TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 9, Pages 700–732 (July 11, 2022)
https://doi.org/10.1090/btran/114

ON REALIZATIONS OF THE SUBALGEBRA $\mathcal{A}^{\mathbb{R}}(1)$ OF THE $\mathbb{R}$-MOTIVIC STEENROD ALGEBRA

P. BHATTACHARYA, B. GUILLOU, AND A. LI

The first author would like to dedicate this work to his baba Jayanta Bhattacharya whose life was lost to COVID-19 on May 2, 2021. He always said, whatever you do do it in depth, and leave no stone unturned. You will be deeply missed!

Abstract. In this paper, we show that the finite subalgebra $\mathcal{A}^{\mathbb{R}}(1)$, generated by Sq^1 and Sq^2, of the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ can be given 128 different $\mathcal{A}^{\mathbb{R}}$-module structures. We also show that all of these $\mathcal{A}$-modules can be realized as the cohomology of a 2-local finite $\mathbb{R}$-motivic spectrum. The realization results are obtained using an $\mathbb{R}$-motivic analogue of the Toda realization theorem. We notice that each realization of $\mathcal{A}^{\mathbb{R}}(1)$ can be expressed as a cofiber of an $\mathbb{R}$-motivic v_1-self-map. The C_2-equivariant analogue of the above results then follows because of the Betti realization functor. We identify a relationship between the $\mathrm{RO}(\mathrm{C}_2)$-graded Steenrod operations on a C_2-equivariant space and the classical Steenrod operations on both its underlying space and its fixed-points. This technique is then used to identify the geometric fixed-point spectra of the C_2-equivariant realizations of $\mathcal{A}^{\mathrm{C}_2}(1)$. We find another application of the $\mathbb{R}$-motivic Toda realization theorem: we produce an $\mathbb{R}$-motivic, and consequently a C_2-equivariant, analogue of the Bhattacharya-Egger spectrum $\mathcal{Z}$, which could be of independent interest.

Contents

1. Introduction

This paper is a continuation of the work that began in [BGL], where we studied periodic self-maps of a certain finite $\mathbb{R}$-motivic spectrum $\mathcal{Y}^{\mathbb{R}}_{(\mathrm{h},1)}$. There, we proved

Received by the editors July 11, 2021, and, in revised form, January 17, 2022.
2020 *Mathematics Subject Classification.* Primary 14F42, 55S10, 55S91.
The first author was supported by NSF grant DMS-2005476. The second and third authors were supported by NSF grant DMS-2003204.

that $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ supports a 1-periodic $v_{(1,\mathrm{nil})}$-self-map

$$v : \Sigma^{2,1}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \longrightarrow \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}, \tag{1.1}$$

whose cofiber realizes the subalgebra $\mathcal{A}^{\mathbb{R}}(1)$ of the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ generated by Sq^1 and Sq^2. Recall from [BGL, Definition 1.7] that a $v_{(1,\mathrm{nil})}$-self-map is a v_1-self-map which is nilpotent on the geometric fixed-points.

The spectrum $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ is an $\mathbb{R}$-motivic lift of the classical spectrum

$$\mathcal{Y} := \Sigma^{-3}\mathbb{C}\mathbb{P}^2 \wedge \mathbb{R}\mathbb{P}^2.$$

From the chromatic point of view, the spectrum $\mathcal{Y}$ is extremely useful because it supports a v_1-self-map of lowest possible periodicity, that is, one. Famously, Mark Mahowald used the spectrum $\mathcal{Y}$ and the low periodicity of its v_1-self-map to prove the height 1 telescope conjecture at the prime 2 [M1, M2]. However, 1-periodic v_1-self-maps of $\mathcal{Y}$ are not unique. In fact, up to homotopy, there are eight different v_1-self-maps supported by $\mathcal{Y}$, all of whose cofibers are realizations of $\mathcal{A}(1)$ (see [DM]). Up to weak equivalence, there are four different finite spectra realizing $\mathcal{A}(1)$, and all of them can be obtained as the cofiber of some v_1-self-map of $\mathcal{Y}$. These four different realizations can be distinguished by their $\mathcal{A}$-module structures. Therefore, it is natural to ask if all of the v_1-self-maps of $\mathcal{Y}$ can be lifted to $\mathbb{R}$-motivic analogues, and whether all of the $\mathbb{R}$-motivic realizations of $\mathcal{A}^{\mathbb{R}}(1)$ can be obtained as the cofiber of such a lift.

The answer to the above question is complicated by the fact that there are multiple $\mathbb{R}$-motivic lifts of the spectrum $\mathcal{Y}$ (see [BGL]). Even if we insist on those lifts which can potentially realize $\mathcal{A}^{\mathbb{R}}(1)$ as a cofiber of a periodic self-map, we are left with two choices: $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ and $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$. We state our first result towards answering these questions after establishing some notations. Further, we shall see that some realizations of $\mathcal{A}^{\mathbb{R}}(1)$ must be given as the cofiber of a map between $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ and $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ rather than as the cofiber of a self-map of either.

Before describing the results of this article, we present some notation that will be used throughout.

Notation 1.2. Throughout this paper, we use the following notations:

- $\mathbf{Sp}^{\mathbb{R}}$ – the ∞-category of $\mathbb{R}$-motivic spectra.
- $\mathbf{Sp}^{\mathrm{C}_2}$ – the ∞-category of genuine C_2-equivariant spectra.
- $\mathrm{H}_{\mathbb{R}}\mathbb{F}_2$ – the $\mathbb{R}$-motivic Eilenberg-Mac Lane spectrum with $\mathbb{F}_2$-coefficients.
- $\mathrm{H}\underline{\mathbb{F}}_2$ – the C_2-equivarient Eilenberg-Mac Lane spectrum at the constant Mackey functor $\underline{\mathbb{F}}_2$.
- $\mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ – the category of cellular $\mathrm{H}_{\mathbb{R}}\mathbb{F}_2$-complete $\mathbb{R}$-motivic spectra with finitely many cells.
- We denote the 1-dimensional trivial $\mathbb{R}$-representation of C_2 by ϵ, the sign representation by σ, and the regular representation by ρ.
- $\mathrm{H}^{n,m}_{\mathbb{R}}(\mathrm{E}) := [\mathrm{E}, \Sigma^{n,m}\mathrm{H}_{\mathbb{R}}\mathbb{F}_2]$ – the $\mathbb{R}$-motivic cohomology of $\mathrm{E} \in \mathbf{Sp}^{\mathbb{R}}$ with constant sheaf $\mathbb{F}_2$, where n is the topological degree and m is the motivic weight.
- $\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}}$ – the category of cellular $\mathrm{H}\underline{\mathbb{F}}_2$-complete C_2-equivariant spectra with finitely many cells.
- $\mathrm{H}^{\star}_{\mathrm{C}_2}(\mathrm{E}) = [\mathrm{E}, \mathrm{H}\underline{\mathbb{F}}_2]^{\mathrm{C}_2}_{-\star}$ – the $\mathrm{RO}(\mathrm{C}_2)$-graded cohomology of $\mathrm{E} \in \mathbf{Sp}^{\mathrm{C}_2}$ with coefficients in the constant Mackey functor. We will often use motivic

bigrading for $\mathrm{H}^\star_{\mathrm{C}_2}(\mathrm{E})$ under the identification

$$(n, m) \rightsquigarrow (n - m)\epsilon + m\sigma.$$

- $\mathbb{M}_2^{\mathbb{R}} := \pi_{*,*}\mathrm{H}_{\mathbb{R}}\mathbb{F}_2$. By a calculation of Voevodsky [V]

$$\mathbb{M}_2^{\mathbb{R}} \cong \mathbb{F}_2[\tau, \rho]$$

with $|\tau| = (0, -1)$ and $|\rho| = (-1, -1)$.
- $\mathbb{M}_2^{\mathrm{C}_2} := \pi_\star \mathrm{H}\underline{\mathbb{F}}_2$. This computation can be found in [C, Appendix] or [HK, Proposition 6.2] and is given by

$$\mathbb{M}_2^{\mathrm{C}_2} \cong \mathbb{F}_2[u_\sigma, a_\sigma] \oplus \Theta\{u_\sigma^{-i} a_\sigma^{-j} : i, j \geq 0\},$$

where $|u_\sigma| = (0, -1)$, $|a_\sigma| = (-1, -1)$ and $|\Theta| = (0, 2)$.
- We follow [DI, BI, BGL] in grading $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}$ as $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s,f,w}$, where s is the stem, f is the Adams filtration, and w is the motivic weight.

Our first result concerns realizations of $\mathcal{A}^{\mathbb{R}}(1)$.

Theorem 1.3. *There exists* 128 *different* $\mathcal{A}^{\mathbb{R}}$*-modules whose underlying* $\mathcal{A}^{\mathbb{R}}(1)$*-module structures are free on one generator, all of which can be realized as* $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{X})$ *for some* $\mathrm{X} \in \mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$.

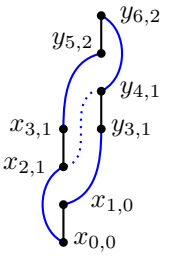


FIGURE 1.4. We depict a singly-generated free $\mathcal{A}^{\mathbb{R}}(1)$-module, where each $\bullet$ represents a $\mathbb{M}_2^{\mathbb{R}}$-generator. The black and blue lines represent the action of motivic Sq^1 and Sq^2, respectively. A dotted line represents that the action hits the τ-multiple of the given $\mathbb{M}_2^{\mathbb{R}}$-generator.

Notation 1.5. For the rest of the paper, we fix an $\mathbb{M}_2^{\mathbb{R}}$-basis

$$\{x_{0,0}, x_{1,0}, x_{2,1}, x_{3,1}, y_{3,1}, y_{4,1}, y_{5,2}, y_{6,2}\}$$

of $\mathcal{A}^{\mathbb{R}}(1)$ as in Figure 1.4, so that

- $\mathrm{Sq}^1(x_{0,0}) = x_{1,0}$
- $\mathrm{Sq}^1(x_{2,1}) = x_{3,1}$
- $\mathrm{Sq}^1(y_{3,1}) = y_{4,1}$
- $\mathrm{Sq}^1(y_{5,2}) = y_{6,2}$
- $\mathrm{Sq}^2(x_{0,0}) = x_{2,1}$
- $\mathrm{Sq}^2(x_{1,0}) = y_{3,1}$
- $\mathrm{Sq}^2(x_{2,1}) = \tau y_{4,1}$
- $\mathrm{Sq}^2(x_{3,1}) = y_{5,2}$
- $\mathrm{Sq}^2(y_{4,1}) = y_{6,2}$.

We now record all 128 $\mathcal{A}^{\mathbb{R}}$-modules of Theorem 1.3 using the basis above.

Theorem 1.6. *For every vector* $(\alpha_{03}, \beta_{03}, \beta_{14}, \beta_{06}, \beta_{25}, \beta_{26}, \gamma_{36}) \in \mathcal{V} = \mathbb{F}_2^7$ *and*

$$\jmath_{24} = \beta_{03}\gamma_{36} + \alpha_{03}(\beta_{25} + \beta_{26}),$$

there exists a unique isomorphism class of $\mathcal{A}^{\mathbb{R}}$*-module structures on* $\mathcal{A}^{\mathbb{R}}(1)$ *determined by the formulas*

(i) $\mathrm{Sq}^4(x_{0,0}) = \beta_{03}(\rho \cdot y_{3,1}) + (1 + \beta_{03} + \beta_{14})(\tau \cdot y_{4,1}) + \alpha_{03}(\rho \cdot x_{3,1})$
(ii) $\mathrm{Sq}^4(x_{1,0}) = y_{5,2} + \beta_{14}(\rho \cdot y_{4,1})$
(iii) $\mathrm{Sq}^4(x_{2,1}) = \beta_{26}(\tau \cdot y_{6,2}) + \beta_{25}(\rho \cdot y_{5,2}) + \jmath_{24}(\rho^2 \cdot y_{4,1})$
(iv) $\mathrm{Sq}^4(x_{3,1}) = (\beta_{25} + \beta_{26})(\rho \cdot y_{6,2})$
(v) $\mathrm{Sq}^4(y_{3,1}) = \gamma_{36}(\rho \cdot y_{6,2})$
(vi) $\mathrm{Sq}^8(x_{0,0}) = \beta_{06}(\rho^2 \cdot y_{6,2})$.

Further, any $\mathcal{A}^{\mathbb{R}}$*-module whose underlying* $\mathcal{A}^{\mathbb{R}}(1)$*-module is free on one generator is isomorphic to one listed above.*

Notation 1.7. For any vector $\overline{\mathrm{v}} \in \mathcal{V}$, we denote the corresponding $\mathcal{A}^{\mathbb{R}}$-module in Theorem 1.6 by $\mathcal{A}^{\mathbb{R}}_{\overline{\mathrm{v}}}(1)$. By $\mathcal{A}^{\mathbb{R}}_1[\overline{\mathrm{v}}]$, we denote an object of $\mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$, whose cohomology is isomorphic to $\mathcal{A}^{\mathbb{R}}_{\overline{\mathrm{v}}}(1)$ as an $\mathcal{A}^{\mathbb{R}}$-module. We let

$$\mathcal{A}^{\mathbb{R}}_1 := \{\mathcal{A}^{\mathbb{R}}_1[\overline{\mathrm{v}}] : \overline{\mathrm{v}} \in \mathcal{V}\}/(\text{weak equivalence})$$

denote the set of equivalence classes of finite $\mathbb{R}$-motivic spectra whose cohomology are free of rank 1 over $\mathcal{A}^{\mathbb{R}}(1)$.

Let $\mathcal{B}^{\mathbb{R}}_{\mathsf{h}}(1)$ and $\mathcal{B}^{\mathbb{R}}_2(1)$ denote the $\mathcal{A}^{\mathbb{R}}$-modules $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$ and $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{Y}^{\mathbb{R}}_{(2,1)})$, respectively. As shown in [BGL, Lemma 4.4], these differ in that the bottom cell of $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ supports a Sq^4, whereas the bottom cell of $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ does not. In [BGL], we used a method due to Smith ([R1, Appendix C]) to produce the $\mathcal{A}^{\mathbb{R}}$-module $\mathcal{A}^{\mathbb{R}}_{\mathbf{0}}(1)$. Then we observed that $\mathcal{A}^{\mathbb{R}}_{\mathbf{0}}(1)$ fits into a short exact sequence

$$\Sigma^{3,1}\mathcal{B}^{\mathbb{R}}_{\mathsf{h}}(1) \hookrightarrow \mathcal{A}^{\mathbb{R}}_{\mathbf{0}}(1) \twoheadrightarrow \mathcal{B}^{\mathbb{R}}_{\mathsf{h}}(1)$$

that can be realized as a cofiber sequence of finite spectra. The connecting map of this cofiber sequence is the map (1.1). In this paper, we extend the above result of [BGL] to prove the following.

Theorem 1.8. *Given* $\overline{\mathrm{v}} = (\alpha_{03}, \beta_{03}, \beta_{14}, \beta_{06}, \beta_{25}, \beta_{26}, \gamma_{36}) \in \mathcal{V}$*, define*

$$\epsilon = \begin{cases} \mathsf{h} & \text{if } \beta_{25} + \beta_{26} + \gamma_{36} = 0 \\ 2 & \text{if } \beta_{25} + \beta_{26} + \gamma_{36} = 1, \end{cases} \quad \text{and} \quad \delta = \begin{cases} \mathsf{h} & \text{if } \alpha_{03} + \beta_{03} = 0 \\ 2 & \text{if } \alpha_{03} + \beta_{03} = 1. \end{cases}$$

Then there exists a short exact sequence

$$\Sigma^{3,1}\mathcal{B}^{\mathbb{R}}_{\epsilon}(1) \hookrightarrow \mathcal{A}^{\mathbb{R}}_{\overline{\mathrm{v}}}(1) \twoheadrightarrow \mathcal{B}^{\mathbb{R}}_{\delta}(1) \tag{1.9}$$

of $\mathcal{A}^{\mathbb{R}}$*-modules. Moreover, this exact sequence can be realized as the cohomology of a cofiber sequence*

$$\mathcal{Y}^{\mathbb{R}}_{(\delta,1)} \longrightarrow \mathcal{A}^{\mathbb{R}}_1[\overline{\mathrm{v}}] \longrightarrow \Sigma^{3,1}\mathcal{Y}^{\mathbb{R}}_{(\epsilon,1)} \tag{1.10}$$

in the category $\mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$.

The map of spectra that underlies the connecting map

$$v\colon \Sigma^{2,1}\mathcal{Y}^{\mathbb{R}}_{(\epsilon,1)} \longrightarrow \mathcal{Y}^{\mathbb{R}}_{(\delta,1)} \tag{1.11}$$

of (1.10) is a v_1-self-map of $\mathcal{Y}$ of periodicity 1.

The algebraic part of Theorem 1.8 is a straightforward consequence of Theorem 1.6 once we identify the $\mathcal{A}^{\mathbb{R}}$-modules $\mathcal{B}^{\mathbb{R}}_{\mathsf{h}}(1)$ and $\mathcal{B}^{\mathbb{R}}_{2}(1)$ (we refer to [BGL, Figure 7] for a complete description). However, the topological assertions in Theorem 1.8, as well as in Theorem 1.3, require a technical tool, which we refer to as the $\mathbb{R}$-*motivic Toda realization theorem.*

Theorem 1.12 ($\mathbb{R}$-motivic Toda realization theorem)**.** *Let* M *be an* $\mathcal{A}^{\mathbb{R}}$*-module whose underlying* $\mathbb{M}^{\mathbb{R}}_2$*-module is free and finite. There exists* $\mathrm{X} \in \mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ *such that* $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{X}) \cong \mathrm{M}$ *as* $\mathcal{A}^{\mathbb{R}}$*-modules if*

$$\mathrm{Ext}^{-2,f,0}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{M},\mathrm{M}) = 0 \tag{1.13}$$

for all $f \geq 3$.

In this paper, we also prove various weaker versions of the $\mathbb{R}$-motivic Toda realization theorem (see Theorem 2.4, Theorem 2.6 and Theorem 2.11), which are perhaps more convenient for application purposes.

A realization theorem is often accompanied by a uniqueness theorem, as is the case with Toda's classical result (see [BE, Proposition 5.1]). The $\mathbb{R}$-motivic analogue can be stated as follows:

Theorem 1.14 ($\mathbb{R}$-motivic unique realization theorem)**.** *Let* $\mathrm{X} \in \mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ *such that*

$$\mathrm{Ext}^{-1,f,0}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{X}),\mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{X})) = 0$$

for any $f \geq 2$. *Then any spectrum* $\mathrm{X}' \in \mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ *for which there exists an* $\mathcal{A}^{\mathbb{R}}$*-module isomorphism* $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{X}') \cong \mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{X})$, *is weakly equivalent to* X. *In other words, the* $\mathcal{A}^{\mathbb{R}}$*-module* $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{X})$ *is uniquely realized in* $\mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ *up to a weak equivalence.*

Proof. The result follows from the fact that the nonzero element

$$\iota \in \mathrm{Ext}^{0,0,0}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{X}'),\mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{X}))$$

representing the isomorphism $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{X}') \cong \mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{X})$ survives the Adams spectral sequence converging to $[\mathrm{X},\mathrm{X}']$. □

The uniqueness theorem applies to the $\mathcal{A}^{\mathbb{R}}$-modules $\mathcal{B}^{\mathbb{R}}_{\mathsf{h}}(1)$ and $\mathcal{B}^{\mathbb{R}}_{2}(1)$ (see Lemma 4.4). However, it does not apply to $\mathcal{A}^{\mathbb{R}}_{1}[\overline{\mathrm{v}}]$ for any $\overline{\mathrm{v}} \in \mathcal{V}$. Potentially, there can be multiple different finite spectra realizing $\mathcal{A}^{\mathbb{R}}_{\overline{\mathrm{v}}}(1)$ up to a weak equivalence (see Remark 4.3), making it difficult to get a precise count of the number of 1-periodic v_1-self-maps on $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ and $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ from Theorem 1.8.

Upon applying the Betti realization functor

$$\beta : \mathbf{Sp}^{\mathbb{R}} \longrightarrow \mathbf{Sp}^{\mathrm{C}_2}$$

we get various C_2-equivariant maps $\beta(v) : \Sigma^{2,1}\mathcal{Y}^{\mathrm{C}_2}_{(\epsilon,1)} \longrightarrow \mathcal{Y}^{\mathrm{C}_2}_{(\delta,1)}$ (where $\epsilon,\delta \in \{2,\mathsf{h}\}$) whose underlying maps are v_1-self-maps of $\mathcal{Y}$. We also get Corollary 1.15 of Theorem 1.3 (see Remark 4.8).

Corollary 1.15. *There exists* 128 *different* $\mathcal{A}^{C_2}$*-modules whose underlying* $\mathcal{A}^{C_2}(1)$*-module structures are free on one generator, all of which can be realized as the* $\mathrm{RO}(C_2)$*-graded cohomology of a* 2*-local finite* C_2*-spectrum.*

Notation 1.16. Let $\mathcal{A}_1^{C_2}[\overline{v}]$ denote the Betti realization $\beta(\mathcal{A}_1^{\mathbb{R}}[\overline{v}])$, where $\overline{v} \in \mathcal{V}$.

Let $\Re : \mathbf{Sp}^{C_2} \to \mathbf{Sp}$ denote the restriction functor (restricting the C_2-action to the trivial group) and $\Phi : \mathbf{Sp}^{C_2} \to \mathbf{Sp}$ denote the geometric fixed-point functor (see Section 3). Note that the geometric fixed-point spectra

$$\Phi(\mathcal{Y}^{C_2}_{(\mathsf{h},1)}) := \Phi(C^{C_2}(\mathsf{h}) \wedge C^{C_2}(\eta_{1,1})) \simeq M_2(1) \vee \Sigma M_2(1)$$

and

$$\Phi(\mathcal{Y}^{C_2}_{(2,1)}) := \Phi(C^{C_2}(2) \wedge C^{C_2}(\eta_{1,1})) \simeq M_2(1) \wedge M_2(1)$$

are both of type 1 (see [BGL]). Further, for degree reasons

$$\Phi(\beta(v)) : \Sigma\Phi(\mathcal{Y}^{C_2}_{(\epsilon,1)}) \longrightarrow \Phi(\mathcal{Y}^{C_2}_{(\delta,1)})$$

cannot be a v_1-self-map, and hence must be nilpotent, using [HS]. Therefore, the fiber $\Phi(\mathcal{A}_1^{C_2}[\overline{v}])$ is a type 1 spectrum, i.e. $\mathcal{A}_1^{C_2}[\overline{v}]$ is of type $(2,1)$ in the sense of [BGL]. Much more can be said about $\Phi(\mathcal{A}_1^{C_2}[\overline{v}])$ than just its type. In this paper, we give a complete description of the $\mathcal{A}$-module structure of $H^*(\Phi(\mathcal{A}_1^{C_2}[\overline{v}]))$ for all $\overline{v} \in \mathcal{V}$ by developing a general method that compares the $\mathrm{RO}(C_2)$-graded squaring operations of a C_2 spectrum with the ordinary squaring operations of its underlying spectrum as well as its geometric fixed-point spectrum (compare [BW, §3]).

Since $\Re(H\underline{\mathbb{F}}_2) \simeq H\mathbb{F}_2$ we have a natural map

$$\Re_* : H^{n,m}_{C_2}(E) \simeq [E, \Sigma^{n,m} H\underline{\mathbb{F}}_2] \longrightarrow [\Re(E), \Sigma^n \Re(H\underline{\mathbb{F}}_2)] \simeq H^n(\Re(E))$$

for any $E \in \mathbf{Sp}^{C_2}$. We use Theorem 1.17 to identify the spectrum underlying $\mathcal{A}_1^{C_2}[\overline{v}]$ (see Theorem 4.9). Here and in Theorem 1.19, we write $\underline{\mathrm{Sq}}^n$ for the C_2-equivariant squaring operations, as in Section 3.2.

Theorem 1.17. *For* $E \in \mathbf{Sp}^{C_2}_{2,\mathrm{fin}}$ *and any class* $u \in H^\star_{C_2}(E)$, $\Re_*(\underline{\mathrm{Sq}}^n(u)) = \mathrm{Sq}^n(\Re_*(u))$.

Using the fact that the projection map

$$\pi^{(0)}_{\mathbb{F}_2} : \Phi(H\underline{\mathbb{F}}_2) \simeq H\mathbb{F}_2[t] \twoheadrightarrow H\mathbb{F}_2$$

is an $\mathbb{E}_\infty$-ring map, as in the discussion preceding (3.3), one defines (also see [BW, (2.7)]) the map

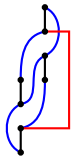
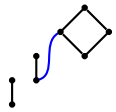
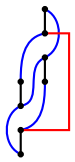
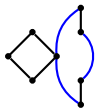
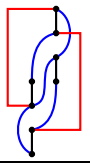
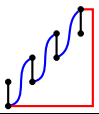
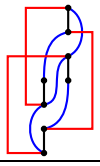
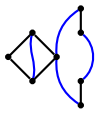
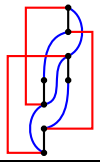
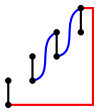
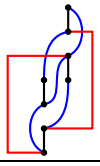
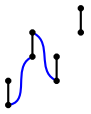
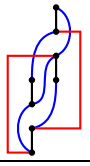
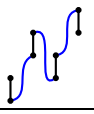
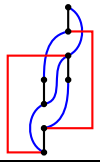
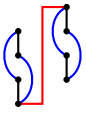
$$\hat{\Phi}_* : H^{n,m}_{C_2}(E) \twoheadrightarrow H^{n-m}(\Phi(E)) \tag{1.18}$$

which compares the $\mathrm{RO}(C_2)$-graded cohomology of a C_2-spectrum E with the ordinary cohomology of its geometric fixed-point spectrum. We show:

Theorem 1.19. *For* $E \in \mathbf{Sp}^{C_2}_{2,\mathrm{fin}}$ *and any class* $u \in H^\star_{C_2}(E)$,

$$\hat{\Phi}_*(\underline{\mathrm{Sq}}^{2n}(u)) = \mathrm{Sq}^n(\hat{\Phi}_*(u)).$$

FIGURE 1.20. Some underlying and fixed $\mathcal{A}$-modules of $\mathcal{A}_1^{C_2}$

$\overline{\nu} \in \mathcal{V}$	$H^*(\Re(\mathcal{A}_1^{C_2}[\overline{\nu}]))$	$H^*(\Phi(\mathcal{A}_1^{C_2}[\overline{\nu}]))$	Cofiber of
$(0,0,1,0,0,0,0)$			$v : \Sigma^{2,1}\mathcal{Y}_{(h,1)} \to \mathcal{Y}_{(h,1)}$
$(1,1,0,0,0,0,1)$			$v : \Sigma^{2,1}\mathcal{Y}_{(2,1)} \to \mathcal{Y}_{(h,1)}$
$(0,1,0,1,0,1,0)$			$v : \Sigma^{2,1}\mathcal{Y}_{(2,1)} \to \mathcal{Y}_{(2,1)}$
$(1,0,0,0,0,1,1)$			$v : \Sigma^{2,1}\mathcal{Y}_{(h,1)} \to \mathcal{Y}_{(2,1)}$
$(0,0,0,1,0,1,0)$			$v : \Sigma^{2,1}\mathcal{Y}_{(2,1)} \to \mathcal{Y}_{(h,1)}$
$(1,0,0,0,0,0,0)$			$v : \Sigma^{2,1}\mathcal{Y}_{(h,1)} \to \mathcal{Y}_{(2,1)}$
$(1,0,0,0,0,0,1)$			$v : \Sigma^{2,1}\mathcal{Y}_{(2,1)} \to \mathcal{Y}_{(2,1)}$
$(1,1,1,1,1,0,1)$			$v : \Sigma^{2,1}\mathcal{Y}_{(2,1)} \to \mathcal{Y}_{(2,1)}$

We find Theorem 1.17 and Theorem 1.19 very handy for computational purposes. These results can be applied to understand the $RO(C_2)$-graded squaring

operations on the cohomology of a wide variety of C_2-spectra whose underlying and geometric fixed-point spectra are known. Alternatively, one can identify the action of the classical Steenrod algebra on the cohomology of the underlying as well as the geometric fixed-points of a C_2-spectrum from the knowledge of $RO(C_2)$-graded Steenrod operations. We apply Theorem 1.17 and Theorem 1.19 to identify the $\mathcal{A}$-module structure of the underlying and the geometric fixed-points of $\mathcal{A}_1^{C_2}[\overline{\nu}]$ (see Theorem 4.9 and Theorem 4.11).

In Figure 1.20, we provide the $\mathcal{A}$-module structure of the underlying and the geometric fixed-points of $\mathcal{A}_1^{C_2}[\overline{\nu}]$ for selected values of $\overline{\nu} \in \mathcal{V}$. We express Sq^1, Sq^2 and Sq^4 using black, blue, and red lines respectively.

Remark 1.21 (Appearance of the Joker). We note that the $\mathcal{A}(1)$-module

often called the Joker, is a subcomplex of the geometric fixed-point of $\mathcal{A}_1^{\mathbb{R}}[\overline{\nu}]$ if and only if $j_{24} = 1$. Further, when $j_{24} = 1$ then in (1.9), ϵ and δ cannot both equal h. This can easily be derived from Theorem 1.6 and Theorem 1.19.

Remark 1.22. In [BGL], the authors construct $\mathcal{A}_1^{\mathbb{R}}[\mathbf{0}]$ as a split summand of $\mathcal{Q}_{\mathbb{R}}^{\wedge 3}$ using a certain idempotent of $\mathbb{Z}_{(2)}[\Sigma_3]$. Let $\tilde{\mathcal{Q}}_{\mathbb{R}} \in \mathbf{Sp}_{2,\mathrm{fin}}^{\mathbb{R}}$ be such that its cohomology as an $\mathcal{A}^{\mathbb{R}}(1)$-module is isomorphic to $\mathcal{Q}_{\mathbb{R}}$, but has the additional relation

$$Sq^4(a) = \rho \cdot c$$

(in the notation of [BGL, Figure 2]), as an $\mathcal{A}^{\mathbb{R}}$-module. If we replace $\mathcal{Q}_{\mathbb{R}}$ by a complex $\tilde{\mathcal{Q}}_{\mathbb{R}}$ in [BGL], we get $\mathcal{A}_1^{\mathbb{R}}[\overline{\nu}]$, where $\overline{\nu} = (1,1,1,1,1,0,1)$ (see the last diagram in Figure 1.20).

Remark 1.23. The classical spectrum $\mathcal{A}_1$ is a type 2 spectrum and supports a v_2^{32}-self-map [BEM]. It remains to be seen if this v_2-self-map can be lifted to $\mathcal{A}_1^{\mathbb{R}}[\overline{\nu}]$ for various $\overline{\nu} \in \mathcal{V}$.

Recently in [BE], the authors introduced a new type 2 spectrum $\mathcal{Z}$ which is notable for admitting a v_2-self-map of lowest possible periodicity, that is 1. The low periodicity of the v_2-self-map makes the spectrum $\mathcal{Z}$ suitable for the analysis of the telescope conjecture which, if true, would imply that the natural map from the telescope of $\mathcal{Z}$ to the K(2)-localization of $\mathcal{Z}$ is a weak equivalence. While the telescope conjecture is true for finite spectra of type 1 [M1,M2,M], it is expected to be false for finite spectra of type ≥ 2 (see [MRS]). In fact, in [BBB+], the authors study the prime 2, height 2 telescope conjecture using the spectrum $\mathcal{Z}$ and lay down several conjectures (see [BBB+, §9]), whose validity would lead to a disproof of the telescope conjecture. In this paper, we also construct an $\mathbb{R}$-motivic analogue of $\mathcal{Z}$ which is likely to shed light on some of these conjectures.

Theorem 1.24. *There exists $\mathcal{Z}_{\mathbb{R}} \in \mathbf{Sp}_{2,\mathrm{fin}}^{\mathbb{R}}$ such that the underlying $\mathcal{A}^{\mathbb{R}}(2)$-module structure of its cohomology is isomorphic to*

$$H_{\mathbb{R}}^{*,*}(\mathcal{Z}_{\mathbb{R}}) \cong_{\mathcal{A}^{\mathbb{R}}(2)} \mathcal{A}^{\mathbb{R}}(2) \otimes_{\Lambda(\tilde{Q}_2^{\mathbb{R}})} \mathbb{M}_2^{\mathbb{R}},$$

where $\tilde{Q}_2^{\mathbb{R}} := [Sq^4, Q_1^{\mathbb{R}}]$.

See Subsection 5.2 for a discussion of the element $\tilde{Q}_2^{\mathbb{R}}$. In future work, we intend to study the properties of $\mathcal{Z}_{\mathbb{R}}$ extensively and hope to prove, among other things, Conjecture 1.25.

Conjecture 1.25. *The spectrum $\mathcal{Z}_{\mathbb{R}}$ is of type $(2,2)$ and admits a $v_{(2,\mathrm{nil})}$-self-map*

$$v : \Sigma^{6,3}\mathcal{Z}_{\mathbb{R}} \longrightarrow \mathcal{Z}_{\mathbb{R}}$$

of periodicity 1.

Organization of the paper. In Section 2, we discuss the $\mathbb{R}$-motivic Toda realization Theorem 1.12 and derive various weak forms that are suitable for applications. In Section 3, we construct the equivariant Steenrod operations using the equivariant extended power construction and prove Theorem 1.17 and Theorem 1.19, which establish comparisons with the classical Steenrod operations. In Section 4, we apply the discussion in Section 2 to obtain the $\mathbb{R}$-motivic topological realizations of $\mathcal{A}^{\mathbb{R}}(1)$ and analyze the properties of their Betti realizations using results from Section 3. In Section 5, we construct the $\mathbb{R}$-motivic spectrum $\mathcal{Z}_{\mathbb{R}}$ using a method of Smith. Finally, the short Section A lists the Adem relations in the $\mathbb{R}$-motivic Steenrod algebra.

2. $\mathbb{R}$-motivic Toda realization theorem

The classical Toda realization theorem [T] (see also [BE, Theorem 3.1]), is recast in the modern literature as a special case of Goerss-Hopkins obstruction theory [GH] (when the chosen operad is trivial). This obstruction theory can be generalized to the $\mathbb{R}$-motivic setting [MG], and Theorem 1.12 would then be a special case of such a generalization.

More recent work of [PV] conceptualizes Goerss-Hopkins obstruction theory in the general setup of stable ∞-categories with t-structures. If we set $\mathcal{C} = \mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$, $\mathrm{A} = \mathbb{S}_{\mathrm{H}_{\mathbb{R}}\mathbb{F}_2}$, and let K to be a finite $\mathcal{A}^{\mathbb{R}}_*$-comodule in [PV, Corollary 4.10], then we get a sequence of obstruction classes

$$\theta_n \in \mathrm{Ext}^{-2,n+2,0}_{\mathcal{A}^{\mathbb{R}}_*}(\mathrm{K},\mathrm{K}) \tag{2.1}$$

for each $n \geq 0$, the vanishing of which guarantees the existence of an $\mathbb{S}_{\mathrm{H}_{\mathbb{R}}\mathbb{F}_2}$-module whose homology is isomorphic to K as an $\mathcal{A}^{\mathbb{R}}_*$-comodule. Since the t-structure in $\mathbf{Sp}^{\mathbb{R}}$ does not change the motivic weight, the obstruction classes in (2.1) lie in the Ext-groups of motivic weight 0.

If M is a finite $\mathbb{M}^{\mathbb{R}}_2$-free $\mathcal{A}^{\mathbb{R}}$-module then $\mathrm{K} := \hom_{\mathbb{M}^{\mathbb{R}}_2}(\mathrm{M}, \mathbb{M}^{\mathbb{R}}_2)$ is a finite $\mathcal{A}^{\mathbb{R}}_*$-comodule,

$$\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}_*}(\mathrm{K},\mathrm{K}) \cong \mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{M},\mathrm{M}),$$

and therefore, Theorem 1.12 follows. Alternatively, one can prove Theorem 1.12 simply by emulating the classical proof (as exposed in [BE, §3]).

The purpose of this section is to deduce, from the $\mathbb{R}$-motivic Toda realization theorem (Theorem 1.12), various weaker forms which are perhaps more convenient for application purposes. Explicit calculation of $\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{M},\mathrm{M})$ can often be difficult, and one can use a sequence of spectral sequences to approximate these ext groups. Each such approximation leads to a corresponding weaker form.

2.1. **Weak $\mathbb{R}$-motivic Toda realization – version (I).** Let M be an $\mathcal{A}^{\mathbb{R}}$-module whose underlying $\mathbb{M}_2^{\mathbb{R}}$-module is free and finitely generated. Let $\mathcal{B}_{\mathrm{M}}$ denote its $\mathbb{M}_2^{\mathbb{R}}$-basis and $\mathcal{D}_{\mathrm{M}}$ denote the collection of bidegrees in which there is an element in $\mathcal{B}_{\mathrm{M}}$. For any element $x \in \mathrm{M}^{s,w}$, we let $\mathtt{t}(x) = s + w$ and define

$$\mathrm{M}_{\geq n} := \mathbb{M}_2^{\mathbb{R}} \cdot \{b \in \mathcal{B}_{\mathrm{M}} : \mathtt{t}(b) \geq n\}$$

as the free sub $\mathbb{M}_2^{\mathbb{R}}$-module of M generated by $\{b \in \mathcal{B}_{\mathrm{M}} : \mathtt{t}(b) \geq n\}$.

Note that the $\mathcal{A}^{\mathbb{R}}$-module structure of M is determined by the action of $\mathcal{A}^{\mathbb{R}}$ on the elements of $\mathcal{B}_{\mathrm{M}}$ and the Cartan formula. This, along with the fact that $\mathtt{t}(a) \geq 0$ for all $a \in \mathcal{A}^{\mathbb{R}}$, implies that $\mathrm{M}_{\geq n}$ are also a sub $\mathcal{A}^{\mathbb{R}}$-module of M. Therefore, we get an $\mathcal{A}^{\mathbb{R}}$-module filtration of M

$$\mathrm{M} = \mathrm{M}_{\geq k} \supset \mathrm{M}_{\geq k+1} \supset \cdots \supset \mathrm{M}_{\geq k+l} = \mathbf{0}$$

such that we for each i there is a short exact sequences

$$0 \longrightarrow \mathrm{M}_{\geq i+1} \longrightarrow \mathrm{M}_{\geq i} \longrightarrow \bigoplus_{\{b \in \mathcal{B}_{\mathrm{M}} : \mathtt{t}(b) = i\}} \Sigma^{|b|} \mathbb{M}_2^{\mathbb{R}} \longrightarrow 0 \tag{2.2}$$

of $\mathcal{A}^{\mathbb{R}}$-modules.

A short exact sequence of $\mathcal{A}^{\mathbb{R}}$-modules gives a long exact sequence in Ext. By splicing the long exact sequences induced by (2.2), we get an "algebraic" Atiyah-Hirzebruch spectral sequence

$$\mathrm{E}_2^{s',w',s,f,w} := \mathcal{B}_{\mathrm{M}}^{s',w'} \otimes \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s,f,w}(\mathrm{M}, \mathbb{M}_2^{\mathbb{R}}) \Rightarrow \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s-s',f,w-w'}(\mathrm{M}, \mathrm{M}) \tag{2.3}$$

and a corresponding weak version of Theorem 1.12, along with a uniqueness criterion, which is a weak form of Theorem 1.14.

Theorem 2.4. *Let* M *denote an $\mathcal{A}^{\mathbb{R}}$-module whose underlying $\mathbb{M}_2^{\mathbb{R}}$-module is free and finite. Suppose*

$$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s-2,f,w}(\mathrm{M}, \mathbb{M}_2^{\mathbb{R}}) = 0$$

for $f \geq 3$ whenever $(s,w) \in \mathcal{D}_{\mathrm{M}}$. Then there exists an $\mathrm{X} \in \mathbf{Sp}_{2,\mathrm{fin}}^{\mathbb{R}}$ such that $\mathrm{H}_{\mathbb{R}}^{*,*}(\mathrm{X}) \cong \mathrm{M}$ as an $\mathcal{A}^{\mathbb{R}}$-module. Further, such a realization is unique if

$$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s-1,f,w}(\mathrm{M}, \mathbb{M}_2^{\mathbb{R}}) = 0$$

for all $f \geq 2$ and $(s,w) \in \mathcal{D}_{\mathrm{M}}$.

2.2. **Weak $\mathbb{R}$-motivic Toda realization – version (II).** For any $\mathcal{A}^{\mathbb{R}}$-module M which is $\mathbb{M}_2^{\mathbb{R}}$-free, the quotient $\mathrm{M}/(\rho)$ is an $\mathcal{A}^{\mathbb{C}}$-module. In particular,

$$\mathcal{A}^{\mathbb{R}}/(\rho) \cong \mathcal{A}^{\mathbb{C}}$$

as a graded Hopf-algebra. Therefore, we have a spectral sequence

$${}^{\rho}\mathrm{E}_2^{s,f,w,i} := \bigoplus_{i \geq 0} \mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}^{s+i,f,w+i}(\mathrm{M}/(\rho), \mathbb{M}_2^{\mathbb{C}}) \Longrightarrow \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s,f,w}(\mathrm{M}, \mathbb{M}_2^{\mathbb{R}}) \tag{2.5}$$

which is often called the *(algebraic) ρ-Bockstein spectral sequence*. Thus we get the following version of the $\mathbb{R}$-motivic Toda realization and uniqueness theorem which is weaker than Theorem 2.4.

Theorem 2.6. *Let* M *denote an $\mathcal{A}^{\mathbb{R}}$-module whose underlying $\mathbb{M}_2^{\mathbb{R}}$-module is free and finite. Suppose*

$$\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}^{s-2+i,f,w+i}(\mathrm{M}/(\rho), \mathbb{M}_2^{\mathbb{C}}) = 0$$

for $f \geq 3$ and all $i \geq 0$ whenever $(s,w) \in \mathcal{D}_{\mathrm{M}}$, then there exists an $\mathrm{X} \in \mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ such that $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{X}) \cong \mathrm{M}$ as an $\mathcal{A}^{\mathbb{R}}$-module. Further, such a realization is unique if

$$\mathrm{Ext}^{s-1+i,f,w+i}_{\mathcal{A}^{\mathbb{C}}}(\mathrm{M}/(\rho), \mathbb{M}^{\mathbb{C}}_2) = 0$$

for all $f \geq 2$, $i \geq 0$ and $(s,w) \in \mathcal{D}_{\mathrm{M}}$.

2.3. **Weak $\mathbb{R}$-motivic Toda realization – version (III).** Similarly to the classical case, the $\mathbb{C}$-motivic Steenrod algebra enjoys an increasing filtration called the May filtration (see [DI]), which is easier to express on its dual. On $\mathcal{A}^{\mathbb{C}}_*$, the May filtration is induced by assigning the May weights

$$\mathsf{m}(\tau_{i-1}) = \mathsf{m}(\xi_i^{2^j}) = 2i-1$$

and extending it multiplicatively. The associated graded is an exterior algebra

$$\mathrm{gr}(\mathcal{A}^{\mathbb{C}}) \cong \Lambda_{\mathbb{M}^{\mathbb{C}}_2}(\xi_{i,j} : i \geq 1, j \geq 0), \tag{2.7}$$

where $\xi_{i,0}$ represents $(\tau_{i-1})_*$ and $(\xi_{i,j+1})_*$ represents $(\xi_i^{2^j})_*$ in the associated graded. When $\mathrm{M} = \mathbb{M}^{\mathbb{R}}_2$ in (2.10), then

$${}^{\mathrm{May}}\mathrm{E}^{*,*,*,*}_{1,\mathbb{M}^{\mathbb{C}}_2} \cong \mathbb{M}^{\mathbb{C}}_2[\mathsf{h}_{i,j} : i \geq 1, j \geq 0], \tag{2.8}$$

where $\mathsf{h}_{i,j}$ represents the class $\xi_{i,j}$. The (s,f,w,m)-degrees of these generators are given by

$$|\mathsf{h}_{i,j}| = \begin{cases} (2^i-2, 1, 2^{i-1}-1, 2i-1) & \text{if } j = 0, \text{ and,} \\ (2^j(2^i-1)-1, 1, 2^{j-1}(2^i-1), 2i-1) & \text{otherwise.} \end{cases}$$

Remark 2.9. After reindexing the May filtration of (2.8) by setting the May weight of $\mathsf{h}_{i,j}$ equal to i, it is consistent with the indexing used in [DI].

When M is a cyclic $\mathcal{A}^{\mathbb{R}}$-module, $\mathrm{M}/(\rho)$ is also cyclic as an $\mathcal{A}^{\mathbb{C}}$-module, thus the May filtration induces a filtration on $\mathrm{M}/(\rho)$. Thus, we get a corresponding May spectral sequence

$${}^{\mathrm{May}}\mathrm{E}^{s,f,w,\mathsf{m}}_{1,\mathrm{M}/(\rho)} := \mathrm{Ext}^{s,f,w,m}_{\mathrm{gr}(\mathcal{A}^{\mathbb{C}})}(\mathrm{gr}(\mathrm{M}/(\rho)), \mathbb{M}^{\mathbb{C}}_2) \Rightarrow \mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{C}}}(\mathrm{M}/(\rho), \mathbb{M}^{\mathbb{C}}_2) \tag{2.10}$$

computing the input of the ρ-Bockstein spectral sequence (2.5). Thus we can formulate a version of $\mathbb{R}$-motivic Toda realization theorem which is even weaker than Theorem 2.6.

Theorem 2.11. *Let* M *denote an cyclic $\mathcal{A}^{\mathbb{R}}$-module whose underlying $\mathbb{M}^{\mathbb{R}}_2$-module is free and finite. Suppose*

$${}^{\mathrm{May}}\mathrm{E}^{s-2+i,f,w+i,*}_{1,\mathrm{M}/(\rho)} = 0.$$

for $f \geq 3$ and all $i \geq 0$ whenever $(s,w) \in \mathcal{D}_{\mathrm{M}}$. Then there exists an $\mathrm{X} \in \mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ such that $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{X}) \cong \mathrm{M}$ as an $\mathcal{A}^{\mathbb{R}}$-module. Further, such a realization is unique if

$${}^{\mathrm{May}}\mathrm{E}^{s-1+i,f,w+i,*}_{1,\mathrm{M}/(\rho)} = 0$$

for $f \geq 2$, $i \geq 0$ and $(s,w) \in \mathcal{D}_{\mathrm{M}}$.

3. A comparison between C_2-equivariant and classical squaring operations

For any C_2-equivariant space $X \in \mathbf{Top}_*^{C_2}$ we can functorially assign two non-equivariant spaces – the underlying space $\Re(X)$, which is obtained by restricting the action of C_2 to the trivial group, and the space of C_2-fixed-points X^{C_2}. For a C_2-equivariant spectrum $E \in \mathbf{Sp}^{C_2}$, restricting the action to the trivial subgroup results in a monoidal functor

$$\Re : \mathbf{Sp}^{C_2} \longrightarrow \mathbf{Sp}$$

that identifies the underlying spectrum. However, there are two different notions of fixed-point spectrum – the categorial fixed-points and the geometric fixed-points.

The categorical fixed-point functor is a lax monodial functor

$$(-)^{C_2} : \mathbf{Sp}^{C_2} \longrightarrow \mathbf{Sp},$$

which is defined so that $\pi_k(E^{C_2}) \cong \pi_k^{C_2}(E)$, but it does not interact well with infinite suspensions. The correction term is explained by the tom Dieck splitting [LMSM, Theorem V.11.1]:

$$(\Sigma^\infty_{C_2} X)^{C_2} \simeq \Sigma^\infty(X^{C_2}) \vee \Sigma^\infty(X_{hC_2}), \tag{3.1}$$

where X_{hC_2} is the homotopy orbit space. Let $\widetilde{EC_2} := \mathrm{Cof}(EC_{2+} \to \mathbb{S})$. The geometric fixed-point functor

$$\Phi : \mathbf{Sp}^{C_2} \longrightarrow \mathbf{Sp},$$

is a symmetric monoidal functor given by $\Phi(E) := (E \wedge \widetilde{EC_2})^{C_2}$. When $X \in \mathrm{Top}_*^{C_2}$,

$$\Phi(\Sigma^\infty_{C_2} X) \simeq \Sigma^\infty X^{C_2} \tag{3.2}$$

is the first component in (3.1). For any $E \in \mathbf{Sp}^{C_2}$, there is a natural map of spectra

$$\iota_E : E^{C_2} \longrightarrow \Phi(E)$$

induced by the map $\mathbb{S} \to \widetilde{EC_2}$.

The Eilenberg-Mac Lane spectrum $H\underline{\mathbb{F}}_2$ is an $\mathbb{E}_\infty^{C_2}$-ring ([LMSM, VII]), i.e. a commutative monoid as a genuine C_2-spectrum. The restriction $\Re(H\underline{\mathbb{F}}_2) \simeq H\mathbb{F}_2$, the categorical fixed-points ${H\underline{\mathbb{F}}_2}^{C_2} \simeq H\mathbb{F}_2$ and the geometric fixed-points $\Phi(H\underline{\mathbb{F}}_2) \simeq H\mathbb{F}_2[t]$ are $\mathbb{E}_\infty$-rings. It follows from the knowledge of $\mathbb{M}_2^{C_2} := \pi_\star^{C_2} H\underline{\mathbb{F}}_2$ that for $n \geq 0$

$$(\Sigma^{n\sigma} H\underline{\mathbb{F}}_2)^{C_2} \simeq \bigvee_{i=0}^n \Sigma^i H\mathbb{F}_2 \hookrightarrow \Phi(H\underline{\mathbb{F}}_2) \simeq \mathrm{colim}_{n\to\infty} \ (\Sigma^{n\sigma} H\underline{\mathbb{F}}_2)^{C_2} \simeq H\mathbb{F}_2[t]$$

is the inclusion of the first $(n+1)$ components. The above map clearly splits. One can endow $(\Sigma^{n\sigma} H\underline{\mathbb{F}}_2)^{C_2}$ with an $\mathbb{E}_\infty$-structure isomorphic to the truncated polynomial algebra $H\mathbb{F}_2[t]/(t^{n+1})$ so that the splitting map

$$\pi^{(n)}_{\mathbb{F}_2} : \Phi(H\underline{\mathbb{F}}_2) \simeq H\mathbb{F}_2[t] \twoheadrightarrow (\Sigma^{n\sigma} H\underline{\mathbb{F}}_2)^{C_2} \simeq H\mathbb{F}_2[t]/(t^{n+1})$$

is an $\mathbb{E}_\infty$-map. Alternatively, $\pi^{(n)}_{\mathbb{F}_2}$ can be obtained as an $\mathbb{E}_\infty$-map by applying the functor H to the map of commutative rings $\mathbb{F}_2[t] \twoheadrightarrow \mathbb{F}_2[t]/(t^{n+1})$. The composition

$$\mathrm{H}\underline{\mathbb{F}}_2{}^{\mathrm{C}_2} \xhookrightarrow{\iota_{\mathbb{F}_2}} \Phi(\mathrm{H}\underline{\mathbb{F}}_2) \xrightarrow{\pi^{(0)}_{\mathbb{F}_2}} \mathrm{H}\underline{\mathbb{F}}_2{}^{\mathrm{C}_2} \tag{3.3}$$

is the identity and exhibits $\Phi(\mathrm{H}\underline{\mathbb{F}}_2)$ as an augmented $\mathrm{H}\mathbb{F}_2$-algebra.

For any C_2-space $\mathrm{X} \in \mathbf{Top}^{\mathrm{C}_2}_*$, the restriction functor induces a natural transformation

$$\Re_* : \mathrm{H}^{i,j}_{\mathrm{C}_2}(\mathrm{X}_+) \longrightarrow \mathrm{H}^i(\Re(\mathrm{X})_+).$$

To compare the cohomology of $\mathrm{X}^{\mathrm{C}_2}$ with the $\mathrm{RO}(\mathrm{C}_2)$-graded cohomology of X, we make use of the splitting (3.3) to define the natural ring map

$$\hat{\Phi}_* : \mathrm{H}^{i,j}_{\mathrm{C}_2}(\mathrm{X}_+) \longrightarrow \mathrm{H}^{i-j}(\mathrm{X}^{\mathrm{C}_2}_+),$$

which sends $u \in \mathrm{H}^{i,j}_{\mathrm{C}_2}(\mathrm{X}_+)$ to the composite (as defined in [BW, (2.7)])

$$\Sigma^\infty \mathrm{X}^{\mathrm{C}_2} \xrightarrow{\Phi(u)} \Sigma^{i-j}\Phi(\mathrm{H}\underline{\mathbb{F}}_2) \xrightarrow{\pi^{(0)}_{\mathbb{F}_2}} \Sigma^{i-j}\mathrm{H}\mathbb{F}_2.$$

The purpose of this section is to compare the $\mathrm{RO}(\mathrm{C}_2)$-graded squaring operations with the classical squaring operations along the maps $\Re_*$ and $\hat{\Phi}_*$. We begin with a brief recollection of the construction of the classical and C_2-equivariant squaring operations.

3.1. **A construction of Steenrod squaring operations.** The construction of the classical mod 2 Steenrod algebra, which is the algebra of stable cohomology operations for ordinary cohomology with $\mathbb{F}_2$-coefficients, involves the $\mathbb{E}_\infty$-structure[1] of $\mathrm{H}\mathbb{F}_2$ and the fact that the tautological line bundle γ over $\mathbb{RP}^\infty$ is $\mathrm{H}\mathbb{F}_2$-orientable. We review here how the mod 2 Steenrod operations are derived from that structure. A similar discussion can be found in [BMMS, Section VIII.2].

Notation 3.4. For any space or spectrum X and $n \geq 1$, we let

$$\mathrm{D}_n(\mathrm{X}) := (\mathrm{E}\Sigma_n)_+ \wedge_{\Sigma_n} (\mathrm{X}^{\wedge n}),$$

where Σ_n acts by permuting the factors of $\mathrm{X}^{\wedge n}$. By convention, $\mathrm{D}_0(\mathrm{X}) = \mathbb{S}$.

An $\mathbb{E}_\infty$-ring structure on a spectrum R gives a collection of maps of the form

$$\Theta^{\mathrm{R}}_n : \mathrm{D}_n(\mathrm{R}) \longrightarrow \mathrm{R}$$

for each $n \geq 0$, which satisfy the usual coherence conditions (see [Ma2]). By assumption, Θ^{R}_0 is the unit map of R and Θ^{R}_1 is the identity map.

The $\mathrm{H}\mathbb{F}_2$-orientibility of γ implies the existence of an $\mathrm{H}\mathbb{F}_2$-Thom class

$$\mathsf{u}_n : \mathrm{Th}(\gamma^{\oplus n}) \simeq \mathbb{RP}^\infty_n \longrightarrow \Sigma^n \mathrm{H}\mathbb{F}_2 \tag{3.5}$$

[1] Technically, we only make use of the $\mathbb{H}_\infty$-ring structure that underlies the $\mathbb{E}_\infty$-structure of $\mathrm{H}\mathbb{F}_2$.

for each $n \geq 0$. These are compatible as n varies, in the sense that the following diagram commutes:

(3.6)
$$\begin{array}{ccccc} \mathrm{Th}(\gamma^{\oplus(m+n)}) & \longrightarrow & \mathrm{Th}(\gamma^{\oplus m}) \wedge \mathrm{Th}(\gamma^{\oplus n}) & \xrightarrow{\mathsf{u}_m \wedge \mathsf{u}_n} & \Sigma^m \mathrm{H}\mathbb{F}_2 \wedge \Sigma^n \mathrm{H}\mathbb{F}_2 \\ & \searrow^{\mathsf{u}_{m+n}} & & & \downarrow{\scriptstyle \mu_{\mathbb{F}_2}} \\ & & & & \mathrm{H}\mathbb{F}_2. \end{array}$$

For any spectra E and F, there is a natural map

$$\delta_n : \mathrm{D}_n(\mathrm{E} \wedge \mathrm{F}) \longrightarrow \mathrm{D}_n(\mathrm{E}) \wedge \mathrm{D}_n(\mathrm{F})$$

induced by the diagonal on $\mathrm{E}\Sigma_n$ and the isomorphism $(\mathrm{E} \wedge \mathrm{F})^{\wedge n} \cong \mathrm{E}^{\wedge n} \wedge \mathrm{F}^{\wedge n}$. Thus, we may define the map τ_n as the composition

(3.7)
$$\begin{array}{ccc} \mathrm{D}_2(\Sigma^n \mathrm{H}\mathbb{F}_2) & \xrightarrow{\qquad\qquad \tau_n \qquad\qquad} & \Sigma^{2n} \mathrm{H}\mathbb{F}_2 \\ \downarrow{\scriptstyle \delta_2} & & \uparrow{\scriptstyle \mu_{\mathbb{F}_2}} \\ \mathrm{D}_2(\mathrm{S}^n) \wedge \mathrm{D}_2(\mathrm{H}\mathbb{F}_2) & \xrightarrow{\simeq} \Sigma^n \mathbb{RP}^\infty_n \wedge \mathrm{D}_2(\mathrm{H}\mathbb{F}_2) & \xrightarrow{\Sigma^n \mathsf{u}_n \wedge \Theta_2^{\mathbb{F}_2}} \Sigma^{2n} \mathrm{H}\mathbb{F}_2 \wedge \mathrm{H}\mathbb{F}_2. \end{array}$$

Definition 3.8. The power operation is the natural transformation

$$\mathcal{P}_2 \colon \mathrm{H}^n(-) \longrightarrow \mathrm{H}^{2n}(\mathrm{D}_2(-))$$

which takes a class $u \in \mathrm{H}^n(\mathrm{E})$ to the composite class

$$\mathcal{P}_2(u) \colon \mathrm{D}_2(\mathrm{E}) \xrightarrow{\mathrm{D}_2(u)} \mathrm{D}_2(\Sigma^n \mathrm{H}\mathbb{F}_2) \xrightarrow{\tau_n} \Sigma^n \mathrm{H}\mathbb{F}_2$$

for any $\mathrm{E} \in \mathbf{Sp}$.

From (3.6), we deduce the commutativity of the diagram

(3.9)
$$\begin{array}{ccccc} \mathrm{D}_2(\Sigma^n \mathrm{H}\mathbb{F}_2 \wedge \Sigma^m \mathrm{H}\mathbb{F}_2) & \longrightarrow & \mathrm{D}_2(\Sigma^n \mathrm{H}\mathbb{F}_2) \wedge \mathrm{D}_2(\Sigma^m \mathrm{H}\mathbb{F}_2) & \xrightarrow{\tau_n \wedge \tau_m} & \Sigma^{2n} \mathrm{H}\mathbb{F}_2 \wedge \Sigma^{2m} \mathrm{H}\mathbb{F}_2 \\ \downarrow{\scriptstyle \mathrm{D}_2(\mu_{\mathbb{F}_2})} & & & & \downarrow{\scriptstyle \mu_{\mathbb{F}_2}} \\ \mathrm{D}_2(\Sigma^{n+m} \mathrm{H}\mathbb{F}_2) & & \xrightarrow[\tau_{n+m}]{\qquad\qquad} & & \Sigma^{2n+2m} \mathrm{H}\mathbb{F}_2. \end{array}$$

As a result, we have

$$\delta_2^*(\mathcal{P}_2(u) \otimes \mathcal{P}_2(v)) = \mathcal{P}_2(u \otimes v)$$

which leads to the Cartan formula for the Steenrod algebra.

If $\mathrm{X} \in \mathbf{Top}_*$ is given the trivial Σ_2-action and $\mathrm{X} \wedge \mathrm{X}$ the permutation action, the diagonal map $\mathrm{X} \to \mathrm{X} \wedge \mathrm{X}$ is Σ_2-equivariant. Consequently, we have an induced map

$$\Delta_{\mathrm{X}} : (\mathrm{B}\Sigma_2)_+ \wedge \mathrm{X} \simeq (\mathrm{E}\Sigma_2)_+ \wedge_{\Sigma_2} \mathrm{X} \longrightarrow \mathrm{D}_2(\mathrm{X}).$$

Since $\mathrm{H}^*(\mathrm{B}\Sigma_2) \cong \mathbb{F}_2[\mathsf{t}]$, we may write (using the Kunneth isomorphism)

$$\Delta_{\mathrm{X}}^*(\mathcal{P}_2(u)) = \sum_{i=0}^{n} \mathsf{t}^{n-i} \otimes \mathrm{Sq}^i(u), \tag{3.10}$$

which defines the natural transformations $\mathrm{Sq}^i \colon \mathrm{H}^n(-) \longrightarrow \mathrm{H}^{n+i}(-)$.

Remark 3.11. The squaring operation $\mathrm{Sq}^i(u)$ for any class $u \in \mathrm{H}^n(\mathrm{X})$ is determined by $\mathrm{Sq}^i(\iota_n)$, where $\iota_n \in \mathrm{H}^*(\mathrm{K}(\mathbb{F}_2, n))$ is the fundamental class, because of the universal property of $\mathrm{K}(\mathbb{F}_2, n)$. A priori, $\mathrm{Sq}^i(u)$ depends on the cohomological degree of u. However, this dependence is eradicated by the fact that the squaring operations are stable, i.e. for any $u \in \mathrm{H}^*(\mathrm{X})$

$$\mathrm{Sq}^i(\sigma_*(u)) = \sigma_*(\mathrm{Sq}^i(u)),$$

where $\sigma_* : \mathrm{H}^*(\mathrm{X}) \cong \mathrm{H}^{*+1}(\Sigma\mathrm{X})$ is the suspension isomorphism. The $\mathrm{H}\mathbb{F}_2$-orientibility of γ implies $\mathrm{Sq}^0(\iota) = \iota$ for the generator $\iota \in \mathrm{H}^1(\mathrm{S}^1)$, which, along with Cartan formula, implies stability.

3.2. **The C_2-equivariant squaring operations.** The construction of the classical squaring operations can be adapted to construct squaring operations on the $\mathrm{RO}(\mathrm{C}_2)$-graded cohomology of a C_2-space.

Remark 3.12. Our ideas are closely related to the construction of the $\mathbb{R}$-motivic squaring operations due to Voevodsky [V]. Certain parts, such as the construction of the power operation Definition 3.21, though different, can be compared to [W1, W2], where the author studies C_2-equivariant power operations on the homology of spaces.

Notation 3.13. For any group G and a family of subgroups $\mathcal{F}$ closed under subconjugacy, there exists a space $\mathrm{E}\mathcal{F}$ determined up to a G-weak equivalence by its universal property

$$\mathrm{E}\mathcal{F}^{\mathrm{H}} \simeq \begin{cases} * & \text{if } \mathrm{H} \in \mathcal{F}, \\ \emptyset & \text{otherwise.} \end{cases}$$

When $\mathrm{G} = \mathrm{C}_2 \times \Sigma_n$ and $\mathcal{F}_n = \{\mathrm{H} \subset \mathrm{G} : \mathrm{H} \cap \Sigma_n = \mathbb{1}\}$, we denote $\mathrm{E}\mathcal{F}_n$ by $\mathrm{E}_{\mathrm{C}_2}\Sigma_n$. Note that there is a natural C_2-equivariant map $\mathrm{E}\Sigma_n \longrightarrow \mathrm{E}_{\mathrm{C}_2}\Sigma_n$.

Notation 3.14. For a based C_2-space or a C_2-spectrum X, we let

$$\mathrm{D}_n^{\mathrm{C}_2}(\mathrm{X}) := (\mathrm{E}_{\mathrm{C}_2}\Sigma_n)_+ \wedge_{\Sigma_n} (\mathrm{X}^{\wedge n})$$

the n-th equivariant extended power construction on X. There is a natural C_2-equivariant map

$$\delta_n^{\mathrm{C}_2} : \mathrm{D}_n^{\mathrm{C}_2}(\mathrm{X} \wedge \mathrm{Y}) \longrightarrow \mathrm{D}_n^{\mathrm{C}_2}(\mathrm{X}) \wedge \mathrm{D}_n^{\mathrm{C}_2}(\mathrm{Y})$$

induced by the diagonal map of $\mathrm{E}_{\mathrm{C}_2}\Sigma_n$ for any pair X and Y of C_2 space or spectra.

For a C_2-equivariant space $\mathrm{X} \in \mathbf{Top}_*^{\mathrm{C}_2}$, the inclusions $\mathrm{X}^{\mathrm{C}_2} \hookrightarrow \mathrm{X}$ and $\mathrm{E}\Sigma_n \longrightarrow \mathrm{E}_{\mathrm{C}_2}\Sigma_n$ together induce a natural map

$$\lambda_{\mathrm{X}} : \mathrm{D}_2(\mathrm{X}^{\mathrm{C}_2}) \longrightarrow \mathrm{D}_2^{\mathrm{C}_2}(\mathrm{X})^{\mathrm{C}_2}, \tag{3.15}$$

which is usually not an equivalence.

Example 3.16. When $\mathrm{X} \simeq \mathrm{S}^0$, $\lambda_{\mathrm{S}^0} : (\mathrm{B}\Sigma_2)_+ \longrightarrow (\mathrm{B}_{\mathrm{C}_2}\Sigma_2)_+^{\mathrm{C}_2} \simeq (\mathrm{B}\Sigma_2)_+ \wedge \mathrm{S}^0_+$ is the inclusion of a summand.

Likewise, when $\mathrm{E} \in \mathbf{Sp}^{\mathrm{C}_2}$, the map $\mathrm{E}^{\mathrm{C}_2} \hookrightarrow \mathrm{E}$ induces a natural map

$$\lambda_{\mathrm{E}} : \mathrm{D}_2(\mathrm{E}^{\mathrm{C}_2}) \longrightarrow \mathrm{D}_2^{\mathrm{C}_2}(\mathrm{E})^{\mathrm{C}_2}.$$

Using the fact that $\widetilde{\mathrm{EC}_2}$ is an $\mathbb{E}_\infty$-ring C_2-spectrum we define a map $\lambda^\Phi_{\mathrm{E}}$ as the composition

(3.17)

$$\begin{array}{ccccc} \mathrm{D}_2(\Phi(\mathrm{E})) & & \xrightarrow{\quad\lambda^\Phi_{\mathrm{E}}\quad} & & \Phi(\mathrm{D}_2^{\mathrm{C}_2}(\mathrm{E})) \\ \downarrow{\scriptstyle \lambda_{\widetilde{\mathrm{EC}_2}\wedge\mathrm{E}}} & & & & \uparrow \\ (\mathrm{D}_2(\widetilde{\mathrm{EC}_2}\wedge\mathrm{E}))^{\mathrm{C}_2} & \longrightarrow & (\mathrm{D}_2(\widetilde{\mathrm{EC}_2})\wedge\mathrm{D}_2(\mathrm{E}))^{\mathrm{C}_2} & \longrightarrow & (\widetilde{\mathrm{EC}_2}\wedge\mathrm{D}_2(\mathrm{E}))^{\mathrm{C}_2}\cong\Phi(\mathrm{D}_2(\mathrm{E})) \end{array}$$

By definition, an $\mathbb{E}_\infty^{\mathrm{C}_2}$-ring structure on a spectrum R consists of a system of maps

$$\Theta^{\mathrm{R}}_n : \mathrm{D}^{\mathrm{C}_2}_n(\mathrm{R}) \longrightarrow \mathrm{R}$$

for each $n \geq 0$, which satisfy certain compatibility criteria [LMSM, §VII.2]. The categorical fixed-point spectrum $\mathrm{R}^{\mathrm{C}_2}$ as well as the geometric fixed-point spectrum $\Phi(\mathrm{R})$ of an $\mathbb{E}_\infty^{\mathrm{C}_2}$-ring spectrum R are $\mathbb{E}_\infty$-ring spectra with structure maps

$$\Theta^{\mathrm{R}^{\mathrm{C}_2}}_n : \mathrm{D}_2(\mathrm{R}^{\mathrm{C}_2}) \xrightarrow{\lambda_{\mathrm{R}}} \mathrm{D}_2^{\mathrm{C}_2}(\mathrm{R})^{\mathrm{C}_2} \xrightarrow{(\Theta^{\mathrm{R}}_n)^{\mathrm{C}_2}} \mathrm{R}^{\mathrm{C}_2}$$

and

$$\Theta^{\Phi(\mathrm{R})}_n : \mathrm{D}_2(\Phi(\mathrm{R})) \xrightarrow{\lambda^\Phi_{\mathrm{R}}} \Phi(\mathrm{D}_2^{\mathrm{C}_2}(\mathrm{R})) \xrightarrow{\Phi(\Theta^{\mathrm{R}}_n)} \Phi(\mathrm{R}),$$

respectively. Further, the natural map

$$\iota_{\mathrm{R}} : \mathrm{R}^{\mathrm{C}_2} \longrightarrow \Phi(\mathrm{R})$$

is an $\mathbb{E}_\infty$-ring map.

Let ω denote the sign representation of Σ_2. The equivariant Eilenberg-Mac Lane spectrum $\mathrm{H}\underline{\mathbb{F}}_2$ does not distinguish between the C_2-equivariant bundles

$$\overline{\epsilon} : \mathrm{E}_{\mathrm{C}_2}\Sigma_2 \times_{\Sigma_2} (\rho) \longrightarrow \mathrm{B}_{\mathrm{C}_2}\Sigma_2$$

$$\overline{\gamma} : \mathrm{E}_{\mathrm{C}_2}\Sigma_2 \times_{\Sigma_2} (\rho\otimes\omega) \longrightarrow \mathrm{B}_{\mathrm{C}_2}\Sigma_2,$$

i.e. there exists a C_2-equivariant Thom isomorphism (see Remark 3.20)

$$\mathrm{Th}(\overline{\gamma})\wedge\mathrm{H}\underline{\mathbb{F}}_2 \simeq \mathrm{Th}(\overline{\epsilon})\wedge\mathrm{H}\underline{\mathbb{F}}_2 \simeq \Sigma^\rho(\mathrm{B}_{\mathrm{C}_2}\Sigma_2)_+\wedge\mathrm{H}\underline{\mathbb{F}}_2. \tag{3.18}$$

The above Thom isomorphism results in an $\mathrm{H}\underline{\mathbb{F}}_2$-Thom class

$$\underline{\mathrm{u}}_n : \mathrm{Th}(\overline{\gamma}^{\oplus n}) \longrightarrow \Sigma^{n\rho}\mathrm{H}\underline{\mathbb{F}}_2$$

for each $n \geq 0$, and these Thom classes can be used to define the C_2-equivariant power operations. Since

$$\mathrm{D}_2^{\mathrm{C}_2}(\mathrm{S}^{n\rho}) \simeq \mathrm{Th}(n\rho\oplus n(\rho\otimes\omega)) \simeq \Sigma^{n\rho}\mathrm{Th}(\overline{\gamma}^{\oplus n}),$$

we define the map $\underline{\tau}_n$ as the composition

(3.19)

$$\begin{array}{ccccc} \mathrm{D}_2^{C_2}(\Sigma^{n\rho}\mathrm{H}\underline{\mathbb{F}}_2) & & \xrightarrow{\quad\underline{\tau}_n\quad} & & \Sigma^{2n\rho}\mathrm{H}\underline{\mathbb{F}}_2 \\ \downarrow & & & & \uparrow{\scriptstyle \mu_{\mathbb{F}_2}} \\ \mathrm{D}_2^{\mathrm{C}_2}(\mathrm{S}^{n\rho})\wedge\mathrm{D}_2^{\mathrm{C}_2}(\mathrm{H}\underline{\mathbb{F}}_2) & \xrightarrow{\simeq} & \Sigma^{n\rho}\mathrm{Th}(\overline{\gamma}^{\oplus n})\wedge\mathrm{D}_2^{\mathrm{C}_2}(\mathrm{H}\underline{\mathbb{F}}_2) & \xrightarrow{\Sigma^n\underline{\mathrm{u}}_n\wedge\Theta^{\mathbb{F}_2}_2} & \Sigma^{2n\rho}\mathrm{H}\underline{\mathbb{F}}_2\wedge\mathrm{H}\underline{\mathbb{F}}_2. \end{array}$$

Remark 3.20. The Thom isomorphism (3.18) does not follow immediately from the general theory of equivariant Thom isomorphisms [CW] because the basespace $B_{C_2}\Sigma_2$ is *not* C_2-connected. In such cases, there is no guarantee of an $H\underline{\mathbb{F}}_2$-Thom class; instead an $H\underline{\mathbb{F}}_2$-orientation is encoded by a family of classes (see [Ma1]). However, there is a map of C_2-equivariant $\mathbb{R}$-vector bundles from $\overline{\gamma}$ to the tautological Atiyah Real line bundle over $BS^{\sigma} = BU_{\mathbb{R}}(1)$, which is C_2-connected. Further, the tautological Atiyah Real line bundle admits a single $H\underline{\mathbb{F}}_2$-Thom class as $H\underline{\mathbb{F}}_2^{\times}$ is C_2-equivariantly contractible. Therefore, $\overline{\gamma}$ also admits a single $H\underline{\mathbb{F}}_2$-Thom class which leads to (3.18) using a standard argument involving the Thom diagonal map.

Definition 3.21. The equivariant power operation is the natural transformation

$$\mathcal{P}_2^{C_2} : H^{n\rho}_{C_2}(-) \longrightarrow H^{2n\rho}_{C_2}(D_2^{C_2}(-)),$$

which takes a class $u \in H^{n\rho}_{C_2}(E)$ to the composite class

$$\mathcal{P}_2^{C_2}(u) : D_2^{C_2}(E) \xrightarrow{D_2^{C_2}(u)} D_2^{C_2}(\Sigma^{n\rho} H\underline{\mathbb{F}}_2) \xrightarrow{\tau_n} \Sigma^{2n\rho} H\underline{\mathbb{F}}_2$$

for any $E \in \mathbf{Sp}^{C_2}$.

When $X \in \mathbf{Top}_*^{C_2}$ is given the trivial Σ_2-action and $X \wedge X$ is given the permutation action, the diagonal map $X \to X \wedge X$ is a $C_2 \times \Sigma_2$-equivariant map. Consequently, we have a C_2-equivariant map

$$\Delta_X^{C_2} : (B_{C_2}\Sigma_2)_+ \wedge X \simeq (E_{C_2}\Sigma_2)_+ \wedge_{\Sigma_2} X \longrightarrow D_2^{C_2}(X).$$

By [HK, Lemma 6.27] (also see [W1, Proposition 3.2]),

$$H^{\star}_{C_2}((B_{C_2}\Sigma_2)_+) \cong \mathbb{M}_2^{C_2}[y, x]/(y^2 = a_{\sigma} y + u_{\sigma} x),$$

where $|y| = (1,1)$ and $|x| = (2,1)$. Since $H^{\star}_{C_2}((B_{C_2}\Sigma_2)_+)$ is $\mathbb{M}_2^{C_2}$-free, we also have a Kunneth isomorphism

$$H^{\star}_{C_2}((B_{C_2}\Sigma_2)_+ \wedge X) \cong H^{\star}_{C_2}((B_{C_2}\Sigma_2)_+) \otimes_{\mathbb{M}_2^{C_2}} H^{\star}_{C_2}(X).$$

Thus, for any $u \in H^{n\rho}_{C_2}(X)$, we may write $(\Delta_X^{C_2})^*(\mathcal{P}_2^{C_2}(u))$ using the formula

$$(\Delta_X^{C_2})^*(\mathcal{P}_2^{C_2}(u)) = \sum_{i=0}^{n} x^{n-i} \otimes \underline{Sq}^{2i}(u) + \sum_{i=0}^{n} y x^{n-i-1} \otimes \underline{Sq}^{2i+1}(u), \tag{3.22}$$

which defines the equivariant squaring operations $\underline{Sq}^i$ for all $i \geq 0$. These can be extended to operations on the entire $RO(C_2)$-graded cohomology ring as in [V, Prop 2.6]).

Remark 3.23. Just like the classical case, one can easily deduce that the $RO(C_2)$-graded squaring operations defined this way are natural, stable and obey the Cartan formula. In fact, Voevodsky [V] uses a similar approach to establish these properties for the $\mathbb{R}$-motivic Steenrod algebra, which can be emulated in the C_2-equivariant case using the Betti realization functor.

3.3. **Comparison theorems.** Since the restriction functor is monoidal, it induces a ring map

$$\Re_* : \mathrm{H}^\star_{\mathrm{C}_2}(\mathrm{X}_+) \longrightarrow \mathrm{H}^*(\Re(\mathrm{X})_+)$$

for any $\mathrm{X} \in \mathbf{Top}_*^{\mathrm{C}_2}$.

Example 3.24. When $\mathrm{X} = *$, the map

$$\Re_* : \pi_\star^{\mathrm{C}_2}\mathrm{H}\underline{\mathbb{F}}_2 \longrightarrow \pi_*\mathrm{H}\mathbb{F}_2 \cong \mathbb{F}_2$$

sends $u_\sigma \mapsto 1$, $a_\sigma \mapsto 0$, and $\Theta \mapsto 0$. This follows from the fact that the cofiber sequence $\mathrm{C}_{2+} \longrightarrow S^0 \xrightarrow{a_\sigma} S^\sigma$ shows that the kernel of $\Re_*$ consists of precisely the a_σ-divisible elements.

Proposition 3.25. *For any* $\mathrm{X} \in \mathbf{Top}_*^{\mathrm{C}_2}$ *and a class* $u \in \mathrm{H}^{n\rho}_{\mathrm{C}_2}(\mathrm{X})$

$$\Re_*(\mathcal{P}_2^{\mathrm{C}_2}(u)) = \mathcal{P}_2(\Re_*(u)).$$

Proof. Since, $\Re(\overline{\gamma}) = 2\gamma$, it follows that $\Re_*(\underline{\mathsf{u}}_n) = \mathsf{u}_{2n}$. This, along with the fact that $\Re(\Theta_2^{\mathbb{F}_2}) = \Theta_2^{\mathbb{F}_2}$ shows $\Re(\underline{\mathsf{t}}_n) = \mathsf{t}_{2n}$, and the result follows. □

Proof of Theorem 1.17. Let $\mathrm{X} \in \mathbf{Top}_*^{\mathrm{C}_2}$ and $u \in \mathrm{H}^{n\rho}_{\mathrm{C}_2}(\mathrm{X})$. Since $\Re(\mathrm{B}_{\mathrm{C}_2}\Sigma_2) \simeq \mathrm{B}\Sigma_2$, $\Re(\Delta^{\mathrm{C}_2}) = \Delta$, $\Re_*(\mathsf{y}) = \mathsf{t}$ and $\Re_*(\mathsf{x}) = \mathsf{t}^2$, it follows that

$$\Delta^*_{\Re(\mathrm{X})}(\mathcal{P}_2(\Re_*(u))) = \sum_{i=-n}^{n} \mathsf{t}^{n-i} \otimes \mathrm{Sq}^i(\Re_*(u))$$

must equal

$$\begin{aligned}\Re_*((\Delta_{\mathrm{X}}^{\mathrm{C}_2})^*(\mathcal{P}_2^{\mathrm{C}_2}(u))) &= \Re_*\Big(\sum_{i=-n}^{n} \mathsf{x}^{n-i} \otimes \underline{\mathrm{Sq}}^{2i}(u) + \sum_{i=-n}^{n} \mathsf{y}\mathsf{x}^{n-i-1} \otimes \underline{\mathrm{Sq}}^{2i+1}(u)\Big) \\ &= \sum_{i=-n}^{n} \mathsf{t}^{2n-2i} \otimes \Re_*(\underline{\mathrm{Sq}}^{2i}(u)) \\ &\qquad + \sum_{i=-n}^{n} \mathsf{t}^{2n-2i-1} \otimes \Re_*(\underline{\mathrm{Sq}}^{2i+1}(u)).\end{aligned}$$

Thus, the result is true for cohomology classes $u \in \mathrm{H}^{n\rho}_{\mathrm{C}_2}(\mathrm{X})$ for any space $\mathrm{X} \in \mathbf{Top}_*^{\mathrm{C}_2}$.

Since the squaring operations are stable, the result extends to arbitrary $\mathrm{RO}(\mathrm{C}_2)$-graded cohomology classes. Moreover, since $\Re$ commutes with suspensions, in the sense that $\Re \circ \Sigma^\infty_{\mathrm{C}_2} \simeq \Sigma^\infty \circ \Re$, and any $\mathrm{E} \in \mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}}$ is equivalent to $\Sigma^{-n}\Sigma^\infty_{\mathrm{C}_2}\mathrm{X}$ for some n and $\mathrm{X} \in \mathbf{Top}_*^{\mathrm{C}_2}$, we conclude the same for any $u \in \mathrm{H}^\star_{\mathrm{C}_2}(\mathrm{E})$. □

Now we draw our attention towards comparing the action of the C_2-equivariant Steenrod algebra $\mathcal{A}^{\mathrm{C}_2}$ on $\mathrm{H}^\star(\mathrm{X}_+)$ to the action of the classical Steenrod algebra $\mathcal{A}$ on $\mathrm{H}^*(\mathrm{X}_+^{\mathrm{C}_2})$, where $\mathrm{X} \in \mathbf{Top}^{\mathrm{C}_2}$. Note that

$$\hat{\Phi}_* : \mathrm{H}^\star_{\mathrm{C}_2}(\mathrm{X}_+) \longrightarrow \mathrm{H}^*(\mathrm{X}_+^{\mathrm{C}_2})$$

is a ring map.

Example 3.26. When $X = *$, the map

$$\hat{\Phi}_*\colon \pi_\star^{C_2} H\underline{\mathbb{F}}_2 \cong \mathbb{F}_2[u_\sigma, a_\sigma] \oplus \Theta\{u_\sigma^{-i} a_\sigma^{-j}\} \longrightarrow \pi_* H\mathbb{F}_2 \cong \mathbb{F}_2$$

sends $a_\sigma \mapsto 1$, $u_\sigma \mapsto 0$, and $\Theta \mapsto 0$. This is essentially because smashing with

$$\widetilde{EC_2} \simeq \operatorname{colim}\{S^0 \xrightarrow{a_\sigma} S^\sigma \xrightarrow{a_\sigma} S^{2\sigma} \longrightarrow \dots\}$$

amounts to inverting a_σ and the projection $\pi_{\mathbb{F}_2}^{(0)}$ kills u_σ.

Remark 3.27. One can deduce from Example 3.16 that in cohomology, the map

$$\lambda_{S^0}^*\colon H^*(B_{C_2}\Sigma_2{}_+^{C_2}) \cong \mathbb{F}_2[\mathsf{t}][\iota]/(\iota^2 - \iota) \twoheadrightarrow H^*((B\Sigma_2)_+) \cong \mathbb{F}_2[\mathsf{t}],$$

is the quotient map sending $\iota \mapsto 0$.

Example 3.28. The map $\hat{\Phi}_*\colon H^\star_{C_2}((B_{C_2}\Sigma_2)_+) \to H^*(B_{C_2}\Sigma_2{}_+^{C_2})$ sends $\mathsf{x} \mapsto \mathsf{t}$ and $\mathsf{y} \to \iota$, $a_\sigma \mapsto 1$ and $u_\sigma \mapsto 0$.

Lemma 3.29. *The composition*

$$H^\star_{C_2}(\mathrm{Th}(\overline{\gamma}^{\oplus n})) \xrightarrow{\hat{\Phi}_*} H^*(\mathrm{Th}(\overline{\gamma}^{\oplus n})^{C_2}) \xrightarrow{\lambda_{S^{\rho}\otimes\omega}} H^*(\mathrm{Th}(\gamma^{\oplus n}))$$

sends $\underline{\mathsf{u}}_n \mapsto \mathsf{u}_n$.

Proof. Let $\zeta_{C_2} : (B_{C_2}\Sigma_2)_+ \to \mathrm{Th}(\overline{\gamma}^{\oplus n})$ denote the zero-section. Under the zero-section map the Thom class is mapped to the Euler class, and therefore $\zeta_{C_2}^*(\underline{\mathsf{u}}_n) = \mathsf{x}^n$. Likewise, the zero-section for the nonequivariant bundle $\zeta : (B\Sigma_2)_+ \to \mathrm{Th}(\gamma^{\oplus n})$ sends $\mathsf{u}_n \mapsto \mathsf{t}^n$. By naturality of $\hat{\Phi}_*$ and λ, we get a commutative diagram

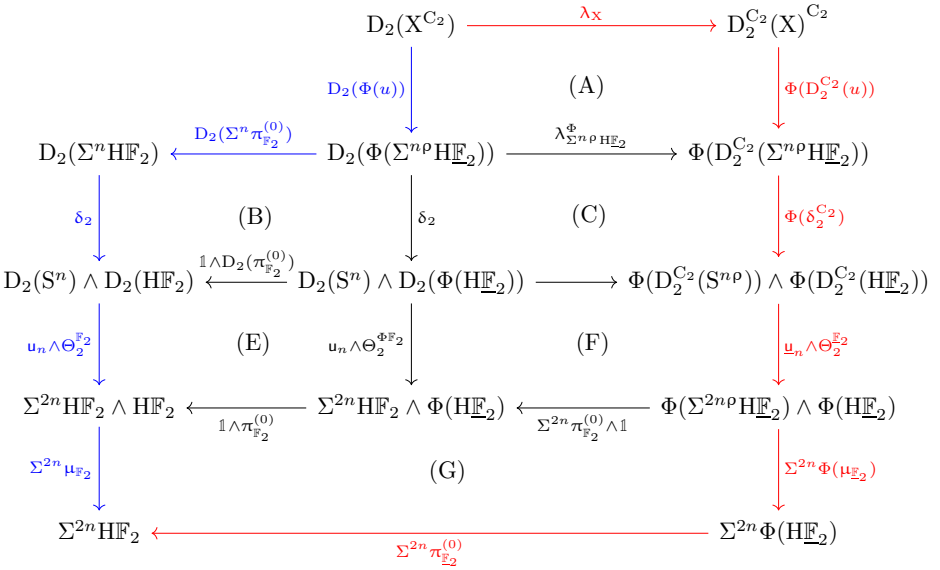
$$\begin{array}{ccccc}
H^\star_{C_2}(\mathrm{Th}(\overline{\gamma}^{\oplus n})) & \xrightarrow{\hat{\Phi}_*} & H^*(\mathrm{Th}(\overline{\gamma}^{\oplus n})^{C_2}) & \xrightarrow{\lambda^*_{S^{\rho}\otimes\omega}} & H^*(\mathrm{Th}(\gamma^{\oplus n})) \\
\downarrow{\scriptstyle \zeta^*_{C_2}} & & \downarrow{\scriptstyle (\zeta_{C_2}^{C_2})^*} & & \downarrow{\scriptstyle \zeta^*} \\
H^\star_{C_2}((B_{C_2}\Sigma_2)_+) & \xrightarrow[\hat{\Phi}_*]{} & H^*(B_{C_2}\Sigma_2{}_+^{C_2}) & \xrightarrow[\lambda^*_{S^0}]{} & H^*((B\Sigma_2)_+).
\end{array}$$

which along with Remark 3.27 and injectivity of ζ^* implies the result. □

Corollary 3.30. *For any space* $X \in \mathbf{Top}_*^{C_2}$ *and a class* $u \in H^{n\rho}_{C_2}(X)$,

$$\mathcal{P}_2(\hat{\Phi}_*(u)) = \lambda_X^*(\hat{\Phi}_*(\mathcal{P}_2^{C_2}(u))). \tag{3.31}$$

Proof. It is enough to show that the following diagram commutes, as the blue path and the red path indicates the left-hand side and the right-hand side of (3.31), respectively.



The squares (A), (B) and (C) commute naturally, the squares (E) and (G) commute because $\pi_{\mathbb{F}_2}^{(0)}$ is an $\mathbb{E}_\infty$-ring map, and (F) commutes because of Lemma 3.29. □

Proof of Theorem 1.19. For any space $X \in \mathbf{Top}_*^{C_2}$ and a class $u \in H_{C_2}^{n\rho}(X)$, we have a commutative diagram

$$\begin{array}{ccc} (B\Sigma_2)_+ \wedge X^{C_2} & \xrightarrow{\Delta_{X^{C_2}}} & D_2(X^{C_2})) \\ {\scriptstyle \lambda_{S^0} \wedge \mathbb{1}_{X^{C_2}}}\downarrow & & \downarrow{\scriptstyle \lambda_X} \\ (B_{C_2}\Sigma_2)_+^{C_2} \wedge X^{C_2} & \xrightarrow[(\Delta_X^{C_2})^{C_2}]{} & D_2^{C_2}(X)^{C_2}. \end{array}$$

Therefore,

$$\begin{aligned}
\sum_{i=0}^{n} \mathsf{t}^{n-i} \otimes \mathrm{Sq}^i(\hat{\Phi}_*(u)) &= \Delta^*_{X^{C_2}}(\mathcal{P}_2(\hat{\Phi}_*(u))) \\
&= \Delta^*_{X^{C_2}}(\lambda^*_X(\hat{\Phi}_*(\mathcal{P}_2^{C_2}(u)))) \\
&= (\lambda^*_{S^0} \otimes \mathbb{1}^*_{X^{C_2}})(((\Delta^{C_2}_X)^{C_2})^*(\hat{\Phi}_*(\mathcal{P}_2^{C_2}(u)))) \\
&= (\lambda^*_{S^0} \otimes \mathbb{1}^*_{X^{C_2}})\hat{\Phi}_*((\Delta^{C_2}_X)^*(\mathcal{P}_2^{C_2}(u))) \\
&= (\lambda^*_{S^0} \otimes \mathbb{1}^*_{X^{C_2}})\hat{\Phi}_*\left(\sum_{i=0}^{n} \mathsf{x}^{n-i} \otimes \mathrm{Sq}^{2i,i}(u) \right. \\
&\qquad \left. + \sum_{i=0}^{n} \mathsf{y}\mathsf{x}^{n-i-1} \otimes \underline{\mathrm{Sq}}^{2i+1}(u)\right) \\
&= (\lambda^*_{S^0} \otimes \mathbb{1}^*_{X^{C_2}})\left(\sum_{i=0}^{n} \mathsf{t}^{n-i} \otimes \hat{\Phi}_*(\underline{\mathrm{Sq}}^{2i}(u)) \right. \\
&\qquad \left. + \sum_{i=0}^{n} \iota\mathsf{t}^{n-i-1} \otimes \hat{\Phi}_*(\underline{\mathrm{Sq}}^{2i+1}(u))\right) \\
&= \sum_{i=0}^{n} \mathsf{t}^{n-i} \otimes \hat{\Phi}_*(\underline{\mathrm{Sq}}^{2i}(u)),
\end{aligned}$$

and hence, the result is true for all $u \in \mathrm{H}^{n\rho}_{C_2}(X)$ for any $X \in \mathbf{Top}^{C_2}_*$.

Since the squaring operations are stable, the result extends to arbitrary $\mathrm{RO}(C_2)$-graded cohomology classes. Moreover, since the geometric fixed-point functor Φ commutes with suspensions (3.2), and any $E \in \mathbf{Sp}^{C_2}_{2,\mathrm{fin}}$ is equivalent to $\Sigma^{-n}\Sigma^\infty_{C_2}X$ for some n and $X \in \mathbf{Top}^{C_2}_*$, we conclude the same for any $u \in \mathrm{H}^\star_{C_2}(E)$. □

4. Topological realization of $\mathcal{A}^{\mathbb{R}}(1)$

We begin by proving Theorem 1.6, which identifies all possible $\mathcal{A}^{\mathbb{R}}$-module structures on $\mathcal{A}^{\mathbb{R}}(1)$ up to isomorphism.

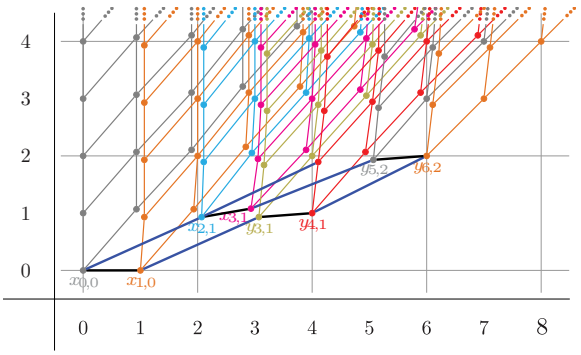


Figure 4.1. This figure displays the free $\mathbb{M}^{\mathbb{R}}_2$-module $\mathcal{A}^{\mathbb{R}}(1)$, with Sq^1 and Sq^2-multiplications drawn in only on the $\mathbb{M}^{\mathbb{R}}_2$-module generators.

Proof of Theorem 1.6. Note that the Cartan formula of $\mathcal{A}^{\mathbb{R}}$ and finiteness of $\mathcal{A}^{\mathbb{R}}(1)$ imply that the $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathcal{A}^{\mathbb{R}}(1)$ is determined once the action of Sq^4 and Sq^8 are specified on its $\mathbb{M}_2^{\mathbb{R}}$-generators. The following are possible Sq^4 and Sq^8-actions on the $\mathbb{M}_2^{\mathbb{R}}$-module generators. As can be seen in Figure 4.1, there is no room for other possible actions.

$$\begin{aligned}
\mathrm{Sq}^4(x_{0,0}) &= \beta_{03}(\rho \cdot y_{3,1}) + \beta_{04}(\tau \cdot y_{4,1}) + \alpha_{03}(\rho \cdot x_{3,1})\\
\mathrm{Sq}^4(x_{1,0}) &= \beta_{14}(\rho \cdot y_{4,1}) + \beta_{15}(y_{5,2})\\
\mathrm{Sq}^4(x_{2,1}) &= \textit{j}_{24}(\rho^2 \cdot y_{4,1}) + \beta_{25}(\rho \cdot y_{5,2}) + \beta_{26}(\tau \cdot y_{6,2})\\
\mathrm{Sq}^4(x_{3,1}) &= \beta_{36}(\rho \cdot y_{6,2})\\
\mathrm{Sq}^4(y_{3,1}) &= \gamma_{36}(\rho \cdot y_{6,2})\\
\mathrm{Sq}^8(x_{0,0}) &= \beta_{06}(\rho^2 \cdot y_{6,2})
\end{aligned}$$

The Adem relation $\mathrm{Sq}^2\mathrm{Sq}^3 = \mathrm{Sq}^5 + \mathrm{Sq}^4\mathrm{Sq}^1 + \rho\mathrm{Sq}^3\mathrm{Sq}^1$ (see Proposition A.1), when applied to $x_{0,0}$ and $x_{2,1}$, yields $\beta_{15} = 1$, $\beta_{03} + \beta_{04} + \beta_{14} = 1$ and $\beta_{25} + \beta_{26} = \beta_{36}$. The equation

$$\textit{j}_{24} = \beta_{03}\gamma_{36} + \alpha_{03}\beta_{36},$$

is forced by the Adem relation $\mathrm{Sq}^4\mathrm{Sq}^4 = \mathrm{Sq}^2\mathrm{Sq}^4\mathrm{Sq}^2 + \tau\mathrm{Sq}^3\mathrm{Sq}^4\mathrm{Sq}^1$ when applied to $x_{0,0}$. This exhausts all constraints imposed by Adem relations in these dimensions. □

In Theorem 1.6, there are exactly seven free variables taking values in $\mathbb{F}_2$, and therefore, there are exactly 128 different $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathcal{A}^{\mathbb{R}}(1)$. Thus, in order to complete the proof of Theorem 1.3, we realize these $\mathcal{A}^{\mathbb{R}}$-modules as spectra using Theorem 2.11, which is a weak form of the $\mathbb{R}$-motivic Toda realization theorem.

Proof of Theorem 1.3. Firstly, note that $\mathcal{A}^{\mathbb{R}}_{\overline{\mathrm{v}}}(1)$ is a cyclic $\mathcal{A}^{\mathbb{R}}$-module for all $\overline{\mathrm{v}} \in \mathcal{V}$, therefore $\mathcal{A}^{\mathbb{C}}_{\overline{\mathrm{v}}}(1) := \mathcal{A}^{\mathbb{R}}_{\overline{\mathrm{v}}}(1)/(\rho)$ admits a May filtration. Secondly, note that

$$\mathrm{gr}(\mathcal{A}^{\mathbb{C}}_{\overline{\mathrm{v}}}(1)) \cong \Lambda_{\mathbb{M}_2^{\mathbb{C}}}(\xi_{1,0}, \xi_{1,1}, \xi_{2,0})$$

as an $\mathrm{gr}(\mathcal{A}^{\mathbb{C}})$-module (see (2.7) for notation). Consequently,

$$^{\mathrm{May}}\mathrm{E}^{*,*,*,*}_{1,\mathcal{A}^{\mathbb{C}}_{\overline{\mathrm{v}}}(1)} \cong {}^{\mathrm{May}}\mathrm{E}^{*,*,*,*}_{1,\mathbb{M}_2^{\mathbb{C}}}/(\mathsf{h}_{1,0}, \mathsf{h}_{1,1}, \mathsf{h}_{2,0}) \cong \frac{\mathbb{M}_2^{\mathbb{C}}[\mathsf{h}_{i,j} : i \geq 1, j \geq 0]}{(\mathsf{h}_{1,0}, \mathsf{h}_{1,1}, \mathsf{h}_{2,0})}. \tag{4.2}$$

In the notation of Section 2.3

$$\mathcal{D}_{\mathcal{A}^{\mathbb{R}}(1)} = \{(0,0), (1,0), (2,1), (3,1), (4,1), (5,2), (6,2)\}.$$

By directly inspecting the (s, f, w)-degree of $^{\mathrm{May}}\mathrm{E}^{*,*,*,*}_{1,\mathcal{A}^{\mathbb{C}}_{\overline{\mathrm{v}}}(1)}$, we see that the condition necessary for existence in Theorem 2.11 is satisfied. Hence, the result. □

Remark 4.3. The vanishing region of $^{\mathrm{May}}\mathrm{E}^{*,*,*,*}_{1,\mathcal{A}^{\mathbb{C}}_{\overline{\mathrm{v}}}(1)}$ does not preclude the possibility of having a nonzero element in $\mathrm{Ext}^{-1,2,0}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{M}, \mathrm{M})$. We suspect (even after running the differentials in (2.3) and (2.5)), that the above group is nonzero for a given $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathcal{A}^{\mathbb{R}}(1)$, and that there are, up to homotopy, multiple realizations as $\mathbb{R}$-motivic spectra.

Our next goal is to prove Theorem 1.8. We begin with the following observation.

Lemma 4.4. *The $\mathcal{A}^{\mathbb{R}}$-modules $\mathcal{B}_2^{\mathbb{R}}(1)$ and $\mathcal{B}_{\mathsf{h}}^{\mathbb{R}}(1)$ are uniquely realizable as objects in $\mathbf{Sp}_{2,\mathrm{fin}}^{\mathbb{R}}$.*

Proof. Both $\mathcal{B}_2^{\mathbb{R}}(1)$ and $\mathcal{B}_{\mathsf{h}}^{\mathbb{R}}(1)$ are cyclic $\mathbb{M}_2^{\mathbb{R}}$-free finite $\mathcal{A}^{\mathbb{R}}$-module and

$$\mathcal{B}_2^{\mathbb{R}}(1)/(\rho) \cong \mathcal{B}_{\mathsf{h}}^{\mathbb{R}}(1)/(\rho)$$

as $\mathcal{A}^{\mathbb{C}}$-module. Let $\mathcal{B}^{\mathbb{C}}(1) := \mathcal{B}_2^{\mathbb{R}}(1)/(\rho)$. It is easy to see that $\mathrm{gr}(\mathcal{B}^{\mathbb{C}}(1))$ is isomorphic to $\Lambda_{\mathbb{M}_2^{\mathbb{C}}}(\xi_{1,0}, \xi_{1,1})$ as an $\mathrm{gr}(\mathcal{A}^{\mathbb{C}})$-module, and therefore

$$ {}^{\mathrm{May}}\mathrm{E}^{*,*,*,*}_{1,\mathcal{B}^{\mathbb{C}}(1)} \cong \mathbb{F}_2[\tau][h_{i,j} : i \geq 1, j \geq 0]/(\mathsf{h}_{1,0}, \mathsf{h}_{1,1}). \tag{4.5}$$

Using this, along with the fact that

$$\mathcal{D}_{\mathcal{B}^{\mathbb{C}}(1)} = \{(0,0), (1,0), (2,1), (3,1), \}$$

shows that the condition necessary for existence as well as uniqueness in Theorem 2.11 is satisfied. Hence, the result. □

Proof of Theorem 1.8. Consider the injective $\mathcal{A}^{\mathbb{R}}$-module map $\Sigma^{3,1}\mathcal{B}_{\epsilon}^{\mathbb{R}}(1) \to \mathcal{A}^{\mathbb{R}}_{\overline{\mathrm{v}}}$ sending the $\mathbb{M}_2^{\mathbb{R}}$ generator in degree $(3,1)$ to $x_{3,1} + y_{3,1}$. It follows from Theorem 1.6 that the quotient is isomorphic to $\mathcal{B}_{\delta}^{\mathbb{R}}(1)$. Thus, we have the exact sequence (1.9).

The topological realization of (1.9), i.e. the cofiber sequence (1.10), would follow immediately from Lemma 4.4 once we show that any one of the $\mathcal{A}^{\mathbb{R}}$-module maps in (1.9) can be realized as a map in $\mathbf{Sp}_{2,\mathrm{fin}}^{\mathbb{R}}$. Thus, it is enough to show that the nonzero class in the E_2-page represented by the projection map $\mathcal{A}^{\mathbb{R}}_{\overline{\mathrm{v}}}(1) \twoheadrightarrow \mathcal{B}_{\delta}^{\mathbb{R}}(1)$ in degree $(0,0,0)$ of the $\mathbb{R}$-motivic Adams spectral sequence

$$\mathrm{E}_2^{s,f,w} := \mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{R}}}(\mathcal{A}^{\mathbb{R}}_{\overline{\mathrm{v}}}(1), \mathcal{B}_{\delta}^{\mathbb{R}}(1)) \Rightarrow \left[\mathcal{Y}^{\mathbb{R}}_{(\delta,1)}, \mathcal{A}_1^{\mathbb{R}}[\overline{\mathrm{v}}]\right]_{s,w} \tag{4.6}$$

is a nonzero permanent cycle.

Using (4.2), the ρ-Bockstein spectral sequence (2.5) for $\mathcal{A}_1^{\mathbb{R}}[\overline{\mathrm{v}}]$ and the Atiyah-Hirzebruch spectral sequence

$$\mathcal{B}^{s',w'}_{\mathcal{B}_{\delta}^{\mathbb{R}}(1)} \otimes \mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{R}}}(\mathcal{A}_1^{\mathbb{R}}[\overline{\mathrm{v}}], \mathbb{M}_2^{\mathbb{R}}) \Rightarrow \mathrm{Ext}^{s+s',f,w+w'}_{\mathcal{A}^{\mathbb{R}}}(\mathcal{A}_1^{\mathbb{R}}[\overline{\mathrm{v}}], \mathcal{B}_{\delta}^{\mathbb{R}}(1)),$$

one can easily check $\mathrm{E}_2^{-1,f,0} = 0$ for all $f \geq 2$, and thus any nonzero element in degree $(0,0,0)$ of the E_2-page in (4.6) is, in fact, a nonzero permanent cycle. □

Our next goal is to analyze the underlying spectrum and geometric fixed-points spectrum of $\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}]$, the Betti realization of $\mathcal{A}_1^{\mathbb{R}}[\overline{\mathrm{v}}]$.

4.1. **The Betti realization of $\mathcal{A}_1^{\mathbb{R}}$.** Under the Betti-realization map

$$\beta_* : \pi_{*,*}\mathrm{H}_{\mathbb{R}}\mathbb{F}_2 \cong \mathbb{F}_2[\rho, \tau] \longrightarrow \pi_\star \mathrm{H}\underline{\mathbb{F}}_2 \tag{4.7}$$

$\rho \mapsto a_\sigma$ and $\tau \mapsto u_\sigma$. Since the functor β is symmetric monoidal and $\beta(\mathrm{H}_{\mathbb{R}}\mathbb{F}_2) = \mathrm{H}\underline{\mathbb{F}}_2$, the i-th $\mathbb{R}$-motivic squaring operations maps to the i-th $\mathrm{RO}(\mathrm{C}_2)$-graded squaring operations under the map

$$\beta_* : \mathcal{A}^{\mathbb{R}} \longrightarrow \mathcal{A}^{\mathrm{C}_2}.$$

Hence, $\mathrm{H}^\star_{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}])$ is $\mathbb{M}_2^{\mathrm{C}_2}$-free (as $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}_1^{\mathbb{R}}[\overline{\mathrm{v}}])$ is $\mathbb{M}_2^{\mathbb{R}}$-free) and its $\mathcal{A}^{\mathrm{C}_2}$-module structure is essentially given by Theorem 1.6 (after replacing Sq^i with $\underline{\mathrm{Sq}}^i$ and $\mathbb{M}_2^{\mathbb{R}}$-basis elements by its image under β_*).

Remark 4.8. The map β_* of (4.7) is only an injection with cokernel the summand $\Theta\{u_\sigma^{-i}a_\sigma^{-j} : i,j \geq 0\}$ of $\mathbb{M}_2^{C_2}$. In general, for an $\mathcal{A}^{\mathbb{R}}(1)$-module $M_{\mathbb{R}}$, the number of $\mathcal{A}^{C_2}$-module structures on $\beta(M_{\mathbb{R}})$ can be strictly larger than the number of $\mathcal{A}^{\mathbb{R}}$-module structures on $M_{\mathbb{R}}$. But this is not the case when $M_{\mathbb{R}} = \mathcal{A}^{\mathbb{R}}_{\overline{\nu}}(1)$ simply for degree reasons, therefore Corollary 1.15 holds.

As discussed in Example 3.24, the restriction map

$$\Re_* : \mathbb{M}_2^{C_2} \longrightarrow \mathbb{F}_2$$

sends $a_\sigma \mapsto 0$, $u_\sigma \mapsto 1$, and $\Theta \mapsto 0$. Thus, when $H^\star_{C_2}(E)$ is $\mathbb{M}_2^{C_2}$-free, $\Re_*$ is simply "setting $a_\sigma = 0$, $u_\sigma = 1$, and $\Theta = 0$". This observation, along with Theorem 1.17, allows us to completely deduce the $\mathcal{A}$-module structure of $H^*(\Re(\mathcal{A}^{\mathbb{R}}_{\overline{\nu}}(1)))$ from Theorem 1.6. Together with the fact that the $\mathcal{A}$-module structures on $\mathcal{A}(1)$ are uniquely-realized, our observations yield Theorem 4.9, where the notation $A_1[i,j]$ is adopted from [BEM].

Theorem 4.9. *For* $\overline{\nu} = (\alpha_{03}, \beta_{03}, \beta_{14}, \beta_{06}, \beta_{25}, \beta_{26}, \gamma_{36}) \in \mathcal{V}$ *(as in Theorem 1.6),*

$$\Re(\mathcal{A}_1^{C_2}[\overline{\nu}]) \simeq A_1[1 + \beta_{03} + \beta_{14}, \beta_{26}].$$

Now we shift our attention towards understand the geometric fixed-points of $\mathcal{A}_1^{C_2}[\overline{\nu}]$. As discussed in Example 3.26, the modified geometric fixed-points functor

$$\hat{\Phi}_* \colon \mathbb{M}_2^{C_2} \longrightarrow \mathbb{F}_2$$

sends $a_\sigma \mapsto 1$, $u_\sigma \mapsto 0$, and $\Theta \mapsto 0$. Thus, when $H^\star_{C_2}(E)$ is $\mathbb{M}_2^{C_2}$-free, $\hat{\Phi}_*$ is simply "setting $a_\sigma = 1$, $u_\sigma = 0$, and $\Theta = 0$". This, along with Theorem 1.6 and Theorem 1.19, gives the following.

Notation 4.10. Because $H^\star_{C_2}(\mathcal{A}_1^{C_2}[\overline{\nu}])$ is $\mathbb{M}_2^{C_2}$-free, the $H\mathbb{F}_2$-cohomology of $\Phi(\mathcal{A}_1^{C_2}[\overline{\nu}])$ consists of eight $\mathbb{F}_2$-generators, all of which are in the image of $\hat{\Phi}_*$. We let

$$s_0 := \hat{\Phi}_*(x_{0,0}), \quad s_{1a} := \hat{\Phi}_*(x_{2,1}), \quad s_{1b} := \hat{\Phi}_*(x_{1,0}), \quad s_2 := \hat{\Phi}_*(y_{3,1})$$
$$t_2 := \hat{\Phi}_*(x_{3,1}), \quad t_{3a} := \hat{\Phi}_*(y_{5,2}), \quad t_{3b} := \hat{\Phi}_*(y_{4,1}), \quad t_4 := \hat{\Phi}_*(y_{6,2}).$$

Note that $|s_{i(-)}| = |t_{i(-)}| = i$.

Theorem 4.11. *Let* $\overline{\nu} = (\alpha_{03}, \beta_{03}, \beta_{14}, \beta_{06}, \beta_{25}, \beta_{26}, \gamma_{36}) \in \mathcal{V}$*, and let*

$$\jmath_{24} = \beta_{03}\gamma_{36} + \alpha_{03}(\beta_{25} + \beta_{26})$$

as in Theorem 1.6. The $\mathcal{A}$-module structure on $H^*(\Phi(\mathcal{A}_1^{C_2}[\overline{\nu}]))$ is determined by the following relations, as depicted in Figure 4.12:

- $\mathrm{Sq}^1(s_0) = s_{1a}$
- $\mathrm{Sq}^1(s_{1b}) = s_2$
- $\mathrm{Sq}^1(t_2) = t_{3a}$
- $\mathrm{Sq}^1(t_{3b}) = t_4$
- $\mathrm{Sq}^2(s_0) = \beta_{03}s_2 + \alpha_{03}t_2$
- $\mathrm{Sq}^2(s_{1a}) = \beta_{25}t_{3a} + \jmath_{24}t_{3b}$
- $\mathrm{Sq}^2(s_{1b}) = t_{3a} + \beta_{14}t_{3b}$
- $\mathrm{Sq}^2(s_2) = \gamma_{36}t_4$
- $\mathrm{Sq}^2(t_2) = (\beta_{25} + \beta_{26})t_4$
- $\mathrm{Sq}^4(s_0) = \beta_{06}t_4$.

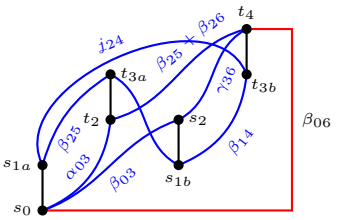


FIGURE 4.12. The $\mathcal{A}$-module $\mathrm{H}^*(\Phi(\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}]))$

5. An $\mathbb{R}$-motivic analogue of the spectrum $\mathcal{Z}$

The type 2 spectrum $\mathcal{Z} \in \mathbf{Sp}_{2,\mathrm{fin}}$, introduced in [BE], is defined by the property that its cohomology as an $\mathcal{A}(2)$-module is

$$\mathcal{B}(2) := \mathcal{A}(2) \otimes_{\Lambda(\mathrm{Q}_2)} \mathbb{F}_2,$$

where $\mathrm{Q}_2 = [\mathrm{Sq}^4, \mathrm{Q}_1]$ is dual to the Milnor generator ξ_3 of the dual Steenrod algebra. They first show that an $\mathcal{A}$-module structure on $\mathcal{A}(2)$ satisfying the criteria in [BE, Lemma 2.7] leads to an $\mathcal{A}$-module structure on $\mathcal{B}(2)$. In [BE], the authors show that among the 1600 possible $\mathcal{A}$-module structures on $\mathcal{A}(2)$ [R2], there are some $\mathcal{A}$-modules that satisfy [BE, Lemma 2.7]. Then they use the classical Toda realization theorem to show that any $\mathcal{A}$-module whose underlying $\mathcal{A}(2)$-module structure is $\mathcal{B}(2)$ can be realized as a 2-local finite spectrum, which they call $\mathcal{Z}$.

We construct $\mathcal{Z}_{\mathbb{R}} \in \mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ by emulating the construction of the classical $\mathcal{Z}$ (as in [BE]) in the $\mathbb{R}$-motivic context. Since there is no a priori $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathcal{A}^{\mathbb{R}}(2)$, we produce one in Section 5.1. In fact, we construct an $\mathbb{R}$-motivic spectrum whose cohomology is the desired $\mathcal{A}^{\mathbb{R}}$-module.

5.1. **A topological realization of $\mathcal{A}^{\mathbb{R}}(2)$.** Let $\mathcal{A}^{\mathbb{R}}(2)$ denote the sub-$\mathbb{M}_2^{\mathbb{R}}$-algebra of the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ generated by $\mathrm{Sq}^1, \mathrm{Sq}^2$, and Sq^4. We will use a method of Smith (exposed in [R1, Appendix C]) to construct an $\mathbb{R}$-motivic spectrum $\mathcal{A}_2^{\mathbb{R}} \in \mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ such that its cohomology as an $\mathcal{A}^{\mathbb{R}}(2)$-module is free on one generator.

Let $\mathsf{h}, \eta_{1,1}$ and $\nu_{3,2}$ denote the first three $\mathbb{R}$-motivic Hopf-elements (these are denoted ω, η, and ν in [DI, Section 8]).

Lemma 5.1. *The $\mathbb{R}$-motivic Toda-bracket $\langle \mathsf{h}, \eta_{1,1}, \nu_{3,2} \rangle$ contains 0.*

Proof. In this argument, it will be convenient to refer to the "coweight", by which we mean the difference $s - w$, as in [GI].

Since h and $\eta_{1,1}$ have coweight 0 while $\nu_{3,2}$ has coweight 1, it follows that the bracket $\langle \mathsf{h}, \eta_{1,1}, \nu_{3,2} \rangle$ is comprised of elements in stem 5 with coweight 2. The only element in stem 5 with coweight 2 is $\rho \cdot \nu_{3,2}^2$ [BI]. Since this element is a $\nu_{3,2}$ multiple, it lies in the indeterminacy, which means that the $\mathbb{R}$-motivic Toda-bracket does contain zero. □

Lemma 5.1 implies that we can construct a 4-cell complex $\mathcal{K}$ whose cohomology as an $\mathcal{A}^{\mathbb{R}}$-module has the structure described in Corollary 5.2 and displayed in Figure 5.3.

Corollary 5.2. *There exists* $\mathcal{K} \in \mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ *such that* $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{K})$ *is* $\mathbb{M}^{\mathbb{R}}_2$*-free on four generators* x_0, x_1, x_3 *and* x_7*, such that* $\mathrm{Sq}^{i+1}(x_i) = x_{2i+1}$ *for* $i \in \{0, 1, 3\}$.

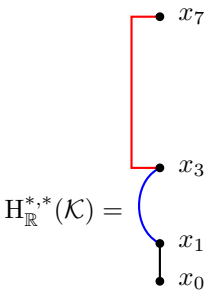


FIGURE 5.3. We depict the $\mathcal{A}^{\mathbb{R}}$-structure of $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{K})$ by marking the Sq^1-action by black straight lines, the Sq^2-action by blue curved lines, and the Sq^4-action by red lines between the $\mathbb{M}^{\mathbb{R}}_2$-generators.

Let $e \in \mathbb{Z}_{(2)}[\Sigma_6]$ denote the idempotent corresponding to the Young tableaux

1	2	3
4	5	
6		

which is constructed as follows. Let $\Sigma_{\mathrm{Row}} \subset \Sigma_6$ denote the subgroup comprised of permutations that preserve each row. Likewise, let Σ_{Col} denote the subgroup comprised of column-preserving permutations. Let

$$\mathsf{R} = \sum_{r \in \Sigma_{\mathrm{Row}}} r \qquad \text{and} \qquad \mathsf{C} = \sum_{c \in \Sigma_{\mathrm{Col}}} (-1)^{\mathrm{sign}(c)} c \tag{5.4}$$

and define

$$e = \frac{1}{\mu} \mathsf{R} \cdot \mathsf{C},$$

where μ is an odd integer defined in [R1, Theorem C.1.3]. We let $\overline{e}$ denote the resulting idempotent in $\mathbb{F}_2[\Sigma_6]$.

Proposition 5.5. *The idempotent* $\overline{e} \in \mathbb{F}_2[\Sigma_6]$ *has the property that* $\overline{e}(\mathrm{V}^{\otimes 6}) = 0$ *if* $\dim_{\mathbb{F}_2} \mathrm{V} < 3$ *and*

$$\dim_{\mathbb{F}_2} \overline{e}(\mathrm{V}^{\otimes 6}) = \begin{cases} 8 & \text{if } \dim_{\mathbb{F}_2} \mathrm{V} = 3, \\ 64 & \text{if } \dim_{\mathbb{F}_2} \mathrm{V} = 4. \end{cases}$$

Proof. Let $\overline{\mathsf{R}}$ and $\overline{\mathsf{C}}$ denote the images of R and C in $\mathbb{F}_2[\Sigma_6]$, respectively. Then $\overline{e} = \overline{\mathsf{R}} \cdot \overline{\mathsf{C}}$. It is straightforward that $\overline{\mathsf{C}}$ vanishes on $\mathrm{V}^{\otimes 6}$ if $\dim \mathrm{V} \leq 2$.

Now suppose that V has basis $\{a, b, c\}$. Then a basis for $\overline{e}(\mathrm{V}^{\otimes 6})$ is given by

$$\left\{ \overline{e}\left(\begin{array}{|c|c|c|}\hline a&b&c\\\hline b&c\\\cline{1-2} c\\\cline{1}\end{array}\right), \overline{e}\left(\begin{array}{|c|c|c|}\hline b&a&c\\\hline a&c\\\cline{1-2} c\\\cline{1}\end{array}\right), \overline{e}\left(\begin{array}{|c|c|c|}\hline a&c&b\\\hline c&b\\\cline{1-2} b\\\cline{1}\end{array}\right), \overline{e}\left(\begin{array}{|c|c|c|}\hline c&a&b\\\hline a&b\\\cline{1-2} b\\\cline{1}\end{array}\right), \overline{e}\left(\begin{array}{|c|c|c|}\hline c&b&a\\\hline b&a\\\cline{1-2} a\\\cline{1}\end{array}\right), \overline{e}\left(\begin{array}{|c|c|c|}\hline b&c&a\\\hline c&a\\\cline{1-2} a\\\cline{1}\end{array}\right), \right.$$
$$\left. \overline{e}\left(\begin{array}{|c|c|c|}\hline a&b&c\\\hline b&a\\\cline{1-2} c\\\cline{1}\end{array}\right), \overline{e}\left(\begin{array}{|c|c|c|}\hline a&c&b\\\hline c&a\\\cline{1-2} b\\\cline{1}\end{array}\right) \right\}.$$

Finally, suppose that $\dim \mathrm{V} = 4$ with basis $\{a, b, c, d\}$. For any subspace $\mathrm{W} \subset \mathrm{V}$ spanned by three of these basis elements, the space $\overline{e}(\mathrm{W}^{\otimes 6})$ has dimension 8, as we have just seen. There are 4 choices of W, which together yield a 32-dimensional subspace of $\mathrm{V}^{\otimes 6}$. Now consider Young tableaux in which all four basis elements appear and only one is repeated. In the case that d is repeated, we generate only two independent elements:

$$\overline{e}\left(\begin{array}{|c|c|c|}\hline a & c & d \\ \hline b & d \\ \cline{1-2} d \\ \cline{1-1}\end{array}\right) \quad \text{and} \quad \overline{e}\left(\begin{array}{|c|c|c|}\hline c & a & d \\ \hline b & d \\ \cline{1-2} d \\ \cline{1-1}\end{array}\right).$$

Allowing any basis element to be the repeating one, this gives an 8-dimensional subspace. Finally, we consider Young tableaux in which all four basis elements appear and two are repeated. In the case that c and d are repeated, we have the four elements

$$\overline{e}\left(\begin{array}{|c|c|c|}\hline a & c & d \\ \hline b & d \\ \cline{1-2} c \\ \cline{1-1}\end{array}\right), \overline{e}\left(\begin{array}{|c|c|c|}\hline a & c & d \\ \hline c & b \\ \cline{1-2} d \\ \cline{1-1}\end{array}\right), \overline{e}\left(\begin{array}{|c|c|c|}\hline b & d & c \\ \hline b & c \\ \cline{1-2} d \\ \cline{1-1}\end{array}\right), \quad \text{and} \quad \overline{e}\left(\begin{array}{|c|c|c|}\hline a & d & c \\ \hline d & b \\ \cline{1-2} c \\ \cline{1-1}\end{array}\right).$$

As there are $\binom{4}{2} = 6$ such choices, this contributes another subspace of dimension $4 \cdot 6 = 24$. □

We define

$$\mathcal{A}_2^{\mathbb{R}} := \Sigma^{-5,-1} e(\mathcal{K}^{\wedge 6}) = \Sigma^{-5,-1}(\mathrm{hocolim}\, \{\mathcal{K}^{\wedge 6} \xrightarrow{e} \mathcal{K}^{\wedge 6} \xrightarrow{e} \dots\}),$$

which is a split summand of $\Sigma^{-5,-1}\mathcal{K}^{\wedge 6}$ as e is an idempotent. We shift the grading by $(-5,-1)$ to make sure that the $\mathcal{A}^{\mathbb{R}}(2)$-module generator of $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}_2^{\mathbb{R}})$ is in $(0,0)$ (see Remark 5.12).

Theorem 5.6. $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}_2^{\mathbb{R}}) \cong \mathcal{A}^{\mathbb{R}}(2)$ *as an* $\mathcal{A}^{\mathbb{R}}(2)$*-module.*

Proof. By [BGL, Corollary 2.2], $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}_2^{\mathbb{R}})$ is a free $\mathcal{A}^{\mathbb{R}}(2)$-module if and only if $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}_2^{\mathbb{R}})$ is free as an $\mathbb{M}_2^{\mathbb{R}}$-module and the Margolis homology $\mathcal{M}(\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}_2^{\mathbb{R}}) \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbb{F}_2, x)$ vanishes for $x \in \{\mathrm{Q}_0^{\mathbb{R}}, \mathrm{Q}_1^{\mathbb{R}}, \overline{\mathrm{P}}_1^1, \mathrm{Q}_2^{\mathbb{R}}, \overline{\mathrm{P}}_2^1\}$, where $\overline{\mathrm{P}}_1^1$ and $\overline{\mathrm{P}}_2^1$ are the elements in $\mathcal{A}^{\mathbb{R}}$ dual to ξ_1 and ξ_2, respectively.

Let $\mathrm{K}_{\mathbb{R}} := \mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{K})$. The $\mathcal{A}^{\mathbb{R}}$-module $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}_2^{\mathbb{R}})$ is $\mathbb{M}_2^{\mathbb{R}}$-projective as it is a summand of

$$\mathrm{H}^{*,*}_{\mathbb{R}}(\Sigma^{-5}\mathcal{K}^{\wedge 6}) \cong \Sigma^{-5}\mathrm{K}_{\mathbb{R}}^{\otimes_{\mathbb{M}_2^{\mathbb{R}}} 6},$$

which is $\mathbb{M}_2^{\mathbb{R}}$-free. However, $\mathbb{M}_2^{\mathbb{R}}$ is a graded local ring, and over a local ring, being projective is equivalent to being free. Hence, $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}_2^{\mathbb{R}})$ is $\mathbb{M}_2^{\mathbb{R}}$-free. Since $\mathrm{Q}_0^{\mathbb{R}}, \mathrm{Q}_1^{\mathbb{R}}, \mathrm{Q}_2^{\mathbb{R}}, \overline{\mathrm{P}}_1^1$, and $\overline{\mathrm{P}}_2^1$ are primitive modulo (ρ, τ), and for $\mathrm{K} := \mathrm{K}_{\mathbb{R}} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbb{F}_2$, $i \in \{0,1,2\}$ and $t \in \{1,2\}$

$$\dim_{\mathbb{F}_2} \mathcal{M}(\mathrm{K}, \mathrm{Q}_i^{\mathbb{R}}) = 2 = \dim_{\mathbb{F}_2} \mathcal{M}(\mathrm{K}, \overline{\mathrm{P}}_t^1),$$

it follows from Proposition 5.5 that

$$\mathcal{M}(\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}_2^{\mathbb{R}}) \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbb{F}_2, x) \cong \mathcal{M}(\overline{e}(\mathrm{K}^{\otimes 6}), x) \cong \overline{e}(\mathcal{M}(\mathrm{K}, x)^{\otimes 6}) = 0$$

for $x \in \{\mathrm{Q}_0^{\mathbb{R}}, \mathrm{Q}_1^{\mathbb{R}}, \overline{\mathrm{P}}_1^1, \mathrm{Q}_2^{\mathbb{R}}, \overline{\mathrm{P}}_2^1\}$. Thus, $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}_2^{\mathbb{R}})$ is free over $\mathcal{A}^{\mathbb{R}}(2)$. Proposition 5.5 also implies that the $\mathbb{M}_2^{\mathbb{R}}$-rank of $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}_2^{\mathbb{R}})$ is 64, and therefore $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}_2^{\mathbb{R}})$ has rank 1 over $\mathcal{A}^{\mathbb{R}}(2)$. □

5.2. **An $\mathbb{R}$-motivic lift of $\mathcal{B}(2)$.** Let $\tilde{\mathrm{Q}}_2^{\mathbb{R}} = [\mathrm{Sq}^4, \mathrm{Q}_1^{\mathbb{R}}]$. Unlike the classical Steenrod algebra, $\mathrm{Q}_2^{\mathbb{R}}$ does not agree with $\tilde{\mathrm{Q}}_2^{\mathbb{R}}$. Instead, as in [V, Example 13.7], these are related by the formula

$$\mathrm{Q}_2^{\mathbb{R}} = [\mathrm{Sq}^4, \mathrm{Q}_1^{\mathbb{R}}] + \rho \mathrm{Sq}^5 \mathrm{Sq}^1.$$

However, one can check that both $\mathrm{Q}_2^{\mathbb{R}}$ and $\tilde{\mathrm{Q}}_2^{\mathbb{R}}$ square to zero, hence generate exterior algebras. We define (left) $\mathcal{A}^{\mathbb{R}}(2)$-modules

$$\mathcal{B}^{\mathbb{R}}(2) := \mathcal{A}^{\mathbb{R}}(2) \otimes_{\Lambda(\mathrm{Q}_2^{\mathbb{R}})} \mathbb{M}_2^{\mathbb{R}}$$

and

$$\tilde{\mathcal{B}}^{\mathbb{R}}(2) := \mathcal{A}^{\mathbb{R}}(2) \otimes_{\Lambda(\tilde{\mathrm{Q}}_2^{\mathbb{R}})} \mathbb{M}_2^{\mathbb{R}}. \tag{5.7}$$

Let $\mathrm{A}_2^{\mathbb{R}}$ denote $\mathrm{H}_{\mathbb{R}}^{*,*}(\mathcal{A}_2^{\mathbb{R}})$. It is easy to check that the left ideal generated by $\mathrm{Q}_2^{\mathbb{R}}$ (likewise $\tilde{\mathrm{Q}}_2^{\mathbb{R}}$) in $\mathcal{A}^{\mathbb{R}}(2)$ is isomorphic to $\Sigma^{7,3}\mathcal{B}^{\mathbb{R}}(2)$ (likewise $\Sigma^{7,3}\tilde{\mathcal{B}}^{\mathbb{R}}(2)$). It follows that there is an exact sequence of $\mathcal{A}^{\mathbb{R}}(2)$-modules

$$0 \longrightarrow \Sigma^{7,3}\mathrm{B}_{\mathbb{R}} \xhookrightarrow{\iota} \mathrm{A}_2^{\mathbb{R}} \xrightarrow{\pi} \!\!\!\rightarrow \mathrm{B}_{\mathbb{R}} \longrightarrow 0, \tag{5.8}$$

where $\mathrm{B}_{\mathbb{R}}$ is either $\mathcal{B}^{\mathbb{R}}(2)$ or $\tilde{\mathcal{B}}^{\mathbb{R}}(2)$. The main purpose of this section is to show that:

Lemma 5.9. *There exists an exact sequence of $\mathcal{A}^{\mathbb{R}}$-modules whose underlying $\mathcal{A}^{\mathbb{R}}(2)$-module exact sequence is isomorphic to (5.8) with $\mathrm{B}_{\mathbb{R}} \cong \tilde{\mathcal{B}}^{\mathbb{R}}(2)$.*

Remark 5.10. In the case of $\mathrm{B}_{\mathbb{R}} = \mathcal{B}^{\mathbb{R}}(2)$, the image of $\Sigma^{7,1}\mathcal{B}^{\mathbb{R}}(2) \longrightarrow \mathrm{A}_2^{\mathbb{R}}$ is a sub-$\mathcal{A}^{\mathbb{R}}(2)$-module, but not a sub-$\mathcal{A}^{\mathbb{R}}$-module. See Remark 5.15 for more details.

Lemma 5.9 and Remark 5.10 are direct consequences of the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{A}_2^{\mathbb{R}}$ which can be deduced from the injection

$$\Sigma^{5,1}\mathrm{A}_2^{\mathbb{R}} \longhookrightarrow \mathrm{K}_{\mathbb{R}}^{\otimes_{\mathbb{M}_2^{\mathbb{R}}} 6},$$

where $\mathrm{K}_{\mathbb{R}} = \mathrm{H}_{\mathbb{R}}^{*,*}(\mathcal{K}_{\mathbb{R}})$. We do not want to entirely leave this calculation to the reader because, without a few tricks, this calculation is likely to require computer assistance as e has 144 elements in its expression (in terms of the standard $\mathbb{F}_2$-generators of $\mathbb{F}_2[\Sigma_3]$) and $\mathrm{K}_{\mathbb{R}}^{\otimes_{\mathbb{M}_2^{\mathbb{R}}} 6}$ has 2^{12} elements in its $\mathbb{M}_2^{\mathbb{R}}$-basis. We begin after setting the following notation.

Notation 5.11. Let x_i denote the $\mathbb{M}_2^{\mathbb{R}}$-generators of $\mathrm{K}_{\mathbb{R}}$ in degree i as in Corollary 5.2. We use the numbered Young diagram (abbrev. NYD)

$$\begin{array}{|c|c|c|}\hline i_1 & i_2 & i_3 \\ \hline i_4 & i_5 \\ \cline{1-2} i_6 \\ \cline{1-1}\end{array}$$

to denote the $\mathbb{M}_2^{\mathbb{R}}$-basis element $x_{i_1} \otimes \cdots \otimes x_{i_6} \in \mathrm{K}_{\mathbb{R}}^{\otimes_{\mathbb{M}_2^{\mathbb{R}}} 6}$, where $i_j \in \{0, 1, 3, 7\}$.

As in Proposition 5.5, let $\overline{\mathsf{R}}$ and $\overline{\mathsf{C}}$ denote the images of R and C (see (5.4)) in $\mathbb{F}_2[\Sigma_6]$, respectively. Since $\overline{e} = \overline{\mathsf{R}} \cdot \overline{\mathsf{C}}$, we record a few properties of $\overline{\mathsf{R}}$ and $\overline{\mathsf{C}}$. Note that $\overline{\mathsf{R}}$ annihilates an NYD if it has repeating digits in a row. Likewise, $\overline{\mathsf{C}}$ annihilates an NYD if there are repeating digits in a column. For instance,

$$\overline{\mathsf{R}}\left(\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 3 & 7 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}\right) = 0 = \overline{\mathsf{C}}\left(\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 3 & 7 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}\right).$$

Remark 5.12. The lowest degree NYD which is not annihilated by $\overline{e}$ is

which lives in degree $(5,1)$. Of course, there are multiple NYD's in bidegree $(5,1)$ not annihilated by $\overline{e}$ but their images are the same. Likewise, the NYD of the highest degree not annihilated by $\overline{e}$ is

which lives in bidegree $(28,11)$.

The lowest degree element $\iota := \overline{e}($  $)$, which serves as the $\mathcal{A}^{\mathbb{R}}$-module generator of $\Sigma^{5,1}\mathrm{A}_2^{\mathbb{R}}$, can also be expressed as

$$\iota = \overline{\mathsf{R}}(\ \text{  }\ )$$

because the other NYDs present in the expression $\overline{\mathsf{C}}(\ $  $)$ are annihilated by $\overline{\mathsf{R}}$.

Since the $\mathbb{R}$-motivic Steenrod algebra is cocommutative we get

$$\overline{\mathsf{R}}(\overline{\mathsf{C}}(\mathrm{Sq}^i(-))) = \overline{\mathsf{R}}(\mathrm{Sq}^i(\overline{\mathsf{C}}(-))) = \mathrm{Sq}^i(\overline{\mathsf{R}}(\overline{\mathsf{C}}(-))).$$

This, along with the Cartan formula, allows us to calculate $a \cdot \iota$ for any $a \in \mathcal{A}^{\mathbb{R}}$, fairly easily. For example,

$$\mathrm{Sq}^1 \cdot \iota = \overline{\mathsf{R}}(\mathrm{Sq}^1\ \text{}\ )$$

$$= \overline{\mathsf{R}}(\ \text{  }\ +\ \text{}\ +\ \text{}\ )$$

$$= \overline{\mathsf{R}}(\ \text{}\ ),$$

$$\mathrm{Sq}^2 \cdot \iota = \overline{\mathsf{R}}(\mathrm{Sq}^2\ \text{}\ )$$

$$= \overline{\mathsf{R}}(\ \text{}\ +\ \text{}\ +\tau(\ \text{}\ +\ \text{}\ +\ \text{}\ ))$$

$$= \overline{\mathsf{R}}(\ \text{}\ ),$$

$$\mathrm{Sq}^4 \cdot \iota = \overline{\mathsf{R}}(\mathrm{Sq}^4(\ \text{}\ ))$$

$$= \overline{\mathsf{R}}(\ \text{}\ +\ \text{}\ +\tau(\ \text{}\ +\ \text{}\ +\ \text{}\ )$$

$$+\tau(\ \text{}\ +\ \text{}\ +\ \text{}\ ))$$

$$= \overline{\mathsf{R}}(\ \text{}\ +\tau\ \text{}\ ).$$

In this way, we calculate

$$\tilde{\mathrm{Q}}_2^{\mathbb{R}} \cdot \iota = \overline{\mathsf{R}}(\ \begin{smallmatrix}3&1&0\\1&0\\7\end{smallmatrix}\ +\ \begin{smallmatrix}3&1&0\\7&1\\0\end{smallmatrix}\ +\ \begin{smallmatrix}7&3&1\\1&0\\0\end{smallmatrix}\ )$$

$$\mathrm{Q}_2^{\mathbb{R}} \cdot \iota = \overline{\mathsf{R}}(\ \begin{smallmatrix}3&1&0\\1&0\\7\end{smallmatrix}\ +\ \begin{smallmatrix}3&1&0\\7&1\\0\end{smallmatrix}\ +\ \begin{smallmatrix}7&3&1\\1&0\\0\end{smallmatrix}\ +\rho\ \begin{smallmatrix}3&1&0\\3&1\\3\end{smallmatrix}\ ),$$

where the details are left to the reader.

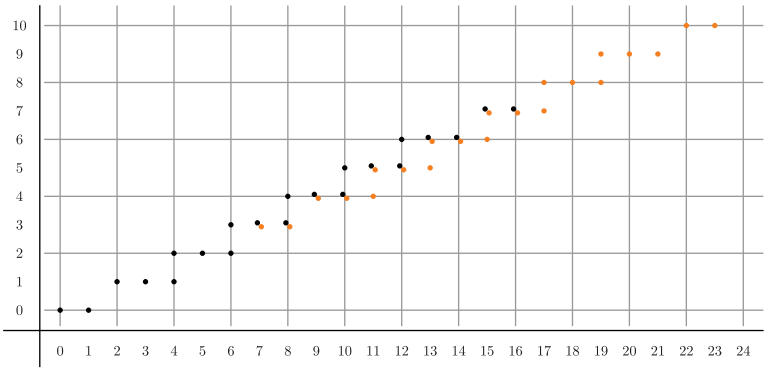


FIGURE 5.13. $\mathbb{M}_2^{\mathbb{R}}$-module generators of $\mathcal{A}^{\mathbb{R}}(2)$. Black dots correspond to generators of $\tilde{\mathcal{B}}^{\mathbb{R}}(2)$ and orange dots to $\Sigma^{7,3}\tilde{\mathcal{B}}^{\mathbb{R}}(2)$.

Remark 5.14. We record (see Figure 5.13), in the notation introduced in Section 2.1, that

$$\mathcal{D}_{\tilde{\mathcal{B}}^{\mathbb{R}}(2)} = \{(0,0),(1,0),(2,1),(3,1),(4,1),(4,2),(5,2),(6,2),(6,3),(7,3),(8,3),$$
$$(8,4),(9,4),(10,4),(10,5),(11,5),(12,5),(12,6),(13,6),(14,6),(15,7),(16,7)\}$$

and $\mathcal{D}_{\mathrm{A}_2^{\mathbb{R}}} = \{(i+7\epsilon, j+3\epsilon) : (i,j) \in \mathcal{D}_{\tilde{\mathcal{B}}^{\mathbb{R}}(2)} \text{ and } \epsilon \in \{0,1\}\}$.

Proof of Lemma 5.9. Recall that the image of $\Sigma^{7,3}\tilde{\mathcal{B}}^{\mathbb{R}}(2)$ in (5.8) is the (left) $\mathcal{A}^{\mathbb{R}}(2)$-submodule of $\mathrm{A}_2^{\mathbb{R}}$ generated by $\tilde{\mathrm{Q}}_2^{\mathbb{R}}$. We must check that this is closed under the action of $\mathcal{A}^{\mathbb{R}}$. Since $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4$ are in $\mathcal{A}^{\mathbb{R}}(2)$, it remains to check that for all $i \geq 3$ and $a \in \mathcal{A}^{\mathbb{R}}(2)$

$$\mathrm{Sq}^{2^i} \cdot (a\tilde{\mathrm{Q}}_2^{\mathbb{R}} \cdot \iota) = b\tilde{\mathrm{Q}}_2^{\mathbb{R}} \cdot \iota$$

for some $b \in \mathcal{A}^{\mathbb{R}}(2)$. For degree reasons (see Remark 5.14), we only need to consider the case when $i = 3$ and $a \in \{1, \mathrm{Sq}^1, \mathrm{Sq}^2\}$. We check

$$\mathrm{Sq}^8 \cdot (\tilde{\mathrm{Q}}_2^{\mathbb{R}} \cdot \iota) = (\mathrm{Sq}^4\mathrm{Sq}^4 + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^2)\tilde{\mathrm{Q}}_2^{\mathbb{R}} \cdot \iota$$
$$\mathrm{Sq}^8 \cdot (\mathrm{Sq}^1\tilde{\mathrm{Q}}_2^{\mathbb{R}} \cdot \iota) = (\mathrm{Sq}^7\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^7)\mathrm{Sq}^1\tilde{\mathrm{Q}}_2^{\mathbb{R}} \cdot \iota$$
$$\mathrm{Sq}^8 \cdot (\mathrm{Sq}^2\tilde{\mathrm{Q}}_2^{\mathbb{R}} \cdot \iota) = (\mathrm{Sq}^4\mathrm{Sq}^4\mathrm{Sq}^2 + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^4 + \tau\mathrm{Sq}^5\mathrm{Sq}^4\mathrm{Sq}^1)\mathrm{Sq}^2\tilde{\mathrm{Q}}_2^{\mathbb{R}} \cdot \iota$$

and thus the result holds. □

Remark 5.15. We notice that

$$\mathrm{Sq}^8 \cdot (\mathrm{Q}_2^{\mathbb{R}} \cdot \iota) = \overline{\mathsf{R}}(\ \begin{smallmatrix}7&3&0\\3&0\\7\end{smallmatrix}\ +\ \begin{smallmatrix}7&3&0\\7&3\\0\end{smallmatrix}\ +\tau\ \begin{smallmatrix}7&3&1\\7&1\\1\end{smallmatrix}\ +\rho(\ \begin{smallmatrix}7&1&0\\7&1\\3\end{smallmatrix}\ +\ \begin{smallmatrix}7&1&0\\3&1\\7\end{smallmatrix}\ +\ \begin{smallmatrix}3&1&0\\7&1\\7\end{smallmatrix}\ ))$$

cannot be equal to $bQ_2^{\mathbb{R}}\cdot\iota$ for any $b\in\mathcal{A}^{\mathbb{R}}(2)$. This is an easy but tedious calculation. For the convenience of the reader, we note that an $\mathbb{F}_2$-basis for the elements in degree $|\mathrm{Sq}^8| = (8,4)$ of $\mathcal{A}^{\mathbb{R}}(2)$ is given by

$$\{\mathrm{Sq}^6\mathrm{Sq}^2, \tau\mathrm{Sq}^7\mathrm{Sq}^1, \tau\mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1, \rho\mathrm{Sq}^7, \rho\mathrm{Sq}^6\mathrm{Sq}^1, \rho\mathrm{Sq}^5\mathrm{Sq}^2, \rho\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1, \rho^2\mathrm{Sq}^5\mathrm{Sq}^1\}.$$

5.3. **The construction of $\mathcal{Z}_{\mathbb{R}}$.** Recall the $\mathcal{A}^{\mathbb{R}}$-module $\tilde{\mathrm{B}}_2^{\mathbb{R}}$, as given in (5.7), and let

$$\mathrm{B}_2^{\mathbb{C}} := \tilde{\mathrm{B}}_2^{\mathbb{R}}/(\rho).$$

Proof of Theorem 1.24. Since $\mathrm{B}_2^{\mathbb{C}}$ is cyclic as an $\mathcal{A}^{\mathbb{C}}$-module, it admits a May filtration, whose associated graded is isomorphic to

$$\mathrm{gr}(\mathrm{B}_2^{\mathbb{C}}) \cong \Lambda(\xi_{1,0}, \xi_{1,1}, \xi_{1,2}, \xi_{2,0}, \xi_{2,1})$$

and whose E_2-page of the corresponding May spectral sequence is isomorphic to

$$ {}^{\mathrm{May}}\mathrm{E}_{1,\mathrm{B}_2^{\mathbb{C}}}^{*,*,*,*} \cong \frac{\mathbb{M}_2^{\mathbb{C}}[\mathsf{h}_{i,j} : i\geq 1, j\geq 0]}{(\mathsf{h}_{1,0}, \mathsf{h}_{1,1}, \mathsf{h}_{1,2}, \mathsf{h}_{2,0}, \mathsf{h}_{2,1})}. \tag{5.16}$$

From this and Remark 5.14, one easily checks that the condition for Theorem 2.11 is satisfied. Thus, there exists $\mathcal{Z}_{\mathbb{R}}\in\mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ such that $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{Z}_{\mathbb{R}})\cong\tilde{\mathrm{B}}_2^{\mathbb{R}}$. □

Remark 5.17. Since, as an $\mathcal{A}(2)$-module

$$\mathrm{H}^*(\Re(\beta(\mathcal{Z}_{\mathbb{R}}))) \cong \Re_*(\beta_*(\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{Z}_{\mathbb{R}}))) \cong \mathcal{B}(2),$$

the underlying spectrum of $\beta(\mathcal{Z}_{\mathbb{R}})$ is indeed one of the spectra $\mathcal{Z}$ considered in [BE], and therefore of type 2.

Appendix A. The $\mathbb{R}$-motivic Adem relations

Voevodsky established the motivic version of the Adem relations [V, Section 10]. However, his formulas contain some typos, so for the convenience of the reader, we here present the Adem relations, in the $\mathbb{R}$-motivic case.

Proposition A.1. *In the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$, the product $\mathrm{Sq}^a\mathrm{Sq}^b$ is equal to*

(1) *(a and b both even)*

$$\sum_{j=0}^{a/2} \tau^{j \bmod 2}\binom{b-1-j}{a-2j}\mathrm{Sq}^{a+b-j}\mathrm{Sq}^j$$

(2) *(a odd and b even)*

$$\sum_{j=0}^{(a-1)/2}\binom{b-1-j}{a-2j}\mathrm{Sq}^{a+b-j}\mathrm{Sq}^j + \rho\binom{b-j}{a-2j}\mathrm{Sq}^{a+b-j-1}\mathrm{Sq}^j.$$

(3) *(a even and b odd)*

$$\sum_{j=0}^{a/2}\binom{b-1-j}{a-2j}\mathrm{Sq}^{a+b-j}\mathrm{Sq}^j + \rho\binom{b-1-j}{a+1-2j}\mathrm{Sq}^{a+b-j-1}\mathrm{Sq}^j.$$

(4) *(a and b both odd)*

$$\sum_{j=0}^{(a-1)/2}\binom{b-1-j}{a-2j}\mathrm{Sq}^{a+b-j}\mathrm{Sq}^j$$

Remark A.2. Given that $\mathrm{Sq}^a = \mathrm{Sq}^1\mathrm{Sq}^{a-1}$ if a is odd and also that $\mathrm{Sq}^1(\tau) = \rho$, cases (2) and (4) follow from (1) and (3), respectively. Note also that (1) is the classical formula, but with τ thrown in whenever needed to balance the weights. In formula (2), the left term appears only when j is even, while the second appears only when j is odd. In formula (3), the second term appears only when j is odd.

Example A.3. Some examples of the $\mathbb{R}$-motivic Adem relation in low degrees are

$$\mathrm{Sq}^2\mathrm{Sq}^2 = \tau\mathrm{Sq}^3\mathrm{Sq}^1, \qquad \mathrm{Sq}^3\mathrm{Sq}^2 = \rho\mathrm{Sq}^3\mathrm{Sq}^1,$$

and

$$\mathrm{Sq}^2\mathrm{Sq}^3 = \mathrm{Sq}^5 + \mathrm{Sq}^4\mathrm{Sq}^1 + \rho\mathrm{Sq}^3\mathrm{Sq}^1.$$

Acknowledgments

The authors have benefited from conversations with Mike Hill, Nick Kuhn, Piotr Pstragowski, Paul VanKoughnett, Dylan Wilson, and Foling Zou. The first author would also like to acknowledge his debt to Mark Behrens for his relentless support. Finally, the authors thank the referee for a careful reading that improved the exposition.

References

[BBB+] Agnès Beaudry, Mark Behrens, Prasit Bhattacharya, Dominic Culver, and Zhouli Xu, *The telescope conjecture at height 2 and the tmf resolution*, J. Topol. **14** (2021), no. 4, 1243–1320, DOI 10.1112/topo.12208. MR4332490

[BW] Mark Behrens and Dylan Wilson, *A C_2-equivariant analog of Mahowald's Thom spectrum theorem*, Proc. Amer. Math. Soc. **146** (2018), no. 11, 5003–5012, DOI 10.1090/proc/14175. MR3856165

[BI] Eva Belmont and Daniel Isaksen, *$\mathbb{R}$-motivic stable stems*, Preprint, arXiv:2001.03606, 2020.

[BE] Prasit Bhattacharya and Philip Egger, *A class of 2-local finite spectra which admit a v_2^1-self-map*, Adv. Math. **360** (2020), 106895, 40, DOI 10.1016/j.aim.2019.106895. MR4031119

[BEM] Prasit Bhattacharya, Philip Egger, and Mark Mahowald, *On the periodic v_2-self-map of A_1*, Algebr. Geom. Topol. **17** (2017), no. 2, 657–692, DOI 10.2140/agt.2017.17.657. MR3623667

[BGL] Prasit Bhattacharya, Bertrand Guillou, and Ang Li, *An $\mathbb{R}$-motivic v_1-self-map of periodicity 1*, Homology Homotopy Appl. **24** (2022), no. 1, 299–324, DOI 10.4310/hha.2022.v24.n1.a15. MR4410466

[BMMS] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, *H_∞ ring spectra and their applications*, Lecture Notes in Mathematics, vol. 1176, Springer-Verlag, Berlin, 1986, DOI 10.1007/BFb0075405. MR836132

[C] Jeffrey L. Caruso, *Operations in equivariant $\mathbf{Z}/p$-cohomology*, Math. Proc. Cambridge Philos. Soc. **126** (1999), no. 3, 521–541, DOI 10.1017/S0305004198003375. MR1684248

[CW] S. R. Costenoble and S. Waner, *The equivariant Thom isomorphism theorem*, Pacific J. Math. **152** (1992), no. 1, 21–39. MR1139971

[DM] Donald M. Davis and Mark Mahowald, *v_1- and v_2-periodicity in stable homotopy theory*, Amer. J. Math. **103** (1981), no. 4, 615–659, DOI 10.2307/2374044. MR623131

[DI] Daniel Dugger and Daniel C. Isaksen, *Low-dimensional Milnor-Witt stems over $\mathbb{R}$*, Ann. K-Theory **2** (2017), no. 2, 175–210, DOI 10.2140/akt.2017.2.175. MR3590344

[GH] P. G. Goerss and M. J. Hopkins, *Moduli spaces of commutative ring spectra*, Structured ring spectra, London Math. Soc. Lecture Note Ser., vol. 315, Cambridge Univ. Press, Cambridge, 2004, pp. 151–200, DOI 10.1017/CBO9780511529955.009. MR2125040

[GI] Bertrand J. Guillou and Daniel C. Isaksen, *The Bredon-Landweber region in C_2-equivariant stable homotopy groups*, Doc. Math. **25** (2020), 1865–1880. MR4184454

[HS] Michael J. Hopkins and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. II*, Ann. of Math. (2) **148** (1998), no. 1, 1–49, DOI 10.2307/120991. MR1652975

[HK] Po Hu and Igor Kriz, *Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence*, Topology **40** (2001), no. 2, 317–399, DOI 10.1016/S0040-9383(99)00065-8. MR1808224

[LMSM] L. G. Lewis Jr., J. P. May, M. Steinberger, and J. E. McClure, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, vol. 1213, Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure, DOI 10.1007/BFb0075778. MR866482

[M1] Mark Mahowald, *bo-resolutions*, Pacific J. Math. **92** (1981), no. 2, 365–383. MR618072

[M2] Mark Mahowald, *The image of J in the EHP sequence*, Ann. of Math. (2) **116** (1982), no. 1, 65–112, DOI 10.2307/2007048. MR662118

[MRS] Mark Mahowald, Douglas Ravenel, and Paul Shick, *The triple loop space approach to the telescope conjecture*, Homotopy methods in algebraic topology (Boulder, CO, 1999), Contemp. Math., vol. 271, Amer. Math. Soc., Providence, RI, 2001, pp. 217–284, DOI 10.1090/conm/271/04358. MR1831355

[Ma1] J. P. May, *Equivariant orientations and Thom isomorphisms*, Tel Aviv Topology Conference: Rothenberg Festschrift (1998), Contemp. Math., vol. 231, Amer. Math. Soc., Providence, RI, 1999, pp. 227–243, DOI 10.1090/conm/231/03362. MR1707345

[Ma2] J. P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics, Vol. 271, Springer-Verlag, Berlin-New York, 1972. MR0420610

[MG] Aaron Mazel-Gee, *Goerss-Hopkins obstruction theory via model infinity-categories*, ProQuest LLC, Ann Arbor, MI, 2016. Thesis (Ph.D.)–University of California, Berkeley. MR3593196

[M] Haynes R. Miller, *On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space*, J. Pure Appl. Algebra **20** (1981), no. 3, 287–312, DOI 10.1016/0022-4049(81)90064-5. MR604321

[PV] Piotr Pstragowski and Paul VanKoughnett, *Abstract Goerss-Hopkins theory*, Adv. Math. **395** (2022), Paper No. 108098, DOI 10.1016/j.aim.2021.108098. MR4363589

[R1] Douglas C. Ravenel, *Nilpotence and periodicity in stable homotopy theory*, Annals of Mathematics Studies, vol. 128, Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith. MR1192553

[R2] Marilyn Jean Roth, *The cyclic module structures of the Hopf subalgebra A2 over the Steenrod algebra and their geometric realization*, ProQuest LLC, Ann Arbor, MI, 1977. Thesis (Ph.D.)–The Johns Hopkins University. MR2626901

[T] Hirosi Toda, *On spectra realizing exterior parts of the Steenrod algebra*, Topology **10** (1971), 53–65, DOI 10.1016/0040-9383(71)90017-6. MR271933

[V] Vladimir Voevodsky, *Reduced power operations in motivic cohomology*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), 1–57, DOI 10.1007/s10240-003-0009-z. MR2031198

[W1] Dylan Wilson, *Power operations for* $\mathrm{H}\underline{\mathbb{F}}_2$ *and a cellular construction of* $\mathrm{BP}\mathbf{R}$, Preprint, arXiv:1611.06958, 2017.

[W2] Dylan Wilson, *C_2-equivariant homology operations: Results and formulas*, Preprint, arXiv:1905.00058, 2019.

Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556
Email address: pbhattac@nd.edu

Department of Mathematics, The University of Kentucky, Lexington, Kentucky 40506–0027
Email address: bertguillou@uky.edu

Department of Mathematics, The University of Kentucky, Lexington, Kentucky 40506–0027
Email address: ang.li1414201@uky.edu