*Homology, Homotopy and Applications, vol. 24(1), 2022, pp.299–324*

# AN $\mathbb{R}$-MOTIVIC $v_1$–SELF-MAP OF PERIODICITY 1

PRASIT BHATTACHARYA, BERTRAND GUILLOU AND ANG LI

(*communicated by Daniel Isaksen*)

*Abstract*

We consider a nontrivial action of $\mathrm{C}_2$ on the type 1 spectrum $\mathcal{Y} := \mathrm{M}_2(1) \wedge \mathrm{C}(\eta)$, which is well-known for admitting a 1-periodic $v_1$–self-map. The resultant finite $\mathrm{C}_2$-equivariant spectrum $\mathcal{Y}^{\mathrm{C}_2}$ can also be viewed as the complex points of a finite $\mathbb{R}$-motivic spectrum $\mathcal{Y}^{\mathbb{R}}$. In this paper, we show that one of the 1-periodic $v_1$–self-maps of $\mathcal{Y}$ can be lifted to a self-map of $\mathcal{Y}^{\mathrm{C}_2}$ as well as $\mathcal{Y}^{\mathbb{R}}$. Further, the cofiber of the self-map of $\mathcal{Y}^{\mathbb{R}}$ is a realization of the subalgebra $\mathcal{A}^{\mathbb{R}}(1)$ of the $\mathbb{R}$-motivic Steenrod algebra. We also show that the $\mathrm{C}_2$-equivariant self-map is nilpotent on the geometric fixed-points of $\mathcal{Y}^{\mathrm{C}_2}$.

## 1. Introduction

In classical stable homotopy theory, the interest in periodic $v_n$–self-maps of finite spectra lies in the fact that one can associate to each $v_n$–self-map an infinite family in the chromatic layer $n$ stable homotopy groups of spheres. Therefore, interest lies in constructing type $n$ spectra and finding $v_n$–self-maps of lowest possible periodicity on a given type $n$ spectrum. This, in general, is a difficult problem, though progress has been made sporadically throughout the history of the subject [**T**, **DM**, **BP**, **BHHM**, **N**, **BEM**, **BE**]. With the modern development of motivic stable homotopy theory, one may ask if there are similar periodic self-maps of finite motivic spectra.

Classically any non-contractible finite $p$-local spectrum admits a periodic $v_n$–self-map for some $n \geqslant 0$. This is a consequence of the thick-subcategory theorem [**HS**, Theorem 7], aided by a vanishing line argument [**HS**, §4.2]. In the classical case all the thick tensor ideals of $\mathbf{Sp}_{p,\mathrm{fin}}$ (the homotopy category of finite $p$-local spectra) are also prime (in the sense of [**B**]). The thick tensor-ideals of the homotopy category of cellular motivic spectra over $\mathbb{C}$ or $\mathbb{R}$ are not completely known (but see [**HO**, **K**]). However, one can gather some knowledge about the prime thick tensor-ideals in $\mathrm{Ho}(\mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}})$ (the homotopy category of 2-local cellular $\mathbb{R}$-motivic spectra) through the Betti realization functor

$$\beta\colon \mathrm{Ho}(\mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}) \longrightarrow \mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}})$$

using the complete knowledge of prime thick subcategories of $\mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}})$ [**BS**].

B. Guillou and A. Li were supported by NSF grants DMS-1710379 and DMS-2003204.
Received October 30, 2020, revised February 22, 2021; published on April 13, 2022.
2010 Mathematics Subject Classification: 14F42, 55Q51, 55Q91.
Key words and phrases: self-map, motivic homotopy, equivariant homotopy.
Article available at `http://dx.doi.org/10.4310/HHA.2022.v24.n1.a15`

The prime thick tensor-ideals of $\mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}})$ are essentially the pull-back of the classical thick subcategories along the two functors, the geometric fixed-point functor

$$\Phi^{\mathrm{C}_2}\colon \mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}}) \longrightarrow \mathrm{Ho}(\mathbf{Sp}_{2,\mathrm{fin}})$$

and the forgetful functor

$$\Phi^{e}\colon \mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}}) \longrightarrow \mathrm{Ho}(\mathbf{Sp}_{2,\mathrm{fin}}).$$

Let $\mathcal{C}_n$ denote the thick subcategory of $\mathrm{Ho}(\mathbf{Sp}_{2,\mathrm{fin}})$ consisting of spectra of type at least $n$. The prime thick subcategories,

$$\mathcal{C}(e,n) = (\Phi^e)^{-1}(\mathcal{C}_n) \text{ and } \mathcal{C}(\mathrm{C}_2,n) = (\Phi^{\mathrm{C}_2})^{-1}(\mathcal{C}_n),$$

are the only prime thick subcategories of $\mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}})$.

**Definition 1.1.** We say a spectrum $X \in \mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}})$ is of *type* $(n,m)$ if $\Phi^e(X)$ is of type $n$ and $\Phi^{\mathrm{C}_2}(X)$ is of type $m$.

For a type $(n,m)$ spectrum $X$, a self-map $f\colon X \to X$ is periodic if and only if at least one of $\{\Phi^e(f), \Phi^{\mathrm{C}_2}(f)\}$ are periodic (see [**BGH**, Proposition 3.17]).

**Definition 1.2.** Let $X \in \mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}})$ be of type $(n,m)$. We say a self-map $f\colon X \to X$ is

(i) a $v_{(n,m)}$*–self-map* of mixed periodicity $(i,j)$ if $\Phi^e(f)$ is a $v_n$–self-map of periodicity $i$ and $\Phi^{\mathrm{C}_2}(f)$ is a $v_m$–self-map of periodicity $j$,

(ii) a $v_{(n,\mathrm{nil})}$*–self-map* of periodicity $i$ if $\Phi^e(f)$ is a $v_n$–self-map of periodicity $i$ and $\Phi^{\mathrm{C}_2}(f)$ is nilpotent, and,

(iii) a $v_{(\mathrm{nil},m)}$*–self-map* of periodicity $j$ if $\Phi^e(f)$ is a nilpotent self-map and $\Phi^{\mathrm{C}_2}(f)$ is a $v_m$–self-map of periodicity $j$.

**Example 1.3.** The sphere spectrum $\mathbb{S}_{\mathrm{C}_2}$ is of type $(0,0)$. The degree 2 map is a $v_{(0,0)}$–self-map. In general, if we consider the $v_n$–self-map of a type $n$ spectrum with trivial action of $\mathrm{C}_2$, then the resultant equivariant self-map is a $v_{(n,n)}$–self-map.

**Example 1.4.** Let $\mathrm{S}^{1,1}_{\mathrm{C}_2}$ denote the $\mathrm{C}_2$-equivariant sphere which is the one-point compactification of the real sign representation. The unstable twist-map

$$\epsilon^u\colon \mathrm{S}^{1,1}_{\mathrm{C}_2} \wedge \mathrm{S}^{1,1}_{\mathrm{C}_2} \longrightarrow \mathrm{S}^{1,1}_{\mathrm{C}_2} \wedge \mathrm{S}^{1,1}_{\mathrm{C}_2}$$

stabilizes to a nonzero element $\epsilon \in \pi_{0,0}(\mathbb{S}_{\mathrm{C}_2})$. Let $\mathsf{h} = 1 - \epsilon \in \pi_{0,0}(\mathbb{S}_{\mathrm{C}_2})$ be the stabilization of the map

$$\mathsf{h}^u = 1 - \epsilon^u\colon \mathrm{S}^{3,2}_{\mathrm{C}_2} \longrightarrow \mathrm{S}^{3,2}_{\mathrm{C}_2}.$$

Note that on the underlying space $\epsilon$ is of degree $-1$, while on the fixed points it is the identity. Therefore $\Phi^e(\mathsf{h})$ is multiplication by 2, whereas $\Phi^{\mathrm{C}_2}(\mathsf{h})$ is trivial. Thus $\mathsf{h}$ is a $v_{(0,\mathrm{nil})}$–self-map, and the cofiber $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ is of type $(1,0)$.

**Example 1.5.** The equivariant Hopf map $\eta_{1,1} \in \pi_{1,1}(\mathbb{S}_{\mathrm{C}_2})$ is the Betti realization of the $\mathbb{R}$-motivic Hopf map $\eta$ [**M2**, **DI4**]. Up to a unit, it is the stabilization of the projection map

$$\eta^u_{1,1} := \pi\colon \mathrm{S}^{3,2}_{\mathrm{C}_2} \simeq \mathbb{C}^2 \setminus \{\mathbf{0}\} \longrightarrow \mathbb{CP}^1 \cong \mathrm{S}^{2,1}_{\mathrm{C}_2},$$

where the domain and the codomain are given the $\mathrm{C}_2$-structure using complex conjugation. On fixed-points, the map $\pi$ is the projection map

$$\pi\colon \mathbb{R}^2 \setminus \{\mathbf{0}\} \longrightarrow \mathbb{RP}^1,$$

which is a degree 2 map. From this we learn that while $\Phi^e(\eta_{1,1})$ is nilpotent, $\Phi^{\mathrm{C}_2}(\eta_{1,1})$ is the periodic $v_0$–self-map. Hence, $\eta_{1,1}$ is a $v_{(\mathrm{nil},0)}$–self-map and the cofiber $\mathrm{C}(\eta_{1,1})$ is of type $(0, 1)$.

**Remark 1.6.** In the $\mathrm{C}_2$-equivariant stable homotopy groups, the usual Hopf map (sometimes referred to as the 'topological Hopf map') is different from $\eta_{1,1}$ of Example 1.5. The 'topological Hopf map' $\eta_{1,0} \in \pi_{1,0}(\mathbb{S}_{\mathrm{C}_2})$ should be thought of as the stabilization of the unstable Hopf map

$$\eta^u_{1,0}\colon \mathrm{S}^{3,0}_{\mathrm{C}_2} \longrightarrow \mathrm{S}^{2,0}_{\mathrm{C}_2},$$

where both domain and codomain are given the trivial $\mathrm{C}_2$-action.

**Definition 1.7.** We say a spectrum $X \in \mathrm{Ho}(\mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}})$ is of type $(n, m)$ if $\beta(X)$ is of type $(n, m)$. We call an $\mathbb{R}$-motivic self-map

$$f\colon X \to X$$

a $v_{(n,m)}$–self-map, where $m$ and $n$ are in $\mathbb{N} \cup \{\mathrm{nil}\}$ (but not both nil), if $\beta(f)$ is a $\mathrm{C}_2$-equivariant $v_{(n,m)}$–self-map.

**Remark 1.8.** The maps 'multiplication by 2' (of Example 1.3), $\mathsf{h}$ (of Example 1.4), and $\eta_{1,1}$ (of Example 1.5) admit $\mathbb{R}$-motivic lifts along $\beta$ and provide us with examples of a $v_{(0,0)}$–self-map, $v_{(0,\mathrm{nil})}$–self-map and $v_{(\mathrm{nil},0)}$–self-map of the $\mathbb{R}$-motivic sphere spectrum $\mathbb{S}_{\mathbb{R}}$, respectively.

A theorem of Balmer and Sanders [**BS**] asserts that $\mathcal{C}(e, n) \subset \mathcal{C}(\mathrm{C}_2, m)$ if and only if $n \geqslant m + 1$. In particular, $\mathcal{C}(e, n)$ is contained in $\mathcal{C}(\mathrm{C}_2, n - 1)$. Consequently, there are no type $(n, m)$ ($\mathrm{C}_2$-equivariant or $\mathbb{R}$-motivic) spectra if $n \geqslant m + 2$. Their result also implies the following:

**Proposition 1.9.** *Let $X \in \mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}})$ be of type $(n + 1, n)$ for some $n$. Then $X$ cannot support a $v_{(n+1,\mathrm{nil})}$–self-map.*

The proposition holds since the cofiber of such a self-map would be of type $(n+2, n)$, contradicting the results of Balmer–Sanders. In particular, neither the $\mathrm{C}_2$-equivariant cofiber $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ nor the $\mathbb{R}$-motivic cofiber $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ supports a $v_{(1,\mathrm{nil})}$–self-map. However, it is possible that $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ as well as $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ can admit a $v_{(1,0)}$–self-map or a $v_{(\mathrm{nil},0)}$–self-map. In fact, $\eta_{1,1} \in \pi_{1,1}(\mathbb{S}_{\mathbb{R}})$ and $\eta_{1,1} \in \pi_{1,1}(\mathbb{S}_{\mathrm{C}_2})$ induce $v_{(\mathrm{nil},0)}$–self-maps of $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ and $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ respectively. In Section 5, we show that:

**Theorem 1.10.** *The spectrum* $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ *does not admit a* $v_{(1,0)}$*–self-map.*

However, it is possible that $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ admits a $v_{(1,0)}$–self-map (see Remark 5.3 for details). In contrast to the classical case, there is no guarantee that a finite $\mathrm{C}_2$-equivariant or $\mathbb{R}$-motivic spectrum will admit *any* periodic self-map, or at least nothing concrete is known yet. This question must be studied!

In this paper, we consider the classical spectrum

$$\mathcal{Y} := \mathrm{M}_2(1) \wedge \mathrm{C}(\eta)$$

that admits, up to homotopy, 8 different $v_1$–self-maps of periodicity 1 [**DM**, Section 2] (see also [**BEM**]). We ask ourselves if the $v_1$–self-maps are equivariant upon providing $\mathcal{Y}$ with interesting $\mathrm{C}_2$-equivariant structures. We will consider four $\mathrm{C}_2$-equivariant lifts of the spectrum $\mathcal{Y}$,

(i) $\mathcal{Y}^{\mathrm{C}_2}_{\mathrm{triv}}$, where the action of $\mathrm{C}_2$ is trivial,

(ii) $\mathcal{Y}^{\mathrm{C}_2}_{(2,1)} := \mathrm{C}^{\mathrm{C}_2}(2) \wedge \mathrm{C}^{\mathrm{C}_2}(\eta_{1,1})$, with $\Phi^{\mathrm{C}_2}(\mathcal{Y}^{\mathrm{C}_2}_{(2,1)}) = \mathrm{M}_2(1) \wedge \mathrm{M}_2(1)$,

(iii) $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},0)} := \mathrm{C}^{\mathrm{C}_2}(\mathsf{h}) \wedge \mathrm{C}^{\mathrm{C}_2}(\eta_{1,0})$, with $\Phi^{\mathrm{C}_2}(\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},0)}) = \Sigma\mathrm{C}(\eta) \vee \mathrm{C}(\eta)$, and,

(iv) $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)} := \mathrm{C}^{\mathrm{C}_2}(\mathsf{h}) \wedge \mathrm{C}^{\mathrm{C}_2}(\eta_{1,1})$, with $\Phi^{\mathrm{C}_2}(\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)}) = \Sigma\mathrm{M}_2(1) \vee \mathrm{M}_2(1)$.

The $\mathrm{C}_2$-spectra $\mathcal{Y}^{\mathrm{C}_2}_{\mathrm{triv}}$, $\mathcal{Y}^{\mathrm{C}_2}_{(2,1)}$ and $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)}$ are of type $(1,1)$, and $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},0)}$ is of type $(1,0)$. There are unique $\mathbb{R}$-motivic lifts of the classes 2, $\mathsf{h}$, $\eta_{1,0}$, and $\eta_{1,1}$, and therefore we have unique $\mathbb{R}$-motivic lifts of $\mathcal{Y}^{\mathrm{C}_2}_{\mathrm{triv}}$, $\mathcal{Y}^{\mathrm{C}_2}_{(2,1)}$, $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},0)}$, and $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)}$ which we will simply denote by $\mathcal{Y}^{\mathbb{R}}_{\mathrm{triv}}$, $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$, $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$, and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$, respectively. In this paper we prove:

**Theorem 1.11.** *The* $\mathbb{R}$*-motivic spectrum* $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ *admits a* $v_{(1,\mathrm{nil})}$*–self-map*

$$v\colon \Sigma^{2,1}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \longrightarrow \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$$

*of periodicity* 1*.*

By applying the Betti realization functor we get:

**Corollary 1.12.** *The* $\mathrm{C}_2$*-equivariant spectrum* $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)}$ *admits a* 1*-periodic* $v_{(1,\mathrm{nil})}$*–self-map*

$$\beta(v)\colon \Sigma^{2,1}\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)} \longrightarrow \mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)}.$$

**Corollary 1.13.** *The geometric fixed-point spectrum of the telescope*

$$\beta(v)^{-1}\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)}$$

*is contractible.*

Classically, the cofiber of a $v_1$–self-map on $\mathcal{Y}$ is a realization of the finite subalgebra $\mathcal{A}(1)$ of the Steenrod algebra $\mathcal{A}$. We see a very similar phenomenon in the $\mathbb{R}$-motivic as well as in the $\mathrm{C}_2$-equivariant cases. The $\mathrm{C}_2$-equivariant Steenrod algebra $\mathcal{A}^{\mathrm{C}_2}$ as well as the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ admit subalgebras analogous to $\mathcal{A}(1)$ (generated by $\mathrm{Sq}^1$ and $\mathrm{Sq}^2$) [**H**, **R2**], which we denote by $\mathcal{A}^{\mathrm{C}_2}(1)$ and $\mathcal{A}^{\mathbb{R}}(1)$, respectively. We observe that:

**Theorem 1.14.** *The spectrum* $\mathrm{C}^{\mathbb{R}}(v) := \mathrm{Cof}(v\colon \Sigma^{2,1}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \to \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$ *is a type* $(2,1)$ *complex whose bigraded cohomology is a free* $\mathcal{A}^{\mathbb{R}}(1)$*-module on one generator.*

**Corollary 1.15.** *The bigraded cohomology of the* $\mathrm{C}_2$*-equivariant spectrum*

$$\mathrm{C}^{\mathrm{C}_2}(\beta(v)) \simeq \beta(\mathrm{C}^{\mathbb{R}}(v))$$

*is a free* $\mathcal{A}^{\mathrm{C}_2}(1)$*-module on one generator.*

Our last main result in this paper is the following.

**Theorem 1.16.** *The spectrum* $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$ *does not admit a* $v_{(1,0)}$*–self-map.*

The above results immediately raise some obvious questions. Pertaining to self-maps one may ask: Does $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ admit a $v_{(1,\mathrm{nil})}$–self-map? Does $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ or $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ admit a $v_{(1,1)}$–self-map? Does $\mathcal{Y}^{\mathbb{R}}_{\mathrm{triv}}$, $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ or $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ admit a $v_{(\mathrm{nil},1)}$–self-map? Or more generally, how many different homotopy types of each kind of periodic self-maps exist? Related to $\mathcal{A}^{\mathbb{R}}(1)$, one may inquire: How many different $\mathcal{A}^{\mathbb{R}}$-module structures can be given to $\mathcal{A}^{\mathbb{R}}(1)$? Can those $\mathcal{A}^{\mathbb{R}}$-modules be realized as a spectrum? Are the realizations of $\mathcal{A}^{\mathbb{R}}(1)$ equivalent to cofibers of periodic self-maps of $\mathcal{Y}^{\mathbb{R}}_{(i,j)}$? We hope to address most, if not all, of the above questions in our upcoming work (see Remark 3.13, Remark 4.13 and Remark 5.3).

### 1.1. Outline of our method

We first construct a spectrum $\mathcal{A}^{\mathbb{R}}_1$ which realizes the algebra $\mathcal{A}^{\mathbb{R}}(1)$ using a method of Smith (outlined in [**R1**, Appendix C]) which constructs new finite spectra (potentially with larger number of cells) from known ones. The idea is as follows. If $X$ is a $p$-local finite spectrum then the permutation group $\Sigma_n$ acts on $X^{\wedge n}$. One may then use an idempotent $e \in \mathbb{Z}_{(p)}[\Sigma_n]$ to obtain a split summand of the spectrum $X^{\wedge n}$. As explained in [**R1**, Appendix C], Young tableaux provide a rich source of such idempotents. For a judicious choice of $e$ and $X$, the spectrum $e(X^{\wedge n})$ can be interesting.

We exploit the relation that $\mathsf{h} \cdot \eta_{1,1} = 0$ in $\pi_{*,*}(\mathbb{S}_{\mathbb{R}})$ [**M2**] to construct an $\mathbb{R}$-motivic analogue of the question mark complex $\mathcal{Q}_{\mathbb{R}}$. The cell-diagram of $\mathcal{Q}_{\mathbb{R}}$ is as described in Figure 1 below. For a choice of idempotent element $e$ in the group ring $\mathbb{Z}_{(2)}[\Sigma_3]$, we

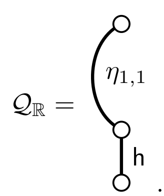



Figure 1: Cell-diagram of the $\mathbb{R}$-motivic question mark complex.

observe that $e(\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})^{\otimes 3})$ is a free $\mathcal{A}^{\mathbb{R}}(1)$-module. This is the cohomology of an $\mathbb{R}$-motivic spectrum $\tilde{e}(\mathcal{Q}_{\mathbb{R}}^{\wedge 3})$, which we call $\Sigma^{1,0}\mathcal{A}^{\mathbb{R}}_1$ (see (6) for details). The observation requires us to develop a criterion that will detect freeness for modules over certain subalgebras of $\mathcal{A}^{\mathbb{R}}$. Writing $\mathbb{M}^{\mathbb{R}}_2$ for the $\mathbb{R}$-motivic cohomology of a point, we prove:

**Theorem 1.17.** *A finitely generated* $\mathcal{A}^{\mathbb{R}}(n)$*-module* M *is free if and only if*

1. M *is free as an* $\mathbb{M}_2^{\mathbb{R}}$*-module, and*
2. $\mathbb{F}_2 \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathrm{M}$ *is a free* $\mathbb{F}_2 \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}^{\mathbb{R}}(n)$*-module.*

The cohomology of $\mathcal{A}_1^{\mathbb{R}}$ provides an $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathcal{A}^{\mathbb{R}}(1)$, which immediately gives us a short exact sequence

$$0 \to \mathrm{H}^{*,*}(\Sigma^{3,1}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \to \mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}}) \to \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \to 0$$

of $\mathcal{A}^{\mathbb{R}}$-modules. Thus, we get a candidate for a $v_{(1,\mathrm{nil})}$–self-map in the $\mathbb{R}$-motivic Adams spectral sequence

$$\overline{v} \in \mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})) \Rightarrow [\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}, \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}]_{*,*},$$

which survives as there is no potential target for a differential supported by $\overline{v}$.

**Organization of the paper**

In Section 2, we review the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$, discuss the structure of its subalgebra $\mathcal{A}^{\mathbb{R}}(n)$, and prove Theorem 1.17. In Section 3, we construct the spectrum $\mathcal{A}_1^{\mathbb{R}}$ that realizes the subalgebra $\mathcal{A}^{\mathbb{R}}(1)$ and prove that it is of type $(2,1)$. In Section 4, we prove Theorem 1.11 and Theorem 1.14; i.e., we show that $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ admits a $v_{1,\mathrm{nil}}$–self-map and that its cofiber has the same $\mathcal{A}^{\mathbb{R}}$-module structure as that of $\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})$. In Section 5, we show the non-existence of a $v_{(1,0)}$–self-map on $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$; i.e., we prove Theorem 1.10 and Theorem 1.16.

## Acknowledgments

The authors are indebted to Nick Kuhn for explaining some of the subtle points of Smith's work exposed in [**R1**, Appendix C], which is the key idea behind Theorem 1.11. The authors also benefited from conversations with Mark Behrens, Dan Isaksen, and Zhouli Xu.

## 2. The $\mathbb{R}$-motivic Steenrod algebra and a freeness criterion

We begin by reviewing the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ following Voevodsky [**V**]. The algebra $\mathcal{A}^{\mathbb{R}}$ is the ring of bigraded homotopy classes of self-maps of the $\mathbb{R}$-motivic Eilenberg–Mac Lane spectrum $\mathrm{H}\mathbb{F}_2^{\mathbb{R}}$:

$$\mathcal{A}^{\mathbb{R}} = [\mathrm{H}\mathbb{F}_2^{\mathbb{R}}, \mathrm{H}\mathbb{F}_2^{\mathbb{R}}]_{*,*}.$$

The unit map $\mathbb{S}_{\mathbb{R}} \to \mathrm{H}\mathbb{F}_2^{\mathbb{R}}$ induces a canonical projection map

$$\epsilon\colon \mathcal{A}^{\mathbb{R}} \longrightarrow \mathbb{M}_2^{\mathbb{R}} := [\mathbb{S}_{\mathbb{R}}, \mathrm{H}\mathbb{F}_2^{\mathbb{R}}]_{*,*} \cong \mathbb{F}_2[\tau, \rho],$$

where $|\tau| = (0,-1)$ and $|\rho| = (-1,-1)$. Further, using the multiplication map $\mathrm{H}\mathbb{F}_2^{\mathbb{R}} \wedge \mathrm{H}\mathbb{F}_2^{\mathbb{R}} \to \mathrm{H}\mathbb{F}_2^{\mathbb{R}}$ one can give $\mathcal{A}^{\mathbb{R}}$ a left $\mathbb{M}_2^{\mathbb{R}}$-module structure as well as a right $\mathbb{M}_2^{\mathbb{R}}$-module structure. Voevodsky shows that $\mathcal{A}^{\mathbb{R}}$ is a free left $\mathbb{M}_2^{\mathbb{R}}$-module. There is an analogue of the classical Adem basis in the motivic setting, and Voevodsky established motivic Adem relations, thereby completely describing the multiplicative structure of $\mathcal{A}^{\mathbb{R}}$.

The motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ also admits a diagonal map, so that its left $\mathbb{M}_2^{\mathbb{R}}$-linear dual is an algebra over $\mathbb{F}_2$. Note that $\mathcal{A}^{\mathbb{R}}$ is an $\mathbb{F}_2$-algebra but not an $\mathbb{M}_2^{\mathbb{R}}$-algebra as $\tau$ is not a central element since

$$\mathrm{Sq}^1(\tau) = \rho \neq \tau \mathrm{Sq}^1. \tag{1}$$

This complication is also reflected in the fact that the pair $(\mathbb{M}_2^{\mathbb{R}}, \hom_{\mathbb{M}_2^{\mathbb{R}}}(\mathcal{A}^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}}))$ is a Hopf algebroid, and not a Hopf algebra like its complex counterpart. The underlying algebra of the dual $\mathbb{R}$-motivic Steenrod algebra is given by

$$\mathcal{A}_*^{\mathbb{R}} \cong \mathbb{M}_2^{\mathbb{R}}[\xi_{i+1}, \tau_i : i \geqslant 0]/(\tau_i^2 = \tau\xi_{i+1} + \rho\tau_{i+1} + \rho\tau_0\xi_{i+1}),$$

where $\xi_i$ and $\tau_i$ live in bidegree $(2^{i+1}-2, 2^i-1)$ and $(2^{i+1}-1, 2^i-1)$, respectively. The complete description of the Hopf algebroid structure can be found in [**V**].

Ricka[1] [**R2**] identified the quotient Hopf algebroids of $\mathcal{A}_*^{\mathbb{R}}$ (see also [**H**]). In particular, there are quotient Hopf algebroids

$$\mathcal{A}^{\mathbb{R}}(n)_* = \mathcal{A}_*^{\mathbb{R}}/(\xi_1^{2^n}, \dots, \xi_n^2, \xi_{n+1}, \dots, \tau_0^{2^{n+1}}, \dots, \tau_n^2, \tau_{n+1}, \dots),$$

which can be thought of as analogues of the quotient Hopf algebras

$$\mathcal{A}(n)_* = \mathcal{A}_*/(\xi_1^{2^{n+1}}, \dots, \xi_{n+1}^2, \xi_{n+2}, \dots)$$

of the classical dual Steenrod algebra $\mathcal{A}_*$. It is an algebraic fact that

$$\tau^{-1}(\mathcal{A}^{\mathbb{R}}(n)_*/(\rho)) \cong \mathbb{F}_2[\tau^{\pm 1}] \otimes \mathcal{A}(n)_* \tag{2}$$

as Hopf algebras (see [**DI2**, Corollary 2.9]). The above isomorphism sends $\tau_i \mapsto \tau^{1-2^i}\xi_{i+1}$ and $\xi_{i+1} \mapsto \tau^{1-2^{i+1}}\xi_{i+1}^2$. The quotient Hopf algebroid $\mathcal{A}^{\mathbb{R}}(n)_*$ is the $\mathbb{M}_2^{\mathbb{R}}$-linear dual of the subalgebra $\mathcal{A}^{\mathbb{R}}(n)$ of $\mathcal{A}^{\mathbb{R}}$, which is generated by the elements $\{\tau, \rho, \mathrm{Sq}^1, \mathrm{Sq}^2, \dots, \mathrm{Sq}^{2^n}\}$.

Although $\tau$ is not in the center (see (1)) of $\mathcal{A}^{\mathbb{R}}$ or $\mathcal{A}^{\mathbb{R}}(n)$, the element $\rho$ is in the center. We make use of this fact to prove the following result.

**Lemma 2.1.** *A finitely-generated $\mathcal{A}^{\mathbb{R}}(n)$-module $\mathrm{M}$ is free if and only if*

1. *$\mathrm{M}$ is free as an $\mathbb{F}_2[\rho]$-module, and,*
2. *$\mathrm{M}/(\rho)$ is a free $\mathcal{A}^{\mathbb{R}}(n)/(\rho)$-module.*

*Proof.* The 'only if' part is trivial. For the 'if' part, choose a basis $\mathcal{B} = \{b_1, \dots, b_n\}$ of $\mathrm{M}/(\rho)$ and let $\tilde{b}_i \in M$ be any lift of $b_i$. Let $\mathrm{F}$ denote the free $\mathcal{A}^{\mathbb{R}}(n)$-module generated by $\mathcal{B}$ and consider the map

$$f\colon \mathrm{F} \to \mathrm{M}$$

that sends $b_i \mapsto \tilde{b}_i$. We show that $f$ is an isomorphism by inductively proving that $f$ induces an isomorphism $\mathrm{F}/(\rho^n) \cong \mathrm{M}/(\rho^n)$ for all $n \geqslant 1$. The case of $n = 1$ is true by assumption.

[1]Ricka actually identified the quotient Hopf algebroids of the $\mathrm{C}_2$-equivariant dual Steenrod algebra. However, the difference between the $\mathbb{R}$-motivic Steenrod algebra and the $\mathrm{C}_2$-equivariant Steenrod algebra lies only in the coefficient rings and results of Ricka easily identifies the quotient Hopf algebroids of the $\mathbb{R}$-motivic Steenrod algebra.

For the inductive argument, first note that the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \mathrm{F}/(\rho^{n-1}) \arrow[r, "\cdot\rho"] \arrow[d, "f_{n-1}"] & \mathrm{F}/(\rho^{n}) \arrow[r] \arrow[d, "f_n"] & \mathrm{F}/(\rho) \arrow[r] \arrow[d, "f_0"] & 0 \arrow[d, equal] \\
0 \arrow[r] & \mathrm{M}/(\rho^{n-1}) \arrow[r, "\cdot\rho"] & \mathrm{M}/(\rho^{n}) \arrow[r] & \mathrm{M}/(\rho) \arrow[r] & 0
\end{tikzcd}$$

is a diagram of $\mathcal{A}^{\mathbb{R}}(n)$-modules (since $\rho$ is in the center) where the horizontal rows are exact. The map $f_0$ is an isomorphism by assumption *(2)*, and $f_{n-1}$ is an isomorphism by the inductive hypothesis; hence, $f_n$ is an isomorphism by the five lemma. $\square$

*Proof of Theorem 1.17.* The result follows immediately from Lemma 2.1 combined with [**HK**, Theorem B] and the fact that $\mathcal{A}^{\mathbb{C}}(n) = \mathcal{A}^{\mathbb{R}}(n)/(\rho)$. $\square$

In order to employ Theorem 1.17, we use the work of Adams and Margolis [**AM**], which provides a freeness criterion for modules over finite-dimensional subalgebras of the classical Steenrod algebra in terms of Margolis homology. Recall that, for an algebra A and an element $x \in \mathrm{A}$ such that $x^2 = 0$, the Margolis homology of M with respect to $x$ is defined as

$$\mathcal{M}(\mathrm{M}, x) = \frac{\ker(x\colon \mathrm{M} \to \mathrm{M})}{\operatorname{img}(x\colon \mathrm{M} \to \mathrm{M})}.$$

In the classical Steenrod algebra, the element $\mathrm{P}^s_t$ is defined to be dual to $\xi_t^{2^s} \in \mathcal{A}_*$. In terms of the Milnor basis,

$$\mathrm{P}^s_t := \mathrm{Sq}(\underbrace{0, \dots, 0}_{t-1}, 2^s).$$

The element $\mathrm{P}^0_t$ is often denoted by $\mathrm{Q}_{t-1}$. One may define the $\mathbb{R}$-motivic analogues of $\mathrm{P}^s_t \in \mathcal{A}$ by setting

$$\overline{\mathrm{Q}}_t := \tau_t^* \quad \text{and} \quad \overline{\mathrm{P}}^s_t := (\xi_t^{2^{s-1}})^*$$

in $\mathcal{A}^{\mathbb{R}}(n)$ for $s \geqslant 1$, recalling that the motivic $\xi_t$ plays the role of the classical $\xi_t^2$. It is easy to see that under the isomorphism (2), $\overline{\mathrm{Q}}_t$ corresponds to $\tau^{1-2^t}\mathrm{Q}_t$ and $\overline{\mathrm{P}}^s_t$ corresponds to $\tau^{2^s(1-2^t)}\mathrm{P}^s_t$.

In the context of Theorem 1.17, freeness over

$$\mathbb{F}_2 \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}^{\mathbb{R}}(n) := \mathcal{A}^{\mathbb{R}}(n)/(\rho, \tau) \cong \mathcal{A}^{\mathbb{C}}(n)/(\tau)$$

can be detected using Margolis homology calculations following [**HK**, Theorem B(i)].

**Corollary 2.2.** *Let* M *be a finitely generated left* $\mathcal{A}^{\mathbb{R}}(n)$*-module and let*

$$\mathrm{M}/(\rho, \tau) := \mathrm{M} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbb{F}_2.$$

*Then* M *is a free* $\mathcal{A}^{\mathbb{R}}(n)$*-module if and only if*

1. M *is free over* $\mathbb{M}_2^{\mathbb{R}}$*,*
2. $\mathcal{M}(\mathrm{M}/(\rho, \tau), \overline{\mathrm{Q}}_i) = 0$ *for* $0 \leqslant i \leqslant n$*, and*
3. $\mathcal{M}(\mathrm{M}/(\rho, \tau), \overline{\mathrm{P}}^s_t) = 0$ *if* $1 \leqslant s \leqslant t$ *and* $s + t \leqslant n + 1$*.*

**Remark 2.3.** The quotient $\mathcal{A}^{\mathbb{R}}(n)/(\rho,\tau)$ fits into a short exact sequence

$$\overline{\mathrm{E}}(n) \hookrightarrow \mathcal{A}^{\mathbb{R}}(n)/(\rho,\tau) \twoheadrightarrow \overline{\mathrm{P}}(n) \tag{3}$$

of connected finite-dimensional Hopf algebras over $\mathbb{F}_2$, where $\overline{\mathrm{E}}(n) := \Lambda_{\mathbb{F}_2}(\overline{\mathrm{Q}}_0,\ldots,\overline{\mathrm{Q}}_n)$ and $\overline{\mathrm{P}}(n) := \mathcal{A}^{\mathbb{R}}(n)/(\rho,\tau,\overline{\mathrm{Q}}_0,\ldots,\overline{\mathrm{Q}}_n)$. The short exact sequence (3) splits from the right. This right splitting map confirms that (3) is a split exact sequence of coalgebras as all of the Hopf algebras involved in (3) admit a cocommutative comultiplication. However, when (3) is viewed as an exact sequence of algebras, it does not split because the algebras involved are not commutative. For example, when $n=1$ then a left splitting map in (3) would imply that $\overline{\mathrm{Q}}_0$ commutes with $\mathrm{Sq}^2$ and contradicts the fact that $\overline{\mathrm{Q}}_1 := [\mathrm{Sq}^2,\overline{\mathrm{Q}}_0]$. Dually, there is a splitting

$$\mathcal{A}^{\mathbb{R}}(n)_*/(\rho,\tau) \cong \frac{\mathbb{F}_2[\xi_1,\ldots,\xi_n]}{(\xi_1^{2^n},\ldots,\xi_n^2)} \otimes \Lambda(\tau_0,\ldots,\tau_n)$$

as an algebra, though it does not split as a coalgebra. This is clear from the fact that

$$\Delta(\tau_k) \equiv \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \tau_i \not\equiv \tau_k \otimes 1 + 1 \otimes \tau_k \mod (\rho,\tau).$$

**Remark 2.4** (A minor correction to [**HK**])**.** Note that Remark 2.3 stands in contradiction to [**HK**, Corollary 4.2]. However, this does not affect [**HK**, Corollary 4.3] which claims $(\overline{\mathrm{P}}_t^t)^2 = 0$. This is because $\overline{\mathrm{P}}(n)$ is a sub-Hopf algebra of $\mathcal{A}^{\mathbb{R}}(n)/(\rho,\tau)$. We also note that the proof of [**HK**, Theorem B(i)] remains unaffected by this change.

## 3. A realization of $\mathcal{A}^{\mathbb{R}}(1)$

Consider the $\mathbb{R}$-motivic question mark complex $\mathcal{Q}_{\mathbb{R}}$, as introduced in Subsection 1.1. Let $\Sigma_n$ act on $\mathcal{Q}_{\mathbb{R}}^{\wedge n}$ by permutation. Any element $e \in \mathbb{Z}_{(2)}[\Sigma_n]$ produces a canonical map

$$\tilde{e}\colon \mathcal{Q}_{\mathbb{R}}^{\wedge n} \longrightarrow \mathcal{Q}_{\mathbb{R}}^{\wedge n}.$$

Now let $e$ be the idempotent

$$e = \tfrac{1+(1\ 2)-(1\ 3)-(1\ 3\ 2)}{3}$$

in $\mathbb{Z}_{(2)}[\Sigma_3]$, and denote by $\overline{e}$ the resulting idempotent of $\mathbb{F}_2[\Sigma_3]$. For an $\mathbb{R}$-motivic spectrum $X$ with action of $\Sigma_n$, we then define

$$\tilde{e}(X) = \underrightarrow{\mathrm{hocolim}}(X \xrightarrow{\tilde{e}} X \xrightarrow{\tilde{e}} \ldots),$$

and we employ the same notation in the $C_2$-equivariant or classical contexts. We will use that for a spectrum $X$ with action of $\Sigma_n$, we have an isomorphism

$$\mathrm{H}^*(\tilde{e}X;\mathbb{F}_2) \cong \overline{e}\mathrm{H}^*(X;\mathbb{F}_2). \tag{4}$$

We record the following important property of $\overline{e}$ which is a special case of [**R1**, Theorem C.1.5].

**Lemma 3.1.** *If $V$ is a finite-dimensional $\mathbb{F}_2$-vector space, then $\overline{e}(V^{\otimes 3}) = 0$ if and only if $\dim V \leqslant 1$.*

The following result, which gives the values of $\overline{e}$ on induced representations, is also straightforward to verify:

**Lemma 3.2.** *Suppose that* $W = \mathrm{Ind}_{C_2}^{\Sigma_3}\mathbb{F}_2$ *is induced up from the trivial representation of a cyclic 2-subgroup. Then* $\overline{e}(W) \cong \mathbb{F}_2$*. Moreover, for the regular representation* $\mathbb{F}_2[\Sigma_3] = \mathrm{Ind}_e^{\Sigma_3}\mathbb{F}_2$*, we have* $\dim \overline{e}(\mathbb{F}_2[\Sigma_3]) = 2$.

We also record the fact that when $\dim_{\mathbb{F}_2} V = 2$ and $\dim_{\mathbb{F}_2} W = 3$ then

$$\dim_{\mathbb{F}_2} \overline{e}(V^{\otimes 3}) = 2 \quad \text{and} \quad \dim_{\mathbb{F}_2} \overline{e}(W^{\otimes 3}) = 8, \tag{5}$$

as we will often use this.

The bottom cell of $\tilde{e}(\mathcal{Q}_{\mathbb{R}}^{\wedge 3})$ is in degree $(1,0)$, and we define

$$\mathcal{A}_1^{\mathbb{R}} := \Sigma^{-1,0}\tilde{e}(\mathcal{Q}_{\mathbb{R}}^{\wedge 3}) = \Sigma^{-1,0}\underrightarrow{\mathrm{hocolim}}(\mathcal{Q}_{\mathbb{R}}^{\wedge 3} \xrightarrow{\tilde{e}} \mathcal{Q}_{\mathbb{R}}^{\wedge 3} \xrightarrow{\tilde{e}} \dots). \tag{6}$$

The purpose of this section is to prove the following theorem.

**Theorem 3.3.** *The spectrum* $\mathcal{A}_1^{\mathbb{R}}$ *is a type* $(2,1)$ *complex whose bi-graded cohomology* $\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})$ *is a free* $\mathcal{A}^{\mathbb{R}}(1)$*-module on one generator.*

### 3.1. $\mathcal{A}_1^{\mathbb{R}}$ is of type $(2,1)$

Let $\mathcal{A}_1^{\mathrm{C}_2} := \beta(\mathcal{A}_1^{\mathbb{R}})$ and $\mathcal{Q}_{\mathrm{C}_2} := \beta(\mathcal{Q}_{\mathbb{R}})$. Note that we have a $\mathrm{C}_2$-equivariant splitting

$$\mathcal{Q}_{\mathrm{C}_2}^{\wedge 3} \simeq \tilde{e}(\mathcal{Q}_{\mathrm{C}_2}^{\wedge 3}) \vee (1-\tilde{e})(\mathcal{Q}_{\mathrm{C}_2}^{\wedge 3}),$$

which splits the underlying spectra as well as the geometric fixed-points, as both $\Phi^e$ and $\Phi^{\mathrm{C}_2}$ are additive functors.

We will identify the underlying spectrum $\Phi^e(\mathcal{A}_1^{\mathrm{C}_2})$ by studying the $\mathcal{A}$-module structure of its cohomology with $\mathbb{F}_2$-coefficients. Firstly, note that

$$\Phi^e(\mathcal{A}_1^{\mathrm{C}_2}) \simeq \Sigma^{-1}\tilde{e}(\Phi^e(\mathcal{Q}_{\mathrm{C}_2}^{\wedge 3})) \simeq \Sigma^{-1}\tilde{e}(\mathcal{Q}^{\wedge 3}),$$

where $\mathcal{Q}$ is the classical question mark complex, whose $\mathrm{H}\mathbb{F}_2$-cohomology as an $\mathcal{A}$-module is well understood. It consists of three $\mathbb{F}_2$-generators $a$, $b$, and $c$ in internal degrees 0, 1, and 3, such that $\mathrm{Sq}^1(a) = b$ and $\mathrm{Sq}^2(b) = c$ are the only nontrivial relations, as displayed in Figure 2.

$\mathrm{H}^*(\mathcal{Q};\mathbb{F}_2) =$

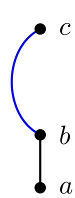



Figure 2: We depict the $\mathcal{A}$-structure of $\mathrm{H}^*(\mathcal{Q};\mathbb{F}_2)$ by marking $\mathrm{Sq}^1$-action by straight (black) lines and $\mathrm{Sq}^2$-action by curved (blue) lines between the $\mathbb{F}_2$-generators. (Colors are only available in the electronic version.)

Because of the Kunneth isomorphism and the fact that the Steenrod algebra is cocommutative, we have an isomorphism of $\mathcal{A}$-modules

$$\mathrm{H}^{*+1}(\Phi^e(\mathcal{A}_1^{\mathrm{C}_2});\mathbb{F}_2) \cong \mathrm{H}^*(\tilde{e}(\mathcal{Q}^{\wedge 3});\mathbb{F}_2) \cong \overline{e}(\mathrm{H}^*(\mathcal{Q};\mathbb{F}_2)^{\otimes 3}),$$

where the second isomorphism is (4).

**Lemma 3.4.** *The underlying* $\mathcal{A}(1)$*-module structure of* $\mathrm{H}^*(\Phi^e(\mathcal{A}_1^{\mathrm{C}_2});\mathbb{F}_2)$ *is free on a single generator.*

*Proof.* Let us denote the $\mathcal{A}$-module $\mathrm{H}^*(\mathcal{Q};\mathbb{F}_2)$ by V. Since $\dim \mathcal{M}(\mathrm{V}, Q_i) = 1$ for $i \in \{0,1\}$, it follows from the Kunneth isomorphism of $Q_i$-Margolis homology groups, cocommutativity of the Steenrod algebra, and Lemma 3.1 that

$$\mathcal{M}(\overline{e}(\mathrm{V}^{\otimes 3}), Q_i) = \overline{e}(\mathcal{M}(\mathrm{V}, Q_i)^{\otimes 3}) = 0$$

for $i \in \{0,1\}$. It follows from [**AM**, Theorem 3.1] that $\mathrm{H}^*(\Phi^e(\mathcal{A}_1^{\mathbb{R}});\mathbb{F}_2)$ is free as an $\mathcal{A}(1)$-module. It is singly generated because of (5). □

We explicitly identify the image of $\overline{e}\colon \mathrm{H}^*(\mathcal{Q};\mathbb{F}_2)^{\otimes 3} \longrightarrow \mathrm{H}^*(\mathcal{Q};\mathbb{F}_2)^{\otimes 3}$ in Figure 3.

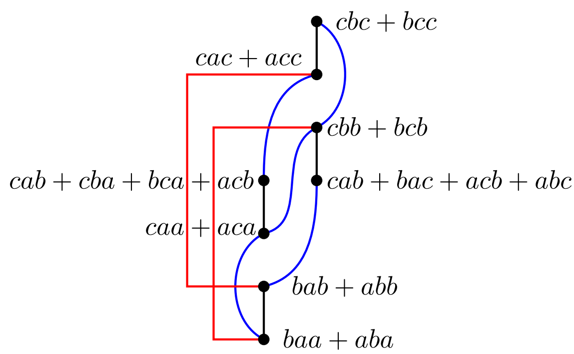



Figure 3: The $\mathcal{A}$-module structure of $\mathrm{H}^*(\Phi^e(\mathcal{A}_1^{\mathrm{C}_2});\mathbb{F}_2)$: Straight (black) lines, curved (blue) lines, and boxed (red) lines represent the $\mathrm{Sq}^1$-action, $\mathrm{Sq}^2$-action, and $\mathrm{Sq}^4$-action, respectively.

**Remark 3.5.** Using the Cartan formula, we can identify the action of $\mathrm{Sq}^4$ on $\Phi^e(\mathcal{A}_1^{\mathrm{C}_2})$. We notice that its $\mathcal{A}$-module structure is isomorphic to $A_1[10]$ of [**BEM**]. Since such an $\mathcal{A}$-module is realized by a unique 2-local finite spectrum, we conclude

$$\Phi^e(\mathcal{A}_1^{\mathrm{C}_2}) \simeq A_1[10]$$

and is of type 2.

Our next goal is to understand the homotopy type of the geometric fixed-point spectrum $\Phi^{C_2}(\mathcal{A}_1^{C_2})$. First observe that the geometric fixed-points of the $C_2$-equivariant question mark complex $\mathcal{Q}_{C_2}$ is the *exclamation mark* complex

$\mathcal{E} :=$  $\simeq \mathbb{S}^0 \vee \Sigma M_2(1)!$

This is because $\Phi^{C_2}(\mathsf{h}) = 0$ and $\Phi^{C_2}(\eta_{1,1}) = 2$. Secondly,

$$H^{*+1}(\Phi^{C_2}(\mathcal{A}_1^{C_2}); \mathbb{F}_2) \cong H^*(\tilde{e}(\mathcal{E}^{\wedge 3}); \mathbb{F}_2) \cong \overline{e}(H^*(\mathcal{E}; \mathbb{F}_2)^{\otimes 3})$$

is an isomorphism of $\mathcal{A}$-modules, where again the second isomorphism is (4). We

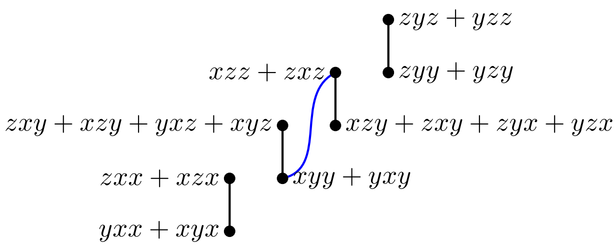



Figure 4: The $\mathcal{A}$-module structure of $H^*(\Phi^{C_2}(\mathcal{A}_1^{C_2}); \mathbb{F}_2)$.

explicitly calculate the $\mathcal{A}$-module structure of $H^*(\Phi^{C_2}(\mathcal{A}_1^{C_2}); \mathbb{F}_2)$ from the above isomorphism and record it in Figure 4 as a subcomplex of $H^*(\mathcal{E}; \mathbb{F}_2)^{\otimes 3}$, with the convention that $x$, $y$ and $z$ are generators in $H^*(\mathcal{E}; \mathbb{F}_2)$ in degree 0, 1 and 2 respectively.

**Lemma 3.6.** *There is an equivalence*

$$\Phi^{C_2}(\mathcal{A}_1^{C_2}) \simeq M_2(1) \vee \Sigma\Big(M_2(1) \wedge M_2(1)\Big) \vee \Sigma^3 M_2(1).$$

*In particular,* $\Phi^{C_2}(\mathcal{A}_1^{C_2})$ *is a type 1 spectrum.*

*Proof.* From Figure 4, it is clear that we have an isomorphism of $\mathcal{A}$-modules

$$H^*(\Phi^{C_2}(\mathcal{A}_1^{C_2}); \mathbb{F}_2) \cong H^*\Big(M_2(1) \vee \Sigma\big(M_2(1) \wedge M_2(1)\big) \vee \Sigma^3 M_2(1); \mathbb{F}_2\Big).$$

It is possible that the $\mathcal{A}$-module $H^*(\Phi^{C_2}(\mathcal{A}_1^{C_2}); \mathbb{F}_2)$ may not realize to a unique finite spectrum (up to weak equivalence). However, other possibilities can be eliminated from the fact that $\mathcal{E}^{\wedge 3}$ splits $\Sigma_3$-equivariantly into four components:

$$\mathcal{E}^{\wedge 3} \simeq \mathbb{S} \vee \left(\bigvee_{i=1}^{3} \Sigma M_2(1)\right) \vee \left(\bigvee_{i=1}^{3} \Sigma^2 M_2(1)^{\wedge 2}\right) \vee \Sigma^3 M_2(1)^{\wedge 3}.$$

The idempotent $\tilde{e}$ annihilates $\mathbb{S} \simeq \mathbb{S}^{\wedge 3}$, and Lemma 3.2 implies that

$$\tilde{e}\left(\bigvee_{i=1}^{3} \Sigma M_2(1)\right) \simeq \Sigma M_2(1) \qquad \text{and}$$

$$\tilde{e}\left(\bigvee_{i=1}^{3}\Sigma^2\mathrm{M}_2(1)\wedge\mathrm{M}_2(1)\right)\simeq\Sigma^2\mathrm{M}_2(1)\wedge\mathrm{M}_2(1).$$

Similarly, we see using (5) that

$$\mathrm{H}^*\left(\tilde{e}\left(\mathrm{M}_2(1)^{\wedge 3}\right)\right)\cong\overline{e}\left(\mathrm{H}^*\left(\mathrm{M}_2(1)\right)^{\otimes 3}\right)\cong\mathrm{H}^*(\Sigma\mathrm{M}_2(1)).$$

Therefore, as an $\mathcal{A}$-module

$$\mathrm{H}^*\left(\tilde{e}\left(\Sigma^3\mathrm{M}_2(1)^{\wedge 3}\right)\right)\cong\mathrm{H}^*(\Sigma^4\mathrm{M}_2(1)).$$

Since, the $\mathcal{A}$-module $\mathrm{H}^*(\mathrm{M}_2(1))$ has a unique lift as a finite spectrum up to homotopy (also see Remark 3.7), we conclude $\tilde{e}\left(\Sigma^3\mathrm{M}_2(1)^{\wedge 3}\right)\simeq\Sigma^4\mathrm{M}_2(1)$.

As $\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2})$ is the desuspension of $\tilde{e}(\mathcal{E}^{\wedge 3})$, the result follows. □

**Remark 3.7.** It is well-known that if $\mathrm{H}^*(X)\cong\mathcal{A}(0)\cong\mathrm{H}^*(\mathrm{M}_2(1))$ as an $\mathcal{A}$-module and $X$ is a 2-local finite spectrum, then $X\simeq\mathrm{M}_2(1)$. Firstly note that the group $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathcal{A}(0),\mathcal{A}(0))$ vanishes in stem equal to $-1$ and cohomological degree at least 2. It follows that the identity map $\mathcal{A}(0)\to\mathcal{A}(0)$, which is a nonzero element in degree $(0,0)$ in the $\mathrm{E}_2$-page of the Adams spectral sequence

$$\mathrm{E}_2^{s,t}:=\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathrm{H}^*(\mathrm{M}_2(1)),\mathrm{H}^*(X))\Rightarrow[X,\mathrm{M}_2(1)]_{t-s},$$

survives to produce a map from $X$ to $\mathrm{M}_2(1)$. This map, by construction, induces an isomorphism in homology. Therefore, by Whitehead's theorem it is an equivalence (also see [**BE**, § 5]).

### 3.2. The cohomology of $\mathcal{A}_1^{\mathbb{R}}$ is free over $\mathcal{A}^{\mathbb{R}}(1)$

Next, we analyze the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})$. We begin by recalling some general properties of the cohomology of motivic spectra.

If $X,Y\in\mathbf{Sp}_{2,\mathrm{fin}}^{\mathbb{R}}$ such that $\mathrm{H}^{*,*}(X)$ is free as a left $\mathbb{M}_2^{\mathbb{R}}$-module, then we have a Kunneth isomorphism [**DI3**, Proposition 7.7]

$$\mathrm{H}^{*,*}(X\wedge Y)\cong\mathrm{H}^{*,*}(X)\otimes_{\mathbb{M}_2^{\mathbb{R}}}\mathrm{H}^{*,*}(Y) \tag{7}$$

as the relevant Kunneth spectral sequence collapses. Further, if $\mathrm{H}^{*,*}(Y)$ is free as a left $\mathbb{M}_2^{\mathbb{R}}$-module, then so is $\mathrm{H}^{*,*}(X\wedge Y)$. The $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(X\wedge Y)$ can then be computed using the Cartan formula. The comultiplication map of $\mathcal{A}^{\mathbb{R}}$ is left $\mathbb{M}_2^{\mathbb{R}}$-linear, coassociative and cocommutative [**V**, Lemma 11.9], which is also reflected in the fact that its $\mathbb{M}_2^{\mathbb{R}}$-linear dual is a commutative and associative algebra. Thus, when $\mathrm{H}^{*,*}(X)$ is a free left $\mathbb{M}_2^{\mathbb{R}}$-module, the elements of $\mathbb{F}_2[\Sigma_n]$ act on

$$\mathrm{H}^{*,*}(X^{\wedge n})\cong\mathrm{H}^{*,*}(X)\otimes_{\mathbb{M}_2^{\mathbb{R}}}\cdots\otimes_{\mathbb{M}_2^{\mathbb{R}}}\mathrm{H}^{*,*}(X)$$

via permutation and commute with the action of $\mathcal{A}^{\mathbb{R}}$. This also implies that $\mathbb{F}_2[\Sigma_n]$ also acts on

$$\mathrm{H}^{*,*}(X^{\wedge n})/(\rho,\tau)\cong\mathrm{H}^{*,*}(X)/(\rho,\tau)\otimes\cdots\otimes\mathrm{H}^{*,*}(X)/(\rho,\tau)$$

and commutes with the action of $\mathcal{A}^{\mathbb{R}}/\!\!/\mathbb{M}_2^{\mathbb{R}}$. From the above discussion we may conclude that

$$\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})\cong\Sigma^{-1}\overline{e}(\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})^{\otimes 3}) \tag{8}$$

is an isomorphism of $\mathcal{A}^{\mathbb{R}}$-modules.

We will also rely upon the following important property of the action of the motivic Steenrod algebra on the cohomology of a motivic space (as opposed to a motivic spectrum):

**Remark 3.8** (Instability condition for $\mathbb{R}$-motivic cohomology)**.** If $X$ is an $\mathbb{R}$-motivic space then $\mathrm{H}^{*,*}(X)$ admits a ring structure, and, for any $u \in \mathrm{H}^{n,i}(X)$, the $\mathbb{R}$-motivic squaring operations obey the rule

$$\mathrm{Sq}^{2i}(u) = \begin{cases} 0 & \text{if } n < 2i, \\ u^2 & \text{if } n = 2i. \end{cases}$$

This is often referred to as the *instability condition.*

To understand the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})$, we first make the following observation regarding $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$ (as $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ is a sub-complex of $\mathcal{Q}_{\mathbb{R}}$) using an argument very similar to [**DI1**, Lemma 7.4].

**Proposition 3.9.** *There are two extensions of $\mathcal{A}^{\mathbb{R}}(0)$ to an $\mathcal{A}^{\mathbb{R}}$-module, and these $\mathcal{A}^{\mathbb{R}}$-modules are realized as the cohomology of $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ and $\mathrm{C}^{\mathbb{R}}(2)$. These are displayed in Figure 5.*

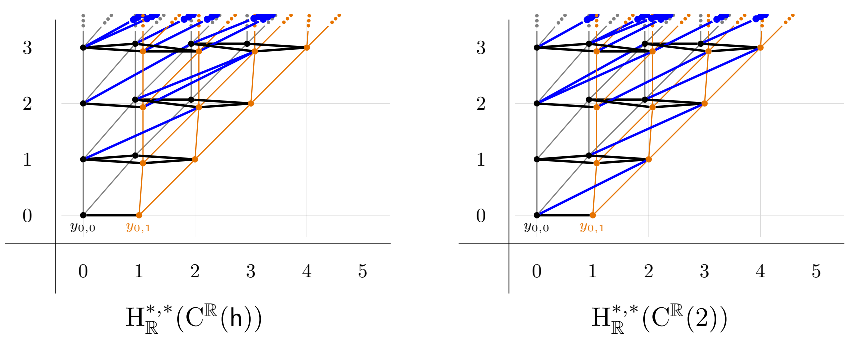



Figure 5: The horizontal axis represents the topological dimension, and the vertical axis represents the motivic weight. Vertical lines of length $(0, 1)$ represent $\tau$-multiplication, diagonal lines of length $(1, 1)$ represent $\rho$-multiplication, horizontal (black) lines represent the $\mathrm{Sq}^1$-action, and slope $1/2$ (blue) lines represent the $\mathrm{Sq}^2$-action. Note: although some of the displayed classes in positive weight do support a $\mathrm{Sq}^4$, we have not displayed this action in order to avoid clutter.

*Proof.* For degree reasons, the only choice in extending $\mathcal{A}^{\mathbb{R}}(0)$ to an $\mathcal{A}^{\mathbb{R}}$-module is the action of $\mathrm{Sq}^2$ on the generator in bidegree $(0, 0)$. We write $y_{0,0}$ for the generator in degree (0, 0) and $y_{1,0}$ for $\mathrm{Sq}^1(y_{0,0})$ in (cohomological) bidegree $(1, 0)$. The two possible choices are

- $\mathrm{Sq}^2(y_{0,0}) = 0$ and
- $\mathrm{Sq}^2(y_{0,0}) = \rho \cdot y_{1,0}$.

We can realize the degree 2 map as an unstable map $\mathrm{S}^{1,0} \longrightarrow \mathrm{S}^{1,0}$, and we will write $\mathrm{C}^{\mathbb{R}}(2)^u$ for the cofiber. We deduce information about the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(2))$ by analyzing the cohomology ring of $\mathrm{S}^{1,1} \wedge \mathrm{C}^{\mathbb{R}}(2)^u$ using the instability condition of Remark 3.8. First, note that in

$$\mathrm{H}^{*,*}(\mathrm{S}^{1,1}) \cong \mathbb{M}_2^{\mathbb{R}} \cdot \iota_{1,1}$$

we have the relation $\iota_{1,1}^2 = \rho \cdot \iota_{1,1}$ [**V**, Lemma 6.8]. Also note that

$$\mathrm{H}^{*,*}((\mathrm{C}^{\mathbb{R}}(2)^u)_+) \cong \mathbb{M}_2^{\mathbb{R}}[x]/(x^3),$$

where $x$ is in cohomological degrees $(1,0)$. Therefore, in

$$\mathrm{H}^{*,*}(\mathrm{S}^{1,1} \wedge \mathrm{C}^{\mathbb{R}}(2)^u) = \mathbb{M}_2^{\mathbb{R}} \cdot \iota_{1,1} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbb{M}_2^{\mathbb{R}}\{x, x^2\}$$

the instability condition implies

$$\mathrm{Sq}^2(\iota_{1,1} \otimes x) = \iota_{1,1}^2 \otimes x^2 = \rho \cdot \iota_{1,1} \otimes x^2.$$

Here the space-level cohomology class $x^2$ corresponds to the spectrum-level class $y_{1,0}$. Therefore, $\mathrm{Sq}^2(y_{0,0}) = \rho \cdot y_{1,0}$ in $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(2))$. This is also reflected in the fact that multiplication by 2 is detected by $h_0 + \rho h_1$ in the $\mathbb{R}$-motivic Adams spectral sequence [**DI1**, §8].

On the other hand $\mathsf{h}$ is the 'zeroth $\mathbb{R}$-motivic Hopf map' detected by the element $h_0$ in the motivic Adams spectral sequence. It follows that $\mathrm{Sq}^2(y_{0,0}) = 0$. □

In order to express the $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathrm{H}^{*,*}(X)$ for a finite spectrum $X$, it is enough to specify the action of $\mathcal{A}^{\mathbb{R}}$ on its left $\mathbb{M}_2^{\mathbb{R}}$-generators as the action of $\tau$ and $\rho$ multiples are determined by the Cartan formula.

**Example 3.10.** Let $\{y_{0,0}, y_{1,0}\} \subset \mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$ denote a left $\mathbb{M}_2^{\mathbb{R}}$-basis of $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$. The data that

- $\mathrm{Sq}^1(y_{0,0}) = y_{1,0}$,
- $\mathrm{Sq}^2(y_{0,0}) = 0$,

completely determines the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$.

**Proposition 3.11.** $\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})$ *is a free* $\mathbb{M}_2^{\mathbb{R}}$*-module generated by* $a, b$ *and* $c$ *in cohomological bidegrees* $(0,0), (1,0)$ *and* $(3,1)$*, and the relations*

1. $\mathrm{Sq}^1(a) = b$,
2. $\mathrm{Sq}^2(b) = c$,
3. $\mathrm{Sq}^4(a) = 0$,

*completely determine the* $\mathcal{A}^{\mathbb{R}}$*-module structure of* $\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})$.

*Proof.* $\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})$ is a free $\mathbb{M}_2^{\mathbb{R}}$-module because the attaching maps of $\mathcal{Q}_{\mathbb{R}}$ induce trivial maps in $\mathrm{H}^{*,*}(-)$. The first two relations can be deduced from the obvious maps

1. $\mathrm{C}^{\mathbb{R}}(\mathsf{h}) \to \mathcal{Q}_{\mathbb{R}}$,
2. $\mathcal{Q}_{\mathbb{R}} \to \Sigma^{1,0}\,\mathrm{C}^{\mathbb{R}}(\eta_{1,1})$,

which are respectively surjective and injective in cohomology.

Let $\mathsf{h}^u \colon \mathrm{S}^{3,2} \to \mathrm{S}^{3,2}$ and $\eta_{1,1}^u \colon \mathrm{S}^{3,2} \to \mathrm{S}^{2,1}$ denote the unstable maps that stabilize to $\mathsf{h}$ and $\eta_{1,1}$, respectively. The unstable $\mathbb{R}$-motivic space $\mathcal{Q}_{\mathbb{R}}^u$ (which stabilizes to $\mathcal{Q}_{\mathbb{R}}$)

can be constructed using the fact that the composite of the unstable maps

$$\mathrm{S}^{4,3} \xrightarrow{\Sigma^{1,1}\eta_{1,1}^u} \mathrm{S}^{3,2} \xrightarrow{\mathsf{h}^u} \mathrm{S}^{3,2}$$

is null. Thus $\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}}^u)$ consists of three generators $a_u$, $b_u$ and $c_u$ in bidegrees $(3,2)$, $(4,2)$ and $(6,3)$. It follows from the instability condition that $\mathrm{Sq}^4(a_u) = 0$. □

*Proof of Theorem 3.3.* From Remark 3.5 and Lemma 3.6, we deduce that $\mathcal{A}_1^{\mathbb{R}}$ is a type $(2,1)$ complex. To show that the bi-graded $\mathbb{R}$-motivic cohomology of $\mathcal{A}_1^{\mathbb{R}}$ is free as an $\mathcal{A}^{\mathbb{R}}(1)$-module, we make use of Corollary 2.2.

Since $\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})$ is a summand of a free $\mathbb{M}_2^{\mathbb{R}}$-module, it is projective as an $\mathbb{M}_2^{\mathbb{R}}$-module. In fact, $\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})$ is free, as projective modules over (graded) local rings are free. Also note that the elements

$$\overline{\mathrm{Q}}_0, \overline{\mathrm{P}}_1^1, \overline{\mathrm{Q}}_1 \in \mathcal{A}^{\mathbb{R}}(1)/(\rho,\tau)$$

are primitive. Hence we have a Kunneth isomorphism in the respective Margolis homologies, in particular we have,

$$\mathcal{M}(\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})/(\rho,\tau), x) = \overline{e}(\mathcal{M}(\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})/(\rho,\tau), x)^{\otimes 3})$$

for $x \in \{\overline{\mathrm{Q}}_0, \overline{\mathrm{P}}_1^1, \overline{\mathrm{Q}}_1\}$. Since $\dim_{\mathbb{F}_2} \mathcal{M}(\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})/(\rho,\tau), x) = 1$ for all $x \in \{\overline{\mathrm{Q}}_0, \overline{\mathrm{P}}_1^1, \overline{\mathrm{Q}}_1\}$, by Lemma 3.1

$$\mathcal{M}(\mathcal{A}_1^{\mathbb{R}}/(\rho,\tau), x) = 0$$

for $x \in \{\overline{\mathrm{Q}}_0, \overline{\mathrm{P}}_1^1, \overline{\mathrm{Q}}_1\}$. Thus, by Corollary 2.2 we conclude that $\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})$ is a free $\mathcal{A}^{\mathbb{R}}(1)$-module. A direct computation shows that

$$\dim_{\mathbb{F}_2} \mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})/(\rho,\tau) = 8,$$

hence $\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})$ is $\mathcal{A}^{\mathbb{R}}(1)$-free of rank one. □

### 3.3. The $\mathcal{A}^{\mathbb{R}}$-module structure

Using the description (8) and Cartan formula we make a complete calculation of the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})$. Let $a, b, c \in \mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})$ as in Proposition 3.11. In Figure 6 we provide a pictorial representation with the names of the generators that are in the image of the idempotent $\overline{e}$. For convenience we relabel the generators in Figure 6, where the indexing on a new label records the cohomological bidegrees of the corresponding generator. The following result is straightforward, and we leave it to the reader to verify.

**Lemma 3.12.** *In $\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})$, the underlying $\mathcal{A}^{\mathbb{R}}(1)$-module structure, along with the relations*

1. $\mathrm{Sq}^4(v_{0,0}) = \tau \cdot w_{4,1}$,
2. $\mathrm{Sq}^4(v_{1,0}) = w_{5,2}$,
3. $\mathrm{Sq}^4(v_{2,1}) = 0$,
4. $\mathrm{Sq}^4(v_{3,1}) = 0 = \mathrm{Sq}^4(w_{3,1})$,
5. $\mathrm{Sq}^8(v_{0,0}) = 0$,

*completely determine the $\mathcal{A}^{\mathbb{R}}$-module structure.*

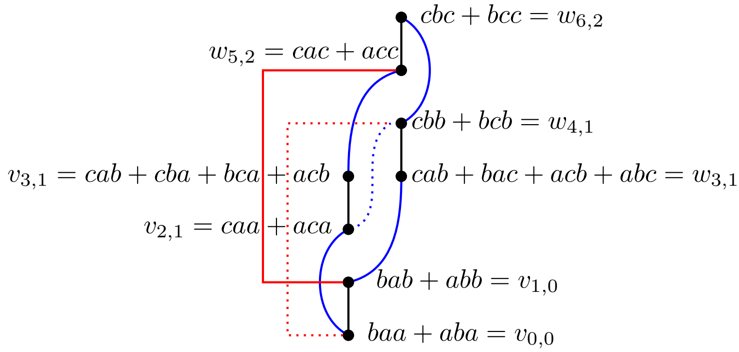



Figure 6: We depict the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathcal{A}_1)$. The straight (black), curved (blue), and boxed (red) lines represent the action of motivic $\mathrm{Sq}^1$, $\mathrm{Sq}^2$, and $\mathrm{Sq}^4$, respectively. Black dots represent $\mathbb{M}_2^{\mathbb{R}}$-generators, and a dotted line represents that the action hits the $\tau$-multiple of the given $\mathbb{M}_2^{\mathbb{R}}$-generator.

**Remark 3.13.** In upcoming work, we show that $\mathcal{A}^{\mathbb{R}}(1)$ admits 128 different $\mathcal{A}^{\mathbb{R}}$-module structures. Whether all of the 128 $\mathcal{A}^{\mathbb{R}}$-module structures can be realized by $\mathbb{R}$-motivic spectra, or not, is currently under investigation.

## 4. An $\mathbb{R}$-motivic $v_1$–self-map

With the construction of $\mathcal{A}_1^{\mathbb{R}}$, one might hope that any one of $\mathcal{Y}_{(i,j)}^{\mathbb{R}}$ fits into an exact triangle

$$\Sigma^{2,1}\mathcal{Y}_{(i,j)}^{\mathbb{R}} \xrightarrow{v} \mathcal{Y}_{(i,j)}^{\mathbb{R}} \longrightarrow \mathcal{A}_1^{\mathbb{R}} \longrightarrow \Sigma^{3,1}\mathcal{Y}_{(i,j)}^{\mathbb{R}} \xrightarrow{\Sigma v} \dots \tag{9}$$

in $\mathrm{Ho}(\mathbf{Sp}_{2,\mathrm{fin}}^{\mathbb{R}})$. The motivic weights prohibit $\mathcal{A}_1^{\mathbb{R}}$ from being the cofiber of a self-map on $\mathcal{Y}_{\mathrm{triv}}$ or $\mathcal{Y}_{(\mathsf{h},0)}$, as the 2-cell in these complexes appears in weight 0, whereas in $\mathcal{A}_1^{\mathbb{R}}$ the 2-cell is in weight 1. We will also see that the spectrum $\mathcal{Y}_{(2,1)}^{\mathbb{R}}$ cannot be a part of (9) because of its $\mathcal{A}^{\mathbb{R}}$-module structure (see Lemma 4.4). If $\mathcal{Y}_{(i,j)} = \mathcal{Y}_{(\mathsf{h},1)}^{\mathbb{R}}$ in (9), then the map $v$ will necessarily be a $v_{(1,\mathrm{nil})}$–self-map because $\mathcal{Y}_{(\mathsf{h},1)}^{\mathbb{R}}$ is of type $(1,1)$ and $\mathcal{A}_1^{\mathbb{R}}$ is of type $(2,1)$. The main purpose of this section is to prove Theorem 1.11 and Theorem 1.14 by showing that $\mathcal{Y}_{(\mathsf{h},1)}^{\mathbb{R}}$ does fit into an exact triangle very similar to (9)

$$\Sigma^{2,1}\mathcal{Y}_{(i,j)}^{\mathbb{R}} \xrightarrow{v} \mathcal{Y}_{(i,j)}^{\mathbb{R}} \longrightarrow \mathrm{C}^{\mathbb{R}}(v) \longrightarrow \Sigma^{3,1}\mathcal{Y}_{(i,j)}^{\mathbb{R}} \xrightarrow{\Sigma v} \dots,$$

where $\mathrm{C}^{\mathbb{R}}(v)$ is of type $(2,1)$ and $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(v)) \cong \mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})$ as $\mathcal{A}^{\mathbb{R}}$-modules.

**Remark 4.1.** The fact that $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(v))$ is isomorphic to $\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})$ as $\mathcal{A}^{\mathbb{R}}$-modules does not imply that $\mathrm{C}^{\mathbb{R}}(v)$ and $\mathcal{A}_1^{\mathbb{R}}$ are equivalent as $\mathbb{R}$-motivic spectra. There are a plethora of examples of Steenrod modules that are realized by spectra of different homotopy types.

We begin by discussing the $\mathcal{A}^{\mathbb{R}}$-module structures of $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$. Using Adem relations, one can show that the element

$$\overline{\mathrm{Q}}_1 := \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1 \in \mathcal{A}^{\mathbb{R}}(1)$$

squares to zero. Let $\Lambda(\overline{\mathrm{Q}}_1)$ denote the exterior subalgebra $\mathbb{M}_2^{\mathbb{R}}[\overline{\mathrm{Q}}_1]/(\overline{\mathrm{Q}}_1^2)$ of $\mathcal{A}^{\mathbb{R}}(1)$. Let $\mathcal{B}^{\mathbb{R}}(1)$ denote the $\mathcal{A}^{\mathbb{R}}(1)$-module

$$\mathcal{B}^{\mathbb{R}}(1) := \mathcal{A}^{\mathbb{R}}(1) \otimes_{\Lambda(\overline{\mathrm{Q}}_1)} \mathbb{M}_2^{\mathbb{R}}.$$

Both $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ are realizations of $\mathcal{B}^{\mathbb{R}}(1)$. In other words:

**Proposition 4.2.** *There is an isomorphism of $\mathcal{A}^{\mathbb{R}}(1)$-modules*

$$\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(i,j)}) \cong \mathcal{B}^{\mathbb{R}}(1)$$

*for $(i,j) \in \{(2,1),(\mathsf{h},1)\}$.*

*Proof.* By direct inspection, $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(i,j)})$ is cyclic as an $\mathcal{A}^{\mathbb{R}}(1)$-module for $(i,j) \in \{(2,1),(\mathsf{h},1)\}$. Thus we have an $\mathcal{A}^{\mathbb{R}}(1)$-module map

$$f_i\colon \mathcal{A}^{\mathbb{R}}(1) \to \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(i,j)}). \tag{10}$$

The result follows from the fact that $\overline{\mathrm{Q}}_1$ acts trivially on $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(i,j)})$ and a dimension counting argument. $\square$

**Remark 4.3.** Let $\{y_{0,0}, y_{1,0}\}$ be the $\mathbb{M}_2^{\mathbb{R}}$-basis of $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$ or $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(2))$, so that $\mathrm{Sq}^1(y_{0,0}) = y_{1,0}$, and let $\{x_{0,0}, x_{2,1}\}$ a basis of $\mathrm{C}^{\mathbb{R}}(\eta_{1,1})$, so that $\mathrm{Sq}^2(x_{0,0}) = x_{2,1}$. If we consider the $\mathbb{M}_2^{\mathbb{R}}$-basis $\{v_{0,0}, v_{1,0}, v_{2,1}, v_{3,1}, w_{3,1}, w_{3,2}, w_{4,2}, w_{5,3}, w_{6,3}\}$ of $\mathcal{A}^{\mathbb{R}}(1)$ from Subsection 3.3, then the maps $f_i$ of (10) are given as in Table 1.

Table 1: The maps $f_2$ and $f_{\mathsf{h}}$.

| $x$ | $f_2(x)$ | $f_{\mathsf{h}}(x)$ |
|---|---|---|
| $v_{0,0}$ | $y_{0,0}x_{0,0}$ | $y_{0,0}x_{0,0}$ |
| $v_{1,0}$ | $y_{1,0}x_{0,0}$ | $y_{1,0}x_{0,0}$ |
| $v_{2,1}$ | $y_{0,0}x_{2,0} + \rho \cdot y_{1,0}x_{0,0}$ | $y_{0,0}x_{2,0}$ |
| $v_{3,1}$ | $y_{1,0}x_{2,0}$ | $y_{1,0}x_{2,0}$ |
| $w_{3,1}$ | $y_{1,0}x_{2,0}$ | $y_{1,0}x_{2,0}$ |
| $w_{4,2}$ | $0$ | $0$ |
| $w_{5,3}$ | $0$ | $0$ |
| $w_{6,3}$ | $0$ | $0$ |

**Lemma 4.4.** *The $\mathcal{A}^{\mathbb{R}}$-module structures on $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(2,1)})$ and $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$ are given as in Figure 7.*

*Proof.* The result is an easy consequence of a calculation using the Cartan formula,

$$\mathrm{Sq}^4(xy) = \mathrm{Sq}^4(x)y + \tau\mathrm{Sq}^3(x)\mathrm{Sq}^1(y) + \mathrm{Sq}^2(x)\mathrm{Sq}^2(y) + \tau\mathrm{Sq}^1(x)\mathrm{Sq}^3(y) + x\mathrm{Sq}^4(y),$$

and the fact that $\mathrm{Sq}^2(y_{0,0}) = \rho y_{1,0}$ in $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(2))$, whereas $\mathrm{Sq}^2(y_{0,0})$ vanishes in $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$ (see Proposition 3.9). $\square$

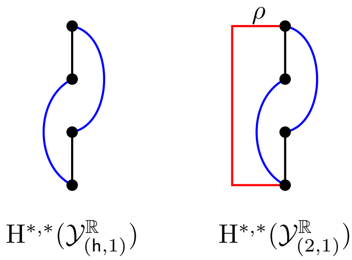



Figure 7: Straight (black), curved (blue), and boxed (red) lines represent the action of $\mathrm{Sq}^1$, $\mathrm{Sq}^2$, and $\mathrm{Sq}^4$, respectively. Black dots represent $\mathbb{M}_2^{\mathbb{R}}$-generators, and in the case of $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$, $\mathrm{Sq}^4$ on the bottom cell is $\rho$ times the top cell.

**Remark 4.5.** Comparing Lemma 4.4 and Lemma 3.12, we see that the $\mathcal{A}^{\mathbb{R}}(1)$-module map $f_2$, as in Remark 4.3, cannot be extended to a map of $\mathcal{A}^{\mathbb{R}}$-modules.

**Corollary 4.6.** *There is an exact sequence of* $\mathcal{A}^{\mathbb{R}}$*-modules*

$$0 \longrightarrow \mathrm{H}^{*,*}(\Sigma^{3,1}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \xrightarrow{\pi^*} \mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}}) \xrightarrow{\iota^*} \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \longrightarrow 0. \tag{11}$$

*Proof.* From the description of the map $f_{\mathsf{h}}$ in Remark 4.3, along with Lemma 3.12 and Lemma 4.4, it is easy to check that $f_{\mathsf{h}}$ extends to an $\mathcal{A}^{\mathbb{R}}$-module map and that

$$\ker f_{\mathsf{h}} \cong \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$$

as $\mathcal{A}^{\mathbb{R}}$-modules. □

The exact sequence (11) corresponds to a nonzero element in the $\mathrm{E}_2$-page of the ℝ-motivic Adams spectral sequence (also see Remark 4.8 and Remark 4.10)

$$\overline{v} \in \mathrm{Ext}^{2,1,1}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \wedge D\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathbb{M}_2^{\mathbb{R}}) \Rightarrow [\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}, \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}]_{2,1}, \tag{12}$$

where $D\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} := \mathrm{F}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}, \mathbb{S}_{\mathbb{R}})$ is the Spanier–Whitehead dual of $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$. If

**Notation 4.7.** Note that we follow [**DI1**, **BI**] in grading $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}$ as $\mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{R}}}$, where $s$ is the stem, $f$ is the Adams filtration, and $w$ is the weight. We will also follow [**GI1**] in referring to the difference $s - w$ as the *coweight.*

**Remark 4.8.** Since $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$ is $\mathbb{M}_2^{\mathbb{R}}$-free, an appropriate universal-coefficient spectral sequence collapses and we get $\mathrm{H}^{*,*}(D\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \cong \hom_{\mathbb{M}_2^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathbb{M}_2^{\mathbb{R}})$. Further, the Kunneth isomorphism of (7) gives us

$$\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \wedge D\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \cong \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathrm{H}^{*,*}(D\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}),$$

and therefore,

$$\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \wedge D\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})) \cong \mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})).$$

Theorem 1.11 follows immediately if we show that the element $\overline{v}$ is a nonzero permanent cycle. The following result implies that a $d_r$-differential (for $r \geqslant 2$) supported by $\overline{v}$ has no potential nonzero target.

**Proposition 4.9.** *For $f \geqslant 3$, $\mathrm{Ext}^{1,f,1}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})) = 0$.*

*Proof.* In order to calculate $\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}))$, we filter the spectrum $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ via the evident maps

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d, equal] & Y_1 \arrow[r] \arrow[d, equal] & Y_2 \arrow[r] \arrow[d, equal] & Y_3. \arrow[d, equal] \\ \mathbb{S}_{\mathbb{R}} & \mathrm{C}^{\mathbb{R}}(\mathsf{h}) & \mathrm{C}^{\mathbb{R}}(\mathsf{h}) \cup_{\mathbb{S}_{\mathbb{R}}} \mathrm{C}^{\mathbb{R}}(\eta_{1,1}) & \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \end{tikzcd}$$

Note that $\mathrm{H}^{*,*}(Y_j)$ are free $\mathbb{M}^{\mathbb{R}}_2$-modules. The above filtration results in cofiber sequences

$$Y_0 \longrightarrow Y_1 \longrightarrow \Sigma^{1,0}\mathbb{S}_{\mathbb{R}},$$

$$Y_1 \longrightarrow Y_2 \longrightarrow \Sigma^{2,1}\mathbb{S}_{\mathbb{R}}, \qquad \text{and}$$

$$Y_2 \longrightarrow Y_3 \longrightarrow \Sigma^{3,1}\mathbb{S}_{\mathbb{R}},$$

which induce short exact sequences of $\mathcal{A}^{\mathbb{R}}$-modules as the connecting map

$$\mathrm{C}^{\mathbb{R}}(Y_j \to Y_{j+1}) \longrightarrow \Sigma Y_j$$

induces the zero map in $\mathrm{H}^{*,*}(-)$. Thus, applying the functor $\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}_{(\mathsf{h},1)}), -)$ to these short-exact sequences, we get long exact sequences, which can be spliced together to obtain an Atiyah–Hirzebruch like spectral sequence

$$\begin{gathered} \mathrm{E}_1^{*,*,*,*} = \mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}_{(\mathsf{h},1)}), \mathbb{M}^{\mathbb{R}}_2)\{g_{0,0}, g_{1,0}, g_{2,1}, g_{3,1}\} \\ \Downarrow \\ \mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})). \end{gathered}$$

An element $x \cdot g_{i,j}$ in the $\mathrm{E}_2$-page contributes to the degree $|x| - (i, 0, j)$ of the abutment. Thus, Proposition 4.9 is a straightforward consequence of Proposition 4.11. □

**Remark 4.10.** Because, $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$ is $\mathbb{M}^{\mathbb{R}}_2$-free and finite, we have

$$\mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \cong \mathrm{hom}_{\mathbb{M}^{\mathbb{R}}_2}(\mathrm{H}^{*,*}(\mathcal{Y}_{(\mathsf{h},1)}), \mathbb{M}^{\mathbb{R}}_2),$$

and therefore, $\mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathbb{M}^{\mathbb{R}}_2) \cong \mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}))$.

**Proposition 4.11.** *For $f \geqslant 3$ and $(i, j) \in \{(0, 0), (1, 0), (2, 1), (3, 1)\}$, we have that*

$$\mathrm{Ext}^{1+i,f,1+j}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})) = 0.$$

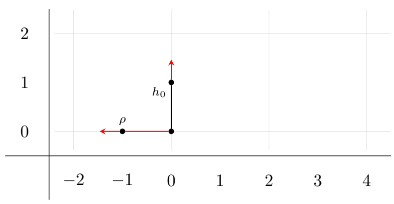
*Proof.* Our desired vanishing concerns only the groups $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}))$ in coweights 0, 1 and 2. These groups can be easily calculated starting from the computations of $\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathbb{M}^{\mathbb{R}}_2)$ in [**DI1**] and [**BI**] and using the short exact sequences in $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}$ arising from the cofiber sequences

$$\Sigma^{1,1} S_{\mathbb{R}} \xrightarrow{\eta_{1,1}} S_{\mathbb{R}} \longrightarrow \mathrm{C}^{\mathbb{R}}(\eta_{1,1}) \quad \text{and}$$

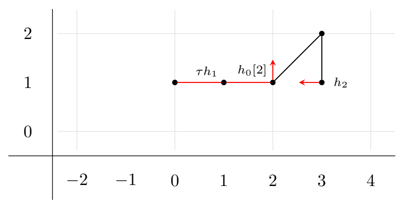
$$\mathrm{C}^{\mathbb{R}}(\eta_{1,1}) \xrightarrow{\mathsf{h}} \mathrm{C}^{\mathbb{R}}(\eta_{1,1}) \longrightarrow \mathrm{C}^{\mathbb{R}}(\mathsf{h}) \wedge \mathrm{C}^{\mathbb{R}}(\eta_{1,1}) = \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}.$$

We display $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathrm{H}_{*,*}(\mathrm{C}^{\mathbb{R}}(\eta_{1,1})))$ in coweights 0, 1 and 2 in the charts below. Here horizontal, vertical, or diagonal lines denote multiplication by $\rho$, $h_0$, and $h_1$, respectively.

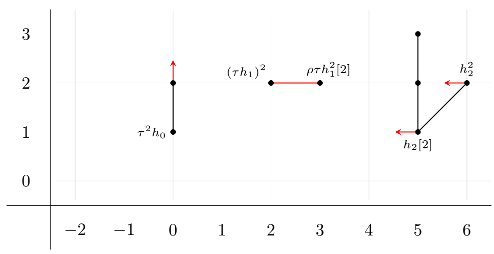
$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}_2, \mathrm{H}_{*,*}(\mathrm{C}^{\mathbb{R}}(\eta_{1,1}))$ in coweight zero




$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}_2, \mathrm{H}_{*,*}(\mathrm{C}^{\mathbb{R}}(\eta_{1,1}))$ in coweight one

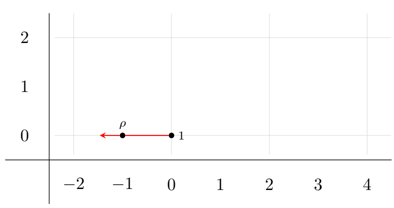



$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}_2, \mathrm{H}_{*,*}(\mathrm{C}^{\mathbb{R}}(\eta_{1,1}))$ in coweight two

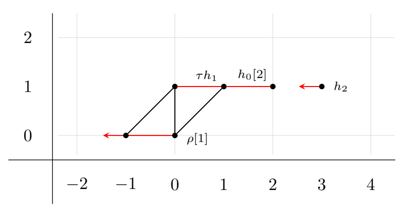



We find that $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}))$ is, in coweights zero, one, and two, also given by the charts below.

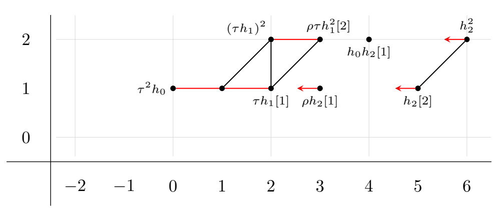
$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}_2, \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}))$ in coweight zero




$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}_2, \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}))$ in coweight one




$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}_2, \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}))$ in coweight two




The result follows from the above charts. ∎

**Remark 4.12.** One can also resolve Proposition 4.11 directly using the $\rho$-Bockstein spectral sequence

$$\begin{array}{c} \mathrm{E}_1 := \mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}_*}(\mathbb{F}_2[\tau], \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{C}}_{(\mathsf{h},1)})) \otimes \mathbb{F}_2[\rho] \\ \Downarrow \\ \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})) \end{array} \tag{13}$$

and identifying a vanishing region for $\mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{C}}_*}(\mathbb{F}_2[\tau], \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{C}}_{(\mathsf{h},1)}))$. Even a rough estimate of the vanishing region using the $\mathrm{E}_1$-page of the $\mathbb{C}$-motivic May spectral sequence leads to Proposition 4.11. Such an approach would avoid explicit calculations of $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}$ as in [**DI1**] and [**BI**].

*Proof of Theorem 1.11.* By Proposition 4.9 every map

$$v \colon \Sigma^{2,1}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \longrightarrow \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$$

detected by $\overline{v}$ of (12) is a nonzero permanent cycle. In order to finish the proof of Theorem 1.11 we must show that $v$ is necessarily a $v_{(1,\mathrm{nil})}$–self-map of periodicity 1. It is easy to see that the underlying map

$$\Phi^e(\beta(v)) \colon \Sigma^2\mathcal{Y} \longrightarrow \mathcal{Y}$$

is a $v_1$–self-map of periodicity 1 as

$$\mathrm{C}(\Phi^e(\beta(v))) \simeq \Phi^e(\beta(\mathrm{C}^{\mathbb{R}}(v))) \simeq \mathcal{A}_1[10]$$

is of type 1 (see Remark 3.5). On the other hand,

$$\Phi^{\mathrm{C}_2}(\beta(v)) \colon \Sigma^2(\Sigma\mathrm{M}_2(1) \vee \mathrm{M}_2(1)) \longrightarrow \Sigma\mathrm{M}_2(1) \vee \mathrm{M}_2(1)$$

is necessarily a nilpotent map because of [**HS**, Theorem 3(ii)] and the fact that a $v_1$–self-map of $\mathrm{M}_2(1)$ has periodicity at least 4 (see [**DM**] for details) which lives in $[\mathrm{M}_2(1), \mathrm{M}_2(1)]_{8k}$ for $k \geqslant 1$. □

*Proof of Theorem 1.14.* Since $v$ is a $v_{(1,\mathrm{nil})}$–self-map and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ is of type $(1,1)$, it follows that $\mathrm{C}^{\mathbb{R}}(v)$ is of type $(2,1)$. Moreover,

$$\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(v)) \cong \mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)$$

as $v$ is detected by $\overline{v}$ of (12) in the $\mathrm{E}_2$-page of the Adams spectral sequence. Thus, $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(v))$ is a free $\mathcal{A}^{\mathbb{R}}(1)$-module on single generator. □

**Remark 4.13.** It is likely that realizing a different $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathcal{A}^{\mathbb{R}}(1)$ as a spectrum (see also Remark 3.13) may lead to a 1-periodic $v_1$–self-map on $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ as well as on $\mathcal{Y}^{\mathrm{C}_2}_{(2,1)}$. We explore such possibilities in upcoming work.

## 5. Nonexistence of $v_{(1,0)}$–self-map on $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$

Let $X$ be a finite $\mathbb{R}$-motivic spectrum and let $f\colon \Sigma^{i,j}X \to X$ be a map such that

$$\Phi^{\mathrm{C}_2}(\beta(f))\colon \Sigma^{i-j}\Phi^{\mathrm{C}_2}(\beta(X)) \longrightarrow \Phi^{\mathrm{C}_2}(\beta(X))$$

is a $v_0$–self-map. Then it must be the case that $i = j$, as $v_0$–self-maps preserve dimension. Note that both $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$ are of type $(1, 0)$.

**Proposition 5.1.** *The $v_1$–self-maps of* $\mathrm{M}_2(1)$ *are not in the image of the underlying homomorphism*

$$\Phi^e \circ \beta\colon [\Sigma^{8k,8k}\mathrm{C}^{\mathbb{R}}(\mathsf{h}), \mathrm{C}^{\mathbb{R}}(\mathsf{h})]^{\mathbb{R}} \longrightarrow [\Sigma^{8k}\mathrm{M}_2(1), \mathrm{M}_2(1)].$$

*Proof.* The minimal periodicity of a $v_1$–self-map of $\mathrm{M}_2(1)$ is 4. Let $v\colon \Sigma^{8k}\mathrm{M}_2(1) \to \mathrm{M}_2(1)$ be a $4k$-periodic $v_1$–self-map. It is well-known that the composite

$$\Sigma^{8k}\mathbb{S} \hookrightarrow \Sigma^{8k}\mathrm{M}_2(1) \xrightarrow{v} \mathrm{M}_2(1) \longrightarrow \Sigma^1\mathbb{S} \tag{14}$$

is not null (and equals $P^{k-1}(8\sigma)$ where $P$ is a periodic operator given by the Toda bracket $\langle \sigma, 16, -\rangle$).

Suppose there exists $f\colon \Sigma^{8k,8k}\mathrm{C}^{\mathbb{R}}(\mathsf{h}) \to \mathrm{C}^{\mathbb{R}}(\mathsf{h})$ such that $\Phi^e \circ \beta(f) = v$. Then (14) implies that the composition

$$\Sigma^{8k,8k}\mathbb{S}_{\mathbb{R}} \hookrightarrow \Sigma^{8k,8k}\mathrm{C}^{\mathbb{R}}(\mathsf{h}) \xrightarrow{v} \mathrm{C}^{\mathbb{R}}(\mathsf{h}) \longrightarrow \Sigma^{1,0}\mathbb{S} \tag{15}$$

is nonzero as the functor $\Phi^e \circ \beta$ is additive. The composite of the maps in (15) is a nonzero element of $\pi_{*,*}(\mathbb{S}_{\mathbb{R}})$ in negative coweight. This contradicts the fact that $\pi_{*,*}(\mathbb{S}_{\mathbb{R}})$ is trivial in negative coweights [**DI1**]. □

**Proposition 5.2.** *The $v_1$–self-maps of $\mathcal{Y}$ are not in the image of the underlying homomorphism*

$$\Phi^e \circ \beta\colon [\Sigma^{2k,2k}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}, \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}]^{\mathbb{R}} \longrightarrow [\Sigma^{8k}\mathcal{Y}, \mathcal{Y}].$$

*Proof.* Let $v\colon \Sigma^{2k}\mathcal{Y} \to \mathcal{Y}$ denote a $v_1$–self-map of periodicity $k$. Notice that the composite

$$\mathrm{S}^{2k} \hookrightarrow \Sigma^{2k}\mathcal{Y} \xrightarrow{v} \mathcal{Y} \longrightarrow \mathcal{Y}_{\geqslant 1}, \tag{16}$$

where $\mathcal{Y}_{\geqslant 1}$ is the first coskeleton, must be nonzero. If not, then $v$ factors through the bottom cell resulting in a map $\mathrm{S}^{2k} \to \Sigma^{2k}\mathcal{Y} \to \mathbb{S}$ which induces an isomorphism in $\mathrm{K}(1)$-homology, contradicting the fact that $\mathbb{S}$ is of type 0.

If $f\colon \Sigma^{2k,2k}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)} \to \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$ were a map such that $\Phi^e \circ \beta(f) = v$, then (16) would force one among the hypothetical composites $(A)$, $(B)$ or $(C)$ in the diagram

$$\begin{tikzcd}
\Sigma^{2k,2k}\mathbb{S}_{\mathbb{R}} \arrow[r, hook] & \Sigma^{2k,2k}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)} \arrow[r] \arrow[dr, dashed] \arrow[ddr, dashed] & \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)} \arrow[r, dashed, "p_3"] & \Sigma^{3,0}\mathbb{S}_{\mathbb{R}} & (A) \\
 & & \mathrm{Fib}(p_3) \arrow[u, hook] \arrow[r, dashed, "p_2"] & \Sigma^{2,0}\mathbb{S}_{\mathbb{R}} & (B) \\
 & & \mathrm{Fib}(p_2) \arrow[u, hook] \arrow[r, dashed, "p_1"] & \Sigma^{1,0}\mathbb{S}_{\mathbb{R}} & (C)
\end{tikzcd}$$

to exist as a nonzero map, thereby contradicting the fact that $\pi_{*,*}(\mathbb{S}_{\mathbb{R}})$ is trivial in negative coweights. $\square$

**Remark 5.3.** The above results do not preclude the existence of a $v_{(1,0)}$–self-map on $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ and $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},0)}$. Forthcoming work [**GI2**] of the second author and Isaksen shows that $8\sigma$ is in the image of $\Phi^e\colon \pi_{7,8}(\mathbb{S}_{\mathrm{C}_2}) \longrightarrow \pi_7(\mathbb{S})$ and suggests that $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ supports a $v_{(1,0)}$–self-map.

## References


[AM] J. F. Adams and H. R. Margolis, *Modules over the Steenrod algebra*, Topology **10** (1971), 271–282, DOI 10.1016/0040-9383(71)90020-6. MR294450

[B] Paul Balmer, *The spectrum of prime ideals in tensor triangulated categories*, J. Reine Angew. Math. **588** (2005), 149–168, DOI 10.1515/crll.2005.2005.588.149. MR2196732

[BS] Paul Balmer and Beren Sanders, *The spectrum of the equivariant stable homotopy category of a finite group*, Invent. Math. **208** (2017), no. 1, 283–326, DOI 10.1007/s00222-016-0691-3. MR3621837

[BGH] Tobias Barthel, J. P. C. Greenlees, and Markus Hausmann, *On the Balmer spectrum for compact Lie groups*, Compos. Math. **156** (2020), no. 1, 39–76, DOI 10.1112/s0010437x19007656.

[BHHM] M. Behrens, M. Hill, M. J. Hopkins, and M. Mahowald, *On the existence of a $v_2^{32}$-self map on $M(1,4)$ at the prime 2*, Homology Homotopy Appl. **10** (2008), no. 3, 45–84. MR2475617

[BP] Mark Behrens and Satya Pemmaraju, *On the existence of the self map $v_2^9$ on the Smith–Toda complex $V(1)$ at the prime 3*, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic $K$-theory, Contemp. Math., vol. 346, Amer. Math. Soc., Providence, RI, 2004, pp. 9–49, DOI 10.1090/conm/346/06284. MR2066495

[BeS] Mark Behrens and Jay Shah, *$C_2$-equivariant stable homotopy from real motivic stable homotopy*, Ann. K-Theory **5** (2020), no. 3, 411–464, DOI 10.2140/akt.2020.5.411. MR4132743

[BE] Prasit Bhattacharya and Philip Egger, *A class of 2-local finite spectra which admit a $v_2^1$-self-map*, Adv. Math. **360** (2020), 106895, 40, DOI 10.1016/j.aim.2019.106895. MR4031119

[BEM] Prasit Bhattacharya, Philip Egger, and Mark Mahowald, *On the periodic $v_2$-self-map of $A_1$*, Algebr. Geom. Topol. **17** (2017), no. 2, 657–692, DOI 10.2140/agt.2017.17.657. MR3623667

[BI] Eva Belmont and Daniel Isaksen, *$\mathbb{R}$-motivic stable stems* (2020), available at https://arxiv.org/abs/2001.03606.

[DM] Donald M. Davis and Mark Mahowald, *$v_1$- and $v_2$-periodicity in stable homotopy theory*, Amer. J. Math. **103** (1981), no. 4, 615–659, DOI 10.2307/2374044. MR623131

[DI1] Daniel Dugger and Daniel C. Isaksen, *Low-dimensional Milnor–Witt stems over* $\mathbb{R}$, Ann. K-Theory **2** (2017), no. 2, 175–210, DOI 10.2140/akt.2017.2.175. MR3590344

[DI2] ______, *The motivic Adams spectral sequence*, Geom. Topol. **14** (2010), no. 2, 967–1014, DOI 10.2140/gt.2010.14.967. MR2629898

[DI3] ______, *Motivic cell structures*, Algebr. Geom. Topol. **5** (2005), 615–652, DOI 10.2140/agt.2005.5.615. MR2153114

[DI4] ______, *Motivic Hopf elements and relations*, New York J. Math. **19** (2013), 823–871. MR3141814

[GI1] Bertrand J. Guillou and Daniel C. Isaksen, *The Bredon–Landweber region in* $C_2$*-equivariant stable homotopy groups*, Doc. Math. **25** (2020), 1865–1880. MR4184454

[GI2] ______, $C_2$*-equivariant stable stems*, in progress.

[HK] Drew Heard and Achim Krause, *Vanishing lines for modules over the motivic Steenrod algebra*, New York J. Math. **24** (2018), 183–199. MR3778499

[HO] Jeremiah Heller and Kyle M. Ormsby, *Primes and fields in stable motivic homotopy theory*, Geom. Topol. **22** (2018), no. 4, 2187–2218, DOI 10.2140/gt.2018.22.2187. MR3784519

[H] Michael A. Hill, *Ext and the motivic Steenrod algebra over* $\mathbb{R}$, J. Pure Appl. Algebra **215** (2011), no. 5, 715–727, DOI 10.1016/j.jpaa.2010.06.017. MR2747214

[HS] Michael J. Hopkins and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. II*, Ann. of Math. (2) **148** (1998), no. 1, 1–49, DOI 10.2307/120991. MR1652975

[K] Achim Krause, *Periodicity in motivic homotopy theory and over* $BP_*BP$, 2018. Thesis (Ph.D.)–Rheinische Friedrich-Wilhelms-Universität Bonn.

[LS] Richard Gustavus Larson and Moss Eisenberg Sweedler, *An associative orthogonal bilinear form for Hopf algebras*, Amer. J. Math. **91** (1969), 75–94, DOI 10.2307/2373270. MR240169

[M1] H. R. Margolis, *Spectra and the Steenrod algebra*, North-Holland Mathematical Library, vol. 29, North-Holland Publishing Co., Amsterdam, 1983. Modules over the Steenrod algebra and the stable homotopy category. MR738973

[M2] Fabien Morel, *An introduction to* $\mathbb{A}^1$*-homotopy theory*, Contemporary developments in algebraic $K$-theory, ICTP Lect. Notes, XV, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004, pp. 357–441. MR2175638

[N] Lee S. Nave, *The Smith–Toda complex* $V((p+1)/2)$ *does not exist*, Ann. of Math. (2) **171** (2010), no. 1, 491–509, DOI 10.4007/annals.2010.171.491. MR2630045

[R1] Douglas C. Ravenel, *Nilpotence and periodicity in stable homotopy theory*, Annals of Mathematics Studies, vol. 128, Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith. MR1192553

[R2] Nicolas Ricka, *Subalgebras of the $\mathbb{Z}/2$-equivariant Steenrod algebra*, Homology Homotopy Appl. **17** (2015), no. 1, 281–305, DOI 10.4310/HHA.2015.v17.n1.a14. MR3350083

[T] Hirosi Toda, *On spectra realizing exterior parts of the Steenrod algebra*, Topology **10** (1971), 53–65, DOI 10.1016/0040-9383(71)90017-6. MR271933

[V] Vladimir Voevodsky, *Reduced power operations in motivic cohomology*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), 1–57, DOI 10.1007/s10240-003-0009-z. MR2031198

Prasit Bhattacharya pbhattac@nd.edu

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA

Bertrand Guillou bertguillou@uky.edu

Department of Mathematics, The University of Kentucky, Lexington, KY 40506, USA

Ang Li ang.li1414201@uky.edu

Department of Mathematics, The University of Kentucky, Lexington, KY 40506, USA