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MEAN-FIELD LINEAR-QUADRATIC STOCHASTIC DIFFERENTIAL GAMES IN AN INFINITE HORIZON*

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Abstract. This paper is concerned with two-person mean-field linear-quadratic non-zero sum stochastic differential games in an infinite horizon. Both open-loop and closed-loop Nash equilibria are introduced. The existence of an open-loop Nash equilibrium is characterized by the solvability of a system of mean-field forward-backward stochastic differential equations in an infinite horizon and the convexity of the cost functionals, and the closed-loop representation of an open-loop Nash equilibrium is given through the solution to a system of two coupled non-symmetric algebraic Riccati equations. The existence of a closed-loop Nash equilibrium is characterized by the solvability of a system of two coupled symmetric algebraic Riccati equations. Two-person mean-field linear-quadratic zero-sum stochastic differential games in an infinite horizon are also considered. Both the existence of open-loop and closed-loop saddle points are characterized by the solvability of a system of two coupled generalized algebraic Riccati equations with static stabilizing solutions. Mean-field linear-quadratic stochastic optimal control problems in an infinite horizon are discussed as well, for which it is proved that the open-loop solvability and closed-loop solvability are equivalent.

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1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a complete filtered probability space, on which a one-dimensional standard Brownian motion $W(\cdot)$ is defined with $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t\geqslant 0}$ being its natural filtration augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$, and $\mathbb{E}[\,\cdot\,]$ denotes the expectation with respect to $\mathbb{P}$. Throughout this paper, we let $\mathbb{R}^{n\times m}$ and $\mathbb{S}^n$ be the set of all $(n\times m)$ (real) matrices and $(n\times n)$ symmetric (real) matrices. We denote $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. For a Euclidean space $\mathbb{H}$, say, $\mathbb{H} = \mathbb{R}^n, \mathbb{R}^{n\times m}$, let $C([0,\infty);\mathbb{H})$ denote the space of $\mathbb{H}$-valued continuous functions $\varphi : [0,\infty) \to \mathbb{H}$, $L^2(\mathbb{H})$ denote the space of $\mathbb{H}$-valued functions $\varphi : [0,\infty) \to \mathbb{H}$ with $\int_0^\infty |\varphi(t)|^2 \mathrm{d}t < \infty$, and $L^2_{\mathbb{F}}(\mathbb{H})$ denote the space of $\mathbb{F}$-progressively measurable processes $\varphi : [0,\infty)\times\Omega \to \mathbb{H}$ with $\mathbb{E}\int_0^\infty |\varphi(t)|^2 \mathrm{d}t < \infty$.

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Consider the following controlled linear *mean-field stochastic differential equation* (MF-SDE, for short) on an infinite horizon $[0,\infty)$:

$$\begin{cases} \mathrm{d}X(t) = \big\{AX(t) + \bar{A}\mathbb{E}[X(t)] + B_1u_1(t) + \bar{B}_1\mathbb{E}[u_1(t)] + B_2u_2(t) + \bar{B}_2\mathbb{E}[u_2(t)] + b(t)\big\}\mathrm{d}t \\ \qquad\qquad +\big\{CX(t)+\bar{C}\mathbb{E}[X(t)]+D_1u_1(t)+\bar{D}_1\mathbb{E}[u_1(t)]+D_2u_2(t)+\bar{D}_2\mathbb{E}[u_2(t)]+\sigma(t)\big\}\mathrm{d}W(t),\ t\geqslant 0, \\ X(0) = x. \end{cases} \tag{1.1}$$

In the above, $X(\cdot)$ is the state process taking values in $\mathbb{R}^n$ with x being the initial state. For $i=1,2$, $u_i(\cdot)$ is the control process of Player i, taking values in $\mathbb{R}^{m_i}$, respectively. The coefficients $A,\bar{A},C,\bar{C}\in\mathbb{R}^{n\times n}$, $B_1,\bar{B}_1,D_1,\bar{D}_1\in\mathbb{R}^{n\times m_1}$, $B_2,\bar{B}_2,D_2,\bar{D}_2\in\mathbb{R}^{n\times m_2}$ are given constant matrices, and the non-homogenous terms $b(\cdot),\sigma(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^n)$. We introduce the following spaces:

$$\begin{cases} \mathscr{X}[0,T] = \Big\{X:[0,\infty)\times\Omega\to\mathbb{R}^n \;\Big|\; X(\cdot) \text{ is } \mathbb{F}\text{-adapted}, t\mapsto X(t,\omega) \text{ is continuous, and} \\ \qquad\qquad\qquad\qquad \mathbb{E}\Big[\sup_{t\in[0,T]}|X(t)|^2\Big]<\infty\Big\}, \quad \text{for } T>0, \\ \mathscr{X}_{loc}[0,\infty) = \bigcap_{T>0}\mathscr{X}[0,T], \qquad \mathscr{X}[0,\infty) = \Big\{X(\cdot)\in\mathscr{X}_{loc}[0,\infty)\;\Big|\;\mathbb{E}\int_0^\infty|X(t)|^2\mathrm{d}t<\infty\Big\}. \end{cases}$$

By a standard argument using contraction mapping theorem, one can show that for any initial state $x\in\mathbb{R}^n$ and control pair $(u_1(\cdot),u_2(\cdot))\in L^2_{\mathbb{F}}(\mathbb{R}^{m_1})\times L^2_{\mathbb{F}}(\mathbb{R}^{m_2})$, state equation (1.1) admits a unique strong solution $X(\cdot)\equiv X(\cdot;x,u_1(\cdot),u_2(\cdot))\in\mathscr{X}_{loc}[0,\infty)$. Next, for $i=1,2$, we introduce the following cost functionals:

$$J_i(x;u_1(\cdot),u_2(\cdot)) = \mathbb{E}\int_0^\infty g_i(t,X(t),u_1(t),u_2(t),\mathbb{E}[X(t)],\mathbb{E}[u_1(t)],\mathbb{E}[u_2(t)])\mathrm{d}t, \tag{1.2}$$

with

$$\begin{aligned} g_i(t,x,u_1,u_2,\bar{x},\bar{u}_1,\bar{u}_2) = &\left\langle\begin{pmatrix} Q_i & S_{i1}^\top & S_{i2}^\top \\ S_{i1} & R_{i11} & R_{i12} \\ S_{i2} & R_{i21} & R_{i22}\end{pmatrix}\begin{pmatrix}x\\u_1\\u_2\end{pmatrix},\begin{pmatrix}x\\u_1\\u_2\end{pmatrix}\right\rangle + 2\left\langle\begin{pmatrix}q_i(t)\\ \rho_{i1}(t)\\ \rho_{i2}(t)\end{pmatrix},\begin{pmatrix}x\\u_1\\u_2\end{pmatrix}\right\rangle \\ &+\left\langle\begin{pmatrix} \bar{Q}_i & \bar{S}_{i1}^\top & \bar{S}_{i2}^\top \\ \bar{S}_{i1} & \bar{R}_{i11} & \bar{R}_{i12} \\ \bar{S}_{i2} & \bar{R}_{i21} & \bar{R}_{i22}\end{pmatrix}\begin{pmatrix}\bar{x}\\\bar{u}_1\\\bar{u}_2\end{pmatrix},\begin{pmatrix}\bar{x}\\\bar{u}_1\\\bar{u}_2\end{pmatrix}\right\rangle, \end{aligned} \tag{1.3}$$

where

$$\begin{cases} Q_i,\ \bar{Q}_i\in\mathbb{S}^n, & S_{i1},\ \bar{S}_{i1}\in\mathbb{R}^{m_1\times n}, & S_{i2},\ \bar{S}_{i2}\in\mathbb{R}^{m_2\times n}, \\ R_{i11},\ \bar{R}_{i11}\in\mathbb{S}^{m_1}, & R_{i22},\ \bar{R}_{i22}\in\mathbb{S}^{m_2}, & R_{i12}=R_{i21}^\top,\ \bar{R}_{i12}=\bar{R}_{i21}^\top\in\mathbb{R}^{m_1\times m_2}, \\ q_i(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^n), & \rho_{i1}(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^{m_1}), & \rho_{i2}(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^{m_2}). \end{cases}$$

Note that for $(x,u_1(\cdot),u_2(\cdot))\in\mathbb{R}^n\times L^2_{\mathbb{F}}(\mathbb{R}^{m_1})\times L^2_{\mathbb{F}}(\mathbb{R}^{m_2})$, the solution $X(\cdot)\equiv X(\cdot\,;x,u_1(\cdot),u_2(\cdot))$ to (1.1) might just be in $\mathscr{X}_{loc}[0,\infty)$ in general. Therefore, in order the cost functionals $J_i(x;u_1(\cdot),u_2(\cdot)), i=1,2$ to be defined, the control pair $(u_1(\cdot),u_2(\cdot))$ has to be restricted in the following set of *admissible control pairs*:

$$\mathscr{U}_{ad}(x) = \Big\{(u_1(\cdot),u_2(\cdot))\in L^2_{\mathbb{F}}(\mathbb{R}^{m_1})\times L^2_{\mathbb{F}}(\mathbb{R}^{m_2})\;\Big|\;X(\cdot\,;x,u_1(\cdot),u_2(\cdot))\in\mathscr{X}[0,\infty)\Big\}, \quad x\in\mathbb{R}^n. \tag{1.4}$$

Note that $\mathscr{U}_{ad}(x)$ depends on the initial state x. For any $(u_1(\cdot),u_2(\cdot))\in\mathscr{U}_{ad}(x)$, the corresponding $X(\cdot)\equiv X(\cdot\,;x,u_1(\cdot),u_2(\cdot))$ is called an *admissible state process* for the initial state x. Then we can loosely formulate the following problem.

Problem (MF-SDG). For any initial state $x\in\mathbb{R}^n$, Player i $(i=1,2)$ wants to find a control $u_i^*(\cdot)$ so that $(u_1^*(\cdot),u_2^*(\cdot))\in\mathscr{U}_{ad}(x)$ such that the cost functionals $u_1(\cdot)\mapsto J_1(x;u_1(\cdot),u_2^*(\cdot))$ and $u_2(\cdot)\mapsto J_2(x;u_1^*(\cdot),u_2(\cdot))$ are minimized, for all $(u_1(\cdot),u_2^*(\cdot)),(u_1^*(\cdot),u_2(\cdot))\in\mathscr{U}_{ad}(x)$, subject to (1.1).

We refer to the above problem as a *mean-field linear-quadratic* (LQ, for short) *two-person (non-zero sum) stochastic differential game in an infinite horizon*. In the special case where $b(\cdot),\sigma(\cdot),q_i(\cdot),\rho_{ij}(\cdot)$ are all zero, we denote the corresponding problem and cost functionals by Problem (MF-SDG)^0 and $J_i^0(x;u_1(\cdot),u_2(\cdot))$, respectively. On the other hand, when

$$J_1(x;u_1(\cdot),u_2(\cdot))+J_2(x;u_1(\cdot),u_2(\cdot))=0,\quad\forall x\in\mathbb{R}^n,\ \forall(u_1(\cdot),u_2(\cdot))\in\mathscr{U}_{ad}(x),\tag{1.5}$$

the corresponding Problem (MF-SDG) is called a *mean-field LQ two-person zero-sum stochastic differential game in an infinite horizon* and is denoted by Problem (MF-SDG)$_0$. To guarantee (1.5), one usually lets

$$\begin{cases}Q_1+Q_2=0, & \bar{Q}_1+\bar{Q}_2=0, & q_1(\cdot)+q_2(\cdot)=0,\\ S_{1j}+S_{2j}=0, & \bar{S}_{1j}+\bar{S}_{2j}=0, & \rho_{1j}(\cdot)+\rho_{2j}(\cdot)=0, \qquad j=1,2,\\ R_{1jk}+R_{2jk}=0, & \bar{R}_{1jk}+\bar{R}_{2jk}=0, & j,k=1,2.\end{cases}\tag{1.6}$$

One may feel that cost functional (1.2) could be a little more general by including terms like $\langle\bar{q}_i(s),\mathbb{E}[X(s)]\rangle$. However, it is not hard to see that (as long as the integrals exist)

$$\mathbb{E}\int_0^\infty\big\langle\bar{q}_i(s),\mathbb{E}[X(s)]\big\rangle\mathrm{d}s=\mathbb{E}\int_0^\infty\big\langle\mathbb{E}[\bar{q}_i(s)],X(s)\big\rangle\mathrm{d}s,$$

which can be absorbed by replacing $q_i(\cdot)$ by $q_i(\cdot)+\mathbb{E}[\bar{q}_i(\cdot)]$ in the cost functionals. Likewise, terms like $\big\langle\bar{\rho}_{ij}(s),\mathbb{E}[u_j(s)]\big\rangle$ are not necessarily included.

We will introduce proper stabilizability conditions for the system so that $\mathscr{U}_{ad}(x)\neq\varnothing$ for all $x\in\mathbb{R}^n$. It is not hard to see that without stabilizabity conditions, one might only has $X(\cdot\,;x,u_1(\cdot),u_2(\cdot))\in\mathscr{X}_{loc}[0,\infty)$ so that $J_i(x;u_1(\cdot),u_2(\cdot))$ might not be well-defined. Then, instead of considering the cost functionals of form (1.2), one might naturally consider the following ergodic type cost functionals:

$$\widetilde{J}_i(x;u_1(\cdot),u_2(\cdot))=\varlimsup_{T\to\infty}\frac{1}{T}\int_0^T g_i\big(t,X(t),u_1(t),u_2(t),\mathbb{E}[X(t)],\mathbb{E}[u_1(t)],\mathbb{E}[u_2(t)]\big)\mathrm{d}t.\tag{1.7}$$

The above type cost functionals are normally used for the case that the running cost rate is bounded so that the right-hand side of the above is always finite (see, for example, Cohen-Fedyashov [18]). However, if no stabilizability conditions are assumed, the state process $X(\cdot)$ could be of exponential growth. In that case, the above type cost functionals are still not useful. Therefore, instead of (1.7), we prefer to study our problems under certain stabilizability condition with cost functionals of for (1.2) restricted on $\mathscr{U}_{ad}(x)$.

The theory of MF-SDEs can be traced back to the work of Kac [25] in the middle of 1950s, where a stochastic toy model for the Vlasov type kinetic equation of plasma was presented. Its rigorous study was initiated by McKean [32] in 1966, which is now known as McKean-Vlasov stochastic differential equations. Since then, many researchers have made contributions to the related topics and their applications; see, for example, Scheutzow [40], Chan [17], Huang–Malhame–Caines [24], Buckdahn–Li–Peng [13], Carmona-Delarue [15], Bensoussan–Yam–Zhang [9], Buckdahn–Li–Peng–Rainer [14], etc.

Optimal control and differential game problems of MF-SDEs have drawn enormous researchers' attention recently. See Ahmed–Ding [3], Lasry–Lions [26], Andersson–Djehiche [4], Buckdahn–Djehiche-Li [10], Li [27],

Meyer-Brandis–Oksendal–Zhou [33], Hosking [21], Bensoussan–Sung–Yam [7], Djehiche–Tembine-Tempone [20], Bensoussan–Sung–Yam–Yung [8], Djehiche-Tembine [19], Huang–Li–Wang [22], Yong [54], Buckdahn–Li–Ma [11, 12], Pham–Wei [37, 38], Li–Sun–Xiong [30], Miller–Pham [34], Moon [35], and the references therein. Next, let us mention a few recent pieces of literature related to our present paper. In Yong [53], an LQ optimal control problem for MF-SDEs in a finite horizon was introduced and investigated. The optimality system of a linear *mean-field forward-backward stochastic differential equation* (MF-FBSDE, for short) is derived, and two Riccati differential equations are obtained to present the feedback representation of the optimal control. Huang–Li–Yong [23] generalized the results in [53] to the infinite horizon case, and the feedback representation of the optimal control is derived *via* two *algebraic Riccati equations* (AREs, for short). Note that in [23], some notions of stabilizability for controlled MF-SDEs are introduced, which are interestingly different from the classical ones, due to the presence of the terms $\mathbb{E}[X(\cdot)]$ and $\mathbb{E}[u(\cdot)]$. Sun [41] continued to investigate the LQ optimal control problem for MF-SDEs in the finite horizon with additional nonhomogeneous terms and concluded that the uniform convexity of the cost functional is sufficient for the open-loop solvability of the LQ optimal control problems for MF-SDEs. Moreover, the uniform convexity of the cost functional is equivalent to the solvability of two coupled differential Riccati equations, and the unique open-loop optimal control admits a state feedback representation in the case that the cost functional is uniformly convex. Li–Sun–Yong [31] studied the closed-loop solvability of the corresponding problem, which is characterized by the existence of a regular solution to the coupled two generalized Riccati equations, together with some constraints on the adapted solution to a linear *backward stochastic differential equation* (BSDE, for short) and a linear terminal value problem of an *ordinary differential equation* (ODE, for short). Li–Li–Yu [28] analyzed the indefinite mean-field type LQ stochastic optimal control problems, where they introduced a relaxed compensator to characterize the open-loop solvability of the problem. Tian–Yu–Zhang [51] considered an LQ zero-sum stochastic differential game with mean-field type, proposed the notions of explicit and implicit strategy laws, and established the closed-loop formulation for saddle points in the mixed-strategy-law. Very recently, Sun–Wang–Wu [44] studied a two-person zero-sum mean-field LQ stochastic differential game over the finite horizon by a Hilbert space method introduced by Mou–Yong [36]. It is shown that the associated two Riccati equations admit unique and strongly regular solutions under the sufficient condition for the existence of an open-loop saddle point when the open-loop saddle point can be represented as linear feedback of the current state. When only the necessary condition for the existence of an open-loop saddle point is satisfied, we can construct an approximate sequence by solving a family of Riccati equations and closed-loop systems. The approximate sequence's convergence turns out to be equivalent to the open-loop solvability of the game, and its limit exactly equals an open-loop saddle point, provided that the game is open-loop solvable.

Ait Rami–Zhou [1] and Ait Rami–Zhou–Moore [2] considered stochastic LQ problems in an infinite horizon, with indefinite control weighting matrices. They introduced a generalized ARE, involving a matrix pseudo-inverse and two additional algebraic equality/inequality constraints, and proved that the problem's attainability is equivalent to the existence of a static stabilizing solution to the generalized ARE. In particular, the associated AREs can be solved by linear matrix inequality and semidefinite programming techniques. In addition to the statements in the previous paragraph about [23], the authors discussed the solvabilities of AREs, by linear matrix inequalities. Sun–Yong [46] first found that both the open-loop and closed-loop solvabilities of the stochastic LQ problems in the infinite horizon are equivalent to the existence of a static stabilizing solution to the associated generalized ARE. We refer the readers to the recent monographs by Bensoussan–Frehse–Yam [6] and by Sun–Yong [48, 49] for more details and references cited therein.

In this paper, we consider two-person mean-field LQ non-zero sum stochastic differential games in an infinite horizon. Both open-loop and closed-loop Nash equilibria are introduced. The existence of an open-loop Nash equilibrium is characterized by the solvability of a system of MF-FBSDEs in an infinite horizon and the convexity of the cost functionals. The closed-loop representation of an open-loop Nash equilibrium is given through the solution to a system of two coupled non-symmetric AREs. The existence of a closed-loop Nash equilibrium is characterized by the solvability of a system of two coupled symmetric AREs. Two-person mean-field LQ zero-sum stochastic differential games in an infinite horizon are also considered. The existence of open-loop and closed-loop saddle points is characterized by the solvability of a system of two coupled generalized AREs with

static stabilizing solutions. As special cases, mean-field LQ stochastic optimal control problems in an infinite horizon are discussed as well, for which it is proved that the open-loop solvability and closed-loop solvability are equivalent. The results obtained in this paper enrich the theory of optimal control and differential games of mean-field type.

Let us briefly highlight the major novelty of this paper:

(i) For MF-SDEs with quadratic performance indexes in an infinite-horizon, problems of two-person non-zero sum differential games, two-person zero-sum differential games and optimal control (which is a single player differential game) are treated in a unified framework. Among other results, the most significant one is the discovery of the system of coupled algebraic Riccati equations (3.35) which is used to characterize the closed-loop Nash equilibrium. From this point of view, the current paper can be regarded as a complementary or a continuation of Sun–Yong [47].

(ii) For MF-SDE LQ optimal control problems in an infinite horizon, we have established the equivalence among the solvability of a system of coupled algebraic Riccati equations, open-loop solvability, and closed-loop solvability. This covers the relevant results found in Sun–Yong [46] where mean-field terms were absent.

(iii) For two-person zero-sum differential games of MF-SDEs with quadratic performance index in $[0,\infty)$, we have proved that if an open-loop saddle point admits a closed-loop representation, and the closed-loop saddle point exists, then the open-loop saddle point must be the outcome of the closed-loop saddle point. It is also shown that such a property fails for non-zero sum differential games. From this angle, the current paper is an extension of Sun–Yong–Zhang [50] where the mean-field terms did not appear.

The rest of the paper is organized as follows. In Section 2, we present some preliminary results about mean-field LQ stochastic optimal control problems in an infinite horizon. Section 3 aims to give results on mean-field LQ non-zero sum stochastic differential games, including open-loop Nash equilibria and their closed-loop representation, and closed-loop Nash equilibria with algebraic Riccati equations. In Section 4, the open-loop and closed-loop saddle points for mean-field LQ zero-sum stochastic differential games are investigated. Some examples are presented in Section 5 illustrating the results developed in the earlier sections. Finally, some concluding remarks are collected in Section 6.

2. Preliminaries

Throughout this paper, besides the notation introduced in the previous section, we let I be the identity matrix or operator with a suitable size. We will use $\langle\cdot\,,\cdot\rangle$ for inner products in possibly different Hilbert spaces, and denote by $|\cdot|$ the norm induced by $\langle\cdot\,,\cdot\rangle$. Let $M^\top$ and $\mathscr{R}(M)$ be the transpose and range of a matrix M, respectively. For $M,N\in\mathbb{S}^n$, we write $M\geqslant N$ (respectively, $M>N$) for $M-N$ being positive semi-definite (respectively, positive definite). Let $M^\dagger$ denote the pseudo-inverse of a matrix $M\in\mathbb{R}^{m\times n}$, which is equal to the inverse M^{-1} of $M\in\mathbb{R}^{n\times n}$ if it exists. See Penrose [39] or Anderson–Moore [5] for some basic properties of the pseudo-inverse. We define the inner product in $L^2_{\mathbb{F}}(\mathbb{H})$ by $\langle\varphi,\phi\rangle=\mathbb{E}\int_0^\infty\langle\varphi(t),\phi(t)\rangle\mathrm{d}t$ so that $L^2_{\mathbb{F}}(\mathbb{H})$ is a Hilbert space.

We now consider the following controlled linear MF-SDE over $[0,\infty)$:

$$\begin{cases}\mathrm{d}X(t)=\big\{AX(t)+\bar{A}\mathbb{E}[X(t)]+Bu(t)+\bar{B}\mathbb{E}[u(t)]+b(t)\big\}\mathrm{d}t\\ \qquad\qquad+\big\{CX(t)+\bar{C}\mathbb{E}[X(t)]+Du(t)+\bar{D}\mathbb{E}[u(t)]+\sigma(t)\big\}\mathrm{d}W(t),\quad t\geqslant 0,\\ X(0)=x,\end{cases}\tag{2.1}$$

with quadratic cost functional

$$\begin{aligned}J(x;u(\cdot))=\mathbb{E}\int_0^\infty\bigg[&\bigg\langle\begin{pmatrix}Q&S^\top\\S&R\end{pmatrix}\begin{pmatrix}X(t)\\u(t)\end{pmatrix},\begin{pmatrix}X(t)\\u(t)\end{pmatrix}\bigg\rangle+2\bigg\langle\begin{pmatrix}q(t)\\\rho(t)\end{pmatrix},\begin{pmatrix}X(t)\\u(t)\end{pmatrix}\bigg\rangle\\&+\bigg\langle\begin{pmatrix}\bar{Q}&\bar{S}^\top\\\bar{S}&\bar{R}\end{pmatrix}\begin{pmatrix}\mathbb{E}[X(t)]\\\mathbb{E}[u(t)]\end{pmatrix},\begin{pmatrix}\mathbb{E}[X(t)]\\\mathbb{E}[u(t)]\end{pmatrix}\bigg\rangle\bigg]\mathrm{d}t,\end{aligned}\tag{2.2}$$

where $A, \bar{A}, C, \bar{C} \in \mathbb{R}^{n\times n}$, $B, \bar{B}, D, \bar{D} \in \mathbb{R}^{n\times m}$, $Q, \bar{Q} \in \mathbb{S}^n$, $S, \bar{S} \in \mathbb{R}^{m\times n}$, $R, \bar{R} \in \mathbb{S}^m$ are given constant matrices, and $b(\cdot), \sigma(\cdot), q(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^n)$, $\rho(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^m)$ are stochastic processes. For any initial state $x \in \mathbb{R}^n$ and control $u(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^m)$, equation (2.1) admits a unique strong solution $X(\cdot) \equiv X(\cdot\,; x, u(\cdot)) \in \mathscr{X}_{loc}[0,\infty)$. We define the *admissible control set* as

$$\mathscr{U}_{ad}(x) = \big\{u(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^m) \;\big|\; X(\cdot) \equiv X(\cdot; x, u(\cdot)) \in \mathscr{X}[0,\infty)\big\}. \tag{2.3}$$

In general, $\mathscr{U}_{ad}(x)$ depends on $x \in \mathbb{R}^n$. Let us pose the following optimal control problem.

Problem (MF-SLQ). For any initial state $x \in \mathbb{R}^n$, find a control $u^*(\cdot) \in \mathscr{U}_{ad}(x)$ such that the cost functional $J(x; u(\cdot))$ of (2.2) is minimized, subject to (2.1). That is to say,

$$J(x; u^*(\cdot)) = \inf_{u(\cdot)\in\mathscr{U}_{ad}(x)} J(x; u(\cdot)) \equiv V(x). \tag{2.4}$$

Any $u^*(\cdot) \in \mathscr{U}_{ad}(x)$ satisfying (2.4) is called an *open-loop optimal control* of Problem (MF-SLQ), and the corresponding $X^*(\cdot) \equiv X(\cdot\,; x, u^*(\cdot))$ is called an *open-loop optimal state process*. The function $V(\cdot)$ is called the *value function* of Problem (MF-SLQ). In the special case where $b(\cdot), \sigma(\cdot), q(\cdot), \rho(\cdot)$ are all zero, we denote the corresponding problem by Problem (MF-SLQ)^0, the cost functional by $J^0(x; u(\cdot))$ and the value function by $V^0(x)$, respectively.

In order Problem (MF-SLQ) to be meaningful, we need to find conditions under which $\mathscr{U}_{ad}(x)$ is non-empty and admits an accessible characterization. For this target, let us first look at the following uncontrolled non-homogeneous linear system on $[0,\infty)$:

$$\begin{cases} \mathrm{d}X(t) = \big\{AX(t) + \bar{A}\mathbb{E}[X(t)] + b(t)\big\}\mathrm{d}t + \big\{CX(t) + \bar{C}\mathbb{E}[X(t)] + \sigma(t)\big\}\mathrm{d}W(t), \quad t \geqslant 0, \\ X(0) = x. \end{cases} \tag{2.5}$$

When $b(\cdot) = \sigma(\cdot) = 0$, the system is said to be *homogeneous* and denoted by $[A, \bar{A}, C, \bar{C}]$. For simplicity, we also denote $[A, C] = [A, 0, C, 0]$ (the linear SDE without mean-fields), and $A = [A, 0] = [A, 0, 0, 0]$ (the linear *ordinary differential equation*, ODE, for short). The following notions can be found in [23].

Definition 2.1. (i) System $[A, \bar{A}, C, \bar{C}]$ is said to be L^2-*globally integrable*, if for any $x \in \mathbb{R}^n$, the solution $X(\cdot) \equiv X(\cdot\,; x)$ of (2.5) with $b(\cdot) = \sigma(\cdot) = 0$ is in $\mathscr{X}[0,\infty)$.

(ii) System $[A, \bar{A}, C, \bar{C}]$ is said to be L^2-*asymptotically stable*, if for any $x \in \mathbb{R}^n$, the solution $X(\cdot) \equiv X(\cdot\,; x) \in \mathscr{X}_{loc}[0,\infty)$ of (2.5) with $b(\cdot) = \sigma(\cdot) = 0$ satisfies $\lim\limits_{t\to\infty} \mathbb{E}|X(t)|^2 = 0$.

According to [23], and *via* a similar argument proving Theorem 3.3 of [50], we have the following result.

Proposition 2.2. *For any $x \in \mathbb{R}^n$ and $b(\cdot), \sigma(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^n)$, linear MF-SDE (2.5) admits a unique solution $X(\cdot) \in \mathscr{X}_{loc}[0,\infty)$. Further, if $[A, \bar{A}, C, \bar{C}]$ is L^2-asymptotically stable and system $[A, C]$ is L^2-globally integrable, then $X(\cdot) \in \mathscr{X}[0,\infty)$, with*

$$\mathbb{E}\int_0^\infty |X(t)|^2\mathrm{d}t \leqslant K\bigg[|x|^2 + \mathbb{E}\int_0^\infty \big(|b(t)|^2 + |\sigma(t)|^2\big)\mathrm{d}t\bigg],$$

for some constant $K > 0$. On the other hand, for any $\varphi(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^n)$, the following linear MF-BSDE:

$$-\mathrm{d}Y(t) = \big\{A^\top Y(t) + \bar{A}^\top\mathbb{E}[Y(t)] + C^\top Z(t) + \bar{C}^\top\mathbb{E}[Z(t)] + \varphi(t)\big\}\mathrm{d}t - Z(t)\mathrm{d}W(t), \quad t \geqslant 0, \tag{2.6}$$

admits a unique adapted solution $(Y(\cdot), Z(\cdot)) \in \mathscr{X}[0,\infty) \times L^2_{\mathbb{F}}(\mathbb{R}^n)$.

For general theory of MF-BSDEs and MF-FBSDEs in a finite horizon, see [13, 15, 16].

Now we return to (2.1). Similar to the above, when $b(\cdot)=\sigma(\cdot)=0$, the system is said to be *homogeneous* and denote it by $[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]$.

Definition 2.3. (i) For any $\mathbf{\Theta}\equiv(\Theta,\bar{\Theta})\in\mathbb{R}^{m\times 2n}$ and $v(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^m)$,

$$u(\cdot)=u^{\mathbf{\Theta},v}(\cdot)\equiv\Theta\big\{X(\cdot)-\mathbb{E}[X(\cdot)]\big\}+\bar{\Theta}\mathbb{E}[X(\cdot)]+v(\cdot)\equiv\mathbf{\Theta}\begin{pmatrix}X(\cdot)-\mathbb{E}[X(\cdot)]\\ \mathbb{E}[X(\cdot)]\end{pmatrix}+v(\cdot) \tag{2.7}$$

is called a *feedback control*. Under such a control, the state equation (2.1) becomes

$$\begin{cases}\mathrm{d}X(t)=\big\{A_\Theta X(t)+\bar{A}_{\mathbf{\Theta}}\mathbb{E}[X(t)]+Bv(t)+\bar{B}\mathbb{E}[v(t)]+b(t)\big\}\mathrm{d}t\\ \qquad\qquad+\big\{C_\Theta X(t)+\bar{C}_{\mathbf{\Theta}}\mathbb{E}[X(t)]+Dv(t)+\bar{D}\mathbb{E}[v(t)]+\sigma(t)\big\}\mathrm{d}W(t),\qquad t\geqslant 0,\\ X(0)=x,\end{cases} \tag{2.8}$$

where

$$A_\Theta=A+B\Theta,\quad \bar{A}_{\mathbf{\Theta}}=\bar{A}+\bar{B}\bar{\Theta}+B(\bar{\Theta}-\Theta),\quad C_\Theta=C+D\Theta,\quad \bar{C}_{\mathbf{\Theta}}=\bar{C}+\bar{D}\bar{\Theta}+D(\bar{\Theta}-\Theta). \tag{2.9}$$

(ii) System $[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]$ is said to be MF-L^2-*stabilizable*, if there exists a $\mathbf{\Theta}\equiv(\Theta,\bar{\Theta})\in\mathbb{R}^{m\times 2n}$ such that system $[A_\Theta,\bar{A}_{\mathbf{\Theta}},C_\Theta,\bar{C}_{\mathbf{\Theta}}]$ is L^2-asymptotically stable and system $[A_\Theta,C_\Theta]$ is L^2-globally integrable. In this case, $\mathbf{\Theta}\equiv(\Theta,\bar{\Theta})$ is called an MF-L^2-*stabilizer* of $[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]$. The set of all MF-$L^2$-stabilizers of $[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]$ is denoted by $\mathscr{S}[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]$.

(iii) Any pair $(\mathbf{\Theta},v(\cdot))\in\mathscr{S}[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]\times L^2_{\mathbb{F}}(\mathbb{R}^m)$ is called a *closed-loop strategy* of Problem (MF-SLQ). The solution $X(\cdot)\equiv X^{\mathbf{\Theta},v}(\cdot)$ of (2.8) is called the *closed-loop state process* corresponding to $(\mathbf{\Theta},v(\cdot))$. The control $u(\cdot)$ defined by (2.7) is called the *outcome* of $(\mathbf{\Theta},v(\cdot))$, or a *closed-loop control* for the initial state $x\in\mathbb{R}^n$.

Note that in the above, A_Θ and C_Θ only depend on Θ, and $\bar{A}_{\mathbf{\Theta}}$ and $\bar{C}_{\mathbf{\Theta}}$ depend on $\mathbf{\Theta}=(\Theta,\bar{\Theta})$. Also, one sees that the corresponding coefficients $B,\bar{B},D,\bar{D}$ of the control process, as well as the nonhomogeneous terms $b(\cdot),\sigma(\cdot)$ are unchanged under (2.7). See [23] for a relevant presentation.

We introduce the following assumption.

(H1) System $[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]$ is MF-L^2-stablizable, *i.e.*, $\mathscr{S}[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]\neq\varnothing$.

We have the following result.

Proposition 2.4. *Let* (H1) *hold. Then for any* $x\in\mathbb{R}^n$, $\mathscr{U}_{ad}(x)\neq\varnothing$ *and* $u(\cdot)\in\mathscr{U}_{ad}(x)$ *if and only if* $u(\cdot)=u^{\mathbf{\Theta},v}(\cdot)$ *given by* (2.7) *for some* $\mathbf{\Theta}\equiv(\Theta,\bar{\Theta})\in\mathscr{S}[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]$ *and* $v(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^m)$, *where* $X(\cdot)\equiv X^{\mathbf{\Theta},v}(\cdot)$ *is the solution to the closed-loop system* (2.8).

Proof. Sufficiency. Let $v(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^m)$ and $X(\cdot)$ be the solution to (2.8). Since system $[A_\Theta,\bar{A}_{\mathbf{\Theta}},C_\Theta,\bar{C}_{\mathbf{\Theta}}]$ is L^2-asymptotically stable and system $[A_\Theta,C_\Theta]$ is L^2-globally integrable, by Proposition 2.2, the solution $X(\cdot)$ to (2.8) is in $\mathscr{X}[0,\infty)$. Hence, setting $u(\cdot)$ by (2.7), we see that $u(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^m)$. By the uniqueness of the solutions, one has that $X(\cdot)\equiv X^{\mathbf{\Theta},v}(\cdot)$ also solves (2.1). Therefore, $u(\cdot)\in\mathscr{U}_{ad}(x)$.

Necessity. Assume that $u(\cdot)\in\mathscr{U}_{ad}(x)$. Let $\mathbf{\Theta}\equiv(\Theta,\bar{\Theta})\in\mathscr{S}[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]$ and the corresponding $X(\cdot)\in\mathscr{X}[0,\infty)$ be the solution to (2.1). Set $v(\cdot)\triangleq u(\cdot)-\Theta\big\{X(\cdot)-\mathbb{E}[X(\cdot)]\big\}-\bar{\Theta}\mathbb{E}[X(\cdot)]\in L^2_{\mathbb{F}}(\mathbb{R}^m)$. By the uniqueness of the solutions again, $X(\cdot)$ coincides with the solution to (2.8). Thus, $u(\cdot)$ admits a representation of the form (2.7). The proof is complete. □

From the above, we can easily show that under (H1), $\mathscr{U}_{ad}(x)=L^2_{\mathbb{F}}(\mathbb{R}^m)$ which is independent of x. Hence, hereafter, once (H1) is assumed, we will denote $\mathscr{U}_{ad}(x)=\mathscr{U}_{ad}$. Now, we introduce the following definitions concerning Problem (MF-SLQ).

Definition 2.5. (i) Problem (MF-SLQ) is said to be *finite* at $x\in\mathbb{R}^n$ if $V(x)>-\infty$, and Problem (MF-SLQ) is said to be finite if it is finite at every $x\in\mathbb{R}^n$.

(ii) An element $u^*(\cdot) \in \mathscr{U}_{ad}(x)$ is called an *open-loop optimal control* of Problem (MF-SLQ) for the initial state $x \in \mathbb{R}^n$ if

$$J(x; u^*(\cdot)) \leqslant J(x; u(\cdot)), \qquad \forall u(\cdot) \in \mathscr{U}_{ad}(x). \tag{2.10}$$

If an open-loop optimal control (uniquely) exists for $x \in \mathbb{R}^n$, Problem (MF-SLQ) is said to be (uniquely) *open-loop solvable* at x. Problem (MF-SLQ) is said to be (uniquely) *open-loop solvable* if it is (uniquely) open-loop solvable at all $x \in \mathbb{R}^n$.

(iii) A pair $(\mathbf{\Theta}^*, v^*(\cdot)) \in \mathscr{S}[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}] \times L^2_{\mathbb{F}}(\mathbb{R}^m)$ is called a *closed-loop optimal strategy* if

$$\begin{aligned} J\big(x; \Theta^*\big\{X^*(\cdot) - \mathbb{E}[X^*(\cdot)]\big\} + \bar{\Theta}^*\mathbb{E}[X^*(\cdot)] + v^*(\cdot)\big) \leqslant J\big(x; \Theta\big\{X(\cdot) - \mathbb{E}[X(\cdot)]\big\} + \bar{\Theta}\mathbb{E}[X(\cdot)] + v(\cdot)\big), \\ \forall (\mathbf{\Theta}, v(\cdot)) \in \mathscr{S}[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}] \times L^2_{\mathbb{F}}(\mathbb{R}^m),\ x \in \mathbb{R}^n, \end{aligned} \tag{2.11}$$

where $X^*(\cdot) \equiv X^{\mathbf{\Theta}^*, v^*}(\cdot)$ and $X(\cdot) \equiv X^{\mathbf{\Theta}, v}(\cdot)$ are the closed-loop state processes corresponding to $(x, \mathbf{\Theta}^*, v^*(\cdot))$ and $(x, \mathbf{\Theta}, v(\cdot))$, respectively. If an optimal closed-loop strategy (uniquely) exists, Problem (MF-SLQ) is said to be (uniquely) *closed-loop solvable*.

(iv) An open-loop optimal controls $u^*(\cdot\,; x) \in \mathscr{U}_{ad}$ of Problem (MF-SLQ), parameterized by $x \in \mathbb{R}^n$, admits a *closed-loop representation*, if there exists a pair $(\mathbf{\Theta}^*, v^*(\cdot)) \in \mathscr{S}[A, \bar{A}, C, \bar{C}; B, \bar{B},\ D, \bar{D}] \times L^2_{\mathbb{F}}(\mathbb{R}^m)$ such that for any initial state $x \in \mathbb{R}^n$,

$$u^*(\cdot) \equiv \Theta^*\big\{X^*(\cdot) - \mathbb{E}[X^*(\cdot)]\big\} + \bar{\Theta}^*\mathbb{E}[X^*(\cdot)] + v^*(\cdot) \tag{2.12}$$

where $X^*(\cdot) \equiv X^{\mathbf{\Theta}^*, v^*}(\cdot) \in \mathscr{X}[0, \infty)$ is the solution to the closed-loop system (2.8) corresponding to $(\mathbf{\Theta}^*, v^*(\cdot))$.

Similar to Proposition 2.5 of [31], we have that $(\mathbf{\Theta}^*, v^*(\cdot)) \in \mathscr{S}[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}] \times L^2_{\mathbb{F}}(\mathbb{R}^m)$ is an optimal closed-loop strategy, if and only if the following condition holds:

$$J\big(x; \Theta^*\big\{X^*(\cdot) - \mathbb{E}[X^*(\cdot)]\big\} + \bar{\Theta}^*\mathbb{E}[X^*(\cdot)] + v^*(\cdot)\big) \leqslant J\big(x; \Theta^*\big\{X(\cdot) - \mathbb{E}[X(\cdot)]\big\} + \bar{\Theta}^*\mathbb{E}[X(\cdot)] + v(\cdot)\big), \tag{2.13}$$

for any $(x, v(\cdot)) \in \mathbb{R}^n \times L^2_{\mathbb{F}}(\mathbb{R}^m)$, where $X^*(\cdot) \equiv X^{\mathbf{\Theta}^*, v^*}(\cdot)$ and $X(\cdot) \equiv X^{\mathbf{\Theta}^*, v}(\cdot)$ are the closed-loop state processes corresponding to $(x, \mathbf{\Theta}^*, v^*(\cdot))$ and $(x, \mathbf{\Theta}^*, v(\cdot))$, respectively. On the other hand, from Proposition 2.4, we see that under (H1), (2.13) is equivalent to the following:

$$J\big(x; \Theta^*\big\{X^*(\cdot) - \mathbb{E}[X^*(\cdot)]\big\} + \bar{\Theta}^*\mathbb{E}[X^*(\cdot)] + v^*(\cdot)\big) \leqslant J\big(x; u(\cdot)\big), \quad \forall (x, u(\cdot)) \in \mathbb{R}^n \times \mathscr{U}_{ad}. \tag{2.14}$$

In general, an open-loop optimal control depends on the initial state $x \in \mathbb{R}^n$, whereas a closed-loop strategy is required to be independent of x. From (2.14), we see that the outcome $u^*(\cdot)$ given by (2.12) for some closed-loop strategy $(\mathbf{\Theta}^*, v^*(\cdot))$ is an open-loop optimal control for the initial state $X^*(0)$. Hence, for Problem (MF-SLQ), the closed-loop solvability implies the open-loop solvability. The converse is also true for stochastic LQ optimal control problems in an infinite horizon without mean fields. That is to say, the open-loop and closed-loop solvabilities are equivalent (see [46]).

It is natural for us to ask: Do we have such equivalence for Problem (MF-SLQ)? To answer this question, we first present the result concerning the characterization of open-loop and closed-loop solvabilities of Problem (MF-SLQ). To simplify notation, in what follows, we denote

$$\widehat{A} = A + \bar{A}, \quad \widehat{B} = B + \bar{B}, \quad \widehat{C} = C + \bar{C}, \quad \widehat{D} = D + \bar{D}, \quad \widehat{Q} = Q + \bar{Q}, \quad \widehat{S} = S + \bar{S}, \quad \widehat{R} = R + \bar{R}. \tag{2.15}$$

Note that for any $\mathbf{\Theta}\equiv(\Theta,\bar\Theta)\in\mathscr{S}[A,\bar A,C,\bar C;B,\bar B,D,\bar D]$ and $v(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^m)$, we have the closed-loop system (2.8). The cost functional (2.2) becomes

$$\begin{aligned}&J^{\mathbf{\Theta}}(x;v(\cdot))\equiv J\big(x;\Theta\big(X(\cdot)-\mathbb{E}[X(\cdot)]\big)+\bar\Theta\mathbb{E}[X(\cdot)]+v(\cdot)\big)\\&=\mathbb{E}\int_0^\infty\bigg[\bigg\langle\begin{pmatrix}Q_\Theta & S_\Theta^\top\\ S_\Theta & R\end{pmatrix}\begin{pmatrix}X\\ v\end{pmatrix},\begin{pmatrix}X\\ v\end{pmatrix}\bigg\rangle+2\bigg\langle\begin{pmatrix}q_{\mathbf{\Theta}}(t)\\ \rho(t)\end{pmatrix},\begin{pmatrix}X\\ v\end{pmatrix}\bigg\rangle+\bigg\langle\begin{pmatrix}\bar Q_{\mathbf{\Theta}} & \bar S_{\mathbf{\Theta}}^\top\\ \bar S_{\mathbf{\Theta}} & \bar R\end{pmatrix}\begin{pmatrix}\mathbb{E}[X]\\ \mathbb{E}[v]\end{pmatrix},\begin{pmatrix}\mathbb{E}[X]\\ \mathbb{E}[v]\end{pmatrix}\bigg\rangle\bigg]\mathrm{d}t,\end{aligned}\tag{2.16}$$

where

$$\begin{cases}Q_\Theta=Q+S^\top\Theta+\Theta^\top S+\Theta^\top R\Theta,\quad \bar Q_{\mathbf{\Theta}}=\bar Q+\widehat S^{\,\top}\bar\Theta+\bar\Theta^\top\widehat S+\bar\Theta^\top\widehat R\bar\Theta-S^\top\Theta-\Theta^\top S-\Theta^\top R\Theta,\\ S_\Theta=S+R\Theta,\qquad \bar S_{\mathbf{\Theta}}=\bar S+\widehat R\bar\Theta-R\Theta,\qquad q_{\mathbf{\Theta}}(\cdot)=q(\cdot)+\Theta^\top\big(\rho(\cdot)-\mathbb{E}[\rho(\cdot)]\big)+\bar\Theta^\top\mathbb{E}[\rho(\cdot)].\end{cases}\tag{2.17}$$

We see that Q_Θ,S_Θ depend on Θ, $\bar Q_{\mathbf{\Theta}},\bar S_{\mathbf{\Theta}},q_{\mathbf{\Theta}}$ depend on $\mathbf{\Theta}\equiv(\Theta,\bar\Theta)$, and $R(\cdot),\bar R(\cdot),\rho(\cdot)$ are unchanged. Similar to (2.15), we will denote

$$\widehat A_{\mathbf{\Theta}}=A_\Theta+\bar A_{\mathbf{\Theta}},\quad \widehat C_{\mathbf{\Theta}}=C_\Theta+\bar C_{\mathbf{\Theta}},\quad \widehat Q_{\mathbf{\Theta}}=Q_\Theta+\bar Q_{\mathbf{\Theta}},\quad \widehat S_{\mathbf{\Theta}}=S_\Theta+\bar S_{\mathbf{\Theta}}.\tag{2.18}$$

In later investigations, we will encounter the comparison between two closed-loop strategies. Therefore, we need the following definition.

Definition 2.6. Let $(\mathbf{\Theta},v(\cdot)),(\mathbf{\Theta}',v'(\cdot))\in\mathscr{S}[A,\bar A,C,\bar C;B,\bar B,D,\bar D]\times L^2_{\mathbb{F}}(\mathbb{R}^n)$.

(i) We say that $(\mathbf{\Theta},v(\cdot))$ and $(\mathbf{\Theta}',v'(\cdot))$ are *intrinsically different* if for some $x\in\mathbb{R}^n$, $X(\cdot\,x,\mathbf{\Theta},v(\cdot))\neq X(\cdot\,;x,\mathbf{\Theta}',v'(\cdot))$.

(ii) We say that $(\mathbf{\Theta},v(\cdot))$ and $(\mathbf{\Theta}',v'(\cdot))$ are *intrinsically the same* if for any $x\in\mathbb{R}^n$, $X(\cdot\,x,\mathbf{\Theta},v(\cdot))=X(\cdot\,;x,\mathbf{\Theta}',v'(\cdot))$.

Remark 2.7. The point that we would like to make here is that sometimes, $(\mathbf{\Theta},v(\cdot))\neq(\mathbf{\Theta}',v'(\cdot))$. But they could be intrinsically the same. Here is such a situation. Let $(\mathbf{\Theta},v(\cdot))\neq(\mathbf{\Theta}',v'(\cdot))$ and let $X(\cdot)$ and $X'(\cdot)$ be the corresponding state processes. Then (note (2.9) and (2.15))

$$\begin{aligned}&A_\Theta-A_{\Theta'}=B(\Theta-\Theta'),\quad \bar A_{\mathbf{\Theta}}-\bar A_{\mathbf{\Theta}'}=\widehat B(\bar\Theta-\bar\Theta')-B(\Theta-\Theta'),\\&C_\Theta-C_{\Theta'}=D(\Theta-\Theta'),\quad \bar C_{\mathbf{\Theta}}-\bar C_{\mathbf{\Theta}'}=\widehat D(\bar\Theta-\bar\Theta')-D(\Theta-\Theta').\end{aligned}$$

Thus, if

$$\begin{aligned}&\mathscr{R}(\Theta-\Theta')\subseteq\mathscr{N}(B)\cap\mathscr{N}(D),\qquad \mathscr{R}(\bar\Theta-\bar\Theta')\subseteq\mathscr{N}(\widehat B)\cap\mathscr{N}(\widehat D)\\&v(t)-v'(t)\in\mathscr{N}(B)\cap\mathscr{N}(D),\qquad \mathbb{E}[v(t)-v'(t)]\in\mathscr{N}(\bar B)\cap\mathscr{N}(\bar D),\qquad\forall t\in[0,\infty),\end{aligned}\tag{2.19}$$

then the closed-loop systems under $(\mathbf{\Theta},v(\cdot))$ and $(\mathbf{\Theta}',v'(\cdot))$ are the same. By definition, this means that if $(\mathbf{\Theta},v(\cdot)),(\mathbf{\Theta}',v'(\cdot))$ are two closed-loop strategies such that (2.19) holds, then they are intrinsically the same. If the above fails, then the two closed-loop strategies will be intrinsically different.

Note that there is another issue when we compare two closed-loop strategies, namely, the corresponding costs could be different. But, we prefer to concentrate on the difference of the corresponding state processes, which will be mainly used later.

Definition 2.8. The following is called a system of generalized AREs:

$$\begin{cases} PA+A^\top P+C^\top PC+Q-(PB+C^\top PD+S^\top)\Sigma^\dagger(B^\top P+D^\top PC+S)=0,\\ \widehat{P}\widehat{A}+\widehat{A}^\top\widehat{P}+\widehat{C}^\top P\widehat{C}+\widehat{Q}-\big(\widehat{P}\widehat{B}+\widehat{C}^\top P\widehat{D}+\widehat{S}^\top\big)\bar{\Sigma}^\dagger\big(\widehat{B}^\top\widehat{P}+\widehat{D}^\top P\widehat{C}+\widehat{S}\big)=0,\\ \Sigma\equiv R+D^\top PD\geqslant 0,\qquad \bar{\Sigma}\equiv\widehat{R}+\widehat{D}^\top P\widehat{D}\geqslant 0,\\ \mathscr{R}(B^\top P+D^\top PC+S)\subseteq\mathscr{R}(\Sigma),\qquad \mathscr{R}\big(\widehat{B}^\top\widehat{P}+\widehat{D}^\top P\widehat{C}+\widehat{S}\big)\subseteq\mathscr{R}\big(\bar{\Sigma}\big), \end{cases} \tag{2.20}$$

with the unknown $(P,\widehat{P})\in\mathbb{S}^n\times\mathbb{S}^n$. A solution pair $(P,\widehat{P})$ to (2.20) is said to be *static stabilizing* if there exists a pair $(\theta,\bar{\theta})\in\mathbb{R}^{m\times 2n}$, such that $\mathbf{\Theta}\equiv(\Theta,\bar{\Theta})\in\mathscr{S}[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]$, where

$$\Theta=-\Sigma^\dagger(B^\top P+D^\top PC+S)+(I-\Sigma^\dagger\Sigma)\theta,\quad \bar{\Theta}=-\bar{\Sigma}^\dagger\big(\widehat{B}^\top\widehat{P}+\widehat{D}^\top P\widehat{C}+\widehat{S}\big)+(I-\bar{\Sigma}^\dagger\bar{\Sigma})\bar{\theta}. \tag{2.21}$$

We now state the following result, which is an extension of a result without mean-field terms in [46] and will play an important role in the next section. We omit the proof here and refer the interested readers to Li–Shi–Yong [29], an arxiv version of the current paper.

Theorem 2.9. *Let* (H1) *hold. Then the following are equivalent:*

(i) *Problem* (MF-SLQ) *is open-loop solvable.*

(ii) *Problem* (MF-SLQ) *is closed-loop solvable.*

(iii) *The system* (2.20) *admits a static stabilizing solution pair* $(P,\widehat{P})\in\mathbb{S}^n\times\mathbb{S}^n$, *the BSDE on* $[0,\infty)$*:*

$$\begin{aligned} -\mathrm{d}\eta(t)=\big\{&A^\top\eta(t)-(B^\top P+D^\top PC+S)^\top\Sigma^\dagger\big[B^\top\eta(t)+D^\top\big(\zeta(t)+P\sigma(t)\big)+\rho(t)\big]\\ &+C^\top\big[\zeta(t)+P\sigma(t)\big]+Pb(t)+q(t)\big\}\mathrm{d}t-\zeta(t)\mathrm{d}W(t),\qquad t\geqslant 0, \end{aligned} \tag{2.22}$$

admits an adapted solution $(\eta(\cdot),\zeta(\cdot))\in\mathscr{X}[0,\infty)\times L^2_{\mathbb{F}}(\mathbb{R}^n)$ *such that*

$$\begin{aligned} B^\top\big[\eta(t)-\mathbb{E}[\eta(t)]\big]+D^\top\big[\zeta(t)-\mathbb{E}[\zeta(t)]\big]+D^\top P\big[\sigma(t)-\mathbb{E}[\sigma(t)]\big]+\rho(t)-\mathbb{E}[\rho(t)]\in\mathscr{R}(\Sigma),\\ \text{a.e. } t\in[0,\infty),\ \text{a.s.}, \end{aligned} \tag{2.23}$$

and the ODE on $[0,\infty)$*:*

$$\begin{aligned} &\dot{\bar{\eta}}(t)+\widehat{A}^\top\bar{\eta}(t)-(\widehat{B}^\top\widehat{P}+\widehat{D}^\top P\widehat{C}+\widehat{S})^\top\bar{\Sigma}^\dagger\big\{\widehat{B}^\top\bar{\eta}(t)+\widehat{D}^\top\mathbb{E}\big[\zeta(t)+P\sigma(t)\big]+\mathbb{E}[\rho(t)]\big\}\\ &\quad+\widehat{C}^\top\mathbb{E}\big[\zeta(t)+P\sigma(t)\big]+\mathbb{E}\big[\widehat{P}b(t)+q(t)\big]=0, \end{aligned} \tag{2.24}$$

admits a solution $\bar{\eta}(\cdot)\in L^2(\mathbb{R}^n)$ *such that*

$$\widehat{B}^\top\bar{\eta}(t)+\widehat{D}^\top\mathbb{E}[\zeta(t)]+\widehat{D}^\top P\mathbb{E}[\sigma(t)]+\mathbb{E}[\rho(t)]\in\mathscr{R}\big(\bar{\Sigma}\big),\quad \text{a.e. } t\in[0,\infty). \tag{2.25}$$

In the above case, the closed-loop optimal strategy $(\mathbf{\Theta}^*,v^*(\cdot))\equiv(\Theta^*,\bar{\Theta}^*,v^*(\cdot))$ *is given by*

$$\begin{cases} \Theta^*=-\Sigma^\dagger(B^\top P+D^\top PC+S)+(I-\Sigma^\dagger\Sigma)\theta,\\ \bar{\Theta}^*=-\bar{\Sigma}^\dagger(\widehat{B}^\top\widehat{P}+\widehat{D}^\top P\widehat{C}+\widehat{S})+(I-\bar{\Sigma}^\dagger\bar{\Sigma})\bar{\theta},\\ v^*(\cdot)=\varphi(\cdot)-\mathbb{E}[\varphi(\cdot)]+\bar{\varphi}(\cdot)+(I-\Sigma^\dagger\Sigma)\big(\nu(\cdot)-\mathbb{E}[\nu(\cdot)]\big)+(I-\bar{\Sigma}^\dagger\bar{\Sigma})\bar{\nu}(\cdot), \end{cases} \tag{2.26}$$

with $\mathbf{\Theta}^* \equiv (\Theta^*, \bar{\Theta}^*) \in \mathscr{S}[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$, *for some* $\theta, \bar{\theta} \in \mathbb{R}^{m\times n}$, $\nu(\cdot), \bar{\nu}(\cdot) \in L^2(\mathbb{R}^m)$, *and*

$$\begin{cases} \varphi(\cdot) \triangleq -\Sigma^\dagger \big\{ B^\top \eta(\cdot) + D^\top \big[\zeta(\cdot) + P\sigma(\cdot)\big] + \rho(\cdot) \big\}, \\ \bar{\varphi}(\cdot) \triangleq -\bar{\Sigma}^\dagger \big\{ \widehat{B}^\top \bar{\eta}(\cdot) + \widehat{D}^\top \mathbb{E}\big[\zeta(\cdot) + P\sigma(\cdot)\big] + \mathbb{E}[\rho(\cdot)] \big\}. \end{cases} \tag{2.27}$$

Every open-loop optimal control $u^*(\cdot)$ *for the initial state* $x \in \mathbb{R}^n$ *admits a closed-loop representation* (2.12), *where* $X^*(\cdot) \in \mathscr{X}[0,\infty)$ *is the solution to the closed-loop system* (2.8) *under* $(\mathbf{\Theta}^*, v^*(\cdot))$. *Further, the value function is given by*

$$\begin{aligned} V(x) = \big\langle \widehat{P}x, x\big\rangle + 2\big\langle \bar{\eta}(0), x\big\rangle + \mathbb{E}\int_0^\infty \Big[&\big\langle P\sigma(t), \sigma(t)\big\rangle + 2\big\langle \eta(t), b(t) - \mathbb{E}[b(t)]\big\rangle + 2\big\langle \bar{\eta}(t), \mathbb{E}[b(t)]\big\rangle \\ &+2\big\langle \zeta(t), \sigma(t)\big\rangle - \big\langle \Sigma\big(\varphi(t) - \mathbb{E}[\varphi(t)]\big), \varphi(t) - \mathbb{E}[\varphi(t)]\big\rangle - \big\langle \bar{\Sigma}\bar{\varphi}(t), \bar{\varphi}(t)\big\rangle \Big] \mathrm{d}t. \end{aligned} \tag{2.28}$$

3. Mean-field LQ non-zero sum stochastic differential games

We now return to our Problem (MF-SDG).

3.1. Notions of Nash equilibria

To simplify the notation, we let $m = m_1 + m_2$ and denote (for $i = 1, 2$)

$$\begin{cases} B = \big(B_1, B_2\big), \quad \bar{B} = \big(\bar{B}_1, \bar{B}_2\big), \quad D = \big(D_1, D_2\big), \quad \bar{D} = \big(\bar{D}_1, \bar{D}_2\big), \\ S_i = \begin{pmatrix} S_{i1} \\ S_{i2} \end{pmatrix}, \quad \bar{S}_i = \begin{pmatrix} \bar{S}_{i1} \\ \bar{S}_{i2} \end{pmatrix}, \quad R_i = \begin{pmatrix} R_{i11} & R_{i12} \\ R_{i21} & R_{i22} \end{pmatrix} \equiv \begin{pmatrix} R_{i1} \\ R_{i2} \end{pmatrix}, \\ \bar{R}_i = \begin{pmatrix} \bar{R}_{i11} & \bar{R}_{i12} \\ \bar{R}_{i21} & \bar{R}_{i22} \end{pmatrix} \equiv \begin{pmatrix} \bar{R}_{i1} \\ \bar{R}_{i2} \end{pmatrix}, \quad q_i(\cdot) = \begin{pmatrix} q_{i1}(\cdot) \\ q_{i2}(\cdot) \end{pmatrix}, \quad \rho_i(\cdot) = \begin{pmatrix} \rho_{i1}(\cdot) \\ \rho_{i2}(\cdot) \end{pmatrix}, \quad u(\cdot) = \begin{pmatrix} u_1(\cdot) \\ u_2(\cdot) \end{pmatrix}. \end{cases} \tag{3.1}$$

Then the state equation (1.1) becomes

$$\begin{cases} \mathrm{d}X(t) = \big\{AX(t) + \bar{A}\mathbb{E}[X(t)] + Bu(t) + \bar{B}\mathbb{E}[u(t)] + b(t)\big\}\mathrm{d}t \\ \qquad\qquad +\big\{CX(t) + \bar{C}\mathbb{E}[X(t)] + Du(t) + \bar{D}\mathbb{E}[u(t)] + \sigma(t)\big\}\mathrm{d}W(t), \qquad t \geqslant 0, \\ X(0) = x, \end{cases} \tag{3.2}$$

which is of the same form as (2.1), and the cost functionals are, for $i = 1, 2$,

$$\begin{aligned} J_i(x; u(\cdot)) = \mathbb{E}\int_0^\infty \Bigg[&\left\langle \begin{pmatrix} Q_i & S_i^\top \\ S_i & R_i \end{pmatrix} \begin{pmatrix} X(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix} \right\rangle + 2\left\langle \begin{pmatrix} q_i(t) \\ \rho_i(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix} \right\rangle \\ &+\left\langle \begin{pmatrix} \bar{Q}_i & \bar{S}_i^\top \\ \bar{S}_i & \bar{R}_i \end{pmatrix} \begin{pmatrix} \mathbb{E}[X(t)] \\ \mathbb{E}[u(t)] \end{pmatrix}, \begin{pmatrix} \mathbb{E}[X(t)] \\ \mathbb{E}[u(t)] \end{pmatrix} \right\rangle \Bigg] \mathrm{d}t. \end{aligned} \tag{3.3}$$

In order the game to make sense, we make a convention that both players at least want to keep the state $X(\cdot) \equiv X(\cdot\,; x, u_1(\cdot), u_2(\cdot))$ in $\mathscr{X}[0,\infty)$ so that both cost functionals are well-defined. To guarantee this, similar to Problem (MF-SLQ), we introduce the following assumption.

(H2) System $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is MF-L^2-stablizable, *i.e.*, $\mathscr{S}[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}] \neq \varnothing$.

Although (H2) looks the same as (H1), the meaning is different. Hypothesis (H2) provides the possibility for both players to make both cost functionals finite cooperatively. In fact, under (H2), for any $x \in \mathbb{R}^n$, by

Proposition 2.4, the following set of all *admissible control pairs* is non-empty:

$$\mathscr{U}_{ad}(x)=\Big\{u(\cdot)=(u_1(\cdot),u_2(\cdot))\in L^2_{\mathbb{F}}(\mathbb{R}^m)\;\Big|\;X(\cdot)\equiv X(\cdot\,;x,u(\cdot))\in\mathscr{X}[0,\infty)\Big\}. \tag{3.4}$$

Further, for any $u_2(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^{m_2})$, making use of Proposition 2.4 again, the following holds:

$$\mathscr{U}^1_{ad}(x)=\Big\{u_1(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^{m_1})\;\Big|\;\exists u_2(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^{m_2}),\ (u_1(\cdot),u_2(\cdot))\in\mathscr{U}_{ad}(x)\Big\}\neq\varnothing. \tag{3.5}$$

Likewise, for any $u_1(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^{m_1})$, one has

$$\mathscr{U}^2_{ad}(x)=\Big\{u_2(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^{m_2})\;\Big|\;\exists u_1(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^{m_1}),\ (u_1(\cdot),u_2(\cdot))\in\mathscr{U}_{ad}(x)\Big\}\neq\varnothing. \tag{3.6}$$

We now present the following definition.

Definition 3.1. A $u^*(\cdot)\equiv(u_1^*(\cdot),u_2^*(\cdot))\in\mathscr{U}_{ad}(x)$ is called an *open-loop Nash equilibrium* of Problem (MF-SDG) for the initial state $x\in\mathbb{R}^n$ if

$$\begin{aligned}&J_1(x;u_1^*(\cdot),u_2^*(\cdot))\leqslant J_1(x;u_1(\cdot),u_2^*(\cdot)),\qquad\forall u_1(\cdot)\in\mathscr{U}^1_{ad}(x),\\&J_2(x;u_1^*(\cdot),u_2^*(\cdot))\leqslant J_2(x;u_1^*(\cdot),u_2(\cdot)),\qquad\forall u_2(\cdot)\in\mathscr{U}^2_{ad}(x).\end{aligned} \tag{3.7}$$

For $\mathbf{\Theta}_i\equiv(\Theta_i,\bar\Theta_i)\in\mathbb{R}^{m_i\times 2n},i=1,2$, we denote $\mathbf{\Theta}\equiv\begin{pmatrix}\mathbf{\Theta}_1\\ \mathbf{\Theta}_2\end{pmatrix}\equiv\begin{pmatrix}\Theta_1&\bar\Theta_1\\ \Theta_2&\bar\Theta_2\end{pmatrix}\in\mathbb{R}^{m\times 2n}$, and let

$$\begin{aligned}\mathscr{S}^1(\mathbf{\Theta}_2)&\triangleq\Big\{\mathbf{\Theta}_1\in\mathbb{R}^{2m_1\times n}\;\Big|\;\mathbf{\Theta}\in\mathscr{S}[A,\bar A,C,\bar C;B,\bar B,D,\bar D]\Big\},\\ \mathscr{S}^2(\mathbf{\Theta}_1)&\triangleq\Big\{\mathbf{\Theta}_2\in\mathbb{R}^{2m_2\times n}\;\Big|\;\mathbf{\Theta}\in\mathscr{S}[A,\bar A,C,\bar C;B,\bar B,D,\bar D]\Big\}.\end{aligned}$$

It is clear that if $\mathbf{\Theta}\in\mathscr{S}[A,\bar A,C,\bar C;B,\bar B,D,\bar D]$, both $\mathscr{S}^1(\mathbf{\Theta}_2)$ and $\mathscr{S}^2(\mathbf{\Theta}_1)$ are nonempty. Similar to the optimal control problem case, any $(\mathbf{\Theta},v(\cdot))\in\mathscr{S}[A,\bar A,C,\bar C;B,\bar B,D,\bar D]\times L^2_{\mathbb{F}}(\mathbb{R}^m)$ is called a *closed-loop strategy* of Problem (MF-SDG). For any initial state $x\in\mathbb{R}^n$ and closed-loop strategy $(\mathbf{\Theta},v(\cdot))\in\mathscr{S}[A,\bar A,C,\bar C;B,\bar B,D,\bar D]\times L^2_{\mathbb{F}}(\mathbb{R}^m)$, we consider the following linear MF-SDE on $[0,\infty)$ (recall (2.9)):

$$\begin{cases}\mathrm{d}X(t)=\big\{A_\Theta X(t)+\bar A_{\mathbf{\Theta}}\mathbb{E}[X(t)]+Bv(t)+\bar B\mathbb{E}[v(t)]+b(t)\big\}\mathrm{d}t\\ \qquad\qquad+\big\{C_\Theta X(t)+\bar C_{\mathbf{\Theta}}\mathbb{E}[X(t)]+Dv(t)+\bar D\mathbb{E}[v(t)]+\sigma(t)\big\}\mathrm{d}W(t),\qquad t\geqslant 0,\\ X(0)=x.\end{cases} \tag{3.8}$$

By Proposition 2.2, (3.8) admits a unique solution $X(\cdot)\in\mathscr{X}[0,\infty)$. If we denote

$$u_i(\cdot)=\Theta_i\big\{X(\cdot)-\mathbb{E}[X(\cdot)]\big\}+\bar\Theta_i\mathbb{E}[X(\cdot)]+v_i(\cdot),\quad i=1,2, \tag{3.9}$$

then (3.8) coincides with the original state equation (3.2). We call $(\mathbf{\Theta}_i,v_i(\cdot))$ a *closed-loop strategy* of Player i, and call (3.8) the *closed-loop system* of the original system under closed-loop strategy $(\mathbf{\Theta},v(\cdot))$. Also, we call $u(\cdot)\equiv(u_1(\cdot),u_2(\cdot))$, with $u_i(\cdot)$ defined by (3.9), the *outcome* of the closed-loop strategy $(\mathbf{\Theta},v(\cdot))$.

With the solution $X(\cdot)\in\mathscr{X}[0,\infty)$ to (3.8), we denote, for $i=1,2$,

$$\begin{aligned}&J_i(x;\mathbf{\Theta},v(\cdot))\equiv J_i(x;\mathbf{\Theta}_1,v_1(\cdot);\mathbf{\Theta}_2,v_2(\cdot))\equiv J_i\big(x;\Theta\big\{X(\cdot)-\mathbb{E}[X(\cdot)]\big\}+\bar\Theta\mathbb{E}[X(\cdot)]+v(\cdot)\big)\\ &\equiv J_i\big(x;\Theta_1\big\{X(\cdot)-\mathbb{E}[X(\cdot)]\big\}+\bar\Theta_1\mathbb{E}[X(\cdot)]+v_1(\cdot);\Theta_2\big\{X(\cdot)-\mathbb{E}[X(\cdot)]\big\}+\bar\Theta_2\mathbb{E}[X(\cdot)]+v_2(\cdot)\big).\end{aligned}$$

Similarly, we can define

$$\begin{aligned} J_i\big(x;\mathbf{\Theta}_1,v_1(\cdot);u_2(\cdot)\big) &\equiv J_i\big(x;\Theta_1\big\{X(\cdot)-\mathbb{E}[X(\cdot)]\big\}+\bar{\Theta}_1\mathbb{E}[X(\cdot)]+v_1(\cdot);u_2(\cdot)\big),\\ J_i\big(x;u_1(\cdot);\mathbf{\Theta}_2,v_2(\cdot)\big) &\equiv J_i\big(x;u_1(\cdot);\Theta_2\big\{X(\cdot)-\mathbb{E}[X(\cdot)]\big\}+\bar{\Theta}_2\mathbb{E}[X(\cdot)]+v_2(\cdot)\big), \end{aligned} \qquad i=1,2.$$

We now introduce the following definition.

Definition 3.2. A closed-loop strategy $(\mathbf{\Theta}^*,v^*(\cdot))\in\mathscr{S}[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]\times L^2_{\mathbb{F}}(\mathbb{R}^m)$ is called a *closed-loop Nash equilibrium* of Problem (MF-SDG) if for any $\mathbf{\Theta}_1\in\mathscr{S}^1(\mathbf{\Theta}_2^*)$, $\mathbf{\Theta}_2\in\mathscr{S}^2(\mathbf{\Theta}_1^*)$, $v_1(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^{m_1})$ and $v_2(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^{m_2})$,

$$\begin{cases} J_1\big(x;\mathbf{\Theta}^*,v^*(\cdot)\big)\leqslant J_1\big(x;\mathbf{\Theta}_1,v_1(\cdot);\mathbf{\Theta}_2^*,v_2^*(\cdot)\big), & \forall x\in\mathbb{R}^n,\\ J_2\big(x;\mathbf{\Theta}^*,v^*(\cdot)\big)\leqslant J_2\big(x;\mathbf{\Theta}_1^*,v_1^*(\cdot);\mathbf{\Theta}_2,v_2(\cdot)\big), & \forall x\in\mathbb{R}^n. \end{cases} \tag{3.10}$$

Note that on the left-hand sides of (3.10), the involved state is $X(\cdot)\equiv X(\cdot\,;x,\mathbf{\Theta}^*,v^*(\cdot))$, depending on $(\mathbf{\Theta}^*,v^*(\cdot))$. Whereas, on the right-hand sides of (3.10), the involved states are $X(\cdot)\equiv X\big(\cdot\,;x,\mathbf{\Theta}_1,v_1(\cdot);\mathbf{\Theta}_2^*,v_2^*(\cdot)\big)$ and $X(\cdot)=X\big(\cdot\,;\mathbf{\Theta}_1^*,v_1^*(\cdot);\mathbf{\Theta}_2,v_2(\cdot)\big)$ respectively, which are different in general. We emphasize that the open-loop Nash equilibrium $(u_1^*(\cdot),u_2^*(\cdot))$ usually depends on the initial state x, whereas a closed-loop Nash equilibrium $\big(\mathbf{\Theta}^*,v^*(\cdot)\big)$ is required to be independent of x. It is easy to see that $\big(\mathbf{\Theta}^*,v^*(\cdot)\big)$ is a closed-loop Nash equilibrium of Problem (MF-SDG) if and only if one of the following hold:

(i) For any $v_1(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^{m_1})$ and $v_2(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^{m_2})$,

$$J_1\big(x;\mathbf{\Theta}^*,v^*(\cdot)\big)\leqslant J_1\big(x;\mathbf{\Theta}^*,v_1(\cdot),v_2^*(\cdot)\big),\quad J_2\big(x;\mathbf{\Theta}^*,v^*(\cdot)\big)\leqslant J_2\big(x;\mathbf{\Theta}^*,v_1^*(\cdot),v_2(\cdot)\big); \tag{3.11}$$

(ii) For any $u_1(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^{m_1})$ and $u_2(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^{m_2})$,

$$J_1\big(x;\mathbf{\Theta}^*,v^*(\cdot)\big)\leqslant J_1\big(x;u_1(\cdot);\mathbf{\Theta}_2^*,v_2^*(\cdot)\big),\quad J_2\big(x;\mathbf{\Theta}^*,v^*(\cdot)\big)\leqslant J_2\big(x;\mathbf{\Theta}_1^*,v_1^*(\cdot);u_2(\cdot)\big). \tag{3.12}$$

If we denote (comparing with (3.10))

$$u_i^*(\cdot)=\Theta_i^*\big\{X^*(\cdot)-\mathbb{E}[X^*(\cdot)]\big\}+\bar{\Theta}_i^*\mathbb{E}[X^*(\cdot)]+v_i^*(\cdot),\quad i=1,2, \tag{3.13}$$

then (3.12) becomes

$$\begin{aligned} J_1\big(x;u_1^*(\cdot);u_2^*(\cdot)\big) &\leqslant J_1\big(x;u_1(\cdot);\Theta_2^*\big\{X^{u_1,v_2^*}(\cdot)-\mathbb{E}[X^{u_1,v_2^*}(\cdot)]\big\}+\bar{\Theta}_2^*\mathbb{E}[X^{u_1,v_2^*}(\cdot)]+v_2^*(\cdot)\big),\\ J_2\big(x;u_1^*(\cdot);u_2^*(\cdot)\big) &\leqslant J_2\big(x;\Theta_1^*\big\{X^{v_1^*,u_2}(\cdot)-\mathbb{E}[X^{v_1^*,u_2}(\cdot)]\big\}+\bar{\Theta}_1^*\mathbb{E}[X^{v_1^*,u_2}(\cdot)]+v_1^*(\cdot);u_2(\cdot)\big), \end{aligned} \tag{3.14}$$

where $X^{u_1,v_2^*}(\cdot)=X(\cdot\,;x,u_1(\cdot);\mathbf{\Theta}_2^*,v_2^*(\cdot))$ and $X^{v_1^*,u_2}(\cdot)=X(\cdot\,;x,\mathbf{\Theta}_1^*,v_1^*(\cdot);u_2(\cdot))$. Clearly, neither of the following holds in general:

$$\begin{aligned} u_1^*(\cdot)&=\Theta_1^*\big\{X^{v_1^*,u_2}(\cdot)-\mathbb{E}[X^{v_1^*,u_2}(\cdot)]\big\}+\bar{\Theta}_1^*\mathbb{E}[X^{v_1^*,u_2}(\cdot)]+v_1^*(\cdot),\\ u_2^*(\cdot)&=\Theta_2^*\big\{X^{u_1,v_2^*}(\cdot)-\mathbb{E}[X^{u_1,v_2^*}(\cdot)]\big\}+\bar{\Theta}_2^*\mathbb{E}[X^{u_1,v_2^*}(\cdot)]+v_2^*(\cdot). \end{aligned}$$

Hence, comparing this with (3.7), we see that the outcome $(u_1^*(\cdot),u_2^*(\cdot))$ of the closed-loop Nash equilibrium $(\mathbf{\Theta}^*,v^*(\cdot))$ given by (3.25) is not necessarily an open-loop Nash equilibrium of Problem (MF-SDG) for $X^*(0)=x$.

On the other hand, if $\big(\mathbf{\Theta}^*, v^*(\cdot)\big)$ is a closed-loop Nash equilibrium of Problem (MF-SDG), we may consider the following stabilized state equation (recall (2.9)):

$$\begin{cases} \mathrm{d}X^{v_1,v_2}(t) = \big\{A_{\Theta^*}X^{v_1,v_2}(t) + \bar{A}_{\mathbf{\Theta}^*}\mathbb{E}[X^{v_1,v_2}(t)] + Bv(t) + \bar{B}\mathbb{E}[v(t)] + b(t)\big\}\mathrm{d}t \\ \qquad\qquad +\big\{C_{\Theta^*}X^{v_1,v_2}(t) + \bar{C}_{\mathbf{\Theta}^*}\mathbb{E}[X^{v_1,v_2}(t)] + Dv(t) + \bar{D}\mathbb{E}[v(t)] + \sigma(t)\big\}\mathrm{d}W(t), \quad t \geqslant 0, \\ X^{v_1,v_2}(0) = x, \end{cases} \tag{3.15}$$

with cost functionals

$$\begin{aligned} J_i^{\mathbf{\Theta}^*}\big(x; v_1(\cdot), v_2(\cdot)\big) \triangleq J_i\big(x; \Theta_1^*\big\{X^{v_1,v_2}(\cdot) - \mathbb{E}[X^{v_1,v_2}(\cdot)]\big\} + \bar{\Theta}_1^*\mathbb{E}[X^{v_1,v_2}(\cdot)] + v_1(\cdot); \\ \Theta_2^*\big\{X^{v_1,v_2}(\cdot) - \mathbb{E}[X^{v_1,v_2}(\cdot)]\big\} + \bar{\Theta}_2^*\mathbb{E}[X^{v_1,v_2}(\cdot)] + v_2(\cdot)\big), \quad i = 1, 2. \end{aligned} \tag{3.16}$$

Then by (3.11), it is easy to see that $(v_1^*(\cdot), v_2^*(\cdot))$ is an open-loop Nash equilibrium of the corresponding mean-field LQ two-person non-zero sum stochastic differential games.

From the above, we see that Problems (MF-SDG) and (MF-SLQ) are essentially different in a certain sense, and we can only say that Problem (MF-SLQ) is formally a special case of Problem (MF-SDG).

3.2. Open-loop Nash equalibria and their closed-loop representation

In this section, we discuss the open-loop Nash equilibria for Problem (MF-SDG) in terms of MF-FBSDEs. We first have the following result, whose proof is similar to that of Theorem 4.1 of [47]. Thus, we omit the proof here and refer the interested readers to [29] for a detailed proof.

Theorem 3.3. *Let* (H2) *hold, and* $x \in \mathbb{R}^n$. *Then* $u^*(\cdot) \equiv (u_1^*(\cdot), u_2^*(\cdot)) \in \mathscr{U}_{ad}(x)$ *is an open-loop Nash equilibrium of Problem (MF-SDG) for* x *if and only if the following two conditions hold:*

(i) *The adapted solution* $(X^*(\cdot), Y_i^*(\cdot), Z_i^*(\cdot)) \in \mathscr{X}[0,\infty) \times \mathscr{X}[0,\infty) \times L^2_{\mathbb{F}}(\mathbb{R}^n)$ *to the following MF-FBSDE:*

$$\begin{cases} \mathrm{d}X^*(t) = \big\{AX^*(t) + \bar{A}\mathbb{E}[X^*(t)] + Bu^*(t) + \bar{B}\mathbb{E}[u^*(t)] + b(t)\big\}\mathrm{d}t \\ \qquad\qquad +\big\{CX^*(t) + \bar{C}\mathbb{E}[X^*(t)] + Du^*(t) + \bar{D}\mathbb{E}[u^*(t)] + \sigma(t)\big\}\mathrm{d}W(t), \\ -\mathrm{d}Y_i^*(t) = \big\{A^\top Y_i^*(t) + \bar{A}^\top\mathbb{E}[Y_i^*(t)] + C^\top Z_i^*(t) + \bar{C}^\top\mathbb{E}[Z_i^*(t)] + Q_iX^*(t) + \bar{Q}_i\mathbb{E}[X^*(t)] \\ \qquad\qquad +S_i^\top u^*(t) + \bar{S}_i^\top\mathbb{E}[u^*(t)] + q_i(t)\big\}\mathrm{d}t - Z_i(t)\mathrm{d}W(t), \qquad t \geqslant 0, \quad i = 1, 2, \\ X^*(0) = x, \end{cases} \tag{3.17}$$

satisfies the following stationarity condition:

$$\begin{aligned} B_i^\top Y_i^*(t) + \bar{B}_i^\top\mathbb{E}[Y_i^*(t)] + D_i^\top Z_i^*(t) + \bar{D}_i^\top\mathbb{E}[Z_i^*(t)] + S_{ii}X^*(t) + \bar{S}_{ii}\mathbb{E}[X^*(t)] \\ +R_{ii}u^*(t) + \bar{R}_{ii}\mathbb{E}[u^*(t)] + \rho_{ii}(t) = 0, \qquad \text{a.e. } t \in [0,\infty), \text{ a.s.}, \quad i = 1, 2. \end{aligned} \tag{3.18}$$

(ii) *The maps* $u_1(\cdot) \mapsto J_1(x; u_1(\cdot), u_2(\cdot))$ *and* $u_2(\cdot) \mapsto J_2(x; u_1(\cdot), u_2(\cdot))$ *are convex, i.e.,*

$$\begin{aligned} \mathbb{E}\int_0^\infty \Big[\big\langle Q_iX_i^0(t), X_i^0(t)\big\rangle + 2\big\langle S_{ii}X_i^0(t), u_i(t)\big\rangle + \big\langle R_{iii}u_i(t), u_i(t)\big\rangle + \big\langle \bar{Q}_i\mathbb{E}[X_i^0(t)], \mathbb{E}[X_i^0(t)]\big\rangle \\ +2\big\langle \bar{S}_{ii}\mathbb{E}[X_i^0(t)], \mathbb{E}[u_i(t)]\big\rangle + \big\langle \bar{R}_{iii}\mathbb{E}[u_i(t)], \mathbb{E}[u_i(t)]\big\rangle\Big]\mathrm{d}t \geqslant 0, \quad \forall u(\cdot) \equiv (u_1(\cdot), u_2(\cdot)) \in \mathscr{U}_{ad}(x), \end{aligned} \tag{3.19}$$

where $X_i^0(\cdot) \in \mathscr{X}[0,\infty)$ is the solution to the following homogeneous controlled MF-SDE:

$$\begin{cases} \mathrm{d}X_i^0(t) = \big\{AX_i^0(t) + \bar{A}\mathbb{E}[X_i^0(t)] + B_i u_i(t) + \bar{B}_i\mathbb{E}[u_i(t)]\big\}\mathrm{d}t \\ \qquad\qquad +\big\{CX_i^0(t) + \bar{C}\mathbb{E}[X_i^0(t)] + D_i u_i(t) + \bar{D}_i\mathbb{E}[u_i(t)]\big\}\mathrm{d}W(t), \quad t \geqslant 0, \\ X_i^0(0) = 0, \quad i = 1,2. \end{cases} \tag{3.20}$$

Note that (3.17) is a system of coupled MF-FBSDEs, with one forward equation and two backward equations, for which the coupling is through the relation (3.18). Our next task is to investigate the solvability of (3.17)–(3.18). To this end, we introduce the following notation (recall (3.1)):

$$\begin{aligned} &\mathbf{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix},\ \bar{\mathbf{A}} = \begin{pmatrix} \bar{A} & 0 \\ 0 & \bar{A} \end{pmatrix},\ \mathbf{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix},\ \bar{\mathbf{C}} = \begin{pmatrix} \bar{C} & 0 \\ 0 & \bar{C} \end{pmatrix} \in \mathbb{R}^{2n\times 2n}; \\ &\mathbf{B} = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix},\ \bar{\mathbf{B}} = \begin{pmatrix} \bar{B} & 0 \\ 0 & \bar{B} \end{pmatrix},\ \mathbf{D} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix},\ \bar{\mathbf{D}} = \begin{pmatrix} \bar{D} & 0 \\ 0 & \bar{D} \end{pmatrix} \in \mathbb{R}^{2n\times 2m}; \\ &\mathbf{Q} = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix},\ \bar{\mathbf{Q}} = \begin{pmatrix} \bar{Q}_1 & 0 \\ 0 & \bar{Q}_2 \end{pmatrix} \in \mathbb{S}^{2n};\quad \mathbf{S} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix},\ \bar{\mathbf{S}} = \begin{pmatrix} \bar{S}_1 & 0 \\ 0 & \bar{S}_2 \end{pmatrix} \in \mathbb{R}^{2m\times 2n}; \\ &\mathbf{R} = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix},\ \bar{\mathbf{R}} = \begin{pmatrix} \bar{R}_1 & 0 \\ 0 & \bar{R}_2 \end{pmatrix} \in \mathbb{S}^{2m};\ q(\cdot) = \begin{pmatrix} q_1(\cdot) \\ q_2(\cdot) \end{pmatrix} \in L^2_{\mathbb{F}}(\mathbb{R}^{2n});\ \rho(\cdot) = \begin{pmatrix} \rho_1(\cdot) \\ \rho_2(\cdot) \end{pmatrix} \in L^2_{\mathbb{F}}(\mathbb{R}^{2m}). \end{aligned} \tag{3.21}$$

With the above notation, MF-FBSDEs (3.17) can be rewritten as

$$\begin{cases} \mathrm{d}X^*(t) = \big\{AX^* + \bar{A}\mathbb{E}[X^*] + Bu^* + \bar{B}\mathbb{E}[u^*] + b\big\}\mathrm{d}t \\ \qquad\qquad +\big\{CX^* + \bar{C}\mathbb{E}[X^*] + Du^* + \bar{D}\mathbb{E}[u^*] + \sigma\big\}\mathrm{d}W(t), \\ -d\mathbf{Y}^*(t) = \big\{\mathbf{A}^\top\mathbf{Y}^* + \bar{\mathbf{A}}^\top\mathbb{E}[\mathbf{Y}^*] + \mathbf{C}^\top\mathbf{Z}^* + \bar{\mathbf{C}}^\top\mathbb{E}[\mathbf{Z}^*] + \mathbf{Q}\mathbf{I}_n X^* + \bar{\mathbf{Q}}\mathbf{I}_n\mathbb{E}[X^*] \\ \qquad\qquad +\mathbf{S}^\top\mathbf{I}_m u^* + \bar{\mathbf{S}}^\top\mathbf{I}_m\mathbb{E}[u^*] + q\big\}\mathrm{d}t - \mathbf{Z}^*\mathrm{d}W(t), \quad t \geqslant 0, \\ X^*(0) = x, \end{cases} \tag{3.22}$$

and the stationarity condition (3.18) can be written as

$$\begin{aligned} \mathbf{J}^\top\big\{\mathbf{B}^\top\mathbf{Y}^* + \bar{\mathbf{B}}^\top\mathbb{E}[\mathbf{Y}^*] + \mathbf{D}^\top\mathbf{Z}^* + \bar{\mathbf{D}}^\top\mathbb{E}[\mathbf{Z}^*] + \mathbf{S}\mathbf{I}_n X^* + \bar{\mathbf{S}}\mathbf{I}_n\mathbb{E}[X^*] + \mathbf{R}\mathbf{I}_m u^* + \bar{\mathbf{R}}\mathbf{I}_m\mathbb{E}[u^*] + \rho\big\} = 0, \\ \text{a.e. } t \in [0,\infty),\ \text{a.s.}, \end{aligned} \tag{3.23}$$

where

$$\begin{cases} \mathbf{Y}^*(\cdot) = \begin{pmatrix} Y_1^*(\cdot) \\ Y_2^*(\cdot) \end{pmatrix} \in \mathscr{X}[0,\infty)^2, \quad \mathbf{Z}^*(\cdot) = \begin{pmatrix} Z_1^*(\cdot) \\ Z_2^*(\cdot) \end{pmatrix} \in L^2_{\mathbb{F}}(\mathbb{R}^{2n}), \quad \mathbf{I}_n = \begin{pmatrix} I_{n\times n} \\ I_{n\times n} \end{pmatrix} \in \mathbb{R}^{2n\times n}, \\ \mathbf{I}_m = \begin{pmatrix} I_{m\times m} \\ I_{m\times m} \end{pmatrix} \in \mathbb{R}^{2m\times m}, \quad \mathbf{J} = \begin{pmatrix} I_{m_1\times m_1} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I_{m_2\times m_2} \end{pmatrix} = \begin{pmatrix} I_{m_1\times m_1} & 0_{m_1\times m_2} \\ 0_{m_2\times m_1} & 0_{m_2\times m_2} \\ 0_{m_1\times m_1} & 0_{m_1\times m_2} \\ 0_{m_2\times m_1} & I_{m_2\times m_2} \end{pmatrix} \in \mathbb{R}^{2m\times m}. \end{cases} \tag{3.24}$$

Now, inspired by [47, 53], we may obtain a closed-loop representation of open-loop Nash equilibria, by which we mean an open-loop Nash equilibrium $u^{**}(\cdot) \equiv (u_1^{**}(\cdot), u_2^{**}(\cdot))$ admits a form (3.9) for some $(\mathbf{\Theta}^{**}, v^{**}(\cdot)) \in \mathscr{S}[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}] \times L^2_{\mathbb{F}}(\mathbb{R}^m)$. We state the result here and put a detailed proof in [29].

Theorem 3.4. *Let* (H2) *hold. Then an open-loop Nash equilibria* $u^{**}(\cdot)\equiv(u_1^{**}(\cdot),u_2^{**}(\cdot))\in\mathscr{U}_{ad}(x)$ *of Problem* (MF-SDG) *admits the following closed-loop representation*

$$u_i^{**}(\cdot)=\Theta_i^{**}\big\{X(\cdot)-\mathbb{E}[X(\cdot)]\big\}+\bar{\Theta}_i^{**}\mathbb{E}[X(\cdot)]+v_i^{**}(\cdot),\quad i=1,2, \tag{3.25}$$

with $\mathbf{\Theta}^{**}\equiv\begin{pmatrix}\mathbf{\Theta}_1^{**}\\ \mathbf{\Theta}_2^{**}\end{pmatrix}\equiv\begin{pmatrix}\Theta_1^{**} & \bar{\Theta}_1^{**}\\ \Theta_2^{**} & \bar{\Theta}_2^{**}\end{pmatrix}\in\mathscr{S}[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]$ *and* $v^{**}(\cdot)\equiv\begin{pmatrix}v_1^{**}(\cdot)\\ v_2^{**}(\cdot)\end{pmatrix}\in L^2_{\mathbb{F}}(\mathbb{R}^m)$ *if and only if the following hold:*

(i) *The convexity condition (3.19) holds for* $i=1,2$.

(ii) *The solution pair* $(\mathbf{P},\widehat{\mathbf{P}})\triangleq\begin{pmatrix}P_1 & \widehat{P}_1\\ P_2 & \widehat{P}_2\end{pmatrix}\in\mathbb{R}^{2n\times 2n}$ *to the system of coupled AREs:*

$$\begin{cases}\mathbf{P}A+\mathbf{A}^\top\mathbf{P}+\mathbf{C}^\top\mathbf{P}C+\mathbf{Q}\mathbf{I}_n-\big(\mathbf{P}B+\mathbf{C}^\top\mathbf{P}D+\mathbf{S}^\top\mathbf{I}_m\big)\mathbf{\Sigma}^{-1}\mathbf{J}^\top\big(\mathbf{B}^\top\mathbf{P}+\mathbf{D}^\top\mathbf{P}C+\mathbf{S}\mathbf{I}_n\big)=0,\\ \widehat{\mathbf{P}}\widehat{A}+\widehat{\mathbf{A}}^\top\widehat{\mathbf{P}}+\widehat{\mathbf{C}}^\top\mathbf{P}\widehat{C}+\widehat{\mathbf{Q}}\mathbf{I}_n-\big(\widehat{\mathbf{P}}\widehat{B}+\widehat{\mathbf{C}}^\top\mathbf{P}\widehat{D}+\widehat{\mathbf{S}}^\top\mathbf{I}_m\big)\bar{\mathbf{\Sigma}}^{-1}\mathbf{J}^\top\big(\widehat{\mathbf{B}}^\top\widehat{\mathbf{P}}+\widehat{\mathbf{D}}^\top\mathbf{P}\widehat{C}+\widehat{\mathbf{S}}\mathbf{I}_n\big)=0,\end{cases} \tag{3.26}$$

where $\widehat{\mathbf{A}}=\mathbf{A}+\bar{\mathbf{A}}$ *and* $\widehat{\mathbf{C}},\widehat{\mathbf{Q}},\widehat{\mathbf{S}}$ *are defined similarly as* (2.15) *and*

$$\mathbf{\Sigma}\triangleq\mathbf{J}^\top\big(\mathbf{R}\mathbf{I}_m+\mathbf{D}^\top\mathbf{P}D\big),\quad \bar{\mathbf{\Sigma}}\triangleq\mathbf{J}^\top\big(\widehat{\mathbf{R}}\mathbf{I}_m+\widehat{\mathbf{D}}^\top\mathbf{P}\widehat{D}\big)\in\mathbb{R}^{m\times m}$$

are both invertible such that $\mathbf{\Theta}^{**}\equiv(\Theta^{**},\bar{\Theta}^{**})\equiv\begin{pmatrix}\Theta_1^{**} & \bar{\Theta}_1^{**}\\ \Theta_2^{**} & \bar{\Theta}_2^{**}\end{pmatrix}\in\mathbb{R}^{m\times 2n}$ *defined by*

$$\Theta^{**}\triangleq-\mathbf{\Sigma}^{-1}\mathbf{J}^\top\big(\mathbf{B}^\top\mathbf{P}+\mathbf{D}^\top\mathbf{P}C+\mathbf{S}\mathbf{I}_n\big),\qquad \bar{\Theta}^{**}\triangleq-\bar{\mathbf{\Sigma}}^{-1}\mathbf{J}^\top\big(\widehat{\mathbf{B}}^\top\widehat{\mathbf{P}}+\widehat{\mathbf{D}}^\top\mathbf{P}\widehat{C}+\widehat{\mathbf{S}}\mathbf{I}_n\big), \tag{3.27}$$

stabilizes the system $[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]$. *Also,*

$$\begin{aligned}v^{**}(\cdot)=&-\mathbf{\Sigma}^{-1}\mathbf{J}^\top\big\{\mathbf{B}^\top(\eta-\mathbb{E}[\eta])+\mathbf{D}^\top(\zeta-\mathbb{E}[\zeta])+\mathbf{D}^\top\mathbf{P}(\sigma-\mathbb{E}[\sigma])+\rho-\mathbb{E}[\rho]\big\}\\ &-\bar{\mathbf{\Sigma}}^{-1}\mathbf{J}^\top\big\{\widehat{\mathbf{B}}^\top\bar{\eta}+\widehat{\mathbf{D}}^\top\mathbb{E}[\zeta]+\widehat{\mathbf{D}}^\top\mathbf{P}\mathbb{E}[\sigma]+\mathbb{E}[\rho]\big\}\in L^2_{\mathbb{F}}(\mathbb{R}^m),\end{aligned} \tag{3.28}$$

where $\left(\eta(\cdot)\triangleq\begin{pmatrix}\eta_1(\cdot)\\ \eta_2(\cdot)\end{pmatrix},\zeta(\cdot)\triangleq\begin{pmatrix}\zeta_1(\cdot)\\ \zeta_2(\cdot)\end{pmatrix}\right)\in\mathscr{X}[0,\infty)^2\times L^2_{\mathbb{F}}(\mathbb{R}^{2n})$ *is an adapted solution to BSDE:*

$$\begin{aligned}-\mathrm{d}\eta(t)=&\Big\{\mathbf{A}^\top\eta(t)-\big(\mathbf{P}B+\mathbf{C}^\top\mathbf{P}D+\mathbf{S}^\top\mathbf{I}_m\big)\mathbf{\Sigma}^{-1}\mathbf{J}^\top\big\{\mathbf{B}^\top\eta(t)+\mathbf{D}^\top\big[\zeta(t)+\mathbf{P}\sigma(t)\big]+\rho(t)\big\}\\ &+\mathbf{C}^\top\big[\zeta(t)+\mathbf{P}\sigma(t)\big]+\mathbf{P}b(t)+q(t)\Big\}\mathrm{d}t-\zeta(t)\mathrm{d}W(t),\qquad t\geqslant 0,\end{aligned} \tag{3.29}$$

and $\bar{\eta}(\cdot)\triangleq\begin{pmatrix}\bar{\eta}_1(\cdot)\\ \bar{\eta}_2(\cdot)\end{pmatrix}\in L^2(\mathbb{R}^{2n})$ *is a solution to ODE:*

$$\begin{aligned}&\dot{\bar{\eta}}+\widehat{\mathbf{A}}^\top\bar{\eta}+\widehat{\mathbf{C}}^\top\mathbb{E}[\zeta+\mathbf{P}\sigma]+\widehat{\mathbf{P}}\mathbb{E}[b]+\mathbb{E}[q]\\ &\quad-\big(\widehat{\mathbf{P}}\widehat{B}+\widehat{\mathbf{C}}^\top\mathbf{P}\widehat{D}+\widehat{\mathbf{S}}^\top\mathbf{I}_m\big)\bar{\mathbf{\Sigma}}^{-1}\big\{\widehat{\mathbf{B}}^\top\bar{\eta}+\widehat{\mathbf{D}}^\top\mathbb{E}[\zeta+\mathbf{P}\sigma]+\mathbb{E}[\rho]\big\}=0,\qquad t\geqslant 0.\end{aligned} \tag{3.30}$$

In such a case, the open-loop Nash equilibrium $u^{**}(\cdot)$ *admits a closed-loop representation* (3.25) *with* $(\mathbf{\Theta}^{**},v^{**}(\cdot))$ *given by the above.*

Note that by the definition of Θ^{**} and $\bar{\Theta}^{**}$ in (3.27), we have

$$\mathbf{\Sigma}\Theta^{**} + \mathbf{J}^\top\big(\mathbf{B}^\top\mathbf{P} + \mathbf{D}^\top\mathbf{P}C + \mathbf{S}\mathbf{I}_n\big) = 0, \qquad \bar{\mathbf{\Sigma}}\bar{\Theta}^{**} + \mathbf{J}^\top\big(\widehat{\mathbf{B}}^\top\widehat{\mathbf{P}} + \widehat{\mathbf{D}}^\top\mathbf{P}\widehat{C} + \widehat{\mathbf{S}}\mathbf{I}_n\big) = 0; \tag{3.31}$$

and we can rewrite (3.26) as

$$\begin{cases} \mathbf{P}A + \mathbf{A}^\top\mathbf{P} + \mathbf{C}^\top\mathbf{P}C + \mathbf{Q}\mathbf{I}_n + \big(\mathbf{P}B + \mathbf{C}^\top\mathbf{P}D + \mathbf{S}^\top\mathbf{I}_m\big)\Theta^{**} = 0, \\ \widehat{\mathbf{P}}\widehat{A} + \widehat{\mathbf{A}}^\top\widehat{\mathbf{P}} + \widehat{\mathbf{C}}^\top\mathbf{P}\widehat{C} + \widehat{\mathbf{Q}}\mathbf{I}_n + \big(\widehat{\mathbf{P}}\widehat{B} + \widehat{\mathbf{C}}^\top\mathbf{P}\widehat{D} + \widehat{\mathbf{S}}^\top\mathbf{I}_m\big)\bar{\Theta}^{**} = 0. \end{cases} \tag{3.32}$$

We may further write (3.31) and (3.32) in the component forms:

$$\begin{cases} P_iA + A^\top P_i + C^\top P_iC + Q_i + \big(P_iB + C^\top P_iD + S_i^\top\big)\Theta^{**} = 0, \\ \widehat{P}_i\widehat{A} + \widehat{A}^\top\widehat{P}_i + \widehat{C}^\top P_i\widehat{C} + \widehat{Q}_i + \big(\widehat{P}_i\widehat{B} + \widehat{C}^\top P_i\widehat{D} + \widehat{S}_i^\top\big)\bar{\Theta}^{**} = 0, \end{cases} \qquad i = 1, 2, \tag{3.33}$$

$$\begin{cases} \begin{pmatrix} R_{111} + D_1^\top P_1 D_1 & R_{112} + D_1^\top P_1 D_2 \\ R_{221} + D_2^\top P_2 D_1 & R_{222} + D_2^\top P_2 D_2 \end{pmatrix}\Theta^{**} + \begin{pmatrix} B_1^\top P_1 + D_1^\top P_1 C + S_{11} \\ B_2^\top P_2 + D_2^\top P_2 C + S_{22} \end{pmatrix} = 0, \\ \begin{pmatrix} \widehat{R}_{111} + \widehat{D}_1^\top P_1\widehat{D}_1 & \widehat{R}_{112} + \widehat{D}_1^\top P_1\widehat{D}_2 \\ \widehat{R}_{221} + \widehat{D}_2^\top P_2\widehat{D}_1 & \widehat{R}_{222} + \widehat{D}_2^\top P_2\widehat{D}_2 \end{pmatrix}\bar{\Theta}^{**} + \begin{pmatrix} \widehat{B}_1^\top\widehat{P}_1 + \widehat{D}_1^\top P_1\widehat{C} + \widehat{S}_{11} \\ \widehat{B}_2^\top\widehat{P}_2 + \widehat{D}_2^\top P_2\widehat{C} + \widehat{S}_{22} \end{pmatrix} = 0. \end{cases} \tag{3.34}$$

In the above, the coefficient matrices of the equations for Θ^{**} and $\bar{\Theta}^{**}$ are not symmetric in general (even if $P_1, P_2, \widehat{P}_1, \widehat{P}_2$ are all symmetric). Hence, the equations for $P_i, \widehat{P}_i, i = 1, 2$ and for $P_i^\top, \widehat{P}_i^\top, i = 1, 2$ are different. Consequently, we do not expect $P_i, \widehat{P}_i, i = 1, 2$ to be symmetric in general.

3.3. Closed-loop Nash equalibria and symmetric algebraic Riccati equations

We now look at closed-loop Nash equilibria for Problem (MF-SDG). First, we present the following result, which is a consequence of Theorem 3.3 and its proof is similar to that of Proposition 5.1 in [47] (see also [29]).

Proposition 3.5. *Let* (H2) *hold. If* $(\mathbf{\Theta}^*, v^*(\cdot))$ *is a closed-loop Nash equilibrium of Problem* (MF-SDG)*, then* $(\mathbf{\Theta}^*, 0)$ *is a closed-loop Nash equilibrium of Problem* (MF-SDG)^0.

Now, we give a necessary condition for closed-loop Nash equilibria of Problem (MF-SDG).

Proposition 3.6. *Let* (H2) *hold, and let* $(\mathbf{\Theta}^*, v^*(\cdot))$ *be a closed-loop Nash equilibrium of Problem* (MF-SDG)*. Then for* $i = 1, 2$*, the following system of coupled AREs admits a solution pair* $(P_i, \widehat{P}_i) \in \mathbb{S}^n \times \mathbb{S}^n$*:*

$$\begin{cases} P_iA + A^\top P_i + C^\top P_iC + Q_i + (\Theta^*)^\top(R_i + D^\top P_iD)\Theta^* \\ \qquad +(P_iB + C^\top P_iD + S_i^\top)\Theta^* + (\Theta^*)^\top(B^\top P_i + D^\top P_iC + S_i) = 0, \\ \widehat{P}_i\widehat{A} + \widehat{A}^\top\widehat{P}_i + \widehat{C}^\top P_i\widehat{C} + \widehat{Q}_i + (\bar{\Theta}^*)^\top(\widehat{R}_i + \widehat{D}^\top P_i\widehat{D})\bar{\Theta}^* \\ \qquad +(\widehat{P}_i\widehat{B} + \widehat{C}^\top P_i\widehat{D} + \widehat{S}_i^\top)\bar{\Theta}^* + (\bar{\Theta}^*)^\top(\widehat{B}^\top\widehat{P}_i + \widehat{D}^\top P_i\widehat{C} + \widehat{S}_i) = 0, \end{cases} \tag{3.35}$$

and the following conditions are satisfied:

$$\begin{cases} B_i^\top P_i + D_i^\top P_iC + S_{ii} + \Sigma_i(R_{ii} + D_i^\top P_iD)\Theta^* = 0, \\ \widehat{B}_i^\top\widehat{P}_i + \widehat{D}_i^\top P_i\widehat{C} + \widehat{S}_{ii} + (\widehat{R}_{ii} + \widehat{D}_i^\top P_i\widehat{D})\bar{\Theta}^* = 0, \end{cases} \tag{3.36}$$

with

$$\Sigma_i \triangleq R_{iii} + D_i^\top P_i D_i \geqslant 0, \qquad \bar{\Sigma}_i \triangleq \widehat{R}_{iii} + \widehat{D}_i^\top P_i \widehat{D}_i \geqslant 0. \tag{3.37}$$

Proof. Suppose that $(\mathbf{\Theta}^*, v^*(\cdot))$ is a closed-loop Nash equilibrium of Problem (MF-SDG). Then by Proposition 3.5, $(\mathbf{\Theta}^*, 0)$ is a closed-loop Nash equilibrium of Problem (MF-SDG)^0. Denote

$$\begin{cases} \mathcal{A}_1 \triangleq A + B_2\Theta_2^*, \qquad \bar{\mathcal{A}}_1 \triangleq \bar{A} + \bar{B}_2\bar{\Theta}_2^* + B_2(\bar{\Theta}_2^* - \Theta_2^*), \qquad \mathcal{C}_1 \triangleq C + D_2\Theta_2^*, \\ \bar{\mathcal{C}}_1 \triangleq \bar{C} + \bar{D}_2\bar{\Theta}_2^* + D_2(\bar{\Theta}_2^* - \Theta_2^*), \qquad \mathcal{Q}_1 \triangleq Q_1 + S_{12}\Theta_2^* + (\Theta_2^*)^\top S_{12} + (\Theta_2^*)^\top R_{122}\Theta_2^*, \\ \bar{\mathcal{Q}}_1 \triangleq \bar{Q}_1 + \widehat{S}_{12}^\top\bar{\Theta}_2^* + (\bar{\Theta}_2^*)^\top\widehat{S}_{12} + (\bar{\Theta}_2^*)^\top\widehat{R}_{122}\bar{\Theta}_2^* - S_{12}^\top\Theta_2^* - (\Theta_2^*)^\top S_{12} - (\Theta_2^*)^\top R_{122}\Theta_2^*, \\ \mathcal{S}_{11} \triangleq S_{11} + R_{112}\Theta_2^*, \qquad \bar{\mathcal{S}}_{11} \triangleq \bar{S}_{11} + \bar{R}_{112}\bar{\Theta}_2^* + R_{112}(\bar{\Theta}_2^* - \Theta_2^*). \end{cases} \tag{3.38}$$

Then, for any $u_1(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^{m_1})$, let us consider the state equation:

$$\begin{cases} \mathrm{d}X_1^0(t) = \big\{\mathcal{A}_1 X_1^0(t) + \bar{\mathcal{A}}_1\mathbb{E}[X_1^0(t)] + B_1u_1(t) + \bar{B}_1\mathbb{E}[u_1(t)]\big\}\mathrm{d}t \\ \qquad\qquad + \big\{\mathcal{C}_1 X_1^0(t) + \bar{\mathcal{C}}_1\mathbb{E}[X_1^0(t)] + D_1u_1(t) + \bar{D}_1\mathbb{E}[u_1(t)]\big\}\mathrm{d}W(t), \quad t \geqslant 0, \\ X_1^0(0) = 0, \end{cases}$$

and cost functional:

$$\begin{aligned} &\bar{J}_1^0(x; u_1(\cdot)) \equiv J_1^0\big(x; u_1(\cdot); \Theta_2^*\big\{X_1^0(\cdot) - \mathbb{E}[X_1^0(\cdot)]\big\} + \bar{\Theta}_2^*\mathbb{E}[X_1^0(\cdot)]\big) \\ &= \mathbb{E}\int_0^\infty \left[\left\langle \begin{pmatrix} \mathcal{Q}_1 & \mathcal{S}_{11}^\top \\ \mathcal{S}_{11} & R_{111} \end{pmatrix} \begin{pmatrix} X_1^0 \\ u_1 \end{pmatrix}, \begin{pmatrix} X_1^0 \\ u_1 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} \bar{\mathcal{Q}}_1 & \bar{\mathcal{S}}_{11}^\top \\ \bar{\mathcal{S}}_{11} & \bar{R}_{111} \end{pmatrix} \begin{pmatrix} \mathbb{E}[X_1^0] \\ \mathbb{E}[u_1] \end{pmatrix}, \begin{pmatrix} \mathbb{E}[X_1^0] \\ \mathbb{E}[u_1] \end{pmatrix} \right\rangle \right] \mathrm{d}t. \end{aligned}$$

It is easy to see that $(\mathbf{\Theta}_1^*, 0)$ is a closed-loop optimal strategy for the above mean-field LQ stochastic optimal control problem. Thanks to Theorem 2.9, the following system of coupled AREs:

$$\begin{cases} P_1\mathcal{A}_1 + \mathcal{A}_1^\top P_1 + \mathcal{C}_1^\top P_1\mathcal{C}_1 + \mathcal{Q}_1 - \big(P_1B_1 + \mathcal{C}_1^\top P_1D_1 + \mathcal{S}_{11}^\top\big)\Sigma_1^\dagger\big(B_1^\top P_1 + D_1^\top P_1\mathcal{C}_1 + \mathcal{S}_{11}\big) = 0, \\ \widehat{P}_1\widehat{\mathcal{A}}_1 + \widehat{\mathcal{A}}_1^\top\widehat{P}_1 + \widehat{\mathcal{C}}_1^\top P_1\widehat{\mathcal{C}}_1 + \widehat{\mathcal{Q}}_1 - \big(\widehat{P}_1\widehat{B}_1 + \widehat{\mathcal{C}}_1^\top P_1\widehat{D}_1 + \widehat{\mathcal{S}}_{11}^\top\big)\bar{\Sigma}_1^\dagger\big(\widehat{B}_1^\top\widehat{P}_1 + \widehat{D}_1^\top P_1\widehat{\mathcal{C}}_1 + \widehat{\mathcal{S}}_{11}\big) = 0, \end{cases} \tag{3.39}$$

admit a static stabilizing solution pair $(P_1, \widehat{P}_1) \in \mathbb{S}^n \times \mathbb{S}^n$, satisfying

$$\begin{cases} B_1^\top P_1 + D_1^\top P_1\mathcal{C}_1 + \mathcal{S}_{11} + \Sigma_1\Theta_1^* = 0, \qquad \Sigma_1 \equiv R_{111} + D_1^\top P_1D_1 \geqslant 0, \\ \widehat{B}_1^\top\widehat{P}_1 + \widehat{D}_1^\top P_1\widehat{\mathcal{C}}_1 + \widehat{\mathcal{S}}_{11} + \bar{\Sigma}_1\bar{\Theta}_1^* = 0, \qquad \bar{\Sigma}_1 \equiv \widehat{R}_{111} + \widehat{D}_1^\top P_1\widehat{D}_1 \geqslant 0, \end{cases} \tag{3.40}$$

where $\widehat{\mathcal{A}}_1 = \mathcal{A} + \bar{\mathcal{A}}_1$, $\widehat{\mathcal{C}}_1 = \mathcal{C}_1 + \bar{\mathcal{C}}_1$, $\widehat{\mathcal{Q}}_1 = \mathcal{Q}_1 + \bar{\mathcal{Q}}_1$, and $\widehat{\mathcal{S}}_{11} = \mathcal{S}_{11} + \bar{\mathcal{S}}_{11}$. Similarly, for any $u_2(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^{m_2})$, we can consider the state equation:

$$\begin{cases} \mathrm{d}X_2^0(t) = \big\{\mathcal{A}_2 X_2^0(t) + \bar{\mathcal{A}}_2\mathbb{E}[X_2^0(t)] + B_2u_2(t) + \bar{B}_2\mathbb{E}[u_2(t)]\big\}\mathrm{d}t \\ \qquad\qquad + \big\{\mathcal{C}_2 X_2^0(t) + \bar{\mathcal{C}}_2\mathbb{E}[X_2^0(t)] + D_2u_2(t) + \bar{D}_2\mathbb{E}[u_2(t)]\big\}\mathrm{d}W(t), \quad t \geqslant 0, \\ X_2^0(0) = 0, \end{cases}$$

and cost functional:

$$\begin{aligned}&\bar{J}_2^0(x;u_2(\cdot))\equiv J_2^0\big(x;\Theta_1^*\big(X_2^0(\cdot)-\mathbb{E}[X_2^0(\cdot)]\big)+\bar{\Theta}_1^*\mathbb{E}[X_2^0(\cdot)];u_2(\cdot)\big)\\&=\mathbb{E}\int_0^\infty\left[\left\langle\begin{pmatrix}\mathcal{Q}_2&\mathcal{S}_{22}^\top\\\mathcal{S}_{22}&R_{222}\end{pmatrix}\begin{pmatrix}X_2^0\\u_2\end{pmatrix},\begin{pmatrix}X_2^0\\u_2\end{pmatrix}\right\rangle+\left\langle\begin{pmatrix}\bar{\mathcal{Q}}_2&\bar{\mathcal{S}}_{22}^\top\\\bar{\mathcal{S}}_{22}&\bar{R}_{222}\end{pmatrix}\begin{pmatrix}\mathbb{E}[X_2^0]\\\mathbb{E}[u_2]\end{pmatrix},\begin{pmatrix}\mathbb{E}[X_2^0]\\\mathbb{E}[u_2]\end{pmatrix}\right\rangle\right]\mathrm{d}t,\end{aligned}$$

where $\mathcal{A}_2,\bar{\mathcal{A}}_2,\mathcal{C}_2,\bar{\mathcal{C}}_2$ and $\mathcal{Q}_2,\bar{\mathcal{Q}}_2,\mathcal{S}_{22},\bar{\mathcal{S}}_{22}$ are defined similar to (3.38). In the same spirit as above, we can see that $(\boldsymbol{\Theta}_2^*,0)\equiv(\Theta_2^*,\bar{\Theta}_2^*,0)$ is a closed-loop optimal strategy for the above mean-field LQ stochastic optimal control problem. Making use of Theorem 2.9 again, the following system of coupled AREs

$$\begin{cases}P_2\mathcal{A}_2+\mathcal{A}_2^\top P_2+\mathcal{C}_2^\top P_2\mathcal{C}_2+\mathcal{Q}_2-\big(P_2B_2+\mathcal{C}_2^\top P_2D_2+\mathcal{S}_{22}^\top\big)\Sigma_2^\dagger\big(B_2^\top P_2+D_2^\top P_2\mathcal{C}_2+\mathcal{S}_{22}\big)=0,\\\widehat{P}_2\widehat{\mathcal{A}}_2+\widehat{\mathcal{A}}_2^\top\widehat{P}_2+\widehat{\mathcal{C}}_2^\top P_2\widehat{\mathcal{C}}_2+\widehat{\mathcal{Q}}_2-\big(\widehat{P}_2\widehat{B}_2+\widehat{\mathcal{C}}_2^\top P_2\widehat{D}_2+\widehat{\mathcal{S}}_{22}^\top\big)\bar{\Sigma}_2^\dagger\big(\widehat{B}_2^\top\widehat{P}_2+\widehat{D}_2^\top P_2\widehat{\mathcal{C}}_2+\widehat{\mathcal{S}}_{22}\big)=0,\end{cases}\tag{3.41}$$

admit a static stabilizing solution pair $(P_2,\widehat{P}_2)\in\mathbb{S}^n\times\mathbb{S}^n$, satisfying

$$\begin{cases}0=B_2^\top P_2+D_2^\top P_2\mathcal{C}_2+\mathcal{S}_{22}+\Sigma_2\Theta_2^*,\qquad\Sigma_2\equiv R_{222}+D_2^\top P_2D_2\geqslant 0,\\0=\widehat{B}_2^\top\widehat{P}_2+\widehat{D}_2^\top P_2\widehat{\mathcal{C}}_2+\widehat{\mathcal{S}}_{22}+\bar{\Sigma}_2\bar{\Theta}_2^*,\qquad\bar{\Sigma}_2\equiv\widehat{R}_{222}+\widehat{D}_2^\top P_2\widehat{D}_2\geqslant 0,\end{cases}\tag{3.42}$$

where $\widehat{\mathcal{A}}_2=\mathcal{A}_2+\bar{\mathcal{A}}_2$, $\widehat{\mathcal{C}}_2=\mathcal{C}_2+\bar{\mathcal{C}}_2$, $\widehat{\mathcal{Q}}_2=\mathcal{Q}_2+\bar{\mathcal{Q}}_2$, and $\widehat{\mathcal{S}}_{22}=\mathcal{S}_{22}+\bar{\mathcal{S}}_{22}$. By (3.40) and (3.42) and putting $\boldsymbol{\Theta}^*\equiv(\Theta^*,\bar{\Theta}^*)=\begin{pmatrix}\Theta_1^*&\bar{\Theta}_1^*\\\Theta_2^*&\bar{\Theta}_2^*\end{pmatrix}\in\mathbb{R}^{m\times 2n}$, we get (3.37) and

$$\begin{cases}0=B_1^\top P_1+D_1^\top P_1\mathcal{C}_1+\mathcal{S}_{11}+\Sigma_1\Theta_1^*=B_1^\top P_1+D_1^\top P_1C+S_{11}+(R_{11}+D_1^\top P_1D)\Theta^*,\\0=\widehat{B}_1^\top\widehat{P}_1+\widehat{D}_1^\top P_1\widehat{\mathcal{C}}_1+\widehat{\mathcal{S}}_{11}+\bar{\Sigma}_1\bar{\Theta}_1^*=\widehat{B}_1^\top\widehat{P}_1+\widehat{D}_1^\top P_1\widehat{C}+\widehat{S}_{11}+\big(\widehat{R}_{11}+\widehat{D}_1^\top P_1\widehat{D}\big)\bar{\Theta}^*.\end{cases}$$

Similarly,

$$\begin{cases}0=B_2^\top P_2+D_2^\top P_2\mathcal{C}_2+\mathcal{S}_{22}+\Sigma_2\Theta_2^*=B_2^\top P_2+D_2^\top P_2C+S_{22}+(R_{22}+D_2^\top P_2D)\Theta^*,\\0=\widehat{B}_2^\top\widehat{P}_2+\widehat{D}_2^\top P_2\widehat{\mathcal{C}}_2+\widehat{\mathcal{S}}_{22}+\bar{\Sigma}_2\bar{\Theta}_2^*=\widehat{B}_2^\top\widehat{P}_2+\widehat{D}_2^\top P_2\widehat{C}+\widehat{S}_{22}+\big(\widehat{R}_{22}+\widehat{D}_2^\top P_2\widehat{D}\big)\bar{\Theta}^*,\end{cases}$$

which implies (3.36). By (3.36), (3.39) and (3.41), we have (by a straightforward calculation)

$$\begin{cases}\begin{aligned}0&=P_1\mathcal{A}_1+\mathcal{A}_1^\top P_1+\mathcal{C}_1^\top P_1\mathcal{C}_1+\mathcal{Q}_1-\big(P_1B_1+\mathcal{C}_1^\top P_1D_1+\mathcal{S}_{11}^\top\big)\Sigma_1^\dagger\big(B_1^\top P_1+D_1^\top P_1\mathcal{C}_1+\mathcal{S}_{11}\big)\\&=P_1A+A^\top P_1+C^\top P_1C+Q_1+(\Theta^*)^\top(R_1+D^\top P_1D)\Theta^*\\&\quad+\big(P_1B+C^\top P_1D+S_1^\top\big)\Theta^*+(\Theta^*)^\top\big(B^\top P_1+D^\top P_1C+S_1\big),\end{aligned}\\\begin{aligned}0&=\widehat{P}_1\widehat{\mathcal{A}}_1+\widehat{\mathcal{A}}_1^\top\widehat{P}_1+\widehat{\mathcal{C}}_1^\top P_1\widehat{\mathcal{C}}_1+\widehat{\mathcal{Q}}_1-\big(\widehat{P}_1\widehat{B}_1+\widehat{\mathcal{C}}_1^\top P_1\widehat{D}_1+\widehat{\mathcal{S}}_{11}^\top\big)\bar{\Sigma}_1^\dagger\big(\widehat{B}_1^\top\widehat{P}_1+\widehat{D}_1^\top P_1\widehat{\mathcal{C}}_1+\widehat{\mathcal{S}}_{11}\big)\\&=\widehat{P}_1\widehat{A}+\widehat{A}^\top\widehat{P}_1+\widehat{C}^\top P_1\widehat{C}+\widehat{Q}_1+(\bar{\Theta}^*)^\top\big(\widehat{R}_1+\widehat{D}^\top P_1\widehat{D}\big)\bar{\Theta}^*\\&\quad+\big(\widehat{P}_1\widehat{B}+\widehat{C}^\top P_1\widehat{D}+\widehat{S}_1^\top\big)\bar{\Theta}^*+(\bar{\Theta}^*)^\top\big(\widehat{B}^\top\widehat{P}_1+\widehat{D}^\top P_1\widehat{C}+\widehat{S}_1\big).\end{aligned}\end{cases}$$

In the same way, we have

$$\begin{cases} 0 = P_2\mathcal{A}_2 + \mathcal{A}_2^\top P_2 + \mathcal{C}_2^\top P_2\mathcal{C}_2 + \mathcal{Q}_2 - \big(P_2B_2 + \mathcal{C}_2^\top P_2D_2 + \mathcal{S}_{22}^\top\big)\Sigma_2^\dagger\big(B_2^\top P_2 + D_2^\top P_2\mathcal{C}_2 + \mathcal{S}_{22}\big) \\ \quad = P_2A + A^\top P_2 + C^\top P_2C + Q_2 + (\Theta^*)^\top(R_2 + D^\top P_2D)\Theta^* \\ \qquad + \big(P_2B + C^\top P_2D + S_2^\top\big)\Theta^* + (\Theta^*)^\top\big(B^\top P_2 + D^\top P_2C + S_2\big), \\ 0 = \widehat{P}_2\widehat{\mathcal{A}}_2 + \widehat{\mathcal{A}}_2^\top\widehat{P}_2 + \widehat{\mathcal{C}}_2^\top P_2\widehat{\mathcal{C}}_2 + \widehat{\mathcal{Q}}_2 - \big(\widehat{P}_2\widehat{B}_2 + \widehat{\mathcal{C}}_2^\top P_2\widehat{D}_2 + \widehat{\mathcal{S}}_{22}^\top\big)\bar{\Sigma}_2^\dagger\big(\widehat{B}_2^\top\widehat{P}_2 + \widehat{D}_2^\top P_2\widehat{\mathcal{C}}_2 + \widehat{\mathcal{S}}_{22}\big) \\ \quad = \widehat{P}_2\widehat{A} + \widehat{A}^\top\widehat{P}_2 + \widehat{C}^\top P_2\widehat{C} + \widehat{Q}_2 + (\bar{\Theta}^*)^\top\big(\widehat{R}^2 + \widehat{D}^\top P_2\widehat{D}\big)\bar{\Theta}^* \\ \qquad + \big(\widehat{P}_2\widehat{B} + \widehat{C}^\top P_2\widehat{D} + \widehat{S}_2^\top\big)\bar{\Theta}^* + (\bar{\Theta}^*)^\top\big(\widehat{B}^\top\widehat{P}_2 + \widehat{D}^\top P_1\widehat{C} + \widehat{S}_2\big). \end{cases}$$

This yields (3.35). The proof is complete. □

Now, we are ready to present the main result of this subsection, which characterizes the closed-loop Nash equilibrium of Problem (MF-SDG).

Theorem 3.7. *Let* (H2) *hold. Problem* (MF-SDG) *admits a closed-loop Nash equilibrium* $(\boldsymbol{\Theta}^*, v^*(\cdot))$ *if and only if the following statements hold:*

(i) *The system* (3.35) *admits a solution pair* $(P_i, \widehat{P}_i) \in \mathbb{S}^n \times \mathbb{S}^n$ *and* (3.36)–(3.37) *are satisfied.*

(ii) *For* $i = 1, 2$, *the following BSDE:*

$$\mathrm{d}\eta_i(t) = -\big\{A_{\Theta^*}^\top\eta_i(t) + C_{\Theta^*}^\top\big[\zeta_i(t) + P_i\sigma(t)\big] + P_ib(t) + q_i(t) + (\Theta^*)^\top\rho_i(t)\big\}\mathrm{d}t + \zeta_i(t)\mathrm{d}W(t), \quad t \geqslant 0, \tag{3.43}$$

admits a solution $(\eta_i(\cdot), \zeta_i(\cdot)) \in \mathscr{X}[0,\infty) \times L^2_{\mathbb{F}}(\mathbb{R}^n)$ *such that*

$$\begin{aligned} 0 = {} & (R_{ii} + D_i^\top P_iD)\big(v^*(t) - \mathbb{E}[v^*(t)]\big) + B_i^\top\big(\eta_i(t) - \mathbb{E}[\eta_i(t)]\big) + D_i^\top\big(\zeta_i(t) - \mathbb{E}[\zeta_i(t)]\big) \\ & + D_i^\top P_i\big(\sigma(t) - \mathbb{E}[\sigma(t)]\big) + \rho_{ii}(t) - \mathbb{E}[\rho_{ii}(t)], \qquad \text{a.e. } t \in [0,\infty),\ \text{a.s.}, \qquad i = 1, 2, \end{aligned} \tag{3.44}$$

and the following ODE admits a solution $\bar{\eta}_i(\cdot) \in L^2(\mathbb{R}^n)$*:*

$$\dot{\bar{\eta}}_i(t) + \widehat{A}_{\bar{\Theta}^*}^\top\bar{\eta}_i(t) + \widehat{C}_{\bar{\Theta}^*}^\top\big(\mathbb{E}[\zeta_i(t)] + P_i\mathbb{E}[\sigma(t)]\big) + \widehat{P}_i\mathbb{E}[b(t)] + \mathbb{E}[q_i(t)] + (\bar{\Theta}^*)^\top\mathbb{E}[\rho_i(t)] = 0, \quad t \geqslant 0, \tag{3.45}$$

satisfying

$$(\widehat{R}_{ii} + \widehat{D}_i^\top P_i\widehat{D})\mathbb{E}[v^*(t)] + \widehat{B}_i^\top\bar{\eta}_i(t) + \widehat{D}_i^\top\mathbb{E}[\zeta_i(t)] + \widehat{D}_i^\top P_i\mathbb{E}[\sigma(t)] + \mathbb{E}[\rho_{ii}(t)] = 0, \quad \text{a.e. } t \in [0,\infty). \tag{3.46}$$

Proof. Necessity. Suppose that $(\boldsymbol{\Theta}^*, v^*(\cdot))$ is a closed-loop Nash equilibrium of Problem (MF-SDG). Then by Proposition 3.6, (i) holds.

For (ii), we first note that system (3.35) is equivalent to

$$\begin{cases} P_iA_{\Theta^*} + A_{\Theta^*}^\top P_i + C_{\Theta^*}^\top P_iC_{\Theta^*} + Q_i + S_i^\top\Theta^* + (\Theta^*)^\top S_i + (\Theta^*)^\top R_i\Theta^* = 0, \\ \widehat{P}_i\widehat{A}_{\bar{\Theta}^*} + \widehat{A}_{\bar{\Theta}^*}^\top\widehat{P}_i + \widehat{C}_{\bar{\Theta}^*}^\top P_i\widehat{C}_{\Theta^*} + \widehat{Q}_i + \widehat{S}_i^\top\bar{\Theta}^* + (\bar{\Theta}^*)^\top\widehat{S}_i + (\bar{\Theta}^*)^\top\widehat{R}_i\bar{\Theta}^* = 0. \end{cases} \tag{3.47}$$

Let $(X^*(\cdot), Y_i^*(\cdot), Z_i^*(\cdot)) \in \mathscr{X}[0,\infty) \times \mathscr{X}[0,\infty) \times L^2_{\mathbb{F}}(\mathbb{R}^n)$ be the solution to MF-FBSDE

$$\begin{cases} \mathrm{d}X^*(t) = \big\{A_{\Theta^*}X^* + \bar{A}_{\boldsymbol{\Theta}^*}\mathbb{E}[X^*] + Bv^* + \bar{B}\mathbb{E}[v^*] + b\big\}\mathrm{d}t \\ \qquad\qquad + \big\{C_{\Theta^*}X^* + \bar{C}_{\boldsymbol{\Theta}^*}\mathbb{E}[X^*] + Dv^* + \bar{D}\mathbb{E}[v^*] + \sigma\big\}\mathrm{d}W(t), \\ -\mathrm{d}Y_i^*(t) = \big\{A_{\Theta^*}^\top Y_i^* + \bar{A}_{\Theta^*}^\top\mathbb{E}[Y_i^*] + C_{\Theta^*}^\top Z_i^* + \bar{C}_{\Theta^*}^\top\mathbb{E}[Z_i^*] + \mathcal{Q}_i^*X^* + \bar{\mathcal{Q}}_i^*\mathbb{E}[X^*] \\ \qquad\qquad + (\mathcal{S}_i^*)^\top v^* + (\bar{\mathcal{S}}_i^*)^\top\mathbb{E}[v^*] + q_i^* + (\bar{\Theta}^* - \Theta^*)^\top\mathbb{E}[\rho_i]\big\}\mathrm{d}t - Z_i^*\mathrm{d}W(t), \qquad t \geqslant 0, \\ X^*(0) = x, \end{cases} \tag{3.48}$$

for $i=1,2$. Proceeding as in the proof of Proposition 3.5, we see that $(X^*(\cdot),Y_i^*(\cdot),Z_i^*(\cdot))$ satisfies

$$\begin{aligned}B_i^\top Y_i^*+\bar B_i^\top\mathbb{E}[Y_i^*]+D_i^\top Z_i^*+\bar D_i^\top\mathbb{E}[Z_i^*]+\mathcal{S}_{ii}^*X^*+\bar{\mathcal{S}}_{ii}^*\mathbb{E}[X^*]+R_{ii}v^*+\bar R_{ii}\mathbb{E}[v^*]+\rho_{ii}=0,\\ \text{a.e. } t\in[0,\infty),\ \text{a.s.},\end{aligned}\tag{3.49}$$

where $\mathcal{S}_{ii}^*\triangleq S_{ii}+R_{ii}\Theta^*$, $\bar{\mathcal{S}}_{ii}^*\triangleq\bar S_{ii}+\widehat R_{ii}\bar\Theta^*-R_{ii}\Theta^*$, $i=1,2$. Now, we define

$$\begin{cases}\alpha_i\triangleq Y_i^*-\mathbb{E}[Y_i^*]-P_i\big(X^*-\mathbb{E}[X^*]\big),\qquad \bar\eta_i\triangleq\mathbb{E}[Y_i^*]-\widehat P_i\mathbb{E}[X^*],\\ \zeta_i\triangleq Z_i^*-P_iC_{\Theta^*}(X^*-\mathbb{E}[X^*])-P_i\widehat C_{\bar\Theta^*}\mathbb{E}[X^*]-P_iD(v^*-\mathbb{E}[v^*])-P_i\sigma-P_i\widehat D\mathbb{E}[v^*],\quad i=1,2.\end{cases}\tag{3.50}$$

We want to show that

$$\alpha_i(\cdot)=\eta_i(\cdot)-\mathbb{E}[\eta_i(\cdot)],\tag{3.51}$$

with $\eta_i(\cdot),\zeta_i(\cdot),\bar\eta(\cdot)$ satisfy (3.43)–(3.46), for $i=1,2$. For this target, applying Itô's formula to $\alpha_i(\cdot)$ yields

$$\begin{aligned}d\alpha_i(t)=&-\big\{A_{\Theta^*}^\top\big(Y_i^*-\mathbb{E}[Y_i^*]\big)+C_{\Theta^*}^\top\big(Z_i^*-\mathbb{E}[Z_i^*]\big)+\mathcal{Q}_i\big(X^*-\mathbb{E}[X^*]\big)+\mathcal{S}_i\big(v^*-\mathbb{E}[v^*]\big)+q_i^*-\mathbb{E}[q_i^*]\big\}\mathrm{d}t\\&+Z_idW(t)-P_i\big\{\mathcal{A}_{\Theta^*}^\top\big(X^*-\mathbb{E}[X^*]\big)+B\big(v^*-\mathbb{E}[v^*]\big)+b-\mathbb{E}[b]\big\}\mathrm{d}t-P_i\big\{C_{\Theta^*}^\top\big(X^*-\mathbb{E}[X^*]\big)\\&+\big(\widehat C+\widehat D\bar\Theta^*\big)\mathbb{E}[\mathrm{X}^*]+D(v^*-\mathbb{E}[v^*])+\widehat D\mathbb{E}[v^*]+\sigma\big\}\mathrm{d}W(t)\\=&-\big\{A_{\Theta^*}^\top\alpha_i+C_{\Theta^*}^\top\big(\zeta_i-\mathbb{E}[\zeta_i]\big)+\big(P_iA_{\Theta^*}+A_{\Theta^*}^\top P_i+C_{\Theta^*}^\top P_iC_{\Theta^*}+\mathcal{Q}_i\big)\big(X^*-\mathbb{E}[X^*]\big)\\&+\big[P_iB+C^\top P_iD+S_i^\top+(\Theta^*)^\top(R_i+D^\top P_iD)\big]\big(v^*-\mathbb{E}[v^*]\big)+\mathcal{C}_{\Theta^*}^\top P_i(\sigma-\mathbb{E}[\sigma])\\&+P_i\big(b-\mathbb{E}[b]\big)+q_i-\mathbb{E}[q_i]+(\Theta^*)^\top\big(\rho_i-\mathbb{E}[\rho_i]\big)\big\}\mathrm{d}t+\zeta_idW(t)\\=&-\big\{A_{\bar\Theta^*}^\top\alpha_i+C_{\bar\Theta^*}^\top\big(\zeta_i-\mathbb{E}[\zeta_i]\big)+C_{\bar\Theta^*}^\top P_i\big(\sigma-\mathbb{E}[\sigma]\big)\\&+(\Theta^*)^\top\big(\rho_i-\mathbb{E}[\rho_i]\big)+P_i(b-\mathbb{E}[b])+q_i-\mathbb{E}[q_i]\big\}\mathrm{d}t+\zeta_idW(t).\end{aligned}\tag{3.52}$$

Since the solution $(\eta_i(\cdot),\zeta_i(\cdot))$ to (3.43) satisfies

$$\begin{aligned}d\big(\eta_i(t)-\mathbb{E}[\eta_i(t)]\big)=&-\big\{A_{\Theta^*}^\top\big(\eta_i-\mathbb{E}[\eta_i]\big)+C_{\Theta^*}^\top\big(\zeta_i-\mathbb{E}[\zeta_i]\big)+C_{\Theta^*}^\top P_i\big(\sigma-\mathbb{E}[\sigma]\big)\\&+(\Theta^*)^\top(\rho_i-\mathbb{E}[\rho_i])+P_i(b-\mathbb{E}[b])+q_i-\mathbb{E}[q_i]\big\}\mathrm{d}t+\zeta_idW(t),\quad t\geqslant0,\end{aligned}$$

by the uniqueness of solutions, we obtain (3.51). Moreover, we get

$$\begin{aligned}-\dot{\bar\eta}_i(t)=&\big[\widehat A+\widehat B\bar\Theta^*\big]^\top\mathbb{E}[Y_i^*]+\big[\widehat C+\widehat D\bar\Theta^*\big]^\top\mathbb{E}[Z_i^*]+\widehat{\mathcal{Q}}_i\mathbb{E}[X^*]\\&+\widehat{\mathcal{S}}_i\mathbb{E}[v^*]+\mathbb{E}[q_i^*]+\widehat P_i\big[\widehat A+\widehat B\bar\Theta^*\big]\mathbb{E}[X^*]+\widehat P_i\widehat B\mathbb{E}[v^*]+\widehat P_i\mathbb{E}[b]\\=&\big[\widehat A+\widehat B\bar\Theta^*\big]^\top\bar\eta_i+\big\{\widehat P_i\big(\widehat A+\widehat B\bar\Theta^*\big)+\big(\widehat A+\widehat B\bar\Theta^*\big)^\top\widehat P_i+\big[\widehat C+\widehat D\Theta^*\big]^\top P_i\big[\widehat C+\widehat D\Theta^*\big]+\widehat Q_i\\&+\widehat S_i^\top\bar\Theta^*+(\bar\Theta^*)^\top\widehat S_i+(\bar\Theta^*)^\top\widehat R_i\bar\Theta^*\big\}\mathbb{E}[X^*]+\big\{\widehat P_i\widehat B+\widehat C^\top P_i\widehat D+\widehat S_i^\top+(\bar\Theta^*)^\top\big[\widehat R_i+\widehat D^\top P_i\widehat D\big]\big\}\mathbb{E}[v^*]\\&+(\bar\Theta^*)^\top\big[\widehat D^\top(P_i\mathbb{E}[\sigma]+\mathbb{E}[\zeta_i])+\mathbb{E}[\rho_i]\big]+\widehat C^\top(P_i\mathbb{E}[\sigma]+\mathbb{E}[\zeta_i])+\widehat P_i\mathbb{E}[b]+\mathbb{E}[q_i]\\=&\big(\widehat A+\widehat B\bar\Theta^*\big)^\top\bar\eta_i+(\bar\Theta^*)^\top\big[\widehat D^\top(P_i\mathbb{E}[\sigma]+\mathbb{E}[\zeta_i])+\mathbb{E}[\rho_i]\big]+\widehat C^\top(P_i\mathbb{E}[\sigma]+\mathbb{E}[\zeta_i])+\widehat P_i\mathbb{E}[b]+\mathbb{E}[q_i],\end{aligned}\tag{3.53}$$

which is (3.45). Further, from (3.49) we have

$$\widehat R_{ii}\mathbb{E}[v^*]+\widehat B_i^\top\mathbb{E}[Y_i^*]+\widehat D_i^\top\mathbb{E}[Z_i^*]+\big[\widehat S_{ii}+\widehat R_{ii}\bar\Theta^*\big]\mathbb{E}[X^*]+\mathbb{E}[\rho_{ii}]=0,\qquad\text{a.e.}t\in[0,\infty),\ i=1,2,\tag{3.54}$$

and

$$R_{ii}(v^*-\mathbb{E}[v^*])+B_i^\top(Y_i^*-\mathbb{E}[Y_i^*])+D_i^\top(Z_i^*-\mathbb{E}[Z_i^*])+(S_{ii}+R_{ii}\Theta^*)(X^*-\mathbb{E}[X^*])+\rho_{ii}-\mathbb{E}[\rho_{ii}]=0,\quad \text{a.e. } t\in[0,\infty),\ \text{a.s.},\ i=1,2. \tag{3.55}$$

Now, (3.36), (3.50) and (3.54) yields

$$\begin{aligned}0&=\widehat{R}_{ii}\mathbb{E}[v^*]+\widehat{B}_i^\top\big(\bar\eta_i+\widehat{P}_i\mathbb{E}[X^*]\big)+\widehat{D}_i^\top\big\{\mathbb{E}[\zeta_i]+P_i\mathbb{E}[\sigma]+P_i\widehat{C}\mathbb{E}[X^*]+P_i\widehat{D}\mathbb{E}[v^*]\big\}+\big[\widehat{S}_{ii}+\widehat{R}_{ii}\bar\Theta^*\big]\mathbb{E}[X^*]+\mathbb{E}[\rho_{ii}]\\&=\big(\widehat{R}_{ii}+\widehat{D}_i^\top P_i\widehat{D}\big)\mathbb{E}[v^*]+\big\{\widehat{B}_i^\top\widehat{P}_i+\widehat{D}_i^\top P_i\widehat{C}+\widehat{S}_{ii}+\big(\widehat{R}_{ii}+\widehat{D}_i^\top P_i\widehat{D}\big)\Theta^*\big\}\mathbb{E}[X^*]\\&\quad+\widehat{B}_i^\top\bar\eta_i+\widehat{D}_i^\top\big(\mathbb{E}[\zeta_i]+P_i\mathbb{E}[\sigma]\big)+\mathbb{E}[\rho_{ii}]\\&=\big(\widehat{R}_{ii}+\widehat{D}_i^\top P_i\widehat{D}\big)\mathbb{E}[v^*]+\widehat{B}_i^\top\bar\eta_i+\widehat{D}_i^\top\big(\mathbb{E}[\zeta_i]+P_i\mathbb{E}[\sigma]\big)+\mathbb{E}[\rho_{ii}],\quad \text{a.e. } t\in[0,\infty),\ i=1,2.\end{aligned}$$

Thus (3.46) holds. Furthermore, (3.36), (3.50) and (3.55) yields

$$\begin{aligned}0&=R_{ii}(v^*-\mathbb{E}[v^*])+B_i^\top\big[\alpha_i+P_i\big(X^*-\mathbb{E}[X^*]\big)\big]+D_i^\top\big\{\zeta_i-\mathbb{E}[\zeta_i]+P_iC_{\Theta^*}\big(X^*-\mathbb{E}[X^*]\big)\\&\quad+P_iD\big(v^*-\mathbb{E}[v^*]\big)+P_i\big(\sigma-\mathbb{E}[\sigma]\big)\big\}+(S_{ii}+R_{ii}\Theta^*)\big(X^*-\mathbb{E}[X^*]\big)+\rho_{ii}-\mathbb{E}[\rho_{ii}]\\&=(R_{ii}+D_i^\top P_iD)(v^*-\mathbb{E}[v^*])+B_i^\top\alpha_i+D_i^\top(\zeta_i-\mathbb{E}[\zeta_i])+D_i^\top P_i\big(\sigma-\mathbb{E}[\sigma]\big)\\&\quad+\big[B_i^\top P_i+D_i^\top P_iC+S_{ii}+(R_{ii}+D_i^\top P_iD)\Theta^*\big]\big(X^*-\mathbb{E}[X^*]\big)+\rho_{ii}-\mathbb{E}[\rho_{ii}]\\&=(R_{ii}+D_i^\top P_iD)\big(v^*-\mathbb{E}[v^*]\big)+B_i^\top\alpha_i+D_i^\top(\zeta_i-\mathbb{E}[\zeta_i])+D_i^\top P_i\big(\sigma-\mathbb{E}[\sigma]\big)+\rho_{ii}-\mathbb{E}[\rho_{ii}],\\&\qquad\qquad\text{a.e. } t\in[0,\infty),\ \text{a.s.},\ i=1,2.\end{aligned} \tag{3.56}$$

Hence, by (3.56), we get (3.44).

Sufficiency. We choose any $x\in\mathbb{R}^n$ and $v(\cdot)\equiv\begin{pmatrix}v_1(\cdot)\\v_2(\cdot)\end{pmatrix}\in L^2_{\mathbb{F}}(\mathbb{R}^m)$. Denote $w(\cdot)\equiv\begin{pmatrix}v_1(\cdot)\\v_2^*(\cdot)\end{pmatrix}$ and let $X^w(\cdot)\equiv X\big(\cdot;x,\mathbf{\Theta}_1^*,v_1(\cdot);\mathbf{\Theta}_2^*,v_2^*(\cdot)\big)$ be the solution to the state equation

$$\begin{cases}\mathrm{d}X^w(t)=\big\{A_{\Theta^*}X^w+\bar A_{\mathbf{\Theta}^*}\mathbb{E}[X^w]+Bw+\bar B\mathbb{E}[w]+b\big\}\mathrm{d}t\\\qquad\qquad+\big\{C_{\Theta^*}X^w+\bar C_{\mathbf{\Theta}^*}\mathbb{E}[X^w]+Dw+\bar D\mathbb{E}[w]+\sigma\big\}\mathrm{d}W(t),\qquad t\geqslant 0,\\X^w(0)=x,\end{cases} \tag{3.57}$$

corresponding to x and $(\mathbf{\Theta}_1^*,v_1(\cdot),\mathbf{\Theta}_2^*,v_2^*(\cdot))$. Also, we have

$$\begin{aligned}&J_1\big(x;\Theta^*\big(X^w(\cdot)-\mathbb{E}[X^w(\cdot)]\big)+\bar\Theta^*\mathbb{E}[X^w(\cdot)]+w(\cdot)\big)\\&=\mathbb{E}\int_0^\infty\Big[\big\langle(Q_1+S_1^\top\Theta^*+(\Theta^*)^\top S_1+(\Theta^*)^\top R_1\Theta^*)(X^w-\mathbb{E}[X^w]),X^w-\mathbb{E}[X^w]\big\rangle\\&\qquad+2\big\langle(S_1+R_1\Theta^*)^\top(w-\mathbb{E}[w]),X^w-\mathbb{E}[X^w]\big\rangle+\big\langle R_1(w-\mathbb{E}[w]),w-\mathbb{E}[w]\big\rangle\\&\qquad+2\big\langle q_1+(\Theta^*)^\top\rho_1,X^w-\mathbb{E}[X^w]\big\rangle+2\big\langle\rho_1,w-\mathbb{E}[w]\big\rangle\\&\qquad+\big\langle\widehat{Q}_1+\widehat{S}_1^\top\bar\Theta^*+(\bar\Theta^*)^\top\widehat{S}_1+(\bar\Theta^*)^\top\widehat{R}_1\bar\Theta^*)\mathbb{E}[X^w],\mathbb{E}[X^w]\big\rangle+\big\langle\widehat{S}_1+\widehat{R}_1\bar\Theta^*\big)^\top\mathbb{E}[w],\mathbb{E}[X^w]\big\rangle\\&\qquad+\big\langle\widehat{R}_1\mathbb{E}[w],\mathbb{E}[w]\big\rangle+2\big\langle\mathbb{E}[q_1]+(\bar\Theta^*)^\top\mathbb{E}[\rho_1],\mathbb{E}[X^w]\big\rangle+2\big\langle\mathbb{E}[\rho_1],\mathbb{E}[w]\big\rangle\Big]\mathrm{d}t.\end{aligned}$$

Applying Itô's formula to

$$\big\langle P_1\big(X^w(\cdot)-\mathbb{E}[X^w(\cdot)]\big)+2\eta_1(\cdot),X^w(\cdot)-\mathbb{E}[X^w(\cdot)]\big\rangle+\big\langle\widehat{P}_1\mathbb{E}[X^w(\cdot)]+2\bar\eta_1(\cdot),\mathbb{E}[X^w(\cdot)]\big\rangle,$$

we have

$$
\begin{aligned}
-\big\langle \widehat{P}_1 x + 2\bar{\eta}_1(0), x\big\rangle &= \mathbb{E}\int_0^\infty \Big[\big\langle \big(P_1 A_{\Theta^*} + A_{\Theta^*}^\top P_1 + C_{\Theta^*}^\top P_1 C_{\Theta^*}\big)(X^w - \mathbb{E}[X^w]), X^w - \mathbb{E}[X^w]\big\rangle \\
&\quad + 2\big\langle \big(P_1 B + C_{\Theta^*}^\top P_1 D\big)(w - \mathbb{E}[w]), X^w - \mathbb{E}[X^w]\big\rangle + \big\langle D^\top P_1 D(w - \mathbb{E}[w]), w - \mathbb{E}[w]\big\rangle \\
&\quad - 2\big\langle (\Theta^*)^\top \rho_1 + q_1, X^w - \mathbb{E}[X^w]\big\rangle + 2\big\langle B^\top \eta_1 + D^\top \zeta_1 + D^\top P_1 \sigma, w - \mathbb{E}[w]\big\rangle \\
&\quad + \big\langle \big(\widehat{P}_1 \widehat{A}_{\bar{\Theta}^*} + \widehat{A}_{\bar{\Theta}^*}^\top \widehat{P}_1 + \widehat{C}_{\bar{\Theta}^*}^\top P_1 \widehat{C}_{\bar{\Theta}^*}\big)\mathbb{E}[X^w], \mathbb{E}[X^w]\big\rangle + 2\big\langle \big(\widehat{P}_1 \widehat{B} + \widehat{C}_{\bar{\Theta}^*}^\top P_1 \widehat{D}\big)\mathbb{E}[w], \mathbb{E}[X^w]\big\rangle \\
&\quad + \big\langle \widehat{D}^\top P_1 \widehat{D}\mathbb{E}[w], \mathbb{E}[w]\big\rangle - 2\big\langle (\bar{\Theta}^*)^\top \mathbb{E}[\rho_1] + \mathbb{E}[q_1], \mathbb{E}[X^w]\big\rangle + 2\big\langle \widehat{B}^\top \bar{\eta}_1 + \widehat{D}^\top \big(P_1 \mathbb{E}[\sigma] + \mathbb{E}[\zeta_1]\big), \mathbb{E}[w]\big\rangle \\
&\quad + \big\langle P_1 \sigma, \sigma\big\rangle + 2\big\langle \eta_1, b - \mathbb{E}[b]\big\rangle + 2\big\langle \zeta_1, \sigma\big\rangle + 2\big\langle \bar{\eta}_1, \mathbb{E}[b]\big\rangle\Big]\mathrm{d}t.
\end{aligned}
$$

Combining the above two equalities, together with conditions (3.44) and (3.46), we obtain

$$
\begin{aligned}
&J_1\big(x; \Theta^*\big(X^w(\cdot) - \mathbb{E}[X^w(\cdot)]\big) + \bar{\Theta}^* \mathbb{E}[X^w(\cdot)] + w(\cdot)\big) - \big\langle \widehat{P}_1 x + 2\bar{\eta}_1(0), x\big\rangle \\
&= \mathbb{E}\int_0^\infty \Big[\big\langle (R_1 + D^\top P_1 D)(w - \mathbb{E}[w]), w - \mathbb{E}[w]\big\rangle + 2\big\langle B^\top \eta_1 + D^\top \zeta_1 + D^\top P_1 \sigma + \rho_1, w - \mathbb{E}[w]\big\rangle \\
&\qquad + \big\langle \big[\widehat{R}_1 + \widehat{D}^\top P_1 \widehat{D}\big]\mathbb{E}[w], \mathbb{E}[w]\big\rangle + 2\big\langle \widehat{B}^\top \bar{\eta}_1 + \widehat{D}^\top (P_1 \mathbb{E}[\sigma] + \mathbb{E}[\zeta_1]) + \mathbb{E}[\rho_1], \mathbb{E}[w]\big\rangle \\
&\qquad + \big\langle P_1 \sigma, \sigma\big\rangle + 2\big\langle \eta_1, b - \mathbb{E}[b]\big\rangle + 2\big\langle \zeta_1, \sigma\big\rangle + 2\big\langle \bar{\eta}_1, \mathbb{E}[b]\big\rangle\Big]\mathrm{d}t \\
&= \mathbb{E}\int_0^\infty \Big[\big\langle (R_{111} + D_1^\top P_1 D_1)(v_1 - \mathbb{E}[v_1]), v_1 - \mathbb{E}[v_1]\big\rangle + 2\big\langle \big(R_{112} + D_1^\top P_1 D_2\big)(v_2^* - \mathbb{E}[v_2^*]), v_1 - \mathbb{E}[v_1]\big\rangle \\
&\qquad + \big\langle \big(R_{122} + D_2^\top P_1 D_2\big)(v_2^* - \mathbb{E}[v_2^*]), v_2^* - \mathbb{E}[v_2^*]\big\rangle \\
&\qquad + 2\big\langle B_1^\top (\eta_1 - \mathbb{E}[\eta_1]) + D_1^\top (\zeta_1 - \mathbb{E}[\zeta_1]) + D_1^\top P_1 (\sigma - \mathbb{E}[\sigma]) + \rho_{11} - \mathbb{E}[\rho_{11}], v_1 - \mathbb{E}[v_1]\big\rangle \\
&\qquad + 2\big\langle B_2^\top (\eta_1 - \mathbb{E}[\eta_1]) + D_2^\top (\zeta_1 - \mathbb{E}[\zeta_1]) + D_2^\top P_1 (\sigma - \mathbb{E}[\sigma]) + \rho_{12} - \mathbb{E}[\rho_{12}], v_2^* - \mathbb{E}[v_2^*]\big\rangle \\
&\qquad + \big\langle (\widehat{R}_{111} + \widehat{D}_1^\top P_1 \widehat{D}_1)\mathbb{E}[v_1], \mathbb{E}[v_1]\big\rangle + 2\big\langle (\widehat{R}_{112} + \widehat{D}_1^\top P_1 \widehat{D}_2)\mathbb{E}[v_2^*], \mathbb{E}[v_1]\big\rangle \\
&\qquad + \big\langle (\widehat{R}_{122} + \widehat{D}_2^\top P_1 \widehat{D}_2)\mathbb{E}[v_2^*], \mathbb{E}[v_2^*]\big\rangle + 2\big\langle \widehat{B}_1^\top \bar{\eta}_1 + \widehat{D}_1^\top \big(P_1 \mathbb{E}[\sigma] + \mathbb{E}[\zeta_1]\big) + \mathbb{E}[\rho_{11}], \mathbb{E}[v_1]\big\rangle \\
&\qquad + 2\big\langle \widehat{B}_2^\top \bar{\eta}_1 + \widehat{D}_2^\top (P_1 \mathbb{E}[\sigma] + \mathbb{E}[\zeta_1]) + \mathbb{E}[\rho_{12}], \mathbb{E}[v_2^*]\big\rangle \\
&\qquad + \big\langle P_1 \sigma, \sigma\big\rangle + 2\big\langle \eta_1, b - \mathbb{E}[b]\big\rangle + 2\big\langle \zeta_1, \sigma\big\rangle + 2\big\langle \bar{\eta}_1, \mathbb{E}[b]\big\rangle\Big]\mathrm{d}t \\
&= \mathbb{E}\int_0^\infty \Big[\big\langle \big(R_{111} + D_1^\top P_1 D_1\big)\big[v_1 - \mathbb{E}[v_1] - (v_1^* - \mathbb{E}[v_1^*])\big], v_1 - \mathbb{E}[v_1] - (v_1^* - \mathbb{E}[v_1^*])\big\rangle \\
&\qquad - \big\langle \big(R_{111} + D_1^\top P_1 D_1\big)(v_1^* - \mathbb{E}[v_1^*]), v_1^* - \mathbb{E}[v_1^*]\big\rangle + \big\langle \big(R_{122} + D_2^\top P_1 D_2\big)(v_2^* - \mathbb{E}[v_2^*]), v_2^* - \mathbb{E}[v_2^*]\big\rangle \\
&\qquad + 2\big\langle B_2^\top (\eta_1 - \mathbb{E}[\eta_1]) + D_2^\top (\zeta_1 - \mathbb{E}[\zeta_1]) \\
&\qquad + D_2^\top \mathrm{P}_1 (\sigma - \mathbb{E}[\sigma]) + \rho_{12} - \mathbb{E}[\rho_{12}], v_2^* - \mathbb{E}[v_2^*]\big\rangle \\
&\qquad + \big\langle \big[\widehat{R}_{111} + \widehat{D}_1^\top P_1 \widehat{D}_1\big](\mathbb{E}[v_1] - \mathbb{E}[v_1^*]), \mathbb{E}[v_1] - \mathbb{E}[v_1^*]\big\rangle - \big\langle \big[\widehat{R}_{111} + \widehat{D}_1^\top P_1 \widehat{D}_1\big]\mathbb{E}[v_1^*], \mathbb{E}[v_1^*]\big\rangle \\
&\qquad + \big\langle \big[\widehat{R}_{122} + \widehat{D}_2^\top P_1 \widehat{D}_2\big]\mathbb{E}[v_2^*], \mathbb{E}[v_2^*]\big\rangle + 2\big\langle \widehat{B}_2^\top \bar{\eta}_1 + \widehat{D}_2^\top P_1 \mathbb{E}[\sigma] + \mathbb{E}[\zeta_1]) + \mathbb{E}[\rho_{12}], \mathbb{E}[v_2^*]\big\rangle \\
&\qquad + \big\langle P_1 \sigma, \sigma\big\rangle + 2\big\langle \eta_1, b - \mathbb{E}[b]\big\rangle + 2\big\langle \zeta_1, \sigma\big\rangle + 2\big\langle \bar{\eta}_1, \mathbb{E}[b]\big\rangle\Big]\mathrm{d}t.
\end{aligned}
$$

Consequently, one gets

$$
\begin{aligned}
&J_1\big(x; \Theta^*\big\{X^w(\cdot) - \mathbb{E}[X^w(\cdot)]\big\} + \bar{\Theta}^* \mathbb{E}[X^w(\cdot)] + w(\cdot)\big) - J_1\big(x; \Theta^*\big\{X^*(\cdot) - \mathbb{E}[X^*(\cdot)]\big\} + \bar{\Theta}^* \mathbb{E}[X^*(\cdot)] + v^*(\cdot)\big) \\
&= \mathbb{E}\int_0^\infty \Big[\big\langle \big(R_{111} + D_1^\top P_1 D_1\big)\big[v_1 - \mathbb{E}[v_1] - (v_1^* - \mathbb{E}[v_1^*])\big], v_1 - \mathbb{E}[v_1] - (v_1^* - \mathbb{E}[v_1^*])\big\rangle \\
&\qquad + \big\langle \big[\widehat{R}_{111} + \widehat{D}_1^\top P_1 \widehat{D}_1\big](\mathbb{E}[v_1] - \mathbb{E}[v_1^*]), \mathbb{E}[v_1] - \mathbb{E}[v_1^*]\big\rangle\Big]\mathrm{d}t \geqslant 0,
\end{aligned}
$$

since (3.37) holds with $i=1$.

Similarly, by (3.37) with $i=2$, for any $\bar{w}(\cdot)\equiv(v_1^*(\cdot),v_2(\cdot))$ and $X^{\bar{w}}(\cdot)\equiv X\big(\cdot\,;x,\mathbf{\Theta}_1^*,v_1^*(\cdot);\mathbf{\Theta}_2^*,v_2(\cdot)\big)$, we can prove that the following holds:

$$\begin{aligned}
&J_2\big(x;\Theta^*\big\{X^{\bar{w}}(\cdot)-\mathbb{E}[X^{\bar{w}}(\cdot)]\big\}+\bar{\Theta}^*\mathbb{E}[X^{\bar{w}}(\cdot)]+\bar{w}(\cdot)\big)-J_2\big(x;\Theta^*\big\{X^*(\cdot)-\mathbb{E}[X^*(\cdot)]\big\}+\bar{\Theta}^*\mathbb{E}[X^*(\cdot)]+v^*(\cdot)\big)\\
&=\mathbb{E}\int_0^\infty\Big[\big\langle\big(R_{222}+D_2^\top\mathrm{P}_2D_2\big)\big[v_2-\mathbb{E}[v_2]-(v_2^*-\mathbb{E}[v_2^*])\big],v_2-\mathbb{E}[v_2]-(v_2^*-\mathbb{E}[v_2^*])\big\rangle\\
&\qquad+\big\langle\big[\widehat{R}_{222}+\widehat{D}_2^\top P_2\widehat{D}_2\big](\mathbb{E}[v_2]-\mathbb{E}[v_2^*]),\mathbb{E}[v_2]-\mathbb{E}[v_2^*]\big\rangle\Big]\mathrm{d}t\geqslant 0.
\end{aligned}$$

By Definition 3.2, this proves the sufficiency. The proof is complete. □

To conclude this section, let us rewrite system (3.35) in a more compact form so that one can see an interesting feature of it. We define

$$\mathbf{P}\triangleq\begin{pmatrix}P_1&0\\0&P_2\end{pmatrix},\quad\widehat{\mathbf{P}}\triangleq\begin{pmatrix}\widehat{P}_1&0\\0&\widehat{P}_2\end{pmatrix}.$$

Note that (3.36) is equivalent to (recalling the notation introduced in (3.21) and (3.24))

$$0=\begin{pmatrix}B_1^\top P_1+D_1^\top P_1C+S_{11}\\B_2^\top P_2+D_2^\top P_2C+S_{22}\end{pmatrix}+\begin{pmatrix}R_{11}+D_1^\top P_1D\\R_{22}+D_2^\top P_2D\end{pmatrix}\Theta^*\equiv\mathbf{J}^\top\big(\mathbf{B}^\top\mathbf{P}+\mathbf{D}^\top\mathbf{P}\mathbf{C}+\mathbf{S}\big)\mathbf{I}_n+\mathbf{\Sigma}\Theta^*,\tag{3.58}$$

and

$$0=\begin{pmatrix}\widehat{B}_1^\top\widehat{P}_1+\widehat{D}_1^\top P_1\widehat{C}+\widehat{S}_{11}\\\widehat{B}_2^\top\widehat{P}_2+\widehat{D}_2^\top P_2\widehat{C}+\widehat{S}_{22}\end{pmatrix}+\begin{pmatrix}\widehat{R}_{11}+\widehat{D}_1^\top P_1\widehat{D}\\\widehat{R}_{22}+\widehat{D}_2^\top P_2\widehat{D}\end{pmatrix}\bar{\Theta}^*\equiv\mathbf{J}^\top\big(\widehat{\mathbf{B}}^\top\widehat{\mathbf{P}}+\widehat{\mathbf{D}}^\top\mathbf{P}\widehat{\mathbf{C}}+\widehat{\mathbf{S}}\big)\mathbf{I}_n+\bar{\mathbf{\Sigma}}\bar{\Theta}^*,\tag{3.59}$$

where

$$\mathbf{\Sigma}\triangleq\mathbf{J}^\top(\mathbf{R}+\mathbf{D}^\top\mathbf{P}\mathbf{D})\mathbf{I}_m\equiv\begin{pmatrix}R_{11}+D_1^\top P_1D\\R_{22}+D_2^\top P_2D\end{pmatrix},\quad\bar{\mathbf{\Sigma}}\triangleq\mathbf{J}^\top\big(\widehat{\mathbf{R}}+\widehat{\mathbf{D}}^\top\mathbf{P}\widehat{\mathbf{D}}\big)\mathbf{I}_m\equiv\begin{pmatrix}\widehat{R}_{11}+\widehat{D}_1^\top P_1\widehat{D}\\\widehat{R}_{22}+\widehat{D}_2^\top P_2\widehat{D}\end{pmatrix}\in\mathbb{R}^{m\times m}.$$

If we assume both $\mathbf{\Sigma}$ and $\bar{\mathbf{\Sigma}}$ are invertible, then we have

$$\Theta^*=-\mathbf{\Sigma}^{-1}\mathbf{J}^\top\big(\mathbf{B}^\top\mathbf{P}+\mathbf{D}^\top\mathbf{P}\mathbf{C}+\mathbf{S}\big)\mathbf{I}_n,\quad\bar{\Theta}^*=-\bar{\mathbf{\Sigma}}^{-1}\mathbf{J}^\top\big(\widehat{\mathbf{B}}^\top\widehat{\mathbf{P}}+\widehat{\mathbf{D}}^\top\mathbf{P}\widehat{\mathbf{C}}+\widehat{\mathbf{S}}\big)\mathbf{I}_n\in\mathbb{R}^{m\times n}.\tag{3.60}$$

On the other hand, (3.35) can be written as

$$\begin{cases}\mathbf{P}\mathbf{A}+\mathbf{A}^\top\mathbf{P}+\mathbf{C}^\top\mathbf{P}\mathbf{C}+\mathbf{Q}+(\mathbf{\Phi}^*)^\top(\mathbf{R}+\mathbf{D}^\top\mathbf{P}\mathbf{D})\mathbf{\Phi}^*\\\qquad+(\mathbf{P}\mathbf{B}+\mathbf{C}^\top\mathbf{P}\mathbf{D}+\mathbf{S}^\top)\mathbf{\Phi}^*+(\mathbf{\Phi}^*)^\top(\mathbf{B}^\top\mathbf{P}+\mathbf{D}^\top\mathbf{P}\mathbf{C}+\mathbf{S})=0,\\\widehat{\mathbf{P}}\widehat{\mathbf{A}}+\widehat{\mathbf{A}}^\top\widehat{\mathbf{P}}+\widehat{\mathbf{C}}^\top\mathbf{P}\widehat{\mathbf{C}}+\widehat{\mathbf{Q}}+(\bar{\mathbf{\Phi}}^*)^\top\big(\widehat{\mathbf{R}}+\widehat{\mathbf{D}}^\top\mathbf{P}\widehat{\mathbf{D}}\big)\bar{\mathbf{\Phi}}^*\\\qquad+\big(\widehat{\mathbf{P}}\widehat{\mathbf{B}}+\widehat{\mathbf{C}}^\top\mathbf{P}\widehat{\mathbf{D}}+\widehat{\mathbf{S}}^\top\big)\bar{\mathbf{\Phi}}^*+(\bar{\mathbf{\Phi}}^*)^\top\big(\widehat{\mathbf{B}}^\top\widehat{\mathbf{P}}+\widehat{\mathbf{D}}^\top\mathbf{P}\widehat{\mathbf{C}}+\widehat{\mathbf{S}}\big)=0,\end{cases}\tag{3.61}$$

with

$$\mathbf{\Phi}^*\triangleq\begin{pmatrix}\Theta^*&0\\0&\Theta^*\end{pmatrix},\quad\bar{\mathbf{\Phi}}^*\triangleq\begin{pmatrix}\bar{\Theta}^*&0\\0&\bar{\Theta}^*\end{pmatrix}.$$

Clearly, both equations in (3.61) are symmetric with solutions $\mathbf{P}, \widehat{\mathbf{P}} \in \mathbb{S}^{2n}$. Recall that system (3.26) (or, equivalently, (3.32)) for closed-loop representation of an open-loop Nash equilibrium are coupled, and are not symmetric. Therefore, the closed-loop representation of open-loop Nash equilibria is different from the outcome of closed-loop Nash equilibria, for Problem (MF-SDG), in general.

4. Mean-field LQ zero-sum stochastic differential games

In this section, we will look at the situation for mean-field LQ zero-sum stochastic differential games. According to (1.6), let us simplify some notation:

$$\begin{cases} Q_1 = -Q_2 \equiv Q, \quad \bar{Q}_1 = -\bar{Q}_2 \equiv \bar{Q}, \quad q_1(\cdot) = -q_2(\cdot) \equiv q(\cdot), \\ S_1 \equiv \begin{pmatrix} S_{11} \\ S_{12} \end{pmatrix} = -\begin{pmatrix} S_{21} \\ S_{22} \end{pmatrix} \equiv -S_2 \equiv \begin{pmatrix} S^1 \\ S^2 \end{pmatrix} \equiv S, \quad \bar{S}_1 \equiv \begin{pmatrix} \bar{S}_{11} \\ \bar{S}_{12} \end{pmatrix} = -\begin{pmatrix} \bar{S}_{21} \\ \bar{S}_{22} \end{pmatrix} \equiv -\bar{S}_2 \equiv \begin{pmatrix} \bar{S}^1 \\ \bar{S}^2 \end{pmatrix} \equiv \bar{S}, \\ R_1 \equiv \begin{pmatrix} R_{111} & R_{112} \\ R_{121} & R_{122} \end{pmatrix} = -\begin{pmatrix} R_{211} & R_{212} \\ R_{221} & R_{222} \end{pmatrix} \equiv -R_2 \equiv R \equiv \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \\ \bar{R}_1 \equiv \begin{pmatrix} \bar{R}_{111} & \bar{R}_{112} \\ \bar{R}_{121} & \bar{R}_{122} \end{pmatrix} = -\begin{pmatrix} \bar{R}_{211} & \bar{R}_{212} \\ \bar{R}_{221} & \bar{R}_{222} \end{pmatrix} \equiv -\bar{R}_2 \equiv \bar{R} \equiv \begin{pmatrix} \bar{R}_{11} & \bar{R}_{12} \\ \bar{R}_{21} & \bar{R}_{22} \end{pmatrix}, \\ \rho_1(\cdot) \equiv \begin{pmatrix} \rho_{11}(\cdot) \\ \rho_{12}(\cdot) \end{pmatrix} = -\begin{pmatrix} \rho_{21}(\cdot) \\ \rho_{22}(\cdot) \end{pmatrix} \equiv -\rho_2(\cdot) \equiv \begin{pmatrix} \rho^1(\cdot) \\ \rho^2(\cdot) \end{pmatrix} \equiv \rho(\cdot). \end{cases} \tag{4.1}$$

Then the cost functional will be the following:

$$\begin{aligned} &J_1(x; u_1(\cdot), u_2(\cdot)) = -J_2(x; u_1(\cdot), u_2(\cdot)) \equiv J(x; u_1(\cdot), u_2(\cdot)) \equiv J(x; u(\cdot)) \\ &\equiv \mathbb{E}\int_0^\infty \bigg[\bigg\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix} \begin{pmatrix} X(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix} \bigg\rangle + 2\bigg\langle \begin{pmatrix} q(t) \\ \rho(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix} \bigg\rangle \\ &\qquad + \bigg\langle \begin{pmatrix} \bar{Q} & \bar{S}^\top \\ \bar{S} & \bar{R} \end{pmatrix} \begin{pmatrix} \mathbb{E}[X(t)] \\ \mathbb{E}[u(t)] \end{pmatrix}, \begin{pmatrix} \mathbb{E}[X(t)] \\ \mathbb{E}[u(t)] \end{pmatrix} \bigg\rangle \bigg] \mathrm{d}t. \end{aligned} \tag{4.2}$$

Similar to Problem (MF-SDG), we assume that (H2) holds. Then, for any $x \in \mathbb{R}^n$, we define $\mathscr{U}_{ad}(x)$ and $\mathscr{U}_{ad}^i(x)$ $(i = 1, 2)$ similar to (3.4)–(3.6). Let us state the following zero-sum problem.

Problem (MF-SDG)$_0$. For any initial state $x \in \mathbb{R}^n$, Player 1 wants to find a control $u_1^*(\cdot)$ to minimize the cost functional $J(x; u_1(\cdot), u_2(\cdot))$, and Player 2 wants to find a control $u_2^*(\cdot)$ to maximize $J(x; u_1(\cdot), u_2(\cdot))$ respectively, subject to (1.1) (or equivalently, (3.2)) such that $(u_1^*(\cdot), u_2^*(\cdot)) \in \mathcal{U}_{ad}(x)$.

Next, we introduce the following definitions.

Definition 4.1. (i) A pair $(u_1^*(\cdot), u_2^*(\cdot)) \in \mathcal{U}_{ad}(x)$ is called an *open-loop saddle point* of Problem (MF-SDG)$_0$ for the initial state $x \in \mathbb{R}^n$ if

$$J(x; u_1^*(\cdot), u_2(\cdot)) \leqslant J(x; u_1^*(\cdot), u_2^*(\cdot)) \leqslant J(x; u_1(\cdot), u_2^*(\cdot)), \qquad \forall (u_1(\cdot), u_2(\cdot)) \in \mathscr{U}_{ad}(x). \tag{4.3}$$

(ii) A closed-strategy $(\mathbf{\Theta}^*, v^*(\cdot)) \in \mathscr{S}[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}] \times L^2_{\mathbb{F}}(\mathbb{R}^m)$ is called a *closed-loop saddle point* of Problem (MF-SDG)$_0$ if for any $x \in \mathbb{R}^n$, $\mathbf{\Theta}_1 \in \mathscr{S}^1(\mathbf{\Theta}_2^*)$, $\mathbf{\Theta}_2 \in \mathscr{S}^2(\mathbf{\Theta}_1^*)$, $v_1(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^{m_1})$ and $v_2(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^{m_2})$,

$$J\big(x; \mathbf{\Theta}_1^*, v_1^*(\cdot); \mathbf{\Theta}_2, v_2(\cdot)\big) \leqslant J\big(x; \mathbf{\Theta}^*, v^*(\cdot)\big) \leqslant J\big(x; \mathbf{\Theta}_1, v_1(\cdot); \mathbf{\Theta}_2^*, v_2^*(\cdot)\big). \tag{4.4}$$

Similar as in Section 3, it is easy to see that $(\mathbf{\Theta}_1^*, v_1^*(\cdot); \mathbf{\Theta}_2^*, v_2^*(\cdot))$ is a closed-loop saddle point of Problem (MF-SDG)$_0$ if and only if one of the following holds:

(i) For any $v_1(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^{m_1})$ and $v_2(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^{m_2})$,

$$J\big(x;\mathbf{\Theta}_1^*, v_1^*(\cdot);\mathbf{\Theta}_2^*, v_2(\cdot)\big) \leqslant J\big(x;\mathbf{\Theta}^*, v^*(\cdot)\big) \leqslant J\big(x;\mathbf{\Theta}_1^*, v_1(\cdot);\mathbf{\Theta}_2^*, v_2^*(\cdot)\big). \tag{4.5}$$

(ii) For any $u_1(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^{m_1})$ and $u_2(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^{m_2})$,

$$J\big(x;\mathbf{\Theta}_1^*, v_1^*(\cdot); u_2(\cdot)\big) \leqslant J\big(x;\mathbf{\Theta}^*, v^*(\cdot)\big) \leqslant J\big(x; u_1(\cdot);\mathbf{\Theta}_2^*, v_2^*(\cdot)\big). \tag{4.6}$$

Definition 4.2. The following maps

$$\begin{cases} V^+(x) \triangleq \inf\limits_{u_1(\cdot)\in\mathscr{U}_{ad}^1(x)} \sup\limits_{u_2(\cdot)\in\mathscr{U}_{ad}^2(x)} J(x;u_1(\cdot),u_2(\cdot)), \\ V^-(x) \triangleq \sup\limits_{u_2(\cdot)\in\mathscr{U}_{ad}^2(x)} \inf\limits_{u_1(\cdot)\in\mathscr{U}_{ad}^1(x)} J(x;u_1(\cdot),u_2(\cdot)), \end{cases}$$

are called the *upper value function* and the *lower value function* of Problem (MF-SDG)$_0$, respectively. In the case that

$$V^+(x) = V^-(x) \equiv V(x),$$

we call the map $x \mapsto V(x)$ the value function of Problem (MF-SDG)$_0$.

Now, let $(\mathbf{\Theta}^*, v^*(\cdot)) \in \mathbb{R}^{2m\times n} \times L^2_{\mathbb{F}}(\mathbb{R}^m)$, and assume the open-loop saddle points of Problem (MF-SDG)$_0$ admit the closed-loop representation. Then by (4.1), we see that $(P_1, \widehat{P}_1) = (-P_2, -\widehat{P}_2) \equiv (P, \widehat{P})$ satisfy the same equations (see (3.33))

$$\begin{cases} PA + A^\top P + C^\top PC + Q + \big(PB + C^\top PD + S^\top\big)\Theta^* = 0, \\ \widehat{P}\widehat{A} + \widehat{A}^\top \widehat{P} + \widehat{C}^\top P\widehat{C} + \widehat{Q} + \big(\widehat{P}\widehat{B} + \widehat{C}^\top P\widehat{D} + \widehat{S}^\top\big)\bar{\Theta}^* = 0, \end{cases} \tag{4.7}$$

and (3.34) is equivalent to

$$\begin{cases} \Sigma_o\Theta^* + B^\top P + D^\top PC + S = 0, \qquad \Sigma_o \equiv R + D^\top PD, \\ \bar{\Sigma}_o\bar{\Theta}^* + \widehat{B}^\top \widehat{P} + \widehat{D}^\top P\widehat{C} + \widehat{S} = 0, \qquad \bar{\Sigma}_o \equiv \widehat{R} + \widehat{D}^\top P\widehat{D}. \end{cases} \tag{4.8}$$

These equations (for Θ^* and $\bar{\Theta}^*$) are solvable if and only if

$$\begin{cases} \mathscr{R}\big(B^\top P + D^\top PC + S\big) \subseteq \mathscr{R}(\Sigma_o), \qquad \Theta^* = -\Sigma_o^\dagger(B^\top P + D^\top PC + S) + \big(I - \Sigma_o^\dagger\Sigma_o\big)\theta, \\ \mathscr{R}\big(\widehat{B}^\top \widehat{P} + \widehat{D}^\top P\widehat{C} + \widehat{S}\big) \subseteq \mathscr{R}(\bar{\Sigma}_o), \qquad \bar{\Theta}^* = -\bar{\Sigma}_o^\dagger\big(\widehat{B}^\top \widehat{P} + \widehat{D}^\top P\widehat{C} + \widehat{S}\big) + (I - \bar{\Sigma}_o^\dagger\bar{\Sigma}_o)\bar{\theta}, \end{cases} \tag{4.9}$$

for some $\theta, \bar{\theta} \in \mathbb{R}^{m\times n}$ are chosen such that $(\Theta^*, \bar{\Theta}^*) \in \mathscr{S}[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$. Putting (4.9) into (4.7), it yields

$$\begin{cases} PA + A^\top P + C^\top PC + Q - \big(PB + C^\top PD + S^\top\big)\Sigma_o^\dagger(B^\top P + D^\top PC + S) = 0, \\ \widehat{P}\widehat{A} + \widehat{A}^\top \widehat{P} + \widehat{C}^\top P\widehat{C} + \widehat{Q} - (\widehat{P}\widehat{B} + \widehat{C}^\top P\widehat{D} + \widehat{S}^\top)\bar{\Sigma}_o^\dagger(\widehat{B}^\top \widehat{P} + \widehat{D}^\top P\widehat{C} + \widehat{S}) = 0, \end{cases} \tag{4.10}$$

which is a system of coupled AREs, and both of them are symmetric. Thus, we have $P, \widehat{P} \in \mathbb{S}^n$. Next, by (4.1) again, from the componentwise form of (3.29), it is easy to see that $(\eta_1(\cdot), \zeta_1(\cdot)) = (-\eta_2(\cdot), -\zeta_2(\cdot))$

$\equiv (\eta_o(\cdot), \zeta_o(\cdot)) \in \mathscr{X}[0,\infty) \times L^2_{\mathbb{F}}(\mathbb{R}^n)$ satisfies

$$\begin{aligned} -\mathrm{d}\eta_o(t) = \Big\{ & A^\top \eta_o(t) - \big(PB + C^\top PD + S^\top\big)\Sigma_o^\dagger \big\{ B^\top \eta_o(t) + D^\top [\zeta_o(t) + P\sigma(t)] + \rho(t) \big\} \\ & + C^\top \big[\zeta_o(t) + P\sigma(t)\big] + Pb(t) + q(t) \Big\} \mathrm{d}t - \zeta_o(t)\mathrm{d}W(t), \quad t \geqslant 0, \end{aligned} \tag{4.11}$$

with constraint

$$B^\top(\eta_o - \mathbb{E}[\eta_o]) + D^\top(\zeta_o - \mathbb{E}[\zeta_o]) + D^\top P(\sigma - \mathbb{E}[\sigma]) + \rho - \mathbb{E}[\rho] \in \mathcal{R}(\Sigma_o), \quad \text{a.e., a.s.} \tag{4.12}$$

Likewise, by (4.1), from the componentwise form of (3.30), we see that $\bar\eta_1(\cdot) = -\bar\eta_2(\cdot) \equiv \bar\eta_o(\cdot) \in L^2(\mathbb{R}^n)$ satisfy the following:

$$\begin{aligned} & \dot{\bar\eta}_o(t) + \widehat{A}^\top \bar\eta_o(t) - (\widehat{P}\widehat{B} + \widehat{C}^\top P\widehat{D} + \widehat{S}^\top)\bar\Sigma_o^\dagger \big(\widehat{B}^\top \bar\eta_o(t) + \mathbb{E}[\bar\zeta_o(t)] + P\mathbb{E}[\sigma(t)] + \mathbb{E}[\rho(t)]\big) \\ & \quad + \widehat{C}^\top \big(\mathbb{E}[\bar\zeta_o(t)] + P\mathbb{E}[\sigma(t)]\big) + \mathbb{E}[q(t)] + \widehat{P}\mathbb{E}[b(t)] = 0, \qquad t \geqslant 0, \end{aligned} \tag{4.13}$$

with constraint

$$\widehat{B}^\top \bar\eta_o + \widehat{D}^\top \mathbb{E}[\zeta_o] + \widehat{D}^\top P\mathbb{E}[\sigma] + \mathbb{E}[\rho] \in \mathscr{R}(\bar\Sigma_o), \quad \text{a.e.} \tag{4.14}$$

Finally, from (3.9), it is direct that the closed-loop representation of an open-loop saddle point is given by

$$\begin{aligned} u^*(\cdot) = {} & \big\{ -\Sigma_o^\dagger(B^\top P + D^\top PC + S) + \big(I - \Sigma_o^\dagger\Sigma_o\big)\theta \big\}\big(X^*(\cdot) - \mathbb{E}[X^*(\cdot)]\big) \\ & + \big\{ -\bar\Sigma_o^\dagger\big(\widehat{B}^\top\widehat{P} + \widehat{D}^\top P\widehat{C} + \widehat{S}\big) + \big(I - \bar\Sigma_o^\dagger\bar\Sigma_o\big)\bar\theta \big\}\mathbb{E}[X^*(\cdot)] \\ & - \Sigma_o^\dagger\big\{ B^\top(\eta_o(\cdot) - \mathbb{E}[\eta_o(\cdot)]) + D^\top\big[\zeta_o(\cdot) - \mathbb{E}[\zeta_o(\cdot)] + P\big(\sigma(\cdot) - \mathbb{E}[\sigma(\cdot)]\big)\big] + \rho(\cdot) - \mathbb{E}[\rho(\cdot)] \big\} \\ & - \bar\Sigma_o^\dagger\big\{ \widehat{B}^\top\bar\eta_o(\cdot) + \widehat{D}^\top\big(P\mathbb{E}[\sigma(\cdot)] + \mathbb{E}[\zeta_o(\cdot)]\big) + \mathbb{E}[\rho(\cdot)] + \bar\rho(\cdot) \big\} \\ & + \big(I - \Sigma_o^\dagger\Sigma_o\big)(\nu(\cdot) - \mathbb{E}[\nu(\cdot)]) + \big(I - \bar\Sigma_o^\dagger\bar\Sigma_o\big)\bar\nu(\cdot), \end{aligned} \tag{4.15}$$

where $\nu(\cdot), \bar\nu(\cdot) \in L^2(\mathbb{R}^m)$, and $X^*(\cdot) \in \mathscr{X}[0,\infty)$ is the solution to (3.8) corresponding to $u^*(\cdot)$.

To summarize, we have the following result.

Theorem 4.3. *Let* (H2) *hold and the initial state $x \in \mathbb{R}^n$ be given. Then any open-loop saddle point $u^*(\cdot)$ of Problem (MF-SDG)$_0$ admits a closed-loop representation if and only if the following hold:*

(i) *The following convexity-concavity condition holds: For $i = 1, 2$,*

$$\begin{aligned} (-1)^{i-1}\mathbb{E}\int_0^\infty \Big[& \big\langle QX_i(t), X_i(t)\big\rangle + 2\big\langle S^iX_i(t), u_i(t)\big\rangle + \big\langle R_{ii}u_i(t), u_i(t)\big\rangle + \big\langle \bar{Q}\mathbb{E}[X_i(t)], \mathbb{E}[X_i(t)]\big\rangle \\ & + 2\big\langle \bar{S}^i\mathbb{E}[X_i(t)], \mathbb{E}[u_i(t)]\big\rangle + \big\langle \bar{R}_{ii}\mathbb{E}[u_i(t)], \mathbb{E}[u_i(t)]\big\rangle \Big]\mathrm{d}t \geqslant 0, \ \forall u(\cdot) \equiv \begin{pmatrix} u_1(\cdot) \\ u_2(\cdot) \end{pmatrix} \in \mathscr{U}_{ad}(x), \end{aligned} \tag{4.16}$$

where $X_i(\cdot) \in \mathscr{X}[0,\infty)$ is the solution to MF-SDE (3.20).

(ii) *System* (4.10) *admits a static stabilizing solution $(P, \widehat{P}) \in \mathbb{S}^n \times \mathbb{S}^n$, such that the solution $(\eta_o(\cdot), \zeta_o(\cdot)) \in \mathscr{X}[0,\infty) \times L^2_{\mathbb{F}}(\mathbb{R}^n)$ to* (4.11) *satisfies* (4.12)*, and the solution $\bar\eta_o(\cdot) \in L^2(\mathbb{R}^n)$ to* (4.13) *satisfies* (4.14).

In the above case, any open-loop saddle point admits the closed-loop representation (4.15).

The following result characterizes the closed-loop saddle points of Problem (MF-SDG)$_0$.

Theorem 4.4. *Problem (MF-SDG)$_0$ admits a closed-loop saddle point $(\Theta^*, \bar\Theta^*, v^*(\cdot))$ if and only if the following statements hold:*

(i) *The following system:*

$$\begin{cases} P_cA+A^\top P_c+C^\top P_cC+Q-(P_cB+C^\top P_cD+S^\top)\Sigma_c^\dagger(B^\top P_c+D^\top P_cC+S)=0,\\ \widehat{P}_c\widehat{A}+\widehat{A}^\top\widehat{P}_c+\widehat{C}^\top P_c\widehat{C}+\widehat{Q}-(\widehat{P}_c\widehat{B}+\widehat{C}^\top P_c\widehat{D}+\widehat{S}^\top)\bar{\Sigma}_c^\dagger(\widehat{B}^\top\widehat{P}_c+\widehat{D}^\top P_c\widehat{C}+\widehat{S})=0,\\ \mathscr{R}(B^\top P_c+D^\top P_cC+S)\subseteq\mathscr{R}(\Sigma_c),\qquad \mathscr{R}(\widehat{B}^\top\widehat{P}_c+\widehat{D}^\top P_c\widehat{C}+\widehat{S})\subseteq\mathscr{R}(\bar{\Sigma}_c), \end{cases} \tag{4.17}$$

admits a static stabilizing solution $(P_c,\widehat{P}_c)\in\mathbb{S}^n\times\mathbb{S}^n$ *such that*

$$\begin{cases} R_{11}+D_1^\top P_cD_1\geqslant 0, & \widehat{R}_{11}+\widehat{D}_1^\top P_c\widehat{D}_1\geqslant 0,\\ R_{22}+D_2^\top P_cD_2\leqslant 0, & \widehat{R}_{22}+\widehat{D}_2^\top P_c\widehat{D}_2\leqslant 0. \end{cases} \tag{4.18}$$

(ii) *The following BSDE on* $[0,\infty)$*:*

$$\begin{cases} -\mathrm{d}\eta_c(t)=\big\{A^\top\eta_c(t)-\big(P_cB+C^\top P_cD+S^\top\big)\Sigma_c^\dagger\big[B^\top\eta_c(t)+D^\top\big(\zeta_c(t)+P_c\sigma(t)\big)+\rho(t)\big]\\ \qquad\qquad +C^\top\big[\zeta_c(t)+P_c\sigma(t)\big]+P_cb(t)+q(t)\big\}\mathrm{d}t-\zeta_c(t)\mathrm{d}W(t),\quad t\geqslant 0,\\ B^\top\big[\eta_c(t)-\mathbb{E}[\eta_c(t)]\big]+D^\top\big[\zeta_c(t)-\mathbb{E}[\zeta_c(t)]\big]+D^\top P_c\big[\sigma(t)-\mathbb{E}[\sigma(t)]\big]+\rho(t)-\mathbb{E}[\rho(t)]\in\mathscr{R}\big(\Sigma_c\big),\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{a.e. } t\in[0,\infty),\ \text{a.s.}, \end{cases} \tag{4.19}$$

admits a solution $(\eta_c(\cdot),\zeta_c(\cdot))\in\mathscr{X}[0,\infty)\times L^2_{\mathbb{F}}(\mathbb{R}^n)$*, and the following ODE:*

$$\begin{cases} \dot{\bar{\eta}}_c(t)+\widehat{A}^\top\bar{\eta}_c(t)-\big(\widehat{P}\widehat{B}+\widehat{C}^\top P_c\widehat{D}+\widehat{S}^\top\big)\bar{\Sigma}_c^\dagger\big\{\widehat{B}^\top\bar{\eta}_c(t)+\widehat{D}^\top\big(\mathbb{E}[\bar{\zeta}_o(t)]+P_c\mathbb{E}[\sigma(t)]\big)+\mathbb{E}[\bar{\rho}(t)]\big\}\\ \qquad +\widehat{C}^\top\big(\mathbb{E}[\bar{\zeta}_o(t)]+P_c\mathbb{E}[\sigma(t)]\big)+\mathbb{E}[q(t)]+\widehat{P}_c\mathbb{E}[b(t)],\qquad t\geqslant 0,\\ \widehat{B}^\top\bar{\eta}_c(t)+\widehat{D}^\top\mathbb{E}[\zeta_c(t)]+\widehat{D}^\top P_c\mathbb{E}[\sigma(t)]+\mathbb{E}[\rho(t)]\in\mathscr{R}(\bar{\Sigma}_c),\quad \text{a.e. } t\in[0,\infty), \end{cases} \tag{4.20}$$

admits a solution $\bar{\eta}_c(\cdot)\in L^2(\mathbb{R}^n)$*. In the above case, the closed-loop saddle point is given by*

$$\begin{cases} \Theta^*=-\Sigma_c^\dagger(B^\top P_c+D^\top P_cC+S)+\big(I-\Sigma_c^\dagger\Sigma_c\big)\theta,\\ \bar{\Theta}^*=-\bar{\Sigma}_c^\dagger\big[\widehat{B}^\top\Pi_c+\widehat{D}^\top P_c\widehat{C}+\widehat{S}\big]+\big(I-\bar{\Sigma}_c^\dagger\bar{\Sigma}_c\big)\bar{\theta}, \end{cases} \tag{4.21}$$

where $\theta,\bar{\theta}\in\mathbb{R}^{m\times n}$ *are chosen such that* $(\Theta^*,\bar{\Theta}^*)\in\mathscr{S}[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]$*, and*

$$\begin{aligned} v^*(\cdot)=&-\Sigma_c^\dagger\big\{B^\top\big(\eta_c(\cdot)-\mathbb{E}[\eta_c(\cdot)]\big)+D^\top\big(\zeta_c(\cdot)-\mathbb{E}[\zeta_c(\cdot)]\big)+D^\top P\big(\sigma(\cdot)-\mathbb{E}[\sigma(\cdot)]\big)+\rho(\cdot)-\mathbb{E}[\rho(\cdot)]\big\}\\ &-\bar{\Sigma}_c^\dagger\big\{\widehat{B}^\top\bar{\eta}_c(\cdot)+\widehat{D}^\top\big(P_c\mathbb{E}[\sigma(\cdot)]+\mathbb{E}[\zeta_c(\cdot)]\big)+\mathbb{E}[\rho(\cdot)]\big\}\\ &+\big(I-\Sigma_c^\dagger\Sigma_c\big)\big(\nu(\cdot)-\mathbb{E}[\nu(\cdot)]\big)+\big(I-\bar{\Sigma}_c^\dagger\bar{\Sigma}_c\big)\bar{\nu}(\cdot), \end{aligned} \tag{4.22}$$

for some $\nu(\cdot)\in L^2_{\mathbb{F}}(\mathbb{R}^m),\bar{\nu}(\cdot)\in L^2(\mathbb{R}^m)$*.*

Proof. Let $(\boldsymbol{\Theta}^*,v^*(\cdot))$ be a closed-loop saddle point of Problem (MF-SDG)$_0$. System (3.35) for $(P_1,\widehat{P}_1)\equiv(-P_2,-\widehat{P}_2)=(P_c,\widehat{P}_c)$ becomes

$$\begin{cases} P_cA+A^\top P_c+C^\top P_cC+Q+(\Theta^*)^\top(R+D^\top P_cD)\Theta^*\\ \quad+(P_cB+C^\top P_cD+S^\top)\Theta^*+(\Theta^*)^\top(B^\top P_c+D^\top P_cC+S)=0,\\ \widehat{P}_c\widehat{A}+\widehat{A}^\top\widehat{P}_c+\widehat{C}^\top P_c\widehat{C}+\widehat{Q}+(\bar{\Theta}^*)^\top(\widehat{R}+\widehat{D}^\top P_c\widehat{D})\bar{\Theta}^*\\ \quad+\big(\widehat{P}_c\widehat{B}+\widehat{C}^\top P_c\widehat{D}+\widehat{S}^\top\big)\bar{\Theta}^*+(\bar{\Theta}^*)^\top\big(\widehat{B}^\top\widehat{P}_c+\widehat{D}^\top P_c\widehat{C}+\widehat{S}\big)=0. \end{cases} \tag{4.23}$$

Thus, (3.36) becomes

$$\begin{cases} B^\top P_c + D^\top P_c C + S + \Sigma_c \Theta^* = 0, \\ \widehat{B}^\top \widehat{P}_c + \widehat{D}^\top P_c \widehat{C} + \widehat{S} + \bar{\Sigma}_c \bar{\Theta}^* = 0. \end{cases}$$

This system is solvable if and only if

$$\begin{cases} \mathscr{R}\big(B^\top P_c + D^\top P_c C + S\big) \subseteq \mathscr{R}(\Sigma_c), & \Theta^* = -\Sigma_c^\dagger (B^\top P_c + D^\top P_c C + S) + \big(I - \Sigma_c^\dagger \Sigma_c\big)\theta, \\ \mathscr{R}\big(\widehat{B}^\top \widehat{P}_c + \widehat{D}^\top P_c \widehat{C} + \widehat{S}\,\big) \subseteq \mathscr{R}(\bar{\Sigma}_c), & \bar{\Theta}^* = -\bar{\Sigma}_c^\dagger \big(\widehat{B}^\top \widehat{P}_c + \widehat{D}^\top P_c \widehat{C} + \widehat{S}\,\big) + \big(I - \bar{\Sigma}_c^\dagger \bar{\Sigma}_c\big)\bar{\theta}, \end{cases} \tag{4.24}$$

where $\theta, \bar{\theta} \in \mathbb{R}^{m\times n}$ are chosen such that $(\Theta^*, \bar{\Theta}^*) \in \mathscr{S}[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$. This proves (4.21). Putting (4.24) into (4.23) yields that (4.17) admits a static stabilizing solution $(P_c, \widehat{P}_c) \in \mathbb{S}^n \times \mathbb{S}^n$. (4.18) can be easily obtained from (3.37) and (4.1). (4.19) and (4.20) can be similarly proved from (ii) of Theorem 3.7.

Finally, from (3.44) we have

$$\begin{aligned} v^*(\cdot) - \mathbb{E}[v^*(\cdot)] = -\Sigma_c^\dagger \big\{ B^\top \big(\eta_c(\cdot) - \mathbb{E}[\eta_c(\cdot)]\big) + D^\top \big(\zeta_c(\cdot) - \mathbb{E}[\zeta_c(\cdot)]\big) + D^\top P_c \big(\sigma(\cdot) - \mathbb{E}[\sigma(\cdot)]\big) \\ + \rho(\cdot) - \mathbb{E}[\rho(\cdot)]\big\} + \big(I - \Sigma_c^\dagger \Sigma_c\big)\big(\nu(\cdot) - \mathbb{E}[\nu(\cdot)]\big), \end{aligned} \tag{4.25}$$

for some $\nu(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^m)$. It follows from (3.46) that we get

$$\mathbb{E}[v^*(\cdot)] = -\bar{\Sigma}_c^\dagger \big\{ \widehat{B}^\top \bar{\eta}_c(\cdot) + \widehat{D}^\top \big(P_c \mathbb{E}[\sigma(\cdot)] + \mathbb{E}[\zeta_c(\cdot)]\big) + \mathbb{E}[\rho(\cdot)]\big\} + \big(I - \bar{\Sigma}_c^\dagger \bar{\Sigma}_c\big)\bar{\nu}(\cdot), \tag{4.26}$$

for some $\bar{\nu}(\cdot) \in L^2(\mathbb{R}^m)$. Combining the above two expressions leads to (4.22). The proof is complete. □

Comparing Theorems 4.3 and 4.4, we have the following result which might not be true for general non-zero sum differential games (see [47]).

Theorem 4.5. *If both the closed-loop representation of open-loop saddle points and the closed-loop saddle points of Problem (MF-SDG)$_0$ exist, then the closed-loop representation coincides with the outcome of the closed-loop saddle points. In the above case, the value function admits the following representation:*

$$\begin{aligned} V(x) = {} & \big\langle \widehat{P} x, x \big\rangle + 2\big\langle \bar{\eta}_c(0), x \big\rangle \\ & + \mathbb{E} \int_0^\infty \Big[\big\langle P\sigma(t), \sigma(t) \big\rangle + 2\big\langle \eta_c(t), b(t) - \mathbb{E}[b(t)] \big\rangle + 2\big\langle \zeta_c(t), \sigma(t) \big\rangle + 2\big\langle \bar{\eta}_c(t), \mathbb{E}[b(t)] \big\rangle \\ & - \big| \big(\Sigma^\dagger\big)^{\frac{1}{2}} \big\{ B^\top (\eta_c(t) - \mathbb{E}[\eta_c(t)]) + D^\top \big[\zeta_c(t) - \mathbb{E}[\zeta_c(t)] + P\big(\sigma(t) - \mathbb{E}[\sigma(t)]\big)\big] + \rho(t) - \mathbb{E}[\rho(t)] \big\} \big|^2 \\ & - \big| \big(\bar{\Sigma}^\dagger\big)^{\frac{1}{2}} \big\{ \widehat{B}^\top \bar{\eta}_c(t) + \widehat{D}^\top \big(\mathbb{E}[\zeta_c(t)] + P\mathbb{E}[\sigma(t)]\big) + \mathbb{E}[\rho(t)] \big\} \big|^2 \Big] \mathrm{d}t, \end{aligned} \tag{4.27}$$

where $\Sigma \equiv \Sigma_o \equiv \Sigma_c \triangleq R + D^\top P D \equiv R + D^\top P_c D$, $\bar{\Sigma} \equiv \bar{\Sigma}_o \equiv \bar{\Sigma}_c \triangleq \widehat{R} + \widehat{D}^\top P \widehat{D} \equiv \widehat{R} + \widehat{D}^\top P_c \widehat{D}$, $\widehat{P} \equiv \widehat{P}_c$.

Proof. Let $(\boldsymbol{\Theta}^*, v^*(\cdot))$ be a closed-loop saddle point of Problem (MF-SDG)$_0$. By (4.6), the outcome

$$u^*(\cdot) = \Theta^* \big(X^*(\cdot) - \mathbb{E}[X^*(\cdot)]\big) + \bar{\Theta}^* \mathbb{E}[X^*(\cdot)] + v^*(\cdot),$$

of $(\Theta^*, \bar{\Theta}^*, v^*(\cdot))$ is an open-loop saddle point of Problem (MF-SDG)$_0$, where $X^*(\cdot) \in \mathscr{X}[0,\infty)$ is the solution to

$$\begin{cases} \mathrm{d}X^*(t) = \big\{A_{\Theta^*}X^* + \bar{A}_{\boldsymbol{\Theta}^*}\mathbb{E}[X^*] + Bv^* + \bar{B}\mathbb{E}[v^*] + b\big\}\mathrm{d}t \\ \qquad\qquad +\big\{C_{\Theta^*}X^* + \bar{C}_{\boldsymbol{\Theta}^*}\mathbb{E}[X^*] + Dv^* + \bar{D}\mathbb{E}[v^*] + \sigma\big\}\mathrm{d}W(t), \quad t \geqslant 0, \\ X^*(0) = x. \end{cases}$$

Thus, similar to the proof of the sufficiency on Theorem 3.3, noting (4.25) and (4.26), we have

$$\begin{aligned} &J\big(x; \Theta^*\big(X^*(\cdot) - \mathbb{E}[X^*(\cdot)]\big) + \bar{\Theta}^*\mathbb{E}[X^*(\cdot)] + v^*(\cdot)\big) - \big\langle \widehat{P}_c x + 2\bar{\eta}_c(0), x\big\rangle \\ &= \mathbb{E}\int_0^\infty \Big[\big\langle \Sigma_c\big(v^* - \mathbb{E}[v^*]\big), v^* - \mathbb{E}[v^*]\big\rangle + 2\big\langle B^\top\eta_c + D^\top(\zeta_c + P_c\sigma) + \rho, v^* - \mathbb{E}[v^*]\big\rangle \\ &\qquad +\big\langle \bar{\Sigma}_c\mathbb{E}[v^*], \mathbb{E}[v^*]\big\rangle + 2\big\langle \widehat{B}^\top\bar{\eta}_c + \widehat{D}^\top\big(\mathbb{E}[\zeta_c] + P_c\mathbb{E}[\sigma]\big) + \mathbb{E}[\rho], \mathbb{E}[v^*]\big\rangle \\ &\qquad +\big\langle P_c\sigma, \sigma\big\rangle + 2\big\langle \eta_c, b - \mathbb{E}[b]\big\rangle + 2\big\langle \zeta_c, \sigma\big\rangle + 2\big\langle \bar{\eta}_c, \mathbb{E}[b]\big\rangle\Big]\mathrm{d}t \\ &= \mathbb{E}\int_0^\infty \Big[\big\langle P_c\sigma, \sigma\big\rangle + 2\big\langle \eta_c, b - \mathbb{E}[b]\big\rangle + 2\big\langle \zeta_c, \sigma\big\rangle + 2\big\langle \bar{\eta}_c, \mathbb{E}[b]\big\rangle \\ &\qquad -\big|(\Sigma_c^\dagger)^{\frac{1}{2}}\big\{B^\top(\eta_c - \mathbb{E}[\eta_c]) + D^\top(\zeta_c - \mathbb{E}[\zeta_c]) + D^\top P_c(\sigma - \mathbb{E}[\sigma]) + \rho - \mathbb{E}[\rho]\big\}\big|^2 \\ &\qquad -\big|(\bar{\Sigma}_c^\dagger)^{\frac{1}{2}}\big\{\widehat{B}^\top\bar{\eta}_c + \widehat{D}^\top\big(P_c\mathbb{E}[\sigma] + \mathbb{E}[\zeta_c]\big) + \mathbb{E}[\rho]\big\}\big|^2\Big]\mathrm{d}t, \end{aligned} \tag{4.28}$$

where $(P_c, \widehat{P}_c)$ satisfies (4.17), $(\eta_c(\cdot), \zeta_c(\cdot))$ satisfies (4.19) and $\bar{\eta}_c(\cdot)$ satisfies (4.20). By Theorem 4.4, $(\Theta^*, \bar{\Theta}^*, v^*(\cdot))$ is given by (4.21)–(4.22).

On the other hand, let $(P, \widehat{P})$ be the solution pair to (4.10), $(\eta_o(\cdot), \zeta_o(\cdot))$ be the solution to (4.11), $\bar{\eta}_o(\cdot)$ be the solution to (4.13), and choose $\theta, \bar{\theta} \in \mathbb{R}^{m\times n}$, $\nu(\cdot), \bar{\nu}(\cdot) \in L^2(\mathbb{R}^m)$, such that the following

$$\begin{cases} \Theta^{**} = -\Sigma_o^\dagger(B^\top P + D^\top PC + S) + \big(I - \Sigma_o^\dagger\Sigma_o\big)\theta, \quad \bar{\Theta}^{**} = -\bar{\Sigma}_o^\dagger\big(\widehat{B}^\top\widehat{P} + \widehat{D}^\top P\widehat{C} + \widehat{S}\big) + \big(I - \bar{\Sigma}_o^\dagger\bar{\Sigma}_o\big)\bar{\theta}, \\ v^{**}(\cdot) = -\Sigma_o^\dagger\big\{B^\top\big(\eta_o(\cdot) - \mathbb{E}[\eta_o(\cdot)]\big) + D^\top\big(\zeta_o(\cdot) - \mathbb{E}[\zeta_o(\cdot)]\big) + D^\top P\big(\sigma(\cdot) - \mathbb{E}[\sigma(\cdot)]\big) + \rho(\cdot) - \mathbb{E}[\rho(\cdot)]\big\} \\ \qquad -\bar{\Sigma}_o^\dagger\big\{\widehat{B}^\top\bar{\eta}_o(\cdot) + \widehat{D}^\top\big(P\mathbb{E}[\sigma(\cdot)] + \mathbb{E}[\zeta_o(\cdot)]\big) + \mathbb{E}[\rho(\cdot)]\big\} + \big(I - \Sigma_o^\dagger\Sigma_o\big)\big(\nu(\cdot) - \mathbb{E}[\nu(\cdot)]\big) + \big(I - \bar{\Sigma}_o^\dagger\bar{\Sigma}_o\big)\bar{\nu}(\cdot), \end{cases}$$

satisfies $(\Theta^{**}, \bar{\Theta}^{**}) \in \mathscr{S}[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$. For any initial state x, define $u^{**}(\cdot) \in \mathscr{U}_{ad}(x)$ as follows:

$$u^{**}(\cdot) = \Theta^{**}\big\{X^{**}(\cdot) - \mathbb{E}[X^{**}(\cdot)]\big\} + \bar{\Theta}^{**}\mathbb{E}[X^{**}(\cdot)] + v^{**}(\cdot),$$

where $X^{**}(\cdot) \in \mathscr{X}[0,\infty)$ is the solution to

$$\begin{cases} \mathrm{d}X^{**}(t) = \big\{A_{\Theta^{**}}X^{**} + \bar{A}_{\boldsymbol{\Theta}^{**}}\mathbb{E}[X^{**}] + Bv^{**} + \bar{B}\mathbb{E}[v^{**}] + b\big\}\mathrm{d}t \\ \qquad\qquad +\big\{C_{\Theta^{**}}X^{**} + \bar{C}_{\boldsymbol{\Theta}^{**}}\mathbb{E}[X^{**}] + Dv^{**} + \bar{D}\mathbb{E}[v^{**}] + \sigma\big\}\mathrm{d}W(t), \quad t \geqslant 0, \\ X^{**}(0) = x. \end{cases}$$

By Theorem 4.3, $u^{**}(\cdot)$ is an open-loop saddle point of Problem (MF-SDG)$_0$ for x. By the same argument as (4.28), noting (4.12) and (4.14), we obtain

$$\begin{aligned}
&J\big(x;\Theta^{**}\big(X^{**}(\cdot)-\mathbb{E}[X^{**}(\cdot)]\big)+\bar{\Theta}^{**}\mathbb{E}[X^{**}(\cdot)]+v^{**}(\cdot)\big)-\big\langle\widehat{P}x+2\bar{\eta}_o(0),x\big\rangle\\
&=\mathbb{E}\int_0^\infty\Big[\big\langle\Sigma_o(v^{**}-\mathbb{E}[v^{**}]),v^{**}-\mathbb{E}[v^{**}]\big\rangle+2\big\langle B^\top\eta_o+D^\top\zeta_o+D^\top P\sigma+\rho,v^{**}-\mathbb{E}[v^{**}]\big\rangle\\
&\qquad+\big\langle\bar{\Sigma}_o\mathbb{E}[v^{**}],\mathbb{E}[v^{**}]\big\rangle+2\big\langle\widehat{B}^\top\bar{\eta}_o+\widehat{D}^\top P\mathbb{E}[\sigma]+\mathbb{E}[\zeta_o]\big)+\mathbb{E}[\rho],\mathbb{E}[v^{**}]\big\rangle\\
&\qquad+\big\langle P\sigma,\sigma\big\rangle+2\big\langle\eta_o,b-\mathbb{E}[b]\big\rangle+2\big\langle\zeta_o,\sigma\big\rangle+2\big\langle\bar{\eta}_o,\mathbb{E}[b]\big\rangle\Big]\mathrm{d}t\\
&=\mathbb{E}\int_0^\infty\Big[\big\langle P\sigma,\sigma\big\rangle+2\big\langle\eta_o,b-\mathbb{E}[b]\big\rangle+2\big\langle\zeta_o,\sigma\big\rangle+2\big\langle\bar{\eta}_o,\mathbb{E}[b]\big\rangle\\
&\qquad-\big|(\Sigma_o^\dagger)^{\frac{1}{2}}\big\{B^\top(\eta_o-\mathbb{E}[\eta_o])+D^\top(\zeta_o-\mathbb{E}[\zeta_o])+D^\top P(\sigma-\mathbb{E}[\sigma])+\rho-\mathbb{E}[\rho]\big\}\big|^2\\
&\qquad-\big|(\bar{\Sigma}_o^\dagger)^{\frac{1}{2}}\big\{\widehat{B}^\top\bar{\eta}_o+\widehat{D}^\top\big(P\mathbb{E}[\sigma]+\mathbb{E}[\zeta_o]\big)+\mathbb{E}[\rho]\big\}\big|^2\Big]\mathrm{d}t.
\end{aligned}\tag{4.29}$$

Since both $u^{**}(\cdot)\equiv(u_1^{**}(\cdot),u_2^{**}(\cdot))$ and $u^*(\cdot)\equiv(u_1^*(\cdot),u_2^*(\cdot))$ are open-loop saddle points of Problem (MF-SDG)$_0$ for x, we have

$$J(x;u_1^*(\cdot),u_2^*(\cdot))\leqslant J(x;u_1^{**}(\cdot),u_2^*(\cdot))\leqslant J(x;u_1^{**}(\cdot),u_2^{**}(\cdot))\leqslant J(x;u_1^*(\cdot),u_2^{**}(\cdot))\leqslant J(x;u_1^*(\cdot),u_2^*(\cdot)).$$

Therefore, $J(x;u^*(\cdot))=J(x;u^{**}(\cdot))$ for all x, which, together with (4.28) and (4.29), yields

$$P_c=P,\quad\widehat{P}_c=\widehat{P},\quad\eta_c(\cdot)=\eta_o(\cdot),\quad\zeta_c(\cdot)=\zeta_o(\cdot),\quad\bar{\eta}_c(\cdot)=\bar{\eta}_o(\cdot).$$

Thus, the value function is given by (4.27). The proof is complete. □

Finally, we have the following corollary for Problem (MF-SLQ), since it is a special case of Problem (MF-SDG)$_0$ when $m_2=0$.

Corollary 4.6. *For Problem* (MF-SLQ)*, if the open-loop optimal control admits a closed-loop representation, then each open-loop optimal control must be an outcome of a closed-loop optimal strategy.*

5. Examples

In this section, we present several examples illustrating the results established in the previous sections. We mainly concentrate the closed-loop representation of the open-loop Nash equilibria/saddle points and the closed-loop Nash equilibria/saddle points.

The following example shows that for the mean-field LQ zero-sum stochastic differential game, it may happen that the closed-loop saddle point uniquely exists, which coincides with the closed-loop representation of the open-loop saddle point. Moreover, the system of generalized AREs admits static stabilizing solutions and the system $[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]$ is MF-L^2-stabilizable.

Example 5.1. Consider the following two-dimensional state equation

$$\begin{cases}\mathrm{d}X(t)=\big\{AX(t)+\bar{A}\mathbb{E}[X(t)]+Bu(t)+\bar{B}\mathbb{E}[u(t)]\big\}\mathrm{d}t\\ \qquad\qquad+\big\{CX(t)+\bar{C}\mathbb{E}[X(t)]+Du(t)+\bar{D}\mathbb{E}[u(t)]\big\}\mathrm{d}W(t),\quad t\geqslant0,\\ X(0)=x,\end{cases}\tag{5.1}$$

with the cost functional

$$J(x;u_1(\cdot),u_2(\cdot))=\mathbb{E}\int_0^\infty\left[\left\langle\begin{pmatrix}Q & S^\top\\ S & R\end{pmatrix}\begin{pmatrix}X(t)\\ u(t)\end{pmatrix},\begin{pmatrix}X(t)\\ u(t)\end{pmatrix}\right\rangle+\left\langle\begin{pmatrix}\bar{Q} & \bar{S}^\top\\ \bar{S} & \bar{R}\end{pmatrix}\begin{pmatrix}\mathbb{E}[X(t)]\\ \mathbb{E}[u(t)]\end{pmatrix},\begin{pmatrix}\mathbb{E}[X(t)]\\ \mathbb{E}[u(t)]\end{pmatrix}\right\rangle\right]\mathrm{d}t. \tag{5.2}$$

Let

$$\begin{cases}A=\begin{pmatrix}-1 & -1\\ 0 & -3\end{pmatrix},\quad \bar{A}=\begin{pmatrix}-1 & 0\\ -1 & -20\end{pmatrix},\quad B=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \bar{B}=\begin{pmatrix}2.5 & 0\\ 0 & 0.5\end{pmatrix},\\ C=\begin{pmatrix}1 & 0.5\\ 0.5 & 1\end{pmatrix},\quad \bar{C}=\begin{pmatrix}0 & -0.5\\ -0.5 & 0\end{pmatrix},\quad D=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \bar{D}=\begin{pmatrix}0.5 & 0\\ 0 & 0.5\end{pmatrix},\\ Q=\begin{pmatrix}2.3070 & 0.7781\\ 0.7781 & 0.3123\end{pmatrix},\quad \bar{Q}=\begin{pmatrix}5.6930 & 0.7219\\ 0.7219 & 22.2317\end{pmatrix},\quad S=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},\quad \bar{S}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},\\ R=\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix},\quad \bar{R}=\begin{pmatrix}0.75 & 0\\ 0 & -0.5\end{pmatrix},\end{cases}$$

and from (2.15), we have

$$\begin{cases}\widehat{A}=\begin{pmatrix}-2 & -1\\ -1 & -23\end{pmatrix},\quad \widehat{B}=\begin{pmatrix}3.5 & 0\\ 0 & 1.5\end{pmatrix},\quad \widehat{C}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \widehat{D}=\begin{pmatrix}1.5 & 0\\ 0 & 1.5\end{pmatrix},\\ \widehat{Q}=\begin{pmatrix}8 & 1.5\\ 1.5 & 22.5440\end{pmatrix},\quad \widehat{S}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},\quad \widehat{R}=\begin{pmatrix}2.75 & 0\\ 0 & -2.5\end{pmatrix}.\end{cases}$$

First, we look at closed-loop representation of open-loop saddle point. According to (4.10), we have the following coupled AREs:

$$\begin{cases}PA+A^\top P+C^\top PC+Q-\big(PB+C^\top PD+S^\top\big)\Sigma_o^\dagger(B^\top P+D^\top PC+S)=0,\\ \widehat{P}\widehat{A}+\widehat{A}^\top\widehat{P}+\widehat{C}^\top P\widehat{C}+\widehat{Q}-(\widehat{P}\widehat{B}+\widehat{C}^\top P\widehat{D}+\widehat{S}^\top)\bar{\Sigma}_o^\dagger(\widehat{B}^\top\widehat{P}+\widehat{D}^\top P\widehat{C}+\widehat{S})=0,\end{cases} \tag{5.3}$$

where

$$\Sigma_o\equiv R+D^\top PD=\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix}+P,\qquad \bar{\Sigma}_o\equiv\widehat{R}+\widehat{D}^\top P\widehat{D}=\begin{pmatrix}2.75 & 0\\ 0 & -2.5\end{pmatrix}+2.25P.$$

Then, solving (5.3) yields

$$P=\begin{pmatrix}1 & 0\\ 0 & 0.1\end{pmatrix},\qquad \widehat{P}=\begin{pmatrix}1 & 0\\ 0 & 0.5\end{pmatrix}.$$

The above solutions satisfy (4.7) and (4.8). Further, since Σ_o and $\bar{\Sigma}_o$ are invertible, we have

$$\begin{cases}\mathscr{R}\big(B^\top P+D^\top PC+S\big)\subseteq\mathscr{R}(\Sigma_o),\qquad \mathscr{R}\big(\widehat{B}^\top\widehat{P}+\widehat{D}^\top P\widehat{C}+\widehat{S}\big)\subseteq\mathscr{R}(\bar{\Sigma}_o),\\ \Theta^*=-\Sigma_o^{-1}(B^\top P+D^\top PC+S)=\begin{pmatrix}-0.6667 & -0.1667\\ 0.0263 & 0.1053\end{pmatrix},\\ \bar{\Theta}^*=-\bar{\Sigma}_o^{-1}\big(\widehat{B}^\top\widehat{P}+\widehat{D}^\top P\widehat{C}+\widehat{S}\big)=\begin{pmatrix}-1 & 0\\ 0 & 0.3956\end{pmatrix}.\end{cases} \tag{5.4}$$

According to Proposition A.5 in [23], the system $[A, \bar{A}, C, \bar{C};\ B, \bar{B}, D, \bar{D}]$ above is MF-L^2-stabilizable, and $(\Theta, \bar{\Theta}) \in \mathscr{S}[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ if and only if there exist $P_0 > 0$ and $\bar{P}_0 > 0$, such that

$$\begin{cases} (A+B\Theta)P_0 + P_0(A+B\Theta)^\top + (C+D\Theta)P_0(C+D\Theta)^\top + (\widehat{C}+\widehat{D}\bar{\Theta})\bar{P}_0(\widehat{C}+\widehat{D}\bar{\Theta})^\top \\ = \left[\begin{pmatrix} -1 & -1 \\ 0 & -3 \end{pmatrix} + \Theta\right] P_0 + P_0 \left[\begin{pmatrix} -1 & -1 \\ 0 & -3 \end{pmatrix} + \Theta\right]^\top + \left[\begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix} + \Theta\right] P_0 \left[\begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix} + \Theta\right]^\top \\ \quad + \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} \bar{\Theta}\right] \bar{P}_0 \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} \bar{\Theta}\right]^\top < 0, \\ (\widehat{A}+\widehat{B}\bar{\Theta})\bar{P}_0 + \bar{P}_0(\widehat{A}+\widehat{B}\bar{\Theta})^\top \\ = \left[\begin{pmatrix} -2 & -1 \\ -1 & -23 \end{pmatrix} + \begin{pmatrix} 3.5 & 0 \\ 0 & 1.5 \end{pmatrix} \bar{\Theta}\right] \bar{P}_0 + \bar{P}_0 \left[\begin{pmatrix} -2 & -1 \\ -1 & -23 \end{pmatrix} + \begin{pmatrix} 3.5 & 0 \\ 0 & 1.5 \end{pmatrix} \bar{\Theta}\right]^\top < 0. \end{cases} \tag{5.5}$$

Substituting the above Θ^* and $\bar{\Theta}^*$ of (5.4) into (5.5), we have

$$P_0 = \begin{pmatrix} 0.5742 & -0.0424 \\ -0.0424 & 0.4258 \end{pmatrix} > 0, \qquad \bar{P}_0 = \begin{pmatrix} 0.8185 & -0.0377 \\ -0.0377 & 0.1815 \end{pmatrix} > 0,$$

such that the negative definiteness conditions of (5.5) hold. Then $(\Theta^*, \bar{\Theta}^*)$ above is an MF-L^2-stabilizer of the system $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$. Now, we solve corresponding systems (4.11)–(4.14) to get $(\eta_o(\cdot), \zeta_o(\cdot))$ and $\bar{\eta}_o(\cdot)$. From (4.15), the closed-loop representation of this open-loop saddle point can be given by

$$\begin{aligned} u^*(\cdot) = &-\Sigma_o^{-1}(B^\top P + D^\top PC + S)\big(X^*(\cdot) - \mathbb{E}[X^*(\cdot)]\big) - \bar{\Sigma}_o^{-1}\big(\widehat{B}^\top \widehat{P} + \widehat{D}^\top P\widehat{C} + \widehat{S}\big)\mathbb{E}[X^*(\cdot)] \\ &-\Sigma_o^{-1}\big\{B^\top(\eta_o(\cdot) - \mathbb{E}[\eta_o(\cdot)]) + D^\top\big(\zeta_o(\cdot) - \mathbb{E}[\zeta_o(\cdot)]\big)\big\} - \bar{\Sigma}_o^{-1}\big\{\widehat{B}^\top \bar{\eta}_o(\cdot) + \widehat{D}^\top \mathbb{E}[\zeta_o(\cdot)]\big\}, \end{aligned} \tag{5.6}$$

where $X^*(\cdot) \in \mathscr{X}[0, \infty)$ is the solution to (5.1) corresponding to $u^*(\cdot)$.

Next, we consider the closed-loop saddle point. Note that the corresponding system of generalized AREs (4.17) reads

$$\begin{cases} 0 = P_cA + A^\top P_c + C^\top P_cC + Q - (P_cB + C^\top P_cD + S^\top)\Sigma_c^\dagger(B^\top P_c + D^\top P_cC + S), \\ 0 = \widehat{P}_c\widehat{A} + \widehat{A}^\top \widehat{P}_c + \widehat{C}^\top P_c\widehat{C} + \widehat{Q} - (\widehat{P}_c\widehat{B} + \widehat{C}^\top P_c\widehat{D} + \widehat{S}^\top)\bar{\Sigma}_c^\dagger(\widehat{B}^\top \widehat{P}_c + \widehat{D}^\top P_c\widehat{C} + \widehat{S}), \end{cases} \tag{5.7}$$

with

$$\begin{cases} \Sigma_c \equiv R + D^\top P_cD = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} + P_c, \\ \bar{\Sigma}_c \equiv \widehat{R} + \widehat{D}^\top P_c\widehat{D} = \begin{pmatrix} 2.75 & 0 \\ 0 & -2.5 \end{pmatrix} + 2.25P_c. \end{cases}$$

Then, solving (5.7) yields

$$P_c = \begin{pmatrix} 1 & 0 \\ 0 & 0.1 \end{pmatrix}, \qquad \widehat{P}_c = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}.$$

Thus,

$$\begin{cases} R_{11} + D_1^\top P_cD_1 = 3 \geqslant 0, & \widehat{R}_{11} + \widehat{D}_1^\top P_c\widehat{D}_1 = 5 \geqslant 0, \\ R_{22} + D_2^\top P_cD_2 = -1.9 \leqslant 0, & \widehat{R}_{22} + \widehat{D}_2^\top P_c\widehat{D}_2 = -2.275 \leqslant 0. \end{cases} \tag{5.8}$$

Also, the range condition

$$\mathscr{R}(B^\top P_c + D^\top P_c C + S) \subseteq \mathscr{R}(\Sigma_c), \qquad \mathscr{R}(\widehat{B}^\top \widehat{P}_c + \widehat{D}^\top P_c \widehat{C} + \widehat{S}) \subseteq \mathscr{R}(\bar{\Sigma}_c)$$

holds automatically since Σ_c and $\bar{\Sigma}_c$ are invertible. We see that

$$\begin{cases} \Theta^* = -\Sigma_c^{-1}(B^\top P_c + D^\top P_c C + S) = \begin{pmatrix} -0.6667 & -0.1667 \\ 0.0263 & 0.1053 \end{pmatrix}, \\ \bar{\Theta}^* = -\bar{\Sigma}_c^{-1}(\widehat{B}^\top \widehat{P}_c + \widehat{D}^\top P_c \widehat{C} + \widehat{S}) = \begin{pmatrix} -1 & 0 \\ 0 & 0.3956 \end{pmatrix}, \end{cases} \tag{5.9}$$

which coincides with (5.4). Hence, by Theorem 4.4, the problem admits a closed-loop saddle point. Solving corresponding backward systems (4.19)–(4.20), we can get $(\eta_c(\cdot), \zeta_c(\cdot))$ and $\bar{\eta}_c(\cdot)$, the closed-loop saddle point is given by (5.9) and

$$v^*(\cdot) = -\Sigma_c^{-1}\big\{B^\top\big(\eta_c(\cdot) - \mathbb{E}[\eta_c(\cdot)]\big) + D^\top\big(\zeta_c(\cdot) - \mathbb{E}[\zeta_c(\cdot)]\big)\big\} - \bar{\Sigma}_c^{-1}\big\{\widehat{B}^\top \bar{\eta}_c(\cdot) + \widehat{D}^\top \mathbb{E}[\zeta_c(\cdot)]\big\}. \tag{5.10}$$

Since $P = P_c$ and $\widehat{P} = \widehat{P}_c$, we get $\Sigma = \Sigma_c, \bar{\Sigma} = \bar{\Sigma}_c$ and $\eta_o(\cdot) = \eta_c(\cdot)$, $\zeta_o(\cdot) = \zeta_c(\cdot)$, $\bar{\eta}_o(\cdot) = \bar{\eta}_c(\cdot)$. This implies that the closed-loop representation of the open-loop saddle point coincides with the closed-loop saddle point.

The following example shows that for the mean-field LQ non-zero sum stochastic differential game, it may happen that the closed-loop representations of open-loop Nash equilibria are different from the closed-loop Nash equilibria. However, the solutions to the system of algebraic Riccati equations for the closed-loop representation of open-loop Nash equilibrium are symmetric.

Example 5.2. Consider the following two-dimensional state equation

$$\begin{cases} \mathrm{d}X(t) = \big\{AX(t) + \bar{A}\mathbb{E}[X(t)] + Bu(t) + \bar{B}\mathbb{E}[u(t)]\big\}\mathrm{d}t \\ \qquad\qquad +\big\{CX(t) + \bar{C}\mathbb{E}[X(t)] + Du(t) + \bar{D}\mathbb{E}[u(t)]\big\}\mathrm{d}W(t), \quad t \geqslant 0, \\ X(0) = x, \end{cases} \tag{5.11}$$

with the cost functional

$$\begin{cases} J_1(x; u_1(\cdot), u_2(\cdot)) = \mathbb{E}\displaystyle\int_0^\infty \Bigg[\left\langle \begin{pmatrix} Q_1 & S_1^\top \\ S_1 & R_1 \end{pmatrix} \begin{pmatrix} X(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix} \right\rangle \\ \qquad\qquad +\left\langle \begin{pmatrix} \bar{Q}_1 & \bar{S}_1^\top \\ \bar{S}_1 & \bar{R}_1 \end{pmatrix} \begin{pmatrix} \mathbb{E}[X(t)] \\ \mathbb{E}[u(t)] \end{pmatrix}, \begin{pmatrix} \mathbb{E}[X(t)] \\ \mathbb{E}[u(t)] \end{pmatrix} \right\rangle \Bigg]\mathrm{d}t, \\ J_2(x; u_1(\cdot), u_2(\cdot)) = \mathbb{E}\displaystyle\int_0^\infty \Bigg[\left\langle \begin{pmatrix} Q_2 & S_2^\top \\ S_2 & R_2 \end{pmatrix} \begin{pmatrix} X(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix} \right\rangle \\ \qquad\qquad +\left\langle \begin{pmatrix} \bar{Q}_2 & \bar{S}_2^\top \\ \bar{S}_2 & \bar{R}_2 \end{pmatrix} \begin{pmatrix} \mathbb{E}[X(t)] \\ \mathbb{E}[u(t)] \end{pmatrix}, \begin{pmatrix} \mathbb{E}[X(t)] \\ \mathbb{E}[u(t)] \end{pmatrix} \right\rangle \Bigg]\mathrm{d}t. \end{cases} \tag{5.12}$$

We let

$$\begin{cases} A=\begin{pmatrix}-1 & -1\\ 0 & -1\end{pmatrix}, \quad \bar{A}=\begin{pmatrix}-1 & 0\\ -1 & -1\end{pmatrix}, \quad B=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \bar{B}=\begin{pmatrix}2.5 & 0\\ 0 & 0.5\end{pmatrix}, \\ C=\begin{pmatrix}1 & 0.5\\ 0.5 & 1\end{pmatrix}, \quad \bar{C}=\begin{pmatrix}0 & -0.5\\ -0.5 & 0\end{pmatrix}, \quad D=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \bar{D}=\begin{pmatrix}0.5 & 0\\ 0 & 0.5\end{pmatrix}, \\ Q_1=\begin{pmatrix}2.85 & 0.9\\ 0.9 & 2.475\end{pmatrix}, \quad Q_2=\begin{pmatrix}1.35 & 0.4\\ 0.4 & 2.5375\end{pmatrix}, \quad \bar{Q}_1=\begin{pmatrix}6.0324 & 0.6\\ 0.6 & 0.325\end{pmatrix}, \\ \bar{Q}_2=\begin{pmatrix}7.15 & 1.6\\ 1.6 & 2.8625\end{pmatrix}, \quad S_1=S_2=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \quad \bar{S}_1=\bar{S}_2=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \\ R_1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad R_2=\begin{pmatrix}1 & 0\\ 0 & 1.5\end{pmatrix}, \quad \bar{R}_1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \bar{R}_2=\begin{pmatrix}0.5 & 0\\ 0 & 0\end{pmatrix}, \end{cases}$$

and from (2.15), we have

$$\begin{cases} \widehat{A}=\begin{pmatrix}-2 & -1\\ -1 & -2\end{pmatrix}, \quad \widehat{B}=\begin{pmatrix}3.5 & 0\\ 0 & 1.5\end{pmatrix}, \quad \widehat{C}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \widehat{D}=\begin{pmatrix}1.5 & 0\\ 0 & 1.5\end{pmatrix}, \quad \widehat{Q}_1=\begin{pmatrix}8.8824 & 1.5\\ 1.5 & 2.8\end{pmatrix}, \\ \widehat{Q}_2=\begin{pmatrix}8.5 & 2\\ 2 & 5.4\end{pmatrix}, \quad \widehat{S}_1=\widehat{S}_2=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \quad \widehat{R}_1=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}, \quad \widehat{R}_2=\begin{pmatrix}1.5 & 0\\ 0 & 1.5\end{pmatrix}. \end{cases}$$

To look at the closed-loop representation of open-loop Nash equilibria, we solve the corresponding (3.33)–(3.34) to get

$$P_1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad P_2=\begin{pmatrix}0.5 & 0\\ 0 & 1\end{pmatrix}, \qquad \widehat{P}_1=\begin{pmatrix}1 & 0\\ 0 & 0.5\end{pmatrix}, \qquad \widehat{P}_2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

which are symmetric. Since

$$\begin{pmatrix}R_{111}+D_1^\top P_1 D_1 & R_{112}+D_1^\top P_1 D_2\\ R_{221}+D_2^\top P_2 D_1 & R_{222}+D_2^\top P_2 D_2\end{pmatrix}=\begin{pmatrix}2 & 0\\ 0 & 2.5\end{pmatrix}$$

and

$$\begin{pmatrix}\widehat{R}_{111}+\widehat{D}_1^\top P_1 \widehat{D}_1 & \widehat{R}_{112}+\widehat{D}_1^\top P_1 \widehat{D}_2\\ \widehat{R}_{221}+\widehat{D}_2^\top P_2 \widehat{D}_1 & \widehat{R}_{222}+\widehat{D}_2^\top P_2 \widehat{D}_2\end{pmatrix}=\begin{pmatrix}4.25 & 0\\ 0 & 3.75\end{pmatrix}$$

are invertible, it follows that

$$\Theta^{**}=\begin{pmatrix}-1 & -0.25\\ -0.2 & -0.8\end{pmatrix}, \quad \bar{\Theta}^{**}=\begin{pmatrix}-1.1765 & 0\\ 0 & -0.8\end{pmatrix}. \tag{5.13}$$

Also, it follows from (3.29)–(3.30), we can get $(\eta(\cdot),\zeta(\cdot))$ and $\bar{\eta}(\cdot)$. Then, from (3.27), we have

$$v^{**}(\cdot)=-\mathbf{\Sigma}^{-1}\mathbf{J}^\top\big\{\mathbf{B}^\top\big(\eta(\cdot)-\mathbb{E}[\eta(\cdot)]\big)+\mathbf{D}^\top\big(\zeta(\cdot)-\mathbb{E}[\zeta(\cdot)]\big)\big\}-\bar{\mathbf{\Sigma}}^{-1}\mathbf{J}^\top\big\{\widehat{\mathbf{B}}^\top\bar{\eta}(\cdot)+\widehat{\mathbf{D}}^\top\mathbb{E}[\zeta(\cdot)]\big\}. \tag{5.14}$$

By Theorem 3.3, making use of (3.27)–(3.30), the problem admits an open-loop Nash equilibrium $u^*(\cdot)$ for any initial state $x\in\mathbb{R}^n$. And it has the closed-loop representation (2.7) with $(\Theta^{**},\bar{\Theta}^{**},v^{**}(\cdot))$ given by (5.13) and (5.14).

Next, we consider the closed-loop Nash equilibria. According to Proposition A.5 in [23], system $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ above is MF-L^2-stabilizable, and $(\Theta, \bar{\Theta}) \in \mathscr{S}[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ if and only if there exist $P_0 > 0$ and $\bar{P}_0 > 0$, such that

$$\begin{cases} (A+B\Theta)P_0 + P_0(A+B\Theta)^\top + (C+D\Theta)P_0(C+D\Theta)^\top + (\widehat{C}+\widehat{D}\bar{\Theta})\bar{P}_0(\widehat{C}+\widehat{D}\bar{\Theta})^\top \\ = \left[\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} + \Theta\right] P_0 + P_0 \left[\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} + \Theta\right]^\top + \left[\begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix} + \Theta\right] P_0 \left[\begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix} + \Theta\right]^\top \\ \quad + \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} \bar{\Theta}\right] \bar{P}_0 \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} \bar{\Theta}\right]^\top < 0, \\ (\widehat{A}+\widehat{B}\bar{\Theta})\bar{P}_0 + \bar{P}_0(\widehat{A}+\widehat{B}\bar{\Theta})^\top \\ = \left[\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} + \begin{pmatrix} 3.5 & 0 \\ 0 & 1.5 \end{pmatrix} \bar{\Theta}\right] \bar{P}_0 + \bar{P}_0 \left[\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} + \begin{pmatrix} 3.5 & 0 \\ 0 & 1.5 \end{pmatrix} \bar{\Theta}\right]^\top < 0. \end{cases} \tag{5.15}$$

For $i = 1, 2$, we solve the corresponding (3.35)–(3.37) to get

$$\begin{cases} P_1 = \begin{pmatrix} 0.9949 & -0.0168 \\ -0.0168 & 0.9201 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0.6255 & -0.0104 \\ -0.0104 & 1.01741 \end{pmatrix}, \\ \widehat{P}_1 = \begin{pmatrix} 1.0023 & -0.0155 \\ -0.0155 & 0.6472 \end{pmatrix}, \qquad \widehat{P}_2 = \begin{pmatrix} 0.8919 & 0.0126 \\ 0.0126 & 0.9964 \end{pmatrix}, \end{cases}$$

$$\begin{cases} \Sigma_1 \triangleq R_{111} + D_1^\top P_1 D_1 = 1.9949, \qquad \Sigma_2 \triangleq R_{222} + D_2^\top P_2 D_2 = 2.5174, \\ \bar{\Sigma}_1 \triangleq \widehat{R}_{111} + \widehat{D}_1^\top P_1 \widehat{D}_1 = 4.2386, \qquad \bar{\Sigma}_2 \triangleq \widehat{R}_{222} + \widehat{D}_2^\top P_2 \widehat{D}_2 = 3.7891, \end{cases}$$

and

$$\Theta^* = \begin{pmatrix} -0.9949 & -0.2393 \\ -0.1979 & -0.8072 \end{pmatrix}, \qquad \bar{\Theta}^* = \begin{pmatrix} -1.1798 & 0.0117 \\ -0.0082 & -0.7971 \end{pmatrix}. \tag{5.16}$$

Substituting the above Θ^* and $\bar{\Theta}^*$ of (5.16) into (5.15), we can get

$$P_0 = \begin{pmatrix} 0.3323 & -0.0865 \\ -0.0865 & 0.3365 \end{pmatrix} > 0, \qquad \bar{P}_0 = \begin{pmatrix} 0.1196 & -0.0327 \\ -0.0327 & 0.2115 \end{pmatrix} > 0.$$

such that the negative definiteness conditions of (5.15) hold. Then $(\Theta^*, \bar{\Theta}^*)$ above is an MF-L^2-stabilizer of the system $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$. Hence, by Theorem 3.7, the problem admits a closed-loop Nash equilibrium.

Now, since

$$B(\Theta^* - \Theta^{**}) = \begin{pmatrix} 0.0051 & 0.0107 \\ 0.0021 & -0.0072 \end{pmatrix} \neq 0,$$

by Remark 2.7, we see that the closed-loop representation of the open-loop Nash equilibrium and the closed-loop Nash equilibrium are intrinsically different.

From this example, we see that the solutions to system (3.33)–(3.34), which come from the open-loop Nash equilibrium, may be symmetric, but they are different from the solution to the system of AREs (3.35)–(3.37), which come from the closed-loop Nash equilibrium. It is obvious that the closed-loop representation of the open-loop Nash equilibrium is different from the closed-loop Nash equilibrium.

The following example shows that for the closed-loop representation of open-loop Nash equilibria of the mean-field LQ non-zero sum stochastic differential game, it may happen that the solutions to the system of AREs may be asymmetric when the system $[A, \bar{A}, C, \bar{C}; B, \bar{B}, D, \bar{D}]$ is MF-L^2-stabilizable. And, the closed-loop representation of the open-loop Nash equilibrium is different from the closed-loop Nash equilibrium.

Example 5.3. Consider the two-dimensional state equation of form (5.11) with cost functionals (5.12), but we let

$$\begin{cases} A = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{C} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{D} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ Q_1 = \begin{pmatrix} 2 & 1 \\ 1 & \frac{3}{2} \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 3 \end{pmatrix}, \quad \bar{Q}_1 = \begin{pmatrix} \frac{83}{44} & 1 \\ 1 & \frac{105}{44} \end{pmatrix}, \quad \bar{Q}_2 = \begin{pmatrix} 3 & \frac{3}{2} \\ \frac{3}{2} & \frac{16}{11} \end{pmatrix}, \\ S_1 = \begin{pmatrix} 0 & 0 \\ \sqrt{\frac{5}{2}} & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\frac{5}{2}} \end{pmatrix}, \quad \bar{S}_1 = \bar{S}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}, \quad \bar{R}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{R}_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}. \end{cases}$$

From (2.15), we have

$$\begin{cases} \widehat{A} = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}, \quad \widehat{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \widehat{C} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \widehat{D} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \\ \widehat{Q}_1 = \begin{pmatrix} \frac{171}{44} & 2 \\ 2 & \frac{171}{44} \end{pmatrix}, \quad \widehat{Q}_2 = \begin{pmatrix} 4 & 2 \\ 2 & \frac{49}{11} \end{pmatrix}, \quad \widehat{S}_1 = \begin{pmatrix} 0 & 0 \\ \sqrt{\frac{5}{2}} & 0 \end{pmatrix}, \quad \widehat{S}_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\frac{5}{2}} \end{pmatrix}, \\ \widehat{R}_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \widehat{R}_2 = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}. \end{cases}$$

To look at the closed-representation of open-loop Nash equilibria, we solve the corresponding (3.33)–(3.34) to get

$$P_1 = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}, \qquad \widehat{P}_1 = \begin{pmatrix} 1 & -\frac{5}{44} \\ 0 & 1 \end{pmatrix}, \qquad \widehat{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We see that P_1 and $\widehat{P}_1$ are not symmetric. Since

$$\begin{pmatrix} R_{111} + D_1^\top P_1 D_1 & R_{112} + D_1^\top P_1 D_2 \\ R_{221} + D_2^\top P_2 D_1 & R_{222} + D_2^\top P_2 D_2 \end{pmatrix} = \begin{pmatrix} 2 & -\frac{1}{2} \\ 0 & \frac{5}{2} \end{pmatrix}$$

and

$$\begin{pmatrix} \widehat{R}_{111} + \widehat{D}_1^\top P_1 \widehat{D}_1 & \widehat{R}_{112} + \widehat{D}_1^\top P_1 \widehat{D}_2 \\ \widehat{R}_{221} + \widehat{D}_2^\top P_2 \widehat{D}_1 & \widehat{R}_{222} + \widehat{D}_2^\top P_2 \widehat{D}_2 \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ 0 & \frac{11}{2} \end{pmatrix}$$

are invertible, it follows that

$$\Theta^{**}=\begin{pmatrix}0 & -0.1581\\ 0 & -0.6325\end{pmatrix},\quad \bar{\Theta}^{**}=\begin{pmatrix}0 & -0.0958\\ 0 & -0.2875\end{pmatrix}. \tag{5.17}$$

Then, similar to Example 5.2, making use of Theorem 3.3, we see that the problem admits an open-loop Nash equilibrium for any initial state $x\in\mathbb{R}^n$. And it has the closed-loop representation of form (2.7).

Next, we consider the closed-loop Nash equilibria. According to Proposition A.5 in [23], the system $[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]$ above is MF-L^2-stabilizable, and $(\Theta,\bar{\Theta})\in\mathscr{S}[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]$ if and only if there exist $P_0>0$ and $\bar{P}_0>0$, such that

$$\begin{cases}(A+B\Theta)P_0+P_0(A+B\Theta)^\top+(C+D\Theta)P_0(C+D\Theta)^\top+(\widehat{C}+\widehat{D}\bar{\Theta})\bar{P}_0(\widehat{C}+\widehat{D}\bar{\Theta})^\top\\ =\left[\begin{pmatrix}-1 & -1\\ 0 & -1\end{pmatrix}+\Theta\right]P_0+P_0\left[\begin{pmatrix}-1 & -1\\ 0 & -1\end{pmatrix}+\Theta\right]^\top+\Theta P_0\Theta^\top+\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\bar{\Theta}\bar{P}_0\bar{\Theta}^\top\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}<0,\\ (\widehat{A}+\widehat{B}\bar{\Theta})\bar{P}_0+\bar{P}_0(\widehat{A}+\widehat{B}\bar{\Theta})^\top=\begin{pmatrix}-2 & -1\\ -1 & -2\end{pmatrix}\bar{P}_0+\bar{P}_0\begin{pmatrix}-2 & -1\\ -1 & -2\end{pmatrix}^\top<0.\end{cases} \tag{5.18}$$

For $i=1,2$, we solve corresponding (3.35)–(3.37) to get

$$\begin{cases}P_1=\begin{pmatrix}1 & -0.4955\\ -0.4955 & 1.7645\end{pmatrix},\qquad P_2=\begin{pmatrix}0.5 & 0\\ 0 & 1.0226\end{pmatrix},\\ \widehat{P}_1=\begin{pmatrix}1.0647 & -0.1861\\ -0.1861 & 1.2327\end{pmatrix},\qquad \widehat{P}_2=\begin{pmatrix}1.0016 & -0.0032\\ -0.0032 & 1.0111\end{pmatrix},\end{cases}$$

$$\begin{cases}\Sigma_1\triangleq R_{111}+D_1^\top P_1D_1=2,\qquad \Sigma_2\triangleq R_{222}+D_2^\top P_2D_2=2.5226,\\ \bar{\Sigma}_1\triangleq \widehat{R}_{111}+\widehat{D}_1^\top P_1\widehat{D}_1=6,\qquad \bar{\Sigma}_2\triangleq \widehat{R}_{222}+\widehat{D}_2^\top P_2\widehat{D}_2=5.5902.\end{cases}$$

and

$$\Theta^{*}=\begin{pmatrix}0 & -0.1553\\ 0 & -0.6268\end{pmatrix},\qquad \bar{\Theta}^{*}=\begin{pmatrix}0 & -0.0934\\ 0 & -0.2828\end{pmatrix}. \tag{5.19}$$

Substituting the above Θ^* and $\bar{\Theta}^*$ of (5.19) into (5.18), we can get

$$P_0=\begin{pmatrix}0.3082 & -0.0910\\ -0.0910 & 0.3148\end{pmatrix}>0,\qquad \bar{P}_0=\begin{pmatrix}0.1901 & -0.0721\\ -0.0721 & 0.1870\end{pmatrix}>0.$$

such that the negative definiteness conditions of (5.18) hold. Then $(\Theta^*,\bar{\Theta}^*)$ above is an MF-L^2-stabilizer of the system $[A,\bar{A},C,\bar{C};B,\bar{B},D,\bar{D}]$. Hence, by Theorem 3.7, the problem admits a closed-loop Nash equilibrium.

Finally, similar to Example 5.2, we easily check that the closed-loop representation of the optimal open-loop strategy and the optimal closed-loop strategy are intrinsically different.

From this example, we see that the solutions to system (3.33)–(3.34), which come from the open-loop saddle point, may be asymmetric, but the solutions to system (3.35)–(3.37), which come from the closed-loop saddle point, are still symmetric. It is obvious that the closed-loop representation of the open-loop Nash equilibrium is different from the closed-loop Nash equilibrium.

6. Concluding remarks

In this paper, we have presented a systematic theory for two-person non-zero sum differential games of mean-field SDEs with quadratic performance indexes in $[0, \infty)$. The case of two-person zero-sum, which is also new, has been treated as a special case. Our results cover several existing ones in the literature for infinite horizon problems, including LQ optimal control problems of mean-field type [23], two-person zero-sum LQ stochastic differential games (without mean-field terms) [50], LQ optimal control problem (without mean-field terms), and the equivalence between the open-loop solvability and the closed-loop solvability for stochastic LQ problem in $[0, \infty)$ [46]. Finally, we have to leave the following question open: Is the existence of the open-loop and closed-loop saddle points equivalent for the mean-field LQ two-person zero-sum stochastic differential game in an infinite horizon? Some related papers can be seen in Sun [42], Sun-Li-Yong [43], Sun-Yong [45], Yu [55], etc. We will research this topic in the future.

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