MELNIKOV METHOD FOR NON-CONSERVATIVE PERTURBATIONS OF THE RESTRICTED THREE-BODY PROBLEM

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Abstract. We consider the planar circular restricted three-body problem (PCRTBP), as a model for the motion of a spacecraft relative to the Earth-Moon system. We focus on the collinear equilibrium points L_1 and L_2. There are families of Lyapunov periodic orbits around either L_1 or L_2, forming Lyapunov manifolds. There also exist homoclinic orbits to the Lyapunov manifolds around either L_1 or L_2, as well as heteroclinic orbits between the Lyapunov manifold around L_1 and the one around L_2. The motion along the homoclinic/heteroclinic orbits can be described via the scattering map, which gives the future asymptotic of a homoclinic orbit as a function of the past asymptotic. In contrast with the more customary Melnikov theory, we do not need to assume that the asymptotic orbits have a special nature (periodic, quasi-periodic, etc.).

We add a non-conservative, time-dependent perturbation, as a model for a thrust applied to the spacecraft for some duration of time, or for some other effect, such as solar radiation pressure. We compute the first order approximation of the perturbed scattering map, in terms of fast convergent integrals of the perturbation along homoclinic/heteroclinic orbits of the unperturbed system. As a possible application, this result can be used to determine the trajectory of the spacecraft upon using the thrust.

1. Introduction

1.1. **Motivation.** A motivation for this work is the following situation from astrodynamics. Consider a spacecraft traveling between the Earth and the Moon. Assume that the spacecraft is coasting along the stable/unstable hyperbolic invariant manifolds associated to the periodic orbits near one of the center-saddle equilibrium points, at some fixed energy level. Such an

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orbit is driven by the gravitational fields of the Earth and the Moon, and does not require using the thrusters. The total energy is preserved along the orbit. One can describe the motion of the spacecraft in terms of some geometrically defined coordinates: an 'action' coordinate describing the size of the periodic orbit associated to the stable/unstable invariant manifold, an 'angle' coordinate describing the asymptotic phase of the motion, and a pair of 'hyperbolic' coordinates describing the position of the spacecraft relative to the corresponding stable/unstable manifold.

Suppose now that we want to make a maneuver in order to jump from the hyperbolic invariant manifold on the given energy level to another hyperbolic invariant manifold on a different energy level. Mathematically, turning on the thrusters amounts to adding a small, non-conservative, time-dependent perturbation to the original system. If such a perturbation is given, we would like to estimate its effect on the orbit of the spacecraft. More precisely, we would like to compute the change in the action and angle coordinates associated to the orbit as a result of applying the perturbation. We want to obtain such an estimate in terms of the original trajectory of the unperturbed system, and of the particular perturbation.

Other non-Hamiltonian perturbations, for instance, due to solar radiation pressure and solar wind, can be considered (see [MNF87]).

In this paper we investigate the general problem of adding a non-Hamiltonian perturbation to a homoclinic/heteroclinic trajectory and computing the effect on the homoclinic/heteroclinic orbits.

Note that adding a non-Hamiltonian perturbation (e.g., a small dissipation) may destroy all the periodic orbits. Nevertheless, the normally hyperbolic manifold, formed by the collection of periodic orbits, persists (see details later). In contrast with the most customary versions of the Melnikov theory, which assume that the asymptotic orbits are periodic or quasiperiodic, and that they preserve their nature under perturbation, we consider homoclinic excursions to a normally hyperbolic manifold. The asymptotic orbits could change their nature under the perturbations. For example, a family of Lyapunov periodic orbits subject to a small dissipation could get transformed into a family of orbits that converge to a critical point (see [dlLK19]).

1.2. **Brief description of the main results and methodology.** While we consider a general set up and derive some general results on the effect of non-conservative, time-dependent perturbation on homoclinic orbits, the main motivation of this work resides with the PCRTBP.

In the PCRTB, the family of periodic orbits about either L_1 or L_2 forms a normally hyperbolic invariant manifold (NHIM). Each NHIM has hyperbolic stable and unstable manifolds. We will assume that the stable and unstable manifolds of the NHIM corresponding to either L_1 or L_2 intersect transversally, and also that the unstable (stable) manifold of the NHIM corresponding to L_1 intersects transversally the stable (unstable) manifold of

the NHIM corresponding to L_2. This assumption can be verified numerically for a wide range of energy levels and mass parameters in the PCRTBP (see, e.g., [KLMR00]). Thus, there exist homoclinic orbits to either one of the NHIM's, as well as heteroclinic orbits between the two NHIM's. There exist scattering maps associated to the transverse homoclinic intersections, as well as to the transverse heteroclinic intersections.

There exist some neighborhoods of L_1 and of L_2 where the Hamiltonian of the PCRTBP can be written in a normal form, via some suitable symplectic action-angle and hyperbolic coordinates (I, θ, y, x). In particular, each NHIM can be parametrized in terms of the action-angle coordinates (I, θ). Therefore, the scattering map can also be described in terms of these coordinates. In the unperturbed case, the scattering map is particularly simple: it is a shift in the angle coordinate (a phase shift).

The fact that we use normal form coordinates to express the Hamiltonian, and we subsequently estimate the scattering map in terms of the action-angle coordinates, is a matter of practical convenience. Normal forms are often used to compute numerically, with high precision, the periodic orbits and the NHIM's around the equilibrium points, as well as the corresponding stable and unstable manifolds, e.g., [Jor99, GLMS01, GSLM01, GJSM01a, GJSM01b].

For applications, it is important to note that the scattering map for the PCRTBP can be computed numerically with high precision; see [CDMR06, DGR16, CGdlL16].

We study the effect of a small, non-Hamiltonian, time-dependent perturbation that is added to the system. Provided that the perturbation is small enough, the NHIM's will persist [Fen72], although periodic orbits inside the NHIM's may disappear. Also, the transverse homoclinic/heteroclinic orbits, hence the scattering map, will survive in the perturbed system.

The main contribution is that we compute the effect of the non-conservative, time-dependent perturbation on the action and angle components of the scattering map. More precisely, we use Melnikov theory to provide explicit estimates – up to first order with respect to the size of the perturbation – for the difference between the perturbed scattering map and the unperturbed one, relative to the action and angle coordinates. The resulting expressions are given in terms of fast convergent improper integrals of the perturbation evaluated along segments of homoclinic/heteroclinic orbits of the unperturbed system. One important aspect in the computation is that, in the perturbed system, the action is a slow variable, while the angle is a fast variable.

We stress that, unlike the usual treatments of Melnikov theory, when one studies orbits homoclinic to hyperbolic fixed points, periodic or quasi-periodic orbits, here we study orbits homoclinic to NHIM's. The asymptotic dynamics inside the NHIM's may change under the perturbation.

The effect of the perturbation on the action-angle components of the scattering map can be interpreted, in the context of astrodynamics, as follows.

The difference in the action coordinates between the perturbed and the unperturbed scattering map can be interpreted as the change in energy due to the maneuver, or equivalently, the change in the 'size' of the periodic orbit which the homoclinic/heteroclinic orbit is asymptotic to. The difference in the angle coordinates between the perturbed and the unperturbed scattering map can be interpreted as the change in asymptotic phase due to the maneuver.

We mention here that there are numerous works on using hyperbolic invariant manifolds to design low-energy space mission, see, for example [KLMR00, GLMS01, GSLM01, GJSM01a, GJSM01b, Bel04, BGT10, PA14], and the references listed there. We hope that our results can be used to optimize the thrust that needs to be applied in order to maneuver between hyperbolic invariant manifolds on different energy levels.

1.3. **Related works.** The study of Hamiltonian systems subject to non-conservative perturbations is of practical interest in physical models, such as in celestial mechanics, where dissipation leads to migration of satellites and spacecrafts [MNF87, GEPC17, dlLK19, CCdlL20].

Computation of the scattering map, similar to the one in this paper, have been done in the case of the pendulum-rotator model subject to Hamiltonian perturbations, e.g., [DdlLS08, GdlL18]. The rotator-pendulum model is a product system and is naturally endowed with action-angle and hyperbolic coordinates. It has two conserved quantities: the action of the rotator and the total energy of the pendulum. The effect of the perturbation on the action component of the scattering map is relatively easy to compute directly. On the other hand, the effect on the angle component of the scattering map is more complicated to compute, since this is a fast variable. To circumvent this difficulty, the papers [DdlLS08, GdlL18] use the symplecticity of the scattering map to estimate indirectly the effect of the perturbation on the angle component of the scattering map.

The perturbed scattering map has been computed in the case of the pendulum-rotator model subject to non-conservative perturbations in [GdlLM21]. Since the perturbations are not Hamiltonian, the symplectic argument used in [DdlLS08] can no longer be applied. Therefore, to determine the effect of the perturbation on the angle component of the scattering map, a direct computation is performed in [GdlLM21].

The PCRTBP model considered in this paper presents some significant differences from the pendulum-rotator model. First, it has only one conserved quantity, namely the total energy. Second, the PCRTBP is not a product system, and does not carry a globally defined system of action-angle and hyperbolic coordinates. Third, in the unperturbed case the stable and unstable manifolds associated to a NHIM do not coincide. For these reasons, we construct locally defined systems of action-angle and hyperbolic coordinates along the unstable manifold as well as along the stable manifold, respectively. At the intersection of the unstable and stable manifolds, the

two coordinate systems do not agree in general. So the computation of the perturbed scattering map has to take into account the 'mismatch' between these coordinate systems. The dynamics in these coordinate systems fails to be of product type, as there is a coupling between the action-angle and the hyperbolic coordinates, which also needs to be taken into account in the computation. All of these features make the computation of the perturbed scattering map for the planar circular restricted three-body problem more intricate than for the rotator-pendulum system. Some of the calculations are simplified taking advantage that some of the variables in the unperturbed system are slow variables, but we can deal with perturbation theory for fast variables by observing that, near the NHIM's, the difference between the variables and their asymptotic values is slow (a technique already used in [GdlLM21].

1.4. **Structure of the paper.** In Section 2 we describe a general set-up for two-degrees of freedom Hamiltonian systems subject to non-conservative, time-dependent perturbations. We also state the main result, Theorem 2.1. It provides an expansion of the perturbed scattering map in terms of the unperturbed scattering map, where the first order term in the expansion is given explicitly in Section 5, in Proposition 5.2 and Proposition 5.7. Section 3 describes how to verify the hypotheses of Theorem 2.1 in the context of the PCRTBP. Section 4 defines some suitable coordinate systems, which we use to describe the geometric objects of interest. The proof of the main result is given in Section 4.

2. Set-up and main result

Consider a (real analytic) $\mathcal{C}^\omega$, symplectic manifold (M, Ω) of dimension 4. Each point $z \in M$ is described via a system of local coordinates $z = z(p, q)$ with $(p, q) \in \mathbb{R}^2 \times \mathbb{R}^2$, such that Ω relative to these coordinates is the standard symplectic form

$$\Omega = dp \wedge dq = \sum_{i=1}^{2} dp_i \wedge dq_i. \tag{2.1}$$

On M we consider a non-autonomous system of differential equations

$$\frac{d}{dt} z = \mathcal{X}^0(z) + \varepsilon \mathcal{X}^1(z, t; \varepsilon), \tag{2.2}$$

where $\mathcal{X}^0 : M \to TM$ is a $\mathcal{C}^\omega$-smooth vector field on M, $\mathcal{X}^1 : M \times \mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \to TM$ is a time-dependent, parameter dependent $\mathcal{C}^r$-smooth vector field on M, with r sufficiently large, and $\varepsilon \in \mathbb{R}$ is a 'smallness' parameter, taking values in the interval $(-\varepsilon_0, \varepsilon_0)$ around 0. The dependence of $\mathcal{X}^1(z, t; \varepsilon)$ on the time t is assumed to be of a general type – not necessarily periodic or quasi-periodic.

The flow of (2.2) will be denoted by Φ_ε^t.

The assumptions that the manifold M is 4-dimensional and that $\mathcal{X}^0$ is analytic are motivated by applications to celestial mechanics (we will consider the special case when the unperturbed system is given by the PCRTBP). We will use the assumption that $\mathcal{X}^0$ is analytic only to be able to quote several normal form theorems. We believe it could be weakened to finitely differentiability at the price of providing some new normal form theorems.

The assumption that $\mathcal{X}^1$ is only $\mathcal{C}^r$ and possibly non-Hamiltonian is also motivated by applications, as $\mathcal{X}^1$ can be chosen to model the thrust applied to a spacecraft for some time. In particular $\mathcal{X}^1$ can have compact support in space and in time. Note that even if the perturbation were analytic, the NHIM's which play a role in our treatment can only be assumed to be finite differentiable. The regularity is limited by ratios between the tangential and normal contraction rates, as well as by the regularity of the perturbation. Since the motion on the unperturbed NHIM is integrable, we have that for ε small enough, the tangential rates are close to zero, so that the limitations of regularity due to the rates become irrelevant.

Below, we will require that the vector fields $\mathcal{X}^0$, $\mathcal{X}^1$ satisfy additional assumptions.

2.1. **The unperturbed system.** We assume that the vector field $\mathcal{X}^0$ represents an autonomous Hamiltonian vector field, that is, $\mathcal{X}^0 = J\nabla H_0$ for some $\mathcal{C}^\omega$ Hamiltonian function $H_0 : M \to \mathbb{R}$, where J is an almost complex structure compatible with the standard symplectic form given by (2.1), and the gradient $\nabla := \nabla_z$ is with respect to the associated Riemannian metric. The Hamilton equation for the unperturbed system is:

$$\frac{d}{dt}z = J\nabla H_0(z). \tag{2.3}$$

2.1.1. *Unperturbed NHIM and action-angle coordinates.* We assume that the Hamiltonian flow associated to H_0 satisfies the conditions **(A-i)** and **(A-ii)** below. In Section 3, we will see that these conditions can be verified in the PCTBP.

(A-i) *There exists an equilibrium point L_1 of saddle-center type, that is, the linearized system $DJ\nabla H_0$ at L_1 has eigenvalues of the type $\pm\lambda, \pm i\omega$, with $\lambda, \omega \neq 0$.*

Consequently, by the Lyapunov Center Theorem [Mos58], there exists a 1-parameter family of periodic orbits $\lambda_0(h)$, parametrized by the energy level $H = h$, for $h \in (H(L_1), h_1)$ with h_1 sufficiently small, such that $\lambda_0(h)$ shrinks to L_1 as $h \to H(L_1)$. This family of periodic orbits determines a 2-dimensional manifold $\Lambda_0 \simeq D \times \mathbb{T} \subseteq M$ which is a normally hyperbolic invariant manifold (NHIM) with boundary for the Hamiltonian flow Φ_0^t of H_0, where D is closed interval with non-empty interior contained in $(H(L_1), h_1)$. The notion of a NHIM is recalled in Definition A.1, Appendix A.

The NHIM Λ_0 is symplectic when endowed with the form $\Omega_{|\Lambda_0}$, where Ω is given by (2.1). Moreover, Λ_0 is foliated by the periodic orbits $\lambda_0(h)$, i.e.,

$\Lambda_0 = \bigcup_{h\in D} \lambda_0(h)$. The flow Φ_0^t on each $\lambda_0(h)$ is a constant speed flow. In particular, the dynamics restricted to the NHIM is integrable. Therefore Λ_0 can be parametrized in terms of symplectic action-angle variables (I, θ), so that each periodic orbit $\lambda_0(h)$ represents a level set I_h of the action variable. The action I_h is uniquely determined by the energy level $H_0 = h$. In fact, as we will see in Section 4.1, there exists a system of action-angle and hyperbolic variables (I, θ, y, x) in a neighborhood of L_1 such the Hamiltonian H_0 can be written in a normal form, and (I, θ) on Λ_0 coincide – up to a phase shift – with the action-angle coordinates described above.

2.1.2. *Homoclinic connections.*

(A-ii) *There exists a relatively compact open set $\mathscr{K}$ in M such that the unstable manifold $W^{\mathrm{u}}_{\mathscr{K}}(\Lambda_0)$ and the stable manifold $W^{\mathrm{s}}_{\mathscr{K}}(\Lambda_0)$ inside $\mathscr{K}$ intersect transversally along a* homoclinic channel $\Gamma_0 \subset \mathscr{K}$.

The definition of a homoclinic channel is given in Appendix B, Definition B.1.

The unstable and stable manifolds of each periodic orbit $\lambda_0(h)$ are contained in the same energy level as $\lambda_0(h)$, i.e., $W^{\mathrm{u}}(\lambda_0(h)), W^{\mathrm{s}}(\lambda_0(h)) \subseteq M_h$, where $M_h = \{z \in M \mid H = h\}$. By **(A-ii)**, these manifolds intersect transversally within the energy level. Hence, each homoclinic orbit is asymptotic, in both forward and backwards time, to the periodic orbit $\lambda_0(h)$. The homoclinic channel Γ_0 consists of a 1-dimensional family of homoclinic orbit segments.

In Section 4.1, we will see that the normal form coordinates (I, θ, y, x) can be extended via the flow Φ_0^t along neighborhoods of $W^{\mathrm{u}}(\Lambda_0)$ and $W^{\mathrm{s}}(\Lambda_0)$, yielding two systems of action-angle and hyperbolic variables $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, y^{\mathrm{u}}, x^{\mathrm{u}})$, $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, y^{\mathrm{s}}, x^{\mathrm{s}})$, respectively. Relative to these two systems of coordinates $W^{\mathrm{u}}(\Lambda_0)$ can be described locally by $y^{\mathrm{u}} = 0$, and $W^{\mathrm{s}}(\Lambda_0)$ can be described locally by $x^{\mathrm{s}} = 0$. The coordinate systems $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, y^{\mathrm{u}}, x^{\mathrm{u}})$ and $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, y^{\mathrm{s}}, x^{\mathrm{s}})$ do not agree with one another in a neighborhood of the homoclinic channel Γ_0.

2.1.3. *Unperturbed scattering map associated to a homoclinic channel.* Let $\Omega^- : W^{\mathrm{u}}(\Lambda_0) \to \Lambda_0$ be the projection mapping defined by $\Omega^-(z_0) = z_0^-$, where $z_0^- \in \Lambda_0$ is the footpoint of the unstable fiber through $z_0 \in W^{\mathrm{u}}(\Lambda_0)$. Similarly, let $\Omega^+ : W^{\mathrm{s}}(\Lambda_0) \to \Lambda_0$ be the projection mapping defined by $\Omega^+(z_0) = z_0^+$, where $z_0^+ \in \Lambda_0$ is the footpoint of the stable fiber through $z_0 \in W^{\mathrm{s}}(\Lambda_0)$. The stable and unstable fibers are defined in Appendix A, equation (A.4).

Consider the homoclinic channel Γ_0 from condition **(A-ii)**. By the definition of a homoclinic channel, $\Omega^{\pm}$ restricted to Γ_0 is a diffeomorphism onto its image. To any homoclinic channel we can associate a *scattering map*, which is defined in Appendix B, Definition B.2. Specifically, the scattering

map σ_0 associated to Γ_0 is given by:

$$\begin{aligned} &\sigma_0 : \Omega^-(\Gamma_0) \subseteq \Lambda_0 \to \Omega^+(\Gamma_0) \subseteq \Lambda_0, \\ &\sigma_0(z_0^-) = z_0^+, \end{aligned} \tag{2.4}$$

provided that there exists a homoclinic point $z_0 \in \Gamma_0$ such that $\Omega^-(z_0) = z_0^-$ and $\Omega^+(z_0) = z_0^+$. For more details on the scattering map, see Appendix B.

The energy preservations along the stable and unstable manifolds of periodic orbits implies that σ_0 leaves each periodic orbit $\lambda_0(h)$ invariant, that is, $\sigma_0(\lambda_0(h)) \subseteq \lambda_0(h)$.

Fixing a point $z_0 \in \Gamma$, we have that $\sigma_0(z_0^-) = z_0^+ = z_0^- + \Delta$, for some Δ depending on z_0. The invariance property of the scattering map (B.2), and the fact that Φ_0^t restricted to $\lambda_0(h)$ is linear implies that

$$\sigma_0(\Phi_0^t(z_0^-)) = \Phi^t(z_0^+) = \Phi^t(z_0^- + \Delta) = \Phi^t(z_0^-) + \Delta,$$

for all t for which $\Phi_0^t(z_0)$ remains in Γ_0. This implies that, in terms of the action-angle coordinates (I, θ), σ_0 is given by a shift in the angle

$$\sigma_0(I, \theta) = (I, \theta + \Delta(I)),$$

for some function Δ that depends differentiably on I. We stress that Δ also depends on the choice of the homoclinic channel Γ_0.

2.2. **The perturbation.** The vector field $\mathcal{X}^1$ is a time-dependent, parameter-dependent vector field on M.

(A-iii) *We assume that $\mathcal{X}^1 = \mathcal{X}^1(z, t; \varepsilon)$ is $\mathcal{C}^r$-differentiable in all variables with uniformly bounded derivatives on $\mathscr{K} \times \mathbb{R} \times (-\varepsilon_0, \varepsilon_0)$, where the set $\mathscr{K}$ is as in the condition **(A-ii)**.*

Above, we assume that r is suitably large. We will not assume that $\mathcal{X}^1$ is Hamiltonian. Thus, our setting can be used to model dissipation or forcing applied to a Hamiltonian system. Note that non-Hamiltonian perturbations are very singular, in the sense that periodic and homoclinic orbits may disappear. On the other hand, the NHIM's and their stable and unstable manifolds persist and can be used as the basis for perturbative calculations.

As a particular case, we will also write our results for the case when the perturbation $\mathcal{X}^1$ in (2.2) is Hamiltonian, given by

$$\mathcal{X}^1(z, t; \varepsilon) = J \nabla_z H_1(z, t; \varepsilon), \tag{2.5}$$

where H_1 is a time-dependent, parameter-dependent $\mathcal{C}^r$-smooth Hamiltonian function on M.

When the perturbation is added to the system, as we will see in Section 5.1, the NHIM for the unperturbed system can be continued to a NHIM for the perturbed system, and the transverse homoclinic/heteroclinic orbits for the unperturbed system can be continued to transverse homoclinic/heteroclinic orbits for the perturbed system, provided that the perturbation is sufficiently small. Hence there exists an associated scattering map for the perturbed system.

The goal is to quantify the effect of the perturbation on the corresponding scattering map.

2.3. **Extended system.** We consider the system (2.2) under the conditions **(A-i)**, and **(A-ii)**. We associate to it the extended system

$$\begin{aligned} \frac{d}{d\tau} z &= \mathcal{X}^0(z) + \varepsilon \mathcal{X}^1(z,t;\varepsilon), \\ \frac{d}{d\tau} t &= 1, \end{aligned} \tag{2.6}$$

which is defined on the extended phase space $\widetilde{M} = M \times \mathbb{R}$. We denote $\tilde{z} = (z,t) \in \widetilde{M}$. The independent variable will be denoted by τ from now on. We will denote by $\tilde{\Phi}_\varepsilon^\tau$ the extended flow of (2.6). We have

$$\tilde{\Phi}_\varepsilon^\tau(z,t) = (\Phi_\varepsilon^\tau(z), t+\tau).$$

In the extended phase space we have the following:

$$\tilde{\Lambda}_0 = \Lambda_0 \times \mathbb{R}$$

is a NHIM with boundary for the extended unperturbed flow $\tilde{\Phi}_0^\tau$, and

$$\tilde{\Gamma}_0 = \Gamma_0 \times \mathbb{R}$$

is a homoclinic channel for $\tilde{\Phi}_0^\tau$.

The scattering map associated to $\tilde{\Gamma}_0$ is given by

$$\tilde{\sigma}_0(I,\theta,t) = (I, \theta + \Delta(I), t).$$

2.4. **Main result.** We describe a general set up for two-degrees of freedom Hamiltonian systems subject to non-conservative, time-dependent perturbations. We stress that some of the results below hold in a more general setting. Theorem 2.1 gives a first order approximation of the perturbed scattering map $\tilde{\sigma}_\varepsilon$, where the 0-th order term is the unperturbed scattering map $\tilde{\sigma}_0$, and the 1-st order term is given by explicit Melnikov-type formulas. The unperturbed scattering map $\tilde{\sigma}_0$ is defined on the unperturbed NHIM $\tilde{\Lambda}_0$. The manifold $\tilde{\Lambda}_0$ is parametrized by (I,θ,t), where (I,θ) are the action-angle coordinates described in Section 2.1.1. The perturbed scattering map $\tilde{\sigma}_\varepsilon$ is defined on the perturbed NHIM $\tilde{\Lambda}_\varepsilon$ which persists from $\tilde{\Lambda}_0$, and whose existence is given by Theorem 2.1-(i). Moreover, the perturbed NHIM $\tilde{\Lambda}_\varepsilon$ can be parametrized in terms of the coordinates (I,θ,t). See Section 5.1.1. Therefore, in Theorem 2.1-(iii)-(2.7), we express both the unperturbed and the perturbed scattering map in terms of the (I,θ,t)-coordinates.

Theorem 2.1. *Consider the system* (2.2).

*Assume that the unperturbed system $\mathcal{X}^0$ satisfies the conditions **(A-i)** and **(A-ii)** and that the perturbation $\mathcal{X}^1$ satisfies the condition **(A-iii)**.*

Then, there exists ε_1, with $0 < \varepsilon_1 < \varepsilon_0$, such that the following hold true:

(i) *For all $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, there is a $\mathcal{C}^\ell$-family of NHIMs $\tilde{\Lambda}_\varepsilon$ for the extended flow $\tilde{\Phi}_\varepsilon^t$, for some $\ell \geqslant 1$;*

(ii) *For all* $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$*, the unstable and stable manifolds of* $\tilde{\Lambda}_\varepsilon$*,* $W^{\mathrm{u}}(\tilde{\Lambda}_\varepsilon)$ *and* $W^{\mathrm{s}}(\tilde{\Lambda}_\varepsilon)$*, respectively, intersect transversally, in the extended phase space* $\widetilde{M}$*, along a homoclinic channel* $\tilde{\Gamma}_\varepsilon$*;*

(iii) *The perturbed scattering map* $\tilde{\sigma}_\varepsilon$ *associated to* $\tilde{\Gamma}_\varepsilon$ *can be written as*

$$\tilde{\sigma}_\varepsilon(I,\theta,t) = \tilde{\sigma}_0(I,\theta,t) + \varepsilon\tilde{\mathcal{S}}(I,\theta,t) + O_{\mathcal{C}^\ell}(\varepsilon^2), \tag{2.7}$$

where $\tilde{\mathcal{S}} = (\tilde{\mathcal{S}}^I, \tilde{\mathcal{S}}^\theta, \tilde{\mathcal{S}}^t)$ *is a mapping from some domain in* $D \times \mathbb{T} \times \mathbb{R}$ *to* $\mathbb{R} \times \mathbb{T} \times \mathbb{R}$ *as follows:*

(iii-a) *the components* $\tilde{\mathcal{S}}^I$ *and* $\tilde{\mathcal{S}}^\theta$ *are given by* (5.17) *and* (5.40)*, respectively, and*

(iii-b) *the component* $\tilde{\mathcal{S}}^t$ *is given by* $\tilde{\mathcal{S}}^t(I,\theta,t) = t$*.*

We recall the notation $O_{\mathcal{C}^k}(\cdot)$ used above: $f = O_{\mathcal{C}^k}(g)$ means that $\|f\|_{\mathcal{C}^k} \leqslant M\|g\|_{\mathcal{C}^k}$ for some $M > 0$, where $k \geqslant 0$, and $\|\cdot\|_{\mathcal{C}^k}$ is the $\mathcal{C}^k$-norm. In the sequel, to simplify the notation we will write $O(\cdot)$ without the subscript indicating the function space topology, whenever this can be inferred from the context.

2.5. **Heteroclinic connections.** Instead of the conditions **(A-i)** and **(A-ii)** we assume that the Hamiltonian flow associated to H_0 satisfies the conditions **(A'-i)** and **(A'-ii)** from below.

Condition **(A'-i)** has two parts **(A'-i-a)** and **(A'-i-b)**.

(A'-i-a) *There exists two equilibrium points* L_1*,* L_2 *of saddle-center type.*

We do not assume that the two equilibrium points are on the same energy level, that is $H(L_1) \neq H(L_2)$ in general. Consequently, for each equilibrium point L_1, L_2 there exists a 1-parameter family of closed orbits $\lambda_0^1(h)$, for $h \in D^1$, and $\lambda_0^2(h)$ for $h \in D^2$, where D^1, D^2 are some closed intervals contained in some neighborhoods of $H(L_1)$, $H(L_2)$, respectively.

(A'-i-b) *There exists an interval of energies* $D \subseteq D^1 \cap D^2$*, with non-empty interior, such that there are periodic orbits* $\lambda_0^1(h)$*,* $\lambda_0^2(h)$ *for all* $h \in D$*. Moreover, there exist normal form coordinates* $(I^1, \theta^1, y^1, x^1)$ *and* $(I^2, \theta^2, y^2, x^2)$ *defined around* $\Lambda_0^1 := \bigcup_{h\in D} \lambda_0^1(h)$ *and* $Lambda_0^2 := \bigcup_{h\in D} \lambda_0^2(h)$*, respectively. These normal form coordinates are as in Section 4.1.*

We remark that the normal form coordinates in **(A'-i-b)** already exist in small neighborhoods of L_1 and L_2, respectively. Condition **(A'-i-b)** requires these normal form coordinates to extend to the specified energy range.

Condition **(A'-ii)** has two parts **(A'-ii-a)** and **(A'-ii-b)**.

(A'-ii-a) *There exists a relatively compact open set* $\mathscr{K}$ *in* M *such that the unstable manifold* $W^{\mathrm{u}}_{\mathscr{K}}(\Lambda_0^1)$ *and the stable manifold* $W^{\mathrm{s}}_{\mathscr{K}}(\Lambda_0^2)$ *inside* $\mathscr{K}$ *intersect transversally along a* heteroclinic channel $\Gamma_0 \subset \mathscr{K}$*.*

The definition of a heteroclinic channel is given in Definition B.3, Appendix B. As a consequence, there exist transverse heteroclinic orbits from Λ_0^1 to Λ_0^2. Each such heteroclinic orbit is asymptotic in backwards time to a

periodic orbit $\lambda_0^1(h)$, and is asymptotic in forward time to a periodic orbit $\lambda_0^2(h)$.

We require an additional non-degeneracy condition **(A'-ii-b)** formulated in terms of normal forms, which will be given in Section 4.1, Remark 4.4. As we will see, **(A'-ii-b)** consists of some explicit and verifiable conditions that the derivatives of certain functions are non-zero. In the case of homoclinic connections we do not need a separate assumption analogous condition to **(A'-ii-b)**, as this is automatically satisfied; see Section 4.1.

2.5.1. *Unperturbed scattering map associated to a heteroclinic channel.* As before, we define $\Omega^{-,1} : W^{\mathrm{u}}(\Lambda_0^1) \to \Lambda_0^1$ by $\Omega^{-,1}(z_0) = z_0^-$, where $z_0^- \in \Lambda_0^1$ is the footpoint of the unstable fiber through $z_0 \in W^{\mathrm{u}}(\Lambda_0^1)$, and $\Omega^{+,2} : W^{\mathrm{s}}(\Lambda_0^2) \to \Lambda_0^2$ by $\Omega^{+,2}(z_0) = z_0^+$, where $z_0^+ \in \Lambda_0^2$ is the footpoint of the stable fiber through $z_0 \in W^{\mathrm{s}}(\Lambda_0^2)$.

Associated to the heteroclinic channel Γ_0 we can define the scattering map as in Definition B.4 in Appendix B. Specifically,

$$\sigma_0 : \Omega^{-,1}(\Gamma_0) \subseteq \Lambda_0^1 \to \Omega^{+,2}(\Gamma_0) \subseteq \Lambda_0^2,$$

is given by

$$\sigma_0(z_0^-) = z_0^+,$$

provided that there exists a $z_0 \in \Gamma_0$ such that $\Omega^{-,1}(z_0) = z_0^-$ and $\Omega^{+,2}(z_0) = z_0^+$.

Since H_0 is constant along heteroclinic orbits, we have that $I(z_0^-) = I(z_0^+)$. Then the scattering map, expressed in terms of the action-angle coordinates, is given by

$$\sigma_0(I, \theta) = (I, \theta + \Delta(I)),$$

for some function Δ that depends smoothly on I.

In this case, we can obtain a result similar to Theorem 2.1. For brevity, we will not provide the precise formulas for the components of the corresponding expansion of the perturbed scattering map, which is analogous to (2.7). Such formulas are analogues of (5.17) and (5.40).

3. Geometric structures in the planar circular restricted three-body problem.

In this section we survey the status of the verification of the conditions **(A-i)** and **(A-ii)**, from Section 2.1, or the conditions **(A'-i)** and **(A'-ii)** from Section 2.5, in the concrete model of the PCRTBP. Some of the verifications in the literature are rigorous and some of them are numerical.

Of course, the verification of the hypothesis of Theorem 2.1 in a concrete model does not affect the validity of the rigorous arguments establishing Theorem 2.1, and the reader interested only in rigorous results may safely skip this section.

We note that, since our hypothesis are mainly transversality conditions, they can be verified with finite precision calculations, which seem to be safe calculations for today's standard and could well be accessible to *"computer*

assisted proofs". We hope that this work could stimulate more extensive verifications.

The PCRTBP is a model describing the motion of an infinitesimal body under the Newtonian gravity exerted by two heavy bodies moving on circular orbits about their center of mass, under the assumption that these orbits are not affected by the gravity of the infinitesimal body.

We can think of the heavy bodies (referred to as primaries) representing the Earth and Moon, and the infinitesimal mass representing a spaceship.

It is convenient to study the motion of the infinitesimal body relative to a co-rotating frame which rotates with the primaries around the center of mass, and to use normalized units. Henceforth, the masses of the heavy bodies are $1-\mu$ and μ, where $\mu \in (0, 1/2]$. Relative to the co-rotating frame, the heavier mass $1-\mu$ is located at $(\mu, 0)$, and the lighter mass μ is located at $(-1+\mu, 0)$. The motion of the infinitesimal body relative to the co-rotating frame is described via the autonomous Hamiltonian

$$H_0(p_1, p_2, q_1, q_2) = \frac{(p_1+q_2)^2 + (p_2-q_1)^2}{2} - V(q_1, q_2), \tag{3.1}$$

where $(p, q) = (p_1, p_2, q_1, q_2) \in \mathbb{R}^4$ represents the momenta and the coordinates of the infinitesimal body with respect to the co-rotating frame,

$$\begin{aligned} V(q_1, q_2) &= \frac{q_1^2+q_2^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}, \\ r_1 &= \left((q_1-\mu)^2 + q_2^2\right)^{1/2}, \\ r_2 &= \left((q_1+1-\mu)^2 + q_2^2\right)^{1/2}. \end{aligned} \tag{3.2}$$

Above $V(q_1, q_2)$ represents the effective potential, and r_1, r_2 represent the distances from the infinitesimal body to the masses $1-\mu$ and μ, respectively. The phase space

$$M = \{(p, q) \in \mathbb{R}^4 \,|\, q \neq (\mu, 0), \text{ and } q \neq (-1+\mu, 0)\}$$

is endowed with the symplectic form

$$\Omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2.$$

Note that the phase space M is not compact.

The equations of motion of the infinitesimal body are given by the Hamilton equations corresponding to (3.1), that is

$$\frac{d}{dt} z = J \nabla H_0(z), \tag{3.3}$$

where $z = z(p_1, p_2, q_1, q_2)$ and J represents the standard almost complex structure.

The Hamiltonian H_0 is an integral of motion, so the flow Φ_0^t of (3.3) leaves invariant each energy hyper-surface

$$M_h := H_0^{-1}(h) = \{(p_1, p_2, q_1, q_2) \in M \,|\, H_0(p_1, p_2, q_1, q_2) = h\}. \tag{3.4}$$

The system has three equilibrium points L_1, L_2, L_3 located on the q_1-axis, and two other equilibrium points L_4, L_5, each lying in the (q_1, q_2)-plane and forming an equilateral triangle with the primaries. The first three of the equilibrium points are attributed to Euler, and the last two are attributed to Lagrange. In our convention, L_1 is located between the primaries, L_2 is on the side of the lighter primary, and L_3 is on the side of the heavier primary. The linearized stability of L_1, L_2, L_3 is of center-saddle type. The linearized stability of L_4, L_5 is of center-center type, for μ less than some critical value $\mu_{\rm cr}$.

We note that condition **(A-i)** is satisfied for each of the equilibrium points L_1, L_2, L_3.

For $i = 1, 2, 3$, for each energy level h, with $H(L_i) < h$ and h sufficiently close to $H(L_i)$, there exists a unique periodic orbit $\lambda_0(h)$ near the equilibrium point L_i, which is referred to as a Lyapunov orbit. The existence of such periodic orbits follows from the Lyapunov Center Theorem (see, e.g., [Mos58]). Moreover, there exists a neighborhood of L_i in the phase space where the Hamiltonian H_0 can be written in a normal form relative to some suitable coordinates (I, θ, y, x); see Section 4.1.

Each Lyapunov orbit is hyperbolic in the energy surface, so it has associated 2-dimensional unstable and stable manifolds denoted $W^{\rm u}(\lambda_0(h))$ and $W^{\rm s}(\lambda_0(h))$.

Numerical evidence, as well as computer assisted proofs, show that these periodic orbits can be continued for energy levels $h > H(L_i)$ that are not close to $H(L_i)$ (see, e.g., [Bro68, CR12]).

Normally hyperbolic invariant manifold for the unperturbed system. For an energy range $h \in D$ sufficiently close to the energy of L_i, the family of Lyapunov orbits

$$\Lambda_0 = \bigcup_{h \in D} \lambda_0(h), \tag{3.5}$$

defines a 2-dimensional NHIM with boundary for the Hamiltonian flow of (3.3). The NHIM carries the symplectic structure $\Omega_{|\Lambda_0}$, and it can be described in terms of the action-angle coordinates (I, θ). The action $I = I_h$ is uniquely defined by the energy h, and θ is symplectically conjugate with I with respect to $\Omega_{|\Lambda_0}$. The variable I is a first integral along the trajectories of the flow on Λ_0, and the action level sets are in fact the Lyapunov orbits $\lambda_0(h)$. The motion restricted to each Lyapunov orbit is a rigid rotation in the variable θ, with the frequency depending on the energy level.

The NHIM Λ_0 and its unstable and stable manifolds $W^{\rm u}(\Lambda_0)$ and $W^{\rm s}(\Lambda_0)$ have simple descriptions in terms of the normal form coordinates (I, θ, y, x) in a neighborhood of L_1: Λ_0 corresponds to $x = y = 0$, $W^{\rm u}(\Lambda_0)$ corresponds to $y = 0$, and $W^{\rm s}(\Lambda_0)$ corresponds to $x = 0$.

Homoclinic connections. We first focus on the dynamics around the equilibrium point L_1. There are analytic arguments (see [LMS85]) showing that, for a discrete set of values of μ that are sufficiently small, and for each h

sufficiently close to $H(L_1)$, the branches of $W^{\mathrm{u}}(\lambda_0(h))$ and $W^{\mathrm{s}}(\lambda_0(h))$ on the side of the heavier primary do not collide with the primary and intersect transversally along some homoclinic orbit $\gamma_0(h)$, not necessarily unique. Numerical evidence, as well as computer assisted proofs, show that this property holds in fact for a large range of values of masses μ and energies h; see, e.g. [KLMR00, GLMS01, GSLM01, GJSM01a, GJSM01b, Cap12].

Each choice of a transverse homoclinic orbit $\gamma_0(h)$ can be continued in energy h to a family of such homoclinic orbits, which forms a homoclinic manifold $\bigcup_{h\in D}\gamma_0(h)$. Moreover, we can ensure that $W^{\mathrm{u}}(\lambda_0(h))$ and $W^{\mathrm{s}}(\lambda_0(h))$ are contained inside some compact subset $\mathscr{K}$ of the phase space.

We define the transverse homoclinic manifold Γ_0 as an open disk contained in $\bigcup_{h\in D}\gamma_0(h)$. It consists of a 1-dimensional family of segments of homoclinic orbits. In order to define the scattering map, we need that the stable (resp. unstable) fibers of Λ_0 are transverse to Γ_0 relative to $W^{\mathrm{s}}(\Lambda_0)$ (resp. $W^{\mathrm{u}}(\Lambda_0)$), as per condition (B.1). This is automatically satisfied provided h is sufficiently close to $H(L_1)$, since the flow is transverse to the corresponding fibers relative to $W^{\mathrm{s}}(\Lambda_0)$ (resp. $W^{\mathrm{u}}(\Lambda_0)$). This can be easily seen, for instance, from (4.6).

To define the scattering map, we also need that $\Omega^{\pm}$ are diffeomorphisms from Γ_0 onto their images; see Definition B.1. We can achieve this by further restricting to a suitable submanifold

$$\Gamma_0 \subseteq \bigcup_{h\in D}\gamma_0(h), \tag{3.6}$$

so that Γ_0 is a homoclinic channel.

In this way, we can ensure condition **(A-ii)**.

In summary, for the PCRTBP, we can verify the existence of the geometric structures of interest and the corresponding conditions **(A-i)** and **(A-ii)**from Section 2.1.

Heteroclinic connections. We now focus on the dynamics around the equilibrium points L_1 and L_2. They satisfy condition **(A'-i-a)**. For energy levels h with $H(L_1) \lesssim h$, there exists a family $\lambda_0^1(h)$ of Lyapunov orbits near L_1, and for $H(L_2) \lesssim h$ there exists a family $\lambda_0^2(h)$ of Lyapunov orbits near L_2. Moreover, there exist normal form coordinates $(I^1, \theta^1, y^1, x^1)$ and $(I^2, \theta^2, y^2, x^2)$ defined around L_1 and L_2, respectively, for some suitable energy ranges. These normal form coordinates are as in Section 4.1.

Numerical evidence shows that families of periodic orbits near L_1 and L_2 can exist simultaneously, for some energy range. Therefore, we consider an interval of energies D such that, for $h \in D$ we have the following: there exists families of periodic orbits $\lambda_0^1(h)$ near L_1, and $\lambda_0^2(h)$ near L_2, the following sets

$$\begin{aligned} \Lambda_0^1 &= \bigcup_{h\in D}\lambda_0^1(h), \\ \Lambda_0^2 &= \bigcup_{h\in D}\lambda_0^2(h), \end{aligned} \tag{3.7}$$

are NHIM's for the Hamiltonian flow of (3.3), and the normal form coordinates $(I^1, \theta^1, y^1, x^1)$ and $(I^2, \theta^2, y^2, x^2)$ are defined in some neighborhoods of Λ_0^1 and Λ_0^2, respectively, granting **(A'-i-b)**.

Numerical evidence, as well as computer assisted proofs, show that there exist transverse heteroclinic connections determined by $W^{\mathrm{u}}(\lambda_0^1(h)) \cap W^{\mathrm{s}}(\lambda_0^2(h))$ and $W^{\mathrm{u}}(\lambda_0^2(h)) \cap W^{\mathrm{s}}(\lambda_0^1(h))$ for certain ranges of energies. See, e.g., [KLMR00, GLMS01, GSLM01, GJSM01a, GJSM01b, WZ03, CM06, BGT10]. In either case, we denote the corresponding family of heteroclinic orbits by $\gamma_0(h)$. We consider a range of energies $h \in D$ for which this additional condition on the existence of transverse heteroclinic connections is satisfied.

The transverse intersection of the unstable manifold $W^{\mathrm{u}}(\Lambda_0^1)$ with the stable manifold $W^{\mathrm{s}}(\Lambda_0^2)$ define a heteroclinic manifold $\bigcup_{h \in D} \gamma_0(h)$, which depends on the choice of the family of heteroclinic orbits $\gamma_0(h)$. One can always restrict to a submanifold

$$\Gamma_0 \subseteq \bigcup_{h \in D} \gamma_0(h), \tag{3.8}$$

that is a heteroclinic channel.

If that is the case, the condition **(A'-ii-a)** is verified.

The remaining condition **(A'-ii-b)**, which will be given in Section 4.1, Remark 4.4, consists of some explicit non-degeneracy conditions on the normal forms. Such condition can also be verified numerically.

Thus, the conditions **(A'-i)** and **(A'-ii)** from Section 2.5 are checkable numerically.

It would be of interest to verify if Theorem 2.1 can be applied when the equilibrium point L_3 is also considered; some numerical results concerning the dynamics around L_3 can be found in [BO06, CCP16, JN20, GLMS01, GSLM01, GJSM01a, GJSM01b].

In summary, in this section we have outlined how the conditions of the Theorem 2.1 can be verified in the PCRTBP. The theoretical results of Theorem 2.1 are independent on the application to the PCRTBP.

4. Coordinate systems and evolution equations

4.1. **New coordinate systems for the unperturbed system.** We consider the case of homoclinic connections described by conditions **(A-i)** and **(A-ii)**. Under these assumptions, the manifolds $W^{\mathrm{u}}(\Lambda_0)$ and $W^{\mathrm{s}}(\Lambda_0)$ intersect transversally along the homoclinic channel Γ_0.

The next proposition states that, in a neighborhood of each $W^{\mathrm{u}}(\Lambda_0)$ and $W^{\mathrm{s}}(\Lambda_0)$, there exists a system of symplectic coordinates such that Λ_0, $W^{\mathrm{u}}(\Lambda_0)$ and $W^{\mathrm{s}}(\Lambda_0)$ have very simple descriptions, and moreover the unperturbed Hamiltonian H_0 can be written in a normal form relative to the corresponding coordinates. As before, for $z \in W^{\mathrm{s,u}}(\Lambda_0)$, we denote $z^{\pm} = \Omega^{\pm}(z)$.

Informally, we have three systems of coordinates near the homoclinic intersection. One is given by the normal form. Two other systems of coordinates are obtained propagating – by the unperturbed dynamics – the coordinate system along the stable and unstable manifolds.

As it is standard, we will think of the collections of coordinates as defining a geometric point. At the same time, we note that the coordinates are functions on a manifold. Therefore, we can compute the evolution of the coordinates using the usual calculus formulas in the ODE. As we will see, in perturbative calculations, it will be enough to use the unperturbed evolution.

Hence near the NHIM we have three coordinate systems, but on the NHIM the three coordinate systems agree.

Near the homoclinic intersection, we have two systems of coordinates. The two systems of coordinates do not agree because the propagation happens in two different ways. Nevertheless, the coordinate change can be computed from the unperturbed dynamics.

Proposition 4.1 (Normal Forms)**.** *There exist three systems of analytic, symplectic coordinates*[1]*, and some analytic functions $h_0 = h_0(I)$, $g_1 = g_1(I)$, and $g_2 = g_2(I, xy)$, defined for some range of energies $h \in D$, as follows:*

- **(N)** *A coordinate system (I, θ, y, x) in a neighborhood $\mathcal{N}$ of Λ_0 such that for $z \in \mathcal{N}$ we have:*
  - **(N-i)** *$z \in \Lambda_0$ if and only if $x(z) = y(z) = 0$;*
  - **(N-ii)** *$z \in W^{\mathrm{u}}(\Lambda_0)$ if and only if $y = 0$, and $z \in W^{\mathrm{s}}(\Lambda_0)$ if and only if $x = 0$;*
  - **(N-iii)** *for $z \in W^{\mathrm{u}}(\Lambda_0)$ we have $I(z) = I(z^-)$ and $\theta(z) = \theta(z^-)$, and for $z \in W^{\mathrm{s}}(\Lambda_0)$ we have $I(z) = I(z^+)$ and $\theta(z) = \theta(z^+)$;*
  - **(N-iv)** *H_0 restricted to $\mathcal{N}$ can be written in a normal form*

$$\begin{aligned} H_0(p,q) =& H_0(I, xy) \\ =& h_0(I) + (xy) g_1(I) + (xy)^2 g_2(I, xy). \end{aligned} \tag{4.1}$$

- **(U)** *A coordinate system $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, y^{\mathrm{u}}, x^{\mathrm{u}})$ in a neighborhood $\mathcal{N}^{\mathrm{u}}$ of $W^{\mathrm{u}}(\Lambda_0)$ such that for $z \in \mathcal{N}^{\mathrm{u}}$ we have:*
  - **(U-i)** *$z \in \Lambda_0$ if and only if $x^{\mathrm{u}}(z) = y^{\mathrm{u}}(z) = 0$;*
  - **(U-ii)** *$z \in W^{\mathrm{u}}(\Lambda_0)$ if and only if $y^{\mathrm{u}} = 0$;*
  - **(U-iii)** *for $z \in W^{\mathrm{u}}(\Lambda_0)$ we have $I^{\mathrm{u}}(z) = I^{\mathrm{u}}(z^-)$, and $\theta^{\mathrm{u}}(z) = \theta^{\mathrm{u}}(z^-)$;*
  - **(U-iv)** *H_0 restricted to $\mathcal{N}^{\mathrm{u}}$ can be written in a normal form*

$$\begin{aligned} H_0(p,q) =& H_0^{\mathrm{u}}(I^{\mathrm{u}}, x^{\mathrm{u}} y^{\mathrm{u}}) \\ =& h_0(I^u) + (x^u y^u) g_1(I^{\mathrm{u}}) + (x^{\mathrm{u}} y^{\mathrm{u}})^2 g_2(I^{\mathrm{u}}, x^{\mathrm{u}} y^{\mathrm{u}}). \end{aligned} \tag{4.2}$$

- **(S)** *A coordinate system $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, y^{\mathrm{s}}, x^{\mathrm{s}})$ in a neighborhood $\mathcal{N}^{\mathrm{s}}$ of $W^{\mathrm{s}}(\Lambda_0)$ such that for $z \in \mathcal{N}^{\mathrm{s}}$ we have:*
  - **(S-i)** *$z \in \Lambda_0$ if and only if $x^{\mathrm{s}}(z) = y^{\mathrm{s}}(z) = 0$;*
  - **(S-ii)** *$z \in W^{\mathrm{s}}(\Lambda_0)$ if and only if $x^{\mathrm{s}} = 0$;*

[1] coordinates obtained from (p, q) via a canonical transformation

(S-iii) *for $z \in W^{\mathrm{s}}(\Lambda_0)$ we have $I^{\mathrm{s}}(z) = I^{\mathrm{s}}(z^+)$, and $\theta^{\mathrm{s}}(z) = \theta^{\mathrm{s}}(z^+)$;*

(S-iv) *H_0 restricted to $\mathscr{N}^{\mathrm{s}}$ can be written in a normal form*

$$\begin{aligned} H_0(p,q) =& H_0^{\mathrm{s}}(I^{\mathrm{s}}, x^{\mathrm{s}} y^{\mathrm{s}}) \\ =& h_0\left(I^{\mathrm{s}}\right) + (x^{\mathrm{s}} y^{\mathrm{s}}) g_1\left(I^{\mathrm{s}}\right) + \left(x^{\mathrm{s}} y^{\mathrm{s}}\right)^2 g_2(I^{\mathrm{s}}, x^{\mathrm{s}} y^{\mathrm{s}}). \end{aligned} \tag{4.3}$$

(C) *The coordinate systems $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, y^{\mathrm{u}}, x^{\mathrm{u}})$ and $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, y^{\mathrm{s}}, x^{\mathrm{s}})$ coincide with (I, θ, y, x) on Λ_0, i.e.,*

$$(I^{\mathrm{u}}, \theta^{\mathrm{u}}, y^{\mathrm{u}}, x^{\mathrm{u}}) = (I^{\mathrm{s}}, \theta^{\mathrm{s}}, y^{\mathrm{s}}, x^{\mathrm{s}}) = (I, \theta, y, x) \text{ on } \Lambda_0. \tag{4.4}$$

Proof. Part **(N)** follows from [Gio01], so we will not give a detailed proof.

We only summarize the procedure to obtain the normal form in a neighborhood of a center-saddle equilibrium point for a 2-degrees of freedom Hamiltonian H_0. First, the Hamiltonian H_0 is expanded in a Taylor series around that equilibrium point (shifted to the origin) as

$$H_0(p,q) = H_2(p,q) + H_3(p,q) + H_4(p,q) + \dots,$$

where $H_j(p,q)$ is an homogeneous polynomial of degree j in the variables (p_1, p_2, q_1, q_2). Then, by making a linear canonical change of coordinates $(p,q) \mapsto (x,y)$, with the eigenvectors of the linearized system given by $J\nabla H_2(0,0)$ as the axes of the new system, the quadratic part H_2 of H_0 can be written in the new coordinates $(x,y) \in \mathbb{R}^4$ as

$$H_2(x,y) = \lambda x_1 y_1 + \frac{\omega}{2}(x_2^2 + y_2^2),$$

where $\pm\lambda_1 = \pm\lambda$ and $\pm\lambda_2 = \pm i\omega$ are the eigenvalues of $J\nabla H_2(0,0)$. Then, via another linear canonical change of coordinates

$$x_1 = \xi_1, \quad y_1 = \eta_1, \quad x_2 = \frac{\xi_2 + i\eta_2}{\sqrt{2}}, \qquad y_2 = \frac{i\xi_2 + \eta_2}{\sqrt{2}},$$

we obtain H_2 written in complex variables as

$$H_2^N(\xi,\eta) = \lambda\xi_1\eta_1 + i\omega\xi_2\eta_2 = \lambda_1\xi_1\eta_1 + \lambda_2\xi_2\eta_2.$$

The next step is to apply a sequence of changes of coordinates to kill all monomials for which the exponent of ξ_j is different from the exponent of η_j. Since the eigenvalue $\lambda_1 = \lambda$ is real and $\lambda_2 = i\omega$ is imaginary, there are no small divisors. The only possible source of divergence is due to the use of Cauchy's estimates for the derivatives required by the normalization procedure. In [Gio01] the accumulation of derivatives is controlled via a KAM technique. The process can be continued to any order.

Thus, in the limit H_0 can be written as an expansion

$$H_0(\xi,\eta) = H_2^N(\xi_1\eta_1, \xi_2\eta_2) + H_3^N(\xi_1\eta_1, \xi_2\eta_2) + H_4^N(\xi_1\eta_1, \xi_2\eta_2) + \dots,$$

where H_j is an homogeneous polynomial in $\xi_1\eta_1, \xi_2\eta_2$ of degree j. The series expansion of H_0 is convergent in a neighborhood $\mathscr{N}$ of the origin, and the coordinate change $x = x(\xi,\eta)$, $y = y(\xi,\eta)$ is canonical and given in terms of convergent series.

There exist periodic orbits $\lambda_0(h)$ around the equilibrium point for all energy levels h sufficiently close to the energy level of the equilibrium point. This implies that the NHIM $\bar{\Lambda}_0 = \bigcup_{h\in\bar{D}} \lambda_0(h)$ is contained in $\mathscr{N}$, for some suitable energy range $h \in \bar{D}$.

To express H_0 in action-angle coordinates one applies the canonical transformation

$$\xi_2 = \sqrt{I}\exp(i\theta), \quad \eta_2 = -i\sqrt{I}\exp(-i\theta).$$

Finally, denote $x = \xi_1$, $y = \eta_1$. We obtain the normal form

$$H_0(I, xy) = \lambda xy + \omega I + H_3(I, xy) + H_4(I, xy) + \dots. \tag{4.5}$$

Moreover, in these coordinates the following hold:

(i) The normally hyperbolic invariant manifold $\bar{\Lambda}_0$ is given by $x = y = 0$, and each periodic orbit in $\bar{\Lambda}_0$ corresponds to a level set of I;
(ii) The local unstable fibers $W^{\mathrm{u}}_{\mathscr{N}}(z)$ for $z \in \bar{\Lambda}_0$ are given by $\theta = \text{const.}$ and $y = 0$;
(iii) The local stable fibers $W^{\mathrm{s}}_{\mathscr{N}}(z)$ for $z \in \bar{\Lambda}_0$ are given by $\theta = \text{const.}$ and $x = 0$;
(iv) The local unstable invariant manifold $W^{\mathrm{u}}_{\mathscr{N}}(\bar{\Lambda}_0)$ is given by $y = 0$;
(v) The local stable invariant manifold $W^{\mathrm{s}}_{\mathscr{N}}(\bar{\Lambda}_0)$ is given by $x = 0$.

The equations of motion are

$$\begin{aligned}
\frac{d}{dt}I =& 0,\\
\frac{d}{dt}\theta =& \omega + \frac{\partial H_3^N}{\partial I} + \frac{\partial H_4^N}{\partial I} + \dots,\\
\frac{d}{dt}y =& -\lambda y - \frac{\partial H_3^N}{\partial x} - \frac{\partial H_4^N}{\partial x} + \dots,\\
\frac{d}{dt}x =& \lambda x + \frac{\partial H_3^N}{\partial y} + \frac{\partial H_4^N}{\partial y} + \dots.
\end{aligned} \tag{4.6}$$

Note that Hamiltonian H_0 on $\mathscr{N}$ has two first integrals I and xy, which are independent and in involution.

This implies that, if $z \in W^{\mathrm{u}}_{\mathscr{N}}(\bar{\Lambda}_0)$ (resp. $z \in W^{\mathrm{s}}_{\mathscr{N}}(\bar{\Lambda}_0)$), since $xy = 0$, we have $I(z) = I(z^-)$ (resp. $I(z) = I(z^+)$).

The Hamiltonian H_0 restricted to $\bar{\Lambda}_0$ is given by

$$h_0(I) := \omega I + H_3^N(I) + H_4^N(I) + \dots,$$

where we denote $H_0(I, 0) = h_0(I)$, and $H_j^N(I, 0) = H_j^N(I)$ for all j. Each Lyapunov orbit λ_h in $\bar{\Lambda}_0$ corresponds to a unique level set of I, so we can write $\lambda_0(h) = \lambda_0(I)$.

By regrouping the terms of the polynomials $H_j^N(I, xy)$ for $j \geqslant 3$, we can rewrite (4.5) in powers of xy up to the second order as

$$H_0(I, \theta, y, x) = h_0(I) + (xy)g_1(I) + (xy)^2 g_2(I, xy), \tag{4.7}$$

for some analytic functions $h_0 = h_0(I)$, $g_1 = g_1(I)$, and $g_2 = g_2(I, xy)$.

For points $z \in W^{\mathrm{u}}_{\mathscr{N}}(\bar{\Lambda}_0)$ or $z \in W^{\mathrm{s}}_{\mathscr{N}}(\bar{\Lambda}_0)$, since $xy = 0$, we have

$$\frac{d}{dt}\theta(z) = \frac{\partial h_0}{\partial I}(I(z)).$$

This implies that if $z \in W^{\mathrm{u}}_{\mathscr{N}}(\bar{\Lambda}_0)$ (resp. $z \in W^{\mathrm{s}}_{\mathscr{N}}(\bar{\Lambda}_0)$), we have $\theta(z) = \theta(z^-)$ (resp. $\theta(z) = \theta(z^+)$).

The coordinate system (I, θ, y, x) constructed above is the coordinate system from part **(N)**.

Now we construct the coordinate system claimed in part **(U)**.

We extend the coordinate system (I, θ, y, x) along the flow to a neighborhood $\mathscr{N}^{\mathrm{u}}$ of $W^{\mathrm{u}}(\bar{\Lambda}_0)$, up to a neighborhood of the homoclinic manifold, as follows. Let $T_{\mathrm{u}} > 0$ be a time such that $\Phi_0^{T_{\mathrm{u}}}(\mathscr{N}) \supseteq \Gamma$. Let $\mathscr{N}^{\mathrm{u}} := \Phi_0^{T_{\mathrm{u}}}(\mathscr{N})$. Each point $z \in \mathscr{N}^{\mathrm{u}}$ is of the form $z = \Phi_0^{T_{\mathrm{u}}}(\zeta)$ with $\zeta \in \mathscr{N}$. We define the coordinates $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, y^{\mathrm{u}}, x^{\mathrm{u}})$ of z to be equal to the coordinates (I, θ, y, x) of ζ, or equivalently

$$(I^{\mathrm{u}}, \theta^{\mathrm{u}}, y^{\mathrm{u}}, x^{\mathrm{u}})(z) = (I, \theta, y, x)(\Phi_0^{-T_{\mathrm{u}}}(z)). \tag{4.8}$$

Since the coordinates $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, y^{\mathrm{u}}, x^{\mathrm{u}})$ of a point z are the coordinates (I, θ, y, x) of $\Phi_0^{-T_{\mathrm{u}}}(z)$, then the coordinate change in symplectic,

The restriction of the coordinates $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, y^{\mathrm{u}}, x^{\mathrm{u}})$ to $\mathscr{N} \cap \mathscr{N}^{\mathrm{u}}$ is given by $(I, \theta, y, x) \circ \Phi_0^{-T_{\mathrm{u}}}$. For every $z \in \bar{\Lambda}_0 \cap \mathscr{N}^{\mathrm{u}}$ we have $I^{\mathrm{u}}(z) = I(z)$ and $\theta^{\mathrm{u}}(z) = \theta \circ \Phi_0^{-T_{\mathrm{u}}}(z) = \theta(z) - \omega(I(z))T_{\mathrm{u}}$. We make a symplectic coordinate change in the action-angle variable

$$(I^{\mathrm{u}}, \theta^{\mathrm{u}}) \mapsto (I^{\mathrm{u}}, \theta^{\mathrm{u}} + \omega(I^{\mathrm{u}})T_{\mathrm{u}}),$$

where we use the same notation for the old and for the new coordinates. Since $x = y = x^{\mathrm{u}} = y^{\mathrm{u}} = 0$ on $\bar{\Lambda}_0$, we obtain that $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, y^{\mathrm{u}}, x^{\mathrm{u}}) = (I, \theta, x, y)$ on $\bar{\Lambda}_0 \cap \mathscr{N}^{\mathrm{u}}$.

We obtain that the normal form expansion of H_0 in the coordinates $(I^{\mathrm{u}}, \theta^{\mathrm{u}})$ is the same as in (4.7):

$$\begin{aligned}
&H(I^{\mathrm{u}}, \theta^{\mathrm{u}}, y^{\mathrm{u}}, x^{\mathrm{u}}) \\
&\quad = h_0(I \circ \Phi_0^{-T_{\mathrm{u}}}) \\
&\qquad + \left((x \circ \Phi_0^{-T_{\mathrm{u}}}) \cdot (y \circ \Phi_0^{-T_{\mathrm{u}}})\right) g_1(I \circ \Phi_0^{-T_{\mathrm{u}}}) \\
&\qquad + \left((x \circ \Phi_0^{-T_{\mathrm{u}}}) \cdot (y \circ \Phi_0^{-T_{\mathrm{u}}})\right)^2 g_2\left(I \circ \Phi_0^{-T_{\mathrm{u}}}, (x \circ \Phi_0^{-T_{\mathrm{u}}}) \cdot (y \circ \Phi_0^{-T_{\mathrm{u}}})\right) \\
&\quad = h_0(I^{\mathrm{u}}) + (x^{\mathrm{u}} y^{\mathrm{u}}) g_1(I) + (x^{\mathrm{u}} y^{\mathrm{u}})^2 g_2(I^{\mathrm{u}}, x^{\mathrm{u}} y^{\mathrm{u}}).
\end{aligned}$$

Now we construct the coordinate system claimed in part **(S)**. Let $T_{\mathrm{s}} > 0$ be a time such that $\Phi_0^{-T_{\mathrm{s}}}(\mathscr{N}) \supseteq \Gamma$. In general, $T_{\mathrm{s}} \neq T_{\mathrm{u}}$.

We extend the coordinate system (I, θ, y, x) along the flow to a neighborhood $\mathscr{N}^{\mathrm{s}}$ of $W^{\mathrm{s}}(\bar{\Lambda}_0)$, up to the homoclinic manifold, as follows. Start with the coordinates (I, θ, y, x) defined on the neighborhood $\mathscr{N}$ of $\bar{\Lambda}_0$. Let $\mathscr{N}^{\mathrm{s}} := \Phi_0^{-T_{\mathrm{s}}}(\mathscr{N})$. Each point $z \in \mathscr{N}^{\mathrm{s}}$ is of the form $z = \Phi_0^{-T_{\mathrm{s}}}(\zeta)$ with

$\zeta \in \mathscr{N}$. We define the coordinates $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, y^{\mathrm{s}}, x^{\mathrm{s}})$ of z to be equal to the coordinates (I, θ, y, x) of ζ, or equivalently

$$(I^{\mathrm{s}}, \theta^{\mathrm{s}}, y^{\mathrm{s}}, x^{\mathrm{s}})(z) = (I, \theta, y, x)(\Phi_0^{T_{\mathrm{s}}}(z)). \tag{4.9}$$

Since the coordinates $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, y^{\mathrm{s}}, x^{\mathrm{s}})$ of a point z are the coordinates $(I, \theta, y, x) \circ \Phi_0^{T_{\mathrm{s}}}$ of z, then the coordinate change is symplectic. Finally, we make the symplectic coordinate change

$$(I^{\mathrm{s}}, \theta^{\mathrm{s}}) \mapsto (I^{\mathrm{s}}, \theta^{\mathrm{s}} - \omega(I^{\mathrm{s}})T_{\mathrm{s}}),$$

where we use the same notation for the old and for the new coordinates.

We obtain that the normal form expansion of H_0 in the coordinates $(I^{\mathrm{u}}, \theta^{\mathrm{u}})$ is the same as in

$$H^N_{\mathscr{N}^{\mathrm{s}}}(I^{\mathrm{s}}, \theta^{\mathrm{s}}, y^{\mathrm{s}}, x^{\mathrm{s}}) = h_0(I^{\mathrm{s}}) + (x^{\mathrm{s}}y^{\mathrm{s}})g_1(I^{\mathrm{s}}) + (x^{\mathrm{s}}y^{\mathrm{s}})^2 g_2(I^{\mathrm{s}}, x^{\mathrm{s}}y^{\mathrm{s}}).$$

Since $x = y = x^{\mathrm{s}} = y^{\mathrm{s}} = 0$ on $\bar{\Lambda}_0 \cap \mathscr{N}^{\mathrm{s}}$, we have $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, y^{\mathrm{s}}, x^{\mathrm{s}}) = (I, \theta, x, y)$ on $\bar{\Lambda}_0 \cap \mathscr{N}^{\mathrm{s}}$,

Finally, we restrict the Hamiltonian to an energy range $h \in D$ such that $\Lambda_0 = \bigcup_{h \in D} \lambda_h \subseteq \mathscr{N} \cap \mathscr{N}^{\mathrm{u}} \cap \mathscr{N}^{\mathrm{s}}$. Moreover, we restrict D such that $W^{\mathrm{u}}(\Lambda_0)$, $W^{\mathrm{s}}(\Lambda_0)$ are contained in $\mathscr{K}$, where the set $\mathscr{K}$ is as in condition **(A-ii)**.

By the above constructions, the coordinates (I, θ, y, x), $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, y^{\mathrm{u}}, x^{\mathrm{u}})$ and $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, y^{\mathrm{s}}, x^{\mathrm{s}})$ satisfy the properties listed in Proposition 4.1. □

We note that H_0 satisfies the following non-degeneracy condition, in terms of the above normal form coordinates:

$$\begin{aligned} \frac{\partial h_0}{\partial I}(I_0) \neq & 0, \\ g_1(I_0) \neq & 0, \end{aligned} \tag{4.10}$$

for all $I_0 = I_0(h)$ with $h \in D$, provided D is contained in a sufficiently small neighborhood of the energy value of the equilibrium point L_1.

Indeed, by condition **(A-i)** we have

$$\begin{aligned} h_0(I) = & \omega I + O(I^3), \\ g_1(I) = & \lambda + O(I), \end{aligned} \tag{4.11}$$

in a sufficiently small neighborhood of the equilibrium point L_1, which imply (4.10).

Remark 4.2. It is important to note that the coordinates $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, x^{\mathrm{u}}, y^{\mathrm{u}})$ and $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, x^{\mathrm{s}}, y^{\mathrm{s}})$ do not generally agree at homoclinic points away from the Lyapunov orbit, where both coordinate systems are well defined. Nevertheless, for any homoclinic point $z \in M_h \cap W^{\mathrm{u}}(\Lambda_0) \cap W^{\mathrm{s}}(\Lambda_0)$, we have that $I^{\mathrm{u}}(z) = I^{\mathrm{s}}(z) = I_h$.

Remark 4.3. The above result on the existence of a convergent normal form in a neighborhood of a center-saddle point, obtained via a convergent canonical coordinate transformation, is valid for 2-degrees of freedom Hamiltonian systems. For higher degree of freedom Hamiltonian systems, a similar result is true under some additional non-resonance conditions (see [Gio01]).

A related approach to the normal form that we use here can be found in [Mos58]. A numerical methodology for the effective computations of normal forms is developed in [Jor99].

Remark 4.4. We now discuss the case when we have heteroclinic connections between two NHIMs Λ_0^1 around L_1 and Λ_0^2 around L_2, as in Section 2.5. The manifolds $W^{\mathrm{u}}(\Lambda_0^1)$ and $W^{\mathrm{s}}(\Lambda_0^2)$ are assumed to intersect transversally.

The construction of the normal form coordinates from Proposition 4.1 only works in a small neighborhood of the equilibrium point. In the case of heteroclinic connections, since L_1 and L_2 are on different energy level, the theory does not guarantee the simultaneous existence two normal form coordinate system around L_1 and L_2 respectively, for some common energy range.

In this case, we need to make a separate assumption that there exist two systems of normal form coordinates around L_1 and L_2, for some common energy range. Indeed, this assumption is already made in **(A'-i-b)**.

Based on this assumption, we can construct, as in the proof of Proposition 4.1, two systems of coordinates

- $(I^{1,\mathrm{u}}, \theta^{1,\mathrm{u}}, y^{1,\mathrm{u}}, x^{1,\mathrm{u}})$ in a neighborhood $\mathscr{N}^{1,\mathrm{u}} \subseteq \mathscr{K}$ of $W^{\mathrm{u}}(\Lambda_0^1)$,
- $(I^{2,\mathrm{s}}, \theta^{2,\mathrm{s}}, y^{2,\mathrm{s}}, x^{2,\mathrm{s}})$ in a neighborhood $\mathscr{N}^{2,\mathrm{s}} \subseteq \mathscr{K}$ of $W^{\mathrm{s}}(\Lambda_0^2)$,

so that that H_0 can be written relative to these coordinate as

$$\begin{aligned} H_0 =& h_0^1(I^{1,\mathrm{u}}) + (x^{1,\mathrm{u}} y^{1,\mathrm{u}}) g_1^1(I^{1,\mathrm{u}}) + (x^{1,\mathrm{u}} y^{1,\mathrm{u}})^2 g_2^1(I^{1,\mathrm{u}}, x^{1,\mathrm{u}} y^{1,\mathrm{u}}), \\ H_0 =& h_0^2(I^{2,\mathrm{s}}) + (x^{2,\mathrm{s}} y^{2,\mathrm{s}}) g_1^2(I^{2,\mathrm{s}}) + (x^{2,\mathrm{s}} y^{2,\mathrm{s}})^2 g_2^2(I^{2,\mathrm{s}}, x^{2,\mathrm{s}} y^{2,\mathrm{s}}). \end{aligned}$$

In the case of heteroclinic connections, we explicitly require the following non-degeneracy condition, analogous to (4.10):

(A'-ii-b) *The Hamiltonian H_0, written in the corresponding normal form coordinates, satisfies:*

$$\begin{aligned} \frac{\partial h_0^1}{\partial I^{1,\mathrm{u}}}(I_0) \neq& 0, \\ g_1^1(I_0) \neq& 0, \\ \frac{\partial h_0^2}{\partial I^{2,\mathrm{s}}}(I_0) \neq& 0, \\ g_1^2(I_0) \neq& 0, \end{aligned}$$

for all $I_0 = I_0(h)$ with $h \in D$.

We remark that these conditions are automatically satisfied in small neighborhoods of L_1 and L_2, respectively. Condition **(A'-ii-b)** requires these conditions to hold on the specified energy range.

4.2. **The scattering map for the unperturbed system.** Consider the scattering map σ_0 associated to Γ_0. We will express the scattering map in terms of the coordinates $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, y^{\mathrm{u}}, x^{\mathrm{u}})$ and $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, y^{\mathrm{s}}, x^{\mathrm{s}})$.

Consider a homoclinic point $z_0 \in W^{\mathrm{u}}(\lambda_h) \cap W^{\mathrm{s}}(\lambda_h)$. Both coordinate systems $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, y^{\mathrm{u}}, x^{\mathrm{u}})$ and $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, y^{\mathrm{s}}, x^{\mathrm{s}})$ are defined around z_0.

By Proposition 4.1 the action coordinate of $z_0 \in \Gamma_0$ is the same as the action coordinate of the unstable and stable foot-points $z_0^-, z_0^+ \in \Lambda_0$, that is $I(z_0^-) = I^{\mathrm{u}}(z_0^-) = I^{\mathrm{u}}(z_0) = I^{\mathrm{s}}(z_0) = I^{\mathrm{s}}(z_0^+) = I(z_0^+)$. Therefore the scattering map σ_0 preserves the I-coordinate. Hence σ_0 is a phase shift on I-level-sets in Λ_0 wherever it is defined:

$$\sigma_0(I, \theta) = (I, \theta + \Delta(I)). \tag{4.12}$$

In general, $\theta^{\mathrm{u}}(z_0) \neq \theta^{\mathrm{s}}(z_0)$. It is easy to see that the phase-shift determined by the unperturbed scattering map is given by the 'mismatch' between the two angle coordinates.

Proposition 4.5. *Let h be a fixed energy level and let $z_0 \in \Gamma_0 \cap M_h$.*

(i) The angle mismatch $\theta^{\mathrm{s}}(z_0) - \theta^{\mathrm{u}}(z_0)$ is a constant that depends only on h, so we write it as $\theta^{\mathrm{s}}(h) - \theta^{\mathrm{u}}(h)$.

(ii) The scattering map σ_0 is given by $(I, \theta) \mapsto \sigma_0(I, \theta) = (I, \theta + \Delta(I))$, where $\Delta(I) = \theta^{\mathrm{s}}(h) - \theta^{\mathrm{u}}(h)$ for $I = I_h$.

Proof. (i) If z_0 is a point in $\Gamma_0 \cap M_h$, then $\Gamma_0 \cap M_h$ consists of points of the form $\Phi_0^t(z)$, $t \in [t_1, t_2]$, for some interval $[t_1, t_2]$ containing 0.

From $\Omega^+(z_0) = z_0^+$ and $\Omega^-(z_0) = z_0^-$, by the equivariance property (A.5) we have $\Omega^+(\Phi_0^t(z_0)) = \Phi_0^t(z_0^+)$ and $\Omega^-(\Phi_0^t(z_0)) = \Phi_0^t(z_0^-)$ for $t \in [t_1, t_2]$. By By Proposition 4.1 **(U-iii)** and **(S-iii)**, and by the fact that the dynamics restricted to λ_h is a rigid rotation in θ, we have

$$\begin{aligned} \theta^{\mathrm{s}}(\Phi_0^t(z_0)) - \theta^{\mathrm{u}}(\Phi_0^t(z_0)) =& \theta(\Phi_0^t(z_0^+)) - \theta(\Phi_0^t(z_0^-)) \\ =& \theta(z_0^+) - \theta(z_0^-), \end{aligned}$$

which depends only on the energy level h.

(ii) In action-angle coordinates, if $z_0^- = (I, \theta)$, then $\sigma_0(z_0^-) = z_0^+ = (I, \theta + \Delta(I))$. From (i) we immediately deduce that $\Delta(I) = \Delta(I(h)) = \theta^{\mathrm{s}}(h) - \theta^{\mathrm{u}}(h)$. □

We now turn the attention to the unperturbed homoclinic channel $\tilde{\Gamma}_0$ in the extended phase space. By the transversality conditions (B.3) on a homoclinic channel, we have that the fibers of $\tilde{\Lambda}_0$ intersect $\tilde{\Gamma}_0$ transversally. Therefore $\tilde{\Gamma}_0$ can be parametrized in terms of the coordinates $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, t)$, as well as in terms of the coordinates $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, t)$:

$$\begin{aligned} \tilde{\Gamma}_0 =& \{(I^{\mathrm{u}}, \theta^{\mathrm{u}}, y^{\mathrm{u}}, x^{\mathrm{u}}, t) \,|\, y^{\mathrm{u}} = 0,\, x^{\mathrm{u}} = x_0^{\mathrm{u}}(I^{\mathrm{u}}, \theta^{\mathrm{u}}, t)\}, \\ =& \{(I^{\mathrm{s}}, \theta^{\mathrm{s}}, x^{\mathrm{s}}, y^{\mathrm{s}}, t) \,|\, y^{\mathrm{s}} = y_0^{\mathrm{s}}(I^{\mathrm{s}}, \theta^{\mathrm{s}}, t),\, x^{\mathrm{s}} = 0\}. \end{aligned} \tag{4.13}$$

Each homoclinic point $\tilde{z}_0 \in \tilde{\Gamma}_0$ is associated to unique $I^{\mathrm{u}} = I^{\mathrm{s}} = I_0$, θ^{u}, and θ^{s}, with $\theta^{\mathrm{s}}(\tilde{z}_0) - \theta^{\mathrm{u}}(\tilde{z}_0) = \Delta(I_0)$.

The corresponding scattering map is given by

$$\tilde{\sigma}_0(I, \theta, t) = (I, \theta + \Delta(I), t).$$

Thus, we have that $\tilde{\Gamma}_0$ is a graph over the $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, t)$-variables, as well as a graph over the $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, t)$-variables. On $\tilde{\Gamma}_0$ we have two coordinate systems

$(I^{\mathrm{u}}, \theta^{\mathrm{u}}, t)$ and $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, t)$, with the corresponding coordinate change given by

$$(I^{\mathrm{s}}, \theta^{\mathrm{s}}, t) = (I^{\mathrm{u}}, \theta^{\mathrm{u}} + \Delta(I^{\mathrm{u}}), t). \tag{4.14}$$

4.3. **Perturbed evolution equations.** In the sequel, we will identify the vector fields $\mathcal{X}^0$ and $\mathcal{X}^1$ with derivative operators acting on functions. In general, given a smooth vector field $\mathcal{X}$ and a smooth function f on a manifold M, and $(z_j)_{j\in\{1,\dots,\dim(M)\}}$ a system of local coordinates, then

$$(\mathcal{X}f)(z) = \sum_j (\mathcal{X})_j(z)(\partial_{z_j} f)(z). \tag{4.15}$$

Consider one of the coordinate systems defined in Section 4.1. To simplify notation, we will denote such a coordinate system by (I, θ, x, y). Below we provide evolution equations of these coordinates, expressing the time-derivative of each coordinate along a solution of the perturbed system. We include the expression for a general perturbation, as well as for the case when the perturbation is Hamiltonian:

$$\begin{aligned} \frac{d}{dt} I &= (\mathcal{X}^0 + \varepsilon\mathcal{X}^1)(I) = -\frac{\partial H_0}{\partial \theta} + \varepsilon \mathcal{X}^1(I) \\ &= -\frac{\partial H_0}{\partial \theta} - \varepsilon \frac{\partial H_1}{\partial \theta}. \end{aligned} \tag{4.16}$$

$$\begin{aligned} \frac{d}{dt} \theta &= (\mathcal{X}^0 + \varepsilon\mathcal{X}^1)(\theta) = \frac{\partial H_0}{\partial I} + \varepsilon \mathcal{X}^1(\theta) \\ &= \frac{\partial H_0}{\partial I} + \varepsilon \frac{\partial H_1}{\partial I}. \end{aligned} \tag{4.17}$$

$$\begin{aligned} \frac{d}{dt} y &= \mathcal{X}^0(y) + \varepsilon\mathcal{X}^1(y) = -\frac{\partial H_0}{\partial x} + \varepsilon \mathcal{X}^1(y) \\ &= -\frac{\partial H_0}{\partial x} - \varepsilon \frac{\partial H_1}{\partial x}. \end{aligned} \tag{4.18}$$

$$\begin{aligned} \frac{d}{dt} x &= \mathcal{X}^0(x) + \varepsilon\mathcal{X}^1(x) = \frac{\partial H_0}{\partial y} + \varepsilon \mathcal{X}^1(x) \\ &= \frac{\partial H_0}{\partial y} + \varepsilon \frac{\partial H_1}{\partial y}. \end{aligned} \tag{4.19}$$

5. Proof of the main result

In this section we prove Theorem 2.1.

5.1. **Perturbed normally hyperbolic invariant manifolds.** In this section we prove the assertions (i) and (ii) of Theorem 2.1. Still, we note that those assertions hold under more general conditions than the ones assumed.

We only give the details in the case when H_0 satisfies the conditions **(A-i)** and **(A-ii)**. The case when H_0 satisfies **(A'-i)** and **(A'-ii)** follows similarly.

5.1.1. *Persistence of the normally hyperbolic invariant manifold under perturbation.* We have that Λ_0 is a NHIM for the flow Φ_0^t of $\mathcal{X}^0$. Then $D\Phi_0^t(z)$ satisfies expansion/contraction rates as in Appendix A, for all $z \in \Lambda_0$, where we denote the constant and the expansion and contraction rates by C, λ_-, λ_+, λ_c, μ_c, μ_-, μ_+, respectively.

It is immediate that $\tilde{\Lambda}_0 = \Lambda_0 \times \mathbb{R}$ is a NHIM for the flow $\tilde{\Phi}_0^\tau$ of the extended system (2.6). We note that Λ_0 is a non-compact manifold with boundary.

The theory of normally hyperbolic invariant manifolds, [Fen72, HPS77, Pes04] (a handy summary of the results of the theory is [DdlLS06]), asserts the persistence of NHIM's under small perturbations. The persistence of non-compact manifolds when the perturbation has uniformly bounded derivatives in all variables is shown in [HPS77, Section 6]. This generality is crucial for infinite dimensional manifolds [BLZ99, BLZ08]. When the unperturbed NHIM has a boundary, in the literature, one can find the results of persistence for inflowing or overflowing manifolds on the boundary (these conditions are automatic when the boundary is invariant). In such ca case, the persistent manifold is not necessarily invariant but only locally invariant, and is not necessarily unique. See Appendix A.

Under the assumption **(A-iii)** of Theorem 2.1, $\mathcal{X}^1 = \mathcal{X}^1(z,t;\varepsilon)$ has uniformly bounded derivatives in all variables. Therefore, there exists ε_1 such that the manifold $\tilde{\Lambda}_0$ persists as a normally hyperbolic manifold $\tilde{\Lambda}_\varepsilon$, for all $|\varepsilon| < \varepsilon_1$, which is locally invariant under the flow $\tilde{\Phi}_\varepsilon^\tau$.

In terms of the normal form coordinates (I, θ, y, x) from Proposition 4.1-**(N)**, the unperturbed NHIM is given by

$$\tilde{\Lambda}_0 = \{(I, \theta, y, x, t) \,|\, x = y = 0\}.$$

Then the perturbed NHIM can be written as a graph over the (I, θ, t)-coordinates over a suitable domain, i.e.,

$$\tilde{\Lambda}_\varepsilon = \{(I, \theta, y, x, t) \,|\, x = x_\varepsilon(I, \theta, t),\, y = y_\varepsilon(I, \theta, t)\}.$$

See Appendix A. Consequently, every point $\tilde{z}_\varepsilon \in \tilde{\Lambda}_\varepsilon$ is determined by its coordinates (I, θ, t).

The NHIM $\tilde{\Lambda}_\varepsilon$ is $O(\varepsilon)$ close in the $\mathcal{C}^\ell$-topology to $\tilde{\Lambda}_0$, where ℓ is as in (A.3). The locally invariant manifolds are in fact invariant manifolds for an extended system, and they depend on the extension. Hence, they do not need to be unique. Nevertheless, given a smooth family of systems, it is possible to choose the invariant manifolds in such a way that the invariant manifolds depend smoothly on parameters, as well as the stable and unstable bundles and the stable and unstable manifolds.

For the perturbed NHIM $\tilde{\Lambda}_\varepsilon$, $|\varepsilon| < \varepsilon_1$, there exists an invariant splitting of the tangent bundle $T\tilde{\Lambda}_\varepsilon$, similar to that in (A.1), so that $D\tilde{\Phi}_\varepsilon^\tau(\tilde{z})$ satisfies expansion/contraction relations similar to those in (A.2) for all $\tilde{z} \in \tilde{\Lambda}_\varepsilon$, for some constants $\tilde{C}$, $\tilde{\lambda}_-$, $\tilde{\lambda}_+$, $\tilde{\lambda}_c$, $\tilde{\mu}_c$, $\tilde{\mu}_-$, $\tilde{\mu}_+$. These constants are independent

of ε, and can be chosen as close as desired to the unperturbed ones, that is, to C, λ_-, λ_+, λ_c, μ_c, μ_-, μ_+, respectively, by choosing ε_1 suitably small.

There exist unstable and stable manifolds $W^{\mathrm{u}}(\tilde{\Lambda}_\varepsilon)$, $W^{\mathrm{s}}(\tilde{\Lambda}_\varepsilon)$ associated to $\tilde{\Lambda}_\varepsilon$, and there exist corresponding projection maps $\Omega^- : W^{\mathrm{u}}(\tilde{\Lambda}_\varepsilon) \to \tilde{\Lambda}_\varepsilon$, and $\Omega^+ : W^{\mathrm{s}}(\tilde{\Lambda}_\varepsilon) \to \tilde{\Lambda}_\varepsilon$.

For $\tilde{z}^+ = \Omega^+(\tilde{z})$, with $\tilde{z} \in W^{\mathrm{s}}(\tilde{\Lambda}_\varepsilon)$ we have

$$d(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}), \tilde{\Phi}_\varepsilon^\tau(\tilde{z}^+)) \leqslant C_{\tilde{z}} e^{\tau \tilde{\lambda}_+}, \quad \text{for all } \tau \geqslant 0, \tag{5.1}$$

and for $\tilde{z}^- = \Omega^-(\tilde{z})$, with $\tilde{z} \in W^{\mathrm{u}}(\tilde{\Lambda}_\varepsilon)$ we have

$$d(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}), \tilde{\Phi}_\varepsilon^\tau(\tilde{z}^-) \leqslant C_{\tilde{z}} e^{\tau \tilde{\mu}_-}, \quad \text{for all } \tau \leqslant 0, \tag{5.2}$$

for some $C_{\tilde{z}} > 0$. The constant $\tilde{C}_{\tilde{z}}$ can be chosen uniformly bounded, provided that we restrict to z to the compact neighborhood $\mathscr{K}$ given by **(A-ii)**, and we use the fact that $\mathcal{X}^1 = \mathcal{X}^1(z, t; \varepsilon)$ is uniformly differentiable in all variables.

To simplify notation, from now on we will drop the symbol $\tilde{}$ from $\tilde{C}$, $\tilde{C}_{\tilde{z}}$ $\tilde{\lambda}_-$, $\tilde{\lambda}_+$, $\tilde{\mu}_-$, $\tilde{\mu}_+$, $\tilde{\lambda}_c$, $\tilde{\mu}_c$.

In the sequel, we will fix a choice $\tilde{\Lambda}_\varepsilon$, and all computations will be performed relative to that choice. Nevertheless the estimate for the perturbed scattering map $\tilde{\sigma}_\varepsilon$ are independent of the choice of the locally invariant manifold $\tilde{\Lambda}_\varepsilon$.

5.1.2. *Persistence of the transverse intersection of the hyperbolic invariant manifolds under perturbation.* For the unperturbed system the unstable and stable manifolds $W^{\mathrm{u}}(\tilde{\Lambda}_0)$, $W^{\mathrm{s}}(\tilde{\Lambda}_0)$ intersect transversally along the 3-dimensional homoclinic channel $\tilde{\Gamma}_0$. By the persistence of transversality under small perturbations, it follows that $W^{\mathrm{u}}(\tilde{\Lambda}_\varepsilon)$, $W^{\mathrm{s}}(\tilde{\Lambda}_\varepsilon)$ intersect transversally along $\tilde{\Gamma}_\varepsilon$, for all $|\varepsilon| < \varepsilon_1$, provided ε_1 is chosen small enough. The condition (B.1) in the definition of a homoclinic/heteroclinic channel is also a transversality-type condition, so it is also persistent under small perturbations. We conclude that (i) and (ii) from Theorem 2.1 hold true for all $|\varepsilon| < \varepsilon_1$, provided ε_1 is chosen small enough.

Recall from Section 4.2 that the unperturbed homoclinic channel $\tilde{\Gamma}_0$ can be described as a graph over the $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, t)$-variables, as well as a graph over the $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, t)$-variables. It follows that for the perturbed system, for $|\varepsilon| < \varepsilon_1$ with ε_1 sufficiently small, the perturbed homoclinic channel $\tilde{\Gamma}_\varepsilon$ can also be described as a graph over the $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, t)$-variables, as well as a graph over the $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, t)$-variables. More precisely, we have

$$\begin{aligned} \tilde{\Gamma}_\varepsilon =& \{(I^{\mathrm{u}}, \theta^{\mathrm{u}}, y^{\mathrm{u}}, x^{\mathrm{u}}, t) \,|\, y^{\mathrm{u}} = y^{\mathrm{u}}_\varepsilon(I^{\mathrm{u}}, \theta^{\mathrm{u}}, t),\, x^{\mathrm{u}} = x^{\mathrm{u}}_\varepsilon(I^{\mathrm{u}}, \theta^{\mathrm{u}}, t)\}, \\ \tilde{\Gamma}_\varepsilon =& \{(I^{\mathrm{s}}, \theta^{\mathrm{s}}, y^{\mathrm{s}}, x^{\mathrm{s}}, t) \,|\, y^{\mathrm{s}} = y^{\mathrm{s}}_\varepsilon(I^{\mathrm{s}}, \theta^{\mathrm{s}}, t),\, x^{\mathrm{s}} = x^{\mathrm{s}}_\varepsilon(I^{\mathrm{s}}, \theta^{\mathrm{s}}, t)\}, \end{aligned} \tag{5.3}$$

with $y^{\mathrm{u}}_\varepsilon(I^{\mathrm{u}}, \theta^{\mathrm{u}}, t) = O(\varepsilon)$, $x^{\mathrm{u}}_\varepsilon(I^{\mathrm{u}}, \theta^{\mathrm{u}}, t) = x^{\mathrm{u}}_0(I^{\mathrm{u}}, \theta^{\mathrm{u}}, t) + O(\varepsilon)$, $y^{\mathrm{s}}_\varepsilon(I^{\mathrm{s}}, \theta^{\mathrm{s}}, t) = y^{\mathrm{s}}_0(I^{\mathrm{s}}, \theta^{\mathrm{s}}, t) + O(\varepsilon)$, and $x^{\mathrm{s}}_\varepsilon(I^{\mathrm{s}}, \theta^{\mathrm{s}}, t) = O(\varepsilon)$.

Therefore, each homoclinic point $\tilde{z}_\varepsilon \in \tilde{\Gamma}_\varepsilon$ is associated to unique coordinate triples $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, t)$ and $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, t)$, which satisfy (4.14).

We now describe how to match a perturbed homoclinic point $\tilde{z}_\varepsilon \in \tilde{\Gamma}_\varepsilon$ to an unperturbed homoclinic point $\tilde{z}_0 \in \tilde{\Gamma}_0$. As we represent both $\tilde{\Gamma}_\varepsilon$ and $\tilde{\Gamma}_0$ as graphs over the same coordinates, we have a canonical way to match the homoclinic points.

To each point $\tilde{z}_0 \in \tilde{\Gamma}_0$, of local coordinates $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, t)$ and $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, t)$, with $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, t) = (I^{\mathrm{u}}, \theta^{\mathrm{u}} + \Delta(I^{\mathrm{u}}), t)$, we assign the point $\tilde{z}_\varepsilon \in \tilde{\Gamma}_\varepsilon$ that has the same local coordinates $(I^{\mathrm{s}}, \theta^{\mathrm{s}}, t)$ and $(I^{\mathrm{u}}, \theta^{\mathrm{u}}, t)$, via the graph representations in (5.3).

More precisely, if $\tilde{z}_0 \in \tilde{\Gamma}_0$ is given by

$$
\begin{aligned}
&(I^{\mathrm{u}}_0, \theta^{\mathrm{u}}_0, 0, x^{\mathrm{u}}_0(I^{\mathrm{u}}, \theta^{\mathrm{u}}, t_0), t_0), \text{ and} \\
&(I^{\mathrm{s}}_0, \theta^{\mathrm{s}}_0, y^{\mathrm{s}}_0(I^{\mathrm{s}}_0, \theta^{\mathrm{s}}_0, t_0), 0, t_0),
\end{aligned}
$$

with $I^{\mathrm{s}}_0 = I^{\mathrm{u}}_0$ and $\theta^{\mathrm{s}}_0 = \theta^{\mathrm{u}}_0 + \Delta(I^{\mathrm{u}}_0)$, then we associate to it $\tilde{z}_\varepsilon \in \tilde{\Gamma}_\varepsilon$ given by

$$
\begin{aligned}
&(I^{\mathrm{u}}_0, \theta^{\mathrm{u}}_0, y^{\mathrm{u}}_\varepsilon(I^{\mathrm{u}}_0, \theta^{\mathrm{u}}_0, t_0), x^{\mathrm{u}}_\varepsilon(I^{\mathrm{u}}_0, \theta^{\mathrm{u}}_0, t_0), t_0) \text{ and} \\
&(I^{\mathrm{s}}_0, \theta^{\mathrm{s}}_0, y^{\mathrm{s}}_\varepsilon(I^{\mathrm{s}}_0, \theta^{\mathrm{s}}_0, t_0), x^{\mathrm{s}}_\varepsilon(I^{\mathrm{s}}_0, \theta^{\mathrm{s}}_0, t_0), t_0),
\end{aligned}
$$

respectively. Note also that the representation of the NHIM in these coordinates will also change.

Therefore, to $\tilde{z}_0 \in \tilde{\Gamma}_0$ we associate $\tilde{z}_\varepsilon \in \tilde{\Gamma}_\varepsilon$ with

$$
\begin{aligned}
I^{\mathrm{s}}(\tilde{z}_\varepsilon) - I^{\mathrm{u}}(\tilde{z}_\varepsilon) =& 0, \\
\theta^{\mathrm{s}}(\tilde{z}_\varepsilon) - \theta^{\mathrm{u}}(\tilde{z}_\varepsilon) =& \theta^{\mathrm{s}}(\tilde{z}_0) - \theta^{\mathrm{u}}(\tilde{z}_0),
\end{aligned}
\tag{5.4}
$$

and with the same time-coordinate t_0.

This choice can be understood geometrically. The intersection Γ_ε is a smooth manifold. By the theory of normally hyperbolic manifolds, it depends smoothly on parameters. This manifold is parameterized by the (I, θ, t) coordinates. To specify a family of points, we impose a condition on these coordinates.

By letting all the choices vary – over all the (I, θ, t) that satisfy (5.4) – we can compute the whole manifold.

The choice (5.4) is one out of many possible choices. We have made it to simplify several calculations.

In the sequel we will compare the scattering map associated to $\tilde{z}_0$ with the scattering map associated to $\tilde{z}_\varepsilon$ satisfying (5.4). The method of calculation we will use is to compute the leading terms of the change of the coordinates of the asymptotic points $z^{\pm}_\varepsilon$. We note that these points exist because of the theory of smooth dependence of the NHIM on parameters. Once we know that these points exist and that they are asymptotic, we compute the change of coordinates by the dynamics. For the slow variables, we can compute the change of coordinates using the unperturbed dynamics. Since the unperturbed system conserves energy, the changes of energy are caused by the perturbation, so they are a slow variable. For the fast variables,

we will need to introduce extra cancellations between the dynamics in the asymptotic trajectory and the homoclinic one.

5.2. **Perturbed scattering map.** In this section we prove the assertion (iii) of Theorem 2.1.

We recall that the existence of the scattering map and its smooth dependence on parameters follows from the standard theory of normally hyperbolic invariant manifolds. The only thing we have to do is to compute the formulas for the derivatives knowing that they exist. This be done by estimating the change of the coordinate functions along the connecting orbits.

We start with the unperturbed system (2.3). We recall that for a given homoclinic channel Γ_0, the corresponding scattering map σ_0, is a phase-shift of the form

$$\sigma_0(I,\theta) = (I, \theta + \Delta(I)).$$

We choose and fix an energy level h of H_0, and a point $z_0 \in \Gamma_0 \cap M_h$. In the $(I^{\mathrm{u,s}}, \theta^{\mathrm{u,s}}, y^{\mathrm{u,s}}, x^{\mathrm{u,s}})$-coordinates, z_0 is given by

$$z_0 = (I_0^{\mathrm{s}}, \theta_0^{\mathrm{s}}, y_0^{\mathrm{s}}, 0) = (I_0^{\mathrm{u}}, \theta_0^{\mathrm{u}}, 0, x_0^{\mathrm{u}}), \tag{5.5}$$

where $I_0^{\mathrm{s}} = I_0^{\mathrm{u}} = I_0$. The effect of the flow Φ_0^τ on z_0 in these coordinates is given by

$$\Phi_0^\tau(z_0) = (I_0^{\mathrm{s}}, \theta_0^{\mathrm{s}} + \omega(I_0)\tau, y^{\mathrm{s}}(\tau), 0) = (I_0^{\mathrm{u}}, \theta_0^{\mathrm{u}} + \omega(I_0)\tau, 0, x^{\mathrm{u}}(\tau)), \tag{5.6}$$

where $y^{\mathrm{s}}(\tau)$ and $x^{\mathrm{u}}(\tau)$ are the y^{s}-component and the x^{u}-component, respectively, of $\Phi_0^\tau(z_0)$ evaluated at time τ, and $\omega(I_0) = \frac{\partial h_0}{\partial I}$.

There exist uniquely defined points z_0^-, z_0^+ in $\lambda_0(h)$ such that $W^{\mathrm{u}}(z_0^-) \cap (\Gamma_0 \cap M_h) = W^{\mathrm{s}}(z_0^+) \cap (\Gamma_0 \cap M_h) = \{z_0\}$. In the (I, θ, y, x)-coordinates, the foot-points $z_0^\pm$ are given by

$$z_0^- = (I_0, \theta_0^-, 0, 0), \quad z_0^+ = (I_0^{\mathrm{s}}, \theta_0^+, 0, 0), \tag{5.7}$$

where $\theta_0^- = \theta_0^{\mathrm{u}}$ and $\theta_0^+ = \theta_0^{\mathrm{s}}$. The effect of the flow Φ_0^τ on $z_0^\pm$ in these coordinates is given by

$$\begin{aligned} \Phi_0^\tau(z_0^-) &= (I_0, \theta_0^- + \omega(I_0)\tau, 0, 0), \\ \Phi_0^\tau(z_0^+) &= (I_0, \theta_0^+ + \omega(I_0)\tau, 0, 0). \end{aligned} \tag{5.8}$$

The scattering map σ_0 takes $z_0^- \in \lambda_0(h)$ into $z_0^+ \in \lambda_0(h)$.

In the extended system (2.6), the corresponding homoclinic point is $\tilde{z}_0 = (z_0, t_0)$ for some $t_0 \in \mathbb{R}$. The scattering map $\tilde{\sigma}_0$ takes $\tilde{z}_0^- = (z_0^-, t_0)$ into $\tilde{z}_0^+ = (z_0^+, t_0)$.

We will compute the effect of the perturbation on the scattering map $\tilde{\sigma}_0$.

When we add the perturbation there exists a homoclinic point $\tilde{z}_\varepsilon \in \tilde{\Gamma}_\varepsilon$ corresponding to $\tilde{z}_0 = (z_0, t_0)$ from the unperturbed case, such that $\tilde{z}_\varepsilon$ satisfies the condition (5.4). Associated to $\tilde{z}_\varepsilon \in \tilde{\Gamma}_\varepsilon$ we have the points $\tilde{z}_\varepsilon^-$, $\tilde{z}_\varepsilon^+$ in $\tilde{\Lambda}_\varepsilon$ such that $W^{\mathrm{u}}(\tilde{z}_\varepsilon^-) \cap \tilde{\Gamma}_\varepsilon = \{\tilde{z}_\varepsilon\}$, and $W^{\mathrm{s}}(\tilde{z}_\varepsilon^+) \cap \tilde{\Gamma}_\varepsilon = \{\tilde{z}_\varepsilon\}$. The scattering map $\tilde{\sigma}_\varepsilon$ takes $\tilde{z}_\varepsilon^- \in \tilde{\Lambda}_\varepsilon$ into $\tilde{z}_\varepsilon^+ \in \tilde{\Lambda}_\varepsilon$.

In the sequel we will make a quantitative comparison between

$$\tilde{z}_0^- \mapsto \tilde{\sigma}_0(z_0^-) := \tilde{z}_0^+,$$

and

$$\tilde{z}_\varepsilon^- \mapsto \tilde{\sigma}_\varepsilon(\tilde{z}_\varepsilon^-) := \tilde{z}_\varepsilon^+.$$

5.2.1. *Estimates.* Below we will refer to the notation in (5.5), (5.6), (5.7), and (5.8). To simplify notation, we denote $I_\varepsilon^{\mathrm{s}} = I^{\mathrm{s}}(\tilde{z}_\varepsilon)$, $I_\varepsilon^{\mathrm{u}} = I^{\mathrm{u}}(\tilde{z}_\varepsilon)$, $I_\varepsilon^{\mathrm{s}+} = I^{\mathrm{s}}(\tilde{z}_\varepsilon^+)$, $I_\varepsilon^{\mathrm{u}-} = I^{\mathrm{u}}(\tilde{z}_\varepsilon^-)$, $\xi_\varepsilon^{\mathrm{s}} = (x^{\mathrm{s}}y^{\mathrm{s}})(\tilde{z}_\varepsilon)$, $\xi^{\mathrm{s}+} = (x^{\mathrm{s}}y^{\mathrm{s}})(\tilde{z}_\varepsilon^+)$, $\xi_\varepsilon^{\mathrm{u}} = (x^{\mathrm{u}}y^{\mathrm{u}})(\tilde{z}_\varepsilon)$, $\xi^{\mathrm{u}-} = (x^{\mathrm{u}}y^{\mathrm{u}})(\tilde{z}_\varepsilon^-)$.

Note that in the following, the coordinates of the scattering map can be considered as functions of the point. Hence, the symbols $O(\varepsilon)$ can be interpreted as relative to the C^r norm.

Lemma 5.1. (i) *Estimates on I:*

$$I_\varepsilon^{\mathrm{s}+} - I_\varepsilon^{\mathrm{s}} = O(\varepsilon),\ I_\varepsilon^{\mathrm{u}-} - I_\varepsilon^{\mathrm{u}} = O(\varepsilon),\ I_\varepsilon^{\mathrm{s}+} - I_\varepsilon^{\mathrm{u}-} = O(\varepsilon). \tag{5.9}$$

(ii) *Estimates on h_0:*

$$\begin{aligned} h_0(I_\varepsilon^{\mathrm{s}+}) - h_0(I_\varepsilon^{\mathrm{s}}) =& (I_\varepsilon^{\mathrm{s}+} - I_\varepsilon^{\mathrm{s}}) \left(\frac{\partial h_0}{\partial I}(I_0) \right) + O(\varepsilon^2), \\ h_0(I_\varepsilon^{\mathrm{u}-}) - h_0(I_\varepsilon^{\mathrm{u}}) =& (I_\varepsilon^{\mathrm{u}-} - I_\varepsilon^{\mathrm{u}}) \left(\frac{\partial h_0}{\partial I}(I_0) \right) + O(\varepsilon^2). \end{aligned} \tag{5.10}$$

(iii) *Estimates on ξ:*

$$\xi_\varepsilon^{\mathrm{s}} = O(\varepsilon),\ \xi_\varepsilon^{\mathrm{u}} = O(\varepsilon),\ \xi_\varepsilon^{\mathrm{s}+} = O(\varepsilon^2),\ \xi_\varepsilon^{\mathrm{u}-} = O(\varepsilon^2). \tag{5.11}$$

(iv) *Estimates on g_1:*

$$\begin{aligned} g_1(I_\varepsilon^{\mathrm{s}+}) =& g_1(I_0) + \frac{\partial g_1}{\partial I}(I_0)(I_\varepsilon^{\mathrm{s}+} - I_0) + O(\varepsilon^2), \\ g_1(I_\varepsilon^{\mathrm{u}-}) =& g_1(I_0) + \frac{\partial g_1}{\partial I}(I_0)(I_\varepsilon^{\mathrm{u}-} - I_0) + O(\varepsilon^2), \end{aligned} \tag{5.12}$$

(v) *Estimates on $\frac{\partial g_1}{\partial I}$:*

$$\begin{aligned} \frac{\partial g_1}{\partial I}(I_\varepsilon^{\mathrm{s}+}) =& \frac{\partial g_1}{\partial I}(I_0) + \frac{\partial^2 g_1}{\partial I^2}(I_0)(I_\varepsilon^{\mathrm{s}+} - I_0) + O(\varepsilon^2), \\ \frac{\partial g_1}{\partial I}(I_\varepsilon^{\mathrm{u}-}) =& \frac{\partial g_1}{\partial I}(I_0) + \frac{\partial^2 g_1}{\partial I^2}(I_0)(I_\varepsilon^{\mathrm{u}-} - I_0) + O(\varepsilon^2), \end{aligned} \tag{5.13}$$

Proof. (i) Due to (5.3), we have $\tilde{z}_\varepsilon = \tilde{z}_0 + O(\varepsilon)$ and $\tilde{z}_\varepsilon^\pm = \tilde{z}_0^\pm + O(\varepsilon)$, we have $I^{\mathrm{s,u}}(\tilde{z}_\varepsilon) = I^{\mathrm{s,u}}(\tilde{z}_0) + O(\varepsilon)$, and $I^{\mathrm{s,u}}(\tilde{z}_\varepsilon^\pm) = I^{\mathrm{s,u}}(\tilde{z}_0^\pm) + O(\varepsilon)$. We note the graph property of $\tilde{\Gamma}_\varepsilon$ (5.3). guarantees that $\tilde{z}_\varepsilon$ and $\tilde{z}_0$ have the same time-coordinate $t_0 \in \mathbb{R}$.

The fact that $I^{\mathrm{s,u}}(\tilde{z}_0) = I^{\mathrm{s,u}}(\tilde{z}_0^\pm) = I_0$ yields the estimates in (i).

(ii) We estimate the term $h_0(I_\varepsilon^{s+}) - h_0(I_\varepsilon^{s})$. Applying the integral form of the Mean Value Theorem we have

$$h_0(I_\varepsilon^{s+}) - h_0(I_\varepsilon^{s}) = (I_\varepsilon^{s+} - I_\varepsilon^{s}) \int_0^1 \frac{\partial h_0}{\partial I}(tI_\varepsilon^{s+} + (1-t)I_\varepsilon^{s})dt. \tag{5.14}$$

We write the integrand of (5.14) as a Taylor expansion

$$\begin{aligned}\frac{\partial h_0}{\partial I}(tI_\varepsilon^{s+} + (1-t)I_\varepsilon^{s}) =& \frac{\partial h_0}{\partial I}(I_0) \\ &+ \frac{\partial^2 h_0}{\partial I^2}(I_0)(t(I_\varepsilon^{s+} - I_0^{s+}) + (1-t)(I_\varepsilon^{s} - I_0^{s})) \\ &+ O(\varepsilon^2)\end{aligned} \tag{5.15}$$

where we used

$$\frac{\partial h_0}{\partial I}(tI_0^{s+} + (1-t)I_0^{s}) = \frac{\partial h_0}{\partial I}(I_0) \text{ and } \frac{\partial^2 h_0}{\partial I^2}(tI_0^{s+} + (1-t)I_0^{s}) = \frac{\partial^2 h_0}{\partial I^2}(I_0).$$

Since $I_\varepsilon^{s+} - I_0^{s+} = O(\varepsilon)$ and $I_\varepsilon^{s} - I_0^{s} = O(\varepsilon)$, we have

$$\frac{\partial h_0}{\partial I}(tI_0^{s+} + (1-t)I_0^{s}) = \frac{\partial h_0}{\partial I}(I_0) + O(\varepsilon). \tag{5.16}$$

Since $I_\varepsilon^{s+} - I_\varepsilon^{s} = O(\varepsilon)$ from (5.14) we obtain the first estimate in (5.10).

The other estimate follows similarly.

(iii) The fact that $\xi^{s,u} = O(\varepsilon)$ follows from $\tilde{z}_\varepsilon = \tilde{z}_0 + O(\varepsilon)$, and $\xi^{s,u}(\tilde{z}_0) = 0$, hence $\xi^{s,u}(\tilde{z}_\varepsilon) = \xi^{s,u}(\tilde{z}_0) + O(\varepsilon) = O(\varepsilon)$.

In the same way we obtain $\xi^{s+} = \xi^{u-} = O(\varepsilon)$. To prove that in fact $\xi^{s+} = \xi^{u-} = O(\varepsilon^2)$, we proceed as follows. For ξ_ε^{s+}, we use the Taylor expansion:

$$\xi_\varepsilon^{s+} = \xi_0^{s+} + D\xi_0^{s+} \cdot (\xi_\varepsilon^{s+} - \xi_0^{s+}) + O(\varepsilon^2),$$

where $\cdot$ denotes the dot product, and we used that $\xi_\varepsilon^{s+} - \xi_0^{s+} = O(\varepsilon)$. We have

$$\xi_0^{s+} = 0$$

and

$$D\xi_0^{s+} \cdot (\xi_\varepsilon^{s+} - \xi_0^{s+}) = x^s(\tilde{z}_0^+)(y_\varepsilon^{s+} - y_0^{s+}) + y^s(\tilde{z}_0^+)(x_\varepsilon^{s+} - x_0^{s+}) = 0,$$

since $y^s(\tilde{z}_0^+) = x^s(\tilde{z}_0^+) = 0$ on $\tilde{\Lambda}_0$. Therefore (5.2.1) implies

$$\xi_\varepsilon^{s+} = O(\varepsilon^2).$$

Similarly we obtain

$$\xi_\varepsilon^{u-} = O(\varepsilon^2).$$

(iv) We write $g_1(I_\varepsilon^{s+})$ as a Taylor expansion, using that $I_\varepsilon^{s+} - I_0^{s+} = O(\varepsilon)$, obtaining

$$g_1(I_\varepsilon^{s+}) = g_1(I_0^{s+}) + \frac{\partial g_1}{\partial I}(I_0^{s+})(I_\varepsilon^{s+} - I_0^{s+}) + O(\varepsilon^2).$$

Since $I_0^{s+} = I_0$, the first equation in (5.12) follows.

Similarly

$$g_1(I_\varepsilon^{u-}) = g_1(I_0^{u-}) + \frac{\partial g_1}{\partial I}(I_0^{u-})(I_\varepsilon^{u-} - I_0^{u-}) + O(\varepsilon^2),$$

and $I_0^{u-} = I_0$ yield the second equation in (5.12).

(v) The proof is similar to that of (iv). □

5.2.2. *Change in action by the scattering map.* We now give the expression of the action-component $\tilde{\mathcal{S}}^I$ of the mapping $\tilde{\mathcal{S}}$ in (2.7), where I is the action-coordinate described in Proposition 4.1-**(N)**.

Proposition 5.2. *The change in I by the scattering map $\tilde{\sigma}_\varepsilon$ is given by the following formula*

(5.17)

$$\begin{aligned} & I\left(\tilde{z}_\varepsilon^+\right) - I\left(\tilde{z}_\varepsilon^-\right) \\ &= -\varepsilon\left(\frac{\partial h_0}{\partial I}(I_0)\right)^{-1} \int_0^{+\infty} \left((\mathcal{X}^1 H_0)(\Phi_0^\tau(\tilde{z}_0^+) - (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0))\right) d\tau \\ &\quad -\varepsilon\left(\frac{\partial h_0}{\partial I}(I_0)\right)^{-1} \int_{-\infty}^{0} \left((\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0^-)) - (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0))\right) d\tau \\ &\quad + O\left(\varepsilon^2\right). \end{aligned}$$

When the perturbation is Hamiltonian $\mathcal{X}^1 = J\nabla H_1$, in (5.17) *we have $\mathcal{X}^1 H_0 = \{H_0, H_1\}$, where $\{\cdot,\cdot\}$ denotes the Poisson bracket.*

Proposition 5.2 implies that

$$\tilde{\sigma}_\varepsilon^I(I,\theta,t) = \tilde{\sigma}_0^I(I,\theta,t) + \varepsilon\tilde{\mathcal{S}}^I(I,\theta,t) + O(\varepsilon^2),$$

where $\varepsilon\tilde{\mathcal{S}}^I$ is given by the first two terms on the right hand-side of (5.17). The expression of $\varepsilon\tilde{\mathcal{S}}^I$ is particularly simple since I is a slow variable.

Below we will refer to the notation in (5.5), (5.6), (5.7), and (5.8).

We note that we can express the right-hand side of (5.17) in terms of the $(I^{u,s}, \theta^{u,s}, y^{u,s}, x^{u,s})$ coordinates, by making the following substitutions:

$$\begin{aligned} \tilde{\Phi}_0^\tau(\tilde{z}_0^+) &= (I_0^s, \theta_0^s + \omega(I_0)\tau, 0, 0, t_0 + \tau), \\ \tilde{\Phi}_0^\tau(\tilde{z}_0^-) &= (I_0^s, \theta_0^u + \omega(I_0)\tau, 0, 0, t_0 + \tau), \\ \tilde{\Phi}_0^\tau(\tilde{z}_0) &= (I_0^s, \theta_0^s + \omega(I_0)\tau, y^s(\tau), 0, t_0 + \tau) \\ &= (I_0^u, \theta_0^u + \omega(I_0)\tau, 0, x^u(\tau), t_0 + \tau). \end{aligned}$$

To prove Proposition 5.2 we will use the following:

Lemma 5.3. *The change in H_0 by the scattering map $\tilde{\sigma}_\varepsilon$ is given by the following equation*

(5.18)
$$\begin{aligned} H_0(\tilde{z}_\varepsilon^+) - H_0(\tilde{z}_\varepsilon^-) = &-\varepsilon \int_0^{+\infty} \Big((\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0^+)) - (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0)) \Big)\, d\tau \\ &-\varepsilon \int_{-\infty}^0 \Big((\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0^-)) - (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0)) \Big)\, d\tau \\ &+ O\left(\varepsilon^2\right). \end{aligned}$$

Proof of Lemma 5.3. First, note that

$$(\mathcal{X}^0 + \varepsilon \mathcal{X}^1) H_0 = \mathcal{X}^0 H_0 + \varepsilon \mathcal{X}^1 H_0 = \{H_0, H_0\} + \varepsilon \mathcal{X}^1 H_0 = \varepsilon \mathcal{X}^1 H_0.$$

Second, applying Lemma D.1 and Lemma D.2 from Appendix D, for $\mathbf{F} = H_0$, we have

(5.19)
$$\begin{aligned} H_0(\tilde{z}_\varepsilon^+) - H_0(\tilde{z}_\varepsilon) = &-\varepsilon \int_0^{+\infty} \Big((\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0^+)) - (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0)) \Big)\, d\tau \\ &+ O\left(\varepsilon^{1+\varrho}\right) \end{aligned}$$

(5.20)
$$\begin{aligned} H_0(\tilde{z}_\varepsilon^-) - H_0(\tilde{z}_\varepsilon) = &\varepsilon \int_{-\infty}^0 \Big((\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0^-)) - (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0)) \Big)\, d\tau \\ &+ O\left(\varepsilon^{1+\varrho}\right). \end{aligned}$$

Subtracting the two equations from above, after cancelling out the common term $H_0(\tilde{z}_\varepsilon)$ representing the value of H_0 at the homoclinic point $\tilde{z}_0$, we obtain (5.18) with an error term of order $O(\varepsilon^{1+\varrho})$. Since the function $H_0(\tilde{z}_\varepsilon^+) - H_0(\tilde{z}_\varepsilon^-)$ can be expanded as a Taylor series in ε, by matching the corresponding terms of this Taylor expansion with the terms in (5.19) minus (5.20), it follows that the error term $O(\varepsilon^{1+\varrho})$ must equal $O(\varepsilon^2)$. □

Proof of Proposition 5.2. By Proposition 4.1 we have

$$H_0(\tilde{z}_\varepsilon^+) = h_0(I_\varepsilon^{s+}) + (\xi_\varepsilon^{s+}) g_1(I_\varepsilon^{s+}) + (\xi_\varepsilon^{s+})^2 g_2(I_\varepsilon^{s+}, \xi_\varepsilon^{s+}), \tag{5.21}$$

$$H_0(\tilde{z}_\varepsilon^-) = h_0(I_\varepsilon^{u-}) + (\xi_\varepsilon^{u-}) g_1(I_\varepsilon^{u-}) + (\xi_\varepsilon^{u-})^2 g_2(I_\varepsilon^{u-}, \xi_\varepsilon^{u-}). \tag{5.22}$$

Subtracting we obtain

$$\begin{aligned} H_0(\tilde{z}_\varepsilon^+) - H_0(\tilde{z}_\varepsilon^-) = &(h_0(I_\varepsilon^{s+}) - h_0(I_\varepsilon^{u-})) \\ &+ ((\xi_\varepsilon^{s+}) g_1(I_\varepsilon^{s+}) - (\xi_\varepsilon^{u-}) g_1(I_\varepsilon^{u-})) \\ &+ O(\varepsilon^2) \end{aligned} \tag{5.23}$$

where the error term $O(\varepsilon^2)$ in the above is due to Lemma 5.1 equation (5.11).

The term $h_0(I_\varepsilon^{s+}) - h_0(I_\varepsilon^{u-})$ in (5.23) is given, by Lemma 5.1 equation (5.10), as

$$h_0(I_\varepsilon^{s+}) - h_0(I_\varepsilon^{u-}) = (I_\varepsilon^{s+} - I_\varepsilon^{u-}) \left(\frac{\partial h_0}{\partial I}(I_0) \right) + O(\varepsilon^2). \tag{5.24}$$

Since, by Lemma 5.1 equation (5.11), we have

$$\xi_\varepsilon^{\mathrm{s}+} = O(\varepsilon^2),\ \xi_\varepsilon^{\mathrm{u}-} = O(\varepsilon^2), \tag{5.25}$$

we obtain

$$(\xi_\varepsilon^{\mathrm{s}+}) g_1(I_\varepsilon^{\mathrm{s}+}) - (\xi_\varepsilon^{\mathrm{u}-}) g_1(I_\varepsilon^{\mathrm{u}-}) = O(\varepsilon^2). \tag{5.26}$$

Thus, from (5.23) and using (5.24) we have

$$H_0(\tilde{z}_\varepsilon^+) - H_0(\tilde{z}_\varepsilon^-) = (I_\varepsilon^{\mathrm{s}+} - I_\varepsilon^{\mathrm{u}-}) \left(\frac{\partial h_0}{\partial I}(I) \right) + O(\varepsilon^2). \tag{5.27}$$

As the left-hand side of (5.27) is given by (5.18), since $\frac{\partial h_0}{\partial I} \neq 0$ by (4.10), solving for $I_\varepsilon^{\mathrm{s}+} - I_\varepsilon^{\mathrm{u}-}$ yields

(5.28)

$$\begin{aligned} I_\varepsilon^{\mathrm{s}+} - I_\varepsilon^{\mathrm{u}-} &= \left(\frac{\partial h_0}{\partial I}(I) \right)^{-1} \left(H_0(\tilde{z}_\varepsilon^+) - H_0(\tilde{z}_\varepsilon^-) \right) \\ &= -\varepsilon \left(\frac{\partial h_0}{\partial I}(I) \right)^{-1} \int_0^{+\infty} \left((\mathcal{X}^1 H_0) \left(\tilde{\Phi}_0^\tau \left(\tilde{z}_0^+ \right) \right) - (\mathcal{X}^1 H_0) \left(\tilde{\Phi}_0^\tau \left(\tilde{z}_0 \right) \right) \right) d\tau \\ &\quad -\varepsilon \left(\frac{\partial h_0}{\partial I}(I) \right)^{-1} \int_{-\infty}^{0} \left((\mathcal{X}^1 H_0) \left(\tilde{\Phi}_0^\tau \left(\tilde{z}_0^- \right) \right) - (\mathcal{X}^1 H_0) \left(\tilde{\Phi}_0^\tau \left(\tilde{z}_0 \right) \right) \right) d\tau \\ &\quad + O\left(\varepsilon^{1+\varrho} \right). \end{aligned}$$

By the same argument as in the proof of Lemma 5.3, the error term $O(\varepsilon^{1+\varrho})$ in (5.28) must equal $O(\varepsilon^2)$. This shows (5.17). □

Proposition 5.4.

(5.29)

$$\begin{aligned} I_\varepsilon^{\mathrm{s}+} - I_\varepsilon^{\mathrm{s}} &= \varepsilon \left(\frac{\partial h_0}{\partial I}(I_0)) \right)^{-1} \int_0^{+\infty} \left((\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0^+)) - (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0)) \right) d\tau \\ &\quad -\varepsilon \left(\frac{\partial h_0}{\partial I}(I_0)) \right)^{-1} g_1(I_0) \int_0^{+\infty} \left((\mathcal{X}^1 \xi^{\mathrm{s}})(\tilde{\Phi}_0^\tau(\tilde{z}_0^+)) - (\mathcal{X}^1 \xi^{\mathrm{s}})(\tilde{\Phi}_0^\tau(\tilde{z}_0)) \right) d\tau \\ &\quad + O\left(\varepsilon^2 \right). \end{aligned}$$

(5.30)

$$\begin{aligned} I_\varepsilon^{\mathrm{u}-} - I_\varepsilon^{\mathrm{u}} &= \varepsilon \left(\frac{\partial h_0}{\partial I}(I_0)) \right)^{-1} \int_{-\infty}^{0} \left((\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0^-)) - (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0)) \right) d\tau \\ &\quad -\varepsilon \left(\frac{\partial h_0}{\partial I}(I_0)) \right)^{-1} g_1(I_0) \int_{-\infty}^{0} \left((\mathcal{X}^1 \xi^{\mathrm{u}})(\tilde{\Phi}_0^\tau(\tilde{z}_0^-)) - (\mathcal{X}^1 \xi^{\mathrm{u}})(\tilde{\Phi}_0^\tau(\tilde{z}_0)) \right) d\tau \\ &\quad + O\left(\varepsilon^2 \right). \end{aligned}$$

Proof. The formula for $H_0(\tilde{z}_\varepsilon^+) - H_0(\tilde{z}_\varepsilon)$ is given by (5.19). The normal form expansion of H_0 at a homoclinic point $\tilde{z}_\varepsilon$ can be written with respect to the

two sets of coordinates as

$$H_0(\tilde{z}_\varepsilon) = h_0(I^{\mathrm{s}}_\varepsilon) + (\xi^{\mathrm{s}}_\varepsilon) g_1(I^{\mathrm{s}}_\varepsilon) + (\xi^{\mathrm{s}}_\varepsilon)^2 g_2(I^{\mathrm{s}}_\varepsilon, \xi^{\mathrm{s}}_\varepsilon) \tag{5.31}$$

$$H_0(\tilde{z}_\varepsilon) = h_0(I^{\mathrm{u}}_\varepsilon) + (\xi^{\mathrm{u}}_\varepsilon) g_1(I^{\mathrm{u}}_\varepsilon) + (\xi^{\mathrm{u}}_\varepsilon)^2 g_2(I^{\mathrm{u}}_\varepsilon, \xi^{\mathrm{u}}_\varepsilon) \tag{5.32}$$

Subtracting (5.31) from (5.21) we obtain

(5.33)

$$\begin{aligned} H_0(\tilde{z}^+_\varepsilon) - H_0(\tilde{z}_\varepsilon) =& h_0(I^{\mathrm{s}+}_\varepsilon) - h_0(I^{\mathrm{s}}_\varepsilon) + (\xi^{\mathrm{s}+}_\varepsilon) g_1(I^{\mathrm{s}+}_\varepsilon) - (\xi^{\mathrm{s}}_\varepsilon) g_1(I^{\mathrm{s}}_\varepsilon) + O(\varepsilon^2) \\ =& \left(\frac{\partial h_0}{\partial I}(I_0) \right) (I^{\mathrm{s}+}_\varepsilon - I^{\mathrm{s}}_\varepsilon) \\ &+ (\xi^{\mathrm{s}+}_\varepsilon)(g_1(I^{\mathrm{s}+}_\varepsilon) - g_1(I^{\mathrm{s}}_\varepsilon)) \\ &+ (\xi^{\mathrm{s}+}_\varepsilon - \xi^{\mathrm{s}}_\varepsilon) g_1(I^{\mathrm{s}}_\varepsilon) + O(\varepsilon^2) \\ =& \left(\frac{\partial h_0}{\partial I}(I_0) \right) (I^{\mathrm{s}+}_\varepsilon - I^{\mathrm{s}}_\varepsilon) \\ &+ (\xi^{\mathrm{s}+}_\varepsilon - \xi^{\mathrm{s}}_\varepsilon) g_1(I_0) + O(\varepsilon^2). \end{aligned}$$

In the above we have used Lemma 5.1, equations (5.10), (5.11), and (5.12).

Applying Lemma D.1 and Lemma D.2 from Appendix D, for $\mathbf{F} = \xi^{\mathrm{s}}$, we have

$$\begin{aligned} \xi^{\mathrm{s}+}_\varepsilon - \xi^{\mathrm{s}}_\varepsilon =& -\varepsilon \int_0^{+\infty} \left((\mathcal{X}^1 \xi^{\mathrm{s}})(\tilde{\Phi}^\tau_0 (\tilde{z}^+_0)) - (\mathcal{X}^1 \xi^{\mathrm{s}})(\tilde{\Phi}^\tau_0 (\tilde{z}_0)) \right) d\tau \\ &+ O\left(\varepsilon^{1+\varrho}\right). \end{aligned} \tag{5.34}$$

Thus, using (5.33), (5.19), and (5.34) we obtain (5.29).

Equation (5.30) follows similarly.

The argument that the error term $O(\varepsilon^{1+\varrho})$ can be replaced by $O(\varepsilon^2)$ follows in the same way as in the proof of Lemma 5.3. □

Let $\tau \in \mathbb{R}$ be some value of the time variable. Applying the formula (5.29) to $\tilde{\Phi}^\tau_\varepsilon(\tilde{z}^+_\varepsilon)$ and $\tilde{\Phi}^\tau_\varepsilon(\tilde{z}_\varepsilon)$ instead $\tilde{z}^+_\varepsilon$ and $\tilde{z}_\varepsilon$, respectively, and the formula (5.30) to $\tilde{\Phi}^\tau_\varepsilon(\tilde{z}^-_\varepsilon)$ and $\tilde{\Phi}^\tau_\varepsilon(\tilde{z}_\varepsilon)$ instead $\tilde{z}^-_\varepsilon$ and $\tilde{z}_\varepsilon$, respectively, we obtain:

Corollary 5.5. *For any time $\tau \in \mathbb{R}$ we have*

$$\begin{aligned} &I^{\mathrm{s}}_\varepsilon(\tilde{\Phi}^\tau(\tilde{z}^+_\varepsilon)) - I^{\mathrm{s}}_\varepsilon(\tilde{\Phi}^\tau(\tilde{z}_\varepsilon)) \\ &\quad = -\varepsilon \left(\frac{\partial h_0}{\partial I}(I_0)) \right)^{-1} \int_0^{+\infty} \Big((\mathcal{X}^1 H_0)(\tilde{\Phi}^{\tau+\varsigma}_0(\tilde{z}^+_0)) \\ &\qquad\qquad - (\mathcal{X}^1 H_0)(\tilde{\Phi}^{\tau+\varsigma}_0(\tilde{z}_0)) \Big) d\varsigma \\ &\qquad + \varepsilon \left(\frac{\partial h_0}{\partial I}(I_0)) \right)^{-1} g_1(I_0) \int_0^{+\infty} \Big((\mathcal{X}^1 \xi^{\mathrm{s}})(\tilde{\Phi}^{\tau+\varsigma}_0(\tilde{z}^+_0)) \\ &\qquad\qquad - (\mathcal{X}^1 \xi^{\mathrm{s}})(\tilde{\Phi}^{\tau+\varsigma}_0(\tilde{z}_0)) \Big) d\varsigma \\ &\qquad + O\left(\varepsilon^2\right). \end{aligned} \tag{5.35}$$

$$
\begin{aligned}
&I^{\mathrm{u}}_{\varepsilon}(\tilde{\Phi}^{\tau}(\tilde{z}^{-}_{\varepsilon}))-I^{\mathrm{u}}_{\varepsilon}(\tilde{\Phi}^{\tau}(\tilde{z}_{\varepsilon}))\\
&\qquad=\varepsilon\left(\frac{\partial h_0}{\partial I}(I_0))\right)^{-1}\int_{-\infty}^{0}\Big((\mathcal{X}^1H_0)(\tilde{\Phi}_0^{\tau+\varsigma}(\tilde{z}_0^{-}))\\
&\qquad\qquad\qquad-(\mathcal{X}^1H_0)(\tilde{\Phi}_0^{\tau+\varsigma}(\tilde{z}_0))\Big)\,d\varsigma\\
&\qquad-\varepsilon\left(\frac{\partial h_0}{\partial I}(I_0))\right)^{-1}g_1(I_0)\int_{-\infty}^{0}\Big((\mathcal{X}^1\xi^{\mathrm{u}})(\tilde{\Phi}_0^{\tau+\varsigma}(\tilde{z}_0^{-}))\\
&\qquad\qquad\qquad-(\mathcal{X}^1\xi^{\mathrm{u}})(\tilde{\Phi}_0^{\tau+\varsigma}(\tilde{z}_0))\Big)\,d\varsigma\\
&\qquad+O\left(\varepsilon^2\right).
\end{aligned}
\tag{5.36}
$$

Corollary 5.6.

$$
\begin{aligned}
&\int_0^{+\infty}\Big((\mathcal{X}^1\xi^{\mathrm{s}})(\tilde{\Phi}_0^{\tau}(\tilde{z}_0^{+}))-(\mathcal{X}^1\xi^{\mathrm{s}})(\tilde{\Phi}_0^{\tau}(\tilde{z}_0))\Big)\,d\tau\\
&\quad-\int_{-\infty}^{0}\Big((\mathcal{X}^1\xi^{\mathrm{u}})(\tilde{\Phi}_0^{\tau}(\tilde{z}_0^{-}))-(\mathcal{X}^1\xi^{\mathrm{u}})(\tilde{\Phi}_0^{\tau}(\tilde{z}_0))\Big)\,d\tau=0.
\end{aligned}
\tag{5.37}
$$

Proof. Let us denote

$$
\begin{aligned}
J^{+}&=\int_0^{+\infty}\Big((\mathcal{X}^1\xi^{\mathrm{s}})(\tilde{\Phi}_0^{\tau}(\tilde{z}_0^{+}))-(\mathcal{X}^1\xi^{\mathrm{s}})(\tilde{\Phi}_0^{\tau}(\tilde{z}_0))\Big)\,d\tau,\\
J^{-}&=-\int_{-\infty}^{0}\Big((\mathcal{X}^1\xi^{\mathrm{u}})(\tilde{\Phi}_0^{\tau}(\tilde{z}_0^{-}))-(\mathcal{X}^1\xi^{\mathrm{u}})(\tilde{\Phi}_0^{\tau}(\tilde{z}_0))\Big)\,d\tau.
\end{aligned}
\tag{5.38}
$$

Recall that, by condition (5.4), we have $I^{\mathrm{u}}_{\varepsilon}=I^{\mathrm{s}}_{\varepsilon}$. Subtracting (5.36) from (5.35), and comparing with (5.17), we should have

$$
\left(\frac{\partial h_0}{\partial I}(I_0))\right)^{-1}g_1(I_0)(J^{+}+J^{-})=0.
$$

By (4.10), we have that $\frac{\partial h_0}{\partial I}(I_0)\neq 0$ and $g_1(I_0)\neq 0$, therefore

$$
J^{+}+J^{-}=0. \tag{5.39}
$$

□

5.2.3. *Change in angle by the scattering map.* We now give the expression of the angle-component $\tilde{\mathcal{S}}^{\theta}$ of the mapping $\tilde{\mathcal{S}}$ in (2.7), where θ is the angle-coordinate described in Proposition 4.1-**(N)**.

Proposition 5.7. *The change in θ by the scattering map $\tilde{\sigma}_\varepsilon$ is given by the following equation*

(5.40)

$$\begin{aligned}
&\theta^{\mathrm{s}}\left(\tilde{z}_\varepsilon^+\right) - \theta^{\mathrm{u}}\left(\tilde{z}_\varepsilon^-\right) \\
&\quad = \Delta(I_0) - \varepsilon \int_0^{+\infty} \left(\mathcal{X}^1\theta^{\mathrm{s}}(\tilde{\Phi}_0^\tau(\tilde{z}_0^+)) - \mathcal{X}^1\theta^{\mathrm{s}}(\tilde{\Phi}_0^\tau(\tilde{z}_0)) \right) d\tau \\
&\qquad - \varepsilon \int_{-\infty}^0 \left(\mathcal{X}^1\theta^{\mathrm{u}}(\tilde{\Phi}_0^\tau(\tilde{z}_0^-)) - \mathcal{X}^1\theta^{\mathrm{u}}(\tilde{\Phi}_0^\tau(\tilde{z}_0)) \right) d\tau \\
&\qquad + \varepsilon \left(\frac{\partial^2 h_0}{\partial I^2}(I_0) \right) \left(\frac{\partial h_0}{\partial I}(I_0) \right)^{-1} \int_0^{+\infty} \Big((\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0^+)) \\
&\qquad\qquad - (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0)) \Big) \tau d\tau \\
&\qquad + \varepsilon \left(\frac{\partial^2 h_0}{\partial I^2}(I_0) \right) \left(\frac{\partial h_0}{\partial I}(I_0) \right)^{-1} \int_{-\infty}^0 \Big((\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0^-)) \\
&\qquad\qquad - (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0)) \Big) \tau d\tau \\
&\qquad + O(\varepsilon^2),
\end{aligned}$$

where $\Delta(I_0)$ is the phase-shift on the action level set I_0 that defines the unperturbed scattering map $\tilde{\sigma}_0$ (see Proposition 4.5).

Proposition 5.7 implies that

$$\tilde{\sigma}_\varepsilon^\theta(I,\theta,t) = \tilde{\sigma}_0^\theta(I,\theta,t) + \Delta(I_0) + \varepsilon\tilde{\mathcal{S}}^\theta(I,\theta,t) + O(\varepsilon^2),$$

where $\varepsilon\tilde{\mathcal{S}}^\theta$ is given by the first four terms on the right hand-side of (5.40). The expression of $\varepsilon\tilde{\mathcal{S}}^\theta$ is more complicated than the one for $\varepsilon\tilde{\mathcal{S}}^I$ since θ is a fast variable.

Proof. We will begin by computing the difference of the θ^{s} evaluated at a homoclinic point and at the footpoint of the stable fiber through the homoclinic point. By Lemma D.2 from Appendix D, we have

$$\theta^{\mathrm{s}}\left(\tilde{z}_\varepsilon^+\right) - \theta^{\mathrm{s}}\left(\tilde{z}_\varepsilon\right) = -\int_0^{+\infty} \frac{d}{d\tau}\left[\theta^{\mathrm{s}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon^+)) - \theta^{\mathrm{s}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon))\right] d\tau.$$

Now by equation (4.17) we have:

$$\frac{d\theta^{\mathrm{s}}}{d\tau} = \frac{\partial H_0}{\partial I^{\mathrm{s}}} + \varepsilon\mathcal{X}^1\theta^{\mathrm{s}}.$$

We can break the integral up into two parts:

$$A = -\int_0^{+\infty} \left(\frac{\partial H_0}{\partial I^{\mathrm{s}}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon^+)) - \frac{\partial H_0}{\partial I^{\mathrm{s}}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon)) \right) d\tau \tag{5.41}$$

and

$$B = -\varepsilon\int_0^{+\infty} \left(\mathcal{X}^1\theta^{\mathrm{s}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon^+)) - \mathcal{X}^1\theta^{\mathrm{s}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon)) \right) d\tau. \tag{5.42}$$

As for the integral B, $\varepsilon\mathcal{X}^1\theta^{\mathrm{s}}$ is $O(\varepsilon)$. So, by Lemma D.4 from Appendix D, we can express the integral in terms of the unperturbed system plus an error term:

$$B = -\varepsilon \int_0^{+\infty} \left(\mathcal{X}^1\theta^{\mathrm{s}}(\tilde{\Phi}_0^\tau(\tilde{z}_0^+)) - \mathcal{X}^1\theta^{\mathrm{s}}(\tilde{\Phi}_0^\tau(\tilde{z}_0)) \right) d\tau + O(\varepsilon^{1+\varrho}).$$

Returning to the integral (5.41), we now use the normal form (4.3) of H_0 given by Proposition 4.1, yielding

$$\frac{\partial H_0}{\partial I^{\mathrm{s}}} = \frac{\partial h_0}{\partial I^{\mathrm{s}}}(I^{\mathrm{s}}) + (\xi^{\mathrm{s}})\frac{\partial g_1}{\partial I^{\mathrm{s}}}(I^{\mathrm{s}}) + (\xi^{\mathrm{s}})^2\frac{\partial g_2}{\partial I^{\mathrm{s}}}(I^{\mathrm{s}}, \xi^{\mathrm{s}}), \tag{5.43}$$

where $\xi^{\mathrm{s}} = x^{\mathrm{s}}y^{\mathrm{s}}$.

Thus, the integral A given by (5.41) breaks into three parts

$$A_1 = -\int_0^{+\infty} \left(\frac{\partial h_0}{\partial I^{\mathrm{s}}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon^+)) - \frac{\partial h_0}{\partial I^{\mathrm{s}}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon)) \right) d\tau, \tag{5.44}$$

$$A_2 = -\int_0^{+\infty} \left(\xi^{\mathrm{s}}\frac{\partial g_1^{\mathrm{s}}}{\partial I^{\mathrm{s}}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon^+)) - \xi^{\mathrm{s}}\frac{\partial g_1^{\mathrm{s}}}{\partial I^{\mathrm{s}}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon)) \right) d\tau, \tag{5.45}$$

$$A_3 = -\int_0^{+\infty} \left((\xi^{\mathrm{s}})^2\frac{\partial g_2^{\mathrm{s}}}{\partial I^{\mathrm{s}}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon^+)) - (\xi^{\mathrm{s}})^2\frac{\partial g_2^{\mathrm{s}}}{\partial I^{\mathrm{s}}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon)) \right) d\tau. \tag{5.46}$$

From Lemma 5.1 equation (5.11) we have $\xi^{\mathrm{s}}(\tilde{\Phi}(\tilde{z}_\varepsilon^+)) = O(\varepsilon^2)$ and $\xi^{\mathrm{s}}(\tilde{\Phi}(\tilde{z}_\varepsilon)) = O(\varepsilon)$ in (5.46). Thus, we can immediately obtain that A_3 is $O(\varepsilon^2)$.

We use the integral form of the Mean Value Theorem to rewrite the integral A_1. Recall that

$$\frac{\partial F}{\partial x}(b) - \frac{\partial F}{\partial x}(a) = (b-a)\int_0^1 \frac{\partial^2 F}{\partial x^2}\left(a + t(b-a)\right) dt$$

Applying this result to A_1 for $b = I^{\mathrm{s}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon^+))$, $a = I^{\mathrm{s}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon))$, and $F = \frac{\partial h_0}{\partial I^{\mathrm{s}}}$, yields

$$A_1 = -\int_0^{+\infty} (I^{\mathrm{s}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon^+)) - I^{\mathrm{s}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon)))C_\varepsilon(\tau)\, d\tau, \tag{5.47}$$

where C_ε stands for the integral

$$C_\varepsilon(\tau) = \int_0^1 \frac{\partial^2 h_0}{\partial (I^{\mathrm{s}})^2}\left[I^{\mathrm{s}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon)) + t\left(I^{\mathrm{s}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon^+)) - I^{\mathrm{s}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon)) \right) \right] dt. \tag{5.48}$$

We evaluate the expression $I^{\mathrm{s}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon^+)) - I^{\mathrm{s}}(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon))$ in (5.47) by invoking Corollary 5.5, obtaining

(5.49)
$$
\begin{aligned}
&-\varepsilon\left(\frac{\partial h_0}{\partial I}(I_0))\right)^{-1}\int_0^{+\infty}\left((\mathcal{X}^1H_0)(\tilde{\Phi}_0^{\tau+\varsigma}(\tilde{z}_0^+)) - (\mathcal{X}^1H_0)(\tilde{\Phi}_0^{\tau+\varsigma}(\tilde{z}_0))\right)d\varsigma\\
&+\varepsilon\left(\frac{\partial h_0}{\partial I^{\mathrm{s}}}(I_0))\right)^{-1} g_1(I_0)\int_0^{+\infty}\left((\mathcal{X}^1\xi^{\mathrm{s}})(\tilde{\Phi}_0^{\tau+\varsigma}(\tilde{z}_0^+)) - (\mathcal{X}^1\xi^{\mathrm{s}})(\tilde{\Phi}_0^{\tau+\varsigma}(\tilde{z}_0))\right)d\varsigma\\
&+O\left(\varepsilon^2\right).
\end{aligned}
$$

The next part of the integrand is

$$
C_\varepsilon(\tau) = \int_0^1 \frac{\partial^2 h_0}{\partial (I^{\mathrm{s}})^2}\left[I^{\mathrm{s}}\left(\tilde{\Phi}_\varepsilon^\tau\left(\tilde{z}_\varepsilon\right)\right) + t\left(I^{\mathrm{s}}\left(\tilde{\Phi}_\varepsilon^\tau\left(\tilde{z}_\varepsilon^+\right)\right) - I^{\mathrm{s}}\left(\tilde{\Phi}_\varepsilon^\tau\left(\tilde{z}_\varepsilon\right)\right)\right)\right]dt.
$$

Using Gronwall's inequality – Lemma C.1 –, we can write

$$
C_\varepsilon(\tau) = C_0(\tau) + O(\varepsilon^\varrho),
$$

where $0 < \varrho < 1$. However, when $\varepsilon = 0$, I^{s} along the flow of the footpoint $\tilde{z}_0^+$ is equal to I^{s} along the flow of the homoclinic point $\tilde{z}_0$. Since $I^{\mathrm{s}}\left(\tilde{z}_0\right) = I_0$ we obtain

$$
C_0(\tau) = \frac{\partial^2 h_0}{\partial I^2}(I_0),
$$

which is a constant.

Putting these expressions together, we can write A_1 as

(5.50)
$$
\begin{aligned}
&\varepsilon\left(\frac{\partial^2 h_0}{\partial I^2}\left(I_0\right)\right)\left(\frac{\partial h_0}{\partial I}(I_0))\right)^{-1}\int_0^{+\infty}\int_0^{+\infty}\Big((\mathcal{X}^1H_0)(\tilde{\Phi}_0^{\tau+\varsigma}(\tilde{z}_0^+))\\
&\qquad\qquad -(\mathcal{X}^1H_0)(\tilde{\Phi}_0^{\tau+\varsigma}(\tilde{z}_0))\Big)d\varsigma d\tau\\
&-\varepsilon\left(\frac{\partial^2 h_0}{\partial I^2}(I_0)\right)\left(\frac{\partial h_0}{\partial I}(I_0))\right)^{-1} g_1(I_0)\int_0^{+\infty}\int_0^{+\infty}\Big((\mathcal{X}^1\xi^{\mathrm{s}})(\tilde{\Phi}_0^{\tau+\varsigma}(\tilde{z}_0^+))\\
&\qquad\qquad -(\mathcal{X}^1\xi^{\mathrm{s}})(\tilde{\Phi}_0^{\tau+\varsigma}(\tilde{z}_0))\Big)d\varsigma d\tau\\
&+O\left(\varepsilon^{1+\varrho}\right).
\end{aligned}
$$

We will now write the double integrals in (5.50) in a simpler form. We show only the details of the computation from the first double integral that appears in (5.50), since the second double integral can be treated in a similar fashion. Denote by $\mathscr{I}^{\mathrm{s}}$ the following improper integral

$$
\mathscr{I}^{\mathrm{s}}(\tau) = -\int_\tau^{+\infty}((\mathcal{X}^1H_0)(\tilde{\Phi}_0^\upsilon(\tilde{z}_0^+)) - (\mathcal{X}^1H_0)(\tilde{\Phi}_0^\upsilon(\tilde{z}_0)))d\upsilon. \tag{5.51}
$$

Since $(\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\upsilon(\tilde{z}_0^+)) - (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\upsilon(\tilde{z}_0)))$ approaches 0 exponentially as $\upsilon \to +\infty$, the above integral is convergent and moreover

$$\frac{d}{d\tau}(\mathscr{I}^{\mathrm{s}}(\tau)) = (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0^+)) - (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0)).$$

That is, $\mathscr{I}^{\mathrm{s}}(\tau)$ is the antiderivative of $\tau \mapsto (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0^+)) - (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0))$ satisfying the condition that it equals 0 at $+\infty$.

Making the change of variable $\upsilon = \sigma + \tau$ with $d\upsilon = d\sigma$ the double integral in (5.50) becomes

(5.52)
$$\int_0^{+\infty} \int_\tau^{+\infty} ((\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\upsilon(\tilde{z}_0^+)) - (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\upsilon(\tilde{z}_0)))d\upsilon d\tau = -\int_0^{+\infty} \mathscr{I}^{\mathrm{s}}(\tau)d\tau.$$

Using Integration by Parts we obtain

$$\begin{aligned} -\int_0^{+\infty} \mathscr{I}^{\mathrm{s}}(\tau)d\tau &= -\tau \mathscr{I}^{\mathrm{s}}(\tau)\Big|_0^{+\infty} \\ &\quad + \int_0^{+\infty} ((\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0^+)) - (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0)))\tau d\tau \\ &= \int_0^{+\infty} ((\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0^+)) - (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0)))\tau d\tau. \end{aligned} \tag{5.53}$$

In the above, the quantity $\tau \mathscr{I}^{\mathrm{s}}(\tau)$ obviously equals 0 at $\tau = 0$, and equals 0 when $\tau \to +\infty$ since, by l'Hospital Rule

$$\lim_{\tau \to +\infty} \frac{\mathscr{I}^{\mathrm{s}}(\tau)}{\tau^{-1}} = \lim_{\tau \to +\infty} -\frac{(\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0^+)) - (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0))}{\tau^{-2}} = 0,$$

since $(\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0^+)) - (\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0))$ approaches 0 at exponential rate as $\tau \to +\infty$.

A similar computation can be done to write the second double integral that appears in (5.50) as a single integral.

Thus, we obtain the following expression of A_1:

$$\begin{aligned} &\varepsilon \left(\frac{\partial^2 h_0}{\partial I^2}(I_0)\right) \left(\frac{\partial h_0}{\partial I}(I_0)\right)^{-1} \int_0^{+\infty} \Big((\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0^+)) \\ &\qquad\qquad -(\mathcal{X}^1 H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0))\Big)\tau d\tau \\ &- \varepsilon \left(\frac{\partial^2 h_0}{\partial I^2}(I_0)\right) \left(\frac{\partial h_0}{\partial I}(I_0))\right)^{-1} g_1(I_0) \int_0^{+\infty} \Big((\mathcal{X}^1 \xi^{\mathrm{s}})(\tilde{\Phi}_0^\tau(\tilde{z}_0^+)) \\ &\qquad\qquad -(\mathcal{X}^1 \xi^{\mathrm{s}})(\tilde{\Phi}_0^\tau(\tilde{z}_0))\Big)\tau d\tau \\ &+ O\left(\varepsilon^{1+\varrho}\right). \end{aligned} \tag{5.54}$$

Finally, we turn to the integral A_2 given by (5.45). Using Lemma 5.1 equations (5.13) and (5.11), (5.34), as well as integration by parts similarly

to above, we express A_2 as

(5.55)
$$
\begin{aligned}
&-\int_0^{+\infty}\left((\xi^{\mathrm{s}}\frac{\partial g_1}{\partial I})(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon^+))-(\xi^{\mathrm{s}}\frac{\partial g_1}{\partial I})(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon))\right)d\tau\\
&=-\left(\frac{\partial g_1}{\partial I}(I_0)\right)\int_0^{+\infty}\left(\xi^{\mathrm{s}}((\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon^+)))-\xi^{\mathrm{s}}((\tilde{\Phi}_\varepsilon^\tau(\tilde{z}_\varepsilon)))\right)d\tau\\
&\quad+O(\varepsilon^{1+\varrho})\\
&=\varepsilon\left(\frac{\partial g_1}{\partial I}(I_0)\right)\int_0^{+\infty}\int_0^{+\infty}\left((\mathcal{X}^1\xi^{\mathrm{s}})(\tilde{\Phi}_0^{\tau+\varsigma}(\tilde{z}_0^+))-(\mathcal{X}^1\xi^{\mathrm{s}})(\tilde{\Phi}_0^{\tau+\varsigma}(\tilde{z}_0))\right)d\varsigma d\tau\\
&\quad+O(\varepsilon^{1+\varrho})\\
&=\varepsilon\left(\frac{\partial g_1}{\partial I}(I_0)\right)\int_0^{+\infty}\left((\mathcal{X}^1\xi^{\mathrm{s}})(\tilde{\Phi}_0^{\tau}(\tilde{z}_0^+))-(\mathcal{X}^1\xi^{\mathrm{s}})(\tilde{\Phi}_0^{\tau}(\tilde{z}_0))\right)\tau d\tau\\
&\quad+O(\varepsilon^{1+\varrho}).
\end{aligned}
$$

In the above, we have used that

$$(\mathcal{X}^0+\varepsilon\mathcal{X}^1)\xi^s=\mathcal{X}^0\xi^s+\varepsilon\mathcal{X}^1\xi^s=\{\xi^s,H_0\}+\varepsilon\mathcal{X}^1\xi^s=\varepsilon\mathcal{X}^1\xi^s+O(\varepsilon^2).$$

Combining (5.54) and (5.55) we obtain

(5.56)
$$
\begin{aligned}
&\theta^{\mathrm{s}}\left(\tilde{z}_\varepsilon^+\right)-\theta^{\mathrm{s}}\left(\tilde{z}_\varepsilon\right)\\
&\qquad=\varepsilon\left(\frac{\partial^2 h_0}{\partial I^2}(I_0)\right)\left(\frac{\partial h_0}{\partial I}(I_0)\right)^{-1}\int_0^{+\infty}\Big((\mathcal{X}^1H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0^+))\\
&\qquad\qquad\qquad\qquad-(\mathcal{X}^1H_0)(\tilde{\Phi}_0^\tau(\tilde{z}_0))\Big)\tau d\tau\\
&\qquad\quad-\varepsilon\left(\frac{\partial^2 h_0}{\partial I^2}(I_0)\right)\left(\frac{\partial h_0}{\partial I}(I_0))\right)^{-1}g_1(I_0)\int_0^{+\infty}\Big((\mathcal{X}^1\xi^{\mathrm{s}})(\tilde{\Phi}_0^\tau(\tilde{z}_0^+))\\
&\qquad\qquad\qquad\qquad-(\mathcal{X}^1\xi^{\mathrm{s}})(\tilde{\Phi}_0^\tau(\tilde{z}_0))\Big)\tau d\tau\\
&\qquad\quad+\varepsilon\left(\frac{\partial g_1}{\partial I}(I_0)\right)\int_0^{+\infty}\left((\mathcal{X}^1\xi^{\mathrm{s}})(\tilde{\Phi}_0^\tau(\tilde{z}_0^+))-(\mathcal{X}^1\xi^{\mathrm{s}})(\tilde{\Phi}_0^\tau(\tilde{z}_0))\right)\tau d\tau\\
&\qquad\quad+O(\varepsilon^2).
\end{aligned}
$$

In the above, we have also replaced the error term $O(\varepsilon^{1+\varrho})$ by $O(\varepsilon^2)$, by calling the same argument as in the proof of Proposition 5.2.

A similar computation yields:

(5.57)

$$\begin{aligned}
&\theta^{\mathrm{u}}\left(\tilde{z}_{\varepsilon}^{-}\right)-\theta^{\mathrm{u}}\left(\tilde{z}_{\varepsilon}\right)\\
&\quad=-\varepsilon\left(\frac{\partial^2 h_0}{\partial I^2}(I_0)\right)\left(\frac{\partial h_0}{\partial I}(I_0)\right)^{-1}\int_{-\infty}^{0}\Big((\mathcal{X}^1 H_0)(\tilde{\Phi}_0^{\tau}(\tilde{z}_0^{-}))\\
&\qquad\qquad -(\mathcal{X}^1 H_0)(\tilde{\Phi}_0^{\tau}(\tilde{z}_0))\Big)\,\tau d\tau\\
&\quad+\varepsilon\left(\frac{\partial^2 h_0}{\partial I^2}(I_0)\right)\left(\frac{\partial h_0}{\partial I}(I_0))\right)^{-1} g_1(I_0)\int_{-\infty}^{0}\Big((\mathcal{X}^1\xi^{\mathrm{u}})(\tilde{\Phi}_0^{\tau}(\tilde{z}_0^{-}))\\
&\qquad\qquad -(\mathcal{X}^1\xi^{\mathrm{u}})(\tilde{\Phi}_0^{\tau}(\tilde{z}_0))\Big)\,\tau d\tau\\
&\quad-\varepsilon\left(\frac{\partial g_1}{\partial I}(I_0)\right)\int_{-\infty}^{0}\Big((\mathcal{X}^1\xi^{\mathrm{u}})(\tilde{\Phi}_0^{\tau}(\tilde{z}_0^{-}))-(\mathcal{X}^1\xi^{\mathrm{u}})(\tilde{\Phi}_0^{\tau}(\tilde{z}_0))\Big)\,\tau d\tau\\
&\quad+O(\varepsilon^2).
\end{aligned}$$

Subtracting (5.57) from (5.56) yields $\theta^{\mathrm{s}}\left(\tilde{z}_{\varepsilon}^{+}\right)-\theta^{\mathrm{u}}\left(\tilde{z}_{\varepsilon}^{-}\right)=\theta^{\mathrm{s}}\left(\tilde{z}_{\varepsilon}\right)-\theta^{\mathrm{u}}\left(\tilde{z}_{\varepsilon}\right)$, plus an epsilon order term consisting of the sum of six integrals, plus an error term of order $O(\varepsilon^2)$. Using the notation (5.38), four of these integrals are

(5.58)

$$\begin{aligned}
&-\varepsilon\left(\frac{\partial^2 h_0}{\partial I^2}(I_0)\right)\left(\frac{\partial h_0}{\partial I}(I_0))\right)^{-1} g_1(I_0)J^{+},\\
&-\varepsilon\left(\frac{\partial^2 h_0}{\partial I^2}(I_0)\right)\left(\frac{\partial h_0}{\partial I}(I_0))\right)^{-1} g_1(I_0)J^{-},\\
&\varepsilon\left(\frac{\partial g_1}{\partial I}(I_0)\right)J^{+},\\
&\varepsilon\left(\frac{\partial g_1}{\partial I}(I_0)\right)J^{-}.
\end{aligned}$$

By Corollary 5.6, since $J^{+}+J^{-}=0$, the sum of the first two expressions in (5.58) equals 0, and the sum of the last two expressions in (5.58) equals 0.

Also, we recall from Section 5.1.2, that for a given unperturbed homoclinic point $\tilde{z}_0$ we selected a perturbed homoclinic point $\tilde{z}_{\varepsilon}$ satisfying condition (5.4), that is $\theta^{\mathrm{s}}(\tilde{z}_{\varepsilon})-\theta^{\mathrm{u}}(\tilde{z}_{\varepsilon})=\theta^{\mathrm{s}}(\tilde{z}_0)-\theta^{\mathrm{u}}(\tilde{z}_0)=\Delta(I(\tilde{z}_0))$.

Combining these results we obtain (5.40). □

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Appendix A. Normally hyperbolic invariant manifolds

We briefly recall the notion of a normally hyperbolic invariant manifold.

Definition A.1. Let M be a $\mathcal{C}^r$-smooth manifold, Φ^t a $\mathcal{C}^r$-flow on M. A submanifold (with or without boundary) Λ of M is a normally hyperbolic invariant manifold (NHIM) for Φ^t if it is invariant under Φ^t, and there exists a splitting of the tangent bundle of TM into sub-bundles over Λ

$$T_zM = E^{\mathrm{u}}_z \oplus E^{\mathrm{s}}_z \oplus T_z\Lambda, \quad \forall z \in \Lambda \tag{A.1}$$

that are invariant under $D\Phi^t$ for all $t \in \mathbb{R}$, and there exist rates

$$\lambda_- \leqslant \lambda_+ < \lambda_c < 0 < \mu_c < \mu_- \leqslant \mu_+$$

and a constant $C > 0$, such that for all $x \in \Lambda$ we have

(A.2)

$$\begin{aligned}
Ce^{t\lambda_-}\|v\| \leqslant \|D\Phi^t(z)(v)\| \leqslant Ce^{t\lambda_+}\|v\| &\text{ for all } t \geqslant 0, \text{ if and only if } v \in E^{\mathrm{s}}_z,\\
Ce^{t\mu_+}\|v\| \leqslant \|D\Phi^t(z)(v)\| \leqslant Ce^{t\mu_-}\|v\| &\text{ for all } t \leqslant 0, \text{ if and only if } v \in E^{\mathrm{u}}_z,\\
Ce^{|t|\lambda_c}\|v\| \leqslant \|D\Phi^t(z)(v)\| \leqslant Ce^{|t|\mu_c}\|v\| &\text{ for all } t \in \mathbb{R}, \text{ if and only if } v \in T_z\Lambda.
\end{aligned}$$

In the case when Φ^t is a Hamiltonian flow, the rates can be chosen so that

$$\lambda_- = -\mu_+,\ \lambda_+ = -\mu_-, \text{ and } \lambda_c = -\mu_c.$$

The regularity of the manifold Λ depends on the rates λ^-, λ^+, μ^-, μ^+, λ_c, and μ_c. More precisely, Λ is $\mathcal{C}^\ell$-differentiable, with $\ell \leqslant r-1$, provided that

$$\begin{aligned}
\ell\mu_c + \lambda_+ &< 0,\\
\ell\lambda_c + \mu_- &> 0.
\end{aligned} \tag{A.3}$$

The manifold Λ has associated unstable and stable manifolds of Λ, denoted $W^{\mathrm{u}}(\Lambda)$ and $W^{\mathrm{s}}(\Lambda)$, respectively, which are $\mathcal{C}^{\ell-1}$-differentiable. They are foliated by 1-dimensional unstable and stable manifolds (fibers) of points, $W^{\mathrm{u}}(z)$, $W^{\mathrm{s}}(z)$, $z \in \Lambda$, respectively, which are as smooth as the flow.

These manifolds are defined by:

$$\begin{aligned}
W^{\mathrm{s}}(\Lambda) =& \{y \,|\, d(\Phi^t_\varepsilon(y), \Lambda) \to 0 \text{ as } t \to +\infty\ \}\\
=& \{y \,|\, d(\Phi^t_\varepsilon(y), \Lambda) \leqslant C_y e^{t\lambda_+}, t \geqslant 0\},\\
W^{\mathrm{u}}(\Lambda) =& \{y \,|\, d(\Phi^t_\varepsilon(y), \Lambda) \to 0 \text{ as } t \to -\infty\ \}\\
=& \{y \,|\, d(\Phi^t_\varepsilon(y), \Lambda) \leqslant C_y e^{t\mu_-}, t \geqslant 0\},\\
W^{\mathrm{s}}(x) =& \{y \,|\, d(\Phi^t(y), \Phi^t(x)) < C_y e^{t\lambda_+},\ t \geqslant 0\},\\
W^{\mathrm{u}}(x) =& \{y \,|\, d(\Phi^t(y), \Phi^t(x)) < C_y e^{t\mu_-},\ t \leqslant 0\}.
\end{aligned} \tag{A.4}$$

The fibers $W^{\mathrm{u}}(x)$, $W^{\mathrm{s}}(x)$ are not invariant by the flow, but *equivariant* in the sense that

$$\begin{aligned}
\Phi^t(W^{\mathrm{u}}(z)) &= W^{\mathrm{u}}(\Phi^t(z)),\\
\Phi^t(W^{\mathrm{s}}(z)) &= W^{\mathrm{s}}(\Phi^t(z)).
\end{aligned} \tag{A.5}$$

Since $W^{\mathrm{s,u}}(\Lambda) = \bigcup_{z\in\Lambda} W^{s,u}(z)$, we can define the projections along the fibers

$$\begin{aligned} &\Omega^+ : W^{\mathrm{s}}(\Lambda) \to \Lambda, \quad \Omega^+(z) = z^+ \text{ iff } z \in W^{\mathrm{s}}(z^+), \\ &\Omega^- : W^{\mathrm{u}}(\Lambda) \to \Lambda, \quad \Omega^-(z) = z^- \text{ iff } z \in W^{\mathrm{u}}(z^-). \end{aligned} \tag{A.6}$$

The point $z^+ \in \Lambda$ is characterized by

$$d(\Phi^t(z), \Phi^t(z^+)) \leqslant C_z e^{t\lambda_+}, \qquad \text{for all } t \geqslant 0. \tag{A.7}$$

and the point $z^- \in \Lambda$ by

$$d(\Phi^t(z), \Phi^t(z^-)) \leqslant C_z e^{t\mu_-}, \qquad \text{for all } t \leqslant 0, \tag{A.8}$$

for some $C_z > 0$.

For our applications, the most important result about NHIM's is that they persist when we perturb the flow. This is the fundamental result of [Fen72, HPS77, Pes04].

The standard assumption for persistence is that the unperturbed NHIM is a compact manifold without a boundary. The persistence of the NHIM also holds when the compactness assumption is replaced with the assumption that the perturbation has uniformly bounded derivatives in all variables [HPS77, Section 6]. There are also proofs of persistence in the infinite dimensional case which do not require compactness, such as in [BLZ99, BLZ08].

We remark that the particular case when the perturbation is periodic or quasi-periodic in time can be reduced to the compact case. More precisely, we can rewrite the system (2.2) as

$$\begin{aligned} \frac{d}{dt} z &= \mathcal{X}^0(z) + \varepsilon \mathcal{X}^1(z,\theta), \\ \frac{d}{dt}\theta &= \omega, \end{aligned}$$

where θ ranges over a torus $\mathbb{T}^d$, and $\omega \in \mathbb{R}^d$ is a rationally independent vector when $d > 1$. If the flow of $\frac{d}{dt} z = \mathcal{X}^0(z)$ admits a compact NHIM Λ_0, the extended system for $\varepsilon = 0$ admits a compact NHIM $\Lambda_0 \times \mathbb{T}^d$ which persists for small enough ε, using the standard theory. The torus $\mathbb{T}^d$ is sometimes called 'the clock manifold'.

In the case when the manifold has a boundary, the persistence result requires a step of extending the flow. This makes that the persistent manifold is not invariant but only locally invariant and not unique (it depends on the extension).

When we are given a family of flows, it is possible to choose the extensions depending smoothly on parameters and obtain that the manifolds depend smoothly on parameters.

The precise meaning of the smooth dependence is that the we can find parametrizations $k_\varepsilon : \Lambda_0 \to \Lambda_\varepsilon$. The parametrizations k_ε can be chosen so that

$$\frac{d}{d\varepsilon} k_\varepsilon(z) \in E^{\mathrm{u}}_z \oplus E^{\mathrm{s}}_z,$$

where the splitting $E^{\mathrm{u}}_z \oplus E^{\mathrm{s}}_z$ corresponds to the invariant manifold of the perturbed system. In this case we obtain Λ_ε as a graph over the central variables on Λ_0; see [DdlLS08].

The maps $k_\varepsilon(x)$ are jointly C^r as functions of x, ε. The proof of this well known result is not very difficult. It suffices to consider an extended flow $\tilde{\Phi}^t(x,\varepsilon) = (\Phi^t_\varepsilon(x), \varepsilon)$, which is a small perturbation of $\tilde{\Phi}^t_0(x,\varepsilon) = (\Phi^t_0(x), \varepsilon)$. The regularity of the NHIM of $\tilde{\Phi}^t$ gives the claimed regularity of the NHIM of Φ^t with respect to parameters.

From the same proof (using the invariant objects of the extended flow), it easily follows the regularity with respect to parameters of the stable and unstable bundles and the stable and unstable manifolds.

Appendix B. Scattering map

Assume that $W^{\mathrm{u}}(\Lambda)$, $W^{\mathrm{s}}(\Lambda)$ have a transverse intersection along a manifold Γ satisfying:

$$\begin{aligned} &T_z\Gamma = T_z W^{\mathrm{s}}(\Lambda) \cap T_z W^{\mathrm{u}}(\Lambda), \text{ for all } z \in \Gamma, \\ &T_z M = T_z\Gamma \oplus T_z W^{\mathrm{u}}(z^-) \oplus T_z W^{\mathrm{s}}(z^+), \text{ for all } z \in \Gamma. \end{aligned} \tag{B.1}$$

Under these conditions the projection mappings $\Omega^\pm$ restricted to Γ are local diffeomorphisms. We can restrict Γ if necessary so that $\Omega^\pm$ are diffeomorphisms from Γ onto open subsets $U^\pm$ in Λ.

Definition B.1. A homoclinic channel is a homoclinic manifold Γ satisfying the strong transversality condition B.1, and such that

$$\Omega^\pm_{|\Gamma} : \Gamma \to U^\pm := \Omega^\pm(\Gamma)$$

are $\mathcal{C}^{\ell-1}$-diffeomorphisms.

Definition B.2. Given a homoclinic channel Γ, the scattering map associated to Γ is defined as

$$\begin{aligned} \sigma :=& \sigma^\Gamma : U^- \subseteq \Lambda \to U^+ \subseteq \Lambda, \\ \sigma =& \Omega^+ \circ (\Omega^-)^{-1}. \end{aligned}$$

Equivalently, $\sigma(z^-) = z^+$, provided that $W^{\mathrm{u}}(z^-)$ intersects $W^{\mathrm{s}}(z^+)$ at a unique point $z \in \Gamma$.

The meaning of the scattering map is that, given a homoclinic excursion, it has two orbits in the manifold is asymptotic to. It is asymptotic to an orbit in the past and to another orbit in the future. The scattering map considers the future asymptotic orbit as a function of the asymptotic in the past. When we consider all the homoclinic orbits in a homoclinic channel we obtain a scattering map from an open domain. The intuition of the scattering map is that if we observe the orbit for long times, we just measure the effect of the homoclinic excursion on the asymptotic behavior. The scattering map is a very economical way of studying these excursions

since it is a map only on the NHIM. Furthermore, as we will see now, it satisfies remarkable geometric properties.

Due to (A.5), the scattering map satisfies the following property

$$\Phi^T \circ \sigma^\Gamma = \sigma^{\Phi^T(\Gamma)} \circ \Phi^T \tag{B.2}$$

for any $T \in \mathbb{R}$.

If M is a symplectic manifold, Φ^t is a Hamiltonian flow on M, and $\Lambda \subseteq M$ is symplectic, then the scattering map is symplectic. If the flow is exact Hamiltonian, the scattering map is exact symplectic. For details see [DdlLS08].

In a similar fashion, we can define heteroclinic channels and associated scattering maps.

Given two NHIM's Λ^1 and Λ^2, we can define the projection mappings $\Omega^{\pm,i} : W^{\mathrm{s,u}}(\Lambda^i) \to \Lambda^i$ for $i = 1, 2$. Assume that $W^{\mathrm{u}}(\Lambda^1)$ intersects transversally $W^{\mathrm{s}}(\Lambda^2)$ along a heteroclinic manifold Γ so that:

$$\begin{aligned} &T_z\Gamma = T_z W^{\mathrm{u}}(\Lambda^1) \cap T_z W^{\mathrm{s}}(\Lambda^2), \text{ for all } z \in \Gamma, \\ &T_z M = T_z\Gamma \oplus T_z W^{\mathrm{u}}(z^-) \oplus T_z W^{\mathrm{s}}(z^+), \text{ for all } z \in \Gamma, \end{aligned} \tag{B.3}$$

where $z^- = \Omega^{-,1}(z) \in \Lambda^1$ and $z^+ = \Omega^{+,2}(z) \in \Lambda^2$.

We can restrict Γ so that $\Omega^{-,1} : \Gamma \to \Lambda^1$ and $\Omega^{+,2} : \Gamma \to \Lambda^2$ are diffeomorphisms onto their corresponding images.

Definition B.3. A heteroclinic channel is a heteroclinic manifold Γ satisfying the strong transversality condition B.3, and such that

$$\begin{aligned} \Omega^{-,1}_{|\Gamma} : \Gamma \to U^- := \Omega^{-,1}(\Gamma) \subseteq \Lambda^1, \\ \Omega^{+,2}_{|\Gamma} : \Gamma \to U^+ := \Omega^{+,2}(\Gamma) \subseteq \Lambda^2, \end{aligned}$$

are $\mathcal{C}^{l-1}$-diffeomorphisms.

Definition B.4. Given a heteroclinic channel Γ, the scattering map associated to Γ is defined as

$$\begin{aligned} \sigma :=& \sigma^\Gamma : U^- \subseteq \Lambda^1 \to U^+ \subseteq \Lambda^2, \\ \sigma =& \Omega^{+,2} \circ (\Omega^{-,1})^{-1}. \end{aligned}$$

From the result of the regularity with respect to parameters of the stable and unstable manifolds and the fact that the scattering map is expressed in terms of transverse intersections, we obtain that the scattering map depends smoothly on parameters. Thus, the goal of this paper is not to prove the derivative of the scattering map with respect to the perturbation parameter exists, only to give explicit formulas knowing that the derivative exists.

Appendix C. Gronwall's inequality

In this section we apply Gronwall's Inequality to estimate the error between the solution of an unperturbed system and the solution of the perturbed system, over a time of logarithmic order with respect to the size of the perturbation.

Lemma C.1. *Consider the following differential equations:*

$$\frac{d}{dt}z(t) = \mathcal{X}^0(z,t) \tag{C.1}$$

$$\frac{d}{dt}z(t) = \mathcal{X}^0(z,t) + \varepsilon\mathcal{X}^1(z,t,\varepsilon) \tag{C.2}$$

Assume that $\mathcal{X}^0, \mathcal{X}^1$ are uniformly Lipschitz continuous in the variable z, C_0 is the Lipschitz constant of $\mathcal{X}^0$, and $\mathcal{X}^1$ is bounded with $\|\mathcal{X}^1\| \leqslant C_1$, for some $C_0, C_1 > 0$. Let z_0 be a solution of the equation (C.1) *and z_ε be a solution of the equation* (C.2) *such that*

$$\|z_0(t_0) - z_\varepsilon(t_0)\| < c\varepsilon. \tag{C.3}$$

Then, for $0 < \varrho_0 < 1$, $k \leqslant \frac{1-\varrho_0}{C_0}$, and $K = c + \frac{C_1}{C_0}$, we have

$$\|z_0(t) - z_\varepsilon(t)\| < K\varepsilon^{\varrho_0}, \text{ for } 0 \leqslant t - t_0 \leqslant k\ln(1/\varepsilon). \tag{C.4}$$

For a proof, see [GdlLM21].

Appendix D. Master lemmas

In this section we recall some abstract Melnikov-type integral operators and some of their properties from [GdlLM21].

Consider a system as in (2.2) and the extended systems as in (2.6).

Assume that, for some $\varepsilon_1 > 0$, and for each $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, there exists a normally hyperbolic invariant manifold $\tilde{\Lambda}_\varepsilon$ for $\tilde{\Phi}^\tau_\varepsilon$, as well as a homoclinic channel $\tilde{\Gamma}_\varepsilon$, which depend $\mathcal{C}^\ell$-smoothly on ε. Associated to $\tilde{\Gamma}_\varepsilon$ we have projections $\Omega^\pm : \tilde{\Gamma}_\varepsilon \to \Omega^\pm(\tilde{\Gamma}_\varepsilon) \subseteq \tilde{\Lambda}_\varepsilon$, which are local diffeomorphisms. We are thinking of $\tilde{\Phi}^\tau_\varepsilon$, $\tilde{\Lambda}_\varepsilon$, $\tilde{\Gamma}_\varepsilon$ as perturbations of $\tilde{\Phi}^\tau_0$, $\tilde{\Lambda}_0$, $\tilde{\Gamma}_0$.

For $\tilde{z}_0 \in \tilde{\Gamma}_0$ let $\tilde{z}_\varepsilon \in \tilde{\Gamma}_\varepsilon$ be the corresponding homoclinic point satisfying (5.4). Because of the smooth dependence of the normally hyperbolic manifold and of its stable and unstable manifolds on the perturbation, $\tilde{z}_\varepsilon$ is $O(\varepsilon)$-close to $\tilde{z}_0$ in the $\mathcal{C}^\ell$-topology, that is

$$\tilde{z}_\varepsilon = \tilde{z}_0 + O(\varepsilon). \tag{D.1}$$

Let $(\tilde{z}_\varepsilon, \varepsilon) \in \widetilde{M} \mapsto \mathbf{F}(\tilde{z}_\varepsilon, \varepsilon) \in \mathbb{R}^k$ be a uniformly $\mathcal{C}^1$-smooth mapping on $\widetilde{M} \times \mathbb{R}$.

We define the integral operators

$$\begin{aligned}
\mathfrak{I}^+(\mathbf{F}, \tilde{\Phi}^\tau_\varepsilon, \tilde{z}_\varepsilon) &= \int_0^{+\infty} \Big(\mathbf{F}(\tilde{\Phi}^\tau_\varepsilon(\tilde{z}^+_\varepsilon)) - \mathbf{F}(\tilde{\Phi}^\tau_\varepsilon(\tilde{z}_\varepsilon))\Big)\, d\tau, \\
\mathfrak{I}^-(\mathbf{F}, \tilde{\Phi}^\tau_\varepsilon, \tilde{z}_\varepsilon) &= \int_{-\infty}^0 \Big(\mathbf{F}(\tilde{\Phi}^\tau_\varepsilon(\tilde{z}^-_\varepsilon)) - \mathbf{F}(\tilde{\Phi}^\tau_\varepsilon(\tilde{z}_\varepsilon)\Big)\, d\tau.
\end{aligned} \tag{D.2}$$

Lemma D.1 (Master Lemma 1)**.** *The improper integrals* (D.2) *are convergent. The operators* $\mathfrak{I}^+(\mathbf{F}, \tilde{\Phi}^\tau_\varepsilon, z_\varepsilon)$ *and* $\mathfrak{I}^-(\mathbf{F}, \tilde{\Phi}^\tau_\varepsilon, z_\varepsilon)$ *are linear in* $\mathbf{F}$*.*

Lemma D.2 (Master Lemma 2)**.**

$$\begin{aligned} \mathbf{F}(\tilde{z}^+_\varepsilon) - \mathbf{F}(\tilde{z}_\varepsilon) =& -\mathfrak{I}^+((\mathcal{X}^0 + \varepsilon\mathcal{X}^1)\mathbf{F}, \tilde{\Phi}^\tau_\varepsilon, \tilde{z}_\varepsilon), \\ \mathbf{F}(\tilde{z}^-_\varepsilon) - \mathbf{F}(\tilde{z}_\varepsilon) =& \mathfrak{I}^-((\mathcal{X}^0 + \varepsilon\mathcal{X}^1)\mathbf{F}, \tilde{\Phi}^\tau_\varepsilon, \tilde{z}_\varepsilon). \end{aligned} \tag{D.3}$$

Lemma D.3 (Master Lemma 3)**.**

$$\begin{aligned} \mathfrak{I}^+(\mathbf{F}, \tilde{\Phi}^\tau_\varepsilon, \tilde{z}_\varepsilon) =& \mathfrak{I}^+(\mathbf{F}, \tilde{\Phi}^\tau_0, \tilde{z}_0) + O(\varepsilon^\varrho), \\ \mathfrak{I}^-(\mathbf{F}, \tilde{\Phi}^\tau_\varepsilon, \tilde{z}_\varepsilon) =& \mathfrak{I}^-(\mathbf{F}, \tilde{\Phi}^\tau_0, \tilde{z}_0) + O(\varepsilon^\varrho), \end{aligned} \tag{D.4}$$

for $0 < \varrho < 1$*. The integrals on the right-hand side are evaluated with* $\mathcal{X}^1 = \mathcal{X}^1(\cdot; 0)$*.*

Lemma D.4 (Master Lemma 4)**.** *If* $\boldsymbol{F} = O_{\mathcal{C}^1}(\varepsilon)$ *then*

$$\begin{aligned} \mathfrak{I}^+(\mathbf{F}, \tilde{\Phi}^\tau_\varepsilon, z_\varepsilon) =& \mathfrak{I}^+(\mathbf{F}, \tilde{\Phi}^\tau_0, z_0) + O(\varepsilon^{1+\varrho}), \\ \mathfrak{I}^-(\mathbf{F}, \tilde{\Phi}^\tau_\varepsilon, z_\varepsilon) =& \mathfrak{I}^-(\mathbf{F}, \tilde{\Phi}^\tau_0, z_0) + O(\varepsilon^{1+\varrho}), \end{aligned} \tag{D.5}$$

for $0 < \varrho < 1$*. The integrals on the right-hand side are evaluated with* $\mathcal{X}^1 = \mathcal{X}^1(\cdot; 0)$*.*

The proofs of the above lemmas can be found in [GdlLM21], and similar arguments can be found in [GdlL18].

References

[Bel04] Edward Belbruno. *Capture Dynamics and Chaotic Motions in Celestial Mechanics: With Applications to the Construction of Low Energy Transfers*. Princeton University Press, 2004.

[BGT10] Edward Belbruno, Marian Gidea, and Francesco Topputo. Weak stability boundary and invariant manifolds. *SIAM Journal on Applied Dynamical Systems*, 9(3):1061–1089, 2010.

[BLZ99] Peter W. Bates, Kening Lu, and Chongchun Zeng. Persistence of overflowing manifolds for semiflow. *Communications on Pure and Applied Mathematics*, 52(8):983–1046, 1999.

[BLZ08] Peter W. Bates, Kening Lu, and Chongchun Zeng. Approximately invariant manifolds and global dynamics of spike states. *Invent. Math.*, 174(2):355–433, 2008.

[BO06] Esther Barrabés and Merce Ollé. Invariant manifolds of L_3 and horseshoe motion in the restricted three-body problem. *Nonlinearity*, 19(9):2065, 2006.

[Bro68] RA Broucke. *Periodic orbits in the restricted three body problem with earth-moon masses*. Jet Propulsion Laboratory, California Institute of Technology, 1968.

[Cap12] Maciej J Capinski. Computer Assisted Existence Proofs of Lyapunov Orbits at L_2 and Transversal Intersections of Invariant Manifolds in the Jupiter–Sun PCR3BP. *SIAM Journal on Applied Dynamical Systems*, 11(4):1723–1753, 2012.

[CCdlL20] Renato Calleja, Alessandra Celletti, and Rafael de la Llave. KAM theory for some dissipative systems. *arXiv preprint arXiv:2007.08394*, 2020.

[CCP16] Marta Ceccaroni, Alessandra Celletti, and Giuseppe Pucacco. Halo orbits around the collinear points of the restricted three-body problem. *Physica D: Nonlinear Phenomena*, 317:28–42, 2016.

[CDMR06] E. Canalias, A. Delshams, J. J. Masdemont, and P. Roldán. The scattering map in the planar restricted three body problem. *Celestial Mech. Dynam. Astronom.*, 95(1-4):155–171, 2006.

[CGdlL16] Maciej J Capiński, Marian Gidea, and Rafael de la Llave. Arnold diffusion in the planar elliptic restricted three-body problem: mechanism and numerical verification. *Nonlinearity*, 30(1):329, 2016.

[CM06] E Canalias and Josep J Masdemont. Homoclinic and heteroclinic transfer trajectories between Lyapunov orbits in the Sun-Earth and Earth-Moon systems. *Discrete & Continuous Dynamical Systems-A*, 14(2):261, 2006.

[CR12] Maciej J. Capiński and Pablo Roldán. Existence of a center manifold in a practical domain around L_1 in the restricted three-body problem. *SIAM J. Appl. Dyn. Syst.*, 11(1):285–318, 2012.

[DdlLS06] Amadeu Delshams, Rafael de la Llave, and Tere M. Seara. A geometric mechanism for diffusion in Hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model. *Mem. Amer. Math. Soc.*, 179(844):viii+141, 2006.

[DdlLS08] Amadeu Delshams, Rafael de la Llave, and Tere M. Seara. Geometric properties of the scattering map of a normally hyperbolic invariant manifold. *Adv. Math.*, 217(3):1096–1153, 2008.

[DGR16] Amadeu Delshams, Marian Gidea, and Pablo Roldan. Arnold's mechanism of diffusion in the spatial circular restricted three-body problem: A semi-analytical argument. *Physica D: Nonlinear Phenomena*, 334:29–48, 2016.

[dlLK19] Rafael de la Llave and Florian Kogelbauer. Global persistence of Lyapunov subcenter manifolds as spectral submanifolds under dissipative perturbations. *SIAM Journal on Applied Dynamical Systems*, 18(4):2099–2142, 2019.

[Fen72] Neil Fenichel. Persistence and smoothness of invariant manifolds for flows. *Indiana Univ. Math. J.*, 21:193–226, 1971/1972.

[GdlL18] Marian Gidea and Rafael de la Llave. Global Melnikov theory in Hamiltonian systems with general time-dependent perturbations. *Journal of NonLinear Science*, 28:1657–1707, 2018.

[GdlLM21] Marian Gidea, Rafael de la Llave, and Maxwell Musser. Global effect of non-conservative perturbations on homoclinic orbits. *Qualitative Theory of Dynamical Systems*, 20(1):1–40, 2021.

[GEPC17] Ioannis Gkolias, Christos Efthymiopoulos, Giuseppe Pucacco, and Alessandra Celletti. Hamiltonian formulation of the spin-orbit model with time-varying non-conservative forces. *Communications in Nonlinear Science and Numerical Simulation*, 51:23–38, 2017.

[Gio01] Antonio Giorgilli. Unstable equilibria of Hamiltonian systems. *Discrete & Continuous Dynamical Systems-A*, 7(4):855, 2001.

[GJSM01a] G. Gómez, À. Jorba, C. Simó, and J. Masdemont. *Dynamics and mission design near libration points. Vol. III*, volume 4 of *World Scientific Monograph Series in Mathematics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2001. Advanced methods for collinear points.

[GJSM01b] G. Gómez, À. Jorba, C. Simó, and J. Masdemont. *Dynamics and mission design near libration points. Vol. IV*, volume 5 of *World Scientific Monograph Series in Mathematics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2001. Advanced methods for triangular points.

[GLMS01] G. Gómez, J. Llibre, R. Martínez, and C. Simó. *Dynamics and mission design near libration points. Vol. I*, volume 2 of *World Scientific Monograph Series in Mathematics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2001.

Fundamentals: the case of collinear libration points, With a foreword by Walter Flury.

[GSLM01] G. Gómez, C. Simó, J. Llibre, and R. Martínez. *Dynamics and mission design near libration points. Vol. II*, volume 3 of *World Scientific Monograph Series in Mathematics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2001. Fundamentals: the case of triangular libration points.

[HPS77] M.W. Hirsch, C.C. Pugh, and M. Shub. *Invariant manifolds*, volume 583 of *Lecture Notes in Math.* Springer-Verlag, Berlin, 1977.

[JN20] Angel Jorba and Begoña Nicolás. Transport and invariant manifolds near L_3 in the Earth-Moon Bicircular model. *Communications in Nonlinear Science and Numerical Simulation*, 89:105327, 2020.

[Jor99] Angel Jorba. A methodology for the numerical computation of normal forms, centre manifolds and first integrals of Hamiltonian systems. *Experimental Mathematics*, 8(2):155–195, 1999.

[KLMR00] Wang Sang Koon, Martin W Lo, Jerrold E Marsden, and Shane D Ross. Heteroclinic connections between periodic orbits and resonance transitions in celestial mechanics. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 10(2):427–469, 2000.

[LMS85] Jaume Llibre, Regina Martínez, and Carles Simó. Tranversality of the invariant manifolds associated to the Lyapunov family of periodic orbits near L_2 in the restricted three-body problem. *J. Differential Equations*, 58(1):104–156, 1985.

[MNF87] Andrea Milani, Anna Maria Nobili, and Paolo Farinella. *Non-gravitational perturbations and satellite geodesy*. CRC Press, 1987.

[Mos58] Jürgen Moser. On the generalization of a theorem of A. Liapounoff. *Communications on Pure and Applied Mathematics*, 11(2):257–271, 1958.

[PA14] Jeffrey S Parker and Rodney L Anderson. *Low-energy lunar trajectory design*, volume 12. John Wiley & Sons, 2014.

[Pes04] Yakov B. Pesin. *Lectures on partial hyperbolicity and stable ergodicity*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2004.

[WZ03] Daniel Wilczak and Piotr Zgliczynski. Heteroclinic connections between periodic orbits in planar restricted circular three-body problem– a computer assisted proof. *Communications in mathematical physics*, 234(1):37–75, 2003.

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