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Gradient and Eigenvalue Estimates on the Canonical Bundle of Kähler Manifolds

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Abstract

We prove certain gradient and eigenvalue estimates, as well as the heat kernel estimates, for the Hodge Laplacian on $(m, 0)$ forms, i.e., sections of the canonical bundle of Kähler manifolds, where m is the complex dimension of the manifold. Instead of the usual dependence on curvature tensor, our condition depends only on the Ricci curvature bound. The proof is based on a new Bochner type formula for the gradient of $(m, 0)$ forms, which involves only the Ricci curvature and the gradient of the scalar curvature.

Keywords Eigenvalue · Gradient estimates · Kahler manifold

Contents

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1 Introduction

In this paper, we prove, under conditions on the Ricci curvature alone, certain gradient and eigenvalue estimates for solutions of some elliptic and parabolic equations involving the Hodge Laplacian on sections of the canonical bundle of Kähler manifolds. Let us recall that a section of the canonical bundle of a Kähler manifold with complex dimension m is an $(m, 0)$ form. Since the appearance of De Rham and Hodge theory, there has been a vast literature on the study of Hodge Laplacian acting on differential forms due to its significance to analysis, geometry and topology. See for example the papers [1,3–9,11,18]. The general paradigm for gradient and eigenvalue estimates on p forms is that results concerning 0 forms e.g. scalar functions involve the Ricci curvature and results for one and higher forms involve the curvature tensor in general. The reason is that the Bochner–Weitzenböck formula for p forms with $p \geq 1$ involve the full curvature tensor. One exception is for spin bundles where the Lichnerowicz formula involves only the scalar curvature. In this paper we manage to find another exception: $(m, 0)$ forms on Kähler manifolds. In this case the Bochner–Weitzenböck formula for gradients involves only the Ricci curvature and the gradient of the scalar curvature. One implication of this result is that gradient estimates on heat kernels and eigenvalue estimates for $(m, 0)$ forms on Calabi–Yau manifolds are free of curvature conditions. We mention that for holomorphic sections in the canonical bundle, it is well known that the Bochner formula for gradient of the sections depends only on the Ricci curvature. See [19] (5.15) e.g. This fact plays an important role in complex differential geometry, especially on the parts involving the Bergman kernel. What we did here is removing the holomorphic condition.

The main results of the paper are Theorem 1.1 below and Lemma 2.1 in the next section. In the theorem, we obtain a qualitatively sharp lower bound for all eigenvalues of the Hodge Laplacian on the canonical bundle of Kähler manifolds. The curvature conditions are bounded Ricci curvature and nonnegative scalar curvature which is positive somewhere. Similar lower bounds for eigenvalues higher than certain Betti numbers and under bounded curvature assumption are proven in Wang–Zhou [18] for general compact manifolds and forms. Finding a lower bound for eigenvalues has been an active research topic for many years. We mention that although an $(m, 0)$ form looks like a scalar function, it behaves more like a vector field since the coefficient are complex valued. For this reason we are unable to extend the classical method of integrating a gradient bound in Li–Yau [13] to bound the first eigenvalue from below, which is the first step in bounding other eigenvalues from below.

Before stating the main results, we need to set up the conventions first. Let $(\mathbf{M}, g_{i\bar{j}})$ be a compact Kähler manifold of complex dimension m. Denote by Δ_d and $\Delta_{\bar{\partial}}$ the real and complex Hodge Laplacian, respectively. Denote by $\Delta = -\nabla^*\nabla$ the complex rough Laplacian, which is $\nabla_i \nabla_{\bar{i}}$ in local normal coordinates, then $\Delta = -\Delta_{\bar{\partial}} = -\frac{1}{2}\Delta_d$ when they act on functions and $(p, 0)$ forms (see e.g. Theorem 6.1 on p119 in [16]). Let Ric stand for the Ricci curvature tensor, and $R_{i\bar{j}}$ is the component of Ric in holomorphic coordinates. Let R be the scalar curvature. The volume of $\mathbf{M}$ is denoted by $|\mathbf{M}|$, and the diameter of $\mathbf{M}$ is denoted by $diam(\mathbf{M})$. Let $K_{\mathbf{M}} = \Lambda^m(T^*\mathbf{M})$ be the canonical bundle of $\mathbf{M}$, and $L^2(\mathbf{M}, K_{\mathbf{M}})$ the space of all smooth sections of $K_{\mathbf{M}}$, i.e., $(m, 0)$ forms on $\mathbf{M}$.

Theorem 1.1 *Let $(\mathbf{M}^m, g_{i\bar{j}})$ be a compact Kähler manifold of complex dimension $m \geq 2$. Denote by λ_k the k-th eigenvalue of the Hodge Laplacian Δ_d acting on $L^2(\mathbf{M}, K_{\mathbf{M}})$, and λ_1^0 the first positive eigenvalue of the scalar Laplacian. Suppose that the Ricci curvature satisfies $|Ric| \leq K$ and the scalar curvature satisfies: $R \geq 0$ and is positive somewhere. Then the following lower bounds are true.*

$$\begin{aligned} &\lambda_1 \geq \min\{\frac{1}{2}\lambda_1^0 + \inf_{\mathbf{M}} R, \quad \frac{\lambda_1^0}{2(\lambda_1^0 + \sup R)} \frac{1}{|\mathbf{M}|} \int_{\mathbf{M}} R\} \equiv c_0 > 0; \\ &\lambda_k \geq c_1 k^{\frac{1}{m}}, \qquad \forall\, k \geq 2, \end{aligned} \tag{1.1}$$

where $c_1 = \min\{c_0, \left(\frac{m}{\Lambda(m+1)}\right)^{\frac{1}{m}}\}$, $\Lambda = 4^{m(m+2)} m^{m+3} C_S^m \left(1 + \frac{K + |\mathbf{M}|^{-\frac{1}{m}}}{c_0}\right)^{m+1} |\mathbf{M}|$, and C_S is the L^2 scalar Sobolev constant, namely, the smallest constant such that

$$\left(\int_{\mathbf{M}} |f|^{\frac{2m}{m-1}}\right)^{\frac{m-1}{m}} \leq C_S \left(\int_{\mathbf{M}} |\nabla f|^2 + |\mathbf{M}|^{-\frac{1}{m}} \int_{\mathbf{M}} |f|^2\right) \tag{1.2}$$

for any smooth function f on $\mathbf{M}$.

Remark 1.2 For Riemann surfaces, i.e., the case $m = 1$, the picture of λ_1 is clearer. We may start from the Euler characteristic $\chi(\mathbf{M}) = b_0 - b_1 + b_2 = 2b_0 - b_1 = 2 - b_1 = 2 - 2g$, where b_k is the kth Betti number, and g is the genus of $\mathbf{M}$. Thus, it follows that $b_1 = g$. So when $g \geq 1$, we have $\lambda_1 = 0$ since $2h^{1,0} = b_1 > 0$, where $h^{1,0}$ is the $(1, 0)$ Hodge number.

When $g = 0$, i.e., $b_1 = 0$, one concludes that $\lambda_1 > 0$. Let a (1,0) form ϕ be an eigenform of λ_1. Since $\partial\phi = 0$ and $\lambda_1 > 0$, it follows that $\partial^*\phi \neq 0$ and $\Delta_d \partial^*\phi = \partial^* \Delta_d \phi = \lambda_1 \partial^*\phi$. In other words, $\partial^*\phi$ is an eigenfunction of Δ_d. Thus, one gets $\lambda_1 \geq \lambda_1^0$.

Remark 1.3 It is well known that

$$\int_{\mathbf{M}} R\, \omega^n = m\pi \int_{\mathbf{M}} \mathbf{c}_1 \wedge [\omega]^{m-1},$$

which is independent of the choice of the Kähler metric ω in the Kähler class. Here $\mathbf{c}_1$ is the first Chern class of $\mathbf{M}$.

Remark 1.4 In general, it is impossible to find a positive lower bound for λ_1, the first eigenvalue. Indeed, λ_1 can be 0 if the scalar curvature is 0. It is a standard fact that the Hodge number $h^{m,0}$ for Calabi–Yau m-manifolds is 1, i.e. the space of harmonic $(m, 0)$ forms is one dimensional, and therefore $\lambda_1 = 0$.

On the other hand, using the fact that the Bochner formula for all $(p, 0)$ forms involve only the Ricci curvature, one can prove an upper bound for the heat kernel of the Hodge Laplacian on $(p, 0)$ forms which depends only on volume of $\mathbf{M}$ and the

bound on Ricci curvature. See for example Theorem 3.1 below. Then it is well known (c.f. Sect. 5 in [14]) that the upper bound for the trace of the heat kernel implies

$$\lambda_j \geq C_1\, j^{1/m}$$

for $j \geq j_0 > 0$. Here j_0 and C_1 depends only on m, p, $|\mathbf{M}|$ and K, the Ricci bound.

Remark 1.5 In the theorem above, the parameters c_0 and C_S can be explicitly estimated in terms of geometric quantities. Following Li–Yau, (see [13] and Li' book [10] Theorem 5.7, and Yang [20]), There is a dimensional constant C_2 such that

$$\lambda_1^0 \geq \frac{\pi^2}{diam(\mathbf{M})^2} \exp(-C_2\, diam(\mathbf{M})\sqrt{K}).$$

As well known by the work of Zhong–Yang [22], if $Ric \geq 0$, then a sharp lower bound holds: $\lambda_1^0 \geq \frac{\pi^2}{diam(\mathbf{M})^2}$.

The Sobolev constant C_S depends on m, lower bound K of the Ricci curvature, diameter upper bound D and volume lower bound V of $\mathbf{M}$ (see e.g., [2,21]). Indeed, by a combination of the results in [10] and [2], the following explicit upper bound of C_S and hence a lower bound of λ_k can be established.

$$C_S \leq 2^{\frac{m-1}{m(2m-1)}} \max\left\{\left(\frac{m-1}{2m-1} IN_\alpha(\mathbf{M})\right)^{-2}, 2^{\frac{2m-3}{m-1}}\right\}, \tag{1.3}$$

with

$$IN_\alpha(\mathbf{M}) \geq \left(\frac{1}{4\omega_n^{n-1}\omega_{n-1}}\right)^{\frac{1}{n}} \left(\frac{|\mathbf{M}|}{\int_0^D [\sqrt{K^{-1}}\sinh(\sqrt{K}r)]^{n-1}dr}\right)^{\frac{n+1}{n}}. \tag{1.4}$$

Here $n = 2m$ is the real dimension of $\mathbf{M}$, and ω_n denotes the volume of the unit n-sphere.

The proof goes as follows. Let $\alpha = \frac{n}{n-1}$, and $\beta = \frac{2\alpha-2}{2-\alpha} = \frac{2}{n-2}$. Using the inequality

$$1 - x^\beta \leq (1-x)^\beta \leq 2^{1-\beta} - x^\beta,\ 0 \leq x, \beta \leq 1,$$

one can see that in the proof of Corollary 9.9 in [10], one can take $C_3 = 2^{1-\beta} = 2^{\frac{n-4}{n-2}}$, $C_4 = 1$, $C_5 = C_6 = 2C_3$, and the inequality (9.8) therein becomes

$$\int_{\mathbf{M}} |\nabla f|^2 \geq \left(\frac{n-2}{2n-2} SN_\alpha(\mathbf{M})\right)^2 \times \left[2^{-\frac{n-2}{n(n-1)}} \left(\int_{\mathbf{M}} |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} - C_5|\mathbf{M}|^{-\frac{2}{n}} \int_{\mathbf{M}} |f|^2,\right]. \tag{1.5}$$

Furthermore, according to Theorem 9.6 in [10] and Theorem 13 in [2], the constant $SN_\alpha(\mathbf{M})$ satisfies the following estimates:

$$IN_\alpha(\mathbf{M}) \leq SN_\alpha(\mathbf{M}) \leq 2^{1/n} IN_\alpha(\mathbf{M}),$$

where the Neumann isoperimetric constant IN_α satisfies (1.4). Combining the estimates above and replacing n by $2m$, one can get the upper bound of C_S in (1.3).

The rest of the paper is organized as follows. In Sect. 2, we will state and prove the Bochner type formula for the gradient of $(m, 0)$ forms. It is well known, c.f. [16], that the Bochner formula for all $(p, 0)$ forms with $p = 1, 2, \cdots, m$ involve only the Ricci curvature. As mentioned, what we will prove here is that for the gradient of $(m, 0)$ forms, the Bochner formula involve only the Ricci curvature and gradient of the scalar curvature. With this formula in hand, we will roughly follow the steps in [9] and [18] to prove Theorem 1.1. One difference is that we need to find a reasonable condition such that the first eigenvalue λ_1 has a positive lower bound, which as mentioned could be 0. It turns out that this is related to the first positive eigenvalue of the scalar Laplacian and the total scalar curvature and hence the Kähler class. Then we need to prove a mean value inequality for the eigenfunctions and their gradients. These will be used in an algebraic iteration process to prove the lower bound.

In Sect. 3, we extend some gradient estimates to the corresponding heat equation. Just like the Hodge Laplacian, heat equation on $(m, 0)$ forms is of interest to several areas. See for example a recent paper [15].

2 New Bochner Formula and Elliptic Estimates

In this section, we will first present the Bochner formula as the main lemma. Then Theorem 1.1 will be proven at the end of the section after a number of intermediate results stated as lemmas.

We will need of the following basic formulas.

Denote by $\{z_1, z_2, \cdots, z_m\}$ a local holomorphic coordinate system. Let the Hermitian metric be

$$h = \sum_{i,j=1,2,\cdots,m} g_{i\bar{j}} dz^i \otimes d\bar{z}^j,$$

where the underlying Remannian metric is given by $g = 2Re(h)$. The Christoffel symbols are given by

$$\Gamma^i_{jk} = g^{i\bar{l}} \frac{\partial g_{j\bar{l}}}{\partial z_k} = g^{i\bar{l}} \frac{\partial g_{k\bar{l}}}{\partial z_j}.$$

The curvature and Ricci curvature tensors are defined by

$$[\nabla_{\frac{\partial}{\partial \bar{z}_i}}, \nabla_{\frac{\partial}{\partial \bar{z}_j}}] \frac{\partial}{\partial z_k} = -\frac{\partial \Gamma^l_{ik}}{\partial \bar{z}_j} \frac{\partial}{\partial z_l} = R^l_{i\bar{j}k} \frac{\partial}{\partial z_l},$$

$$[\nabla_{\frac{\partial}{\partial z_i}}, \nabla_{\frac{\partial}{\partial \bar{z}_j}}]dz^l = \frac{\partial \Gamma^l_{ik}}{\partial \bar{z}_j} dz^k = -R^l_{i\bar{j}k} dz^k, \tag{2.1}$$

$$g_{s\bar{l}} R^s_{i\bar{j}k} = R_{i\bar{j}k\bar{l}}, \ g^{l\bar{s}} R_{i\bar{j}k\bar{s}} = R^l_{i\bar{j}k}, \tag{2.2}$$

and

$$R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = R^k_{i\bar{j}k} = -\partial_i \partial_{\bar{j}} \log \det(g). \tag{2.3}$$

Here and in the following, Einstein summation convention is used, namely, repeated indices are implicitly summed over. Readers can also refer to the book of Morrow–Kodaira [16] Chapter 3 for more details and formulas for Kähler geometry. Notice that the curvature tensor in [16] differs from the one defined in (2.1) by a negative sign.

Thus, for any (p, q) form

$$\phi = \sum_{\substack{i_1 < i_2 \cdots < i_p \\ j_1 < j_2 < \cdots < j_q}} \phi_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}, \tag{2.4}$$

the following Ricci identity holds in local normal coordinates.

$$\begin{aligned} &(\nabla_a \nabla_{\bar{b}} - \nabla_{\bar{b}} \nabla_a)\phi \\ &= -\sum_{t=1}^{p} R^l_{a\bar{b}i_t} \phi_{i_1 \cdots i_{t-1} l i_{t+1} \cdots i_p \bar{j}_1 \cdots \bar{j}_q} - \sum_{t=1}^{q} R^{\bar{l}}_{a\bar{b}\bar{j}_t} \phi_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_{t-1} \bar{l}\ \bar{j}_{t+1} \cdots \bar{j}_q} \\ &= -\sum_{t=1}^{p} R_{a\bar{b}i_t\bar{l}} \phi_{i_1 \cdots i_{t-1} l i_{t+1} \cdots i_p \bar{j}_1 \cdots \bar{j}_q} - \sum_{t=1}^{q} R_{a\bar{b}\bar{j}_t l} \phi_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_{t-1} \bar{l}\ \bar{j}_{t+1} \cdots \bar{j}_q}. \end{aligned} \tag{2.5}$$

Similar identities also hold for p covariant and q contravariant tensors. The Kähler form and Ricci form are

$$\omega = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j \text{ , and } \rho = \frac{\sqrt{-1}}{2} R_{i\bar{j}} dz^i \wedge d\bar{z}^j, \tag{2.6}$$

respectively.

Also, during the course of proofs below, there are situations in which complex gradient and Laplacian on real functions need to be transferred into the real ones, e.g. using min-max definition to get the first nonzero eigenvalue of the Laplacian on functions. For this purpose, be aware that

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right),$$

and

$$g_{i\bar{j}} = \frac{1}{2}\left(h_{ij} + \sqrt{-1}h_{i,m+j}\right), \ \textit{and} \ g^{i\bar{j}} = 2\left(h^{ij} - \sqrt{-1}h^{i,m+j}\right)$$

where $z_i = x_i + \sqrt{-1}y_i$, $h_{ij} = h(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$, and $h_{i,m+j} = h(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j})$. Thus, for a real function f, we have

$$\begin{aligned}|\nabla f|^2 =&|\nabla_i f|^2 + |\nabla_{\bar{i}} f|^2 = 2g^{i\bar{j}}\nabla_i f \nabla_{\bar{j}} f\\ =&h^{ij}\nabla_{\frac{\partial}{\partial x_i}} f \nabla_{\frac{\partial}{\partial x_j}} f + h^{m+i,m+j}\nabla_{\frac{\partial}{\partial y_i}} f \nabla_{\frac{\partial}{\partial y_j}} f\\ &+ h^{i,m+j}\nabla_{\frac{\partial}{\partial x_i}} f \nabla_{\frac{\partial}{\partial y_j}} f + h^{m+i,j}\nabla_{\frac{\partial}{\partial y_i}} f \nabla_{\frac{\partial}{\partial x_j}} f,\end{aligned}$$

and

$$\begin{aligned}\Delta f =&\frac{1}{2}g^{i\bar{j}}\left(\nabla_i\nabla_{\bar{j}} + \nabla_{\bar{j}}\nabla_i\right) f\\ =&\frac{1}{2}\left(h^{ij}\nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}} + h^{m+i,m+j}\nabla_{\frac{\partial}{\partial y_i}}\nabla_{\frac{\partial}{\partial y_i}} + h^{i,m+j}\nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial y_j}} + h^{m+i,j}\nabla_{\frac{\partial}{\partial y_i}}\nabla_{\frac{\partial}{\partial x_j}}\right) f.\end{aligned}$$

In other words, the norm of the complex and real gradients of f are the same, and the complex Laplacian is half of the real Laplacian.

The next lemma presents a Bochner type formula involving only the Ricci curvature and the gradient of the scalar curvature. As mentioned, usually such results are expected to hold only for scalar functions. Also it does not follow from the Bochner formula for $(m, 1)$ forms where only the Ricci curvature appears. The reason is that the covariant derivative of an $(m, 0)$ form is not necessarily an $(m, 1)$ form. Later we will show that, in many situations, the gradient of the scalar curvature can be integrated out, leaving the dependence only on Ricci curvature.

Lemma 2.1 *[Main Lemma] Let ϕ be a smooth $(m, 0)$ form on a Kähler manifold* $\mathbf{M}$ *of complex dimension m. Then*

$$\begin{aligned}\Delta|\nabla\phi|^2 =&|\nabla^2\phi|^2 + <\nabla\Delta\phi, \nabla\phi> + <\nabla\phi, \nabla\Delta\phi> + R|\nabla\phi|^2 + <\phi\nabla^{1,0}R, \nabla^{1,0}\phi>\\ &+ <\nabla^{1,0}\phi, \phi\nabla^{1,0}R> + 3Ric(\nabla^{1,0}\phi, \nabla^{1,0}\phi) - Ric(\nabla^{0,1}\phi, \nabla^{0,1}\phi)\\ \equiv&|\nabla^2\phi|^2 + <\nabla\Delta\phi, \nabla\phi> + <\nabla\phi, \nabla\Delta\phi> + R|\nabla\phi|^2 + <\phi\nabla_j R, \nabla_j\phi>\\ &+ <\nabla_j\phi, \phi\nabla_j R> + 3 < R_{j\bar{k}}\nabla_k\phi, \nabla_j\phi> - < R_{k\bar{j}}\nabla_{\bar{k}}\phi, \nabla_{\bar{j}}\phi>;\end{aligned} \tag{2.7}$$

where $R = g^{i\bar{j}}R_{i\bar{j}}$ is the scalar curvature of $\mathbf{M}$*, and $\nabla^{1,0}\phi = \nabla_i\phi_{12\cdots m}dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m \otimes dz^i$ and $\nabla^{0,1}\phi = \nabla_{\bar{i}}\phi_{12\cdots m}dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m \otimes d\bar{z}^i$.*

In particular, for Kähler–Einstein manifold $R_{i\bar{j}} = \mu g_{i\bar{j}}$, $\mu = \frac{R}{m}$, the Bochner formula becomes

$$\begin{aligned}\Delta|\nabla\phi|^2 =& |\nabla^2\phi|^2 + <\nabla\Delta\phi, \nabla\phi> + <\nabla\phi, \nabla\Delta\phi> + m\mu|\nabla\phi|^2\\ &+3\mu|\nabla_j\phi|^2 - \mu|\nabla_{\bar{j}}\phi|^2.\end{aligned} \tag{2.8}$$

Furthermore, if $\mathbf{M}$ *is a Calabi–Yau manifold, i.e.,* $\mu = 0$*, then the following curvature free identity holds:*

$$\Delta|\nabla\phi|^2 = |\nabla^2\phi|^2 + < \nabla\Delta\phi, \nabla\phi > + < \nabla\phi, \nabla\Delta\phi > . \tag{2.9}$$

Proof We may compute at the center of a local normal coordinate system with

$$\phi = \phi_{12\cdots m} dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m.$$

To simplify the presentation, we will drop the dz^i below.

First of all,

$$\begin{aligned} \Delta|\nabla\phi|^2 &= \nabla_i\nabla_{\bar{i}}(\nabla_j\phi_{12\cdots m}\overline{\nabla_j\phi_{12\cdots m}}) + \nabla_i\nabla_{\bar{i}}(\nabla_{\bar{j}}\phi_{12\cdots m}\overline{\nabla_{\bar{j}}\phi_{12\cdots m}}) \\ &\equiv I + II. \end{aligned} \tag{2.10}$$

By direct computation, the first term on the RHS can be written as

$$\begin{aligned} I =& \nabla_i\nabla_{\bar{i}}(\nabla_j\phi_{12\cdots m}\overline{\nabla_j\phi_{12\cdots m}}) \\ =& \nabla_i\nabla_{\bar{i}}\nabla_j\phi_{12\cdots m}\overline{\nabla_j\phi_{12\cdots m}} + \nabla_j\phi_{12\cdots m}\overline{\nabla_{\bar{i}}\nabla_i\nabla_j\phi_{12\cdots m}} \\ &+ \nabla_{\bar{i}}\nabla_j\phi_{12\cdots m}\overline{\nabla_i\nabla_j\phi_{12\cdots m}} + \nabla_i\nabla_j\phi_{12\cdots m}\overline{\nabla_i\nabla_j\phi_{12\cdots m}}. \end{aligned} \tag{2.11}$$

Therefore

$$\begin{aligned} I =& \nabla_i\nabla_{\bar{i}}\nabla_j\phi_{12\cdots m}\overline{\nabla_j\phi_{12\cdots m}} + \nabla_j\phi_{12\cdots m}\overline{\nabla_{\bar{i}}\nabla_j\nabla_i\phi_{12\cdots m}} \\ &+ |\nabla_{\bar{i}}\nabla_j\phi|^2 + |\nabla_i\nabla_j\phi|^2. \end{aligned} \tag{2.12}$$

By (2.5), the following identity holds

$$\nabla_j\nabla_{\bar{i}}\phi - \nabla_{\bar{i}}\nabla_j\phi = -R_{j\bar{i}}\phi_{12\cdots m}, \tag{2.13}$$

where $\phi_{i_1 i_2\cdots i_m}$ with repeating subindices are zero by the definition of (p, q) forms (2.4). Also, it follows from (2.4) and (2.5) that

$$\begin{aligned} (\nabla_{\bar{i}}\nabla_j - \nabla_j\nabla_{\bar{i}})\nabla_i\phi =& -R_{\bar{i}ji\bar{l}}\nabla_l\phi_{12\cdots m} - \sum_k R_{\bar{i}jk\bar{l}}\nabla_i\phi_{1\cdots k-1lk+1\cdots m} \\ =& -R_{\bar{i}ji\bar{l}}\nabla_l\phi_{12\cdots m} + R_{j\bar{i}}\nabla_i\phi_{12\cdots m}. \end{aligned} \tag{2.14}$$

Applying (2.13) and (2.14) to the righthand side of (2.12), we find

$$\begin{aligned} I =& \nabla_i\left(\nabla_j\nabla_{\bar{i}}\phi_{12\cdots m} + R_{\bar{i}j}\phi_{12\cdots m}\right)\overline{\nabla_j\phi_{12\cdots m}} \\ &+ \nabla_j\phi_{12\cdots m}\left(\overline{\nabla_j\nabla_{\bar{i}}\nabla_i\phi_{12\cdots m} - R_{\bar{i}ji\bar{l}}\nabla_l\phi_{12\cdots m} + R_{j\bar{i}}\nabla_i\phi_{12\cdots m}}\right) \\ &+ |\nabla_i\nabla_{\bar{j}}\phi|^2 + |\nabla_i\nabla_j\phi|^2. \end{aligned} \tag{2.15}$$

Recall that for $(m,0)$ forms the complex Laplacian is $\Delta = \nabla_i \nabla_{\bar{i}}$. Moreover, by the definition of the Ricci curvature (2.3) and Bianchi identity, one can see that

$$R_{\bar{i}ji\bar{l}} = -R_{i\bar{i}j\bar{l}} = -R_{j\bar{l}};$$

Therefore we can apply these properties and (2.13) on the second term of the RHS of identity (2.15) to deduce:

$$\begin{aligned}
I =& \left(\nabla_j \Delta\phi_{12\cdots m} + \nabla_i(R_{\bar{i}j}\phi_{12\cdots m})\right)\overline{\nabla_j\phi_{12\cdots m}} \\
&+ \nabla_j\phi_{12\cdots m}\left(\overline{\nabla_j\Delta\phi_{12\cdots m} + \nabla_j(R\phi_{12\cdots m}) + 2R_{j\bar{l}}\nabla_l\phi_{12\cdots m}}\right) \\
&+ |\nabla_i\nabla_{\bar{j}}\phi|^2 + |\nabla_i\nabla_j\phi|^2 \\
=&\nabla_j\Delta\phi_{12\cdots m}\overline{\nabla_j\phi_{12\cdots m}} + \nabla_j\phi_{12\cdots m}\overline{\nabla_j\Delta\phi_{12\cdots m}} + (\nabla_j R)\,\phi_{12\cdots m}\overline{\nabla_j\phi_{12\cdots m}} \\
&+ 3R_{\bar{i}j}\nabla_i\phi_{12\cdots m}\overline{\nabla_j\phi_{12\cdots m}} + \nabla_j\phi_{12\cdots m}\overline{\phi_{12\cdots m}}(\nabla_{\bar{j}}R) \\
&+ R|\nabla_i\phi|^2 + |\nabla_i\nabla_{\bar{j}}\phi|^2 + |\nabla_i\nabla_j\phi|^2.
\end{aligned}$$

Here we have also used the identities

$$\nabla_i R_{\bar{i}j} = \nabla_j R, \quad \text{and} \quad \overline{R_{\bar{i}j}} = R_{\bar{j}i} = R_{i\bar{j}}. \tag{2.16}$$

Similarly, the second term on the RHS of (2.10) is

$$\begin{aligned}
II =&\nabla_i\nabla_{\bar{i}}(\nabla_{\bar{j}}\phi_{12\cdots m}\overline{\nabla_{\bar{j}}\phi_{12\cdots m}}) \\
=&\nabla_i\nabla_{\bar{i}}\nabla_{\bar{j}}\phi_{12\cdots m}\overline{\nabla_{\bar{j}}\phi_{12\cdots m}} + \nabla_{\bar{j}}\phi_{12\cdots m}\overline{\nabla_{\bar{i}}\nabla_i\nabla_{\bar{j}}\phi_{12\cdots m}} \\
&+ \nabla_{\bar{i}}\nabla_{\bar{j}}\phi_{12\cdots m}\overline{\nabla_{\bar{i}}\nabla_{\bar{j}}\phi_{12\cdots m}} + \nabla_i\nabla_{\bar{j}}\phi_{12\cdots m}\overline{\nabla_i\nabla_{\bar{j}}\phi_{12\cdots m}}.
\end{aligned}$$

Thus

$$\begin{aligned}
II =&\nabla_{\bar{j}}\Delta\phi_{12\cdots m}\overline{\nabla_{\bar{j}}\phi_{12\cdots m}} \\
&- R_{i\bar{j}i\bar{l}}\nabla_{\bar{l}}\phi_{12\cdots m}\overline{\nabla_{\bar{j}}\phi_{12\cdots m}} - \sum_k R_{i\bar{j}k\bar{l}}\nabla_{\bar{i}}\phi_{1\cdots k-1lk+1\cdots m}\overline{\nabla_{\bar{j}}\phi_{12\cdots m}} \\
&+ \nabla_{\bar{j}}\phi_{12\cdots m}\left(\overline{\nabla_{\bar{j}}\nabla_{\bar{i}}\nabla_i\phi_{12\cdots m} - \nabla_{\bar{i}}(\sum_k R_{i\bar{j}k\bar{l}}\phi_{1\cdots k-1lk+1\cdots m})}\right) \\
&+ |\nabla_{\bar{i}}\nabla_{\bar{j}}\phi|^2 + |\nabla_i\nabla_{\bar{j}}\phi|^2 \\
=&\nabla_{\bar{j}}\Delta\phi_{12\cdots m}\overline{\nabla_{\bar{j}}\phi_{12\cdots m}} + \nabla_{\bar{j}}\phi_{12\cdots m}\left(\overline{\nabla_{\bar{j}}\Delta\phi_{12\cdots m} + \nabla_{\bar{j}}(R\phi_{12\cdots m})}\right) \\
&- \nabla_{\bar{j}}\phi_{12\cdots m}\overline{\phi_{12\cdots m}}\nabla_j R \\
&- R_{\bar{i}j}\nabla_{\bar{j}}\phi_{12\cdots m}\overline{\nabla_{\bar{i}}\phi_{12\cdots m}} + |\nabla_{\bar{i}}\nabla_{\bar{j}}\phi|^2 + |\nabla_i\nabla_{\bar{j}}\phi|^2 \\
=&\nabla_{\bar{j}}\Delta\phi_{12\cdots m}\overline{\nabla_{\bar{j}}\phi_{12\cdots m}} + \nabla_{\bar{j}}\phi_{12\cdots m}\overline{\nabla_{\bar{j}}\Delta\phi_{12\cdots m}} + R|\nabla_{\bar{i}}\phi|^2
\end{aligned}$$

$$- R_{i\bar{j}}\nabla_{\bar{j}}\phi_{12\cdots m}\overline{\nabla_{\bar{i}}\phi_{12\cdots m}} + |\nabla_{\bar{i}}\nabla_{\bar{j}}\phi|^2 + |\nabla_i\nabla_{\bar{j}}\phi|^2.$$

Note that $|\nabla\phi|^2 = |\nabla_i\phi|^2 + |\nabla_{\bar{i}}\phi|^2$. Combining the computations above for I and II with (2.10) finishes the proof of the Lemma. □

Next we turn to gradient and eigenvalue estimates for the Hodge Laplacian on $(m, 0)$ forms and the proof of Theorem 1.1 which is divided into a number of lemmas. The following is a flow chart of the proof: 1. mean value inequality for eigenforms; 2. mean value inequality for the gradient of linear combinations of eigenforms; 3. bounds of sum of the first k eigenvalue by a power of the last one via integration of the mean value inequality; 4. positive lower bound of λ_1 under a mild condition; 5. lower bound for all eigenvalues.

As a start, applying Moser's iteration, we will prove a mean value inequality for $(m, 0)$ eigenforms of the Hodge Laplacian. Only the scalar curvature is involved since it is well known that, formula (2.20) below i.e., the Bochner formula for $(m, 0)$ forms, involves only the scalar curvature. c.f. [16] Chapter 3.

Lemma 2.2 *Let $(\mathbf{M}, g_{i\bar{j}})$ be a compact Kähler manifold of complex dimension $m \geq 2$. Suppose that the scalar curvature R of $\mathbf{M}$ satisfies $R \geq -mK$, for some $K \geq 0$. Let $\lambda \geq 0$ be an eigenvalue of the Hodge Laplacian on $(m, 0)$ forms, and $\phi = \phi_{12\cdots m}dz^1 \wedge \cdots \wedge dz^m$ an eigenform associated with λ, i.e.*

$$\Delta_d\phi = \lambda\phi. \tag{2.17}$$

Assume further that the following Sobolev inequality is satisfied,

$$\left(\int_{\mathbf{M}} u^{\frac{2m}{m-1}}\right)^{\frac{m-1}{m}} \leq C_S\left(\int_{\mathbf{M}} |\nabla u|^2 + |\mathbf{M}|^{-\frac{1}{m}}\int_{\mathbf{M}} u^2\right) \tag{2.18}$$

for all smooth functions on $\mathbf{M}$. Then we have the mean value inequality

$$\max_{\mathbf{M}} |\phi|^2 \leq 2^{m(m-1)}m^m C_S^m(\lambda + K + |\mathbf{M}|^{-\frac{1}{m}})^m \int_{\mathbf{M}} |\phi|^2. \tag{2.19}$$

Proof Recall that $\Delta = -\frac{1}{2}\Delta_d$ on functions. Hence, we have

$$\begin{aligned} -\Delta_d|\phi|^2 &= 2\Delta|\phi|^2 \\ &= 2 < \nabla_j\phi, \nabla_j\phi > +2 < \nabla_{\bar{j}}\phi, \nabla_{\bar{j}}\phi > \\ &\quad + 2 < \nabla_j\nabla_{\bar{j}}\phi, \phi > +2 < \phi, \nabla_{\bar{j}}\nabla_j\phi > . \end{aligned}$$

For the last term on the right side, using (2.5) for $(m, 0)$ forms yields the known identity:

$$-\Delta_d|\phi|^2(x) = 2|\nabla\phi|^2(x) - < \Delta_d\phi, \phi > (x) - < \phi, \Delta_d\phi > (x) + 2R(x)|\phi|^2(x). \tag{2.20}$$

From (2.17) and Kato's inequality, we have

$$\begin{aligned}\Delta|\phi|^2 = -\frac{1}{2}\Delta_d|\phi|^2 &= |\nabla\phi|^2 - \lambda|\phi|^2 + R|\phi|^2 \\ &\geq |\nabla|\phi||^2 - (\lambda + mK)|\phi|^2.\end{aligned} \tag{2.21}$$

Hence,

$$\int_{\mathbf{M}} 2|\phi|^{2p-2}\Delta|\phi|^2 \geq \int_{\mathbf{M}} \left[|\phi|^{2p-2}\,|\nabla|\phi||^2 - (\lambda + mK)|\phi|^{2p}\right].$$

Using integration by parts on the LHS, one gets that for $p \geq 1$,

$$\int_{\mathbf{M}} -\frac{4(p-1)}{p^2}|\nabla|\phi|^p|^2 \geq \int_{\mathbf{M}} \left[\frac{1}{p^2}\left|\nabla|\phi|^p\right|^2 - (\lambda + mK)|\phi|^{2p}\right],$$

i.e.,

$$\int_{\mathbf{M}} |\nabla|\phi|^p|^2 \leq \frac{p^2(\lambda + mK)}{4p-3}\int_{\mathbf{M}} |\phi|^{2p}.$$

Let $\alpha = \frac{m}{m-1}$, according to the Sobolev inequality (2.18), we have

$$\begin{aligned}\left(\int_{\mathbf{M}} |\phi|^{2p\alpha}\right)^{1/\alpha} &\leq C_S\left[\frac{p^2(\lambda + mK)}{4p-3} + |\mathbf{M}|^{-\frac{1}{m}}\right]\int_{\mathbf{M}} |\phi|^{2p} \\ &\leq pC_S(\lambda + mK + |\mathbf{M}|^{-\frac{1}{m}})\int_{\mathbf{M}} |\phi|^{2p} = p\tilde{C}\int_{\mathbf{M}} |\phi|^{2p},\end{aligned}$$

where $\tilde{C} = C_S(\lambda + mK + |\mathbf{M}|^{-\frac{1}{m}})$. Setting $p = \alpha^{k-1}$, $k = 1, 2, 3, \cdots$, it implies that

$$\begin{aligned}\left(\int_{\mathbf{M}} |\phi|^{2\alpha^k}\right)^{1/\alpha^k} &\leq \tilde{C}^{\alpha^{-(k-1)}}\alpha^{(k-1)\alpha^{-(k-1)}}\left(\int_{\mathbf{M}} |\phi|^{2\alpha^{k-1}}\right)^{1/\alpha^{k-1}} \\ &\leq \tilde{C}^{\alpha^{-(k-1)}}2^{(k-1)\alpha^{-(k-1)}}\left(\int_{\mathbf{M}} |\phi|^{2\alpha^{k-1}}\right)^{1/\alpha^{k-1}}\end{aligned}$$

Therefore,

$$\max_{\mathbf{M}} |\phi|^2 \leq \tilde{C}^{\sum_{k=1}^{\infty}\alpha^{-(k-1)}}2^{\sum_{k=1}^{\infty}(k-1)\alpha^{-(k-1)}}\int_{\mathbf{M}} |\phi|^2 = 2^{m(m-1)}\tilde{C}^m\int_{\mathbf{M}} |\phi|^2$$

This finishes the proof. □

In particular, by normalizing ϕ so that $||\phi||_{L^2} = 1$, Lemma 2.2 implies that

$$|\phi|^2 \leq 2^{m(m-1)}m^m C_S^m(\lambda + K + |\mathbf{M}|^{-\frac{1}{m}})^m. \tag{2.22}$$

Next, from Lemma 2.1 and Moser iteration, we are able to prove the following gradient estimate for $(m, 0)$ eigenforms of the Hodge Laplacian. This time the Ricci curvature is involved.

Lemma 2.3 *Let* $(\mathbf{M}, g_{i\bar{j}})$ *and* ϕ *be the same as in Lemma 2.2. Assume that* $|Ric| \leq K$. *Then*

$$\max_{\mathbf{M}} |\nabla\phi|^2 \leq 4^{m(m+2)} m^{m+1} C_S^m \left[\lambda + K + |\mathbf{M}|^{-\frac{1}{m}}\right]^m (\lambda + K) \int_{\mathbf{M}} |\phi|^2. \quad (2.23)$$

In particular, when $||\phi||_{L^2} = 1$, *we have*

$$|\nabla\phi|^2 \leq 4^{m(m+2)} m^{m+1} C_S^m \left[\lambda + K + |\mathbf{M}|^{-\frac{1}{m}}\right]^m (\lambda + K). \quad (2.24)$$

Proof Let us first deal with the case where $K > 0$. Let

$$v = |\nabla\phi|^2 + A|\phi|^2$$

where $A > 0$ is a constant to be chosen later. Once we can bound v, then the bound on $|\nabla\phi|$ follows. The reason for the appearance of $A|\phi|^2$ term is to help dealing with the term involving ∇R in the Bochner formula. By (2.21) and Lemma 2.1, we have

$$\begin{aligned}
\Delta v \geq & -[\lambda + (m+4)K]\,|\nabla\phi|^2 + \phi_{12\cdots m}\nabla_j R\overline{\nabla_j\phi_{12\cdots m}} + \overline{\phi_{12\cdots m}}\nabla_j\phi_{12\cdots m}\nabla_{\bar{j}} R \\
& + |\nabla^2\phi|^2 + A|\nabla\phi|^2 - (\lambda + mK)A|\phi|^2 \\
\geq & -[\lambda + (m+4)K]\,v + \phi_{12\cdots m}\nabla_j R\overline{\nabla_j\phi_{12\cdots m}} + \overline{\phi_{12\cdots m}}\nabla_j\phi_{12\cdots m}\nabla_{\bar{j}} R \\
& + |\nabla^2\phi|^2.
\end{aligned} \quad (2.25)$$

For $p \geq 1$, multiple both sides above by v^{2p-1} and then take integrals over $\mathbf{M}$. Since the second and third terms on the RHS of the inequality are conjugate complex functions, we get

$$\begin{aligned}
& \int_{\mathbf{M}} \phi_{12\cdots m}\nabla_j R\overline{\nabla_j\phi_{12\cdots m}} v^{2p-1} + \overline{\phi_{12\cdots m}}\nabla_j\phi_{12\cdots m}\nabla_{\bar{j}} R v^{2p-1} \\
& = 2Re\left[\int_{\mathbf{M}} \phi_{12\cdots m}\nabla_j R\overline{\nabla_j\phi_{12\cdots m}} v^{2p-1}\right] \\
& = -2Re\int_{\mathbf{M}} \left(R\nabla_j\phi\overline{\nabla_j\phi} v^{2p-1} + R\phi\overline{\nabla_{\bar{j}}\nabla_j\phi} v^{2p-1} + R\phi\overline{\nabla_j\phi}\nabla_j v^{2p-1}\right) \\
& = -2Re\int_{\mathbf{M}} \left(R|\nabla_j\phi|^2 v^{2p-1} + (-\frac{\lambda}{2}R + R^2)|\phi|^2 v^{2p-1}\right. \\
& \quad \left. + (2p-1)v^{2p-2} R\phi\overline{\nabla_j\phi}\nabla_j v\right) \\
& \geq -2\int_{\mathbf{M}} \left[\left(mK + (\frac{\lambda m K}{2} + m^2K^2)A^{-1}\right) v^{2p} + \frac{2p-1}{p} mKA^{-1/2} v^p |\nabla v^p|\right]
\end{aligned}$$

$$\geq -\int_{\mathbf{M}} \left(2mK + (\lambda mK + 4pm^2K^2)A^{-1}\right) v^{2p} - \int_{\mathbf{M}} \frac{2p-1}{2p^2}|\nabla v^p|^2, \tag{2.26}$$

where Cauchy-Schwarz inequality has been applied to the last step above. When going from the 4th to the 5th line in the preceding paragraph, we also used the inequality

$$|\phi\overline{\nabla_j\phi}| \leq A^{-1/2}v.$$

Integrating (2.25) and using the last paragraph gives

$$\begin{aligned}\int_{\mathbf{M}} \frac{2p-1}{p^2}|\nabla v^p|^2 &= -\int_{\mathbf{M}} v^{2p-1}\Delta v \\ &\leq \int_{\mathbf{M}} \left[(\lambda + 4mK) + A^{-1}mK(\lambda + 4pmK)\right] v^{2p} + \int_{\mathbf{M}} \frac{2p-1}{2p^2}|\nabla v^p|^2,\end{aligned}$$

i.e.,

$$\int_{\mathbf{M}} |\nabla v^p|^2 \leq 2p\left[(\lambda + 4mK) + A^{-1}mK(\lambda + 4pmK)\right]\int_{\mathbf{M}} v^{2p}. \tag{2.27}$$

Since the Ricci curvature is bounded, Sobolev inequality as in (2.18) holds. Then combining (2.18) and (2.27) gives

$$\left(\int_{\mathbf{M}} v^{2\alpha p}\right)^{\frac{1}{\alpha}} \leq 2p^2 Q\int_{\mathbf{M}} v^{2p}, \tag{2.28}$$

where $\alpha = \frac{m}{m-1}$, and $Q = C_S\left[\lambda + 4mK + A^{-1}mK(\lambda + 4mK) + |\mathbf{M}|^{-\frac{1}{m}}\right]$. Letting $p = \alpha^k$, $k = 0, 1, 2, \cdots$, and performing Moser iteration as in Lemma 2.2 yield

$$\begin{aligned}\max_{\mathbf{M}} v^2 \leq& 2^{\sum_{k=1}^{\infty}\alpha^{-(k-1)}}\left(\frac{m}{m-1}\right)^{\sum_{k-1}^{\infty}2(k-1)\alpha^{-(k-1)}} Q^{\sum_{k=1}^{\infty}\alpha^{-(k-1)}}\int_{\mathbf{M}} v^2 \\ =& 2^m\left(\frac{m}{m-1}\right)^{2m(m-1)} Q^m\int_{\mathbf{M}} v^2,\end{aligned} \tag{2.29}$$

which infers

$$\max_{\mathbf{M}} v \leq 2^m 4^{m(m-1)} C_S^m\left[\lambda + 4mK + A^{-1}mK(\lambda + 4mK) + |\mathbf{M}|^{-\frac{1}{m}}\right]^m\int_{\mathbf{M}} v. \tag{2.30}$$

Since

$$\begin{aligned}\int_{\mathbf{M}} v &= \int_{\mathbf{M}} (|\nabla\phi|^2 + A|\phi|^2)\\ &= \int_{\mathbf{M}} (-\nabla_{\bar{i}}\nabla_i\phi\cdot\bar{\phi} - \nabla_i\nabla_{\bar{i}}\phi\cdot\bar{\phi} + A|\phi|^2)\\ &= \int_{\mathbf{M}} (-2\Delta\phi\cdot\bar{\phi} - R|\phi|^2 + A|\phi|^2)\\ &\leq (\lambda + mK + A)\int_{\mathbf{M}} |\phi|^2,\end{aligned}$$

gradient estimate (2.23) follows immediately by setting $A = \lambda + mK$.

The case where $K = 0$, i.e., $\mathbf{M}$ is Calabi–Yau, is much easier. One just need to set $v = |\nabla\phi|^2$ and apply the Moser iteration directly to (2.9) as in the proof of Lemma 2.2. □

By modifying the proof of the preceding two Lemmas 2.2 an 2.3, one can actually show a gradient bound for a linear combination of eigenforms, which is needed for lower bound estimate of higher eigenvalues. The main idea comes from [18], comparing with which we also need to deal with the gradient of the scalar curvature.

Lemma 2.4 *Let $(\mathbf{M}, g_{i\bar{j}})$ be a compact Kähler manifold of complex dimension $m \geq 2$. Assume that $|Ric| \leq K$. Let $\phi_1, \phi_2, \cdots, \phi_k$ be orthonormal $(m,0)$ forms satisfying $\Delta_d\phi_j = \lambda_j\phi_j$, $j = 1, 2, \cdots, k$. Then for any sequence of real numbers b_j, $j = 1, 2, \cdots, k$, with $\sum_{j=1}^k b_j^2 \leq 1$, the $(m,0)$ form $w = \sum_{j=1}^k b_j\phi_j$, the following estimate holds*

$$|\nabla w|^2 + (\lambda_k + K)|w|^2 \leq 4^{m(m+2)}m^{m+1}C_S^m(\lambda_k + K + |\mathbf{M}|^{-\frac{1}{m}})^m(\lambda_k + K) \quad (2.31)$$

Proof Eventually, only the gradient bound will be used later. We will give a proof of the bound for $|w|$ first as an illustration for the main idea. We may assume that $\lambda_k > 0$, for otherwise we have $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$, which implies that w is also an eigenform associated with the zero eigenvalue, hence (2.31) follows from Lemmas 2.2 and 2.3 directly.

First of all, we have, for the complex rough Laplacian $\Delta = -\frac{1}{2}\Delta_d$, that

$$\Delta w = \sum_j b_j(-\frac{\lambda_j}{2})\phi_j = -\frac{\lambda_k}{2}\eta,$$

where

$$\eta = \sum_j b_j(\lambda_j\lambda_k^{-1})\phi_j.$$

It then follows that

$$\begin{aligned}\Delta|w|^2 =& -\frac{\lambda_k}{2}(w\bar{\eta}+\eta\bar{w})+|\nabla w|^2+R|w|^2\\ \geq& -\lambda_k|w||\eta|-mK|w|^2,\end{aligned}$$

i.e.,

$$\Delta v \geq -\lambda_k|\eta|v^{1/2}-mKv,$$

where $v=|w|^2$. Multiplying both sides by v^{2p-1}, $p\geq 1$, and integrating over $\mathbf{M}$ give

$$\begin{aligned}\frac{2p-1}{p^2}\int_{\mathbf{M}}|\nabla v^p|^2 =& -\int_{\mathbf{M}} v^{2p-1}\Delta v\\ \leq& \lambda_k\int_{\mathbf{M}}|\eta|v^{2p-\frac{1}{2}}+mK\int_{\mathbf{M}}v^{2p}\\ \leq& \lambda_k\left(\int_{\mathbf{M}}|\eta|^{4p}\right)^{\frac{1}{4p}}\left(\int_{\mathbf{M}}v^{2p}\right)^{\frac{4p-1}{4p}}+mK\int_{\mathbf{M}}v^{2p}\\ \leq& (\lambda_k+mK)\max_{b_1,\cdots,b_k}\int_{\mathbf{M}}v^{2p}.\end{aligned}\tag{2.32}$$

Here and later in the proof the maximum is taken for all real numbers $b_1,\ldots,b_k$ whose square sum is less than or equal to 1. The last step above holds because the coefficients in η satisfy $\sum_j b_j^2(\lambda_j\lambda_k^{-1})^2\leq\sum_j b_j^2\leq 1$. Thus, it follows from the Sobolev inequality (2.18) and (2.32) that

$$\begin{aligned}C_S^{-1}\left(\max_{b_1,\cdots,b_k}\int_{\mathbf{M}}v^{\frac{2pm}{m-1}}\right)^{\frac{m-1}{m}} \leq& \max_{b_1,\cdots,b_k}\int_{\mathbf{M}}|\nabla v^p|^2+|\mathbf{M}|^{-\frac{1}{m}}\max_{b_1,\cdots,b_k}\int_{\mathbf{M}}v^{2p}\\ \leq& p(\lambda_k+mK+|\mathbf{M}|^{-\frac{1}{m}})\max_{b_1,\cdots,b_k}\int_{\mathbf{M}}v^{2p}.\end{aligned}\tag{2.33}$$

Using Moser iteration as in (2.28) and (2.29) yields

$$|w|^2\leq 2^{m(m-1)}m^mC_S^m\left[\lambda_k+K+|\mathbf{M}|^{-\frac{1}{m}}\right]^m.\tag{2.34}$$

For the bound of $|\nabla w|^2$. One just need to notice that from the Bochner formula (2.1)

$$\begin{aligned}\Delta|\nabla w|^2 =&|\nabla^2 w|^2-\frac{\lambda_k}{2}(<\nabla\eta,\nabla w>+<\nabla w,\nabla\eta>)\\ &+R|\nabla w|^2+<w\nabla_j R,\nabla_j w>\\ &+<\nabla_j w,w\nabla_j R>+3<R_{j\bar{k}}\nabla_k w,\nabla_j w>-<R_{k\bar{j}}\nabla_{\bar{k}}w,\nabla_{\bar{j}}w>\end{aligned}$$

Let $v = |\nabla w|^2 + A|w|^2$ this time, then

$$\begin{aligned}\Delta v \geq & |\nabla^2 w|^2 - \lambda_k Re(\nabla_i \eta \nabla_{\bar{i}} \bar{w} + \nabla_{\bar{i}} \eta \nabla_i \bar{w}) \\ & - (m+4)K|\nabla w|^2 + 2Re < w\nabla_j R, \nabla_j w > \\ & - \lambda_k A|w||\eta| - mKA|w|^2 + A|\nabla w|^2 \\ \geq & - \lambda_k Re(\nabla_i \eta \nabla_{\bar{i}} \bar{w} + \nabla_{\bar{i}} \eta \nabla_i \bar{w}) - (mK+4)v - \lambda_k v^{\frac{1}{2}} A^{\frac{1}{2}} |\eta| \\ & + 2Re < w\nabla_j R, \nabla_j w > .\end{aligned}$$

Integrating both sides over $\mathbf{M}$ after multiplying by v^{2p-1}, we get

$$\begin{aligned}\frac{2p-1}{p^2} \int_{\mathbf{M}} |\nabla v|^2 = & - \int_{\mathbf{M}} v^{2p-1} \Delta v \\ \leq & \int_{\mathbf{M}} \big[\lambda_k Re(\nabla_i \eta \nabla_{\bar{i}} \bar{w} + \nabla_{\bar{i}} \eta \nabla_i \bar{w}) v^{2p-1} + (m+4)K v^{2p} \\ & + \lambda_k A^{\frac{1}{2}} |\eta| v^{2p-\frac{1}{2}} - 2Re < w\nabla_j R, \nabla_j w > v^{2p-1}\big].\end{aligned} \tag{2.35}$$

Using the method in (2.32), one concludes

$$\int_{\mathbf{M}} \lambda_k A^{\frac{1}{2}} |\eta| v^{2p-\frac{1}{2}} \leq \lambda_k \max_{b_1, \cdots, b_k} \int_{\mathbf{M}} v^{2p}. \tag{2.36}$$

For the term

$$\lambda_k Re \int_{\mathbf{M}} (\nabla_i \eta \nabla_{\bar{i}} \bar{w} + \nabla_{\bar{i}} \eta \nabla_i \bar{w}) v^{2p-1},$$

using integration by parts and Hölder inequality as in (2.32) induce

$$\begin{aligned}& \lambda_k Re \int_{\mathbf{M}} (\nabla_i \eta \nabla_{\bar{i}} \bar{w} + \nabla_{\bar{i}} \eta \nabla_i \bar{w}) v^{2p-1} \\ & = \lambda_k Re \int_{\mathbf{M}} \left[\eta \Delta \bar{w} v^{2p-1} + \eta \nabla_{\bar{i}} \bar{w} \nabla_i v^{2p-1} + \eta \overline{\Delta w} v^{2p-1} + \eta \nabla_i \bar{w} \nabla_{\bar{i}} v^{2p-1}\right] \\ & = \lambda_k Re \int_{\mathbf{M}} \left[\lambda_k |\eta|^2 v^{2p-1} + R\eta \bar{w} v^{2p-1} + \frac{2p-1}{p} v^{p-1} < \eta \nabla \bar{w}, \nabla v^p >\right] \\ & \leq 2p\lambda_k(\lambda_k + mK) A^{-1} \max_{b_1, \cdots, b_k} \int_{\mathbf{M}} v^{2p} + \frac{2p-1}{4p^2} \int_{\mathbf{M}} |\nabla v^p|^2,\end{aligned} \tag{2.37}$$

where one can see that the second term on the right hand side above can be absorbed in the left hand side of (2.35).

For the term

$$-2Re\int_{\mathbf{M}} < w\nabla_j R, \nabla_j w > v^{2p-1},$$

when integration by parts is applied, the only term needing extra attention is

$$\int_{\mathbf{M}} Rw\Delta\bar{w}\cdot v^{2p-1} = \int_{\mathbf{M}} Rw(\overline{\Delta w + Rw})v^{2p-1}.$$

However, using the argument in (2.32) again, we have

$$\begin{aligned}\int_{\mathbf{M}} Rw(\overline{\Delta w + Rw})v^{2p-1} = &\int_{\mathbf{M}} \left(\lambda_k Rw\bar{\eta} + R^2|w|^2\right)v^{2p-1}\\ \leq &mK(\lambda_k + mK)A^{-1}\max_{b_1,\cdots,b_k}\int_{\mathbf{M}} v^{2p}.\end{aligned}$$

Thus,

$$\begin{aligned}&-2Re\int_{\mathbf{M}} < w\nabla_j R, \nabla_j w > v^{2p-1}\\ &= 2Re\int_{\mathbf{M}}\left[R|\nabla_j w|^2 v^{2p-1} + Rw\Delta\bar{w}\cdot v^{2p-1} + wR\nabla_{\bar{j}}\bar{w}\nabla_j v^{2p-1}\right]\\ &\leq \left[2mK + (8p-2)mK(\lambda_k + mK)A^{-1}\right]\max_{b_1,\cdots,b_k}\int_{\mathbf{M}} v^{2p} + \frac{2p-1}{4p^2}\int_{\mathbf{M}}|\nabla v^p|^2.\end{aligned}$$

Putting everything above together in (2.35) and setting $A = \lambda_k + mK$ give

$$\frac{2p-1}{2p^2}\int_{\mathbf{M}}|\nabla v|^2 \leq \left[2p\lambda_k + \lambda_k + (m+4)K + 8pmK\right]\max_{b_1,\cdots,b_k}\int_{\mathbf{M}} v^{2p},$$

i.e.,

$$\int_{\mathbf{M}}|\nabla v|^2 \leq 24p^2(\lambda_k + mK)\max_{b_1,\cdots,b_k}\int_{\mathbf{M}} v^{2p}. \tag{2.38}$$

Therefore, by Moser iteration as in (2.28) and (2.29), we get

$$\max_{b_1,\cdots,b_k}||v||_{L^\infty(\mathbf{M})} \leq 24^m 4^{m(m-1)}C_S^m\left[\lambda_k + mK + |\mathbf{M}|^{-\frac{1}{m}}\right]^m \max_{b_1,\cdots,b_k}\int_{\mathbf{M}} v \tag{2.39}$$

On the other hand, since $\phi_1, \phi_2, \cdots, \phi_k$ are orthonormal, we have

$$\begin{aligned}\int_{\mathbf{M}} v &= \int_{\mathbf{M}}(|\nabla w|^2 + A|w|^2)\\ &= \int_{\mathbf{M}}\left(2w\overline{\Delta w} + R|w|^2 + A|w|^2\right)\end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathbf{M}} \Big[< \sum_j b_j \phi_j, \sum_l \lambda_l b_l \phi_l > + (\lambda_k + 2mK)|w|^2 \Big] \\
&= \sum_j \lambda_j b_j^2 + (\lambda_k + 2mK) \\
&\leq 2(\lambda_k + mK).
\end{aligned}$$

Plugging this in (2.39) completes the proof. □

Using an argument as in the proof of Theorem 2.2 (2) in [18], we have

Lemma 2.5 *Under the assumptions in Lemma 2.4, we have*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \leq 4^{m(m+2)} m^{m+3} C_S^m \left[\lambda_k + K + |\mathbf{M}|^{-\frac{1}{m}} \right]^{m+1} |\mathbf{M}|. \quad (2.40)$$

Proof Since the complex situation is somewhat different from the real case, we give a detailed proof. For each $x \in \mathbf{M}$, since the complex rank of the k vectors at x, $(\nabla_{\bar{1}}\phi_1, \cdots, \nabla_{\bar{m}}\phi_1)$, $(\nabla_{\bar{1}}\phi_2, \cdots, \nabla_{\bar{m}}\phi_2)$, $\cdots$, $(\nabla_{\bar{1}}\phi_k, \cdots, \nabla_{\bar{m}}\phi_k)$ is no more than m, it is possible to find a unitary matrix $\left(a_{ij}\right)_{k\times k}$ such that $\psi_i = \sum_{j=1}^k a_{ij}\phi_j$, $1 \leq i \leq k$ satisfy that

$$\nabla_{\bar{l}}\psi_i(x) = 0, \ 1 \leq l \leq m, \ m+1 \leq i \leq k.$$

Here, for simplicity, we used $\nabla_{\bar{j}}\phi_i$ to denote the coefficient $\nabla_{\bar{j}}(\phi_i)_{1,2,\cdots,m}$ of $\nabla^{0,1}\phi_i$.

Then we derive from Lemma 2.4 that

$$\begin{aligned}
\sum_{j=1}^k \sum_{l=1}^m |\nabla_{\bar{l}}\phi_j|^2(x) &= \sum_{i=1}^m \sum_{l=1}^m |\nabla_{\bar{l}}\psi_i|^2(x) \\
&\leq m \max_i |\nabla\psi_i|^2(x) \\
&\leq 4^{m(m+2)} m^{m+2} C_S^m (\lambda_k + K + |\mathbf{M}|^{-\frac{1}{m}})^m (\lambda_k + K).
\end{aligned}$$

Thus integrating both sides gives (2.40) since

$$\int_{\mathbf{M}} \sum_{l=1}^m |\nabla_{\bar{l}}\phi_j|^2(x) = \frac{\lambda_j}{2},$$

which follows from the fact that $\Delta_d = -2\Delta$. □

As indicated in [18], it can be shown that (2.40) induces a lower bound of λ_k. Explicitly, first by induction one gets that

Lemma 2.6 *For $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$, if the inequality*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \leq c_0 \lambda_k^{m+1} \quad (2.41)$$

holds for any $k \geq 1$, then one has

$$\lambda_k \geq c_1 k^{\frac{1}{m}}, \tag{2.42}$$

where $c_1 = \min\{\lambda_1, \left(\frac{m}{c_0(m+1)}\right)^{\frac{1}{m}}\}$, and $m \geq 1$ is an integer.

Proof The conclusion follows from induction on k. Firstly, it is obvious that (2.42) holds for $k = 1$.

Assume that (2.42) holds for all $1 \leq i < k$. We show that (2.42) also holds for k. There will be two cases.

Case 1: If $\lambda_k = 0$, then $\lambda_1 = 0$. Hence $c_1 = 0$, and (2.42) is trivial for k.

Case 2: If $\lambda_k > 0$, then we may argue by contradiction. Suppose that

$$\lambda_k < c_1 k^{\frac{1}{m}}. \tag{2.43}$$

Dividing both sides of (2.41) by λ_k, and using the induction hypothesis for $\lambda_1, \cdots, \lambda_{k-1}$, we get

$$\left(\frac{1}{k}\right)^{\frac{1}{m}} + \left(\frac{2}{k}\right)^{\frac{1}{m}} + \cdots + \left(\frac{k-1}{k}\right)^{\frac{1}{m}} + \left(\frac{k}{k}\right)^{\frac{1}{m}} \leq \frac{\lambda_1}{\lambda_k} + \frac{\lambda_2}{\lambda_k} + \cdots + \frac{\lambda_{k-1}}{\lambda_k} + 1 \leq c_0 \lambda_k^m.$$

Plugging in (2.43) yields

$$\left(\frac{1}{k}\right)^{\frac{1}{m}} + \left(\frac{2}{k}\right)^{\frac{1}{m}} + \cdots + \left(\frac{k-1}{k}\right)^{\frac{1}{m}} + \left(\frac{k}{k}\right)^{\frac{1}{m}} < \frac{m}{m+1} k. \tag{2.44}$$

However, by induction again, it is straightforward to check that

$$\left(\frac{1}{k}\right)^{\frac{1}{m}} + \left(\frac{2}{k}\right)^{\frac{1}{m}} + \cdots + \left(\frac{k-1}{k}\right)^{\frac{1}{m}} + \left(\frac{k}{k}\right)^{\frac{1}{m}} \geq \frac{m}{m+1} k, \tag{2.45}$$

for any $k \geq 1$, which contradicts with (2.44). In fact, (2.45) obviously holds for $k = 1$. If furthermore, it holds for $k - 1$, i.e.,

$$\left(\frac{1}{k-1}\right)^{\frac{1}{m}} + \left(\frac{2}{k-1}\right)^{\frac{1}{m}} + \cdots + \left(\frac{k-2}{k-1}\right)^{\frac{1}{m}} + \left(\frac{k-1}{k-1}\right)^{\frac{1}{m}} \geq \frac{m}{m+1}(k-1),$$

then

$$\left(\frac{1}{k}\right)^{\frac{1}{m}} + \left(\frac{2}{k}\right)^{\frac{1}{m}} + \cdots + \left(\frac{k-1}{k}\right)^{\frac{1}{m}} + \left(\frac{k}{k}\right)^{\frac{1}{m}} \geq \frac{m}{m+1}(k-1)\left(\frac{k-1}{k}\right)^{\frac{1}{m}} + 1.$$

By direct computation, one can see that function $f(x) = \frac{m}{m+1}(1-x)^{\frac{1}{m}+1} + x - \frac{m}{m+1}$ has a global minimum at $x = 0$. It implies that

$$f(\frac{1}{k}) \geq f(0) = 0,$$

which is equivalent to

$$\frac{m}{m+1}(k-1)\left(\frac{k-1}{k}\right)^{\frac{1}{m}} + 1 \geq \frac{m}{m+1}k.$$

Therefore, (2.45) also holds for k. This finishes the proof of the lemma.

An immediate consequence of (2.40) and Lemma 2.6 is that

Corollary 2.7 *Under the assumptions in Lemma 2.4, if $\lambda_{k_0} \geq c_2 > 0$ for some k_0, then one has*

$$\lambda_k \geq c_3(k - k_0 + 1)^{\frac{1}{m}}, \forall\, k \geq k_0, \tag{2.46}$$

where $c_3 = \min\{c_2, \left(\frac{m}{\Lambda(m+1)}\right)^{\frac{1}{m}}\}$ with $\Lambda = 4^{m(m+2)}m^{m+3}C_S^m\left(1 + \frac{K+|\mathbf{M}|^{-\frac{1}{m}}}{c_2}\right)^{m+1}$ $|\mathbf{M}|$.

On the other hand, the following lower bound for the first eigenvalue λ_1 holds.

Lemma 2.8 *Let $\mathbf{M}$ be a compact Kähler manifold of complex dimension m. If the scalar curvature R is non-negative and positive somewhere, then*

$$\lambda_1 \geq \min\{\frac{1}{2}\lambda_1^0 + \inf_{\mathbf{M}} R, \quad \frac{\lambda_1^0}{2(\lambda_1^0 + \sup R)}\frac{1}{|\mathbf{M}|}\int_{\mathbf{M}} R\}$$

where λ_1^0 is the first nontrivial eigenvalue of the scalar Laplacian.

Proof Assume that ϕ is an eigenform of λ_1 with $\int_{\mathbf{M}} |\phi|^2 = 1$. Recall from (2.21) that

$$\Delta|\phi|^2 = -\frac{1}{2}\Delta_d|\phi|^2 = |\nabla\phi|^2 - \lambda_1|\phi|^2 + R|\phi|^2. \tag{2.47}$$

Integrating the above identity and using Kato's inequality, we find that

$$\begin{aligned}\lambda_1 &= \int_{\mathbf{M}} |\nabla\phi|^2 + \int_{\mathbf{M}} R|\phi|^2 \geq \int_{\mathbf{M}} |\nabla|\phi||^2 + \int_{\mathbf{M}} R|\phi|^2 \\ &\geq \lambda_{1,R},\end{aligned} \tag{2.48}$$

where $\lambda_{1,R}$ is the first eigenvalue of the scalar Schrödinger operator $-\Delta^{\mathbb{R}} + R$ with $\Delta^{\mathbb{R}}$ being the real Laplacian.

If $|\phi|$ is a constant function, then (2.48) also gives us

$$\lambda_1 \geq \frac{1}{|\mathbf{M}|}\int_{\mathbf{M}} R. \tag{2.49}$$

Next we assume that $|\phi|$ is not a constant. For simplicity, we write $f = |\phi(x)|$ and $a = \frac{1}{|\mathbf{M}|}\int_{\mathbf{M}} f$, the average of $|\phi|$ over $\mathbf{M}$. We consider two cases.

Case 1. Suppose

$$\int_{\mathbf{M}} (f-a)^2 \geq 1/2. \tag{2.50}$$

Then the first line of (2.48) implies

$$\begin{aligned}\lambda_1 &\geq \int_{\mathbf{M}} |\nabla f|^2 + \int_{\mathbf{M}} Rf^2 \\ &\geq \lambda_1^0 \int_{\mathbf{M}} (f-a)^2 + \inf_{\mathbf{M}} R.\end{aligned}$$

Therefore

$$\lambda_1 \geq \frac{1}{2}\lambda_1^0 + \inf_{\mathbf{M}} R. \tag{2.51}$$

Case 2. Suppose

$$\int_{\mathbf{M}} (f-a)^2 < 1/2. \tag{2.52}$$

Then we can expand the square to reach:

$$\int_{\mathbf{M}} f^2 - 2a\int_{\mathbf{M}} f + a^2|\mathbf{M}| < 1/2,$$

which shows, since $\int_{\mathbf{M}} f^2 = 1$, that

$$a^2 > \frac{1}{2|\mathbf{M}|}. \tag{2.53}$$

From the first line of (2.48) again,

$$\begin{aligned}\lambda_1 &\geq \int_{\mathbf{M}} |\nabla f|^2 + \int_{\mathbf{M}} R(f-a+a)^2 \\ &\geq \int_{\mathbf{M}} \left(\lambda_1^0 + R\right)(f-a)^2 + 2a\int_{\mathbf{M}} R(f-a) + a^2\int_{\mathbf{M}} R\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbf{M}} \left(\lambda_1^0 + R\right)(f-a)^2 + 2a \int_{\mathbf{M}} \frac{R}{\sqrt{\lambda_1^0 + R}} \sqrt{\lambda_1^0 + R}(f-a) + a^2 \int_{\mathbf{M}} R \\
&\geq -a^2 \int_{\mathbf{M}} \frac{R^2}{\lambda_1^0 + R} + a^2 \int_{\mathbf{M}} R \\
&= a^2 \int_{\mathbf{M}} \frac{\lambda_1^0 R}{\lambda_1^0 + R}.
\end{aligned}
$$

Here we just used Cauchy-Schwarz inequality in the 3rd last line. From (2.53), this shows

$$
\lambda_1 \geq a^2 \frac{\lambda_1^0}{\lambda_1^0 + \sup R} \int_{\mathbf{M}} R > \frac{\lambda_1^0}{2(\lambda_1^0 + \sup R)} \frac{1}{|\mathbf{M}|} \int_{\mathbf{M}} R. \tag{2.54}
$$

Combining this with (2.49) and (2.51), we find that

$$
\lambda_1 \geq \min\{\frac{1}{2}\lambda_1^0 + \inf_{\mathbf{M}} R, \quad \frac{\lambda_1^0}{2(\lambda_1^0 + \sup R)} \frac{1}{|\mathbf{M}|} \int_{\mathbf{M}} R\}.
$$

This completes the proof of the lemma.

Remark 2.9 It is not hard to generalize from the above lemma that Hodge numbers $h^{m,0} = h^{0,m} = 0$ whenever the total scalar curvature is positive and the negative part of the scalar curvature is sufficiently small in L^∞ norm. Indeed in this case $\lambda_1 > 0$ and hence there is no nonzero harmonic $(m, 0)$ forms. The same conclusion for $R > 0$ was first obtained by Kobayashi–Wu [7].

Now we are ready to give

Completion of proof of Theorem 1.1 Theorem 1.1 is a direct consequence of Corollary 2.7 with $k_0 = 1$ and Lemma 2.8. □

Another consequence of Lemma 2.8 is a more explicit lower bound of λ_1 for Fano manifolds. If $\mathbf{M}$ is a Fano manifold with $\mathbf{c}_1 = \alpha[\omega]$ for some positive real number α. Here $\mathbf{c}_1$ is the first Chern class and ω is the Kähler form. By the $\partial\bar{\partial}$ lemma, there is a real smooth function L such that $R_{i\bar{j}} = \pi\alpha g_{i\bar{j}} + \partial_i \partial_{\bar{j}} L$. Thus, we get $R = m\pi\alpha + \Delta L$, and

$$
\frac{1}{|\mathbf{M}|} \int_{\mathbf{M}} R = m\pi\alpha. \tag{2.55}
$$

On the other hand, the relation between $\mathbf{c}_1$ and $[\omega]$ also implies that

$$
\alpha^m |\mathbf{M}| = \alpha^m \int_{\mathbf{M}} [\omega]^m = \int_{\mathbf{M}} \mathbf{c}_1^m.
$$

Since $\mathbf{c}_1 > 0$, the RHS is a positive integer. Hence, it induces that

$$
\alpha^m |\mathbf{M}| \geq 1,
$$

which, from (2.55), is equivalent to

$$\frac{1}{|\mathbf{M}|}\int_{\mathbf{M}} R \geq m\pi\,|\mathbf{M}|^{-\frac{1}{m}}. \tag{2.56}$$

Therefore, substituting the bound above in Lemma 2.8, we get

Corollary 2.10 *Let* $\mathbf{M}$ *be a Fano manifold of complex dimension* m. *Suppose that* $\mathbf{c}_1 = [w]$, *and* $0 \leq R \leq K$ *then*

$$\lambda_1 \geq \min\{\frac{1}{2}\lambda_1^0, \quad \frac{\lambda_1^0 m\pi}{2(\lambda_1^0 + K)}|\mathbf{M}|^{-\frac{1}{m}}\}.$$

We end this section with a lower bound estimate for the first eigenvalue of 1 forms on some Kähler surfaces, which is similar in spirit to Theorem 1.1.

Proposition 2.11 *Suppose* $\mathbf{M}$ *is a simply connected compact Kähler manifold of complex dimension 2 whose scalar curvature is nonnegative and positive somewhere. Let* $\lambda_1^{(1)}$ *be the first eigenvalue of the Hodge Laplacian on* $(1,0)$ *forms. Then*

$$\lambda_1^{(1)} \geq \min\{\frac{1}{2}\lambda_1^0, \quad \frac{\lambda_1^0}{2(\lambda_1^0 + \sup R)}\frac{1}{|\mathbf{M}|}\int_{\mathbf{M}} R\} > 0,$$

where λ_1^0 *is the first nonzero eigenvalue of the scalar Laplacian.*

Proof Let α be an eigenform for $\lambda_1^{(1)}$, i.e.

$$\Delta_d \alpha = \lambda_1^{(1)}\alpha.$$

Then applying ∂ on both sides yields

$$\Delta_d \partial\alpha = \lambda_1^{(1)}\partial\alpha.$$

If $\partial\alpha \neq 0$, then it is a $(2,0)$ eigenform. By Lemma 2.8, the stated lower bound for $\lambda_1^{(1)}$ is true. If $\partial\alpha = 0$, then we notice that $\partial^*\alpha$ can not be a constant function. Otherwise we would have

$$\Delta_d \alpha = \partial\partial^*\alpha + \partial^*\partial\alpha = 0.$$

Hence α is a nontrivial harmonic 1 form whose existence means that the first Betti number of $\mathbf{M}$ is not 0, contradicting with the assumption that $\mathbf{M}$ is simply connected. Consequently $\partial^*\alpha$ is a nonconstant solution to the scalar equation:

$$-2\Delta\partial^*\alpha = \Delta_d\partial^*\alpha = \lambda_1^{(1)}\partial^*\alpha.$$

Since when acting on real functions, $2\Delta = -\Delta_d$ is the real Laplacian, we have

$$\lambda_1^{(1)} \geq \lambda_1^0.$$

This completes the proof of the proposition. □

Remark 2.12 From the proof above, we can see that the "simply connectedness" assumption can be dropped. Instead, $\lambda_1^{(1)}$ will be considered as the first nonzero eigenvalue.

3 The Heat Kernel Estimates

In this section we prove pointwise and gradient bound for the heat kernel of the Hodge Laplacian on $(m, 0)$ forms. Again, the main feature of the results is that the bounds of the heat kernel and its gradient only rely on the Ricci curvature bound instead of the full curvature bound.

We consider time dependent, smooth $(m, 0)$ forms ϕ which satisfy the heat equation

$$(\partial_t - \Delta)\phi = (\partial_t + \frac{1}{2}\Delta_d)\phi = 0 \tag{3.1}$$

on $\mathbf{M} \times (0, T]$. Here, for consistency with the usual heat equation, we put $1/2$ in front of the Hodge Laplacian and Δ is the complex Laplacian. Let $\mathbf{G} = \mathbf{G}(x, t, y)$ be the heat kernel, i.e. fundamental solution of (3.1) such that

$$\lim_{t\to 0^+} \int \mathbf{G}(x, t, y)\psi(y)dg(y) = \psi(x)$$

for all smooth $(m, 0)$ forms ψ. We have

Theorem 3.1 *Let $(\mathbf{M}, g_{i\bar{j}})$ be a compact Kähler manifold of complex dimension m. Suppose that the Ricci curvature satisfies $|Ric| \leq K$. Then there exist a positive constant A_1 depending on the Sobolev constant C_S in (1.2) and dimensional constants a_2 and a_3 such that*

$$\begin{aligned} |\mathbf{G}(x, t, y)| &\leq \frac{A_1}{t^m} e^{a_2 K t} e^{-a_3 d^2(x,y)/t}, \\ |\nabla_x \mathbf{G}(x, t, y)| &\leq \frac{A_1}{t^{m+(1/2)}} e^{a_2 K t} e^{-a_3 d^2(x,y)/t} \ when\ m \geq 2. \end{aligned} \tag{3.2}$$

In order to prove the theorem, we need two intermediate results. The first result is a pointwise bound for $|\phi|$ via semigroup domination property. Note we can also use Moser's iteration to get a similar result.

Proposition 3.2 *Let $(\mathbf{M}^m, g_{i\bar{j}})$ be a compact Kähler manifold of complex dimension m with $Ric \geq -K$, and $\phi = \phi(x, t)$ be a smooth $(m, 0)$ form satisfying the heat*

equation (3.1) on $[0, T]$. *Then we have,*

$$|\phi|^2(x,t) \le \frac{A(m)}{t^m} e^{mKt} \, \||\phi|(\cdot,0)\|^2_{L^2(\mathbf{M})}, \ t \in (0,T], \tag{3.3}$$

and

$$\sup_{B_{\frac{\sqrt{T}}{2}} \times [\frac{3}{4}T, T]} |\phi|^2(x,t) \le \frac{A(m)}{T^m} e^{mKT} \, \||\phi|(\cdot,0)\|^2_{L^2(B_{\sqrt{T}})}, \tag{3.4}$$

where $A(m)$ *is a constant only depending on* m *and* C_S, *the Sobolev constant in (1.2), and* B_r *denotes the geodesic ball with radius* r *centered at some point* $O \in \mathbf{M}$.

Proof Let ϕ be a solution to (3.1) Then

$$\partial_t |\phi|^2 = \Delta\phi \cdot \bar{\phi} + \phi \cdot \overline{\Delta\phi} = \Delta\phi \cdot \bar{\phi} + \phi \cdot \nabla_{\bar{i}} \nabla_i \bar{\phi},$$

and

$$\begin{aligned} \Delta|\phi|^2 &= \nabla_i \left(\nabla_{\bar{i}} \phi \cdot \bar{\phi} + \phi \cdot \nabla_{\bar{i}} \bar{\phi} \right) \\ &= \Delta\phi \cdot \bar{\phi} + \nabla_{\bar{i}} \phi \nabla_i \bar{\phi} + \nabla_i \phi \nabla_{\bar{i}} \bar{\phi} + \phi \nabla_i \nabla_{\bar{i}} \bar{\phi} \\ &= \Delta\phi \cdot \bar{\phi} + \phi \cdot \nabla_{\bar{i}} \nabla_i \bar{\phi} + R|\phi|^2 + |\nabla\phi|^2. \end{aligned}$$

Therefore, $|\phi|$, the norm of ϕ satisfies the scalar equation:

$$(\partial_t - \Delta)|\phi|^2 = -R|\phi|^2 - |\nabla\phi|^2. \tag{3.5}$$

From (3.5) and the lower bound of the scalar curvature coming from Ric lower bound assumption, we have

$$(\partial_t - \Delta)|\phi|^2 \le mK|\phi|^2 - |\nabla|\phi||^2.$$

It implies that

$$(\partial_t - \Delta)(e^{-mKt}|\phi|^2) \le 0. \tag{3.6}$$

Let $G = G(x,t,y)$ be the heat kernel of the standard scalar heat equation. The maximum principle infers that

$$e^{-mKt}|\phi|^2(x,t) \le \int_{\mathbf{M}} G(x,t,y)|\phi(y,0)|^2 dy. \tag{3.7}$$

According to [14], since $\mathbf{M}$ is compact, there exist a constant C_1 depending on m and $|\mathbf{M}|$, and a dimensional positive constant C_2 such that

$$G(x,t,y) \le \frac{C_1}{|B(x,\sqrt{t})|} e^{-C_2 d^2(x,y)/t}. \tag{3.8}$$

Since the Sobolev constant C_S is a finite number, there exists another positive number, $A(m)$, depending on C_S and the dimension such that

$$G(x,t,y) \le \frac{A(m)}{t^m} e^{-C_2 d^2(x,y)/t}. \tag{3.9}$$

Substituting this to (3.7) gives (3.3).

For (3.4), one just need a cut-off function $\psi(x,t) = \xi(d(x,O))\eta(t)$, where $\xi : \mathbb{R} \to [0,1]$ satisfies $\xi(u) = 1$ in $[0, \frac{\sqrt{T}}{2}]$, $\xi(u) = 0$ on $[\sqrt{T}, \infty]$, $-\frac{8}{\sqrt{T}} \le \xi' \le 0$, $|\xi''| \le \frac{8}{T}$, and $\eta : \mathbb{R} \to [0,1]$ satisfies $\eta(s) = 1$ in $[\frac{3T}{4}, T]$, $\xi(u) = 0$ on $[0, \frac{T}{2}]$, $0 \le \eta'(s) \le \frac{8}{T}$. Then it is not hard to check that (3.4) follows by multiplying (3.5) by ψ^2 and applying Duhamel's formula and the heat kernel bound (3.9). □

Further more, we can get similar gradient estimate for ϕ, assuming two sided bound of the Ricci curvature. Here we need to integrate out the gradient of the scalar curvature appeared in the Bochner formula.

Proposition 3.3 *Let $(\mathbf{M}^m, g_{i\bar{j}})$ be a compact Kähler manifold of complex dimension $m \ge 2$ with $|\operatorname{Ric}| \le K$, $\phi = \phi(x,t)$ a smooth $(m,0)$ form satisfying the heat equation (3.1) on $[0,T]$. Then we have*

$$\sup_{B_{\frac{\sqrt{T}}{4}} \times [\frac{15}{16}T, T]} |\nabla\phi|^2 \le \frac{A(m)e^{C(m)KT}}{T^{m+1}} ||\phi(\cdot,0)||^2_{L^2(B_{\sqrt{T}/2})}.$$

Here B_r denotes a geodesic ball of radius r centered at some point $O \in \mathbf{M}$, $A(m)$ is a constant depending on m and the Sobolev constant C_S in (2.18), and $C(m)$ is just a dimensional constant.

Proof From the Bochner formula (2.7), we have

$$\begin{aligned}(\partial_t - \Delta)|\nabla\phi|^2 = &-|\nabla^2\phi|^2 - R|\nabla\phi|^2 - \phi\nabla_j R\nabla_{\bar{j}}\bar{\phi} - \bar{\phi}\nabla_j\phi\nabla_{\bar{j}}R \\ &- 3R_{j\bar{k}}\nabla_j\phi\nabla_{\bar{j}}\bar{\phi} + R_{j\bar{k}}\nabla_{\bar{j}}\phi\nabla_k\bar{\phi}.\end{aligned} \tag{3.10}$$

Let

$$v = |\nabla\phi|^2 + A|\phi|^2$$

with A to be determined.

Combine (3.5) and (3.10), we have

$$\begin{aligned}(\partial_t - \Delta)v^p \le &- p|\nabla^2\phi|^2 v^{p-1} + p(m+4)Kv^p - p\phi\nabla_j R\nabla_{\bar{j}}\bar{\phi}v^{2p-1} \\ &- p\bar{\phi}\nabla_j\phi\nabla_{\bar{j}}Rv^{p-1} \\ &- p(p-1)v^{p-2}|\nabla v|^2.\end{aligned} \tag{3.11}$$

For positive numbers $\delta < \sigma$, let $\eta(s)$ a cut-off function satisfying $\eta = 0$ on $[0, T-\sigma r^2]$, $\eta = 1$ on $[T-\delta r^2, T]$, and $0 \le \eta'(s) \le \frac{2}{(\sigma-\delta)r^2}$. For positive numbers $\mu < \nu$, let $\xi(u)$ be a cut-off function such that $\xi = 1$ on $[0, \mu r]$, $\xi = 0$ on $[\nu r, \infty]$, and $-\frac{2}{(\nu-\mu)r} \le \xi' \le 0$.

Set $\psi = \xi(x)\eta(t)$. Multiplying both sides of (3.11) by $\psi^2 v^p$ and taking integral over $\mathbf{M} \times [0, T]$ yield

$$\begin{aligned}
&\iint_{\mathbf{M}\times[0,T]} \psi^2 v^p (\partial_t - \Delta) v^p \\
&\le \iint_{\mathbf{M}\times[0,T]} p\Big[-\frac{1}{m}|\Delta\phi|^2 + (m+4)K - (\phi\nabla_j R\nabla_{\bar{j}}\bar{\phi} + \bar{\phi}\nabla_j\phi\nabla_{\bar{j}}R)\Big]\psi^2 v^{2p-1} \\
&\quad - \iint_{\mathbf{M}\times[0,T]} p(p-1)\psi^2 v^{2p-2}|\nabla v|^2.
\end{aligned} \tag{3.12}$$

On the other hand

$$\begin{aligned}
\iint_{\mathbf{M}\times[0,T]} \psi^2 v^p (\partial_t - \Delta) v^p =& \frac{1}{2}\int_{\mathbf{M}} \psi^2 v^{2p}\Big|_T - \iint_{\mathbf{M}\times[0,T]} \psi v^{2p}\frac{\partial\psi}{\partial t} \\
&+ \iint_{\mathbf{M}\times[0,T]} \Big(|\nabla(\psi v^p)|^2 - v^{2p}|\nabla\psi|^2\Big).
\end{aligned} \tag{3.13}$$

Therefore, combining (3.12) and (3.13) shows that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbf{M}} \psi^2 v^{2p}\Big|_T + \iint_{\mathbf{M}\times[0,T]} |\nabla(\psi v^p)|^2 \\
&\le \iint_{\mathbf{M}\times[0,T]} p\Big[-\frac{1}{m}|\Delta\phi|^2 + (m+4)K - (\phi\nabla_j R\nabla_{\bar{j}}\bar{\phi} + \bar{\phi}\nabla_j\phi\nabla_{\bar{j}}R)\Big]\psi^2 v^{2p-1} \\
&\quad - \iint_{\mathbf{M}\times[0,T]} p(p-1)\psi^2 v^{2p-2}|\nabla v|^2 + \iint_{\mathbf{M}\times[0,T]} \Big(\psi\frac{\partial\psi}{\partial t} + |\nabla\psi|^2\Big) v^{2p}
\end{aligned} \tag{3.14}$$

Similar to (2.26), using integration by parts and Cauchy–Schwarz inequality give

$$\begin{aligned}
&-\int_{\mathbf{M}} \Big(\phi\nabla_j R\nabla_{\bar{j}}\bar{\phi} + \bar{\phi}\nabla_j\phi\nabla_{\bar{j}}R\Big)\psi^2 v^{2p-1} \\
&\le \int_{\mathbf{M}} \Big[C(m)(K + pK^2A^{-1})\psi^2 v^{2p} + \frac{1}{m}|\Delta\phi|^2\psi^2 v^{2p-1} \\
&\quad + \frac{1}{2}|\nabla(\psi v^p)|^2 + \frac{1}{p}\psi^2 v^{2p-2}|\nabla v|^2\Big].
\end{aligned} \tag{3.15}$$

Also,

$$\psi\frac{\partial\psi}{\partial t}+|\nabla\psi|^2=\psi\xi\eta'+\eta^2|\nabla\xi|^2\le 4\left(\frac{1}{(\sigma-\delta)r^2}+\frac{1}{(\nu-\mu)^2r^2}\right). \quad (3.16)$$

Thus, by plugging (3.15) and (3.16) into (3.14), and setting $A=K$, it follows that

$$\begin{aligned}&\int_{\mathbf{M}}\psi^2v^{2p}\Big|_T+\iint_{\mathbf{M}\times[0,T]}|\nabla(\psi v^p)|^2\\&\le p^2C(m)\left(K+\frac{1}{(\sigma-\delta)r^2}+\frac{1}{(\nu-\mu)^2r^2}\right)\iint_{\mathbf{M}\times[0,T]}(\psi v^p)^2 \quad (3.17)\\&:=p^2L\iint_{\mathbf{M}\times[0,T]}(\psi v^p)^2,\end{aligned}$$

where $L=C(m)\left(K+\frac{1}{(\sigma-\delta)r^2}+\frac{1}{(\nu-\mu)^2r^2}\right)$. By the definition of ψ, it is not hard to deduce from above that

$$\int_{B_{\mu r}}v^{2p}|_{T'}+\iint_{B_{\mu r}\times[T-\delta r^2,T]}|\nabla v^p|^2\le p^2L\iint_{B_{\nu r}\times[T-\sigma r^2,T]}v^{2p}, \quad (3.18)$$

for any $T'\in[T-\delta r^2,T]$.

For $Ric\ge -K$, the following local Sobolev inequality holds (see e.g. [17]).

$$\left(\int_{B_r}|u|^{\frac{2m}{m-1}}\right)^{\frac{m-1}{m}}\le C(r)\left[\int_{B_r}|\nabla u|^2+r^{-2}\int_{B_r}|u^2|\right],$$

where $u\in C^\infty(\mathbf{M})$, and $C(r)=e^{C(m)(1+\sqrt{K}r)}|B_r|^{-\frac{1}{m}}r^2$. It implies that

$$\begin{aligned}&\iint_{B_{\mu r}\times[T-\delta r^2,T]}|\nabla v^p|^2\ge C(\mu r)^{-1}\int_{T-\delta r^2}^T\left(\int_{B_{\mu r}}v^{2p\alpha}\right)^{1/\alpha}ds-(\mu r)^{-2}\\&\iint_{B_{\nu r}\times[T-\sigma r^2,T]}v^{2p}. \quad (3.19)\end{aligned}$$

Moreover, from (3.18), we can see that

$$\begin{aligned}&\left(\int_{T-\delta r^2}^T\left(\int_{B_{\mu r}}v^{2p\alpha}\right)^{1/\alpha}ds\right)\cdot\left(\iint_{B_{\nu r}\times[T-\sigma r^2,T]}v^{2p}\right)^{1/m}\\&\ge(p^2L)^{-1/m}\left(\int_{T-\delta r^2}^T\left(\int_{B_{\mu r}}v^{2p\alpha}\right)^{1/\alpha}ds\right)\cdot\left(\sup_{[T-\delta r^2,T]}\int_{B_{\mu r}}v^{2p}\right)^{1/m}\\&\ge(p^2L)^{-1/m}\int_{T-\delta r^2}^T\left(\int_{B_{\mu r}}v^{2p\alpha}\right)^{1/\alpha}\left(\int_{B_{\mu r}}v^{2p}\right)^{1/m}ds\end{aligned}$$

$$\geq (p^2L)^{-1/m}\left(\iint_{B_{\mu r}\times[T-\delta r^2,T]} v^{2p(1+\frac{1}{m})}\right).$$

Multiplying both sides of (3.18) by $\left(\iint_{B_{\nu r}\times[T-\sigma r^2,T]} v^{2p}\right)^{1/m}$, and applying the estimates above, we derive

$$\begin{aligned}
&\iint_{B_{\mu r}\times[T-\delta r^2,T]} v^{2p(1+\frac{1}{m})}\\
&\leq C(\mu r)\left[p^2L+(\mu r)^{-2}\right](p^2L)^{\frac{1}{m}}\cdot\left(\iint_{B_{\nu r}\times[T-\sigma r^2,T]} v^{2p}\right)^{1+\frac{1}{m}}\\
&\leq p^3e^{C(m)(1+\sqrt{K}\mu r)}|B_{\mu r}|^{-\frac{1}{m}}\mu^{-2}\left(\mu^2r^2L+1\right)(L)^{\frac{1}{m}}\left(\iint_{B_{\nu r}\times[T-\sigma r^2,T]} v^{2p}\right)^{1+\frac{1}{m}}.
\end{aligned}\tag{3.20}$$

Now, suppose that $a\in[\frac{1}{2},1]$, $\tau\in(0,1]$. Let $\beta=1+\frac{1}{m}$, and for $k=1,2,\cdots$, set $p=\beta^{k-1}, \sigma=a^2+\frac{(a+\tau)^2-a^2}{2^{k-1}}, \delta=a^2+\frac{(a+\tau)^2-a^2}{2^k}, I_k=[T-(a^2+\frac{(a+\tau)^2-a^2}{2^k})r^2,T]$, $\mu=a+\frac{\tau}{2^k}$, $\nu=a+\frac{\tau}{2^{k-1}}$, and $r_k=(a+\frac{\tau}{2^k})r$, then (3.20) becomes

$$\begin{aligned}
||v^2||_{\beta^k,B_{r_k}\times I_k}\leq&\beta^{3(k-1)\beta^{-k}}e^{C(m)(1+\sqrt{K}(a+2^{-k}\tau)r)\beta^{-k}}|B_{ar}|^{-\frac{1}{m}\beta^{-k}}\\
&\cdot\left[C(m)2^{2k}e^{\sqrt{K}r}\tau^{-1}\right]^{(1+\frac{1}{m})\beta^{-k}}r^{-\frac{2}{m}\beta^{-k}}||v^2||_{\beta^{k-1},B_{r_{k-1}}\times I_{k-1}}.
\end{aligned}$$

Using iteration on k, then letting $k\to 0$, we have

$$\max_{B_{ar}\times[T-(ar)^2,T]} v^2\leq\frac{C_0(m)e^{C(m)\sqrt{K}r}}{r^2|B_{ar}|}\tau^{-(m+1)}||v^2||_{1,B_{(a+\tau)r}\times[T-[(a+\tau)r]^2,T]},$$

where $C_0(m)=2^{8m(m+1)}m^{2(m+1)}$.

Next, we may use a method in [12] to reduce the L^2 mean value inequality above to L^1 mean value inequality. Let $r_j=(\sum_{k=0}^{j}2^{-k})r$, $Q_j=B_{r_k}\times[T-r_j^2,T]$ for $j=0,1,2,\cdots$. Then $Q_0=B_r\times[T-r^2,T]\subset Q_1\subset\cdots\subset Q_j\subset\cdots\subset B_{2r}\times[T-(2r^2),T]$, and

$$\sup_{Q_j} v^2\leq A_0 2^{(j+1)(m+1)}\iint_{Q_{j+1}} v^2\leq A_0 2^{(j+1)(m+1)}\left(\sup_{Q_{j+1}} v^2\right)^{\frac{1}{2}}\iint_{B_{2r}\times[T-(2r)^2,T]} v,$$

where $A_0 = \frac{C(m)e^{C(m)\sqrt{K}r}}{r^2|B_r|}$. Denote by $\hat{A} = A_0 \iint_{B_{2r}\times[T-(2r)^2,T]} v$ and run iteration, we get

$$\sup_{Q_0} v^2 \le \hat{A}^{\sum_{k=0}^{j} 2^{-k}} 2^{(m+1)\sum_{k=0}^{j}(k+1)2^{-k}} (\sup_{Q_j} v^2)^{2^{-j}}.$$

Letting $j \to \infty$, one can see that

$$\sup_{Q_0} v^2 \le C(m)\hat{A}^2,$$

i.e.,

$$\sup_{B_r\times[T-r^2,T]} v \le \frac{C(m)e^{C(m)\sqrt{K}r}}{r^2|B_r|} \iint_{B_{2r}\times[T-(2r)^2,T]} v. \tag{3.21}$$

At last, let us bound the right hand side of the previous inequality. Multiplying (3.5) by the spatial cutoff function ξ^2 and doing the standard energy estimate, taking $r = \frac{1}{4}\sqrt{T}$, we can deduce with the assistance of (3.4) that

$$\iint_{B_{\sqrt{T}/2}\times[3T/4,T]} \left(|\nabla\phi|^2 + |\phi|^2\right)\xi^2 \le \quad C(m)e^{mKT}||\phi(x,0)||^2_{L^2(B_{\sqrt{T}})}. \tag{3.22}$$

Now the conclusion of the proposition follows from (3.21), (3.22) and the volume comparison. □

Now we are ready to give a

Proof of Theorem 3.1 *Step 1*. The pointwise bound for $\mathbf{G}$ follows quickly from (3.6). Fixing y, let $\phi(x,t) = \mathbf{G}(x,t,y)$. Then, (3.6) infers that

$$(\partial_t - \Delta)(e^{-mKt}|\phi(x,t)|) \le 0.$$

Hence $e^{-mKt}|\phi(x,t)|$ is dominated by $G(x,t,y)$, the heat kernel of the scalar Laplacian. So the pointwise bound in (3.2) for $|\mathbf{G}|$ follows from (3.9).

Step 2 . We prove the gradient bound.

$$|\nabla_x \mathbf{G}(x,t,y)| \le \frac{A_1}{t^{m+(1/2)}} e^{a_2 Kt}. \tag{3.23}$$

where A_1 depends only on K, m and C_S and a_2 depends only on m.

Let ϕ be as in Step 1. We apply Proposition 3.3 on the region $B_{\sqrt{t}/2}(x)\times[t/2,t]$ with $t/2$ taken as the initial time. Note that we are free to adjust the total time interval by a fixed factor. This gives

$$|\nabla\phi(x,t)|^2 \le \frac{A_0 e^{a_2 Kt}}{t^{m+1}} \int_{B_{\sqrt{t}/2}(x)} |\mathbf{G}(z,t/2,y)|^2 dz. \tag{3.24}$$

Here A_0 depends only on K, m and C_S. From the pointwise bound proven in Step 1, after routine computation, this implies

$$|\nabla\phi(x,t)|^2 \leq \frac{A_1 e^{a_2 K t}}{t^{2m+1}}$$

proving (3.23). *Step 3* . Completion of the proof of the gradient bound in (3.2).

Since the manifold is compact, the exponential term $e^{-a_3 d^2(x,y)/t}$ is mute for $t \geq 1$. So, by Step 2, we only need to deal with the case when $t \in (0, 1]$ and $d^2(x, y) \geq 4t$.

Now from (3.24) and using the bound on $|\mathbf{G}|$ and the property that $d(z, y) \geq d(x, y)/2$ for $z \in B_{\sqrt{t}/2}(x)$, we deduce

$$\begin{aligned} |\nabla\phi(x,t)|^2 &\leq \frac{A_0 e^{a_2 K t}}{t^{m+1}} \int_{B_{\sqrt{t}/2}(x)} \frac{A_1^2}{t^{2m}} e^{a_2 K t} e^{-4a_3 d^2(z,y)/t} dz \\ &\leq \frac{A_0 e^{a_2 K t}}{t^{m+1}} \left| B_{\sqrt{t}/2}(x) \right| \frac{A_1^2}{t^{2m}} e^{a_2 K t} e^{-a_3 d^2(x,y)/t}. \end{aligned} \tag{3.25}$$

The desired gradient bound then follows by volume comparison, after a suitable adjustment of constants. □

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