslant 0,
\end{aligned}
$$

where $\boldsymbol{\xi}^\top\equiv(\xi^\top,(\mathbb{E}[\widetilde{\mathbf{Q}}\xi])^\top)$ and $\boldsymbol{\eta}^\top\equiv(\eta^\top,(\mathbb{E}[\widetilde{\mathbf{R}}\eta])^\top)$.

Thirdly, for any $s\in[t,T]$, any $\eta\in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m)$,

$$
\mathbb{E}\langle\mathcal{R}\eta,\eta\rangle-\delta\mathbb{E}|\eta|^2=\mathbb{E}\langle\mathbf{R}\boldsymbol{\eta},\boldsymbol{\eta}\rangle-\delta\mathbb{E}\left\langle\begin{pmatrix}I&O\\ O&O\end{pmatrix}\boldsymbol{\eta},\boldsymbol{\eta}\right\rangle=\mathbb{E}\left\langle\left[\mathbf{R}-\delta\begin{pmatrix}I&O\\ O&O\end{pmatrix}\right]\boldsymbol{\eta},\boldsymbol{\eta}\right\rangle\geqslant 0.
$$

$\square$

Remark 4.3. *Due to the condition* (4.6)*, the inverse of operator* $\mathcal{R}(\cdot)$ *exists. In other words,* $\bar{u}(\cdot)$ *can be solved from the algebraic equation (the first two lines) in the system* (4.8)*. Therefore, system* (4.8) *is essentially a coupled mean-field FBSDE.*

Next, we try to present the explicit expression of the optimal control $\bar{u}(\cdot)$ by means of the coupled mean-field FBSDE (4.8). For this, the following lemma is necessary.

Lemma 4.4. *Under* (4.6), $\mathbb{E}\Big[I+2\widetilde{\mathbf{R}}R^{-1}\bar{\mathbf{R}}^\top+\widetilde{\mathbf{R}}R^{-1}\widetilde{\mathbf{R}}^\top\mathbb{E}[\widehat{\mathbf{R}}]\Big]$ *is invertible.*

Proof. We only need to show that all the eigenvalues of $\mathbb{E}\Big[I+2\widetilde{\mathbf{R}}R^{-1}\bar{\mathbf{R}}^\top+\widetilde{\mathbf{R}}R^{-1}\widetilde{\mathbf{R}}^\top\mathbb{E}[\widehat{\mathbf{R}}]\Big]$ are nonzero. From $\mathbf{R}=\begin{pmatrix}R & \bar{\mathbf{R}}^\top\\ \bar{\mathbf{R}} & \widehat{\mathbf{R}}\end{pmatrix}\geqslant\delta\begin{pmatrix}I & O\\ O & O\end{pmatrix}$, we know R is positive definite, and $\widehat{\mathbf{R}}-\bar{\mathbf{R}}R^{-1}\bar{\mathbf{R}}^\top$ is positive semi-definite.

Firstly, if $\widehat{\mathbf{R}}=0$, then $\bar{\mathbf{R}}$ is necessary 0. In this case, $I+2\widetilde{\mathbf{R}}R^{-1}\bar{\mathbf{R}}^\top+\widetilde{\mathbf{R}}R^{-1}\widetilde{\mathbf{R}}^\top\mathbb{E}[\widehat{\mathbf{R}}]\equiv I$ is positive.

Secondly, if $\widehat{\mathbf{R}}>0$,

$$
\begin{aligned}
&\mathbb{E}\Big[I+2\widetilde{\mathbf{R}}R^{-1}\bar{\mathbf{R}}^\top+\widetilde{\mathbf{R}}R^{-1}\widetilde{\mathbf{R}}^\top\mathbb{E}[\widehat{\mathbf{R}}]\Big]\\
&=\mathbb{E}\Big[\Big((\mathbb{E}[\widehat{\mathbf{R}}])^{-1}+2\widetilde{\mathbf{R}}R^{-1}\bar{\mathbf{R}}^\top(\mathbb{E}[\widehat{\mathbf{R}}])^{-1}+\widetilde{\mathbf{R}}R^{-1}\widetilde{\mathbf{R}}^\top\Big)\mathbb{E}[\widehat{\mathbf{R}}]\Big]\\
&=\mathbb{E}\Big[\Big\langle R\Big(R^{-1}\widetilde{\mathbf{R}}^\top+R^{-1}\bar{\mathbf{R}}^\top(\mathbb{E}[\widehat{\mathbf{R}}])^{-1}\Big),R^{-1}\widetilde{\mathbf{R}}^\top+R^{-1}\bar{\mathbf{R}}^\top(\mathbb{E}[\widehat{\mathbf{R}}])^{-1}\Big\rangle\Big]\mathbb{E}[\widehat{\mathbf{R}}]\\
&\quad+\mathbb{E}\Big[I-\bar{\mathbf{R}}R^{-1}\bar{\mathbf{R}}^\top(\mathbb{E}[\widehat{\mathbf{R}}])^{-1}\Big]\\
&=\mathbb{E}\Big[\Big\langle R\Big(R^{-1}\widetilde{\mathbf{R}}^\top+R^{-1}\bar{\mathbf{R}}^\top(\mathbb{E}[\widehat{\mathbf{R}}])^{-1}\Big),R^{-1}\widetilde{\mathbf{R}}^\top+R^{-1}\bar{\mathbf{R}}^\top(\mathbb{E}[\widehat{\mathbf{R}}])^{-1}\Big\rangle\Big]\mathbb{E}[\widehat{\mathbf{R}}]\\
&\quad+\mathbb{E}[\widehat{\mathbf{R}}-\bar{\mathbf{R}}R^{-1}\bar{\mathbf{R}}^\top](\mathbb{E}[\widehat{\mathbf{R}}])^{-1}.
\end{aligned}
\tag{4.9}
$$

From R and $\widehat{\mathbf{R}}$ being positive definite, we know all the eigenvalues of the following matrix

$$
\mathbb{E}\Big[\Big\langle R\Big(R^{-1}\widetilde{\mathbf{R}}^\top+R^{-1}\bar{\mathbf{R}}^\top(\mathbb{E}[\widehat{\mathbf{R}}])^{-1}\Big),R^{-1}\widetilde{\mathbf{R}}^\top+R^{-1}\bar{\mathbf{R}}^\top(\mathbb{E}[\widehat{\mathbf{R}}])^{-1}\Big\rangle\Big]\mathbb{E}[\widehat{\mathbf{R}}]
$$

are positive.

On the other hand, $\mathbb{E}[\widehat{\mathbf{R}}-\bar{\mathbf{R}}R^{-1}\bar{\mathbf{R}}^\top]$ being positive semi-definite, and $(\mathbb{E}[\widehat{\mathbf{R}}])^{-1}$ being positive definite imply us that $\mathbb{E}[\widehat{\mathbf{R}}-\bar{\mathbf{R}}R^{-1}\bar{\mathbf{R}}^\top](\mathbb{E}[\widehat{\mathbf{R}}])^{-1}$ has the non-negative eigenvalues.

Therefore, from (4.9), we get $\mathbb{E}\Big[I+2\widetilde{\mathbf{R}}R^{-1}\bar{\mathbf{R}}^\top+\widetilde{\mathbf{R}}R^{-1}\widetilde{\mathbf{R}}^\top\mathbb{E}[\widehat{\mathbf{R}}]\Big]$ is invertible. $\square$

Corollary 4.5. *Under* (4.6)*. The optimal control* $\bar{u}(\cdot)$ *has the following explicit expression:*

$$
\bar{u}=\Big(2\bar{\mathbf{R}}^\top+\widetilde{\mathbf{R}}^\top\mathbb{E}[\widehat{\mathbf{R}}]\Big)\Big(\mathbb{E}\Big[I+2\widetilde{\mathbf{R}}R^{-1}\bar{\mathbf{R}}^\top+\widetilde{\mathbf{R}}R^{-1}\widetilde{\mathbf{R}}^\top\mathbb{E}[\widehat{\mathbf{R}}]\Big]\Big)^{-1}\mathbb{E}[\widetilde{\mathbf{R}}R^{-1}\Lambda]-\Lambda.
$$

where

$$
\Lambda\equiv B^\top\bar{Y}+\widetilde{\mathbf{B}}^\top\mathbb{E}[\bar{\mathbf{B}}\bar{Y}]+D^\top\bar{Z}+\widetilde{\mathbf{D}}^\top\mathbb{E}[\bar{\mathbf{D}}\bar{Z}]+S\bar{X}+\widetilde{R}^\top\mathbb{E}[\bar{\mathbf{S}}\bar{X}]+\widetilde{S}\mathbb{E}[\widetilde{\mathbf{Q}}\bar{X}]+\widetilde{R}^\top\mathbb{E}[\widehat{\mathbf{S}}]\mathbb{E}[\widetilde{\mathbf{Q}}\bar{X}]+\rho.
$$

Proof. To get the explicit expression of the optimal control $\bar{u}(\cdot)$, we need to solve the following algebraic equation:

$$
R\bar{u}+2\bar{\mathbf{R}}^\top\mathbb{E}[\widetilde{\mathbf{R}}\bar{u}]+\widetilde{\mathbf{R}}^\top\mathbb{E}[\widehat{\mathbf{R}}]\mathbb{E}[\widetilde{\mathbf{R}}\bar{u}]=-\Lambda
\tag{4.10}
$$

Firstly, multiplying $\widetilde{\mathbf{R}}R^{-1}$ on the both sides of the above equality, and taking expectation, we get

$$
\mathbb{E}\Big[I+2\widetilde{\mathbf{R}}R^{-1}\bar{\mathbf{R}}^\top+\widetilde{\mathbf{R}}R^{-1}\widetilde{\mathbf{R}}^\top\mathbb{E}[\widehat{\mathbf{R}}]\Big]\mathbb{E}[\widetilde{\mathbf{R}}\bar{u}]=-\mathbb{E}[\widetilde{\mathbf{R}}R^{-1}\Lambda].
$$

Due to $\mathbb{E}\Big[I+2\widetilde{\mathbf{R}}R^{-1}\bar{\mathbf{R}}^\top+\widetilde{\mathbf{R}}R^{-1}\widetilde{\mathbf{R}}^\top\mathbb{E}[\widehat{\mathbf{R}}]\Big]$ is invertible, we know

$$\mathbb{E}[\widetilde{\mathbf{R}}\bar{u}]=-\Big(\mathbb{E}\Big[I+2\widetilde{\mathbf{R}}R^{-1}\bar{\mathbf{R}}^\top+\widetilde{\mathbf{R}}R^{-1}\widetilde{\mathbf{R}}^\top\mathbb{E}[\widehat{\mathbf{R}}]\Big]\Big)^{-1}\mathbb{E}[\widetilde{\mathbf{R}}R^{-1}\Lambda].$$

Therefore, the optimal control $\bar{u}(\cdot)$ is

$$\begin{aligned}\bar{u}&=-\Big(2\bar{\mathbf{R}}^\top\mathbb{E}[\widetilde{\mathbf{R}}\bar{u}]+\widetilde{\mathbf{R}}^\top\mathbb{E}[\widehat{\mathbf{R}}]\mathbb{E}[\widetilde{\mathbf{R}}\bar{u}]\Big)-\Lambda\\&=-\Big(2\bar{\mathbf{R}}^\top+\widetilde{\mathbf{R}}^\top\mathbb{E}[\widehat{\mathbf{R}}]\Big)\mathbb{E}[\widetilde{\mathbf{R}}\bar{u}]-\Lambda\\&=\Big(2\bar{\mathbf{R}}^\top+\widetilde{\mathbf{R}}^\top\mathbb{E}[\widehat{\mathbf{R}}]\Big)\Big(\mathbb{E}\Big[I+2\widetilde{\mathbf{R}}R^{-1}\bar{\mathbf{R}}^\top+\widetilde{\mathbf{R}}R^{-1}\widetilde{\mathbf{R}}^\top\mathbb{E}[\widehat{\mathbf{R}}]\Big]\Big)^{-1}\mathbb{E}[\widetilde{\mathbf{R}}R^{-1}\Lambda]-\Lambda.\end{aligned}$$

□

5 Conclusion

It is the mean-field LQ Problem (MF-LQ) that inspires us to study the LQ optimal control problem with operator coefficients (i.e., Problem (OLQ)). As we known, (4.3) and (4.4) is a new form of mean-field LQ problems. Besides, all the coefficients are allowed to be random in our study. As a start, we only study the open-loop case. The closed-loop cases of the control problems, as well as differential games are under our investigation.

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