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Recursive Utility Processes, Dynamic Risk Measures and Quadratic Backward Stochastic Volterra Integral Equations

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Abstract

For an $\mathcal{F}_T$-measurable payoff of a European type contingent claim, the recursive utility process/dynamic risk measure can be described by the adapted solution to a backward stochastic differential equation (BSDE). However, for an $\mathcal{F}_T$-measurable stochastic process (called a position process, not necessarily $\mathbb{F}$-adapted), mimicking BSDE's approach will lead to a time-inconsistent recursive utility/dynamic risk measure. It is found that a more proper approach is to use the adapted solution to a backward stochastic Volterra integral equation (BSVIE). The corresponding notions are called equilibrium recursive utility and equilibrium dynamic risk measure, respectively. Motivated by this, the current paper is concerned with BSVIEs whose generators are allowed to have quadratic growth (in $Z(t,s)$). The existence and uniqueness for both the so-called adapted solutions and adapted M-solutions are established. A comparison theorem for adapted solutions to the so-called Type-I BSVIEs is established as well. As consequences of these results, some general continuous-time equilibrium dynamic risk measures and equilibrium recursive utility processes are constructed.

Keywords Backward stochastic Volterra integral equation · Quadratic generator · Comparison theorem · Equilibrium dynamic risk measure · Equilibrium recursive utility process · Time-consistency

Mathematics Subject Classification 60H10 · 60H20 · 91B30 · 91B70 · 91G80

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which a one-dimensional standard Brownian motion $W = \{W(t); 0 \le t < \infty\}$ is defined, with $\mathbb{F} = \{\mathcal{F}_t\}_{t \ge 0}$ being the natural filtration of W augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Let ξ be a (random) payoff at some future time T of a certain European type contingent claim, and $c(\cdot)$ be a consumption rate. Following [17], we let $Y(\cdot)$ solve the following equation:

$$Y(t) = \mathbb{E}_t\Big[\xi + \int_t^T \Big(f(c(s), Y(s)) + A(Y(s))Z(s)^2\Big)ds\Big], \quad t \in [0, T], \quad (1.1)$$

hereafter, $\mathbb{E}_t[\,\cdot\,] = \mathbb{E}[\,\cdot \mid \mathcal{F}_t]$ is the conditional expectation operator, and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given map, called the *aggregator*,

$$Z(t)^2 = \frac{d}{dt}\langle Y \rangle(t),$$

with $t \mapsto \langle Y \rangle(t)$ being the quadratic variation process of $Y(\cdot)$, and $A(Y(t))$ is called the *variance multiplier*. Such defined $Y(\cdot)$ is called a *recursive utility process* (which has also been called *stochastic differential utility process*) of the payoff ξ and the consumption rate $c(\cdot)$. The main feature of such a process $Y(\cdot)$ is that the current value $Y(t)$ depends on the future values $Y(s)$, $t < s \le T$ of the process. This notion was firstly introduced by Duffie and Epstein [17] in 1992. It is easy to see that $(Y(\cdot), Z(\cdot))$ solves (1.1) if and only if it is an adapted solution to the following backward stochastic differential equation (BSDE, for short):

$$Y(t) = \xi + \int_t^T g(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s), \quad t \in [0, T], \quad (1.2)$$

with

$$g(s, y, z) = f(c(s), y) + A(y)z^2. \quad (1.3)$$

Thanks to the discovery of the relation between (1.1) and (1.2), recursive utility process was later extended to the adapted solution of general BSDEs (see [18,27,28]).

Now, if instead of ξ, we have an $\mathcal{F}_T$-measurable process $\psi(t)$, not necessarily $\mathbb{F}$-adapted, which is called a *position process* (see [36] for a study of discrete-time cases). It could also be called an *anticipated wealth flow process*. For example, it could be an anticipated received dividend process of a stock (which depends on the uncertain performance of the company), anticipated received mortgage payments (for a bank, say, with an uncertainty of default or prepayment), anticipated claim payments of an insurance policy, the random maintenance costs of an owned facility, etc. The feature of such kind of process is that at time t, the actually anticipated value of the process is not $\mathcal{F}_t$-measurable. To "calculate" the recursive utility for such a process at the current time t, mimicking (1.1), we might formally solve the following BSDE:

$$
\begin{aligned}
Y(t;r) = \psi(t) + \int_r^T g(s, Y(t;s), Z(t;s))ds \\
- \int_r^T Z(t;s)dW(s), \qquad r \in [t,T],
\end{aligned}
\tag{1.4}
$$

with the current time t being a parameter. Intuitively, $Y(t;r)$ should represent the utility of the process $\psi(\cdot)$ at a future time r, estimated/predicted at the current time t. Therefore, the utility at the current time t should be given by $Y(t;t)$. However, by taking $r = t$ in the above, we obtain

$$
\begin{aligned}
Y(t;t) = \psi(t) + \int_t^T g(s, Y(t;s), Z(t;s))ds \\
- \int_t^T Z(t;s)dW(s), \quad t \in [0,T],
\end{aligned}
\tag{1.5}
$$

which is not an equation for the process $t \mapsto Y(t;t)$ since $Y(t;s)$ appears on the right-hand side of the above. A careful observation shows that $Y(t;r)$ obtained through (1.4) has some time-inconsistent nature, by which we mean the following: If everything is ideal, the value $Y(t;r)$, which is supposed to be the utility of the process $\psi(\cdot)$ at a future time r estimated/predicted at the current time t should be equal to $Y(r;r)$, the realistic utility at future time r. But this seems to have very little hope. In another word, $t \mapsto Y(t;t)$ determined by a family of BSDEs as above seems not to be a good description of the recursive utility process for the position process $\psi(\cdot)$.

Suggested by (1.4)–(1.5), we propose the following modified equation:

$$
\begin{aligned}
Y(t) = \psi(t) + \int_t^T g(s, Y(s), Z(t,s))ds \\
- \int_t^T Z(t,s)dW(s), \qquad t \in [0,T].
\end{aligned}
\tag{1.6}
$$

Note that the above modification is simply to force $Y(t;s) = Y(s;s)$ in (1.5), then rename $Y(t;t)$ to be $Y(t)$. The advantage of such a modification is that as long as a solution $(Y(\cdot), Z(\cdot\,,\cdot))$ of (1.6) exists, $Y(\cdot)$ is time-consistent. Then, $Y(\cdot)$ could serve as a good description of the recursive utility for the process $\psi(\cdot)$ (by suitably selecting the aggregator $g(s,y,z)$). However, a couple of natural questions arise: (i) Is there any convincing mathematical justification for the model (1.6), and (ii) By "brutally" forcing $Y(t;s) = Y(s;s)$, is the resulting equation (1.6) well-posed? For question (i), we will sketch a convincing argument in the appendix at the end of the paper, justifying our modification. We will borrow some ideas from the study of time-inconsistent optimal control problems [49]. For question (ii), it turns out that (1.6) is nothing but a so-called *backward stochastic Volterra integral equation* (BSVIE, for short), which has been studied since the early 2000 for various cases, and the current paper is actually a continuation of those investigations. With the well-posedness of (1.6) (see below for details), the map $t \mapsto Y(t)$ will be called an *equilibrium recursive* *utility process* of $\psi(\cdot)$. Interestingly, our mathematical justification presented in the appendix will perfectly justify the word "equilibrium".

BSVIEs have been studied since 2002 [29]. Let us now elaborate a little more on BSVIEs. Let

$$g : [0,T]^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}, \qquad \psi : [0,T] \times \Omega \to \mathbb{R}$$

be two given random fields. We consider the following BSVIE:

$$\begin{aligned} Y(t) = \psi(t) + \int_t^T g(t,s,Y(s),Z(t,s),Z(s,t))ds \\ - \int_t^T Z(t,s)dW(s), \quad t \in [0,T]. \end{aligned} \tag{1.7}$$

By an *adapted solution* to BSVIE (1.7), we mean an $(\mathbb{R} \times \mathbb{R})$-valued random field $(Y,Z) = \{(Y(t), Z(t,s)); 0 \le s,t \le T\}$ such that

(i) $Y(\cdot)$ is $\mathbb{F}$-progressively measurable (not necessarily continuous),
(ii) for each fixed $0 \le t \le T$, $Z(t,\cdot)$ is $\mathbb{F}$-progressively measurable, and
(iii) Equation (1.7) is satisfied in the usual Itô sense for Lebesgue measure almost every $t \in [0,T]$.

Condition (ii) implies that for any $t \in [0,T)$, the random variable $Z(t,s)$ is $\mathcal{F}_s$-measurable for any $s \in [t,T]$. In (1.7), g and ψ are called the *generator* and the *free term*, respectively. Let us point out that in this paper, we only study the BSVIEs with $Y(\cdot)$ being one-dimensional. The case that $Y(\cdot)$ being higher dimensional will be significantly different in general, and will be investigated in the near future. However, the Brownian motion $W(\cdot)$ assumed to be one-dimensional is just for convenience of our presentation.

When $Z(s,t)$ is absent, (1.7) is reduced to the form

$$\begin{aligned} Y(t) = \psi(t) + \int_t^T g(t,s,Y(s),Z(t,s))ds \\ - \int_t^T Z(t,s)dW(s), \quad t \in [0,T], \end{aligned} \tag{1.8}$$

which is a natural extension of BSDEs, and is a little more general than (1.6) since g depends on both t and s. BSVIEs of form (1.8), referred to as *Type-I BSVIEs*, were firstly studied by Lin [29], followed by several other researchers: Aman and N'Zi [3], Wang and Zhang [44], Djordjević and Janković [15,16], Hu and Øksendal [20].

BSVIEs of the form (1.7) (containing $Z(s,t)$) were firstly introduced by Yong [46,48], motivated by the study of optimal control for forward stochastic Volterra integral equations (FSVIEs, for short). We call (1.7) a *Type-II BSVIE* to distinguish it from Type-I BSVIEs. Type-II BSVIE (1.7) has a remarkable feature that its adapted solution, similarly defined as that for Type-I BSVIEs, might not be unique due to lack of restriction on the term $Z(s,t)$ (with $0 \le t \le s \le T$). Suggested by the natural form of the adjoint equation in the Pontryagin type maximum principle, Yong [48] introduced the notion of *adapted M-solutions*: A pair $(Y(\cdot), Z(\cdot\,,\cdot))$ is called an adapted M-solution to (1.7), if in addition to (i)–(iii) stated above, the following condition is also satisfied:

$$Y(t) = \mathbb{E}[Y(t)] + \int_0^t Z(t,s)dW(s), \quad \text{a.e. } t \in [0,T], \text{ a.s.} \tag{1.9}$$

Under usual Lipschitz conditions, well-posedness was established in [48] for the adapted M-solutions to Type-II BSVIEs of form (1.7). This important development has triggered extensive research on BSVIEs and their applications. For instance, Anh, Grecksch and Yong [4] investigated BSVIEs in Hilbert spaces; Shi et al. [37] studied well-posedness of BSVIEs containing mean-fields (of the unknowns); Ren [35], Wang and Zhang [45] discussed BSVIEs with jumps; Overbeck and Röder [32] even developed a theory of path-dependent BSVIEs; Numerical aspect was considered by Bender and Pokalyuk [6]; relevant optimal control problems were studied by Shi et al. [38], Agram and Øksendal [2], Wang and Zhang [43], and Wang [40]; Wang and Yong [41] established various comparison theorems for both adapted solutions and adapted M-solutions to BSVIEs in multi-dimensional Euclidean spaces.

Recently, inspired by the Four-Step Scheme in the theory of forward-backward stochastic differential equations (FBSDEs, for short) [31] and the time-inconsistent stochastic optimal control problems [49], Wang and Yong [42] established a representation of adapted solutions to Type-I BSVIEs and adapted M-solutions to Type-II BSVIEs in terms of the solution to a system of (non-classical) partial differential equations and the solution to a (forward) stochastic differential equation.

We point out that in all the above-mentioned works on BSVIEs, the generator $g(t,s,y,z,z')$ of the BSVIE (1.7) satisfies a uniform Lipschitz condition in (y,z,z') so that the generator has a linear growth in (z,z'). However, when the generator $g(s,y,z)$ of BSVIE (1.6) is given by (1.3), it has a quadratic growth in z. Hence, a theory needs to be established for BSVIEs with the generators $g(t,s,y,z,z')$ growing quadratically in z, which are called quadratic BSVIEs (QBSVIEs, for short, if the quadratic growth of the generator in z needs to be emphasized). We point out that at the moment, we are not able to handle the case that $z' \mapsto g(t,s,y,z,z')$ is quadratic, and it is also lack of motivation for that case.

Recall that for BSDE (1.2), when $(y,z) \mapsto g(s,y,z)$ satisfies a uniform Lipschitz condition, with $g(\cdot\,,0,0)$ being L^p-integrable (with some $p > 1$), for any $\mathcal{F}_T$-measurable L^p-integrable random variable ξ, it admits a unique adapted solution $(Y(\cdot), Z(\cdot))$ [31,33,50] which could be called a recursive utility process for ξ. On the other hand, when $z \mapsto g(s,y,z)$ has an up to quadratic growth, the BSDE (1.2) is called a *quadratic BSDE* (QBSDE, for short). In 2000, Kobylanski [24] established the well-posedness of QBSDE with ξ being bounded. Since then, some efforts have been made by researchers to relax the assumptions on the generator as well as on the terminal value ξ. Among relevant works, we would like to mention Briand and Hu [7,8], Hu and Tang [21], Briand and Richou [9], and Zhang [51, Chapter 7]. Further, BSDEs with superquadratic growth was investigated by Delbaen et al. [10], where some general negative results concerning the well-posedness can be found. Therefore, one can say that the theory of recursive utility for terminal payoff ξ has reached a pretty mature stage.

The purpose of this paper is to establish the well-posedness of QBSVIEs under certain conditions. The method introduced by Yong [48] and techniques found in Briand–Hu [7,8] will be combined and further developed. In addition, a comparison theorem for adapted solutions of Type-I QBSVIEs will be established. Consequently, equilibrium recursive utility processes and continuous-time equilibrium dynamic risk measures will be investigated. See Yong [47] and Wang–Yong [41], Agram [1] for some earlier works. See also Di Persio [14] for stochastic differential utility, and Kromer–Overbeck [26] for dynamical capital allocation by means of BSVIEs.

The rest of this paper is organized as follows. In Sect. 2, we introduce some preliminary notations and definitions, and present some lemmas which are of frequent use in the sequel. Section 3 is devoted to the study of existence and uniqueness of adapted solutions for Type-I QBSVIEs, and Sect. 4 is devoted to the study of existence and uniqueness of adapted M-solutions for Type-II QBSVIE. A comparison theorem for adapted solutions to Type-I QBSVIEs (1.8) will be established in Sect. 5, and an application of Type-I BSVIEs to continuous-time equilibrium dynamic risk measures will be presented in Sect. 6. Some conclusion remarks will be collected in Sect. 7. Finally, a mathematical justification of the BSVIE model is sketched in the appendix.

2 Preliminaries

For $0 \le a < b \le T$, we denote by $\mathcal{B}([a,b])$ the Borel σ-field on $[a,b]$ and define the following sets:

$$\Delta[a,b] \triangleq \Big\{(t,s) \,\Big|\, a \le t \le s \le b\Big\}, \qquad \Delta^c[a,b] \triangleq \Big\{(t,s) \,\Big|\, a \le s < t \le b\Big\},$$
$$[a,b]^2 \triangleq \Big\{(t,s) \,\Big|\, a \le t, s \le b\Big\} = \Delta[a,b] \cup \Delta^c[a,b], \qquad \Delta^*[a,b] \triangleq \overline{\Delta^c[a,b]}.$$

Note that $\Delta^*[a,b]$ is a little different from the complement $\Delta^c[a,b]$ of $\Delta[a,b]$ in $[a,b]^2$, since both $\Delta[a,b]$ and $\Delta^*[a,b]$ contain the diagonal line segment. In the sequel we shall deal with various spaces of functions and processes, which we collect here first for the convenience of the reader:

$$\begin{aligned}
&L^1(a,b) \\
&\quad = \Big\{h : [a,b] \to \mathbb{R} \mid h(\cdot) \text{ is } \mathcal{B}([a,b])\text{-measurable, } \textstyle\int_a^b |h(s)|ds < \infty\Big\}, \\
&L^\infty_{\mathcal{F}_b}(\Omega) \\
&\quad = \Big\{\xi : \Omega \to \mathbb{R} \mid \xi \text{ is } \mathcal{F}_b\text{-measurable and bounded}\Big\}, \\
&L^\infty_{\mathcal{F}_b}(a,b) \\
&\quad = \Big\{\varphi : [a,b] \times \Omega \to \mathbb{R} \mid \varphi(\cdot) \text{ is } \mathcal{B}([a,b]) \otimes \mathcal{F}_b\text{-measurable and bounded}\Big\}, \\
&L^2_{\mathbb{F}}(a,b) \\
&\quad = \Big\{\varphi : [a,b] \times \Omega \to \mathbb{R} \mid \varphi(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable, } \mathbb{E}\textstyle\int_a^b |\varphi(s)|^2 ds < \infty\Big\},
\end{aligned}$$

$$
\begin{aligned}
&L^\infty_{\mathbb{F}}(a,b)\\
&\quad=\Big\{\varphi(\cdot)\in L^2_{\mathbb{F}}(a,b)\;\Big|\;\varphi(\cdot)\text{ is bounded}\Big\},\\
&L^2_{\mathbb{F}}(\Omega;C[a,b])\\
&\quad=\Big\{\varphi:[a,b]\times\Omega\to\mathbb{R}\mid\varphi(\cdot)\text{ is continuous, }\mathbb{F}\text{-adapted, }\mathbb{E}\big[\sup_{a\le s\le b}|\varphi(s)|^2\big]<\infty\Big\},\\
&L^\infty_{\mathbb{F}}(\Omega;C[a,b])\\
&\quad=\Big\{\varphi(\cdot)\in L^2_{\mathbb{F}}(\Omega;C[a,b])\;\Big|\;\sup_{a\le t\le b}|\varphi(t)|\in L^\infty_{\mathcal{F}_b}(\Omega)\Big\},\\
&L^\infty_{\mathcal{F}_b}(\Omega;C^U[a,b])\\
&\quad=\Big\{\varphi(\cdot)\in L^\infty_{\mathcal{F}_b}(a,b)\;\Big|\;\text{there exists a modulus of continuity }\rho:[0,\infty)\to[0,\infty)\\
&\qquad\qquad\text{such that }|\varphi(t)-\varphi(s)|\le\rho(|t-s|),\;\;(t,s)\in[a,b],\;\;\text{a.s.}\Big\},\\
&L^2_{\mathbb{F}}(\Delta[a,b])\\
&\quad=\Big\{\varphi:\Delta[a,b]\times\Omega\to\mathbb{R}\mid\varphi(t,\cdot)\text{ is }\mathbb{F}\text{-progressively measurable on }[t,b],\text{ a.e. }t\in[a,b],\\
&\qquad\qquad\mathbb{E}\int_a^b\int_t^b|\varphi(t,s)|^2dsdt<\infty\Big\},\\
&L^2_{\mathbb{F}}([a,b]^2)\\
&\quad=\Big\{\varphi:[a,b]^2\times\Omega\to\mathbb{R}\mid\varphi(t,\cdot)\text{ is }\mathbb{F}\text{-progressively measurable on }[a,b],\text{ a.e. }t\in[a,b],\\
&\qquad\qquad\mathbb{E}\int_a^b\int_a^b|\varphi(t,s)|^2dsdt<\infty\Big\},\\
&\mathcal{H}^2_\Delta[a,b]=L^2_{\mathbb{F}}(a,b)\times L^2_{\mathbb{F}}(\Delta[a,b]),\quad\mathcal{H}^2[a,b]=L^2_{\mathbb{F}}(a,b)\times L^2_{\mathbb{F}}([a,b]^2).
\end{aligned}
$$

Now, we recall the definitions of adapted solutions and adapted M-solutions for Type-I BSVIE (1.8) and Type-II BSVIE (1.7), respectively (see [48]).

Definition 2.1 (i) A pair of processes $(Y(\cdot),Z(\cdot,\cdot))\in\mathcal{H}^2_\Delta[0,T]$ is called an *adapted solution* of BSVIE (1.8) if (1.8) is satisfied in the usual Itô sense for Lebesgue measure almost every $t\in[0,T]$.

(ii) A pair of processes $(Y(\cdot),Z(\cdot,\cdot))\in\mathcal{H}^2[0,T]$ is called an *adapted solution* of BSVIE (1.7) if (1.7) is satisfied in the usual Itô sense for Lebesgue measure almost every $t\in[0,T]$. Further, it is called an *adapted M-solution* of BSVIE (1.7) on $[r,T]$ if, in addition, the following holds:

$$
Y(s)=\mathbb{E}_r[Y(s)]+\int_r^s Z(s,t)dW(t),\quad\text{a.e. }s\in[r,T]. \tag{2.1}
$$

Here, we recall that $\mathbb{E}_r=[\,\cdot\,|\,\mathcal{F}_r]$.

Let $\mathcal{M}^2[r,T]$ be the set of all $(y(\cdot), z(\cdot,\cdot)) \in \mathcal{H}^2[r,T]$ satisfying (2.1). Clearly, $\mathcal{M}^2[r,T]$ is a closed subspace of $\mathcal{H}^2[r,T]$. Further, for any $(y(\cdot), z(\cdot,\cdot)) \in \mathcal{M}^2[r,T]$, we have

$$\mathbb{E}|y(s)|^2 = \mathbb{E}\big|\mathbb{E}_r[y(s)]\big|^2 + \mathbb{E}\int_r^s |z(s,t)|^2 dt \geq \mathbb{E}\int_r^s |z(s,t)|^2 dt, \quad \text{a.e. } s \in [r,T].$$

It follows that

$$\begin{aligned}
&\|(y(\cdot), z(\cdot,\cdot))\|^2_{\mathcal{H}^2[r,T]} \\
&\quad \equiv \mathbb{E}\Big[\int_r^T |y(s)|^2 ds + \int_r^T\int_r^T |z(s,t)|^2 dt ds\Big] \\
&\quad = \mathbb{E}\Big[\int_r^T |y(s)|^2 ds + \int_r^T\int_r^s |z(s,t)|^2 dt ds + \int_r^T\int_s^T |z(s,t)|^2 dt ds\Big] \\
&\quad \leq \mathbb{E}\Big[2\int_r^T |y(s)|^2 ds + 2\int_r^T\int_s^T |z(s,t)|^2 dt ds\Big] \\
&\quad \equiv 2\|(y(\cdot), z(\cdot,\cdot))\|^2_{\mathcal{M}^2[r,T]} \leq 2\|(y(\cdot), z(\cdot,\cdot))\|^2_{\mathcal{H}^2[r,T]},
\end{aligned}$$

which implies that $\|\cdot\|_{\mathcal{M}^2[r,T]}$ is an equivalent norm of $\|\cdot\|_{\mathcal{H}^2[r,T]}$ on $\mathcal{M}^2[r,T]$.

Next, we recall the following definition (see [23] for relevant details).

Definition 2.2 A uniformly integrable $\mathbb{F}$-martingale $M = \{M(t) : 0 \leq t \leq T\}$ with $M(0) = 0$ is called a *BMO martingale* on $[0,T]$ if

$$\|M(\cdot)\|^2_{\mathrm{BMO}(0,T)} \triangleq \sup_{\tau\in\mathscr{T}[0,T]} \Big\|\mathbb{E}_\tau\big[|M(T) - M(\tau)|^2\big]\Big\|_\infty < \infty,$$

where $\mathscr{T}[0,T]$ is the set of all $\mathbb{F}$-stopping times τ valued in $[0,T]$.

Sometimes, the norm $\|\cdot\|_{\mathrm{BMO}(0,T)}$ is written as $\|\cdot\|_{\mathrm{BMO}_{\mathbb{P}}(0,T)}$, indicating the dependence on the probability $\mathbb{P}$.

Next, let $X = \{X_t, \mathcal{F}_t; 0 \leq t \leq T\}$ be a measurable, adapted process satisfying

$$\mathbb{P}\Big[\int_0^T |X(s)|^2 ds < \infty\Big] = 1.$$

Recall the *Doléan-Dade exponential* of X:

$$\mathcal{E}\{X\}_t \triangleq e^{\int_0^t X(s)dW(s) - \frac{1}{2}\int_0^t |X(s)|^2 ds}, \quad t \in [0,T], \tag{2.2}$$

and define a probability measure $\overline{\mathbb{P}}$ on $\mathcal{F}_T$ by

$$d\overline{\mathbb{P}} = \mathcal{E}\{X\}_T d\mathbb{P}. \tag{2.3}$$

Then, we have the following lemma which is a combination of the Girsanov's theorem (see Karatzas–Shreve [22] for a proof) and a result found in Kazamaki [23].

Lemma 2.3 *If $t \mapsto \int_0^t X(s)dW(s)$ is a BMO martingale on $[0,T]$, then $\mathcal{E}\{X\}_t$ is a uniformly integrable martingale and the process $\overline{W} = \{\overline{W}(t), \mathcal{F}_t \mid 0 \le t \le T\}$ defined by*

$$\overline{W}(t) \triangleq W(t) - \int_0^t X(s)ds, \quad 0 \le t \le T \tag{2.4}$$

is a standard Brownian motion on $(\Omega, \mathcal{F}_T, \overline{\mathbb{P}})$.

Next, we introduce the following spaces. Let $0 \le a < b < c \le T$, and

$$\begin{aligned}
&\overline{\mathrm{BMO}}(a,b)\\
&\quad= \Big\{\varphi : [a,b]\times\Omega \to \mathbb{R} \,\Big|\, \varphi(\cdot) \in L^2_{\mathbb{F}}(a,b),\\
&\quad\|\varphi(\cdot)\|^2_{\overline{\mathrm{BMO}}(a,b)} \triangleq \sup_{\tau\in\mathscr{T}[a,b]} \Big\|\mathbb{E}_\tau\Big[\int_\tau^b |\varphi(s)|^2 ds\Big]\Big\|_\infty < \infty\Big\},\\
&\overline{\mathrm{BMO}}(\Delta[a,b])\\
&\quad= \Big\{\varphi : \Delta[a,b]\times\Omega \to \mathbb{R} \,\Big|\, \varphi(\cdot,\cdot) \in L^2_{\mathbb{F}}(\Delta[a,b]),\\
&\quad\|\varphi(\cdot,\cdot)\|^2_{\overline{\mathrm{BMO}}(\Delta[a,b])} \triangleq \operatorname*{esssup}_{t\in[a,b]} \sup_{\tau\in\mathscr{T}[t,b]} \Big\|\mathbb{E}_\tau\Big[\int_\tau^b |\varphi(t,s)|^2 ds\Big]\Big\|_\infty < \infty\Big\},\\
&\overline{\mathrm{BMO}}\big([a,b]\times[b,c]\big)\\
&\quad= \Big\{\varphi : [a,b]\times[b,c]\times\Omega \to \mathbb{R} \,\Big|\, \varphi(\cdot,\cdot) \in L^2_{\mathbb{F}}([a,b]\times[b,c]),\\
&\quad\|\varphi(\cdot,\cdot)\|^2_{\overline{\mathrm{BMO}}([a,b]\times[b,c])} \triangleq \operatorname*{esssup}_{t\in[a,b]} \sup_{\tau\in\mathscr{T}[b,c]} \Big\|\mathbb{E}_\tau\Big[\int_\tau^c |\varphi(t,s)|^2 ds\Big]\Big\|_\infty < \infty\Big\}.
\end{aligned}$$

We note that for $\varphi(\cdot) \in \overline{\mathrm{BMO}}(a,b)$, if we let $\varphi(s) \equiv 0$, $s \in [0,a)$, then $\int_0^s \varphi(r)dW(r);\ 0 \le s \le b$ is a BMO martingale on $[0,b]$. Similarly, for $\varphi(\cdot,\cdot) \in \overline{\mathrm{BMO}}(\Delta[a,b])$, if we let $\varphi(t,s) \equiv 0$, $s \in [0,t)$, then $\int_0^s \varphi(t,r)dW(r);\ 0 \le s \le b$ is a BMO martingale on $[0,b]$ for almost all $t \in [a,b)$. The situation for $\overline{\mathrm{BMO}}\big([a,b]\times[b,c]\big)$ is also similar. The following lemma plays a basic role in our subsequent arguments. we refer the reader to [23, Theorem 3.3] for the proof and details.

Lemma 2.4 *For $K > 0$, there are constants $c_1, c_2 > 0$ depending only on K such that for any BMO martingale $M(\cdot)$, we have for any one-dimensional BMO martingale $N(\cdot)$ such that $\|N(\cdot)\|_{\mathrm{BMO}(0,T)} \le K$,*

$$c_1\|M(\cdot)\|_{\mathrm{BMO}_{\mathbb{P}}(0,T)} \le \|\overline{M}(\cdot)\|_{\mathrm{BMO}_{\overline{\mathbb{P}}}(0,T)} \le c_2\|M(\cdot)\|_{\mathrm{BMO}_{\mathbb{P}}(0,T)},$$

where $\overline{M}(\cdot) \triangleq M(\cdot) - \langle M, N\rangle(\cdot)$ and $d\overline{\mathbb{P}} = \bar{\mathcal{E}}\{N(\cdot)\}_T d\mathbb{P}$.

We now consider the following BSDE:

$$Y(t)=\xi+\int_t^T f(s,Y(s),Z(s))ds-\int_t^T Z(s)dW(s),\quad t\in[0,T]. \tag{2.5}$$

Let us introduce the following hypothesis.

(A0) Let the generator $f:[0,T]\times\mathbb{R}\times\mathbb{R}\times\Omega\to\mathbb{R}$ be $\mathcal{B}([0,T]\times\mathbb{R}\times\mathbb{R})\otimes\mathcal{F}_T$-measurable such that $s\mapsto f(s,y,z)$ is $\mathbb{F}$-progressively measurable for all $(y,z)\in\mathbb{R}\times\mathbb{R}$. There exist constants β,γ,L and a function $h(\cdot)\in L^1(0,T)$ such that

$$|f(s,y,z)|\le h(s)+\beta|y|+\frac{\gamma}{2}|z|^2,\quad (s,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}; \tag{2.6}$$

$$|f(s,y_1,z_1)-f(s,y_2,z_2)|\le L|y_1-y_2|+L(1+|z_1|+|z_2|)|z_1-z_2|,$$
$$(s,y_i,z_i)\in[0,T]\times\mathbb{R}\times\mathbb{R},\ i=1,2. \tag{2.7}$$

Lemma 2.5 *Let (A0) hold. Then, for any $\xi\in L^\infty_{\mathcal{F}_T}(\Omega)$, BSDE (2.5) admits a unique adapted solution $(Y(\cdot),Z(\cdot))\in L^\infty_{\mathbb{F}}(\Omega;C[0,T])\times\overline{\text{BMO}}(0,T)$. Moreover,*

$$e^{\gamma|Y(t)|}\le\mathbb{E}_t\Big[e^{\gamma e^{\beta(T-t)}|\xi|+\gamma\int_t^T|h(s)|e^{\beta(s-t)}ds}\Big]. \tag{2.8}$$

Proof By [51, Theorem 7.3.3], BSDE (2.5) admits a unique adapted solution $(Y(\cdot),Z(\cdot))\in L^\infty_{\mathbb{F}}(\Omega;C[0,T])\times L^2_{\mathbb{F}}(0,T)$. Then, by [51, Theorem 7.2.1], we see that the adapted solution $(Y(\cdot),Z(\cdot))\in L^\infty_{\mathbb{F}}(\Omega;C[0,T])\times\overline{\text{BMO}}(0,T)$. Further, by [8, Proposition 1], we have inequality (2.8). □

3 Adapted Solution to Type-I QBSVIE

In this section, we will establish the existence and uniqueness of the adapted solution to Type-I QBSVIE. Keep in mind that we may just use "BSVIE", instead of "Type-I QBSVIE", for convenience. First, let us look at the following simple example.

Example 3.1 Consider the one-dimensional BSVIE:

$$Y(t)=\psi(t)+\int_t^T\frac{Z(t,s)^2}{2}ds-\int_t^T Z(t,s)dW(s), \tag{3.1}$$

where $\psi(\cdot)\in L^\infty_{\mathcal{F}_T}(0,T)$, and $W(\cdot)$ is a one-dimensional standard Brownian motion. In order to solve Eq. (3.1), we introduce a family of BSDEs parameterized by $t\in[0,T]$:

$$\eta(t,s)=\psi(t)+\int_s^T\frac{\zeta(t,r)^2}{2}dr-\int_s^T\zeta(t,r)dW(r),\quad s\in[t,T]. \tag{3.2}$$

By Lemma 2.5, BSDE (3.2) admits a unique adapted solution $(\eta(t,\cdot),\zeta(t,\cdot)) \in L^\infty_{\mathbb{F}}(\Omega; C[t,T]) \times \overline{\text{BMO}}(t,T)$. Let

$$Y(t) = \eta(t,t) \text{ and } Z(t,s) = \zeta(t,s), \quad (t,s) \in \Delta[0,T],$$

then

$$Y(t) = \psi(t) + \int_t^T \frac{Z(t,s)^2}{2} ds - \int_t^T Z(t,s)dW(s), \quad t \in [0,T],$$

which implies that $(Y(\cdot), Z(\cdot,\cdot))$ is an adapted solution to BSVIE (3.1). The uniqueness of the solutions to BSVIE (3.1) can be obtained by the following Theorem 3.2. Moreover, the first term $Y(\cdot)$ of the unique solution to BSVIE (3.1) could be solved explicitly:

$$Y(t) = \ln\{\mathbb{E}[e^{\psi(t)}|\mathcal{F}_t]\}, \quad t \in [0,T]. \tag{3.3}$$

Clearly, from the expression (3.3), we see that as long as

$$\sup_{t\in[0,T]} \mathbb{E}\Big[e^{\psi(t)}\Big] < \infty,$$

by a usual approximation technique, one could find that BSVIE (3.1) will still have the adapted solution with $Y(\cdot)$ given by (3.3). Some general exploration in this direction will be carried out elsewhere.

From the above example, we see that BSVIE (3.1) can be fully characterized by a family of BSDEs (3.2). The main reason is that the generator of equation (3.1) is independent of y. This suggests us first consider a special case of Type-I QBSVIE (1.8).

3.1 A Special Case

Consider the following BSVIE:

$$Y(t) = \psi(t) + \int_t^T g(t,s,Z(t,s))ds - \int_t^T Z(t,s)dW(s), \tag{3.4}$$

where the generator $g : \Delta[0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ and the free term $\psi : [0,T] \times \Omega \to \mathbb{R}$ are given maps. We adopt the following assumption concerning $g(\cdot)$, which is comparable with (A0).

(A1) Let the generator $g : \Delta[0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ be $\mathcal{B}(\Delta[0,T] \times \mathbb{R}) \otimes \mathcal{F}_T$-measurable such that $s \mapsto g(t,s,z)$ is $\mathbb{F}$-progressively measurable on $[t,T]$, for all $(t,z) \in [0,T) \times \mathbb{R}$. There exist two constants γ, L and a function $h(\cdot) \in L^1(0,T;\mathbb{R})$ such that

$$|g(t,s,z)| \le h(s) + \frac{\gamma}{2}|z|^2, \quad (t,s,z) \in \Delta[0,T] \times \mathbb{R};$$
$$|g(t,s,z_1) - g(t,s,z_2)| \le L(1 + |z_1| + |z_2|)|z_1 - z_2|,$$
$$(t,s,z_i) \in \Delta[0,T] \times \mathbb{R}, \ i = 1, 2.$$

Now, we state the following existence and uniqueness result of BSVIE (3.4).

Theorem 3.2 *Let* (A1) *hold. Then for any* $\psi(\cdot) \in L^\infty_{\mathcal{F}_T}(0,T)$*, BSVIE* (3.4) *admits a unique adapted solution* $(Y(\cdot), Z(\cdot,\cdot)) \in L^\infty_{\mathbb{F}}(0,T) \times \overline{\text{BMO}}(\Delta[0,T])$.

Proof We first show the existence of the adapted solution to BSVIE (3.4). Consider the following BSDEs parameterized by $t \in [0,T]$:

$$\eta(t,s) = \psi(t) + \int_s^T g(t,r,\zeta(t,r))dr - \int_s^T \zeta(t,r)dW(r), \quad s \in [t,T]. \quad (3.5)$$

For almost all $t \in [0,T]$, by Lemma 2.5, under (A1), BSDE (3.5) admits a unique adapted solution $(\eta(t,\cdot), \zeta(t,\cdot)) \in L^\infty_{\mathbb{F}}(\Omega; C[t,T]) \times \overline{\text{BMO}}(t,T)$. Let

$$Y(t) = \eta(t,t), \quad Z(t,s) = \zeta(t,s), \quad (t,s) \in \Delta[0,T],$$

then $(Y(\cdot), Z(\cdot,\cdot)) \in L^\infty_{\mathbb{F}}(0,T) \times \overline{\text{BMO}}(\Delta[0,T])$ and

$$Y(t) = \psi(t) + \int_t^T g(t,s,Z(t,s))ds - \int_t^T Z(t,s)dW(s), \quad t \in [0,T],$$

which implies that $(Y(\cdot), Z(\cdot,\cdot))$ is an adapted solution for BSVIE (3.4).

The uniqueness is followed from the next theorem. □

Consider the following BSVIEs: For $i = 1, 2$,

$$Y_i(t) = \psi_i(t) + \int_t^T g_i(t,s,Z_i(t,s))ds - \int_t^T Z_i(t,s)dW(s), \quad t \in [0,T]. \quad (3.6)$$

We have the following comparison theorem.

Theorem 3.3 *Let* $g_1(\cdot)$ *and* $g_2(\cdot)$ *satisfy (A1),* $\psi_1(\cdot), \psi_2(\cdot) \in L^\infty_{\mathcal{F}_T}(0,T)$*. Let* $(Y_i(\cdot), Z_i(\cdot,\cdot)) \in L^\infty_{\mathbb{F}}(0,T) \times \overline{\text{BMO}}(\Delta[0,T])$ *be the adapted solution of corresponding BSVIE* (3.6)*. Suppose*

$$\psi_1(t) \le \psi_2(t), \quad g_1(t,s,z) \le g_2(t,s,z),$$
$$a.s., \ a.e. \ (t,s,z) \in \Delta[0,T] \times \mathbb{R}, \quad (3.7)$$

then we have

$$Y_1(t) \le Y_2(t), \quad a.s., \ a.e. \ t \in [0,T]. \quad (3.8)$$

In particular, if $g_1(\cdot) = g_2(\cdot)$ and $\psi_1(\cdot) = \psi_2(\cdot)$, the comparison implies the uniqueness of adapted solution to BSVIEs (3.4).

Proof We note that

$$Y_1(t) - Y_2(t) = \psi_1(t) - \psi_2(t) + \int_t^T [g_1(t, s, Z_1(t, s)) - g_2(t, s, Z_2(t, s))]\, ds \\ - \int_t^T [Z_1(t, s) - Z_2(t, s)]\, dW(s). \tag{3.9}$$

Define the process $\theta(\cdot, \cdot)$ such that

$$\theta(t, s) = 0, \quad (t, s) \in \Delta^*[0, T]; \tag{3.10}$$

$$|\theta(t, s)| \leq C(1 + |Z_1(t, s)| + |Z_2(t, s)|), \quad (t, s) \in \Delta[0, T]; \tag{3.11}$$

$$g_1(t, s, Z_1(t, s)) - g_1(t, s, Z_2(t, s)) \\ = \big[Z_1(t, s) - Z_2(t, s)\big]\theta(t, s), \quad (t, s) \in \Delta[0, T]. \tag{3.12}$$

Hereafter, $C > 0$ stands for a generic constant which could be different from line to line. Then, for almost all $t \in [0, T]$, $W(t; \cdot)$ defined by

$$W(t; s) \triangleq W(s) - \int_0^s \theta(t, r) dr, \quad s \in [0, T] \tag{3.13}$$

is a Brownian motion on $[0, T]$ under the equivalent probability measure $\overline{\mathbb{P}}_t$ defined by

$$d\overline{\mathbb{P}}_t \triangleq \mathcal{E}\{\theta(t, \cdot)\}_T\, d\mathbb{P}.$$

The corresponding expectation is denoted by $\mathbb{E}^{\bar{\mathbb{P}}_t}$. Thus, by (3.9) and (3.13), we have

$$Y_1(t) - Y_2(t) = \psi_1(t) - \psi_2(t) + \int_t^T [g_1(t, s, Z_2(t, s)) - g_2(t, s, Z_2(t, s))]\, ds \\ - \int_t^T [Z_1(t, s) - Z_2(t, s)]\, dW(t; s).$$

Taking the conditional expectation with respect to $\overline{\mathbb{P}}_t$ on the both sides of the above equation and then by (3.7), we have

$$Y_1(t) - Y_2(t) = \mathbb{E}_t^{\bar{\mathbb{P}}_t}\Big[\psi_1(t) - \psi_2(t) + \int_t^T [g_1(t, s, Z_2(t, s)) \\ - g_2(t, s, Z_2(t, s))]\, ds\Big] \leq 0, \quad \text{a.s.}$$

Hence, (3.8) follows. □

Remark 3.4 Theorems 3.2 and 3.3 are both concerned with the BSVIE (3.4), a very special case of Type-I BSVIE (1.8), in which, the generator $g(\cdot)$ is independent of the variable y. This makes the BSVIE (3.4) much easier to handle. Even though, Theorems 3.2 and 3.3 serve as a crucial bridge to the proof of the results for general Type-I BSVIEs.

3.2 The General Case

In this subsection, we will consider the following Type-I BSVIE:

$$\begin{aligned} Y(t) = \psi(t) + &\int_t^T g(t, s, Y(s), Z(t, s))ds \\ &- \int_t^T Z(t, s)dW(s), \qquad t \in [0, T]. \end{aligned} \tag{3.14}$$

We first introduce the following assumption, which is comparable to (A0).

(A2) Let the generator $g : \Delta[0, T] \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ be $\mathcal{B}(\Delta[0, T] \times \mathbb{R} \times \mathbb{R}) \otimes \mathcal{F}_T$-measurable such that $s \mapsto g(t, s, y, z)$ is $\mathbb{F}$-progressively measurable on $[t, T]$ for all $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}$. There exist two constants L and γ such that:

$$\begin{aligned} &|g(t, s, y, z)| \leq L(1 + |y|) + \frac{\gamma}{2}|z|^2, \quad \forall (t, s, y, z) \in \Delta[0, T] \times \mathbb{R} \times \mathbb{R}; \\ &|g(t, s, y_1, z_1) - g(t, s, y_2, z_2)| \leq L\big\{|y_1 - y_2| + (1 + |z_1| + |z_2|)|z_1 - z_2|\big\}, \\ &\qquad\qquad\qquad\qquad \forall (t, s, y_i, z_i) \in \Delta[0, T] \times \mathbb{R} \times \mathbb{R}, \ i = 1, 2. \end{aligned}$$

At the same time, we introduce the following additional assumption which will be used to establish a better regularity for the adapted solutions.

(A3) Let $g : [0, T]^2 \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ be measurable such that for every $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, $s \mapsto g(t, s, y, z)$ is $\mathbb{F}$-progressively measurable. There exists a modulus of continuity $\rho : [0, \infty) \to [0, \infty)$ (a continuous and monotone increasing function with $\rho(0) = 0$) such that

$$\begin{aligned} &|g(t, s, y, z) - g(t', s, y, z)| \leq \rho(|t - t'|)(1 + |y| + |z|^2), \\ &\qquad\qquad \forall\, t, t', s \in [0, T], \ (y, z) \in \mathbb{R} \times \mathbb{R}. \end{aligned}$$

Note that in (A3), the generator $g(t, s, y, z)$ is defined for (t, s) in the square domain $[0, T]^2$ instead of the triangle domain $\Delta[0, T]$, and the uniform continuity of the map $t \mapsto f(t, y, z)$ (uniform for (y, z) in any bounded set) is assumed. Now, we state the main result of this subsection.

Theorem 3.5 *Let* (A2) *hold. Then for any* $\psi(\cdot) \in L^\infty_{\mathcal{F}_T}(0, T)$, *BSVIE* (3.14) *admits a unique adapted solution* $(Y(\cdot), Z(\cdot, \cdot)) \in L^\infty_{\mathbb{F}}(0, T) \times \overline{\text{BMO}}(\Delta[0, T])$.

We will prove Theorem 3.5 by means of contraction mapping theorem. For any $(U(\cdot), V(\cdot,\cdot)) \in L^\infty_{\mathbb{F}}(0,T) \times \overline{\mathrm{BMO}}(\Delta[0,T])$, consider the following BSVIE:

$$Y(t) = \psi(t) + \int_t^T g(t,s,U(s),Z(t,s))ds - \int_t^T Z(t,s)dW(s). \tag{3.15}$$

By Theorem 3.2, BSVIE (3.15) admits a unique adapted solution $(Y(\cdot), Z(\cdot,\cdot)) \in L^\infty_{\mathbb{F}}(0,T) \times \overline{\mathrm{BMO}}\,(\Delta[0,T])$. Thus, the map

$$\Gamma(U(\cdot), V(\cdot,\cdot)) \triangleq (Y(\cdot), Z(\cdot,\cdot)), \quad (U(\cdot), V(\cdot,\cdot)) \in L^\infty_{\mathbb{F}}(0,T) \times \overline{\mathrm{BMO}}(\Delta[0,T]) \tag{3.16}$$

is well-defined. In order to prove Theorem 3.5, we present the following lemma.

Lemma 3.6 *Let* (A2) *hold and* $\varepsilon \in (0, \frac{1}{2L}]$. *Then for any* $\psi(\cdot) \in L^\infty_{\mathcal{F}_T}(0,T)$, *the map* $\Gamma(\cdot,\cdot)$ *defined by* (3.16) *satisfies the following:*

$$\Gamma(\mathcal{B}_\varepsilon) \subseteq \mathcal{B}_\varepsilon, \tag{3.17}$$

where $\mathcal{B}_\varepsilon$ *is defined by the following:*

$$\begin{aligned}\mathcal{B}_\varepsilon \triangleq \Big\{ &(U(\cdot), V(\cdot,\cdot)) \in L^\infty_{\mathbb{F}}(T-\varepsilon,T) \times \overline{\mathrm{BMO}}(\Delta[T-\varepsilon,T]) \;\Big|\\ &\|U(\cdot)\|_{L^\infty_{\mathbb{F}}(T-\varepsilon,T)} \le 2\|\psi(\cdot)\|_\infty + 1, \quad \|V(\cdot\,,\cdot)\|^2_{\overline{\mathrm{BMO}}(\Delta[T-\varepsilon,T])} \le A \Big\},\end{aligned} \tag{3.18}$$

with

$$A = \frac{2}{\gamma^2} e^{\gamma\|\psi(\cdot)\|_\infty} + \frac{1}{\gamma} e^{2(\gamma+1)\|\psi(\cdot)\|_\infty + \gamma + 2}.$$

Proof For any $(U(\cdot), V(\cdot,\cdot)) \in \mathcal{B}_\varepsilon$, consider a family of BSDEs (parameterized by $t \in [0,T]$):

$$\begin{aligned}\eta(t,s) = \psi(t) &+ \int_s^T g(t,r,U(r),\zeta(t,r))dr\\ &- \int_s^T \zeta(t,r)dW(r), \quad s \in [t,T].\end{aligned} \tag{3.19}$$

Note that $U(\cdot)$ is bounded. For almost all $t \in [T-\varepsilon, T]$, by Lemma 2.5, the above BSDE admits a unique adapted solution $(\eta(t,\cdot), \zeta(t,\cdot)) \in L^\infty_{\mathbb{F}}(\Omega; C[t,T]) \times \overline{\mathrm{BMO}}(t,T)$. Let

$$Y(t) = \eta(t,t), \quad Z(t,s) = \zeta(t,s), \quad (t,s) \in \Delta[T-\varepsilon,T]. \tag{3.20}$$

Then by Theorem 3.2, $(Y(\cdot), Z(\cdot,\cdot)) \in L^\infty_{\mathbb{F}}(0,T) \times \overline{\text{BMO}}(\Delta[0,T])$ is the unique adapted solution to BSVIE (3.15). The rest of the proof is divided into two steps.

Step 1 *Estimate of* $\|Y(\cdot)\|_\infty$.

For BSDE (3.19), by (A2), we have

$$|g(t,r,U(r),\zeta)| \leq L\big(1+|U(r)|\big) + \frac{\gamma}{2}|\zeta|^2.$$

Thus, note that $\varepsilon \in (0, \frac{1}{2L}]$, by Lemma 2.5 with $h(s) = L(1+|U(s)|)$, $\gamma = \gamma$ and $\beta = 0$, we have

$$\begin{aligned} e^{\gamma|\eta(t,s)|} &\leq \mathbb{E}_s\Big[e^{\gamma\big(|\psi(t)|+L\int_s^T(1+|U(r)|)dr\big)}\Big] \leq e^{\gamma\big[\|\psi(\cdot)\|_\infty + L\varepsilon\big(1+\|U(\cdot)\|_{L^\infty_{\mathbb{F}}(T-\varepsilon,T)}\big)\big]} \\ &\leq e^{\gamma(2\|\psi(\cdot)\|_\infty+1)}, \quad T-\varepsilon \leq t \leq s \leq T, \end{aligned} \tag{3.21}$$

which is equivalent to

$$|\eta(t,s)| \leq 2\|\psi(\cdot)\|_\infty + 1, \quad T-\varepsilon \leq t \leq s \leq T. \tag{3.22}$$

Consequently, noting $Y(t) = \eta(t,t)$, one has

$$\|Y(\cdot)\|_{L^\infty_{\mathbb{F}}(T-\varepsilon,T)} \leq 2\|\psi(\cdot)\|_\infty + 1.$$

Step 2 *Estimate of* $\|Z(\cdot\,,\cdot)\|^2_{\overline{\text{BMO}}(\Delta[T-\varepsilon,T])}$.

Define

$$\phi(y) \triangleq \gamma^{-2}\big(e^{\gamma|y|} - \gamma|y| - 1\big); \quad y \in \mathbb{R}. \tag{3.23}$$

Then, we have

$$\phi'(y) = \gamma^{-1}[e^{\gamma|y|} - 1]\text{sgn}(y), \quad \phi''(y) = e^{\gamma|y|}, \tag{3.24}$$

which leads to $\phi''(y) = \gamma|\phi'(y)| + 1$. Applying Itô's formula to $s \mapsto \phi(\eta(t,s))$, we have

$$\begin{aligned} &\phi(\psi(t)) - \phi(\eta(t,s)) \\ &\quad = -\int_s^T \phi'(\eta(t,r))g(t,r,U(r),\zeta(t,r))dr + \frac{1}{2}\int_s^T \phi''(\eta(t,r))|\zeta(t,r)|^2 dr \\ &\qquad + \int_s^T \phi'(\eta(t,r))\zeta(t,r)dW(r), \quad s \in [t,T]. \end{aligned} \tag{3.25}$$

Taking conditional expectation on the both sides of (3.25) and by (A2), we have

$$\begin{aligned}&\phi(\eta(t,s))+\frac{1}{2}\mathbb{E}_s\Big[\int_s^T\phi''(\eta(t,r))|\zeta(t,r)|^2dr\Big]\\&\le\phi(\|\psi(\cdot)\|_\infty)+L\mathbb{E}_s\Big[\int_s^T|\phi'(\eta(t,r))|\big(1+|U(r)|\big)dr\Big]\\&\quad+\frac{\gamma}{2}\mathbb{E}_s\Big[\int_s^T|\phi'(\eta(t,r))|\,|\zeta(t,r)|^2dr\Big].\end{aligned}$$

Combining this with (3.24), one obtains

$$\begin{aligned}&\phi(\eta(t,s))+\frac{1}{2}\mathbb{E}_s\Big[\int_s^T|\zeta(t,r)|^2dr\Big]\\&\le\phi(\|\psi(\cdot)\|_\infty)+L\mathbb{E}_s\Big[\int_s^T|\phi'(\eta(t,r))|(1+|U(r)|)dr\Big].\end{aligned}\tag{3.26}$$

Then, noting that $\phi(\eta(t,s))\ge 0$, we simply drop it to get

$$\begin{aligned}\mathbb{E}_s\Big[\int_s^T|Z(t,r)|^2dr\Big]&\le 2\phi(\|\psi(\cdot)\|_\infty)+2L\mathbb{E}_s\Big[\int_s^T|\phi'(\eta(t,r))|(1+|U(r)|)dr\Big]\\&\le\frac{2}{\gamma^2}e^{\gamma\|\psi(\cdot)\|_\infty}+\frac{2L}{\gamma}\varepsilon e^{\gamma(2\|\psi(\cdot)\|_\infty+1)}e^{2(\|\psi(\cdot)\|_\infty+1)}\\&\le\frac{2}{\gamma^2}e^{\gamma\|\psi(\cdot)\|_\infty}+\frac{1}{\gamma}e^{2(\gamma+1)\|\psi(\cdot)\|_\infty+\gamma+2}.\end{aligned}$$

Hence,

$$\|Z(\cdot\,,\cdot)\|^2_{\overline{\mathrm{BMO}}(\Delta[T-\varepsilon,T])}\le\frac{2}{\gamma^2}e^{\gamma\|\psi(\cdot)\|_\infty}+\frac{1}{\gamma}e^{2(\gamma+1)\|\psi(\cdot)\|_\infty+\gamma+2}=A.\tag{3.27}$$

This proves our claim. □

The next result is concerned with the local solution of BSVIE (3.14).

Proposition 3.7 *Let* (A2) *hold and the map* $\Gamma(\cdot\,,\cdot)$ *be defined by* (3.16). *Then there is* $\varepsilon>0$ *such that* $\Gamma(\cdot\,,\cdot)$ *is a contraction on* $\mathcal{B}_\varepsilon$*, where* $\mathcal{B}_\varepsilon$ *is defined by* (3.18). *This implies that BSVIE* (3.14) *admits a unique adapted solution on* $[T-\varepsilon,T]$.

Proof Let $\varepsilon\in(0,\frac{1}{2L}]$. For any $(U(\cdot),V(\cdot\,,\cdot)),(\widetilde U(\cdot),\widetilde V(\cdot\,,\cdot))\in\mathcal{B}_\varepsilon$, set

$$(Y(\cdot),Z(\cdot,\cdot))=\Gamma(U(\cdot),V(\cdot\,,\cdot))\quad\text{and}\quad(\widetilde Y(\cdot),\widetilde Z(\cdot))=\Gamma(\widetilde U(\cdot),\widetilde V(\cdot\,,\cdot));\tag{3.28}$$

that is,

$$\eta(t,s)=\psi(t)+\int_s^T g(t,r,U(r),\zeta(t,r))ds-\int_s^T\zeta(t,r)dW(r),\tag{3.29}$$

$$\widetilde{\eta}(t,s)=\psi(t)+\int_s^T g(t,r,\widetilde{U}(r),\widetilde{\zeta}(t,r))dr-\int_s^T\widetilde{\zeta}(t,r)dW(r), \tag{3.30}$$

and

$$Y(t)=\eta(t,t),\ \widetilde{Y}(t)=\widetilde{\eta}(t,t),\quad Z(t,r)=\zeta(t,r),\ \widetilde{Z}(t,r)=\widetilde{\zeta}(t,r). \tag{3.31}$$

By Lemma 3.6, $(Y(\cdot),Z(\cdot\,,\cdot))$ and $(\widetilde{Y}(\cdot),\widetilde{Z}(\cdot\,,\cdot))\in\mathcal{B}_\varepsilon$. By (A2), for almost all $t\in[T-\varepsilon,T]$, we can define the process $\theta(t,\cdot)$ in an obvious way such that:

$$\theta(t,s)=0,\quad (t,s)\in[T-\varepsilon,T]\times[0,t], \tag{3.32}$$

$$|\theta(t,s)|\le L(1+|\zeta(t,s)|+|\widetilde{\zeta}(t,s)|),\quad (t,s)\in\Delta[T-\varepsilon,T], \tag{3.33}$$

$$g(t,s,\widetilde{U}(s),\zeta(t,s))-g(t,s,\widetilde{U}(s),\widetilde{\zeta}(t,s))=[\zeta(t,s)-\widetilde{\zeta}(t,s)]\theta(t,s). \tag{3.34}$$

Note that $(Y(\cdot),\zeta(\cdot\,,\cdot)),(\widetilde{Y}(\cdot),\widetilde{\zeta}(\cdot\,,\cdot))\in\mathcal{B}_\varepsilon$. Thus, by (3.32)–(3.33),

$$\begin{aligned}\|\theta(\cdot,\cdot)\|^2_{\overline{\mathrm{BMO}}(\Delta[T-\varepsilon,T])}&\le 3L^2T+3L^2\|\zeta(\cdot,\cdot)\|^2_{\overline{\mathrm{BMO}}(\Delta[T-\varepsilon,T])}\\&\quad+3L^2\|\widetilde{\zeta}(\cdot,\cdot)\|^2_{\overline{\mathrm{BMO}}(\Delta[T-\varepsilon,T])}]\\&\le 3L^2T+6L^2A.\end{aligned} \tag{3.35}$$

Thus, for almost all $t\in[T-\varepsilon,T]$, $\int_0^s\theta(t,r)dW(r);\,0\le s\le T$ is a BMO martingale and

$$\left\|\int_0^\cdot\theta(t,r)dW(r)\right\|^2_{\mathrm{BMO}(0,T)}\le 3L^2T+6L^2A. \tag{3.36}$$

By Lemma 2.3, $W(t;\cdot)$ defined by

$$W(t;s)\triangleq W(s)-\int_0^s\theta(t,r)dr,\quad s\in[0,T] \tag{3.37}$$

is a Brownian motion on $[0,T]$ under the equivalent probability measure $\overline{\mathbb{P}}_t$, which is defined by

$$d\overline{\mathbb{P}}_t\triangleq\mathcal{E}\{\theta(t,\cdot)\}_T d\mathbb{P}. \tag{3.38}$$

Denote the expectation in $\bar{\mathbb{P}}_t$ by $\mathbb{E}^{\bar{\mathbb{P}}_t}$. Combining (3.29), (3.30), and (3.34)–(3.37), we have

$$\begin{aligned}&\eta(t,s)-\widetilde{\eta}(t,s)+\int_s^T[\zeta(t,r)-\widetilde{\zeta}(t,r)]dW(t;r)\\&=\int_s^T\big[g(t,r,U(r),\zeta(t,r))-g(t,r,\widetilde{U}(r),\zeta(t,r))\big]dr.\end{aligned} \tag{3.39}$$

Taking square and then taking conditional expectation with respect to $\bar{\mathbb{P}}_t$ on the both sides of the above equation, we have (noting $T-\varepsilon \le t \le s \le T$)

$$
\begin{aligned}
&|\eta(t,s)-\widetilde{\eta}(t,s)|^2+\mathbb{E}_s^{\bar{\mathbb{P}}_t}\Big[\int_s^T |\zeta(t,r)-\widetilde{\zeta}(t,r)|^2dr\Big]\\
&=\mathbb{E}_s^{\bar{\mathbb{P}}_t}\Big\{\Big[\int_s^T \Big(g(t,r,U(r),\zeta(t,r))-g(t,r,\widetilde{U}(r),\zeta(t,r))\Big)dr\Big]^2\Big\}\\
&\le \mathbb{E}_s^{\bar{\mathbb{P}}_t}\Big\{\Big[\int_s^T \Big(L|U(r)-\widetilde{U}(r)|\Big)dr\Big]^2\Big\} \qquad (3.40)\\
&\le L^2(T-t)^2\|U(\cdot)-\widetilde{U}(\cdot)\|^2_{L^\infty_{\mathbb{F}}(T-\varepsilon,T)}\\
&\le L^2\varepsilon^2\|U(\cdot)-\widetilde{U}(\cdot)\|^2_{L^\infty_{\mathbb{F}}(T-\varepsilon,T)}.
\end{aligned}
$$

Let $s=t$, by (3.31) and (3.40), we have

$$
\|Y(\cdot)-\widetilde{Y}(\cdot)\|^2_{L^\infty_{\mathbb{F}}(T-\varepsilon,T)} \le L^2\varepsilon^2\|U(\cdot)-\widetilde{U}(\cdot)\|^2_{L^\infty_{\mathbb{F}}(T-\varepsilon,T)}. \qquad (3.41)
$$

Also, by (3.31), (3.40), (3.36), and Lemma 2.4, there is a constant C (which is depending on $\|\psi(\cdot)\|_\infty$ and is independent of t) such that

$$
\begin{aligned}
&\sup_{s\in[t,T]}\mathbb{E}_s\Big[\int_s^T |Z(t,r)-\tilde{Z}(t,r)|^2dr\Big]=\sup_{s\in[t,T]}\mathbb{E}_s\Big[\int_s^T |\zeta(t,r)-\widetilde{\zeta}(t,r)|^2dr\Big]\\
&\le C\sup_{s\in[t,T]}\mathbb{E}_s^{\bar{\mathbb{P}}_t}\Big[\int_s^T |\zeta(t,r)-\tilde{\zeta}(t,r)|^2dr\Big]\le CL^2\varepsilon^2\|U(\cdot)-\widetilde{U}(\cdot)\|^2_{L^\infty_{\mathbb{F}}(T-\varepsilon,T)}.
\end{aligned} \qquad (3.42)
$$

Thus,

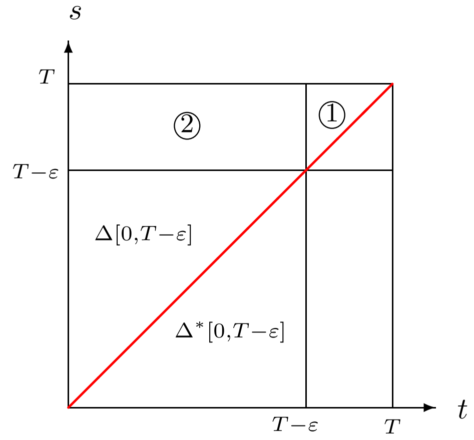
$$
\|Z(\cdot,\cdot)-\widetilde{Z}(\cdot,\cdot)\|^2_{\overline{\mathrm{BMO}}(\Delta[T-\varepsilon,T])} \le CL^2\varepsilon^2\|U(\cdot)-\widetilde{U}(\cdot)\|^2_{L^\infty_{\mathbb{F}}(T-\varepsilon,T)}. \qquad (3.43)
$$

Combining (3.41)–(3.43), we see that for some small $\varepsilon>0$, the map $\Gamma(\cdot,\cdot)$ is a contraction on the set $\mathcal{B}_\varepsilon$. Hence, BSVIE (3.14) admits a unique adapted solution on $[T-\varepsilon,T]$. □

Let us make some comments on the above local existence of the unique adapted solution.
We have seen that $(Y(s),Z(t,s))$ is defined for $(t,s)\in\Delta[T-\varepsilon,T]$, the region marked ① in the Fig. 1. Now, for any $t\in[0,T-\varepsilon]$, we can rewrite our Type-I BSVIE as follows:

$$
\begin{aligned}
Y(t)=\psi^{T-\varepsilon}(t)&+\int_t^{T-\varepsilon} g(t,s,Y(s),Z(t,s))ds\\
&-\int_t^{T-\varepsilon} Z(t,s)dW(s),\quad t\in[0,T-\varepsilon],
\end{aligned} \qquad (3.44)
$$

Fig. 1 Type-I



where

$$\psi^{T-\varepsilon}(t) = \psi(t) + \int_{T-\varepsilon}^{T} g(t,s,Y(s),Z(t,s))ds - \int_{T-\varepsilon}^{T} Z(t,s)dW(s), \quad t \in [0, T-\varepsilon]. \tag{3.45}$$

If $\psi^{T-\varepsilon}(\cdot) \in L^{\infty}_{\mathcal{F}_{T-\varepsilon}}(0, T-\varepsilon)$, then (3.44) is a BSVIE on $[0, T-\varepsilon]$. However, unlike BSDEs, having $(Y(s), Z(t,s))$ defined on $\Delta[T-\varepsilon, T]$, $\psi^{T-\varepsilon}(t); t \in [0, T-\varepsilon]$ has still not been defined yet. Since, on the right-hand side of (3.45), although $Y(s)$ with $s \in [T-\varepsilon, T]$ has already been determined, $Z(t,s)$ has not been defined for $(t,s) \in [0, T-\varepsilon] \times [T-\varepsilon, T]$, the region marked ② in the Fig. 1, which is needed to define $\psi^{T-\varepsilon}(t)$. Moreover, we need that $\psi^{T-\varepsilon}(t)$ is $\mathcal{F}_{T-\varepsilon}$-measurable (not just $\mathcal{F}_T$-measurable). Hence, (3.45) is actually a *stochastic Fredholm integral equation* (SFIE, for short) to be solved to determine $\psi^{T-\varepsilon}(t); t \in [0, T-\varepsilon]$.

Now, we are at the position to prove Theorem 3.5.

Proof of Theorem 3.5 The proof will be divided into three steps.

Step 1 *Estimate of* $|Y(\cdot)|^2$.

For given $\psi(\cdot) \in L^{\infty}_{\mathcal{F}_T}(0, T)$, we can find a constant $\widetilde{C} > 0$ such that $\|\psi(\cdot)\|^2_{\infty} \le \widetilde{C}$ and (by (A2))

$$|2xg(t,s,y,0)| \le \widetilde{C} + \widetilde{C}|x|^2 + \widetilde{C}|y|^2, \quad \forall (t,s,x,y) \in \Delta[0,T] \times \mathbb{R} \times \mathbb{R}. \tag{3.46}$$

Let us consider the following (integral form of) ordinary differential equation:

$$\alpha(t) = \widetilde{C} + \int_t^T \widetilde{C}\alpha(s)ds + \int_t^T \widetilde{C}[\alpha(s)+1]ds, \quad t \in [0, T]. \tag{3.47}$$

It is easy to see that the unique solution to the above ordinary differential equation is given by

$$\alpha(t) = \Big(\widetilde{C} + \frac{1}{2}\Big)e^{2\widetilde{C}(T-t)} - \frac{1}{2}, \quad t \in [0, T],$$

which is a (continuous) decreasing function. Thus,

$$\|\psi(\cdot)\|_\infty^2 \le \widetilde{C} = \alpha(T) \le \alpha(0).$$

By Proposition 3.7, there exists an $\varepsilon > 0$ (depending on $\|\psi(\cdot)\|_\infty$) such that $\Gamma(\cdot\,,\cdot)$ defined by (3.16) is a contraction on $\mathcal{B}_\varepsilon$. Therefore, a Picard iteration sequence converges to the unique adapted solution $(Y(\cdot), Z(\cdot,\cdot))$ of the BSVIE on $[T-\varepsilon, T]$. Namely, if we define:

$$\begin{cases} (Y^0(\cdot), Z^0(\cdot,\cdot)) = 0, \\ (Y^{k+1}(\cdot), Z^{k+1}(\cdot,\cdot)) = \Gamma(Y^k(\cdot), Z^k(\cdot,\cdot)), \quad k \ge 0; \end{cases} \tag{3.48}$$

that is,

$$\begin{aligned} &(Y^0(\cdot), Z^0(\cdot,\cdot)) = 0, \\ &\eta^{k+1}(t,s) = \psi(t) + \int_s^T g(t,r,Y^k(r),\zeta^{k+1}(t,r))dr - \int_s^T \zeta^{k+1}(t,r)dW(r), \\ &Y^{k+1}(t) = \eta^{k+1}(t,t), \quad Z^{k+1}(t,s) = \zeta^{k+1}(t,s), \quad (t,s) \in \Delta[T-\varepsilon, T], \end{aligned}$$

then

$$\lim_{k\to\infty} \|(Y^k(\cdot), Z^k(\cdot,\cdot)) - (Y(\cdot), Z(\cdot,\cdot))\|_{L^\infty_{\mathbb{F}}(T-\varepsilon,T)\times\overline{\mathrm{BMO}}(\Delta[T-\varepsilon,T])} = 0. \tag{3.49}$$

Next, for almost all $t \in [T-\varepsilon, T]$, similar to (3.33), (3.34), (3.37), and (3.38), there exists a process $\theta^{k+1}(t,\cdot)$ such that

$$g(t,r,Y^k(r),\zeta^{k+1}(t,r)) - g(t,r,Y^k(r),0) = \zeta^{k+1}(t,r)\theta^{k+1}(t,r), \tag{3.50}$$

and

$$W^{k+1}(t;s) \triangleq W(s) - \int_0^s \theta^{k+1}(t,r)dr, \quad s \in [0, T] \tag{3.51}$$

is a Brownian motion on $[0, T]$ under the corresponding equivalent probability measure $\mathbb{P}_t^{k+1}$ defined by

$$\mathbb{P}_t^{k+1} = \mathcal{E}\{\theta^{k+1}(t,\cdot)\}_T d\mathbb{P}.$$

For simplicity, we denote $\mathbb{P}_t^{k+1}$ by $\mathbb{P}^{k+1}$ here, suppressing the subscript t. The corresponding expectation is denoted by $\mathbb{E}^{k+1}$. It follows that

$$\begin{aligned}\eta^{k+1}(t,s) &= \psi(t) + \int_s^T g(t,r,Y^k(r),\zeta^{k+1}(t,r))dr - \int_s^T \zeta^{k+1}(t,r)dW(r),\\ &= \psi(t) + \int_s^T g(t,r,Y^k(r),0)dr - \int_s^T \zeta^{k+1}(t,r)dW^{k+1}(t;r). \end{aligned} \tag{3.52}$$

Applying the Itô formula to the map $s \mapsto |\eta^{k+1}(t,s)|^2$ and taking conditional expectation $\mathbb{E}_\tau^{k+1} = \mathbb{E}^{k+1}[\,\cdot \mid \mathcal{F}_\tau]$ for any $\tau \in [T-\varepsilon, s]$, by (3.46), we have

$$\begin{aligned}&\mathbb{E}_\tau^{k+1}\Big[|\eta^{k+1}(t,s)|^2\Big] + \mathbb{E}_\tau^{k+1}\Big[\int_s^T |\zeta^{k+1}(t,r)|^2 dr\Big]\\ &= \mathbb{E}_\tau^{k+1}\Big[|\psi(t)|^2\Big] + \mathbb{E}_\tau^{k+1}\Big[\int_s^T 2\eta^{k+1}(t,r) g(t,r,Y^k(r),0)dr\Big] \\ &\le \widetilde{C} + \widetilde{C}\int_s^T \mathbb{E}_\tau^{k+1}\Big[|\eta^{k+1}(t,r)|^2\Big]dr + \widetilde{C}\int_s^T \Big\{\mathbb{E}_\tau^{k+1}\Big[|Y^k(r)|^2\Big] + 1\Big\}dr.\end{aligned} \tag{3.53}$$

We now prove the following inequality by induction:

$$|Y^k(t)|^2 \le \alpha(t), \quad t \in [T-\varepsilon, T], \quad \text{for any } k \ge 0. \tag{3.54}$$

In fact, by (3.48), it is obvious to see $|Y^0(t)|^2 = 0 \le \alpha(t)$. Suppose $|Y^k(t)|^2 \le \alpha(t)$ for any $t \in [T-\varepsilon, T]$, then

$$\begin{aligned}&\mathbb{E}_\tau^{k+1}\Big[|\eta^{k+1}(t,s)|^2\Big]\\ &\quad \le \widetilde{C} + \widetilde{C}\int_s^T \mathbb{E}_\tau^{k+1}\Big[|\eta^{k+1}(t,r)|^2\Big]dr + \widetilde{C}\int_s^T [\alpha(r)+1]dr.\end{aligned} \tag{3.55}$$

In light of (3.47), by the comparison theorem of ordinary differential equations, we have

$$\mathbb{E}_\tau^{k+1}\Big[|\eta^{k+1}(t,s)|^2\Big] \le \alpha(s). \tag{3.56}$$

Let $\tau = s$ and $s = t$, we have

$$|Y^{k+1}(t)|^2 \le \alpha(t), \quad t \in [T-\varepsilon, T]. \tag{3.57}$$

Thus, by induction, (3.54) holds. Then by (3.49), we have

$$|Y(t)|^2 \le \alpha(t), \quad t \in [T-\varepsilon, T]. \tag{3.58}$$

Step 2 *A related stochastic Fredholm integral equation is solvable on* $[0, T-\varepsilon]$.

We now solve SFIE (3.45) on $[0, T-\varepsilon]$. Let us introduce a family of BSDEs parameterized by $t \in [0, T-\varepsilon]$:

$$\eta(t,s) = \psi(t) + \int_s^T g(t,r,Y(r),\zeta(t,r))dr - \int_s^T \zeta(t,r)dW(r), \quad s \in [T-\varepsilon, T]. \tag{3.59}$$

By Lemma 2.5, the above BSDE admits a unique adapted solution $(\eta(t,\cdot), \zeta(t,\cdot))$ on $[T-\varepsilon, T]$. Note that (3.58), similar to (3.56), we have

$$|\eta(t,s)|^2 \leq \alpha(s), \quad s \in [T-\varepsilon, T]. \tag{3.60}$$

Similar to (3.27), we have

$$\operatorname*{esssup}_{t\in[0,T-\varepsilon]} \|\zeta(t,\cdot)\|^2_{\overline{\mathrm{BMO}}([T-\varepsilon,T])} < \infty. \tag{3.61}$$

Let $\psi^{T-\varepsilon}(t) = \eta(t, T-\varepsilon)$ and $Z(t,s) = \zeta(t,s)$, we have $(\psi^{T-\varepsilon}(\cdot), Z(\cdot,\cdot)) \in L^\infty_{\mathcal{F}_{T-\varepsilon}}(0, T-\varepsilon) \times \overline{\mathrm{BMO}}([0, T-\varepsilon] \times [T-\varepsilon, T])$ and $(\psi^{T-\varepsilon}(\cdot), Z(\cdot,\cdot))$ is a solution to SFIE (3.45). Moreover, by (3.60), we have

$$|\psi^{T-\varepsilon}(t)|^2 = |\eta(t, T-\varepsilon)|^2 \leq \alpha(T-\varepsilon) \leq \alpha(0), \quad t \in [0, T-\varepsilon]. \tag{3.62}$$

Next, we will prove the solution to SFIE (3.45) is unique. Let

$$\begin{aligned} &(\psi^{T-\varepsilon}(\cdot), Z(\cdot,\cdot)),\ (\widetilde{\psi}^{T-\varepsilon}(\cdot), \widetilde{Z}(\cdot,\cdot)) \\ &\quad \in L^\infty_{\mathcal{F}_{T-\varepsilon}}(0, T-\varepsilon) \times \overline{\mathrm{BMO}}([0, T-\varepsilon] \times [T-\varepsilon, T]). \end{aligned}$$

be two solutions to SFIE (3.45). Then

$$\begin{aligned} \psi^{T-\varepsilon}(t) - \widetilde{\psi}^{T-\varepsilon}(t) &= \int_{T-\varepsilon}^T \Big[g(t,s,Y(s),Z(t,s)) - g(t,s,Y(s),\widetilde{Z}(t,s))\Big]ds \\ &\quad - \int_{T-\varepsilon}^T \Big[Z(t,s) - \widetilde{Z}(t,s)\Big]dW(s), \quad t \in [0, T-\varepsilon]. \end{aligned} \tag{3.63}$$

For almost all $t \in [0, T-\varepsilon]$, similar to (3.33), (3.34), (3.37), and (3.38), there is a process $\widetilde{\theta}(t,\cdot)$ such that:

$$g(t,s,Y(s),Z(t,s)) - g(t,s,Y(s),\widetilde{Z}(t,s)) = [Z(t,s) - \widetilde{Z}(t,s)]\widetilde{\theta}(t,s), \tag{3.64}$$

and

$$\overline{W}(t;s) \triangleq W(s) - \int_0^s \widetilde{\theta}(t,r)dr, \quad s \in [0, T] \tag{3.65}$$

is a Brownian motion on $[0, T]$ under the corresponding equivalent probability measure $\overline{\mathbb{P}}_t$. The corresponding expectation is denoted by $\mathbb{E}^{\bar{\mathbb{P}}_t}$. Combining (3.63)–(3.65), we have

$$\begin{aligned}&\psi^{T-\varepsilon}(t)-\widetilde{\psi}^{T-\varepsilon}(t)\\&\quad=-\int_{T-\varepsilon}^{T}\Big[Z(t,s)-\widetilde{Z}(t,s)\Big]d\overline{W}(t;s),\quad t\in[0,T-\varepsilon].\end{aligned}\tag{3.66}$$

Taking conditional expectation $\mathbb{E}^{\bar{\mathbb{P}}_t}_{T-\varepsilon}[\,\cdot\,]\equiv\mathbb{E}^{\bar{\mathbb{P}}_t}[\,\cdot\,|\,\mathcal{F}_{T-\varepsilon}]$ on the both sides of the equation (3.66), we have

$$\mathbb{E}^{\bar{\mathbb{P}}_t}_{T-\varepsilon}\Big[\psi^{T-\varepsilon}(t)-\widetilde{\psi}^{T-\varepsilon}(t)\Big]=0,\quad t\in[0,T-\varepsilon].\tag{3.67}$$

Note that $\psi^{T-\varepsilon}(t)$ is $\mathcal{F}_{T-\varepsilon}$-adapted for any $t\in[0,T-\varepsilon]$. It follows that

$$\psi^{T-\varepsilon}(t)=\widetilde{\psi}^{T-\varepsilon}(t),\quad \text{a.s.},\ t\in[0,T-\varepsilon].\tag{3.68}$$

By (3.66)–(3.68), we have

$$\int_{T-\varepsilon}^{T}\big[Z(t,s)-\widetilde{Z}(t,s)\big]\,d\overline{W}(t;s)=0,\quad t\in[0,T-\varepsilon],\tag{3.69}$$

which implies

$$Z(t,s)=\widetilde{Z}(t,s),\quad \text{a.s.},\ (t,s)\in[0,T-\varepsilon]\times[T-\varepsilon,T].\tag{3.70}$$

Combining (3.68)–(3.70), SFIE (3.45) admits a unique solution.

Step 3 *Complete the proof by induction.*

Combining Steps 1 and 2, we have uniquely determined

$$\begin{cases}Y(t), & t\in[T-\varepsilon,T],\\ Z(t,s), & (t,s)\in\Delta[T-\varepsilon,T]\bigcup\Big([0,T-\varepsilon]\times[T-\varepsilon,T]\Big).\end{cases}\tag{3.71}$$

Now, we consider BSVIE (3.44) on $[0, T-\varepsilon]$. By (3.62), we see that the above procedure can be repeated. We point out that the introduction of $\alpha(\cdot)$ is to uniformly control the terminal state $\psi(T-\varepsilon)$, $\psi(T-2\varepsilon)$, etc. Then we can use induction to finish the proof of the existence and uniqueness of adapted solution to BSVIE (3.14).□

Remark 3.8 When the terminal condition $\psi(\cdot)$ is bounded, the well-posedness of QBSVIE (3.14) is established by Theorem 3.5. If $\psi(\cdot)$ is unbounded, the unboundedness of $\psi(\cdot)$ will bring some essential difficulties in establishing the solvability of QBSVIE (3.14). At the moment, we are not able to overcome the difficulties. We hope to come back in our future publications.

We now would like to look at some better regularity for the adapted solution of BSVIEs under additional conditions. More precisely, we introduce the following assumption.

Theorem 3.9 *Let* (A2)–(A3) *hold. Then for any* $\psi(\cdot) \in L^\infty_{\mathcal{F}_T}(\Omega; C^U[0,T])$, *BSVIE* (3.14) *admits a unique adapted solution* $(Y(\cdot), Z(\cdot,\cdot)) \in L^\infty_{\mathbb{F}}(\Omega; C[0,T]) \times \overline{\text{BMO}}(\Delta[0,T])$.

Proof Without loss of generality, let us assume that

$$|\psi(t') - \psi(t)| \le \rho(|t - t'|), \quad \forall\, t, t' \in [0,T],$$

with the same modulus of continuity $\rho(\cdot)$ given in (A3).

By Theorem 3.5, BSVIE (3.14) admits a unique adapted solution $(Y(\cdot), Z(\cdot,\cdot)) \in L^\infty_{\mathbb{F}}(0,T) \times \overline{\text{BMO}}(\Delta[0,T])$. We just need to prove that $Y(\cdot) \in L^\infty_{\mathbb{F}}(\Omega; C[0,T])$, i.e., $Y(\cdot)$ is continuous. Consider the following family of BSDEs (parameterized by $t \in [0,T]$):

$$\begin{aligned}\eta(t,s) = \psi(t) + \int_s^T g(t,r,Y(r),\zeta(t,r))dr \\ - \int_s^T \zeta(t,r)dW(r), \quad s \in [0,T].\end{aligned} \tag{3.72}$$

By Lemma 2.5, for any $t \in [0,T]$, BSDE (3.72) admits a unique adapted solution $(\eta(t,\cdot), \zeta(t,\cdot)) \in L^\infty_{\mathbb{F}}(\Omega; C[0,T]) \times \overline{\text{BMO}}(0,T)$. By Theorem 3.5, we have $Y(t) = \eta(t,t)$, $Z(t,s) = \zeta(t,s)$ for any $(t,s) \in \Delta[0,T]$. Now, let $0 \le t < t' \le T$. Similar to (3.33), (3.34), (3.37), and (3.38), there is a process $\theta(t,t';\cdot)$ such that

$$\begin{aligned}&g(t',s,Y(s),\zeta(t,s)) - g(t',s,Y(s),\zeta(t',s)) \\ &= [\zeta(t,s) - \zeta(t',s)]\theta(t,t';s),\end{aligned} \tag{3.73}$$

and

$$W(t,t';s) \triangleq W(s) - \int_0^s \theta(t,t';r)dr, \quad s \in [0,T] \tag{3.74}$$

is a Brownian motion on $[0,T]$ under the corresponding equivalent probability measure $\mathbb{P}_{t,t'}$. The corresponding expectation is denoted by $\mathbb{E}^{\mathbb{P}_{t,t'}}$. Combining (3.72), (3.73), and (3.74), we have

$$\begin{aligned}\eta(t,s) - \eta(t',s) = \psi(t) - \psi(t') - \int_s^T [\zeta(t,r) - \zeta(t',r)]dW(t,t';r) \\ + \int_s^T [g(t,r,Y(r),\zeta(t,r)) - g(t',r,Y(r),\zeta(t,r))]dr.\end{aligned}$$

Taking conditional expectation $\mathbb{E}_s^{\mathbb{P}_{t,t'}}[\,\cdot\,] \equiv \mathbb{E}_s^{\mathbb{P}_{t,t'}}[\,\cdot\,|\mathcal{F}_s]$ on the both sides of the above equation, we have

$$\eta(t,s) - \eta(t',s) = \mathbb{E}_s^{\mathbb{P}_{t,t'}}\Big[\psi(t) - \psi(t') + \int_s^T \Big(g(t,r,Y(r),\zeta(t,r)) - g(t',r,Y(r),\zeta(t,r))\Big)dr\Big].$$

Combining this with (A3), by Lemma 2.4, we have

$$\begin{aligned}
&|\eta(t,s) - \eta(t',s)| \\
&\le \mathbb{E}_s^{\mathbb{P}_{t,t'}}\Big[|\psi(t) - \psi(t')| + \int_s^T |g(t,r,Y(r),\zeta(t,r)) - g(t',r,Y(r),\zeta(t,r))|dr\Big] \\
&\le \rho(|t-t'|) + \rho(|t-t'|)\mathbb{E}_s^{\mathbb{P}_{t,t'}}\Big[\int_s^T (1 + |Y(s)| + |\zeta(t,r)|)dr\Big] \\
&\le C(1 + \|Y(\cdot)\|_{L^\infty_{\mathbb{F}}(0,T)})\rho(|t-t'|) + C\rho(|t-t'|)\mathbb{E}_s^{\mathbb{P}_{t,t'}}\Big[\int_s^T |\zeta(t,r)|^2 dr\Big] \\
&\le C(1 + \|Y(\cdot)\|_{L^\infty_{\mathbb{F}}(0,T)})\rho(|t-t'|) + C\rho(|t-t'|)\|\zeta(t,\cdot)\|_{\overline{\mathrm{BMO}}_{\mathbb{P}_{t,t'}}(t,T)} \\
&\le C(1 + \|Y(\cdot)\|_{L^\infty_{\mathbb{F}}(0,T)})\rho(|t-t'|) + C\rho(|t-t'|)\|\zeta(t,\cdot)\|_{\overline{\mathrm{BMO}}_{\mathbb{P}}(t,T)} \\
&\le C(1 + \|Y(\cdot)\|_{L^\infty_{\mathbb{F}}(0,T)} + \|\zeta(\cdot,\cdot)\|_{\overline{\mathrm{BMO}}_{\mathbb{P}}(\Delta[0,T])})\rho(|t-t'|) \\
&= C(1 + \|Y(\cdot)\|_{L^\infty_{\mathbb{F}}(0,T)} + \|Z(\cdot,\cdot)\|_{\overline{\mathrm{BMO}}_{\mathbb{P}}(\Delta[0,T])})\rho(|t-t'|),
\end{aligned}$$

where $C > 0$ is a generic constant (which could be different from line to line). This leads to

$$\lim_{|t-t'|\to 0}\Big[\sup_{s\in[0,T]} |\eta(t,s) - \eta(t',s)|\Big] = 0, \text{ a.s.}$$

On the other hand, since $\eta(t,\cdot) \in L^\infty_{\mathbb{F}}(\Omega; C[0,T])$ for any $t \in [0,T]$, one has

$$\lim_{|s-s'|\to 0} |\eta(t,s) - \eta(t,s')| = 0, \quad \forall t \in [0,T], \text{ a.s.} \tag{3.75}$$

It follows that $(t,s) \mapsto \eta(t,s)$ is continuous, i.e.,

$$\lim_{(t',s')\to(t,s)} |\eta(t',s') - \eta(t,s)| = 0, \quad \forall (t,s) \in [0,T]^2, \text{ a.s.}$$

Consequently, $t \mapsto \eta(t,t) = Y(t)$ is continuous. □

4 Adapted M-Solution to Type-II QBSVIE

We now consider the following one-dimensional Type-II QBSVIE:

$$Y(t)=\psi(t)+\int_t^T g(t,s,Y(s),Z(t,s),Z(s,t))ds-\int_t^T Z(t,s)dW(s), \quad t\in[0,T]. \quad (4.1)$$

Since $Z(s,t)$ is presented in the generator $g(\cdot)$, we shall consider the adapted M-solution. Let us first introduce the following assumption:

(A4) Let the generator $g:\Delta[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\Omega\to\mathbb{R}$ be $\mathcal{B}(\Delta[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})\otimes\mathcal{F}_T$-measurable such that $s\mapsto g(t,s,y,z,z')$ is $\mathbb{F}$-progressively measurable on $[t,T]$ for all $(t,y,z,z')\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$. There exist two constants L and γ such that:

$$\begin{aligned}
&|g(t,s,y,z,z')|\le L(1+|y|)+\frac{\gamma}{2}|z|^2, \quad \forall(t,s,y,z,z')\in\Delta[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R};\\
&|g(t,s,y_1,z_1,z_1')-g(t,s,y_2,z_2,,z_2')|\\
&\quad\le L\Big(|y_1-y_2|+(1+|z_1|+|z_2|)|z_1-z_2|+|z_1'-z_2'|\Big),\\
&\qquad\forall(t,s,y_i,z_i,z_i')\in\Delta[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R},\ i=1,2.
\end{aligned}$$

Note that in (A4), we have assumed that $z'\mapsto g(t,s,y,z,z')$ is bounded. This will allow us to use the results for Type-I QBSVIEs. Therefore, the following result can be regarded as a byproduct of the results for Type-I QBSVIEs from the previous section. The case that allowing $z'\mapsto g(t,s,y,z,z')$ to be unbounded seems to be more difficult and might be treated in our future investigations. Now, here is the main result of this section.

Theorem 4.1 *Let* (A4) *hold. Then for any* $\psi(\cdot)\in L^\infty_{\mathcal{F}_T}(0,T)$*, Type-II QBSVIE* (4.1) *admits a unique adapted M-solution* $(Y(\cdot),Z(\cdot,\cdot))\in\mathcal{M}^2[0,T]\bigcap\big(L^\infty_{\mathbb{F}}(0,T)\times\overline{\mathrm{BMO}}(\Delta[0,T])\big)$.

Proof For any $(y(\cdot),z(\cdot,\cdot))\in\mathcal{M}^2[0,T]$, consider the following BSVIE:

$$\begin{aligned}
Y(t)=\psi(t)+\int_t^T g(t,s,Y(s),Z(t,s),z(s,t))ds\\
-\int_t^T Z(t,s)dW(s), \quad t\in[0,T].
\end{aligned} \quad (4.2)$$

In light of (A4), by Theorem 3.5, BSVIE (4.2) admits a unique adapted solution $(Y(\cdot),Z(\cdot,\cdot))\in L^\infty_{\mathbb{F}}(0,T)\times\overline{\mathrm{BMO}}(\Delta[0,T])$. Determine $Z(s,t)$; $(t,s)\in\Delta[0,T]$ by martingale representation theorem, i.e.,

$$Y(s)=\mathbb{E}[Y(s)]+\int_0^s Z(s,t)dW(t), \quad s\in[0,T].$$

This means that BSVIE (4.2) admits a unique adapted M-solution $(Y(\cdot), Z(\cdot,\cdot)) \in \mathcal{M}^2[0,T]$. Thus the map

$$\widetilde{\Gamma}(y(\cdot), z(\cdot,\cdot)) \triangleq (Y(\cdot), Z(\cdot,\cdot)), \qquad (y(\cdot), z(\cdot,\cdot)) \in \mathcal{M}^2(0,T) \tag{4.3}$$

is well-defined. In order to prove BSVIE (4.1) admits a unique adapted M-solution, we need to prove that $\widetilde{\Gamma}(\cdot\,,\cdot)$ has a fixed point in $\mathcal{M}^2[0,T]$. The proof is divided into two steps.

Step 1 There is an $\varepsilon > 0$ such that $\widetilde{\Gamma}(\cdot,\cdot)$ is a contraction on $\mathcal{M}^2[T-\varepsilon, T]$ and hence BSVIE (4.1) admits a unique adapted M-solution on $[T-\varepsilon, T]$.

For any $(y(\cdot), z(\cdot,\cdot)), (\widetilde{y}(\cdot), \widetilde{z}(\cdot,\cdot)) \in \mathcal{M}^2[T-\varepsilon, T]$, with $\varepsilon > 0$ undetermined, set

$$(Y(\cdot), Z(\cdot,\cdot)) = \widetilde{\Gamma}(y(\cdot), z(\cdot,\cdot)), \quad (\widetilde{Y}(\cdot), \widetilde{Z}(\cdot\,,\cdot)) = \widetilde{\Gamma}(\widetilde{y}(\cdot), \widetilde{z}(\cdot,\cdot)); \tag{4.4}$$

that is, for $t \in [T-\varepsilon, T]$,

$$Y(t) = \psi(t) + \int_t^T g(t,s,Y(s),Z(t,s),z(s,t))ds - \int_t^T Z(t,s)dW(s), \tag{4.5}$$

$$\widetilde{Y}(t) = \psi(t) + \int_t^T g(t,s,\widetilde{Y}(s),\widetilde{Z}(t,s),\widetilde{z}(s,t))ds - \int_t^T \widetilde{Z}(t,s)dW(s), \tag{4.6}$$

and

$$Y(s) = \mathbb{E}[Y(s)|\mathcal{F}_{T-\varepsilon}] + \int_{T-\varepsilon}^s Z(s,t)dW(t),\ \ s \in [T-\varepsilon, T], \tag{4.7}$$

$$\widetilde{Y}(s) = \mathbb{E}[\widetilde{Y}(s)|\mathcal{F}_{T-\varepsilon}] + \int_{T-\varepsilon}^s \widetilde{Z}(s,t)dW(t),\ \ s \in [T-\varepsilon, T]. \tag{4.8}$$

Similar to Lemma 3.6, noting that $z' \mapsto g(t,s,y,z,z')$ is bounded, there is an $\varepsilon > 0$ such that $\widetilde{\Gamma}(y(\cdot), z(\cdot\,,\cdot)) \in \mathcal{B}_\varepsilon$ for any $(y(\cdot), z(\cdot\,,\cdot)) \in \mathcal{M}^2(T-\varepsilon, T)$, where $\mathcal{B}_\varepsilon$ is defined by (3.18). Thus, we have

$$(Y(\cdot), Z(\cdot\,,\cdot)),\ (\widetilde{Y}(\cdot), \widetilde{Z}(\cdot\,,\cdot)) \in \mathcal{B}_\varepsilon. \tag{4.9}$$

By (A4), for any $t \in [T-\varepsilon, T]$, there is a process $\theta(t,\cdot)$ such that:

$$\theta(t,s) = 0, \quad t \in [T-\varepsilon, T],\ \ s \in [0,t], \tag{4.10}$$

$$|\theta(t,s)| \le L(1 + |Z(t,s)| + |\widetilde{Z}(t,s)|), \quad (t,s) \in \Delta[T-\varepsilon, T], \tag{4.11}$$

$$\begin{aligned} &g(t,s,\widetilde{Y}(s),Z(t,s),\widetilde{z}(s,t)) - g(t,s,\widetilde{Y}(s),\widetilde{Z}(t,s),\widetilde{z}(s,t)) \\ &\quad = [Z(t,s) - \widetilde{Z}(t,s)]\theta(t,s), \quad (t,s) \in \Delta[T-\varepsilon, T]. \end{aligned} \tag{4.12}$$

Similar to (3.36), we have

$$\|\theta(\cdot,\cdot)\|^2_{\mathrm{BMO}(\Delta[T-\varepsilon,T])} \le 3L^2T + 6L^2A. \tag{4.13}$$

For almost all $t \in [T-\varepsilon, T]$, by Lemma 2.3, $W(t;\cdot)$ defined by

$$W(t;s) \triangleq W(s) - \int_0^s \theta(t,r)dr, \quad s \in [0,T] \tag{4.14}$$

is a Brownian motion on $[0,T]$ under the equivalent probability measure $\overline{\mathbb{P}}_t$, which is defined by

$$d\overline{\mathbb{P}}_t \triangleq \mathcal{E}\{\theta(t,\cdot)\}_T d\mathbb{P}. \tag{4.15}$$

The corresponding expectation is denoted by $\mathbb{E}^{\bar{\mathbb{P}}_t}$. Combining (4.5)–(4.6) and (4.12)–(4.14), we have

$$\begin{aligned} &Y(t) - \widetilde{Y}(t) + \int_t^T [Z(t,s) - \widetilde{Z}(t,s)]dW(t,s) \\ &= \int_t^T \big[g(t,s,Y(s),Z(t,s),z(s,t)) - g(t,s,\widetilde{Y}(s),Z(t,s),\widetilde{z}(s,t))\big]ds. \end{aligned} \tag{4.16}$$

Taking square and then taking the conditional expectation $\mathbb{E}_t^{\bar{\mathbb{P}}_t}[\,\cdot\,] = \mathbb{E}^{\bar{\mathbb{P}}_t}[\,\cdot\,|\,\mathcal{F}_t]$, we have

$$\begin{aligned} &|Y(t) - \widetilde{Y}(t)|^2 + \mathbb{E}_t^{\bar{\mathbb{P}}_t}\Big[\int_t^T |Z(t,s) - \widetilde{Z}(t,s)|^2 ds\Big] \\ &= \mathbb{E}_t^{\bar{\mathbb{P}}_t}\Big[\int_t^T \Big(g(t,s,Y(s),Z(t,s),z(s,t)) - g(t,s,\widetilde{Y}(s),Z(t,s),\widetilde{z}(s,t))\Big)ds\Big]^2 \\ &\le L^2 \mathbb{E}_t^{\bar{\mathbb{P}}_t}\Big[\int_t^T \Big(|Y(s) - \widetilde{Y}(s)| + |z(s,t) - \widetilde{z}(s,t)|\Big)ds\Big]^2. \end{aligned} \tag{4.17}$$

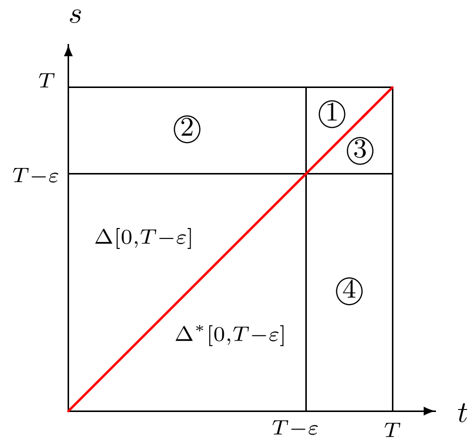
By $(Y(\cdot), Z(\cdot,\cdot)), (\widetilde{Y}(\cdot), \widetilde{Z}(\cdot,\cdot)) \in \mathcal{B}_\varepsilon$ and Lemma 2.4, there is a constant $C > 0$ (which is depending on $\|\psi(\cdot)\|_\infty$ and is independent of t) such that

$$\begin{aligned} &|Y(t) - \widetilde{Y}(t)|^2 + \mathbb{E}_t\Big[\int_t^T |Z(t,s) - \widetilde{Z}(t,s)|^2 ds\Big] \\ &\le C\mathbb{E}_t\Big[\int_t^T \Big(|Y(s) - \widetilde{Y}(s)| + |z(s,t) - \widetilde{z}(s,t)|\Big)ds\Big]^2 \\ &\le C(T-t)\mathbb{E}_t\Big[\int_t^T \Big(|Y(s) - \widetilde{Y}(s)|^2 + |z(s,t) - \widetilde{z}(s,t)|^2\Big)ds\Big]. \end{aligned} \tag{4.18}$$

Thus, integrating the above on $[T-\varepsilon, T]$, we obtain

$$\mathbb{E}\int_{T-\varepsilon}^T |Y(t) - \widetilde{Y}(t)|^2 dt + \mathbb{E}\int_{T-\varepsilon}^T \int_t^T |Z(t,s) - \widetilde{Z}(t,s)|^2 ds dt$$

Fig. 2 Type-II



$$\leq C\varepsilon\mathbb{E}\int_{T-\varepsilon}^{T}\int_{t}^{T}\Big[|Y(s)-\widetilde{Y}(s)|^2+|z(s,t)-\widetilde{z}(s,t)|^2\Big]dsdt, \tag{4.19}$$

with a possible different constant $C>0$. By the variation of constants formula, we obtain

$$\begin{aligned}&\mathbb{E}\int_{T-\varepsilon}^{T}|Y(t)-\widetilde{Y}(t)|^2dt+\mathbb{E}\int_{T-\varepsilon}^{T}\int_{t}^{T}|Z(t,s)-\widetilde{Z}(t,s)|^2dsdt\\&\leq C\varepsilon\mathbb{E}\int_{T-\varepsilon}^{T}\int_{t}^{T}|z(s,t)-\widetilde{z}(s,t)|^2dsdt\leq C\varepsilon\mathbb{E}\int_{T-\varepsilon}^{T}|y(t)-\widetilde{y}(t)|^2dt. \end{aligned}\tag{4.20}$$

The constant appears above is generic (only depends on the constants L, γ, T, and $\|\psi(\cdot)\|_\infty$, and is independent of $\varepsilon>0$). Therefore, when ε is small enough, $\widetilde{\Gamma}(\cdot,\cdot)$ is a contraction on $\mathcal{M}^2(T-\varepsilon,T)$. Consequently, BSVIE (4.1) admits a unique adapted solution on $[T-\varepsilon,T]$. Further, by (4.9), the unique adapted M-solution $(Y(\cdot),Z(\cdot,\cdot))$ also belongs to $L^\infty_{\mathbb{F}}(T-\varepsilon,T)\times\overline{\text{BMO}}(\Delta[T-\varepsilon,T])$.

The above determined $Y(t)$ for $t\in[T-\varepsilon,T]$ and determined $Z(t,s)$ for $(t,s)\in\Delta[T-\varepsilon,T]$ (the region marked ① in the Fig. 2) by using Type-I BSVIEs, and for $(t,s)\in\Delta^*[T-\varepsilon,T]$ (the region marked ③ in the Fig. 2) by using martingale representation.

Step 2 BSVIE (4.1) admits a unique adapted M-solution on $[0,T]$.

By Step 1, BSVIE (4.1) admits a unique solution on $[T-\varepsilon,T]$. For almost every $s\in[T-\varepsilon,T]$, $\mathbb{E}_{T-\varepsilon}[Y(s)]\in L^2_{\mathcal{F}_{T-\varepsilon}}(\Omega)$, by martingale representation theorem, there is a unique $Z(\cdot,\cdot)\in L^2(T-\varepsilon,T;L^2_{\mathbb{F}}(0,T-\varepsilon))$ such that:

$$\mathbb{E}_{T-\varepsilon}[Y(s)]=\mathbb{E}[Y(s)|+\int_0^{T-\varepsilon}Z(s,t)dW(t),\quad s\in[T-\varepsilon,T]. \tag{4.21}$$

Hence, we have uniquely determined $(Y(t), Z(t,s))$ for $(t,s) \in [T-\varepsilon, T] \times [0, T]$ (the region marked ①, ③ and ④) and the following is well-defined:

$$g^{T-\varepsilon}(t,s,z) = g(t,s,Y(s),z,Z(s,t)), \quad (t,s) \in [0, T-\varepsilon] \times [T-\varepsilon, T]. \tag{4.22}$$

Note that $[0, T-\varepsilon] \times [T-\varepsilon, T]$ is the region marked ② in the above Fig. 2. Now, consider the following SFIE:

$$\begin{aligned} \psi^{T-\varepsilon}(t) &= \psi(t) + \int_{T-\varepsilon}^{T} g^{T-\varepsilon}(t,s,Z(t,s))ds \\ &\quad - \int_{T-\varepsilon}^{T} Z(t,s)dW(s), \quad t \in [0, T-\varepsilon]. \end{aligned} \tag{4.23}$$

Similar to the Step 2 of the proof of Theorem 3.5, SFIE (4.23) admits a unique solution $(\psi^{T-\varepsilon}(\cdot), Z(\cdot,\cdot))$ on $[0, T-\varepsilon] \times [T-\varepsilon, T]$ and the following estimate holds:

$$|\psi^{T-\varepsilon}(t)|^2 \leq \alpha(0), \quad t \in [0, T-\varepsilon], \tag{4.24}$$

where $\alpha(\cdot)$ solves an equation similar to (3.47). The above uniquely determined

$$\begin{cases} Y(t), & t \in [T-\varepsilon, T], \\ Z(t,s), & (t,s) \in \Big([T-\varepsilon, T] \times [0, T]\Big) \bigcup \Big([0, T-\varepsilon] \times [T-\varepsilon, T]\Big). \end{cases} \tag{4.25}$$

Now, we consider

$$\begin{aligned} Y(t) &= \psi^{T-\varepsilon}(t) + \int_{t}^{T-\varepsilon} g(t,s,Y(s),Z(t,s),Z(s,t))ds \\ &\quad - \int_{t}^{T-\varepsilon} Z(t,s)dW(s) \end{aligned} \tag{4.26}$$

on $[0, T-\varepsilon]$. Since $\psi^{T-\varepsilon}(\cdot) \in L^{\infty}_{\mathcal{F}_{T-\varepsilon}}(0, T-\varepsilon)$, (4.26) is a BSVIE on $[0, T-\varepsilon]$. Then the above procedure can be repeated. Since the step-length $\varepsilon > 0$ can be fixed, we then could use induction to complete the proof. □

5 A Comparison Theorem for Type-I BSVIEs

Consider the following BSVIEs: For $i = 1, 2$,

$$\begin{aligned} Y^i(t) &= \psi^i(t) + \int_t^T g^i(t,s,Y^i(s),Z^i(t,s))ds \\ &\quad - \int_t^T Z^i(t,s)dW(s), \quad t \in [0, T]. \end{aligned} \tag{5.1}$$

We assume that the generators $g^i(\cdot)$, $i = 1, 2$ of BSVIEs (5.1) satisfy (A2). Then by Theorem 3.5, BSVIE (5.1) admits a unique adapted solution $(Y^i(\cdot), Z^i(\cdot,\cdot)) \in L^\infty_{\mathbb{F}}(0,T) \times \overline{\text{BMO}}(\Delta[0,T])$ for any $\psi^i(\cdot) \in L^\infty_{\mathcal{F}_T}(0,T)$. In order to study the comparison theorem of the solutions to BSVIE (5.1), we introduce the following BSVIE:

$$\bar{Y}(t) = \bar{\psi}(t) + \int_t^T \bar{g}(t,s,\bar{Y}(s),\bar{Z}(t,s))ds - \int_t^T \bar{Z}(t,s)dW(s), \quad t \in [0,T], \tag{5.2}$$

with the generator $\bar{g}(\cdot)$ also satisfies (A2). Further, we adopt the following assumption.

(C) Let the generator $\bar{g} : \Delta[0,T] \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ satisfy that $y \mapsto \bar{g}(t,s,y,z)$ is nondecreasing for any $(t,s,z) \in \Delta[0,T] \times \mathbb{R}$.

We present the comparison theorem for BSVIE (5.1) now.

Theorem 5.1 *Let $g^1(\cdot)$, $g^2(\cdot)$ and $\bar{g}(\cdot)$ satisfy (A2) and let $\bar{g}(\cdot)$ satisfy (C). Suppose*

$$g^1(t,s,y,z) \le \bar{g}(t,s,y,z) \le g^2(t,s,y,z), \quad \forall (y,z) \in \mathbb{R} \times \mathbb{R}, \ a.s., \ a.e. \ (t,s) \in \Delta[0,T]. \tag{5.3}$$

Then for any $\psi^1(\cdot), \psi^2(\cdot) \in L^\infty_{\mathcal{F}_T}(0,T)$ satisfying

$$\psi^1(t) \le \psi^2(t), \ a.s., \ a.e. \ t \in [0,T], \tag{5.4}$$

the corresponding unique adapted solutions $(Y^i(\cdot), Z^i(\cdot,\cdot))$, $i = 1, 2$ of BSVIEs (5.1) satisfy

$$Y^1(t) \le Y^2(t), \quad a.s., \ a.e. \ t \in [0,T]. \tag{5.5}$$

If, in addition, the generators $g^1(\cdot)$, $g^2(\cdot)$ and $\bar{g}(\cdot)$ satisfy (A3), and

$$g^1(t,s,y,z) \le \bar{g}(t,s,y,z) \le g^2(t,s,y,z), \quad \forall (t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}, \ a.s., \ a.e. \ s \in [0,T]. \tag{5.6}$$

Then for any $\psi^1(\cdot), \psi^2(\cdot) \in L^\infty_{\mathcal{F}_T}(\Omega; C^U[0,T])$ satisfying

$$\psi^1(t) \le \psi^2(t), \quad t \in [0,T], \ a.s., \tag{5.7}$$

the corresponding unique adapted solutions $(Y^i(\cdot), Z^i(\cdot,\cdot))$, $i = 1, 2$ of BSVIEs (5.1) satisfy

$$Y^1(t) \le Y^2(t), \quad t \in [0,T], \quad a.s. \tag{5.8}$$

Proof Let $\bar{\psi}(\cdot) \in L^{\infty}_{\mathcal{F}_T}(0, T)$ such that

$$\psi^1(t) \leq \bar{\psi}(t) \leq \psi^2(t), \quad \text{a.s., a.e. } t \in [0, T]. \tag{5.9}$$

Without loss of generality, let

$$\|\psi(\cdot)\|_{\infty} \leq L, \tag{5.10}$$

where $\psi(\cdot) = \psi^1(\cdot), \psi^2(\cdot), \bar{\psi}(\cdot)$. By Theorem 3.5, BSVIE (5.1) admits a unique adapted solution $(Y^1(\cdot), Z^1(\cdot,\cdot)) \in L^{\infty}_{\mathbb{F}}(0, T) \times \overline{\text{BMO}}(\Delta[0, T])$ for $i = 1$. Set $\widetilde{Y}_0(\cdot) = Y^1(\cdot)$ and consider

$$\begin{aligned}\widetilde{Y}_1(t) = \bar{\psi}(t) + \int_t^T \bar{g}(t, s, \widetilde{Y}_0(s), \widetilde{Z}_1(t, s))ds \\ - \int_t^T \widetilde{Z}_1(t, s)dW(s), \quad t \in [0, T].\end{aligned} \tag{5.11}$$

By Theorem 3.2, there is a unique adapted solution $(\widetilde{Y}_1(\cdot), \widetilde{Z}_1(\cdot,\cdot)) \in L^{\infty}_{\mathbb{F}}(0, T) \times \overline{\text{BMO}}(\Delta[0, T])$ to the above BSVIE. By (5.3), we have

$$g^1(t, s, \widetilde{Y}_0(s), z) \leq \bar{g}(t, s, \widetilde{Y}_0(s), z), \quad \forall z \in \mathbb{R}, \text{ a.s., a.e. } (t, s) \in \Delta[0, T]. \tag{5.12}$$

Combining this and (5.9), by Theorem 3.3, for almost all $t \in [0, T]$, there exists a measurable set $\Omega_t^1 \subseteq \Omega$ satisfying $\mathbb{P}(\Omega_t^1) = 0$ such that

$$\widetilde{Y}_0(t) = Y^1(t) \leq \widetilde{Y}_1(t), \ \omega \in \Omega \backslash \Omega_t^1, \text{ a.e. } t \in [0, T]. \tag{5.13}$$

Next, we consider the following BSVIE

$$\begin{aligned}\widetilde{Y}_2(t) = \bar{\psi}(t) + \int_t^T \bar{g}(t, s, \widetilde{Y}_1(s), \widetilde{Z}_2(t, s))ds \\ - \int_t^T \widetilde{Z}_2(t, s)dW(s), \quad t \in [0, T].\end{aligned} \tag{5.14}$$

Let $(\widetilde{Y}_2(\cdot), \widetilde{Z}_2(\cdot,\cdot))$ be the unique solution to the above equation. Since $y \mapsto \bar{g}(t, s, y, z)$ is nondecreasing, by (5.13), we have

$$\bar{g}(t, s, \widetilde{Y}_0(s), z) \leq \bar{g}(t, s, \widetilde{Y}_1(s), z), \quad \forall z \in \mathbb{R}, \text{ a.s., a.e. } (t, s) \in \Delta[0, T]. \tag{5.15}$$

Similar to the above, for almost everywhere $t \in [0, T]$, there exists a measurable set $\Omega_t^2 \subseteq \Omega$ satisfying $\mathbb{P}(\Omega_t^2) = 0$ such that

$$\widetilde{Y}_1(t) \leq \widetilde{Y}_2(t), \ \omega \in \Omega \backslash \Omega_t^2, \text{ a.e. } t \in [0, T]. \tag{5.16}$$

By induction, we can construct a sequence $(\widetilde{Y}_k(\cdot), \widetilde{Z}_k(\cdot,\cdot))$ and Ω_t^k satisfying $\mathbb{P}(\Omega_t^k) = 0$ such that

$$\begin{aligned}\widetilde{Y}_{k+1}(t) &= \bar{\psi}(t) + \int_t^T \bar{g}(t,s,\widetilde{Y}_k(s),\widetilde{Z}_{k+1}(t,s))ds \\ &\quad - \int_t^T \widetilde{Z}_{k+1}(t,s)dW(s), \quad t \in [0,T], \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} Y^1(t) &= \widetilde{Y}_0(t) \le \widetilde{Y}_1(t) \le \widetilde{Y}_2(t) \le \cdots, \\ \omega &\in \Omega \backslash \Big(\bigcup_{k\ge 1} \Omega_t^k \Big), \text{ a.e. } t \in [0,T]. \end{aligned} \tag{5.18}$$

Note that $\mathbb{P}[\Omega \backslash (\bigcup_{k\ge 1} \Omega_t^k)] = 0$. We may assume that

$$|\psi(t)| \le \alpha(0), \quad t \in [0,T], \tag{5.19}$$

where $\psi(\cdot) = \psi^1(\cdot), \psi^2(\cdot), \bar{\psi}(\cdot)$ and $\alpha(\cdot)$ solves an ODE of form (3.47). By Proposition 3.7, there is an $\varepsilon > 0$ such that $\widetilde{Y}_k(\cdot)$ is Cauchy in $L^\infty_{\mathbb{F}}(T-\varepsilon, T)$ and

$$\lim_{k\to\infty} \|\widetilde{Y}_k(\cdot) - \bar{Y}(\cdot)\|_{L^\infty_{\mathbb{F}}(T-\varepsilon,T)} = 0. \tag{5.20}$$

Combining (5.18) and (5.20), we have

$$Y^1(t) \le \bar{Y}(t), \quad \text{a.s., a.e. } t \in [T-\varepsilon, T]. \tag{5.21}$$

Next, consider the following SFIEs:

$$\begin{aligned} \psi^{1,T-\varepsilon}(t) &= \psi^1(t) + \int_{T-\varepsilon}^T g^1(t,s,Y^1(s),Z^1(t,s))ds \\ &\quad - \int_{T-\varepsilon}^T Z^1(t,s)dW(s), \quad t \in [0, T-\varepsilon]; \end{aligned} \tag{5.22}$$

$$\begin{aligned} \bar{\psi}^{T-\varepsilon}(t) &= \bar{\psi}(t) + \int_{T-\varepsilon}^T \bar{g}(t,s,\bar{Y}(s),\bar{Z}(t,s))ds \\ &\quad - \int_{T-\varepsilon}^T \bar{Z}(t,s)dW(s), \quad t \in [0, T-\varepsilon]. \end{aligned} \tag{5.23}$$

Similar to the Step 2 in Theorem 3.5, the above SFIEs (5.22) and (5.23) admit unique solutions $(\psi^{1,T-\varepsilon}(\cdot), Z^1(\cdot,\cdot)), (\bar{\psi}^{T-\varepsilon}(\cdot), \bar{Z}(\cdot,\cdot)) \in L^\infty_{\mathcal{F}_{T-\varepsilon}}(0, T-\varepsilon) \times \overline{\text{BMO}}([0, T-\varepsilon] \times [T-\varepsilon, T])$, respectively. Similar to (3.62), we have

$$|\psi^{1,T-\varepsilon}(t)| \le \alpha(0), \quad |\bar{\psi}^{T-\varepsilon}(t)| \le \alpha(0), \quad t \in [0, T-\varepsilon]. \tag{5.24}$$

For almost all $t \in [0, T - \varepsilon]$, similar to (3.33)–(3.34) and (3.37)–(3.38), there is a process $\theta(t, \cdot)$ such that:

$$\begin{aligned} & g^1(t, s, Y^1(s), Z^1(t, s)) - g^1(t, s, Y^1(s), \bar{Z}(t, s)) \\ & \quad = \big[Z^1(t, s) - \bar{Z}(t, s)\big]\theta(t, s), \end{aligned} \tag{5.25}$$

and

$$W(t; s) \triangleq W(s) - \int_0^s \theta(t, r)dr, \quad s \in [0, T] \tag{5.26}$$

is a Brownian motion on $[0, T]$ under the corresponding equivalent probability measure $\overline{\mathbb{P}}_t$. The corresponding expectation is denoted by $\mathbb{E}^{\bar{\mathbb{P}}_t}$. Combining (5.22)–(5.23) and (5.25)–(5.26), we have

$$\begin{aligned} & \psi^{1,T-\varepsilon}(t) - \bar{\psi}^{T-\varepsilon}(t) \\ & \quad = \psi^1(t) - \bar{\psi}(t) + \int_{T-\varepsilon}^T \big[g^1(t, s, Y^1(s), \bar{Z}(t, s)) - \bar{g}(t, s, \bar{Y}(s), \bar{Z}(t, s))\big]ds \\ & \qquad - \int_{T-\varepsilon}^T \big[Z^1(t, s) - \bar{Z}(t, s)\big]dW(t; s), \quad t \in [0, T - \varepsilon]. \end{aligned} \tag{5.27}$$

Since $y \mapsto \bar{g}(t, s, y, z)$ is nondecreasing for any $(t, s, z) \in \Delta[0, T] \times \mathbb{R}$, by (5.21), we have

$$\bar{g}(t, s, Y^1(s), z) \le \bar{g}(t, s, \bar{Y}(s), z), \quad (t, s, z) \in [0, T] \times [T - \varepsilon, T] \times \mathbb{R}. \tag{5.28}$$

Taking conditional expectation $\mathbb{E}_t^{\bar{\mathbb{P}}_t}[\,\cdot\,] \equiv \mathbb{E}^{\bar{\mathbb{P}}_t}[\,\cdot \,|\, \cdot\,]$, on the both sides of (5.27), by (5.3), (5.28) and (5.21), we have

$$\begin{aligned} & \psi^{1,T-\varepsilon}(t) - \bar{\psi}^{T-\varepsilon}(t) \\ & = \mathbb{E}_t^{\bar{\mathbb{P}}_t}\Big[\psi^1(t) - \bar{\psi}(t) + \int_{T-\varepsilon}^T \big[g^1(t, s, Y^1(s), \bar{Z}(t, s)) - \bar{g}(t, s, \bar{Y}(s), \bar{Z}(t, s))\big]ds\Big] \\ & \le \mathbb{E}_t^{\bar{\mathbb{P}}_t}\Big[\psi^1(t) - \bar{\psi}(t) + \int_{T-\varepsilon}^T \big[g^1(t, s, Y^1(s), \bar{Z}(t, s)) - \bar{g}(t, s, Y^1(s), \bar{Z}(t, s))\big]ds\Big] \\ & \le 0, \quad t \in [0, T - \varepsilon]. \end{aligned} \tag{5.29}$$

Now, we consider the following BSVIEs:

$$\begin{aligned} y^1(t) = & \ \psi^{1,T-\varepsilon}(t) + \int_t^{T-\varepsilon} g^1(t, s, y^1(s), z^1(t, s))ds \\ & - \int_t^{T-\varepsilon} z^1(t, s)dW(s), \quad t \in [0, T - \varepsilon]; \end{aligned} \tag{5.30}$$

$$\bar{y}(t) = \bar{\psi}^{T-\varepsilon}(t) + \int_t^{T-\varepsilon} \bar{g}(t, s, \bar{y}(s), \bar{z}(t, s))ds - \int_t^{T-\varepsilon} \bar{z}(t, s)dW(s), \quad t \in [0, T-\varepsilon]. \tag{5.31}$$

By Theorem 3.5, the above Eqs. (5.30), (5.31) admit unique solutions $(y^1(\cdot), z^1(\cdot, \cdot))$, $(\bar{y}(\cdot), \bar{z}(\cdot, \cdot)) \in L^\infty_{\mathbb{F}}(0, T-\varepsilon) \times \overline{\text{BMO}}(\Delta[0, T-\varepsilon])$, respectively. By the Step 3 in the proof of Theorem 3.5, we have

$$y^1(t) = Y^1(t), \; z^1(t, s) = Z^1(t, s), \quad (t, s) \in \Delta[0, T-\varepsilon]; \tag{5.32}$$

$$\bar{y}(t) = \bar{Y}(t), \; \bar{z}(t, s) = \bar{Z}(t, s), \quad (t, s) \in \Delta[0, T-\varepsilon]. \tag{5.33}$$

Hence, by induction, we have

$$Y^1(t) \le \bar{Y}(t), \quad \text{a.s., a.e. } t \in [0, T]. \tag{5.34}$$

Similarly, we can prove that

$$\bar{Y}(t) \le Y^2(t), \quad \text{a.s., a.e. } t \in [0, T]. \tag{5.35}$$

Thus, the inequality (5.5) holds.

Next, by what we have proved,

$$Y^1(t) \le Y^2(t), \quad \text{a.s.,} \quad t \in [0, T]. \tag{5.36}$$

Let $\{t_k\}_{k\ge1} \subseteq [0, T]$ be all the rational numbers in $[0, T]$. For any fixed t_k, by (5.36), there is a $\Omega_k \subseteq \Omega$ satisfying $\mathbb{P}(\Omega_k) = 0$ such that:

$$Y_1(t_k) \le Y_2(t_k), \qquad \omega \in \Omega \backslash \Omega_{t_k}. \tag{5.37}$$

Let $\widetilde{\Omega} = \bigcup_{k\ge1} \Omega_{t_k}$, then $\mathbb{P}(\widetilde{\Omega}) = 0$. By (5.37), we have

$$Y_1(t) \le Y_2(t), \quad t \in \{t_k\}_{k\ge1}, \; \omega \in \Omega \backslash \widetilde{\Omega}. \tag{5.38}$$

By Theorem 3.9, there is a $\bar{\Omega} \subseteq \Omega$ satisfying $\mathbb{P}(\bar{\Omega}) = 0$ such $Y_i(\cdot, \omega)$, $i = 1, 2$ are continuous for any $\omega \in \Omega \backslash \bar{\Omega}$. For any fixed $\omega \in \Omega \backslash (\widetilde{\Omega} \cup \bar{\Omega})$, by (5.38), we have

$$Y_1(t, \omega) \le Y_2(t, \omega), \quad t \in \{t_k\}_{k\ge1}. \tag{5.39}$$

Since $Y_i(\cdot, \omega)$, $i = 1, 2$ are continuous on $[0, T]$ and $\{t_k\}_{k\ge1} \subseteq [0, T]$ is dense on $[0, T]$, we have

$$Y_1(t, \omega) \le Y_2(t, \omega), \quad t \in [0, T]. \tag{5.40}$$

Note that $\mathbb{P}(\Omega\backslash(\widetilde{\Omega}\cup\bar{\Omega}))=0$, we have

$$Y_1(t)\le Y_2(t),\quad t\in[0,T],\ \text{a.s.} \tag{5.41}$$

This completes the proof. □

6 Continuous-Time Equilibrium Dynamic Risk Measures

We have seen the so-called equilibrium recursive utility process in the introduction section, which serves as a very important motivation of studying BSVIEs. In this section, we will look another closely related application of BSVIEs.

Static risk measures have been studied by many researchers. Among many of them, we mention Artzner–Delbaen–Eber–Heath [5], Föllmer–Schied [19], and the references cited therein. For discrete-time dynamic risk measures, we mention Riedel [36] and Detlefsen–Scandolo [13], and the references cited therein.

We now look at continuous-time dynamic risk measures. Any $\xi\in L^\infty_{\mathcal{F}_T}(\Omega)$ represents the payoff of certain European type contingent claim at the maturity time T. According to El Karoui–Peng–Quenez [18], we introduce the following definition.

Definition 6.1 A map $\rho:[0,T]\times L^\infty_{\mathcal{F}_T}(\Omega)\to\mathbb{R}$ is called a *dynamic risk measure* if the following are satisfied:

(i) (Adaptiveness) For any $\xi\in L^\infty_{\mathcal{F}_T}(\Omega)$, $t\mapsto\rho(t;\xi)$ is $\mathbb{F}$-adapted;
(ii) (Monotonicity) For any $\xi,\bar{\xi}\in L^\infty_{\mathcal{F}_T}(\Omega)$ with $\xi\ge\bar{\xi}$, one has $\rho(t;\xi)\le\rho(t;\bar{\xi})$, for all $t\in[0,T]$;
(iii) (Translation Invariant) For any $\xi\in L^\infty_{\mathcal{F}_T}(\Omega)$ and $c\in\mathbb{R}$, $\rho(t;\xi+c)=\rho(t;\xi)-c$.

Further, ρ is said to be *convex* if the following holds:

(iv) (Convexity): $\xi\mapsto\rho(t;\xi)$ is convex;

and ρ is said to be *coherent* if the following are satisfied:

(v) (Positive Homogeneity): For any $\xi\in L^\infty_{\mathcal{F}_T}(\Omega)$ and $\lambda\ge0$, $\rho(t;\lambda\xi)=\lambda\rho(t;\xi)$;
(vi) (Subadditivity): For any $\xi,\bar{\xi}\in L^\infty_{\mathcal{F}_T}(\Omega)$, $\rho(t;\xi+\bar{\xi})\le\rho(t;\xi)+\rho(t;\bar{\xi})$.

Each item in the above definition can be naturally explained. For example, (ii) means that between two gains, the one dominantly larger one has a smaller risk; (vi) means that combining two investments will have smaller risk. The following is a combination of the results from [18] and [24] (see also [7–9]).

Proposition 6.2 *Let $g:[0,T]\times\mathbb{R}\to\mathbb{R}$ be measurable such that $z\mapsto g(t,z)$ is convex and grow at most quadratically. Then for any $\xi\in L^\infty_{\mathcal{F}_T}(\Omega)$, the following BSDE:*

$$Y(t)=-\xi+\int_t^T g(s,Z(s))ds-\int_t^T Z(s)dW(s),\quad t\in[0,T], \tag{6.1}$$

admits a unique adapted solution $(Y(\cdot), Z(\cdot)) \equiv (Y(\cdot\,;\xi), Z(\cdot\,;\xi))$. *Let* $\rho : [0,T] \times L^\infty_{\mathcal{F}_T}(\Omega) \to \mathbb{R}$ *be defined by the following:*

$$\rho(t,\xi) = Y(t;\xi), \quad (t,\xi) \in [0,T] \times L^\infty_{\mathcal{F}_T}(\Omega).$$

Then ρ *is a dynamic convex risk measure.*

One of the most interesting examples is the following.

$$Y(t) = -\xi + \int_t^T \frac{1}{2\gamma}|Z(s)|^2 ds - \int_t^T Z(s)dW(s), \quad t \in [0,T].$$

The above admits a unique adapted solution $(Y(\cdot), Z(\cdot))$, and

$$\rho(t,\xi) \equiv Y(t) = \gamma \ln \mathbb{E}\Big[e^{-\frac{\xi}{\gamma}} \;\Big|\; \mathcal{F}_t\Big] \triangleq e_{\gamma,t}(\xi), \quad t \in [0,T],$$

is called a *dynamic entropic risk measure* for ξ.

Now, if we have an $\mathcal{F}_T$-measurable wealth flow process $\psi(\cdot)$ instead of just a terminal payoff ξ, then formally, the corresponding dynamic risk should be measured via the following parameterized BSDE:

$$\begin{aligned} Y(t,r) = -\psi(t) &+ \int_r^T g(s, Y(t,s), Z(t,s))ds \\ &- \int_r^T Z(t,s)dW(s)), \quad (r,t) \in \Delta[0,T], \end{aligned}$$

and the current dynamic risk should be $Y(t;t)$. But, similar to the introduction section, simply taking $r = t$ in the above leads to the following:

$$Y(t,t) = -\psi(t) + \int_t^T g(s, Y(t,s), Z(t,s))ds - \int_t^T Z(t,s)dW(s)), \quad t \in [0,T],$$

which is not a closed form equation for the pair $(Y(t,t), Z(t,s))$ of processes. As we indicated in the introduction, $Y(t,r)$ above has some hidden *time-inconsistency* nature. One expects that the dynamic risk measure should be time-consistent. Namely, the value of the risk today (for a process $\psi(\cdot)$) should match the one that one expected yesterday. Therefore, it is natural to use BSVIEs to describe/measure the dynamic risk of the process $\psi(\cdot)$. We now make this precise.

We call $\psi(\cdot) \in L^\infty_{\mathcal{F}_T}(0,T)$ a position process (a name borrowed from [36]), and $\psi(t)$ could represent the total (nominal) value of certain portfolio process which might be a combination of certain (say, European type) contingent claims (which are mature at time T, thus they are usually only $\mathcal{F}_T$-measurable), some current cash flows (such as dividends to be received, premia to be paid), positions of stocks, mutual funds, and bonds, and so on, at time the current time t. Thus, the position process $\psi(\cdot)$ is merely $\mathcal{F}_T$-measurable (not necessarily $\mathbb{F}$-adapted). Now, mimicking Definition 6.1, we introduce the following.

Definition 6.3 A map $\rho : [0, T] \times L^\infty_{\mathcal{F}_T}(0, T) \to L^\infty_{\mathbb{F}}(0, T)$ is called an *equilibrium dynamic risk measure* if the following hold:

(i) (Past Independence) For any $\psi_1(\cdot), \psi_2(\cdot) \in L^\infty_{\mathcal{F}_T}(0, T)$, if

$$\psi_1(s) = \psi_2(s), \quad \text{a.s., a.e. } s \in [t, T],$$

for some $t \in [0, T)$, then

$$\rho(t; \psi_1(\cdot)) = \rho(t; \psi_2(\cdot)), \quad \text{a.s.}$$

(ii) (Monotonicity) For any $\psi_1(\cdot), \psi_2(\cdot) \in L^\infty_{\mathcal{F}_T}(0, T)$, if

$$\psi_1(s) \le \psi_2(s), \quad \text{a.s., a.e. } s \in [t, T],$$

for some $t \in [0, T)$, then

$$\rho(s; \psi_1(\cdot)) \ge \rho(s; \psi_2(\cdot)), \quad \text{a.s., } \; s \in [t, T].$$

(iii) (Translation Invariance) There exists a deterministic integrable function $r(\cdot)$ such that for any $\psi(\cdot) \in L^\infty_{\mathcal{F}_T}(0, T)$,

$$\rho(t; \psi(\cdot) + c) = \rho(t; \psi(\cdot)) - ce^{\int_t^T r(s)ds}, \quad \text{a.s., } \; t \in [0, T].$$

Further, ρ is said to be *convex* if the following holds:

(iv) (Convexity) For any $\psi_1(\cdot), \psi_2(\cdot) \in L^\infty_{\mathcal{F}_T}(0, T)$ and $\lambda \in [0, 1]$,

$$\rho(t; \lambda\psi_1(\cdot) + (1-\lambda)\psi_2(\cdot)) \le \lambda\rho(t; \psi_1(\cdot)) + (1-\lambda)\rho(t; \psi_2(\cdot)), \quad \text{a.s., } \; t \in [0, T].$$

And ρ is said to be *coherent* if the following are satisfied:

(v) (Positive Homogeneity) For any $\psi(\cdot) \in L^\infty_{\mathcal{F}_T}(0, T)$ and $\lambda > 0$,

$$\rho(t; \lambda\psi(\cdot)) = \lambda\rho(t; \psi(\cdot)), \quad \text{a.s., } \; t \in [0, T].$$

(vi) (Subadditivity) For any $\psi_1(\cdot), \psi_2(\cdot) \in L^\infty_{\mathcal{F}_T}(0, T)$,

$$\rho(t; \psi_1(\cdot) + \psi_2(\cdot)) \le \rho(t; \psi_1(\cdot)) + \rho(t; \psi_2(\cdot)), \quad \text{a.s., } \; t \in [0, T].$$

The word "equilibrium" indicates the time-consistency of the risk measure ρ which is some kind of modification of the naive one. Similar situation has happened in the study of time-inconsistent optimal control problems (see [49]). The meaning of each item is similar to the static case. In (iii), the function $r(\cdot)$ is the riskless interest rate.

Let us now look at the following Type-I BSVIE:

$$Y(t) = -\psi(t) + \int_t^T g(t, s, Y(s), Z(t, s))ds$$

$$- \int_t^T Z(t,s)dW(s), \quad t \in [0,T]. \tag{6.2}$$

We have the following result.

Proposition 6.4 *Let the generator be given by*

$$g(t,s,y,z) \equiv r(s)y + g_0(t,s,z);\ (t,s,y,z) \in \Delta[0,T] \times \mathbb{R} \times \mathbb{R},$$

satisfying (A2)*, where* $r(\cdot)$ *is a non-negative deterministic function. Then the following are true:*

(i) *The map* $\psi(\cdot) \mapsto \rho(t;\psi(\cdot))$ *is translation invariant.*
(ii) *Suppose* $z \mapsto g_0(t,s,z)$ *is convex, so is* $\psi(\cdot) \mapsto \rho(t;\psi(\cdot))$.
(iii) *Suppose* $z \mapsto g_0(t,s,z)$ *is positively homogeneous and sub-additive, so is* $\psi(\cdot) \mapsto \rho(t;\psi(\cdot))$.

By Theorem 5.1, the proof of Proposition 6.4 is very similar to [47, Corollary 3.4, Proposition 3.5], we omit them here. By Proposition 6.4, we can construct a large class of equilibrium dynamic risk measures by choosing suitable generator $g(\cdot)$ of BSVIE (6.2). More precisely, we have the following result.

Theorem 6.5 *Let the generator* $g(t,s,y,z) \equiv r(s)y + g_0(t,s,z);\ (t,s,y,z) \in \Delta \times \mathbb{R} \times \mathbb{R}$ *satisfy* (A2)*, where* $r(\cdot)$ *is a non-negative deterministic function and* $z \mapsto g_0(t,s,z)$ *is convex, then* $\psi(\cdot) \mapsto \rho(t;\psi(\cdot))$ *is an equilibrium dynamic convex risk measure. If* $z \mapsto g_0(t,s,z)$ *is positively homogeneous and sub-additive, then* $\psi(\cdot) \mapsto \rho(t;\psi(\cdot))$ *is an equilibrium dynamic coherent risk measure.*

From Proposition 6.4, the proof of the above result is obvious. According to the above results, we can have some examples of equilibrium dynamic risk measures by the choices of $g_0(t,s,z)$: If

$$g_0(t,s,z) = \bar{g}(t,s)|z|, \qquad \bar{g}(t,s) \geq 0,$$

then, it is sub-additive and positively homogeneous in z. The corresponding equilibrium dynamic risk measure is coherent. If

$$g_0(t,s,z) = \bar{g}(t,s)\sqrt{1+|z|^2}, \qquad \bar{g}(t,s) \geq 0,$$

then, it is convex in z. The corresponding equilibrium dynamic risk measure is convex. If

$$g_0(t,s,z) = \bar{g}(t,s)|z|^2, \qquad \bar{g}(t,s) \geq 0,$$

then one has an entropy type equilibrium dynamic risk measure.

7 Concluding Remarks

Recursive utility process (or stochastic differential utility process) and dynamic risk measures for terminal payoff can be described by the adapted solutions to proper BSDEs. For $\mathcal{F}_T$-measurable position process $\psi(\cdot)$, instead of the terminal payoff ξ, one could also try to find its recursive utility process and/or dynamic risk. One possibility is again to use BSDEs. However, one immediately finds that the resulting processes (recursive utility or dynamic risk measure) are kind of time-inconsistent nature. Type-I BSVIEs turn out to be a proper tool for describing them. This serves one of major motivations of studying BSVIEs. Recall from [46,48], we know that mathematical extension of BSDEs and optimal control of forward stochastic Volterra integral equations are other two motivations. To meet the needs for the equilibrium recursive utility processes and equilibrium dynamic risk measures, we have to allow the generator of the BSVIE to have a quadratic growth in $Z(t, s)$. We have developed a theory of Type-I QBSVIEs, including the well-posedness, regularity and a comparison theorem, etc. in this paper. As a byproduct, we also have obtained the well-posedness of Type-II QBSVIEs. Then a theory of equilibrium recursive utility and equilibrium dynamic risk measures are successfully established with the results of Type-I QBSVIEs.

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8 Appendix

In this appendix, we will sketch an argument supporting the BSVIE model for the equilibrium recursive utility process/equilibrium dynamic risk measure of a position process $\psi(\cdot)$. The idea is adopted from [49]. Let $\psi(\cdot)$ be a continuous $\mathcal{F}_T$-measurable process. Let $\Pi = \{t_k \mid 0 \leq k \leq N\}$ be a partition of $[0, T]$ with $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$. The mesh size of Π is denoted by $\|\Pi\| \triangleq \max_{0 \leq i \leq N-1} |t_{i+1} - t_i|$. Let

$$\psi^{\Pi}(t) = \sum_{k=1}^{N} \psi_k \mathbf{1}_{(t_{k-1}, t_k]}(t),$$

with

$$\psi_k = \psi(t_k) \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}), \qquad k = 1, 2, \ldots, N.$$

We assume that

$$\lim_{\|\Pi\| \to 0} \sup_{t \in [0,T]} \mathbb{E}|\psi^{\Pi}(t) - \psi(t)|^2 = 0.$$

We first try to specify the time-consistent recursive utility process for $\psi^{\Pi}(\cdot)$, making use of BSDEs. Then let $\|\Pi\| \to 0$ to get our BSVIE time-consistent recursive utility process model for $\psi(\cdot)$.

For $\{\psi^\Pi(t) \mid t \in (t_{N-1}, t_N]\} = \{\psi_N\}$, its recursive utility at $t \in [t_{N-1}, t_N]$ is given by $Y^N(t)$, where $(Y^N(\cdot), Z^N(\cdot))$ is the adapted solution to the following BSDE:

$$\begin{aligned} Y^N(t) = \psi_N + &\int_t^T g(s, Y^N(s), Z^N(s))ds \\ &- \int_t^T Z^N(s)dW(s), \qquad t \in [t_{N-1}, t_N]. \end{aligned} \tag{8.1}$$

Here, $g : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an aggregator. Next, for $\{\psi^\Pi(t) \mid t \in (t_{N-2}, t_N]\}$, the recursive utility at $t \in (t_{N-2}, t_{N-1}]$ is denoted by $Y^{N-1}(t)$ and we should have

$$\begin{aligned} &Y^{N-1}(t) \\ &= \psi_{N-1} + \int_{t_{N-1}}^T g(s, Y^N(s), Z^{N-1}(s))ds + \int_t^{t_{N-1}} g(s, Y^{N-1}(s), Z^{N-1}(s))ds \\ &\quad - \int_t^T Z^{N-1}(s)dW(s), \qquad t \in (t_{N-2}, t_{N-1}]. \end{aligned} \tag{8.2}$$

Note that due to the time-consistent requirement, we have to use the already determined $Y^N(\cdot)$ in the drift term over $[t_{N-1}, T]$. On the other hand, since ψ_{N-1} is still merely $\mathcal{F}_T$-measurable, (8.2) has to be solved in $[t, T]$ although $t \in (t_{N-2}, t_{N-1}]$. Hence, in the martingale term, $Z^{N-1}(\cdot)$ has to be free to choose over the entire $[t_{N-2}, T]$ and the already determined $Z^N(\cdot)$ cannot be forced to use there (on $[t_{N-1}, T]$). Whereas, in the drift term over $[t_{N-1}, T]$, it seems to be fine to either use already determined $Z^N(\cdot)$ or to freely choose $Z^{N-1}(\cdot)$, since the time-inconsistent requirement is not required for Z part. However, we use $Z^{N-1}(\cdot)$ in the drift, which will enable us to avoid a technical difficulty for BSVIEs later.

Similarly, the recursive utility on $(t_{N-3}, t_{N-2}]$ should be

$$\begin{aligned} &Y^{N-2}(t) \\ &= \psi_{N-2} + \int_{t_{N-1}}^T g(s, Y^N(s), Z^{N-2}(s))ds + \int_{t_{N-2}}^{t_{N-1}} g(s, Y^{N-1}(s), Z^{N-2}(s))ds \\ &\quad + \int_t^{t_{N-2}} g(s, Y^{N-2}(s), Z^{N-2}(s))ds - \int_t^T Z^{N-2}(s)dW(s), \quad t \in (t_{N-3}, t_{N-2}]. \end{aligned}$$

This procedure can be continued inductively. In general, we have

$$\begin{aligned} Y^k(t) = \psi_k + \sum_{i=k+1}^N \int_{t_{i-1}}^{t_i} g(s, Y^i(s), Z^k(s))ds + \int_t^{t_k} g(s, Y^k(s), Z^k(s))ds \\ - \int_t^T Z^k(s)dW(s), \quad t \in (t_{k-1}, t_k]. \end{aligned}$$

Let us denote

$$Y^{\Pi}(t)=\sum_{k=1}^{N}Y^{k}(t)\mathbf{1}_{(t_{k-1},t_k]}(t),\qquad Z^{\Pi}(t,s)=\sum_{k=1}^{N}Z^{k}(s)\mathbf{1}_{(t_{k-1},t_k]}(t).$$

Then

$$Y^{\Pi}(t)=\psi^{\Pi}(t)+\int_t^T g(s,Y^{\Pi}(s),Z^{\Pi}(t,s))ds-\int_t^T Z^{\Pi}(t,s)dW(s),\quad t\in[0,T].$$

Let $\|\Pi\|\to 0$, by the stability of adapted solutions to BSVIEs [48], we obtain

$$Y(t)=\psi(t)+\int_t^T g(s,Y(s),Z(t,s))ds-\int_t^T Z(t,s)dW(s),\quad t\in[0,T],\qquad (8.3)$$

which is the BSVIE that we expected. Moreover, it is found that if $Y(\cdot)$ is a utility process for $\psi(\cdot)$, the current utility $Y(t)$ depends on the (realistic) future utilities $Y(r);\ t\le r\le T$, which is the main character of recursive utility process. Finally, we note that if we restrict $Z^{N-1}(\cdot)$ on $[t_{N-1},T]$ in (8.2), etc., then we will end up with the following BSVIE:

$$Y(t)=\psi(t)+\int_t^T g(s,Y(s),Z(s,s))ds-\int_t^T Z(t,s)dW(s),\quad t\in[0,T],$$

which is technically difficult since in general, $s\mapsto Z(s,s)$ is not easy to define.

Finally, we would like to point out a fact about BSVIEs and BSDEs. Let us first look at the following general BSDE:

$$Y(t)=\xi+\int_t^T g(s,Y(s),Z(s))ds-\int_t^T Z(s)dW(s),\qquad t\in[0,T].\qquad (8.4)$$

Under standard conditions, for any ξ in a proper space, the above BSDE admits a unique solution $(Y(\cdot),Z(\cdot))\equiv(Y(\cdot\,;T,\xi),Z(\cdot\,;T,\xi))$. By the uniqueness of adapted solutions of BSDEs, we have

$$\begin{aligned}Y(t;T,\xi)&=Y(t;\tau,Y(\tau;T,\xi)),\\ Z(t;T,\xi)&=Z(t;\tau,Y(\tau;T,\xi)),\end{aligned}\qquad \forall 0\le t<\tau\le T.$$

This can be referred to as a (backward) *semi-group property* of BSDEs [34]. However, there is no way to talk about the (backward) semi-group property for BSVIEs. To illustrate this point, let us look at the following simple BSVIE:

$$Y(t)=tW(T)-\int_t^T Z(t,s)dW(s),\qquad t\in[0,T].$$

We can directly check that the adapted solution is given by

$$Y(t) = tW(t), \quad Z(t,s) = t, \qquad (t,s) \in \Delta[0,T].$$

We see that the above $Y(\cdot)$ really could not be related to any (backward) semi-group property. The point that we want to make is that time-consistency and semi-group property are irrelevant.

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