Acta Mathematicae Applicatae Sinica, English Series
Vol. 36, No. 2 (2020) 262–282
https://doi.org/10.1007/s10255-020-0925-4
http://www.ApplMath.com.cn & www.SpringerLink.com

Acta Mathematicae Applicatae Sinica, English Series

Controlled Stochastic Partial Differential Equations for Rabbits on a Grassland

Suzanne LENHART[1], **Xiao TANG**[2], **Jie XIONG**[2,†], **Jiong-min YONG**[3]

[1]Department of Mathematics, University of Tennessee, Knoxville, TN 37996-1320, USA

[2]Department of Mathematics, Southern University of Science and Technology, Shenzhen, 518055, China
(†E-mail: xiongj@sustech.edu.cn)

[3]Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA

Abstract A new approach to modeling populations incorporating stochasticity, a random environment, and individual behavior is illustrated with a specific example of two interacting populations: rabbits and grass. The derivation of the system of stochastic partial differential equations (SPDEs) to show how the individual mechanisms of both populations are included. This model also has an unusual feature of a nonlocal term. The harvesting of the rabbit population is introduced as a control variable.

Keywords Stochastic partial differential equation; tightness; optimal control; population model
2000 MR Subject Classification 60H15; 60F05

1 Introduction

Managing pest or invasive species is an important ecological issue[5, 18, 21, 24]. The use of agent-based models (ABMs) has been expanding especially in modeling complex biological populations[6, 22], including pest species[23]. These models include stochasticity in individual behavior and movement and sometimes in the environmental features. ABMs have been applied to pest or invasive species, and there is a need to understand how to build effective controls for managing such populations. Some ABMs have been approximated by using difference equations, differential equations or polynomial dynamical systems[3, 11, 16, 25]. To capture spatial heterogeneity better, partial differential equations, interacting particle systems and mean-field approximation have been used with some success[1, 4, 12, 20]. The small amount of work on optimal control of ABMs has shown the difficulty of generalization to broad class of ABMs; some work has used aggregated models to approximate the ABMs and to get approximate optimal controls from the aggregated models [3, 7], but spatial heterogeneity can lead to poor approximations. It seems to us that alternative modeling approaches with features of stochasticity, random environments and individual behavior may better represent some biosystems and have a successful framework for handling optimal control of these systems.

Such an alternate modeling approach is introduced in this paper. We will illustrate the approach by modeling a rabbit population in an environment with grass; the rabbits are viewed

Manuscript received December 28, 2018. Accepted on September 24, 2019.
The authors thankfully acknowledge the input from the Optimal Control of Agent-based Model Working Group at the National Institute for Mathematical and Biological Synthesis, which is sponsored by the National Science Foundation through NSF Award DBI-1300426, with additional support from the University of Tennessee, Knoxville. The work of Lenhart and Xiong is partially supported by the Department of Mathematics of the University of Tennessee. Xiong is partially supported by Southern University of Science and Technology Start up fund Y01286120 and National Science Fundation of China grants 61873325 and 11831010. Yong is partially supported by NSF under grants DMS-1007514 and DMS-1406776.
†Corresponding author.

as the pest population to be controlled by harvesting. This particular biological system is motivated by the Rabbits-Grass-Weeds ABM in Netlogo [26, 27], which is available in Netlogo's library of sample models. This version of the model (without weeds) has two populations: a resource (grass) and a consumer pest (rabbits), similar to a predator-prey system. This model with rabbits and grass has features of individual behavior that are common in simple ABMs.

The goal of this paper is to derive an aggregate model for the rabbit population and the grass population. We will use specific mechanisms of the movement of individual rabbits. The grass population represents a type of plant that disperses due to wind. The two interacting population will be represented by the solution of a pair of controlled SPDEs with an unusual nonlocal term. The optimal control problem for this pair of SPDEs is studied in our companion paper[17] under a more general framework. This modeling approach and the corresponding optimal control results are developed here for a simple biological scenario, but these tools have the potential to be applied to more complex and realistic scenarios.

This paper serves as the main motivation of its companion[17]. Due to the length of the combined paper and the interest in both features (modeling and control), we decided to separate the modeling part (the current paper) from the optimal control part in [17]. The main contribution and novelty of this paper include the following. First of all, we establish an aggregate model from individual mechanisms directly from the applied problem. Secondly, the aggregate model is a pair of coupled SPDEs with non-local term and one of the SPDE degenerates to a PDE. To the best of our knowledge, such a type of SPDE pair is new in the literature. We studied them using refined Sobolev space techniques.

The rest of this paper is organized as follows. The next section introduces the model formulation with the assumptions on the rabbit population and the grass environment. A smoothing technique is needed for technical reasons. In Section 3, we present some mathematical results needed in the later proofs. In Section 4, we construct an aggregate model by taking n, the initial number of rabbits, to infinity. Namely, we give the proof of our existence result for the smoothed aggregate system, Theorem 2.1. Then we also let the smoothing constant δ tend to 0; namely we present the proof of our existence result for the limiting aggregate system, Theorem 2.2. Finally, we make some concluding remarks.

2 Model Formulation

Now, we proceed to introducing the dynamic model for a population of rabbits living on a grassland. Suppose the grass occupies and grows on $\mathbb{R}^2$. Let $n \geq 1$ be the initial number of rabbits living on $\mathbb{R}^2$. Let $\varepsilon > 0$ and $h > 0$ be the discretizing parameters for the space and the time variable, respectively. Denote $\varepsilon\mathbb{Z}^2 = \{\varepsilon i \mid i \in \mathbb{Z}^2\}$ and let $Q_\varepsilon(x_i)$ be the closed square centered at $x_i \in \mathbb{R}^2$ with side-length 2ε.

To model the movements of the rabbits, we adopt the following rule: *The rabbit would choose to move to a neighboring square with relatively more grass taking into consideration of the grass level at the current location, and is also subject to random noise in the movement.* More precisely, the drift coefficients of the model for the movements of the rabbits should depend (increasingly) on the gradient of a smooth version of the grass density field. Taking into account the above rule, we suppose that the kth rabbit, $k = 1, 2, \cdots$, moves in $\mathbb{R}^2$ during its lifetime according to the following stochastic differential equation (SDE):

$$dX_k^{n,\delta}(t) = b\big(t, X_k^{n,\delta}(t), \zeta^{n,\delta}(t, X_k^{n,\delta}(t)), \nabla\zeta^{n,\delta}(t, X_k^{n,\delta}(t))\big)dt + dB_k(t), \tag{2.1}$$

where $X_k^{n,\delta}(t)$ is the location of the kth rabbit at time t. The rabbit has knowledge about the grass level at nearby locations, represented by the term,

$$\zeta^{n,\delta}(t, x) = \int_{\mathbb{R}^{\nvDash}} p_\delta(x - x')z^{n,\delta}(t, x')dx', \tag{2.2}$$

where

$$p_\delta(x-x') = \frac{1}{\sqrt{2\pi\delta}} e^{-\frac{(x-x')^2}{2\delta}},$$

and $z^{n,\delta}(t,x)$ is the density of the grass in the square centered at x. Here, δ is a smoothing parameter which is needed due to a technical reason. As we shall see in (2.3) below, the approximate grass density field $z^{n,\delta}$ is not differentiable, and hence, we need its smoothed version $\zeta^{n,\delta}$ for the model (2.1) to make sense.

The proportion $u^n(t,x_i)$ of rabbits in the square $Q_\varepsilon(x_i)$ will be killed by the controller. This control u^n will first enter into the ABM (4.7) for the rabbit population, and then its limit will enter into the aggregated model (2.9) for the density of the rabbit population. Note that the selection of $u(t,x)$ is the objective of the companion paper [17]. Namely, it is the optimal control of an optimization problem.

For simplicity, we assume that a rabbit will not die before being killed by the controller. However, we do assume that a rabbit will give birth to a rabbit in exponential time of parameter $\alpha(\zeta^{n,\delta}(t,x))$ when a rabbit is at x at time t, namely,

$$\lim_{h\to 0+} h^{-1}\mathbb{P}(\tau_k < t+h|\mathcal{F}_t,\ X_k^{n,\delta}(t)=x) = \alpha(\zeta^{n,\delta}(t,x)),$$

where τ_k is the first time after t that the kth rabbit give birth. The birth clock is reset after each birth. This birth rate will enter into the aggregated model (2.9) for the density of the rabbit population.

For the growth of the grass, we use the following model:

$$\begin{aligned} z^{n,\delta}(t+h,x_i)\varepsilon^2 = & \left(1+\mu(t,x_i)h+\xi^n(t,x_i)\sqrt{h}\right) z^{n,\delta}(t,x_i)\varepsilon^2 \\ & -\gamma(x_i,Y^{n,\delta}(t))z^{n,\delta}(t,x_i)\varepsilon^2 h \\ & +h\sum_{j\sim i}\left(z^{n,\delta}(t+h,x_j)-z^{n,\delta}(t+h,x_i)\right), \end{aligned} \tag{2.3}$$

where $j\sim i$ means that x_j is in a neighboring square of x_i, i.e.,

$$x_j \in x_i \pm \varepsilon\{(1,0),(0,1),(-1,0),(0,-1)\}.$$

On the right hand side of the equation above, the first term represents the growth of the grass in $Q_\epsilon(x_i)$ with a natural growth rate $\mu(t,x_i)$, and with some random disturbance modeled by the random field $\xi^n(t,x)$ which may represent changes in features of the habitat. The second term represents the grass in $Q_\epsilon(x_i)$ consumed by the rabbits with the proportionality constant $\gamma(x_i,Y^{n,\delta}(t))\in(0,1)$, depending on the location and the nearby distribution of the rabbits, where $Y^{n,\delta}(t)$ is the empirical distribution of the rabbits in $\mathbb{R}^2$. i.e., for any $f\in C_b(\mathbf{R}^2)$, we have

$$\langle Y^{n,\delta}(t),f\rangle = \frac{1}{n}\sum_k f(X_k^{n,\delta}(t)),$$

where the sum is over all k such that the kth particle is alive at time t. Note that the harvest rate u^n does not enter this definition explicitly. However, it affects the dynamic of the population and enters into the dynamic model (4.7). A typical example of $\gamma(x,\nu)$ is as follows:

$$\gamma(x,\nu) = F\left(\int_{\mathbb{R}^2}\theta(x-x')\nu(dx')\right), \qquad \forall x\in\mathbb{R}^2, \tag{2.4}$$

where $\theta(\cdot)\in C_b(\mathbb{R}^2)$ is supported on a compact neighborhood of 0, and $F:\mathbb{R}\to\mathbb{R}$ is a suitable increasing function. The third term on the right hand side of (2.3) models the dispersal of the grass population by the wind or other environmental effects.

To study the limit of $z^{n,\delta}$, we first consider that for its corresponding measure-valued process $Z^{n,\delta}$ defined as:

$$\langle Z^{n,\delta}(jh), f \rangle = \sum_{i\geq 1} f(x_i) z^{n,\delta}(jh, x_i)\varepsilon^2, \qquad \forall f(\cdot) \in C_b(\mathbb{R}^2).$$

We also extend the definition by $Z^{n,\delta}(t) = Z^{n,\delta}(jh)$ for $jh \leq t < (j+1)h$.

We assume that the random field $\xi^n(kh, x)$, independent of the Brownian motions $B_k(t)$, is Gaussian with mean 0 and covariance structure

$$\begin{aligned} \operatorname{cov}\left(\xi^n(kh,x), \xi^n(jh,x')\right) &\equiv \mathbb{E}\big(\left[\xi^n(kh,x) - \mathbb{E}\xi^n(kh,x)\right]\left[\xi^n(jh,x') - \mathbb{E}\xi^n(jh,x')\right]\big) \\ &= \rho(x,x')1_{j=k} = \sum_{k=1}^{\infty} \phi^k(x)\phi^k(x')1_{j=k}, \end{aligned} \tag{2.5}$$

with $\phi^k(\cdot) \in C^1(\mathbb{R}^2)$, $k \geq 1$, satisfying

$$\sum_{k=1}^{\infty} \big(|\phi^k(\cdot)| + |\phi^k_x(\cdot)|\big) \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2). \tag{2.6}$$

Throughout this paper, we make the following Assumption (S) together with (2.6).

Assumption (S): The drift coefficient $b(t,x,z,p)$ is bounded and Lipschitz continuous in (x,z,p) uniformly in t. The birth rate $\alpha(z)$ is bounded and Lipschitz continuous.

Let $\overline{\mathbb{R}^2}$ be the one-point compactification of $\mathbb{R}^2$. We regard $Y^{n,\delta}$ and $Z^{n,\delta}$ as two families of stochastic processes taking values in $\mathcal{M}_F(\overline{\mathbb{R}^2})$, the space of finite Borel measures on $\overline{\mathbb{R}^2}$. Our aim is to derive the aggregate model for the rabbit population on grassland by taking the initial number n of rabbits to ∞.

Let $D([0,T], \mathcal{M}_F(\overline{\mathbb{R}^2}))$ be the space of càdlàg mappings from $[0,T]$ to $\mathcal{M}_F(\overline{\mathbb{R}^2})$ endowed with Skorohod topology. Denote by $C_0(\mathbb{R}^2)$ the collection of all continuous functions on $\mathbb{R}^2$ with 0 as limit at ∞, endowed with supremum norm. We now state our main results of this paper. The first result gives the formulation of the aggregate model with smoothing and the existence of its solution. The second result give the corresponding system and solution as the smoothing parameter δ goes to 0.

Theorem 2.1. *Let $\delta > 0$ be fixed and let $(\epsilon, h) = (\epsilon_n, h_n) \to 0$ and $\epsilon_n/h_n \to 0$. Suppose that u^n, as a random sequence in $L^2([0,T], C_0(\mathbb{R}^2))$, is tight. Then the sequence $\{(Y^{n,\delta}, Z^{n,\delta}) \bigm| n \geq 1\}$ is tight in $D([0,T], \mathcal{M}_F(\overline{\mathbb{R}^2}))^2$. Let (Y^δ, Z^δ, u) be a limit point of $(Y^{n,\delta}, Z^{n,\delta}, u^n)$. Then (Y^δ, Z^δ) is a solution to the following SPDEs on $\mathcal{M}_F(\overline{\mathbb{R}^2})^2$:*

$$\begin{aligned} \langle Y^\delta(t), f \rangle = \langle Y(0), f \rangle + \int_0^t \big(&\langle Y^\delta(s), L_{s,\zeta^\delta(s),\nabla\zeta^\delta(s)} f \rangle \\ &+ \langle (\alpha(\zeta^\delta(s)) - u(t)) Y^\delta(t), f \rangle \big) ds \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \langle Z^\delta(t), f \rangle = \langle Z(0), f \rangle + &\int_0^t \langle Z^\delta(s), \mu(s)f + \Delta f - \gamma(\cdot, Y^\delta(s)) f \rangle\, ds \\ &+ \int_0^t \langle Z^\delta(s), f W(\cdot, ds) \rangle, \end{aligned} \tag{2.8}$$

where the Gaussian random measure W is white in time and colored in space with covariance function ρ given in (2.5),

$$\zeta^\delta(s,x) = \int_{\mathbb{R}^2} p_\delta(x-x')Z^\delta(s,dx'),$$

and the generator L is given by

$$L_{s,p,q}f(x) = \frac{1}{2}\Delta f(x) + b(s,x,p,q)\cdot\nabla f(x), \qquad \forall\, f\in C_b^2(\mathbb{R}^2).$$

Note that the smoothing constant δ is introduced in the individual grass model (2.2) for a technical reason only. We now remove this extra smoothing by taking $\delta\to 0$. Denote by $\|\cdot\|_\infty$ and $\|\cdot\|_p$ the supremum and L^p norm, respectively.

Theorem 2.2. *The family $\{(Y^\delta,Z^\delta)\mid \delta>0\}$ is tight in $C([0,T],\ \mathcal{M}_F(\overline{\mathbb{R}^2}))^2$. Let (Y,Z) be a limit point of the family (Y^δ,Z^δ) as $\delta\to 0$. Then it admits a random density field (y,z) and the following holds*

$$\begin{cases} \partial_t y(t,x) = \dfrac{1}{2}\Delta y(t,x) - \nabla\cdot\big(b(t,x,z(t,x),\nabla z(t,x))y(t,x)\big) \\ \qquad\qquad + \left(\alpha(z(t,x)) - u(t,x)\right)y(t,x), \\ dz(t,x) = \big(\Delta z(t,x) + \left[\mu(t,x) - \gamma(\cdot,Y(t,\cdot))\right]z(t,x)\big)dt + z(t,x)W(x,dt), \\ \qquad\qquad\qquad (t,x)\in[0,T]\times\mathbb{R}^2, \\ y(0,x) = y_0(x), \qquad z(0,x) = z_0(x), \qquad x\in\mathbb{R}^2. \end{cases} \tag{2.9}$$

Further, suppose that γ satisfies the following:

$$\|\gamma(\cdot,Y_1) - \gamma(\cdot,Y_2)\|_\infty \le K\|y_1-y_2\|_2, \qquad \forall y_1,y_2\in L^2(\mathbb{R}^2), \tag{2.10}$$

where Y_i is the measure with density y_i, $i=1,\ 2$. Then system (2.9) has a unique solution. Consequently, the family (Y^δ,Z^δ) has a limit (Y,Z) as $\delta\to 0$.

Remark 2.3. Note that if $\gamma(\cdot\,,\cdot)$ is of form (2.4) with $F(\cdot)$ being Lipschitz and $\theta(\cdot)\in L^1(\mathbb{R}^2)$, then

$$\begin{aligned} &\|\gamma(\cdot\,,Y_1(\cdot\,,\cdot)) - \gamma(\cdot\,,Y_2(\cdot\,,\cdot))\|_2^2 \\ &= \Big\|F\Big(\textstyle\int_{\mathbb{R}^2}\theta(x-x')y_1(x')dx'\Big) - F\Big(\int_{\mathbb{R}^2}\theta(x-x')y_2(x')dx'\Big)\Big\|_2^2 \\ &\le K\Big\|\textstyle\int_{\mathbb{R}^2}\theta(x-x')[y_1(x')-y_2(x')]dx'\Big\|_2^2 \\ &= K\textstyle\int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\theta(x-x')\left[y_1(x')-y_2(x')\right]dx'\Big|^2dx \\ &\le K\|\theta(\cdot)\|_1^2\|y_1(\cdot)-y_2(\cdot)\|_2^2, \end{aligned}$$

where the last inequality follows from Cauchy-Schwarz inequality. Thus, (2.10) holds in such a case.

3 Some Mathematical Preparations

In this section, we present some technical details needed later in our proofs of the two main theorems. Firstly, we introduce two spaces $\mathbb{H}_p^n((0,T)\times\mathbb{R}^2)$ and $\mathcal{H}_p^2((0,T)\times\mathbb{R}^2)$. Secondly, some properties of the Gaussian random measure W are introduced. Then, we consider a linear stochastic partial differential equation, and show this equation admits a unique solution in $\mathcal{H}_p^2((0,T)\times\mathbb{R}^2)$ under appropriate conditions. Finally, we define a Hilbert space $\mathbb{L}_\varepsilon^2$ and the discrete Laplacian operator Δ_ε.

Note that for any $\zeta \geq 1$ and $\alpha \in (0,1)$,

$$\int_0^\infty \frac{e^{-\zeta t}-1}{t^{1+\alpha}}\,dt = -\frac{\Gamma(1-\alpha)}{\alpha}\,\zeta^\alpha,$$

where $\Gamma(\cdot)$ is the standard Gamma function. Thus, for any $z \leq 0$, taking $\zeta = 1-z$ in the above, we obtain

$$(1-z)^\alpha = -\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{e^{-t}e^{zt}-1}{t^{\alpha+1}}\,dt. \tag{3.1}$$

Now, let

$$T_t\varphi(x) = \frac{1}{4\pi t}\int_{\mathbb{R}^2}\varphi(y)e^{-\frac{|x-y|^2}{4t}}\,dy, \qquad \forall \varphi(\cdot)\in C_0^\infty(\mathbb{R}^2), \tag{3.2}$$

which is the C_0-semigroup defined on $C_0^\infty(\mathbb{R}^2)$ with the generator Δ. Inspired by (3.1), for any $\alpha \in (0,1)$, we define

$$(1-\Delta)^\alpha\varphi(x) = -\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{e^{-t}T_t\varphi(x)-\varphi(x)}{t^{\alpha+1}}\,dt, \qquad \forall\varphi(\cdot)\in C_0^\infty(\mathbb{R}^2).$$

Then, for any $n \geq 0$, we introduce the following norm:

$$\|\varphi(\cdot)\|_{n,p} = \|(1-\Delta)^{\frac{n}{2}}\varphi(\cdot)\|_{L^p(\mathbb{R}^2)}, \qquad \forall\varphi(\cdot)\in C_0^\infty(\mathbb{R}^2).$$

Let $H_p^n(\mathbb{R}^2)$ be the completion of $C_0^\infty(\mathbb{R}^2)$ under the norm $\|\cdot\|_{n,p}$, which coincides with the usual Sobolev space $W^{n,p}(\mathbb{R}^2)$. We denote $\|\cdot\|_p = \|\cdot\|_{0,p}$. Clearly, $H_p^0(\mathbb{R}^2) = L^p(\mathbb{R}^2)$ is the usual L^p space. Next, we let

$$H_p^{1,2}((0,T)\times\mathbb{R}^2) = \Big\{\varphi : [0,T]\times\mathbb{R}^2 \to \mathbb{R} \;\Big|\; \int_0^T \big(\|\varphi_t(t,\cdot)\|_p^p + \|\varphi(t,\cdot)\|_{2,p}^p\big)dt < \infty\Big\}.$$

With the given complete filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, we let

$$\begin{aligned}&\mathbb{H}_p^n((0,T)\times\mathbb{R}^2) = L_{\mathbb{F}}^p(0,T;H_p^n(\mathbb{R}^2))\\ &\equiv \Big\{\varphi : [0,T]\to H_p^n(\mathbb{R}^2) \;\Big|\; t\mapsto\varphi(t) \text{ is } \mathbb{F}\text{-adapted}, \|\varphi\|_{\mathbb{H}_p^n} < \infty\Big\},\end{aligned}$$

where

$$\|\varphi\|_{\mathbb{H}_p^n} = \Big(\mathbb{E}\int_0^T \|\varphi(t)\|_{n,p}^p dt\Big)^{\frac{1}{p}}.$$

Denote by $\mathcal{H}_p^2((0,T)\times\mathbb{R}^2)$ the collection of all functions φ of the form

$$\varphi(t,x) = \varphi(0,x) + \int_0^t b(s,x)ds + \int_0^t \sum_{k\geq 1}\sigma^k(s,x)dB^k(s),$$

such that $\varphi,\ b,\ \sigma \in \mathbb{H}_p^n((0,T)\times\mathbb{R}^2)$ for $n = 0,1,2$, respectively, and

$$\|\varphi\|_{\mathcal{H}_p^2} \equiv \Big\{\mathbb{E}\int_0^T\int_{\mathbb{R}^2}\Big[|\varphi|+|\varphi_x|+|\varphi_{xx}|+|b|+\Big(\sum_{k=1}^\infty|\sigma^k|^2\Big)^{\frac{1}{2}}+\Big(\sum_{k=1}^\infty|\sigma_x^k|^2\Big)^{\frac{1}{2}}\Big]^p dxds\Big\}^{1/p} < \infty.$$

Note that the Gaussian random measure W can be represented by a sequence of independent Brownian motions as follows:

$$W(x,dt) \equiv \sum_{k=1}^{\infty} \phi^k(x) dB^k(t), \qquad x \in \mathbb{R}^2.$$

Also, we understand that for any predictable random fields $f_i : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, $i = 1,2$, satisfying

$$\mathbb{E} \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f_i(s,x) f_i(s,y) \rho(x,y) dx dy ds < \infty, \qquad i = 1,2,$$

we have

$$\int_0^t f_1(s,x) W(x,ds) = \sum_{k=1}^{\infty} \int_0^t f_1(s,x) \phi^k(x) dB^k(s),$$

and

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^t f_1(s,x)W(x,ds)\Big)\Big(\int_0^t f_2(s,x)W(x,ds)\Big)\Big] &= \mathbb{E}\sum_{k=1}^{\infty}\int_0^t f_1(s,x)f_2(s,x)\phi^k(x)^2 ds \\ &= \sum_{k=1}^{\infty}\phi^k(x)^2\mathbb{E}\int_0^t f_1(s,x)f_2(s,x)ds.\end{aligned}$$

Further, we have the Burkholder-Davis-Gundy's inequality: For any $p > 0$,

$$\begin{aligned}&\mathbb{E}\Big(\sup_{t\in[0,T]}\Big|\int_0^t \int_{\mathbb{R}^2}\sum_{k=1}^{\infty} f_1(s,x)\phi^k(x)dxdB^k(s)\Big|^p\Big) \\ &\le K\mathbb{E}\int_0^T \Big(\sum_{k=1}^{\infty}\Big|\int_{\mathbb{R}^2} f_1(s,x)\phi^k(x)dx\Big|^2\Big)^{\frac{p}{2}} ds \\ &\le K\Big(\sum_{k=1}^{\infty}\int_{\mathbb{R}^2}|\phi^k(x)|^2dx\Big)^{\frac{p}{2}}\mathbb{E}\int_0^T\Big(\int_{\mathbb{R}^2}|f_1(s,x)|^2dx\Big)^{\frac{p}{2}}ds.\end{aligned}$$

Throughout this paper, $K > 0$ will be a generic constant which can be different from line to line. Note that under (2.6), one has

$$\sum_{k=1}^{\infty}\big(|\phi^k(\cdot)| + |\phi^k_x(\cdot)|\big) \in \bigcap_{p\ge 1} L^p(\mathbb{R}^2).$$

Now, we consider the following linear stochastic partial differential equation:

$$\begin{cases} dz(t,x) = \big\{\Delta z(t,x) + f_0(t,x)z(t,x) + f(t,x)\big\}dt \\ \qquad\qquad + \displaystyle\sum_{k=1}^{\infty}\big[g_0^k(t,x)z(t,x) + g^k(t,x)\big]dB^k(t), \qquad (t,x) \in [0,T]\times\mathbb{R}^2, \\ z(0,x) = z_0(x), \qquad x \in \mathbb{R}^2. \end{cases} \tag{3.3}$$

We adopt the following definition (see [13]).

Definition 2.2. *A function $z(\cdot,\cdot) \in \mathcal{H}^2_p((0,T)\times\mathbb{R}^2)$ is called a solution to (3.3) if for all $\varphi(\cdot) \in C_0^\infty(\mathbb{R}^2)$,*

$$\int_{\mathbb{R}^2} z(t,x)\varphi(x)dx = \int_{\mathbb{R}^2} z_0(x)\varphi(x)dx$$

$$+\int_0^t\int_{\mathbb{R}^2}\Big(\big[\Delta z(s,x)+f_0(s,x)z(s,x)+f(s,x)\big]\varphi(x)dx$$
$$+\int_0^t\int_{\mathbb{R}^2}\sum_{k=1}^{\infty}\big[g_0^k(s,x)z(s,x)+g^k(s,x)\big]\varphi(x)dxdB^k(s).$$

The following result is a special case of Theorem 5.1 in [13] *(p.207).*

Proposition 2.3. *Suppose that for* $p\geq 2$,

$$\begin{cases}\mathbb{E}\Big\{\displaystyle\int_0^T\int_{\mathbb{R}^2}|f(t,x)|^p dxds+\int_0^T\Big[\int_{\mathbb{R}^2}\sum_{k=1}^{\infty}\Big(|g^k(t,x)|^2+|\nabla g^k(t,x)|^2\Big)^{\frac{p}{2}}dx\Big]ds\Big\}<\infty,\\ |f_0(t,x)|+\displaystyle\sum_{k=1}^{\infty}\Big(|g_0^k(t,x)|^2+|\nabla g_0^k(t,x)|^2\Big)\leq K,\qquad (t,x)\in[0,T]\times\mathbb{R}^2.\end{cases}\tag{3.4}$$

Then for any $z_0(\cdot)\in H_p^{2-2/p}(\mathbb{R}^2)$, (3.3) *admits a unique solution* $z(\cdot,\cdot)$ *in* $\mathcal{H}_p^2((0,T)\times\mathbb{R}^2)$, *and the following estimate holds:*

$$\begin{aligned}\|z\|_{\mathcal{H}_p^2}\leq K\Big\{\|z_0\|_{2-2/p,p}+\Big(\mathbb{E}\int_0^T\int_{\mathbb{R}^2}|f(t,x)|^p dxdt\Big)^{\frac{1}{p}}\\ +\Big[\mathbb{E}\int_0^T\int_{\mathbb{R}^2}\Big(\sum_{k=1}^{\infty}|g^k(t,x)|^2+|\nabla g^k(t,x)|^2\Big)^{\frac{p}{2}}dxds\Big]^{\frac{1}{p}}\Big\}.\end{aligned}\tag{3.5}$$

Let

$$\mathbb{L}_\varepsilon^2=\{\varphi:\varepsilon\mathbb{Z}^2\to\mathbb{R}\ \big|\ \|\varphi\|_{\mathbb{L}_\varepsilon^2}^2=\sum_{x_i\in\varepsilon\mathbb{Z}^2}|\varphi(x_i)|^2<\infty\},$$

which is a Hilbert space. The definition of $\mathbb{L}_\varepsilon^\infty$ is similar. Define

$$\Delta_\varepsilon\varphi(x_i)=\varepsilon^{-2}\sum_{j\sim i}\big(\varphi(x_j)-\varphi(x_i)\big)$$

which is called the *discrete Laplacian operator*, recalling that the notation $j\sim i$ means that j is a neighbor of i.

Lemma 3.1. $-\Delta_\varepsilon$ *is a positive definite self-adjoint operator on* $\mathbb{L}_\varepsilon^2$ *and for any* $h>0$,

$$T_j^\varepsilon=(I-h\Delta_\varepsilon)^{-j},\qquad j\geq 0$$

is a discrete-time contraction semigroup on $\mathbb{L}_\varepsilon^2$.

Proof. It is easy to verify that for any $\varphi\in\mathbb{L}_\varepsilon^2$,

$$\langle\Delta_\varepsilon\varphi,\gamma\rangle_{\mathbb{L}_\varepsilon^2}=-\frac{1}{2}\varepsilon^{-2}\sum_{j\sim i}\left(\varphi(x_j)-\varphi(x_i)\right)\left(\gamma(x_j)-\gamma(x_i)\right).$$

Hence, $-\Delta_\varepsilon$ is a positive definite self-adjoint operator on $\mathbb{L}_\varepsilon^2$. The semigroup property and the contraction of $\{T_j^\varepsilon\}$ are straightforward. □

4 Tightness for Smoothed System (δ fixed and $n \to \infty$)

In this section, we will prove the tightness of $(Y^{n,\delta}, Z^{n,\delta})$ in path space $D([0,T], \mathcal{M}_F(\overline{\mathbb{R}^2}))^2$, as $n \to \infty$, and a limit (Y^δ, Z^δ) has a smooth density (y^δ, z^δ). For the simplicity of notation, we drop the parameter $\delta > 0$ below when there is no confusion by doing so. Let $t = jh$. Then, equation of z^n in (2.3) becomes

$$\begin{aligned}(I - h\Delta_\varepsilon)z^n((j+1)h) &= (1 + h\mu(jh))\, z^n(jh) - \gamma(\cdot, Y^n(jh))z^n(jh)h + \sqrt{h}\xi^n(jh)z^n(jh)\\ &= z^n(jh) + \big[h\mu(jh) - \gamma(\cdot\,, Y^n(jh))h + \sqrt{h}\xi^n(jh)\big]\, z^n(jh).\end{aligned}$$

Hence,

$$\begin{aligned}z^n(jh) &= (I - h\Delta_\varepsilon)^{-j} z^n(0) + \sum_{k=0}^{j-1} h(I - h\Delta_\varepsilon)^{k-j}\,(\mu(kh)z^n(kh))\\ &\quad -\sum_{k=0}^{j-1} h(I - h\Delta_\varepsilon)^{k-j}\gamma(\cdot, Y^n(kh))z^n(kh)\\ &\quad +\sqrt{h}\sum_{k=0}^{j-1}(I - h\Delta_\varepsilon)^{k-j}\,(\xi^n(kh)z^n(kh))\\ &\equiv I_j^1 + I_j^2 + I_j^3 + I_j^4.\end{aligned} \tag{4.1}$$

We have the following lemma.

Lemma 4.1. *Suppose the following holds:*

$$\sup_{n\in\mathbb{N}, \varepsilon>0} \|z^n(0)\|^2_{\mathbb{L}^2_\varepsilon} + \sup_{k\in\mathbb{N}, \varepsilon>0} \|\mu(kh)\|^2_{\mathbb{L}^2_\varepsilon} \le K < \infty. \tag{4.2}$$

Then

$$\sup_{n\in\mathbb{N},\, t\le T,\, \varepsilon>0} \mathbb{E}\|z^n(t)\|^2_{\mathbb{L}^2_\varepsilon} < \infty. \tag{4.3}$$

Proof. Under condition (4.2), it is easy to show that (since $T_j^\varepsilon \equiv (I - h\Delta_\varepsilon)^{-j}$ is a contraction semi-group on $\mathbb{L}^2_\varepsilon$)

$$\|I_j^1\|^2_{\mathbb{L}^2_\varepsilon} = \|T_j^\varepsilon z^n(0)\|_{\mathbb{L}^2_\varepsilon} \le \|z^n(0)\|_{\mathbb{L}^2_\varepsilon} \le K,$$

and

$$\begin{aligned}\|I_j^2\|_{\mathbb{L}^2_\varepsilon} + \|I_j^3\|_{\mathbb{L}^2_\varepsilon} &\le h\sum_{k=0}^{j-1} \|T^\varepsilon_{k-j}\big[\mu(kh) + \gamma(\cdot\,, Y^n(kh))\big] z^n(kh)\|_{\mathbb{L}^2_\varepsilon}\\ &\le h\sum_{k=0}^{j-1}\big(\|\mu(kh)\|_{\mathbb{L}^\infty_\varepsilon} + \|\gamma(\cdot\,, Y^n(kh))\|_{\mathbb{L}^\infty_\varepsilon}\big)\|z^n(kh)\|_{\mathbb{L}^2_\varepsilon} \le K_1 h\sum_{k=0}^{j-1}\|z^n(kh)\|_{\mathbb{L}^2_\varepsilon},\end{aligned}$$

where K_1 is a constant depending on K only, recalling that $\gamma \in (0,1)$ is the proportion of grass consumed by rabbits. To estimate I_j^4, we note that

$$(I - h\Delta_\varepsilon)^{-j} f(x_i) = \sum_\ell p^{(j)}(x_i - x_\ell) f(x_\ell),$$

where $p^{(j)}$ is the j-step transition function of the discrete Markov chain on $\varepsilon\mathbb{Z}^2$ with semigroup $\{(I-h\Delta_\varepsilon)^{-j}:\ j\geq 0\}$. Then,

$$I_j^4(x_i)=\sqrt{h}\sum_{k=0}^{j-1}\sum_{\ell}p^{(j-k)}(x_i-x_\ell)z^n(kh,x_\ell)\xi^n(kh,x_\ell),\quad x_i\in\varepsilon\mathbb{Z}^2.$$

Thus, for any $z\in\mathbb{L}_\varepsilon^2$, we have

$$\left\langle I_j^4,z\right\rangle_{\mathbb{L}_\varepsilon^2}=\sqrt{h}\sum_{k=0}^{j-1}\sum_{\ell}\left\langle p^{(j-k)}(\cdot-x_\ell),z\right\rangle_{L_\varepsilon^2}z^n(kh,x_\ell)\xi^n(kh,x_\ell).$$

The right hand side of the above can be regarded as a stochastic integral on discrete space $\{0,1,\cdots,j\}\times\varepsilon\mathbb{Z}^2$ of the process

$$k\mapsto\sum_i p^{(j-k)}(x_i-\cdot)z(x_i)z^n(kh,\cdot)$$

with respect to the Gaussian random measure determined by the process $k\mapsto\xi^n(kh,\cdot)$. Then,

$$\begin{aligned}\mathbb{E}\left\langle I_j^4,z\right\rangle_{\mathbb{L}_\varepsilon^2}^2=h\mathbb{E}\sum_{k=0}^{j-1}\sum_{\ell_1,\ell_2}&\left\langle p^{(j-k)}(\cdot-x_{\ell_1}),z\right\rangle_{\mathbb{L}_\varepsilon^2}z^n(kh,x_{\ell_1})\\&\times\left\langle p^{(j-k)}(\cdot-x_{\ell_2}),z\right\rangle_{\mathbb{L}_\varepsilon^2}z^n(kh,x_{\ell_2})\rho(x_{\ell_1},x_{\ell_2}).\end{aligned}$$

Summing on z over a complete orthonormal system of $\mathbb{L}_\varepsilon^2$, we get

$$\begin{aligned}\mathbb{E}\|I_j^4\|_{\mathbb{L}_\varepsilon^2}^2&=h\varepsilon^2\mathbb{E}\sum_{k=0}^{j-1}\sum_{\ell_1,\ell_2}p^{(2(j-k))}(x_{\ell_1}-x_{\ell_2})z^n(kh,x_{\ell_1})z^n(kh,x_{\ell_2})\rho(x_{\ell_1},x_{\ell_2})\\&\leq Kh\|\rho\|_\infty\varepsilon^2\sum_{k=0}^{j-1}\mathbb{E}\sum_{\ell_1,\ell_2}p^{(2(j-k))}(x_{\ell_1}-x_{\ell_2})z^n(kh,x_{\ell_1})^2\\&=K_1h\|\rho\|_\infty\sum_{k=0}^{j-1}\mathbb{E}\|z^n(kh)\|_{\mathbb{L}_\varepsilon^2}^2,\end{aligned}$$

where the inequality follows from the property of the transition probability and the boundedness of the covariance function ρ. The conclusion then follows from (4.1) and a discrete version of the Gronwall inequality (cf. Xiong [29]). □

To prove the tightness of Z^n in $D([0,T],\mathcal{M}_F(\overline{\mathbb{R}^2}))$, we need the following uniform estimate.

Lemma 4.2. *For any $f\in C_0^\infty(\mathbb{R}^2)$, we have*

$$\sup_n\left\{\mathbb{E}\left[\sup_{t\leq T}|\left\langle Z^n(t),f\right\rangle|^2\right]\right\}<\infty.$$

Proof. We note that

$$
\begin{aligned}
&\langle Z^n(t+h), f\rangle = \sum_i z^n(t+h,x_i) f(x_i)\varepsilon^2 \\
&= \langle Z^n(t), f\rangle + \varepsilon^2 h \sum_i \mu(t,x_i) z^n(t,x_i) f(x_i) + \varepsilon^2\sqrt{h}\sum_i \xi^n(t,x_i) z^n(t,x_i) f(x_i) \\
&\quad - h\,\langle Z^n(t), \gamma(\cdot, Y^n(t)) f\rangle + h\sum_i \sum_{j\sim i} (f(x_j)-f(x_i)) z^n(t+h,x_i) \\
&= \langle Z^n(t), f\rangle + h\,\langle Z^n(t), \mu(t) f\rangle - h\,\langle Z^n(t), \gamma(\cdot, Y^n(t)) f\rangle \\
&\quad + h\,\langle Z^n(t+h), \Delta_\varepsilon f\rangle + \varepsilon^2\sqrt{h}\sum_i \xi^n(t,x_i) z^n(t,x_i) f(x_i).
\end{aligned}
$$

Summing up with respect to $t = 0, h, 2h, \cdots, (k-1)h$, we have

$$
\begin{aligned}
\langle Z^n(t), f\rangle &= \langle Z^n(0), f\rangle + M^{n,f}(t) \\
&\quad + \sum_{i=0}^{k-1} \left(\langle Z^n(ih), \mu(ih) f - \gamma(\cdot, Y^n(ih)) f\rangle + \langle Z^n((i+1)h), \Delta_\varepsilon f\rangle\right) h \\
&\equiv \langle Z^n(0), f\rangle + M^{n,f}(t) + A^{n,f}(t),
\end{aligned} \tag{4.4}
$$

where

$$
A^{n,f}(t) = \sum_{i=0}^{k-1} \left(\langle Z^n(ih), \mu(ih) f - \gamma(\cdot, Y^n(ih)) f\rangle + \langle Z^n((i+1)h), \Delta_\varepsilon f\rangle\right) h, \tag{4.5}
$$

and

$$
M^{n,f}(t) = \varepsilon^2\sqrt{h}\sum_{i=0}^{k-1}\sum_j \xi^n(ih,x_j) z^n(ih,x_j) f(x_j),
$$

which is a martingale with conditional quadratic variation

$$
\begin{aligned}
\langle\, M^{n,f}\rangle(t) &= \varepsilon^4 h \sum_{i=0}^{k-1}\sum_{j_1,j_2} \rho(x_{j_1}, x_{j_2}) z^n(ih, x_{j_1}) z^n(ih, x_{j_2}) f(x_{j_1}) f(x_{j_2}) \\
&= h\sum_{i=0}^{k-1}\sum_j \left\langle Z^n(ih), \phi^j f\right\rangle^2 .
\end{aligned} \tag{4.6}
$$

Applying Hölder's inequality to $A^{n,f}$ and Doob's inequality to $M^{n,f}$, we then get

$$
\mathbb{E}\sup_{s\le t} \|\,\langle Z^n(s), f\rangle\,\|^2 \le 3\,\langle Z^n(0), f\rangle^2 + K_1(f) h \sum_{i=0}^{k-1} \mathbb{E}\|z^n(ih)\|_2^2 + K_2 \mathbb{E}\,\langle\, M^{n,f}\,\rangle(t) \le K_3 < \infty.
$$

Then our conclusion follows. □

We need the following definition and the theorem about real-valued stochastic processes taken from the book of Jacod and Shiryaev ([8], p.317, Corollary 3.33 and p.322, Theorem 4.13).

Definition 4.3. *A sequence of probability measures $\{\lambda_n\}$ on $D([0,T],\mathbb{R})$ is C-tight if it is tight and all cluster points are supported on $C([0,T],\mathbb{R})$.*

Theorem 4.4. *For each n, let λ_n be a probability measure on $D([0,T],\mathbb{R})$ induced by a real-valued semimartingale $\xi^n(0)+M^n(t)+A^n(t)$ on a stochastic basis $(\Omega_n,\mathcal{F}^n,P^n,(\mathcal{F}^n_t))$, where $\xi^n(0)$ is an $\mathcal{F}^n_0$-measurable random variable, M^n is a martingale and A^n is of finite variation. If $\{\xi^n(0)\}$ is tight in $\mathbb{R}$, $\{\langle M^n\rangle\}$ and $\{A^n\}$ are C-tight, then $\{\lambda_n\}$ is tight.*

Lemma 4.5. *For each $f\in C_0^\infty(\mathbb{R}^2)$, the sequence $\{A^{n,f}\}$ is C-tight in $D([0,T],\mathbb{R})$.*

Proof. Note that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\le jh<kh\le T}\frac{|A^{n,f}(kh)-A^{n,f}(jh)|}{(k-j)h}\Big]\\&\le\mathbb{E}\Big(\sup_{s\in[0,T]}|\langle Z^n(s),\mu(s)f+\Delta_\varepsilon f-\gamma(\cdot,Y^n(s))f\rangle|\Big)\le K_1(f)<\infty.\end{aligned}$$

The C-tightness then follows easily. □

Lemma 4.6. *Z^n is tight in $D([0,T],\mathcal{M}_F(\overline{\mathbb{R}^2}))$.*

Proof. Similar to last lemma, we can prove the C-tightness of $\langle M^{n,f}\rangle$. By the last theorem, we get the tightness of $\{\langle Z^n,f\rangle\}$. By Jakubowski's criterion (see [9], or Theorem 3.6.4 of [2]), it is then well-known that Z^n is tight in $D([0,T],\mathcal{M}_F(\overline{\mathbb{R}^2}))$. □

Next, we consider the tightness of Y^n. Note that

$$\langle Y^n(t+h),f\rangle=\frac{1}{n}\sum_{k\sim t}f(X^n_k(t+h))(1+1_{\tau_k<t+h})-\langle Y^n(t),u^n(t)f\rangle h,\tag{4.7}$$

where $k\sim t$ means that the rabbit k is alive at time t before harvest occurs.

Lemma 4.7. *The estimate*

$$\sup_{n\ge0,\ t\le T}\mathbb{E}\left\langle Y^n(t),1\right\rangle^2<\infty$$

holds.

Proof. Denote $m(k)=\langle Y^n(kh),1\rangle$. Then,

$$\begin{aligned}m(i+1)&\le m(i)+\frac{1}{n}\sum_{k\sim ih}(1_{\tau_k<(i+1)h}-\alpha(\zeta^n(ih))h)+\frac{1}{n}\sum_{k\sim ih}\alpha(\zeta^n(ih))h\\&\equiv m(i)+J_1(i)+J_2(i),\end{aligned}$$

where the harvested rabbits are deleted in the first inequality above (and hence, an inequality). Note that $\mathbb{E}J_1(i)^2\le\frac{Kh}{n}\mathbb{E}m(i)$ and $J_2(i)\le Km(i)h$. Thus,

$$m(i+1)-(1+Kh)m(i)\le J_1(i).\tag{4.8}$$

By iteration, we get

$$m(i)\le(1+Kh)^im(0)+\sum_{k=0}^{i-1}J_1(k)(1+Kh)^{i-1-k}.$$

Then,

$$\mathbb{E}m(i)\le(1+Kh)^im(0)\equiv K_1,$$

and

$$\mathbb{E}m(i)^2 \le 2K_1^2 + \sum_{k=0}^{i-1} \frac{Kh}{n} \mathbb{E}m(k)(1+Kh)^{2(i-1-k)} \le 2K_1^2 + K_2 n^{-1}.$$

This concludes the proof of the lemma. □

Now, we are ready to present

Proof of Theorem 2.1. For any $f \in C_0^2(\mathbb{R}^2)$, it is not difficult to show that

$$\begin{aligned}\langle Y^n(t+h), f\rangle = \langle Y^n(t), f\rangle &+ \int_t^{t+h} \left\langle Y^n(s), L_{s,\zeta^n(s),\nabla\zeta^n(s)} f\right\rangle ds \\ &+ \frac{1}{n}\sum_{k\sim t} \int_t^{t+h} b^T \nabla f(X_k^n(s)) dB_k(s) \\ &+ I^n(t) + \langle Y^n(t), (\alpha(\zeta^n(t)) - u^n(t))\, f\rangle\, h,\end{aligned}$$

where

$$I^n(t) = \frac{1}{n}\sum_i \sum_{k\sim t} \left(f(x_i) - f(X_k^n(t+h))\right) I_{Q_\varepsilon(x_i)}(X_k^n(t+h)).$$

Since f is Lipschitz continuous,

$$|I^n(t)| \le K\epsilon \langle Y^n(t), 1\rangle .$$

Note that

$$\begin{aligned}\mathbb{E}\sup_{t\le T}\Big|\frac{1}{n}\sum_k \int_0^t b^T \nabla f((X_k^n(s)) I_{k\sim s} dB_k(s)\Big|^2 &\le \frac{4}{n^2}\sum_k \int_0^t |b^T \nabla f((X_k^n(s))|^2 I_{k\sim s} ds \\ &= \frac{4}{n}\int_0^t \left\langle Y^n(s), (b^T\nabla f)^2\right\rangle ds \to 0,\end{aligned}$$

and

$$\sum_{j=0}^{[T/h]} |I^n(jh)| \le K\varepsilon h^{-1} \int_0^T \langle Y^n(t), 1\rangle\, dt \to 0.$$

Similar to the case of Z^n, we get the tightness of Y^n.

Let (Y^δ, Z^δ) be a limit point. Then, Y^δ solves the following PDE

$$\frac{d}{dt}\left\langle Y^\delta(t), f\right\rangle = \left\langle Y^\delta(t), L_{t,\zeta^\delta(t),\nabla\zeta^\delta(t)} f\right\rangle + \left\langle Y^\delta(t), \left(\alpha(\zeta^\delta(t)) - u(t)\right) f\right\rangle .$$

Finally, we pass to limit on (4.4)

$$\langle\, Z^\delta(t), f\,\rangle = \langle Z(0), f\rangle + M^{\delta,f}(t) + A^{\delta,f}(t)$$

where

$$A^{\delta,f}(t) = \int_0^t \left(\langle\, Z^\delta(s), \mu(s) f + \Delta f - \gamma(\cdot\,, Y^\delta(s)) f\,\rangle\right) ds$$

and $M^{\delta,f}$ is a martingale with conditional quadratic variation

$$\langle\, M^{\delta,f}\,\rangle(t) = \sum_{i=0}^\infty \int_0^t \langle\, Z^\delta(s), f\phi_i\,\rangle^2 ds. \tag{4.9}$$

By martingale representation, we see that

$$M^{\delta,f}(t)=\sum_{i=0}^{\infty}\int_0^t \langle\, Z^\delta(s), f\phi_i\,\rangle\, dB^j(s)=\int_0^t\int_{\mathbb{R}}\langle\, Z^\delta(s), fW(\cdot\,,ds)\,\rangle\,.$$

Note that when the sum in (4.9) contains only finite many terms, the martingale representation is standard. In the current case, it can be obtained by extending the finite dimensional case using the same proof, or by taking limit from the finite dimensional result. We omit the detail here. This completes the proof. □

The existence of density y^δ for Y^δ follows from convolution. In fact, y^δ can be represented in the following convolution form:

$$\begin{aligned} y^\delta(t,x) &= \int_{\mathbb{R}^2} y^\delta(0,r)p_t(x-x')dx' \\ &\quad+\int_0^t\int_{\mathbb{R}^2}\big(\alpha(\zeta^\delta(s,x'))-u(s,x')\big)\,p_{t-s}(x-x')Y^\delta(s,dx')ds \\ &\quad+\int_0^t\int_{\mathbb{R}^2}\nabla p_{t-s}(x-x')b(s,x',\zeta^\delta(s,x'),\nabla\zeta^\delta(s,x'))Y^\delta(s,dx')ds. \end{aligned}\tag{4.10}$$

The existence of the density of z^δ follows from Lemma 4.1 and Fatou's lemma. The equations for (y^δ, z^δ) can be written as

$$\left\{\begin{aligned} \partial_t y^\delta(t,x) &= \tfrac{1}{2}\Delta y^\delta(t,x) - \nabla\cdot\big(b(t,x,\zeta^\delta(t,x),\nabla\zeta^\delta(t,x))y^\delta(t,x)\big) \\ &\quad+\big(\alpha(\zeta^\delta(t,x))-u(t,x)\big)\,y^\delta(t,x), \\ dz^\delta(t,x) &= \big[\Delta z^\delta(t,x)+\big(\mu(t)-\gamma(\cdot\,,Y^\delta(t))\big)z^\delta(t,x)\big]dt + z^\delta(t,x)W(x,dt). \end{aligned}\right.\tag{4.11}$$

5 Limiting System as $\delta\to 0$

In this section, we study the tightness of (y^δ, z^δ). Namely, this section is devoted to the proof of Theorem 2.2.

Note that

$$d\,\langle\, Y^\delta(t), f\,\rangle=\Big(\langle\, Y^\delta(t),\frac{1}{2}\Delta f+b(t,\cdot,\zeta^\delta(t),\nabla\zeta^\delta(t))\nabla f\,\rangle+\langle\, Y^\delta(t),\big(\alpha(\zeta^\delta(t))-u(t)\big)\,f\,\rangle\Big)dt.$$

Similar to Lemma 4.5, it is easy to get the tightness of $Y^\delta(t)$ in path space $C([0,T],\mathcal{M}_F(\overline{\mathbb{R}^2}))$. Without loss of generality, we may and will assume that Y^δ convergence to Y a.s. It then follows from Theorem 5.7 of Krylov [13] with $n=0$ that z^δ converges to z a.s. in $\mathcal{H}^2_p(\mathbb{R}^2)$ as $\delta\to 0$, where z is a solution to the second equation in (2.9). Finally, we come back to y^δ. Using the convolution representation (4.10), we get a.s.,

$$\begin{aligned} y^\delta(t,x) &\to \int_{\mathbb{R}^2} y_0(x')p_t(x-x')dx' + \int_0^t\int_{\mathbb{R}^2}\left(\alpha(z(s,x'))-u(s,x')\right)p_{t-s}(x-x')Y(s,dx')ds \\ &\quad+\int_0^t\int_{\mathbb{R}^2}\nabla p_{t-s}(x-x')b(s,x',z(s,x'),\nabla z(s,x'))Y(s,dx')ds \\ &\equiv y(t,x). \end{aligned}$$

Thus, $y(\cdot\,,\cdot)$ is a solution to the first equation in (2.9).

Finally, we prove the uniqueness of the solution. Let (y^1, z^1) and (y^2, z^2) be two solutions. Denote $\bar{y} = y^1 - y^2$, and $\bar{z} = z^1 - z^2$. Then,

$$\begin{aligned}\bar{y}(t,x) = &\int_0^t\int_{\mathbb{R}^2} \nabla p_{t-s}(x-x') \\ &\times\big(b(s,x',z^1(s,x'),\nabla z^1(s,x')) - b(s,x',z^2(s,x'),\nabla z^2(s,x'))\big)y^1(s,x')dx'ds \\ &+\int_0^t\int_{\mathbb{R}^2} \nabla p_{t-s}(x-x')b(s,x',z^2(s,x'),\nabla z^2(s,x'))\bar{y}(s,x')dx'ds \\ &+\int_0^t\int_{\mathbb{R}^2} \left(\alpha(z^1(s,x')) - \alpha(z^2(s,x'))\right) p_{t-s}(x-x')\bar{y}(s,x')dx'ds.\end{aligned}$$

Note that

$$\partial_{x_1} p_{t-s}(x-x') \le K(t-s)^{-1/2} q_{t-s}(x-x') \tag{5.1}$$

where $q_t(x) = p_{2t}(x)$. So,

$$\begin{aligned}|\bar{y}(t,x)| \le\, &K_1\int_0^t\int_{\mathbb{R}^2} (t-s)^{-1/2} q_{t-s}(x-x')|\nabla\bar{z}(s,x')||y^1(s,x')|dx'ds \\ &+K_2\int_0^t\int_{\mathbb{R}^2} (t-s)^{-1/2} q_{t-s}(x-x')|\bar{y}(s,x')|dx'ds \\ &+K_3\int_0^t\int_{\mathbb{R}^2} p_{t-s}(x-x')|\bar{z}(s,x')|\bar{y}(s,x')dx'ds.\end{aligned}$$

As $y^1(\cdot,\cdot)$ is bounded, we have,

$$\begin{aligned}|\bar{y}(t,x)|^2 \le\, &K_4\int_0^t\int_{\mathbb{R}^2} (t-s)^{-1/2} q_{t-s}(x-x')|\nabla\bar{z}(s,x')|^2 dx'ds \\ &+K_5\int_0^t\int_{\mathbb{R}^2} (t-s)^{-1/2} q_{t-s}(x-x')|\bar{y}(s,x')|^2 dx'ds \\ &+K_6\int_0^t\int_{\mathbb{R}^2} p_{t-s}(x-x')|\bar{z}(s,x')|^2\bar{y}(s,x')dx'ds.\end{aligned}$$

Note that in the above, we have used the fact that

$$\int_0^t\int_{\mathbb{R}^2} (t-s)^{-1/2} q_{t-s}(x-x')dx'ds = \int_0^t (t-s)^{-1/2} ds = \sqrt{t} \le \sqrt{T}.$$

Therefore,

$$\mathbb{E}\|\bar{y}(t)\|_2^2 \le K_7\int_0^t (t-s)^{-1/2}\mathbb{E}\|\bar{z}(s)\|_{1,2}^2 ds + K_5\int_0^t (t-s)^{-1/2}\mathbb{E}\|\bar{y}(s)\|_2^2 ds. \tag{5.2}$$

Now, we estimate the partial derivative $\partial_{x_1}\bar{z}(s,x)$ with respect to x_1. Using the convolution form, we get for $i = 1, 2$,

$$\begin{aligned}z^i(t,x) = T_t z_0(x) &+ \int_0^t\int_{\mathbb{R}^2} p_{t-s}(x-x')\gamma(x',Y^i(s,\cdot))z^i(s,x')dx'ds \\ &+\int_0^t\int_{\mathbb{R}^2} p_{t-s}(x-x')z^i(s,x')W(x',ds)dx'.\end{aligned}$$

So,

$$\bar{z}(t,x) = \int_0^t\int_{\mathbb{R}^2} p_{t-s}(x-x')\Big(\gamma(x',Y^1(s,\cdot))z^1(s,x') - \gamma(x',Y^2(s,\cdot))z^2(s,x')\Big)dx'ds$$
$$+ \int_0^t\int_{\mathbb{R}^2} p_{t-s}(x-x')\bar{z}(s,x')W(x',ds)dx'.$$

Then,

$$\|\partial_{x_1}\bar{z}(t)\|_2^2 = I_1(t) + I_2(t)$$

where

$$I_1(t) = \Big\|\int_0^t\int_{\mathbb{R}^2}\partial_{x_1}p_{t-s}(x-x')\widehat{I}(s,x')dx'ds\Big\|_2^2,$$

$$I_2(t) = \Big\|\int_0^t\int_{\mathbb{R}^2}\partial_{x_1}p_{t-s}(x-x')\bar{z}(s,x')W(x',ds)dx'\Big\|_2^2,$$

with

$$\widehat{I}(s,x') = \gamma(x',Y^1(s,\cdot))z^1(s,x') - \gamma(x',Y^2(s,\cdot))z^2(s,x').$$

The term $I_1(t)$ can be dealt with using argument similar to that for $\bar{y}$ so we focus on $I_2(t)$. Indeed,

$$I_2(t) = \Big\langle\int_0^t\int_{\mathbb{R}^2}\partial_{x_1}p_{t-s}(x-x')\bar{z}(s,x')W(x',ds)dx',$$
$$\int_0^t\int_{\mathbb{R}^2}p_{t-s}(x-x')\partial_{x_1}\bar{z}(s,x')W(x',ds)dx'\Big\rangle_2.$$

We write

$$I_2(t) = I_{21}(t) + I_{22}(t),$$

where

$$I_{21}(t) = \mathbb{E}\Big\langle\sum_{j=1}^\infty\int_0^t\int_{\mathbb{R}^2}\partial_{x_1}p_{t-s}(x-x')\bar{z}(s,x')\phi_j(x')dx'dB^j(s),$$
$$\sum_{j=1}^\infty\int_0^t\int_{\mathbb{R}^2}p_{t-s}(x-x')\partial_{x_1}\bar{z}(s,x')\phi_j(x')dx'dB^j(s)\Big\rangle_2,$$

and

$$I_{22}(t) = \mathbb{E}\Big\langle\sum_{j=1}^\infty\int_0^t\int_{\mathbb{R}^2}\partial_{x_1}p_{t-s}(x-x')\bar{z}(s,x')\phi_j(x')dx'dB^j(s),$$
$$\sum_{j=1}^\infty\int_0^t\int_{\mathbb{R}^2}p_{t-s}(x-x')\bar{z}(s,x')\partial_{x_1}\phi_j(x')dx'dB^j(s)\Big\rangle_2.$$

The term $I_{22}(t)$ can be dealt with similarly as that for $\bar{Y}$ so we focus on $I_{21}(t)$.

Making use of (5.1), we have

$$
\begin{aligned}
I_{21}(t) \le\, & K\mathbb{E}\sum_{j=1}^{\infty}\int_{\mathbb{R}^2} dx \int_0^t (t-s)^{-1/2}\int_{\mathbb{R}^2} q_{t-s}(x-x')\bar{z}(s,x')\phi_j(x')dx' \\
& \times \int_{\mathbb{R}^2} p_{t-s}(x-x')\partial_{x_1}\bar{z}(s,x')\phi_j(x')dx'ds \\
\le\, & K\int_0^t (t-s)^{-1/2}\int_{\mathbb{R}^2}\Big(\sum_{j=1}^{\infty}\int_{\mathbb{R}^2} q_{t-s}(x-x')\bar{z}(s,x')^2\phi_j(x')^2dx'\Big)^{1/2} \\
& \cdot\Big(\sum_{j=1}^{\infty}\int_{\mathbb{R}^2} p_{t-s}(x-x')\left|\partial_{x_1}\bar{z}(s,x')\phi_j(x')\right|^2dx'\Big)^{1/2}dxds \\
\le\, & K\int_0^t (t-s)^{-1/2}\Big(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} q_{t-s}(x-x')\bar{z}(s,x')^2\rho(x',x')dx'dx\Big)^{1/2} \\
& \cdot\Big(\sum_{j=1}^{\infty}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} p_{t-s}(x-x')\left|\partial_{x_1}\bar{z}(s,x')\phi_j(x')\right|^2dx'dx\Big)^{1/2}ds \\
\le\, & K\int_0^t (t-s)^{-1/2}\|\bar{z}(s)\|_2\Big(\sum_{j=1}^{\infty}\|\bar{z}(s)\phi_j|_{1,2}^2\Big)^{1/2}ds \\
\le\, & K\int_0^t (t-s)^{-1/2}\mathbb{E}\|\bar{z}(s)\|_2\|\bar{z}(s)\|_{1,2}ds \\
\le\, & K\int_0^t (t-s)^{-1/2}\mathbb{E}\|\bar{z}(s)\|_{1,2}^2ds.
\end{aligned}
$$

To summarize, we get

$$
\mathbb{E}\|\bar{z}(t)\|_{1,2}^2 \le K\int_0^t (t-s)^{-1/2}\left(\mathbb{E}\|\bar{y}(s)\|_2^2 + \mathbb{E}\|\bar{z}(s)\|_{1,2}^2\right)ds.
$$

Combining with (5.2), we obtain

$$
\mathbb{E}\left(\|\bar{y}(t)\|_2^2 + \|\bar{z}(t)\|_{1,2}^2\right) \le K\int_0^t (t-s)^{-1/2}\left(\mathbb{E}\|\bar{y}(s)\|_2^2 + \mathbb{E}\|\bar{z}(s)\|_{1,2}^2\right)ds.
$$

Iterating this inequality once, we get

$$
\mathbb{E}\left(\|\bar{y}(t)\|_2^2 + \|\bar{z}(t)\|_{1,2}^2\right) \le K\int_0^t \mathbb{E}\left(\|\bar{y}(s)\|_2^2 + \|\bar{z}(s)\|_{1,2}^2\right)ds.
$$

Gronwall's inequality implies $(y^1, z^1) = (y^2, z^2)$. □

6 Numerical Example

In order to give an intuitive explanation for our model (2.9), we present a specific numerical example in this section. Firstly, we restrict the space variable $x = (x_1, x_2)$ to a limited domain

$\mathbb{R}_x = [0,1] \times [0,1]$. Secondly, we choose the appropriate functions

$$
\begin{aligned}
&b(t,x,z(t,x),\nabla z(t,x)) = \begin{bmatrix} b_1 z(t,x) + b_2 z_{x_1}(t,x) \\ b_3 z(t,x) + b_4 z_{x_2}(t,x) \end{bmatrix}, \qquad b_1,b_2,b_3,b_4 \in \mathbb{R}, \\
&\alpha(z(t,x)) = \alpha_1 z(t,x), \qquad u(t,x) = u_1 \frac{y(t,x)}{y(t,x)+z(t,x)}, \\
&\mu(t,x) = \mu_1, \qquad \gamma(\cdot, Y(t,\cdot)) = \gamma_1 \int_{\mathbb{R}_x} \theta(x-x')y(t,x')dx', \qquad (6.1)\\
&\theta(x-x') = \frac{1}{50(x_1-x_1')^2 + 50(x_2-x_2')^2 + 1}, \qquad \alpha_1,u_1,\mu_1,\gamma_1 \in (0,1), \\
&W(x,dt) = \sum_{k=1}^{m} \Phi_k(x) dB_k(x), \qquad \Phi_k(x) = \phi_1 \frac{\sqrt{x_1^2+x_2^2}}{k^2}, \quad \phi_1 \in \mathbb{R}.
\end{aligned}
$$

Taking the coefficients $b_1 = b_3 = 0.4$, $b_2 = b_4 = 0.6$, $\alpha_1 = 0.2$, $u_1 = 0.8$, $\mu_1 = 0.3$, $\gamma_1 = 0.2$, $\phi_1 = 0.2$, and giving initial value $y(0,x) = 10$, $z(0,x) = 4$, we can obtain Figures 6.1-6.3. Note that choosing the b_2 and b_4 positive makes advection in one direction for rabbits (depending on the grass).

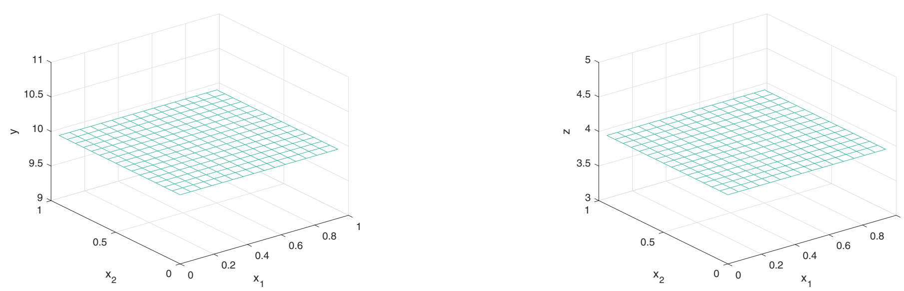

Figure 6.1. Spatial distribution at the initial moment. Rabbits (left) and grass (right).

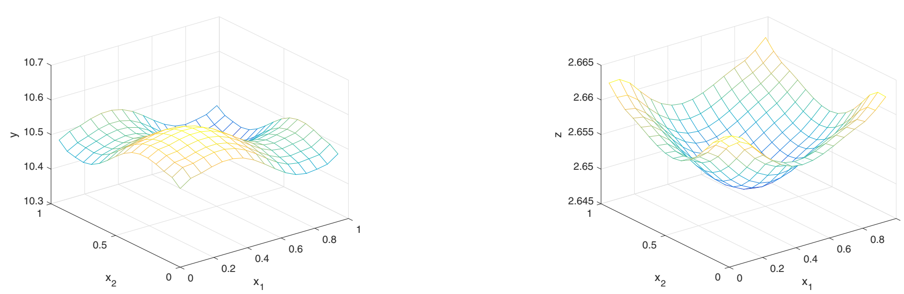

Figure 6.2. Spatial distribution at $t = 3$. Rabbits (left) and grass (right).

Here, we assume that rabbits and grass are evenly distributed in the space domain at the initial moment (see Figure 6.1). Over time, the distributions of rabbit and grass are no longer uniform due to the advection movement of the rabbits (see Figures 6.2-6.3). In particular, we can find that the rabbit movement is affected by the gradient of the grass. This phenomenon is consistent with our model assumption in section 2. Figures 6.1-6.3 present the distributions of rabbits and grass in space. Next, we consider the trajectorys of rabbits and grass over the time.

Fixing all the original coefficients except the control coefficient u_1, and taking $u_1 = 0.8$, 0.2 in turn, we can obtain the time-varying trajectories of rabbits and grass at a small closed square.

Without loss of generality, we choose the closed square centered at $x = (\frac{5}{16}, \frac{3}{16})$. Figures 6.4 and 6.5 show that rabbits and grass grow alternately over the time. At the same time, the rabbit's reaction is slightly delayed. These two phenomena are consistent with our real experience. In addition, we can find that the control coefficient can affect the cycle trajectories of rabbits and grass from Figures 6.4 and 6.5.

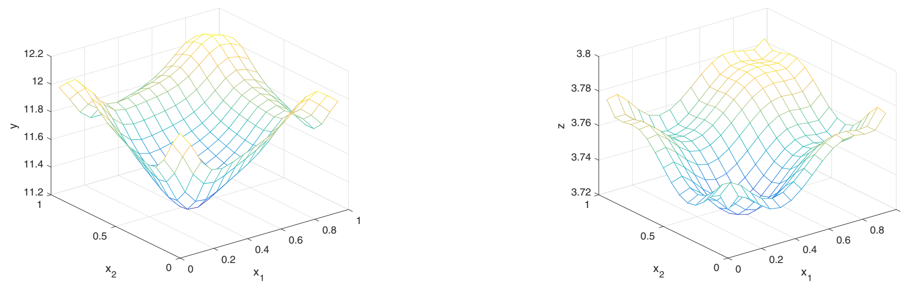

Figure 6.3. Spatial distribution at $t = 6$. Rabbits (left) and grass (right).

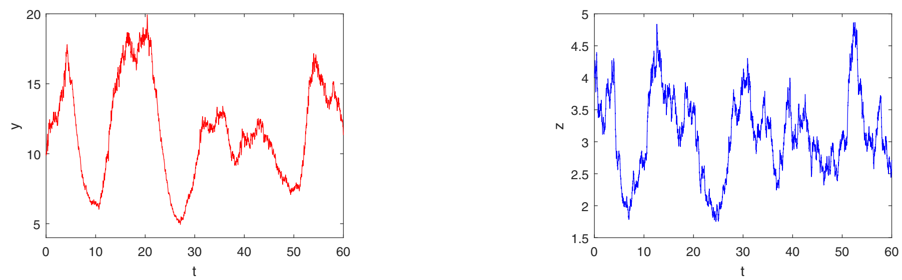

Figure 6.4. Time plots of rabbits and grass at $x = (\frac{5}{16}, \frac{3}{16})$. Rabbits (left) and grass (right). Control coefficient $u_1 = 0.8$.

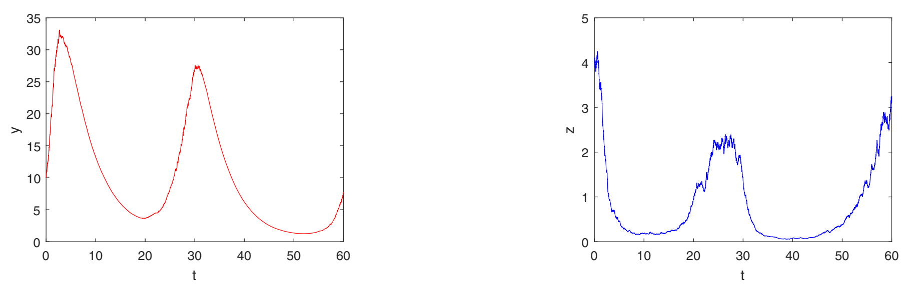

Figure 6.5. Time plots of rabbits and grass at $x = (\frac{5}{16}, \frac{3}{16})$. Rabbits (left) and grass (right). Control coefficient $u_1 = 0.2$.

In addition to the control coefficient u_1, we can also observe the effects of noise on rabbits and grasses. Fixing all the original coefficients except the noise intensity coefficient ϕ_1, and taking $\phi_1 = 0,\ 2$ in turn, we can obtain Figures 6.6 and 6.7. Comparing Figure 6.4 with Figure 6.6, we can see that the time-varying trajectories of rabbits and grass are smoother and more stable in the absence of noise. However, Figure 6.7 shows that the time-varying trajectories will tend to zero when the noise intensity is large. This means that large noise (disaster) may lead to the destruction of rabbits and grass.

Based on the above analysis, we can show that our model (2.9) is a good approach to modeling populations incorporating individual behavior and stochasticity. Figures 6.1-6.3 show that our model is easy to observe the spatial distributions of populations and the individual's movement behavior. Figures 6.4, 6.6 and 6.7 show that our model can reflect well the influence of noise on the evolution trajectories of populations. Finally, Figures 6.4 and 6.5 imply that we can better manage the evolution of populations by adjusting the control coefficient.

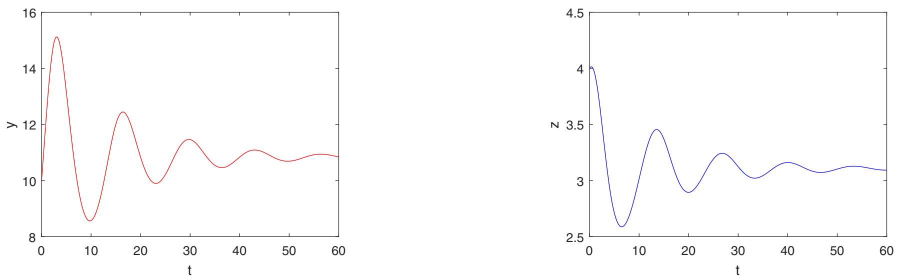

Figure 6.6. Time plots of rabbits and grass at $x = (\frac{5}{16}, \frac{3}{16})$. Rabbits (left) and grass (right). Noise intensity coefficient $\phi_1 = 0$.

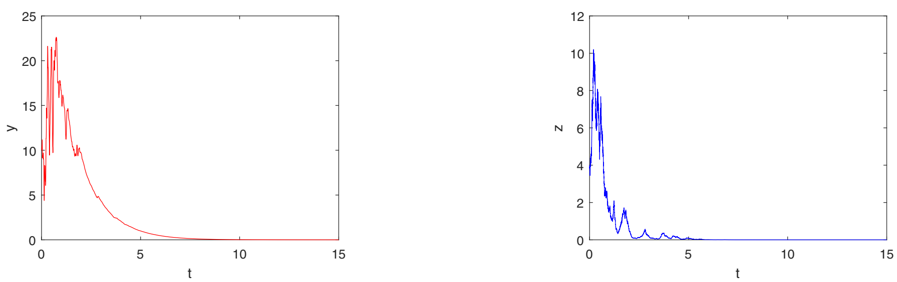

Figure 6.7. Time plots of rabbits and grass at $x = (\frac{5}{16}, \frac{3}{16})$. Rabbits (left) and grass (right). Noise intensity coefficient $\phi_1 = 2$.

7 Conclusions

We obtained a system of stochastic differential equations based on the movement of individual rabbits and the dispersal and the growth of the grass. As the number of the rabbits becomes large, we derived the limit model as a pair of stochastic partial differential equations representing the interactions. The harvest rate is introduced as a control variable in our model. The existence and uniqueness of the solution to this pair of controlled SPDEs were established. The optimal control problem for this SPDE model has been presented in the companion paper [17].

This paper laid the foundation of this tool of deriving such a system of SPDEs from population interaction and movement rules. This tool can be applied to model other populations with different rules of interacting with each other and with the environment. A variety of features of a random environment could be included.

There are interesting questions to investigate in the future. It might be interesting to compare the aggregation from a PDE model with that from a SPDE model. Would the SPDE model give some additional information beyond the PDE model? Also, one could consider the possible asymptotic behavior of the solution to the SPDE model without control or with constant control. Later, it would be interesting to investigate numerical simulations for specific examples.

References

[1] Capasso, V., Morale, D. Asymptotic behavior of a system of stochastic particles subject to nonlocal interactions. *Stoch. Anal. Appl.*, 27: 574–603 (2009)
[2] Dawson, D.A. Measure-valued Markov processes. École d'Été de Probabilités de Saint-Flour XXI-1991. *Lecture Notes in Math.*, 1541: 1–260 (1993)
[3] Federico, P., Gross, L.J., Lenhart, S., Ryan, D. Optimal control in individual based models: implications from aggregated methods. *American Naturalist*, 181: 64–77 (2013)
[4] Edwards, M., Huet, S., Goreaud, S.F., Deffuant, C. Comparing an individual-based model of behavior diffusion with its mean field aggregate approximation. *J. Artificial Societies Soc. Simulation*, 6: 4 (2003)
[5] Epanchin-Niell, R., Hastings, A. Controlling established invaders: integrating economics and spread dynamics to determine optimal management. *Ecology Letters*, 13: 528–541 (2010)
[6] Hinkelmann, F., Murragarra, D., Jarrah, A., Laubenbacher, R. A mathematical framework for agent-based models of complex biological networks. *Bulletin of Math. Biology*, 73: 1583–1602 (2011)
[7] Huet, S., Deffuant, G. Differential equation models derived from an individual-based model can help to understand emergent effects. *J. Artificial Societies Soc. Simulation*, 11: 2–10 (2008)
[8] Jacod, J., Shiryaev, A.N. Limit Theorems for Stochastic Processes. Springer-Verlag, 1987
[9] Jakubowski, A. On the Skorohod topology. *Ann. Inst. H. Poincar*, B22: 263–285 (1986)
[10] Kallianpur, G., Xiong, J. Stochastic Differential Equations on Infinite Dimensional Spaces. *IMS Lecture notes-monograph series*, 26, 1995
[11] Kim, P.S., Lee, P.P., Levy, D. Modeling imatinib-treated chronic myelogenous leukemia: reducing the complexity of agent-based models. *Bull. Math. Biol.*, 70: 728–744 (2008)
[12] Kim, P.S., Lee, P.P., Levy, D. A PDE model for imatinib-treated chronic myelogenous leukemia. *Bull. Math. Biol.*, 70: 1994–2016 (2009)
[13] Krylov, N.V. An analytic approach to SPDEs. Stochastic partial differential equations: six perspectives. *Math. Surveys Monogr.*, 64: 185–242 (1999)
[14] Kurtz, T., Xiong, J. Particle representations for a class of nonlinear SPDEs. *Stoch. Proce. Appl.*, 83: 103–126 (1999)
[15] Ladyzenskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N. Linear and Quasi-linear Equations of Parabolic Type. *Amer. Math. Soc.*, 23, 1988
[16] Laubenbacher, R., Hinkelmann, F., Oremland, M. Agent-based models and optimal control in biology: a discrete approach. Mathematical Concepts and Methods in Modern Biology, Academic Press, Boston, 2013, 143–178
[17] Lenhart, S., Xiong, J., Yong, J.M. Optimal controls for stochastic partial differential equations with an application in population modeling. *SIAM Control. Optim.*, 54: 495–535 (2016)
[18] Loehle, C. Control theory and the management of ecosystems. *Journal of Applied Ecology*, 43: 957–966 (2006)
[19] Mikulevicius, R., Rozovskii, R. A note on Krylov's L_p-theory for systems of SPDEs. *Electronic J. Probab.*, 6: 1–35 (2001)
[20] Ovaskainen, O., Cornell, S.J. Space and stochasticity in population dynamics. *Proc. Natl. Acad. Sci. U.S.A.*, 103: 12781–12786 (2006)
[21] Pimentel, D., Zuniga, R., Morrison, D. Update on the environmental and economic cost associated with alien-invasive species in the United States. *Ecol. Econ.*, 52: 273–288 (2005)
[22] Railsback, S.F., Grimm, V. Agent-based and individual-based modeling: a practical introduction. Princeton University Press, Princeton, NJ, 2012
[23] Salinas, R.A., Stiver, W.H., Corn, J.L., Lenhart, S., Collins, C., Madden, M., Vercauteren, K.C., Schmit, B.B., Kasari, E., Odoi, A., Hickling, G., McCallum, H. An individual-based model for feral hogs in Great Smoky Mountains National Park. *Natural Resource Modeling*, 28: 18–29 (2015)
[24] Simberloff, D., Rejmanek, M. Encyclopedia of Biological Invasions. University of California Press, Oakland, 2011
[25] Veliz-Cuba, A., Jarrah, A.S., Laubenbacher, R. Polynomial algebra of discrete models in systems biology. *Bioinformatics*, 26: 1637–1643 (2010)
[26] Wilensky, U. Center for Connected Learning and Computer-Based Modeling. NetLogo, Northwestern University, 1999
[27] Wilensky, U. Rabbits Grass Weeds Model, Center for Connected Learning and Computer-Based Modeling. NetLogo, Northwestern University, 2001
[28] Xiong, J. A nonlinear differential equation associated with a class of interacting diffusion systems. *Trends in Contemporary Infinite Dimensional Analysis and Quantunm Probability*, 433–457 (2000)
[29] Xiong, J. An Introduction to Stochastic Filtering Theory. Oxford University Press, 2008
[30] Yong, J.M., Zhou, X.Y. Stochastic Controls: Hamiltonian Systems and HJB Equations. Springer-Verlag, New York, 1999