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The Wald–Zoupas prescription for asymptotic charges at null infinity in general relativity

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Abstract

We use the formalism developed by Wald and Zoupas to derive explicit covariant expressions for the charges and fluxes associated with the Bondi–Metzner–Sachs symmetries at null infinity in asymptotically flat spacetimes in vacuum general relativity. Our expressions hold in non-stationary regions of null infinity, are local and covariant, conformally-invariant, and are independent of the choice of foliation of null infinity and of the chosen extension of the symmetries away from null infinity. While similar expressions have appeared previously in the literature in Bondi–Sachs coordinates (to which we compare our own), such a choice of coordinates obscures these properties. Our covariant expressions can be used to obtain charge formulae in any choice of coordinates at null infinity. We also include detailed comparisons with other expressions for the charges and fluxes that have appeared in the literature: the Ashtekar–Streubel flux formula, the Komar formulae, and the linkage and twistor charge formulae. Such comparisons are easier to perform using our explicit expressions, instead of those which appear in the original work by Wald and Zoupas.

Keywords: asymptotic charges, general relativity, null infinity

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1. Introduction

There has been a considerable revival of interest in asymptotic symmetries and their associated charges in general relativity in recent years. At null infinity in asymptotically flat spacetimes, these symmetries are elements of an infinite-dimensional symmetry group called the Bondi–Metzner–Sachs (BMS) group [1, 2]. A major reason for this revival of interest has been the conjecture by Strominger that these symmetries give rise to an infinite number of conservation laws which constrain classical gravitational scattering in asymptotically flat spacetimes⁴. In a series of papers, conservation laws associated with these symmetries have also been shown to be related to soft graviton theorems [6–11] and gravitational memory effects [6, 7, 12–16] and have even been speculated to have a possible bearing on the black hole information paradox [17–20].

In this paper we consider the problem of deriving explicit general expressions for the charges associated with asymptotic symmetries at null infinity in vacuum general relativity. We list a set of criteria that these should satisfy to be physically reasonable and, using the prescription provided by Wald and Zoupas in [21], provide an explicit computation of the general expressions for these quantities. While such charge expressions have previously appeared in the literature [22], they have been obtained entirely in Bondi–Sachs coordinates. Using a fixed coordinate system from the outset makes the covariance and conformal-invariance of the resulting expressions difficult to verify. In this paper we obtain explicit expressions for these charges in terms of quantities defined on null infinity in a completely covariant manner. These covariant expressions can be used to obtain formulae in *any* choice of coordinates one wishes to choose (see section 6).

We will consider four-dimensional asymptotically flat spacetimes defined in a coordinate independent manner through Penrose’s conformal completion (see definition 2.1). In this picture, the null infinity of the original (physical) spacetime is represented by a smooth null surface $\mathcal{I}$ in the conformally-completed (unphysical) spacetime. This null surface has the topology of $\mathcal{I} \cong \mathbb{R} \times \mathbb{S}^2$ where $\mathbb{R}$ represents the null directions and $\mathbb{S}^2$ the angular directions at infinity. It then follows from the definition of asymptotic flatness that there is a certain *universal* structure associated with $\mathcal{I}$ (see section 3). This structure is universal in the sense that it is independent of which asymptotically flat physical spacetime is being considered and thus provides a ‘fixed background’ at infinity which is common to all physical spacetimes. The infinitesimal diffeomorphisms (i.e. vector fields) which preserve this universal structure form the asymptotic symmetry algebra given by the BMS algebra (section 4).

Then, one wishes to define certain ‘conserved quantities’ (similar to mass, energy or angular momentum) associated with any given physical spacetime and an asymptotic symmetry. In general dynamical spacetimes such quantities will not be conserved due to the presence of gravitational radiation. However, one can proceed as follows: let $S \cong \mathbb{S}^2$ be some cross-section (or ‘cut’) of null infinity $\mathcal{I}$; the choice of S represents an ‘instant of time’ on $\mathcal{I}$. Then, for a vector field ξ^a representing an asymptotic symmetry and to every cross-section S we define a *charge* $Q[\xi; S]$, which is to represent the ‘not really conserved quantity’ at that ‘instant of time’. The change in this charge with ‘time’, i.e. between two cross-sections, is then a *flux* $\mathcal{F}[\xi; \Delta\mathcal{I}]$ where $\Delta\mathcal{I}$ is the region of null infinity between those two cross-sections. However, without any further physical criteria one could make up arbitrary expressions for such a charge and flux formula. Even if one requires that any notion of charge associated with time translations and

⁴ This conjecture has been recently proven under certain regularity conditions on the spacetimes near spatial infinity, see [3–5] for details.

rotations coincide with the mass and angular momentum usually defined in Kerr spacetimes, it is not apparent how to generalize the charge expressions to all BMS symmetries and to non-stationary spacetimes.

Next we list some criteria which any physically reasonable notion of asymptotic charges and fluxes should satisfy: (1) any charge or flux quantities one defines must be independent of any choice of coordinates; this issue is already resolved since we work directly with the covariant formulation of null infinity mentioned above, instead of fixing some coordinates like Bondi–Sachs coordinates. (2) We want to associate the charges and fluxes with the physical spacetime and not with the choice of conformal-completion used to obtain the unphysical conformally-completed spacetime, or with any additional structure used to compute these quantities. Thus, we require that the charges and fluxes be independent of the conformal factor used to obtain the Penrose conformal-completion. Below we will also use a foliation of $\mathcal{I}$ to simplify our computations, and so we will also demand that the charges and fluxes be independent of this additional choice of foliation. (3) We also want to associate the charges and fluxes with the BMS symmetries at $\mathcal{I}$ and so, we demand that they be independent of any arbitrary extension of these asymptotic symmetries into the spacetime (these extensions are considered ‘pure gauge’ and are discussed in section 4.1). (4) The charges and fluxes are local and covariant in the following sense: the value of the charge $Q[\xi; S]$ is obtained as an integral over S of a two-form which is constructed from the available fields and the BMS symmetry, and finitely-many of their derivatives at S . The flux $\mathcal{F}[\xi; \Delta\mathcal{I}]$ is the integral over $\Delta\mathcal{I}$ of a three-form constructed in a similar way from the fields and symmetry in the region $\Delta\mathcal{I}$. (5) Finally, since we want the fluxes $\mathcal{F}[\xi; \Delta\mathcal{I}]$ to characterize physical dynamical processes, like gravitational radiation, we also require that the flux associated with *any* BMS symmetry between *any* two cross-sections vanishes when the physical spacetime is stationary.

A prescription for obtaining such asymptotic charges and fluxes in any local and covariant Lagrangian theory of gravity was given by Wald and Zoupas [21]. We will detail this procedure in section 5 for our case of interest, namely asymptotically flat spacetimes in vacuum general relativity. This case was also considered in detail by Wald and Zoupas [21], where it was shown, rather indirectly, that the resulting charge expression matches the one given by Dray and Streubel [23], which was obtained by generalizing Penrose’s formula motivated by twistor theory (see appendix C3). Explicit expressions for these charges in terms of quantities on null infinity were obtained by Flanagan and Nichols [22], but their analysis was done in Bondi coordinates (where the covariance and conformal-invariance of the expressions is not manifest) and also in stationary regions of null infinity⁵.

In this paper we compute the formulae for the charges and fluxes, using the Wald–Zoupas (WZ) prescription, written explicitly in terms of fields defined on null infinity in full generality. We call these the ‘WZ charge’ and ‘WZ flux’ respectively. To simplify our computations we will choose an arbitrary but fixed foliation of null infinity, but show that our formulae are independent of this choice of foliation. This computation can be done easily for the flux, since the WZ flux expression is written entirely in terms of quantities at null infinity; our expression for the WZ flux in terms of a choice of foliation is given in equation (5.33). The computation of the charges, however, is more complicated: unlike the flux, a portion of the expression for the WZ charge involves the limit of the integral of a quantity that is defined in a neighborhood of null infinity. To compute this limit, we make use of Bondi coordinates, much like [22]; however,

⁵The method used in [24] only gives a ‘perturbed charge’ expression which is not integrable in phase space; see equation (5.41) and the subsequent discussion. Charge expressions similar to those in [22] were also obtained in [25], but their charge expression differs from the one obtained using the Wald–Zoupas method, except in the case where the asymptotic symmetry is a pure supertranslation; see also footnote 11 of [22].

we show that the value of this limit is independent of the choices that are made in using Bondi coordinates, such as the choice of foliation, conformal factor, and extension of the BMS vector field off of null infinity. As such, we can convert the value of the limit back into a covariant form, allowing us to write an expression for the WZ charge (see equation (5.44)) in terms of quantities defined on null infinity. Since this procedure is somewhat involved, we then check explicitly that the change of the charge is consistent with the flux formula in equation (5.33). These covariant expressions are equal to those given by [21], and are therefore independent of foliation; however, picking a choice of foliation allows us to write the charge in terms of quantities defined entirely at null infinity. These explicit formulae can then be easily compared to the expressions one would obtain in *any* coordinate system of ones choosing, see e.g. section 6. These expressions can also be compared to other formulations of BMS charges, some of which we detail in appendix C.

The rest of this paper is organized as follows. We start by reviewing the definition of asymptotic-flatness at null infinity in section 2, and summarize the various asymptotic fields arising at null infinity that we will use in subsequent computations. In section 3, we discuss the universal structure at null infinity (that is, the structure common to all asymptotically-flat spacetimes at null infinity) and the restrictions on metric perturbations which follow from the definition of asymptotic flatness. In section 4, we review the BMS algebra and its relevant properties. In section 5, we describe the WZ prescription for obtaining the charges and fluxes associated with BMS symmetries and obtain manifestly covariant expressions for these quantities which we call the ‘WZ charge’ and the ‘WZ flux’. Finally, we discuss the construction of Bondi–Sachs and conformal Gaußian null coordinate frames near null infinity in section 6 and express the BMS symmetries and the WZ charge in these coordinates. We end with a short discussion of our results in section 7. While in the body of the paper, we work in a conformal frame where the Bondi condition (see equation (2.6)) holds, we include a summary of our main results in more general conformal frames in appendix A. In appendix B we rewrite the WZ charge and flux expressions in the Geroch–Held–Penrose (GHP) formalism at null infinity. In appendix C, we compare the WZ charge and flux formulae to some of the other charge and flux formulae that have been proposed for BMS symmetries, namely, the Ashtekar–Streubel flux formula, the Komar charge formulae and their linkage versions and Penrose’s twistor charge formula. Appendix D collects some useful results on symmetric tracefree tensors on a two-sphere.

Our notation and conventions are as follows. We follow the conventions of Wald [26] for the metric signature, Riemann tensor, and differential forms, and use abstract index notation with Latin indices $a, b, c, \dots$ for tensor fields. Quantities defined on the physical spacetime are denoted by a ‘hat’, while the ones on the conformally-completed, unphysical spacetime are denoted without a ‘hat’ (e.g. $\hat{g}_{ab}$ is the physical metric while g_{ab} is the unphysical metric). In addition, we use ‘ $\hat{=}$ ’ to denote equality at null infinity and $\leftarrow$ to denote the pullback to null infinity. We also use both the indexed and index-free notations for differential forms, and in the index-free notation we denote differential forms in bold. We use ‘ $\equiv$ ’ to translate between indexed and index-free notation, writing, e.g. $\epsilon_4 \equiv \epsilon_{abcd}$. Since ‘ $\equiv$ ’ is used for this translation, we use ‘ $:=$ ’ for definitions.

2. Asymptotic-flatness at null infinity

In this section, we recall the covariant definition of asymptotically-flat spacetimes and define the asymptotic fields and their equations at null infinity that will appear in our later analysis.

Definition 2.1 (Asymptotic flatness). A *physical* spacetime $(\hat{M}, \hat{g}_{ab})$, which satisfies the vacuum Einstein equation $\hat{G}_{ab} = 0$, is asymptotically-flat at null infinity if there exists an

unphysical spacetime (M, g_{ab}) with a boundary $\mathcal{S} = \partial M$ and an embedding of $\hat{M}$ into M (we use this embedding to identify $\hat{M}$ as a submanifold of M), such that

- (a) There exists a smooth function Ω (the *conformal factor*) on M satisfying $\Omega \hat{=} 0$ and $\nabla_a \Omega \neq 0$ such that $g_{ab} = \Omega^2 \hat{g}_{ab}$ is smooth on M including at $\mathcal{S}$.
- (b) $\mathcal{S}$ is topologically $\mathbb{R} \times \mathbb{S}^2$.
- (c) Defining $n_a := \nabla_a \Omega$, the vector field $\omega^{-1} n^a$ is complete on $\mathcal{S}$ for any smooth function ω on M such that $\omega > 0$ on M and $\nabla_a(\omega^4 n^a) \hat{=} 0$.

Detailed expositions on the motivations for this definition may be found in [26, 27]. The differentiability conditions on the unphysical spacetime can be significantly weakened, but we restrict to the smooth case for simplicity.

Using the conformal transformation relating the unphysical Ricci tensor R_{ab} to the physical Ricci tensor $\hat{R}_{ab}$ (see, e.g. appendix D of [26]), the vacuum Einstein equation can be written as

$$S_{ab} = -2\Omega^{-1}\nabla_{(a}n_{b)} + \Omega^{-2}n^c n_c g_{ab}, \quad (2.1)$$

where S_{ab} is given by

$$S_{ab} = R_{ab} - \frac{1}{6}Rg_{ab}. \quad (2.2)$$

It follows from equation (2.1) and the smoothness of Ω and the unphysical metric g_{ab} at $\mathcal{S}$ that $n_a n^a \hat{=} 0$. This implies that $\mathcal{S}$ is a smooth null hypersurface in M with normal $n_a = \nabla_a \Omega$ and that the vector field $n^a = g^{ab} n_b$ is a null geodesic generator of $\mathcal{S}$.

Further, the Bianchi identity $\nabla_{[a} R_{bc]de} = 0$ on the unphysical Riemann tensor along with equation (2.1) gives the following equations for the unphysical Weyl tensor C_{abcd} (see [27] for details)

$$\nabla_{[a}(\Omega^{-1} C_{bc]de}) = 0, \quad (2.3a)$$

$$\nabla^d C_{abcd} = -\nabla_{[a} S_{b]c}. \quad (2.3b)$$

Given a physical spacetime $(\hat{M}, \hat{g}_{ab})$ there is a lot of freedom in constructing its conformal-completion to obtain the unphysical spacetime (M, g_{ab}) . Firstly, there is the choice of the embedding of $\hat{M}$ into M , but since this is simply used to identify points of $\hat{M}$ with those of M this freedom is innocuous and one can always fix a choice of this embedding map. Another freedom is the choice of the conformal factor Ω . Note that for the same physical spacetime another choice of the conformal factor $\Omega' = \omega\Omega$, with $\omega|_{\mathcal{S}} > 0$ also satisfies definition 2.1 with a different unphysical metric $g'_{ab} = \Omega'^2 \hat{g}_{ab} = \omega^2 g_{ab}$. Since we are interested in studying the asymptotic properties of the physical spacetime—the conformal-completion is used only to bring the asymptotic boundary ‘at infinity’ of the physical spacetime to a finite boundary $\mathcal{S}$ in the unphysical spacetime—all physical quantities (such as symmetries, charges and fluxes) must be independent of the choice of conformal factor made.

All of our computations which follow can be done using an arbitrary conformal factor; however it is convenient to fix some of the conformal freedom by imposing the *Bondi condition* as follows. On $\mathcal{S}$, let Φ be defined by

$$\Phi := \frac{1}{4} \nabla_a n^a \Big|_{\mathcal{S}}. \quad (2.4)$$

Under a change of conformal factor $\Omega \mapsto \omega\Omega$ and $g_{ab} \mapsto \omega^2 g_{ab}$ we have

$$\Phi \mapsto \omega^{-1}(\Phi + \mathcal{L}_n \ln \omega). \quad (2.5)$$

Now, without loss of generality, we can choose ω to be a solution of $\Phi + \mathcal{L}_n \ln \omega \hat{=} 0$ to set $\Phi \hat{=} 0$ [26, 27].⁶ In this choice of the conformal factor, equation (2.1) implies the *Bondi condition*

$$\nabla_a n_b \hat{=} 0, \quad (2.6)$$

and

$$n_a n^a = O(\Omega^2). \quad (2.7)$$

We will henceforth work in a conformal frame where the Bondi condition holds. Having imposed the Bondi condition, the remaining freedom in the conformal factor is of the form $\Omega \mapsto \omega\Omega$ where

$$\omega|_{\mathcal{S}} > 0, \quad \mathcal{L}_n \omega \hat{=} 0. \quad (2.8)$$

We reiterate that the choice to work in a conformal frame where the Bondi condition holds is made purely for convenience and is not essential for the calculations in this paper. The main results of this paper, expressed in general conformal frames where the Bondi condition is not imposed, are included in appendix A. In the body of the paper, our statements about conformal invariance will pertain to conformal transformations that satisfy equation (2.8). More general conformal transformations for which $\mathcal{L}_n \omega \neq 0$ will only be considered in appendices A and B.

Finally, let q_{ab} denote the pullback of g_{ab} to $\mathcal{S}$. This defines a degenerate metric on $\mathcal{S}$ such that

$$q_{ab} n^b \hat{=} 0, \quad \mathcal{L}_n q_{ab} \hat{=} 0, \quad (2.9)$$

where the second condition follows from equation (2.6). Thus, q_{ab} defines a Riemannian metric on the space of generators of $\mathcal{S}$ which is diffeomorphic to $\mathbb{S}^2$.

2.1. Auxiliary foliation and null normal at $\mathcal{S}$

For carrying out explicit computations on $\mathcal{S}$, it is convenient to pick an additional ‘auxiliary’ structure on $\mathcal{S}$ given by a choice of foliation of $\mathcal{S}$ by a one-parameter family of *cross-sections* which are diffeomorphic to $\mathbb{S}^2$. Note that such a foliation always exists since $\mathcal{S} \cong \mathbb{R} \times \mathbb{S}^2$, though there is no unique choice for this foliation. The results in this paper can be obtained without reference to any choice of foliation, but it is far simpler to use some choice of foliation and then verify that the results are independent of the choice of foliation made.

For any choice of foliation, we obtain a one-form l_a on $\mathcal{S}$ which is normal to each cross-section of the foliation. Then there exists a *unique* vector field l^a defined at $\mathcal{S}$ such that $l_a \hat{=} g_{ab} l^b$ is the normal to the chosen foliation and

$$l^a l_a \hat{=} 0, \quad l^a n_a \hat{=} -1. \quad (2.10)$$

We will call such a vector field associated with the chosen foliation the *auxiliary normal*. Note that the vector field l^a can be extended arbitrarily away from $\mathcal{S}$, and our computations will be independent of which extension is chosen.

⁶ Note that while this condition can always be imposed on $\mathcal{S}$, it cannot be imposed at spatial infinity where solutions, ω , to the equation $\Phi + \mathcal{L}_n \ln \omega \hat{=} 0$ diverge [26].

Using the auxiliary null normal l^a , we now define a tensor Q_{ab} at $\mathcal{S}$ by

$$Q_{ab} := g_{ab} + 2l_{(a}n_{b)}. \quad (2.11)$$

Note that $q_{ab} = Q_{ab}$, and so Q_{ab} is a choice of pushforward of the intrinsic (degenerate) metric on $\mathcal{S}$. By the definition of Q_{ab} , we have

$$Q_{ab}l^b \cong 0, \quad Q_{ab}n^b \cong 0. \quad (2.12)$$

We also define the symmetric trace-free part of a tensor with respect to Q_{ab} by:

$$\text{STF} A_{ab} := \left(Q_{(a}{}^c Q_{b)}{}^d - \frac{1}{2} Q_{ab} Q^{cd} \right) A_{cd}. \quad (2.13)$$

Next, we define a volume form on $\mathcal{S}$ and on the cross-sections of the chosen foliation by

$$\varepsilon_{abc} := l^d \varepsilon_{dabc}, \quad \varepsilon_{ab} := \varepsilon_{abcd} l^c n^d \cong \varepsilon_{abc} n^c. \quad (2.14)$$

Remark 2.1 (Orientation conventions). The orientation of these volume forms are such that $\mathcal{S}$ is outward-facing as a manifold in M , and such that the cross-sections have a future orientation within $\mathcal{S}$. Note that this convention for the orientations for ε_2 and ε_3 is somewhat unexpected. In Minkowski spacetime, setting $l_a = -\nabla_a u$ and $n_a = \nabla_a \Omega$ (with $\Omega = 1/r$ and $u = t - r$), we have that $\varepsilon_3 = -\sin \theta du \wedge d\theta \wedge d\phi$ and $\varepsilon_2 = -\sin \theta d\theta \wedge d\phi$, which is the opposite of the ‘expected’ sign for ε_2 . Despite this sign difference, we still have that

$$\int_S \varepsilon_2 > 0; \quad (2.15)$$

this, in fact, is the *definition* of an orientation. In Minkowski, this can be seen by noting that this means that the bounds of the integrals are in the opposite order (say, the θ integral is from π to 0, instead of 0 to π).

Finally, note that, in our choice of orientation, if we consider a cross-section S as a limit of spheres S' within a spacelike hypersurface Σ in the unphysical spacetime, then the orientation for S' that is compatible with that of S is the *inward*-facing one within Σ .

The *shear* of the auxiliary normal on $\mathcal{S}$ is defined by

$$\sigma_{ab} \cong \text{STF} \nabla_a l_b \cong \frac{1}{2} \text{STF} \mathcal{L}_l g_{ab}. \quad (2.16)$$

The twist $\varepsilon^{ab} \nabla_a l_b$ vanishes on $\mathcal{S}$ since l_a is normal to cross-sections of a foliation of $\mathcal{S}$. We will mostly not need the expansion of l_a in our analysis (it is introduced in section 6.2 where needed).

The change in l_a along the null generators of $\mathcal{S}$ is given by

$$\tau_a \cong Q_a{}^b n^c \nabla_c l_b. \quad (2.17)$$

By the normalization conditions for l^a and n^a and the Bondi condition we have

$$\tau_a \cong n^b \nabla_b l_a \cong \mathcal{L}_n l_a. \quad (2.18)$$

We also can define a ‘sphere derivative’ along the cross-sections for any tensor field that is orthogonal to l^a and n^a on all indices (at $\mathcal{S}$):

$$\mathcal{D}_a T^{b_1 \dots b_r}{}_{c_1 \dots c_s} := Q^d{}_a Q^{b_1}{}_{e_1} \dots Q^{b_r}{}_{e_r} Q^{f_1}{}_{c_1} \dots Q^{f_s}{}_{c_s} \nabla_d T^{e_1 \dots e_r}{}_{f_1 \dots f_s}. \quad (2.19)$$

One can easily show that this derivative operator is compatible with both Q_{ab} and ε_{ab} at $\mathcal{S}$:

$$\mathcal{D}_a Q_{bc} \hat{=} 0, \quad \mathcal{D}_a \varepsilon_{bc} \hat{=} 0. \quad (2.20)$$

With our choice of ε_3 and ε_2 , we can derive several useful formulas from Stokes' theorem. Consider some portion $\Delta\mathcal{S}$ of $\mathcal{S}$ whose boundary is given by two cross-sections S_1 and S_2 in our foliation, with S_2 to the future of S_1 . Then, for any scalar α we have

$$\int_{\Delta\mathcal{S}} \varepsilon_3 \mathcal{L}_n \alpha = \int_{S_2} \varepsilon_2 \alpha - \int_{S_1} \varepsilon_2 \alpha. \quad (2.21)$$

Also, for any vector v^a which is orthogonal to both n_a and l_a we have

$$\int_{\Delta\mathcal{S}} \varepsilon_3 \nabla_a v^a \hat{=} \int_{\Delta\mathcal{S}} \varepsilon_3 (\mathcal{D}_a + \tau_a) v^a \hat{=} 0, \quad (2.22)$$

where the second expression is obtained using the Bondi condition along with equations (2.11) and (2.18), and the vanishing of the expression follows from the fact that $\varepsilon_3 \nabla_a v^a$ is an exact three-form and $l_a v^a \hat{=} 0$. Similarly, for any cross-section S of $\mathcal{S}$ we also have

$$\int_S \varepsilon_2 \mathcal{D}_a v^a \hat{=} 0. \quad (2.23)$$

We note that equation (2.22) was erroneously given in equation (2.13) of [4] as an integration-by-parts identity on the sphere, instead of $\Delta\mathcal{S}$ (all instances when equation (2.22) was used in [4], however, were correct, as they were applied to $\Delta\mathcal{S}$).

The final quantities that we will need are components of the Weyl tensor at $\mathcal{S}$. By the peeling theorem, we have $C_{abcd} \hat{=} 0$ at $\mathcal{S}$, and thus $\Omega^{-1} C_{abcd}$ admits a limit to $\mathcal{S}$ (see theorem 11 of [27]). In the choice of a foliation of $\mathcal{S}$ we define the Weyl tensor fields

$$\mathcal{R}_{ab} := (\Omega^{-1} C_{cdef}) Q_a^c n^d Q_b^e n^f, \quad \mathcal{S}_a := (\Omega^{-1} C_{cdef}) l^c n^d Q_a^e n^f, \quad (2.24a)$$

$$\mathcal{P} := (\Omega^{-1} C_{cdef}) l^c n^d l^e n^f, \quad \mathcal{P}^* := \frac{1}{2} (\Omega^{-1} C_{cdef}) l^c n^d \varepsilon^{ef}, \quad (2.24b)$$

$$\mathcal{I}_a := (\Omega^{-1} C_{cdef}) n^c l^d Q_a^e l^f, \quad \mathcal{I}_{ab} := (\Omega^{-1} C_{cdef}) Q_a^c l^d Q_b^e l^f. \quad (2.24c)$$

Note that, due to the symmetries of the Weyl tensor, $\mathcal{R}_{ab}$ and $\mathcal{I}_{ab}$ are symmetric and traceless. The relation of these tensors to the Weyl scalar components in the Newman–Penrose notation is given in equation (B.12).

For the fields defined in equation (2.24), equation (2.3a) implies the following evolution equations along $\mathcal{S}$, which can be verified to be conformally-invariant (equation (B.13) in the GHP formalism):

$$\mathcal{L}_n \mathcal{S}_a = (\mathcal{D}^b + \tau^b) \mathcal{R}_{ab}, \quad (2.25a)$$

$$\mathcal{L}_n \mathcal{P} = (\mathcal{D}^a + 2\tau^a) \mathcal{S}_a - \sigma^{ab} \mathcal{R}_{ab}, \quad (2.25b)$$

$$\mathcal{L}_n \mathcal{P}^* = -\varepsilon^{ab} (\mathcal{D}_a + 2\tau_a) \mathcal{S}_b + \varepsilon_b^c \sigma^{ab} \mathcal{R}_{ac}, \quad (2.25c)$$

$$\mathcal{L}_n \mathcal{I}_a = \frac{1}{2} (\mathcal{D}_b + 3\tau_b) (Q_a^b \mathcal{P} - \varepsilon_a^b \mathcal{P}^*) - 2\sigma_a^b \mathcal{S}_b, \quad (2.25d)$$

$$\mathcal{L}_n \mathcal{I}_{ab} = \text{STF}(\mathcal{D}_a + 4\tau_a)\mathcal{J}_b - \frac{3}{2}\sigma_{ac}(Q_b{}^c\mathcal{P} - \varepsilon_b{}^c\mathcal{P}^*). \quad (2.25e)$$

Next, we list the conformal transformation weights of these various quantities. Any quantity α is said to have *conformal weight* w if, under $\Omega \mapsto \omega\Omega$, it transforms as $\alpha \mapsto \omega^w\alpha$. First, we clearly have that

$$g_{ab} : w = 2, \quad \varepsilon_{abcd} : w = 4. \quad (2.26)$$

Next, while in the spacetime

$$n_a \mapsto \omega n_a + \Omega \nabla_a \omega, \quad (2.27)$$

we therefore have that $n_a : w = 1$ on $\mathcal{S}$ itself, and so

$$l_a : w = 1, \quad (Q_{ab}, \varepsilon_{ab}) : w = 2, \quad \varepsilon_{abc} : w = 3. \quad (2.28)$$

Moreover, we have that

$$\sigma_{ab} : w = 1, \quad (2.29)$$

while, under a conformal transformation, τ_a transforms as

$$\tau_a \mapsto \tau_a + \mathcal{D}_a \ln \omega. \quad (2.30)$$

Finally, the fields constructed from the Weyl tensor have the following conformal weights:

$$(\mathcal{R}_{ab}, \mathcal{I}_{ab}) : w = -1, \quad (\mathcal{S}_a, \mathcal{J}_a) : w = -2, \quad (\mathcal{P}, \mathcal{P}^*) : w = -3. \quad (2.31)$$

Finally, we remark that there is a particularly convenient choice of conformal factor and foliation, namely one where q_{ab} is given by the unit two-sphere metric, and $\tau^a \hat{=} 0$. This is the *Bondi frame*, and in particular it is a restriction of the remaining conformal freedom $\Omega \mapsto \omega\Omega$ that is more restrictive than the Bondi condition. Even though Bondi frame can always be achieved by a conformal transformation and a change of foliation, we do not enforce Bondi frame, since we want to consider charges on arbitrary cross-sections of $\mathcal{S}$ (and fluxes between these cross-sections) and want our expressions to be manifestly conformally invariant (apart from the choice of Bondi condition).

2.2. News tensor

In this section, we define the *News tensor*, which characterizes the radiative degrees of freedom of the gravitational field at null infinity. This tensor can be defined in terms of a foliation, or invariantly from the universal structure that exists on $\mathcal{S}$; we review both definitions and show that they are equal. Moreover, we review how the News tensor vanishes in asymptotically stationary regions of $\mathcal{S}$, which motivates its usage as a characterization of radiation.

We define the News tensor as the (projected) Lie derivative along n^a of the shear σ_{ab} :

$$N_{ab} := 2Q_a{}^c Q_b{}^d \mathcal{L}_n \sigma_{cd} = 2 \text{STF } \mathcal{L}_n \sigma_{ab}. \quad (2.32)$$

Note that it follows from the above definition that

$$N_{ab} n^b \hat{=} g^{ab} N_{ab} \hat{=} Q^{ab} N_{ab} \hat{=} 0. \quad (2.33)$$

It is straightforward to verify that the News is conformally invariant.

We can also write the News N_{ab} in terms of S_{ab} as follows. Consider the quantity $\text{STF } \mathcal{L}_n \mathcal{L}_l g_{ab}$. Despite involving covariant derivatives of l_a , when evaluated at $\mathcal{S}$, this quantity is independent of the choice of l_a away from $\mathcal{S}$. By choosing a particular extension of l_a , one can show from equations (2.16) and (2.18) that

$$\text{STF } \mathcal{L}_n \mathcal{L}_l g_{ab} \hat{=} 2 \text{STF}(\mathcal{L}_n \sigma_{ab} - \tau_a \tau_b) \hat{=} N_{ab} - 2 \text{STF}(\tau_a \tau_b), \quad (2.34)$$

where the second equality follows from equation (2.32). Using $[\mathcal{L}_n, \mathcal{L}_l] = \mathcal{L}_{[n, l]}$, together with equation (2.18), we therefore find that

$$N_{ab} \hat{=} \text{STF}[\mathcal{L}_l \mathcal{L}_n g_{ab} + 2(\mathcal{D}_a + \tau_a)\tau_b]. \quad (2.35)$$

Using equation (2.1), we therefore find that

$$N_{ab} \hat{=} \text{STF}[S_{ab} + 2(\mathcal{D}_a + \tau_a)\tau_b]. \quad (2.36)$$

Further, from equations (2.3b) and (2.36), the News is related to the Weyl tensor components (defined in equation (2.24)) by:

$$Q_a{}^c Q_b{}^d \mathcal{L}_n N_{cd} \hat{=} 2\mathcal{R}_{ab}, \quad \mathcal{D}^b N_{ab} \hat{=} 2\mathcal{S}_a, \quad (2.37)$$

and

$$\mathcal{P}^* = \varepsilon^{ab} \left[\mathcal{D}_a (\mathcal{D}_c - \tau_c) \sigma_b{}^c - \frac{1}{2} N_{ac} \sigma_b{}^c \right]. \quad (2.38)$$

In the GHP notation these relations can be found in equations (B.16) and (B.18).

As defined by equation (2.32), it is not obvious that the News tensor is independent of our choice of foliation. However, we show below that this definition of the News coincides with the covariant definition given by Geroch [27] which makes no reference to any foliation of $\mathcal{S}$. While most of the literature exclusively uses only one of these two definitions, we will find it convenient to use either interchangeably, and (since there does not appear to be any proof that we could find), we will now show that these two definitions yield the same tensor.

Consider the *Geroch News tensor* defined in [27] as

$$N_{ab} := \underset{\leftarrow}{S}_{ab} - \rho_{ab}, \quad (2.39)$$

where $\underset{\leftarrow}{S}_{ab}$ denotes the pullback of S_{ab} to $\mathcal{S}$ and ρ_{ab} is the *unique* symmetric tensor field on $\mathcal{S}$ constructed from the intrinsic universal structure on $\mathcal{S}$ defined in theorem 5 of [27].

First note that the Geroch News tensor is conformally invariant, and satisfies the conditions equation (2.33) (see [27]). Moreover, projections of equation (68) of [27] show that equation (2.37) also holds for the Geroch News tensor. Thus, if λ_{ab} is the difference of the Geroch News tensor and the one defined in equation (2.32), then λ_{ab} is a symmetric tensor field on $\mathcal{S}$ which satisfies

$$\lambda_{ab} n^b \hat{=} Q^{ab} \lambda_{ab} \hat{=} Q_a{}^c Q_b{}^d \mathcal{L}_n \lambda_{cd} \hat{=} \mathcal{D}^b \lambda_{ab} \hat{=} 0. \quad (2.40)$$

Thus, λ_{ab} is a tensor field on S that is symmetric, traceless and divergence-free, and therefore vanishes by proposition D.2. Consequently, the News tensor defined in equation (2.32) is equivalent to the covariant definition by Geroch.

Finally, we review the key property of the News tensor, namely that it characterizes the presence of gravitational radiation in a spacetime. This can be seen by the following result,

which is due to Geroch (see pp 53–54 of [27]): consider an asymptotically flat spacetime $(\hat{M}, \hat{g}_{ab})$, and some portion $\Delta\mathcal{I}$ of null infinity. If $\Delta\mathcal{I}$ is asymptotically stationary, in the sense that there exists a vector field t^a in a neighborhood of $\Delta\mathcal{I}$ that is a timelike Killing vector with respect to $\hat{g}_{ab}$, then $N_{ab} \hat{=} 0$ on $\Delta\mathcal{I}$. Since any notion of gravitational radiation should vanish in asymptotically stationary regions of null infinity, this motivates the News tensor as indicating the presence of radiation. It should be noted, however, that it is not known whether the converse of this statement is true, namely that all regions $\Delta\mathcal{I}$ where the News tensor vanishes are asymptotically stationary.

3. Universal structure and metric perturbations

In this section, we summarize the *universal structure* at null infinity. This is the structure that is common to the conformal completion of *all* spacetimes that satisfy the definition of asymptotic flatness given in definition 2.1 and is thus independent of the specific physical spacetime under consideration.

If (M, g_{ab}, Ω) and (M', g'_{ab}, Ω') are the unphysical spacetimes corresponding to *any* two asymptotically-flat physical spacetimes then, *a priori*, M and M' are distinct manifolds each with their own boundary $\mathcal{I}$ and $\mathcal{I}'$. However, we argue below that there exists a smooth diffeomorphism from a neighbourhood of $\mathcal{I}$ in M to a neighbourhood of $\mathcal{I}'$ in M' which can be used to identify these unphysical spacetimes (in this neighbourhood) and which maps $\mathcal{I}$ to $\mathcal{I}'$. Since we are only interested in the asymptotic properties near null infinity, we can work with just one manifold M and one null boundary $\mathcal{I}$ to represent null infinity for any two (and thus, all) asymptotically-flat spacetimes. Further, *without any loss of generality*, this diffeomorphism can also be chosen so that $\Omega' = \Omega$ in a neighborhood of $\mathcal{I}$ and $g'_{ab} \hat{=} g_{ab}$.

This can be achieved by setting up a suitable, geometrically-defined coordinate system in a neighborhood of $\mathcal{I}$ and identifying the two unphysical spacetimes in these coordinates, as we now explain (see also the argument on p 22 in [27]). On the null infinity $\mathcal{I}$ of any asymptotically-flat spacetime, we define a parameter u along the null generators of $\mathcal{I}$ such that $n^a \nabla_a u \hat{=} 1$. We then pick some cross-section $S_0 \cong \mathbb{S}^2$ with constant $u = u_0$. On S_0 define a coordinate system x^A (with $A = 1, 2$), and extend these coordinates to all of $\mathcal{I}$ by parallel transport along n^a :

$$n^a \nabla_a x^A \hat{=} 0. \quad (3.1)$$

This gives us a coordinate system (u, x^A) on $\mathcal{I}$. Next, since $\Omega \hat{=} 0$ and $n_a \hat{=} \nabla_a \Omega \not\hat{=} 0$, we can use Ω as a coordinate transverse to $\mathcal{I}$. As discussed above, there is considerable freedom in the choice of the conformal factor at $\mathcal{I}$ which we need to fix to specify the choice of coordinate used. First, as before, we pick the conformal factor so that the Bondi condition (equation (2.6)) is satisfied, which leaves us the freedom to change the conformal factor on the cross-sections of $\mathcal{I}$. To fix this freedom we proceed as follows. Consider the induced metric q_{ab} on the cross-section S_0 chosen above. It follows from the *uniformization theorem* (for instance see chapter 8 of [28])⁷ that any metric on S_0 is conformal to the unit round metric on $\mathbb{S}^2$ (that is, the metric with constant Ricci scalar equaling 2). Thus, we can always choose the conformal factor so that the metric q_{ab} on this cross-section S_0 is also the unit round metric of $\mathbb{S}^2$ and from equation (2.9), this holds on any cross-section. Thus, we choose as our

⁷ The uniformization theorem is a *global* result depending on the topology of the two-dimensional space. Locally, all metrics of a particular signature on a two-surface are conformally-equivalent, problem 2, chapter 3 of [26].

transverse coordinate the choice of conformal factor Ω which satisfies the Bondi condition and makes the metric on the cross-sections of $\mathcal{I}$ to be the unit round metric on $\mathbb{S}^2$. This gives us a coordinate system (Ω, u, x^A) at $\mathcal{I}$ for *any* asymptotically-flat spacetime. Since this construction can be done for any asymptotically-flat spacetime, we can, without any loss of generality, identify the null infinities of all asymptotically-flat spacetimes by identifying their points in these coordinates. This shows that there exists a diffeomorphism between neighbourhoods of null infinities of different spacetimes such that we can identify their boundaries with one ‘abstract’ manifold $\mathcal{I}$ and also that we can choose the same conformal factor Ω in this neighbourhood.

Next, we show that the unphysical metric at $\mathcal{I}$ can be chosen to be the same for the conformal completion of any physical spacetime. Consider a foliation of $\mathcal{I}$ by cross-sections S_u of constant u . Then, by equation (3.1) the null generator n^a , in these coordinates, can be written as $n^a \hat{=} \partial/\partial u$. The one-form on $\mathcal{I}$ normal to this foliation is given by $l_a \hat{=} -\nabla_a u$. Using $n_a = \nabla_a \Omega$, $l_a \hat{=} -\nabla_a u$ and $n^a l_a \hat{=} -1$, we obtain the following expression for the line element of the unphysical metric on $\mathcal{I}$ in the coordinate system (Ω, u, x^A)

$$ds^2 \hat{=} 2 d\Omega du + s_{AB} dx^A dx^B, \quad (3.2)$$

where s_{AB} is the unit round metric on $\mathbb{S}^2$ in the chosen coordinates x^A .⁸ Note that the form of the unphysical metric equation (3.2) is completely independent of which physical spacetime is under consideration, that is, *all* asymptotically-flat spacetimes have the same universal unphysical metric at null infinity. Different choices of the physical spacetime are only reflected in the unphysical metric *away* from $\mathcal{I}$. Note that the definition of asymptotic-flatness (definition 2.1) includes an embedding map from the physical spacetime manifold $\hat{M}$ into the unphysical manifold M . The existence of the universal structure at $\mathcal{I}$ described above implies that one can embed *any* physical spacetime into an unphysical spacetime by identifying some physical spacetime coordinates with the coordinates (Ω, u, x^A) constructed above so that the unphysical metric g_{ab} takes the form equation (3.2). In the conformal-completion formalism this is the version of the statement that ‘asymptotically-flat spacetimes behave like the Minkowski spacetime to leading order at infinity’ and the difference between two physical spacetimes only shows up at ‘sub-leading order’.

Note that since the manifold $\mathcal{I}$ is universal, the choice of the foliation can also be made independently of the physical spacetime. It follows that the auxiliary normal $l_a \hat{=} -\nabla_a u$ is also universal, and (from equation (3.2)) we also have $l^a \hat{=} -\partial/\partial \Omega$ which is the auxiliary null normal and is also universal.

Many of the choices made in constructing the coordinates at $\mathcal{I}$ are irrelevant to this argument and are made just for convenience. For instance, the choice of the unit-metric s_{AB} is irrelevant. In *any* asymptotically-flat spacetime, we can instead choose the freedom in the conformal factor at $\mathcal{I}$ so that the induced metric on the cross-sections $q_{ab} = q_{ab}^{(0)}$ where $q_{ab}^{(0)}$ is *any fixed* metric on $\mathbb{S}^2$. Similarly, one could have chosen a different foliation of $\mathcal{I}$ if one wishes. These choices simply correspond to the freedom of choosing the embedding map from the physical spacetime into the unphysical spacetime. Then the rest of the construction proceeds as before and $g_{ab}|_{\mathcal{I}}$ is universal.

⁸ The precise choice of coordinates x^A on the cross-sections is irrelevant; one could pick polar coordinates $x^A = (\theta, \phi)$ to put the unit round sphere metric in the standard form $s_{AB} dx^A dx^B = d\theta^2 + \sin^2 \theta d\phi^2$, but any other coordinate system is just as good.

3.1. Metric perturbations near $\mathcal{I}$

Next we consider linearized perturbations of the physical metric and derive the conditions on the perturbations arising from requiring asymptotic flatness. We consider a one-parameter family of physical metrics $\hat{g}_{ab}(\lambda)$ with $\hat{g}_{ab} = \hat{g}_{ab}(\lambda = 0)$ being any chosen background space-time metric, and define the physical metric perturbation by

$$\delta \hat{g}_{ab} := \left. \frac{d}{d\lambda} \hat{g}_{ab}(\lambda) \right|_{\lambda=0}. \quad (3.3)$$

We also use the notation δ to denote perturbations of other quantities defined in a similar way as above. We emphasize that the δ denotes changes of quantities when the physical metric is varied; the quantities appearing in the universal structure described above including the unphysical metric at $\mathcal{I}$, the conformal factor and the choice of foliation do not vary under δ . Note, this does not mean that the conformal factor and foliation cannot be changed—our expressions for the charges and fluxes are independent of the conformal factor and foliation—it means that these quantities can always be held fixed when the physical metric is varied.

Taking the one-parameter family of physical metrics $\hat{g}_{ab}(\lambda)$ to all be asymptotically-flat, we want to consider the behaviour of the metric perturbations at null infinity. Let $g_{ab}(\lambda)$ be the one-parameter family of unphysical metrics obtained by the conformal completion of $\hat{g}_{ab}(\lambda)$. As discussed above, without loss of generality we can take all the unphysical metrics $g_{ab}(\lambda)$ to be defined on the same manifold M , with a common boundary $\mathcal{I}$ describing null infinity. Further, the conformal factor Ω can also be chosen to be independent of the parameter λ . Thus, we get

$$g_{ab}(\lambda) := \Omega^2 \hat{g}_{ab}(\lambda), \quad \delta g_{ab} = \Omega^2 \delta \hat{g}_{ab}, \quad (3.4)$$

where δg_{ab} is the perturbation of the unphysical metric. Moreover, since the unphysical metric at $\mathcal{I}$ is universal, we have that $\delta g_{ab} \hat{=} 0$ and thus

$$\delta g_{ab} \hat{=} \Omega \gamma_{ab}, \quad (3.5)$$

for some γ_{ab} which is smooth at $\mathcal{I}$.

Since the conformal factor is chosen to satisfy the Bondi condition equation (2.6) in any spacetime, varying the Bondi condition we get

$$\delta(\nabla_a n_b) \hat{=} 0 \implies \gamma_{ab} n^b \hat{=} 0. \quad (3.6)$$

Thus, since γ_{ab} is smooth at $\mathcal{I}$, there exists a smooth γ_a such that

$$\gamma_{ab} n^b = \Omega \gamma_a. \quad (3.7)$$

Thus, the perturbations δg_{ab} of the unphysical metric are given by the tensor fields γ_{ab} and γ_a with

$$\gamma_{ab} = \Omega^{-1} \delta g_{ab}, \quad \gamma_a = \Omega^{-2} \delta g_{ab} n^b = \Omega^{-1} \gamma_{ab} n^b. \quad (3.8)$$

These tensor fields are constrained by the linearized vacuum Einstein equations. A particularly important component of these equations is given by [21]

$$\nabla^b \gamma_{ab} - 3\gamma_a - \nabla_a \gamma^b_b \hat{=} 0. \quad (3.9)$$

Further, using the auxiliary foliation of $\mathcal{S}$ we can relate the unphysical metric perturbation characterized by γ_{ab} to the perturbation of the shear σ_{ab} as follows. Varying the definition of the shear (equation (2.16)), noting that the foliation is kept fixed and that $\delta g_{ab} \hat{=} 0$, from equation (3.8) it is straightforward to compute that

$$\delta \sigma_{ab} = -\frac{1}{2} \text{STF } \gamma_{ab}. \quad (3.10)$$

This equation shows that, although γ_{ab} is a subleading quantity, its value is determined by the variation of an object that is constructed from quantities at $\mathcal{S}$: it only depends on quantities off of $\mathcal{S}$ through the dependence of σ_{ab} on ∇_a .

4. Asymptotic symmetries at $\mathcal{S}$: the BMS Lie algebra

Consider a smooth vector field ξ^a in the physical spacetime $\hat{M}$, and let $\hat{g}_{ab}(\lambda)$ be the one-parameter family of physical metrics generated by diffeomorphisms along ξ^a . The physical metric perturbation corresponding to this family is given by

$$\delta_\xi \hat{g}_{ab} = \mathcal{L}_\xi \hat{g}_{ab}. \quad (4.1)$$

The corresponding perturbation of the unphysical metric is

$$\delta_\xi g_{ab} = \Omega^2 \mathcal{L}_\xi \hat{g}_{ab} = \mathcal{L}_\xi g_{ab} - 2\Omega^{-1} \xi^c n_c g_{ab}. \quad (4.2)$$

For ξ^a to be an asymptotic symmetry the infinitesimal diffeomorphism generated by ξ^a must preserve the universal structure discussed in section 3. We now obtain the conditions on ξ^a for it to be an asymptotic symmetry.

Firstly, since the unphysical spacetime M is smooth up to and including $\mathcal{S}$, ξ^a must extend to a smooth vector field on M including at $\mathcal{S}$. Secondly, since the unphysical metric is smooth at $\mathcal{S}$, the perturbation $\delta_\xi g_{ab}$ in equation (4.2) is also smooth at $\mathcal{S}$. This condition implies that

$$\xi^a n_a \hat{=} 0. \quad (4.3)$$

That is, as expected, an asymptotic symmetry ξ^a must be tangent to $\mathcal{S}$ and thus preserves the asymptotic boundary. For convenience we define

$$\alpha_{(\xi)} := \Omega^{-1} \xi^a n_a, \quad (4.4)$$

which is smooth at $\mathcal{S}$.

Next, δ_ξ preserves the universal structure, which implies that (equation (3.8))

$$\gamma_{ab}^{(\xi)} := \Omega^{-1} \delta_\xi g_{ab}, \quad \gamma_a^{(\xi)} := \Omega^{-1} \gamma_{ab}^{(\xi)} n^b, \quad (4.5)$$

must be smooth at $\mathcal{S}$. Using equations (4.2) and (4.4), the smoothness of $\gamma_{ab}^{(\xi)}$ implies

$$\mathcal{L}_\xi g_{ab} \hat{=} 2\alpha_{(\xi)} g_{ab}, \quad (4.6)$$

while, the smoothness of $\gamma_a^{(\xi)}$ implies

$$\mathcal{L}_\xi n^a \hat{=} -\alpha_{(\xi)} n^a, \quad \mathcal{L}_n \alpha_{(\xi)} \hat{=} 0. \quad (4.7)$$

The pullback of equation (4.6) gives

$$\mathcal{L}_\xi q_{ab} \hat{=} 2\alpha_{(\xi)} q_{ab}. \quad (4.8)$$

Intrinsically on $\mathcal{I}$, an asymptotic symmetry is, thus, given by a vector field ξ^a tangent to $\mathcal{I}$ which satisfies equations (4.7) and (4.8) for some smooth function $\alpha_{(\xi)}$. Such vector fields are the BMS symmetries on null infinity.

In any choice of foliation of $\mathcal{I}$ we can further characterize the BMS symmetries as follows. Since ξ^a is tangent to $\mathcal{I}$, we can write

$$\xi^a \hat{=} \beta n^a + X^a, \quad (4.9)$$

where

$$\beta := -l_a \xi^a, \quad X^a := Q^a_b \xi^b. \quad (4.10)$$

We note their conformal weights:

$$\beta : w = 1, \quad X^a : w = 0. \quad (4.11)$$

That is, β is a smooth function of conformal weight one and X^a is a smooth vector field tangent to the cross-sections of the chosen foliation on $\mathcal{I}$. The conditions on β and X^a follow from equations (4.6) and (4.7) as we derive next. Note that equations (4.6) and (4.7) only depend on the vector field ξ^a at $\mathcal{I}$ and are independent of how this vector field is extended away from $\mathcal{I}$.

The only non-trivial component of equation (4.6) is given by its projection tangent to the cross-sections of the foliation which gives

$$Q_a^c Q_b^d \mathcal{L}_\xi g_{cd} \hat{=} 2 \mathcal{D}_{(a} X_{b)} \hat{=} 2 \alpha_{(\xi)} Q_{ab}, \quad (4.12)$$

which shows both that

$$\text{STF } \mathcal{D}_a X_b \hat{=} 0, \quad (4.13)$$

and that

$$\alpha_{(\xi)} \hat{=} \frac{1}{2} \mathcal{D}_a X^a. \quad (4.14)$$

Next, we consider equation (4.7) on $\mathcal{I}$ using equation (4.9) to get

$$\mathcal{L}_\xi n^a = -\mathcal{L}_n \xi^a \hat{=} -n^a \mathcal{L}_n \beta - \mathcal{L}_n X^a \hat{=} -\alpha_{(\xi)} n^a. \quad (4.15)$$

Projecting along the cross-sections we get

$$Q^a_b \mathcal{L}_n X^b \hat{=} 0, \quad (4.16)$$

whereas $X^a l_a \hat{=} 0$, the Bondi condition, and equation (2.18) imply that

$$l_a \mathcal{L}_n X^a \hat{=} -X_a \mathcal{L}_n l^a \hat{=} -X_a \tau^a, \quad n_a \mathcal{L}_n X^a \hat{=} 0, \quad (4.17)$$

so that

$$\mathcal{L}_n X^a \hat{=} n^a X_b \tau^b. \quad (4.18)$$

Contracting equation (4.15) with l_a , we find that

$$\mathcal{L}_n \beta + l_a \mathcal{L}_n X^a \hat{=} \mathcal{L}_n \beta - X_a \tau^a \hat{=} \alpha_{(\xi)}, \quad (4.19)$$

where the second equality follows from equation (4.18). As such, we find that equation (4.19) becomes

$$\mathcal{L}_n \beta \hat{=} \alpha_{(\xi)} - X_a \tau^a \hat{=} \frac{1}{2}(\mathcal{D}_a - 2\tau_a)X^a. \quad (4.20)$$

In summary, in a chosen foliation of $\mathcal{S}$ any BMS symmetry can be written as $\xi^a \hat{=} \beta n^a + X^a$ where X^a is tangent to the cross-sections of the foliation and the following conditions are satisfied:

$$\mathcal{L}_n \beta \hat{=} \frac{1}{2}(\mathcal{D}_a - 2\tau_a)X^a, \quad (4.21a)$$

$$\mathcal{L}_n X^a \hat{=} n^a X_b \tau^b, \quad (4.21b)$$

$$\text{STF } \mathcal{D}_a X_b \hat{=} 0, \quad (4.21c)$$

$$\alpha_{(\xi)} \hat{=} \frac{1}{2}\mathcal{D}_a X^a. \quad (4.21d)$$

Note that, using equations (2.8), (2.27), (2.30), (4.4) and (4.11), one can show that these equations are invariant under conformal transformations that preserve Bondi condition, that is, with $\mathcal{L}_n \omega \hat{=} 0$ (for the form of these equations when Bondi condition does not hold, see equation (A.7)).

Let $\xi_1^a \hat{=} \beta_1 n^a + X_1^a$ and $\xi_2^a \hat{=} \beta_2 n^a + X_2^a$ be two BMS symmetries (with equation (4.21) holding for each) then their Lie bracket can be computed to give

$$\begin{aligned} \xi^a \hat{=} [\xi_1, \xi_2]^a \hat{=} \beta n^a + X^a \quad \text{with} \\ \beta \hat{=} \mathcal{L}_{X_1} \beta_2 - \frac{1}{2}\beta_2 \mathcal{D}_a X_1^a - (1 \leftrightarrow 2) \quad X^a \hat{=} \mathcal{L}_{X_1} X_2^a. \end{aligned} \quad (4.22)$$

It can be checked that ξ^a is also a BMS symmetry, i.e. the β and X^a in equation (4.22) also satisfy the conditions equation (4.21). Thus, the BMS symmetries form a Lie algebra $\mathfrak{b}$.

The structure of the BMS algebra can be analyzed using equation (4.22). Consider a BMS symmetry of the form $\xi_1^a \hat{=} f_1 n^a$ where f_1 is a smooth function on $\mathcal{S}$ satisfying $\mathcal{L}_n f_1 \hat{=} 0$ (from equation (4.21a)). Then, from equation (4.22) we see that the Lie bracket of $\xi_1^a \hat{=} f_1 n^a$ with any other BMS symmetry is also of the form $\xi^a \hat{=} f n^a$ with $\mathcal{L}_n f \hat{=} 0$, that is the set of such vector fields is invariant under the Lie bracket. Further, the Lie bracket of any two such symmetries vanishes. Thus, BMS symmetries of the form $f n^a$ form a preferred infinite-dimensional abelian subalgebra $\mathfrak{s}$ which is a *Lie ideal* of the BMS algebra $\mathfrak{b}$ consisting of *supertranslations*. The quotient algebra $\mathfrak{b}/\mathfrak{s}$ can then be parameterized by X^a . From equation (4.21c) we see that this consists of conformal Killing fields on the cross-sections of $\mathcal{S}$. Since the cross-sections are diffeomorphic to $\mathbb{S}^2$ we get that $\mathfrak{b}/\mathfrak{s}$ is isomorphic to the Lorentz algebra $\mathfrak{so}(1, 3)$. Note that the Lorentz algebra is only identified as a quotient algebra—since the Lie bracket of a supertranslation and X^a is non-zero, there is no invariant choice of Lorentz subalgebra within the BMS algebra $\mathfrak{b}$. Thus the BMS algebra is the semi-direct sum

$$\mathfrak{b} \cong \mathfrak{s} \ltimes \mathfrak{so}(1, 3), \quad (4.23)$$

of the Lie ideal $\mathfrak{s}$ of supertranslations with the Lorentz algebra; the $\ltimes$ indicates the non-trivial Lie bracket between the two factors.

There is another finite-dimensional Lie ideal within the BMS algebra given by supertranslations $f n^a \in \mathfrak{s}$ which satisfy the additional condition

$$\text{STF}(\mathcal{D}_a + \tau_a)(\mathcal{D}_b - \tau_b)f \hat{=} 0. \quad (4.24)$$

It can be checked that this equation is conformally-invariant. Further, the space of solutions to this equation is four-dimensional and is preserved under the Lie bracket of the BMS algebra (see remark 4.1 below). This four-dimensional Lie ideal $\mathfrak{t}$ can be viewed as the space of *translations*. In fact, if the physical spacetime $(\hat{M}, \hat{g}_{ab})$ possesses any Killing vectors ξ^a , then they can be extended to $\mathcal{I}$; moreover, if the Killing field is of the form $\xi^a \hat{=} f n^a$, then ξ^a must be a translation in the sense of f obeying equation (4.24) (see [29]).

A special case of the translations are those that are given by a time-translation in some conformal frame. A time-translation in a given conformal frame is given by $f = 1$, a definition that is motivated by the behavior of the time-translation vector field in Minkowski (and, similarly, the time-translation vector field in Kerr). Since f transforms by $f \mapsto \omega f$ under a conformal transformation, a conformally-invariant notion of a time-translation is given by $f > 0$, since $\omega > 0$.

Remark 4.1 (Characterization of BMS symmetries through spherical harmonics).

Consider the Bondi frame—where the conformal factor is chosen so that the metric on the cross-sections is the unit two-sphere metric and the foliation is chosen so that $\tau_a \hat{=} 0$. A general supertranslation can then be expanded in spherical harmonics on $\mathbb{S}^2$. The condition equation (4.24) implies that the translations are spanned by the $\ell = 0, 1$ harmonics which is indeed a four-dimensional space. The fact that the translations are preserved under Lie brackets can also be shown using some spherical harmonics technology (see the appendix in [30, 31]). Note that since supertranslations have conformal weight one, this spherical harmonic decomposition only holds in the Bondi frame. Similarly, the Lorentz vector fields satisfying

$$\text{STF } \mathcal{D}_a X_b \hat{=} 0, \quad (4.25)$$

are spanned by vector spherical harmonics with $\ell = 1$ (see, e.g. [22]) which is the six-dimensional space of the Lorentz algebra $\mathfrak{so}(1, 3)$.

Remark 4.2 (Supertranslation ‘ambiguity’ in the Lorentz algebra). As noted above, while the supertranslations form a subalgebra of the BMS algebra, there is no preferred Lorentz subalgebra. Instead, the Lorentz algebra arises as the quotient algebra $\mathfrak{b}/\mathfrak{s}$: the set of equivalence classes $[\xi^a]$ of $\mathfrak{b}$, where ξ_1^a and ξ_2^a are members of the same equivalence class if $\xi_1^a - \xi_2^a \hat{=} f n^a \in \mathfrak{s}$.

For a given cross-section S , one can consider the algebra $\mathfrak{l}_S$ of BMS vector fields of the form $\xi^a|_S = X^a$ (that is, vector fields tangent to S). This is a subalgebra of $\mathfrak{b}$, and in fact it is straightforward to show that $\mathfrak{l}_S \cong \mathfrak{b}/\mathfrak{s}$. The issue, therefore, is not that there is no Lorentz subalgebra of $\mathfrak{b}$ but that there is an uncountably infinite number of them, one per cross-section of $\mathcal{I}$.

The fact that the members of the quotient algebra $\mathfrak{b}/\mathfrak{s}$ are only defined up to supertranslations is known as the *supertranslation ambiguity*. A similar situation occurs with the Poincaré algebra, which, while possessing a Lie ideal in the form of the algebra of translations, possesses no unique Lorentz subalgebra. For each origin in Minkowski space, there is a Lorentz subalgebra that consists of infinitesimal rotations and boosts that fix that point, which is analogous to the Lorentz subalgebra $\mathfrak{l}_S$ associated with some cross-section S of $\mathcal{I}$.

Much like how there is a Lorentz subalgebra associated with each cross-section of $\mathcal{I}$, there is moreover a Poincaré subalgebra, given by considering the semidirect product of this Lorentz subalgebra with the translation subalgebra picked out by equation (4.24). This Poincaré subalgebra can be shown to contain any *exact* Killing vector field that might exist in the physical spacetime (see theorem 1 of [29]). Similarly, in appendix C3, we review a construction (based

on twistors) of a Poincaré subalgebra, once again depending explicitly on a choice of cross-section S . Finally, there is no invariant notion of a set of charges conjugate to Lorentz vector fields, which would form an invariant notion of angular momentum. Such a set of charges only exists for a given cross-section, so that (in particular) one cannot have an invariant notion of the flux of angular momentum between two cross-sections. This is similar to how, since the Poincaré algebra has no unique notion of a Lorentz subalgebra, with each origin determining a particular Lorentz subalgebra, the notion of angular momentum in flat space is origin-dependent.

Remark 4.3 (Extended BMS algebra). There have been various recent proposals for extending the BMS algebra. These include proposals to extend the Lorentz quotient algebra, $\mathfrak{b}/\mathfrak{s}$, to the Virasoro algebra [32, 33] and to the algebra of all diffeomorphisms of $\mathbb{S}^2$ [34–36]. The Virasoro vector fields are however singular at isolated points of $\mathbb{S}^2$ and hence do not preserve the smoothness of $\mathcal{I}$. Similarly, as detailed in section 3, it follows from the definition of asymptotic flatness that the conformal class of the induced metric on cross-sections of $\mathcal{I}$ is always universal. As a result, arbitrary diffeomorphisms of $\mathbb{S}^2$ do not arise as symmetries in this context. It has also been shown that the extension to all diffeomorphisms of $\mathbb{S}^2$ cannot be implemented in a fully covariant manner [30]. For these reasons, we do not work with these extended symmetries in this paper.

4.1. Extensions of BMS symmetries away from $\mathcal{I}$

Up to this point, although we started this discussion with vector fields ξ^a defined throughout the unphysical spacetime, we have worked mostly with the value of ξ^a on $\mathcal{I}$ itself. The extension of the BMS symmetries away from $\mathcal{I}$ into the spacetime is arbitrary—one could choose some coordinate system or gauge conditions to obtain a particular extension, but then one needs to verify that any physical quantities (like charges and fluxes) associated with the BMS symmetries are independent of such choices. We collect some of the relevant results on the extensions of BMS symmetries away from $\mathcal{I}$ below.

First, we show that the extension of ξ^a away from $\mathcal{I}$ is determined up to $O(\Omega^2)$:

Proposition 4.1 (Equivalent representatives of a BMS symmetry). *If ξ^a and ξ'^a are vector fields in M which represent the same BMS symmetry, i.e. $\xi^a \hat{=} \xi'^a \in \mathfrak{b}$ then $\xi'^a = \xi^a + O(\Omega^2)$.*

Proof. Since $\xi'^a \hat{=} \xi^a$, let $\xi'^a = \xi^a + \Omega Z^a$ from which we obtain

$$\alpha_{(\xi')} - \alpha_{(\xi)} = \Omega^{-1} n_a (\xi'^a - \xi^a) = n_a Z^a. \quad (4.26)$$

Since $\mathcal{L}_\xi g_{ab} - 2\alpha_{(\xi)} g_{ab} \hat{=} 0$ for any BMS vector field we have

$$0 \hat{=} (\mathcal{L}_{\xi'} g_{ab} - 2\alpha_{(\xi')} g_{ab}) - (\mathcal{L}_\xi g_{ab} - 2\alpha_{(\xi)} g_{ab}) \hat{=} 2n_{(a} Z_{b)} - 2n_c Z^c g_{ab}. \quad (4.27)$$

Taking the trace gives $n_a Z^a \hat{=} 0$. Then we have $n_{(a} Z_{b)} \hat{=} 0$ which implies $Z^a \hat{=} 0$. Thus $Z^a = O(\Omega)$ and $\xi'^a = \xi^a + O(\Omega^2)$. $\square$

The perturbation $\gamma_{ab}^{(\xi)}$ generated by a BMS symmetry $\xi^a|_{\mathcal{I}}$ will, in general, depend on the extension of the symmetry away from $\mathcal{I}$ and hence is *not* well-defined for the BMS symmetries. However, using the above lemma it can be shown that $\text{STF} \gamma_{ab}^{(\xi)}$ on $\mathcal{I}$ is, in fact,

independent of the extension of the BMS symmetry ξ^a away from $\mathcal{I}$, and thus is well-defined for any BMS symmetry.

Corollary 4.1. *If ξ^a and ξ'^a are any two extensions of a given BMS symmetry $\xi^a|_{\mathcal{I}}$ away from $\mathcal{I}$ then*

$$\text{STF } \gamma_{ab}^{(\xi)} \cong \text{STF } \gamma_{ab}^{(\xi')}. \quad (4.28)$$

Proof. From proposition 4.1, we have $\xi'^a = \xi^a + \Omega^2 W^a$, from which (using equation (4.4)) we compute

$$\alpha_{(\xi')} = \Omega^{-1} n_a \xi'^a = \Omega^{-1} n_a \xi^a + \Omega n_a W^a = \alpha_{(\xi)} + \Omega n_a W^a. \quad (4.29)$$

Similarly

$$\mathcal{L}_{\xi'} g_{ab} = \mathcal{L}_{\xi} g_{ab} + 4\Omega n_{(a} W_{b)} + O(\Omega^2). \quad (4.30)$$

Thus, we find that

$$\begin{aligned} \Omega \gamma_{ab}^{(\xi')} &= \mathcal{L}_{\xi'} g_{ab} - 2\alpha_{(\xi')} g_{ab} = \mathcal{L}_{\xi} g_{ab} - 2\alpha_{(\xi)} g_{ab} + 4\Omega \left(n_{(a} W_{b)} - \frac{1}{2} n_c W^c g_{ab} \right) + O(\Omega^2) \\ &\implies \gamma_{ab}^{(\xi')} = \gamma_{ab}^{(\xi)} + 4 \left(n_{(a} W_{b)} - \frac{1}{2} n_c W^c g_{ab} \right) + O(\Omega). \end{aligned} \quad (4.31)$$

Evaluating the STF of both sides on $\mathcal{I}$ of the above equation we find the desired result. $\square$

For later computations it will be useful to have an explicit expression for $\text{STF } \gamma_{ab}^{(\xi)}$ at $\mathcal{I}$ for any BMS symmetry ξ^a in terms of the fields defined on $\mathcal{I}$. We show below that

$$\text{STF } \gamma_{ab}^{(\xi)} \cong -2 \text{STF} \left[\frac{1}{2} \beta N_{ab} + (\mathcal{D}_a + \tau_a)(\mathcal{D}_b - \tau_b)\beta + \mathcal{L}_X \sigma_{ab} - \frac{1}{2} (\mathcal{D}_c X^c) \sigma_{ab} \right]. \quad (4.32)$$

Comparing the above formula to equation (3.10) we find that under a BMS symmetry the shear transforms as

$$\delta_{\xi} \sigma_{ab} = \text{STF} \left[\frac{1}{2} \beta N_{ab} + (\mathcal{D}_a + \tau_a)(\mathcal{D}_b - \tau_b)\beta + \mathcal{L}_X \sigma_{ab} - \frac{1}{2} (\mathcal{D}_c X^c) \sigma_{ab} \right]. \quad (4.33)$$

This is the infinitesimal version of the transformation of the shear found by Sachs [2]. It is also the same as that given in, for example, equation (2.18b) of [22] for the transformation of the shear C_{AB} in Bondi coordinates (which is related to σ_{ab} by equation (6.10)).

In the remainder of this section we detail the computations which lead to equation (4.32). Let $\xi^a|_{\mathcal{I}} = \beta n^a + X^a$ be a BMS symmetry on $\mathcal{I}$. In the following, let l^a be any vector field in a neighbourhood of $\mathcal{I}$ which coincides with the chosen auxiliary normal at $\mathcal{I}$ —the result of the computation can be checked to be independent of how the auxiliary normal is extended away from $\mathcal{I}$.

As shown above in corollary 4.1, $\text{STF } \gamma_{ab}^{(\xi)}|_{\mathcal{I}}$ is independent of how the BMS symmetry $\xi^a|_{\mathcal{I}}$ is extended away from $\mathcal{I}$. Thus, we can choose to extend the BMS symmetry away from $\mathcal{I}$ as follows. We first extend β and X^a away from $\mathcal{I}$ to satisfy

$$\mathcal{L}_l \beta \cong 0, \quad \mathcal{L}_l X^a \cong 0. \quad (4.34)$$

With this choice any extension of the BMS symmetry $\xi^a|_{\mathcal{S}}$ can be written as

$$\xi^a = \beta n^a + X^a + \Omega Z^a \quad (4.35)$$

where Z^a is some smooth vector field. Next from proposition 4.1 we see that $Z^a|_{\mathcal{S}}$ can be determined in terms of β and X^a . To do this, first note that, by equation (4.4),

$$\alpha_{(\xi)} \hat{=} -\mathcal{L}_l(\xi^a n_a) \hat{=} -n_a \mathcal{L}_l \xi^a \hat{=} n_a Z^a. \quad (4.36)$$

Next, we compute $l^b \mathcal{L}_\xi g_{ab}$, using equation (4.6), together with equation (4.34):

$$l^b \mathcal{L}_\xi g_{ab} \hat{=} -\nabla_a \beta + \mathcal{L}_X l_a - Z_a + n_a l^b Z_b \hat{=} 2\alpha_{(\xi)} l_a. \quad (4.37)$$

Contracting this equation with l^a , using equation (4.34) and $l^a \mathcal{L}_X l_a \hat{=} -l_a \mathcal{L}_X l^a \hat{=} 0$, we find that $l_a Z^a \hat{=} 0$. Therefore, we can rearrange equation (4.37) to solve for Z_a : ple

$$Z_a \hat{=} -\nabla_a \beta - 2\alpha_{(\xi)} l_a + \mathcal{L}_X l_a. \quad (4.38)$$

Further, since l_a is the normal and X^a is tangent to the cross-sections of the chosen foliation, we have $Q_a^b \mathcal{L}_X l_b \hat{=} 0$. So

$$Z_a \hat{=} -\mathcal{D}_a \beta + l_a (\mathcal{L}_n \beta - 2\alpha_{(\xi)} + l_b \mathcal{L}_X n^b) \hat{=} -\mathcal{D}_a \beta - \alpha_{(\xi)} l_a, \quad (4.39)$$

where we have used equations (4.21a) and (4.21b). In summary, with the choice equation (4.34), we can write any extension of a given BMS symmetry $\xi^a|_{\mathcal{S}}$ as

$$\xi^a = \beta n^a + X^a - \Omega(\mathcal{D}^a \beta + \alpha_{(\xi)} l^a) + \Omega^2 W^a, \quad (4.40)$$

for some smooth W^a .

With this choice of extension of the BMS symmetry, we compute $\text{STF} \gamma_{ab}^{(\xi)}|_{\mathcal{S}}$ in terms of β and X^a . Using the expansion in equation (4.40) we have

$$\begin{aligned} \Omega \gamma_{ab}^{(\xi)} &= \mathcal{L}_\xi g_{ab} - 2\alpha_{(\xi)} g_{ab} \\ &= \beta \mathcal{L}_n g_{ab} + \mathcal{L}_X g_{ab} - 2\alpha_{(\xi)} g_{ab} - 2n_a [\mathcal{D}_b \beta + \alpha_{(\xi)} l_b] \\ &\quad - \Omega[2\nabla_a \nabla_b \beta + \alpha_{(\xi)} \mathcal{L}_l g_{ab} + 2l_a \nabla_b \alpha_{(\xi)} - 4n_a W_b] + \Omega^2 \mathcal{L}_W g_{ab}. \end{aligned} \quad (4.41)$$

We now want to solve this equation for $\text{STF} \gamma_{ab}^{(\xi)}|_{\mathcal{S}}$. This can be done by taking the $\mathcal{L}_l$ of the above equation, evaluating on $\mathcal{S}$, and then taking the STF. Using $\mathcal{L}_l \Omega \hat{=} l^a n_a = -1$ and that $\mathcal{L}_l n_a \propto n_a$ at $\mathcal{S}$, a long but straightforward computation gives

$$\text{STF} \gamma_{ab}^{(\xi)} \hat{=} -\text{STF}[\beta \mathcal{L}_l \mathcal{L}_n g_{ab} + \mathcal{L}_l \mathcal{L}_X g_{ab} + 2\mathcal{D}_a \mathcal{D}_b \beta - \alpha_{(\xi)} \mathcal{L}_l g_{ab}], \quad (4.42)$$

where we have used equation (4.34). Note that by equation (4.34), we find that

$$\text{STF} \mathcal{L}_l \mathcal{L}_X g_{ab} \hat{=} \text{STF} \mathcal{L}_X \mathcal{L}_l g_{ab} \hat{=} 2 \text{STF} \mathcal{L}_X \sigma_{ab}, \quad (4.43)$$

where the final equality can be shown using equations (2.16) and (4.21c), together with the fact that $X^a n_a \hat{=} X^a l_a \hat{=} 0$. Combining equations (4.42) and (4.43), together with equations (2.16), (2.35) and (4.21d), we find that

$$\text{STF} \gamma_{ab}^{(\xi)} = -2 \text{STF} \left[\frac{1}{2} \beta N_{ab} + (\mathcal{D}_a + \tau_a)(\mathcal{D}_b - \tau_b) \beta + \mathcal{L}_X \sigma_{ab} - \frac{1}{2} (\mathcal{D}_c X^c) \sigma_{ab} \right], \quad (4.44)$$

as claimed in equation (4.32).

5. Asymptotic charges and fluxes: the Wald–Zoupas prescription

The prescription of Wald and Zoupas [21] provides a method of determining charges and fluxes at null infinity, and can be applied to any local and covariant theory. However, for simplicity, we will be specializing to the case of vacuum general relativity. Note that the addition of matter, for example, does not significantly complicate this discussion (see for example, [21, 37]).

We start with the Einstein–Hilbert Lagrangian four-form L :

$$L = \frac{1}{16\pi} \hat{R} \hat{\epsilon}_4. \quad (5.1)$$

where $\hat{R}$ is the Ricci scalar and $\hat{\epsilon}_4$ is the four-form volume element of the physical metric. The dynamical field is the physical metric $\hat{g}_{ab}$, and varying this dynamical field we obtain

$$\delta L = E^{ab} \delta \hat{g}_{ab} + d\theta(\delta \hat{g}), \quad (5.2)$$

where the three-form θ is the *symplectic potential*, and E^{ab} is a tensor-valued four-form which gives the equations of motion in the form $E^{ab} = 0$. For general relativity, we have [38]

$$E^{ab} = -\frac{1}{16\pi} \hat{\epsilon}_4 \hat{G}^{ab}, \quad (5.3a)$$

$$\theta(\delta \hat{g}) \equiv -\frac{1}{8\pi} \hat{\epsilon}_{abc} {}^{[d} \hat{g}^{e]f} \hat{\nabla}_e \delta \hat{g}_{df}. \quad (5.3b)$$

The *symplectic current* is defined by taking a second, independent variation of the symplectic potential and antisymmetrizing in the perturbations:

$$\omega(\delta_1 \hat{g}, \delta_2 \hat{g}) := \delta_1 \theta(\delta_2 \hat{g}) - \delta_2 \theta(\delta_1 \hat{g}). \quad (5.4)$$

Computing $d\omega$ from this equation, using the fact that d and δ commute, using the second variation of equation (5.2), and using the fact that δ_1 and δ_2 commute, one finds that (for example, [38])

$$d\omega(\delta_1 \hat{g}, \delta_2 \hat{g}) = \delta_2 E^{ab} \delta_1 \hat{g}_{ab} - \delta_1 E^{ab} \delta_2 \hat{g}_{ab}, \quad (5.5)$$

which vanishes whenever the perturbations satisfy the linearized equations of motion $\delta E^{ab} = 0$. When the dynamical fields $\hat{g}_{ab}$ satisfy the equations of motion, and $\delta \hat{g}_{ab}$ satisfy the linearized equations of motion, one can show that (see [39–41])

$$\omega(\delta \hat{g}, \mathcal{L}_\xi \hat{g}) = d[\delta \mathcal{Q}_\xi - \xi \cdot \theta(\delta \hat{g})], \quad (5.6)$$

for all infinitesimal diffeomorphisms generated by ξ^a , where the two-form $\mathcal{Q}_\xi$ is the *Noether charge* associated with ξ^a . In general relativity, we have that [21, 38]

$$\omega(\delta_1 \hat{g}, \delta_2 \hat{g}) \equiv \frac{1}{16\pi} \hat{\epsilon}_{dabc} \hat{P}^{defghi} \left[\delta_2 \hat{g}_{ef} \hat{\nabla}_g \delta_1 \hat{g}_{hi} - (1 \leftrightarrow 2) \right], \quad (5.7)$$

where

$$\hat{P}^{abcdef} := \hat{g}^{ae} \hat{g}^{fb} \hat{g}^{cd} - \frac{1}{2} \hat{g}^{ad} \hat{g}^{be} \hat{g}^{fc} - \frac{1}{2} \hat{g}^{ab} \hat{g}^{cd} \hat{g}^{ef} - \frac{1}{2} \hat{g}^{bc} \hat{g}^{ae} \hat{g}^{fd} + \frac{1}{2} \hat{g}^{bc} \hat{g}^{ad} \hat{g}^{ef}. \quad (5.8)$$

Moreover, the Noether charge is given by [40]

$$\mathcal{Q}_\xi \equiv -\frac{1}{16\pi} \hat{\epsilon}_{cdab} \hat{\nabla}^c \xi^d. \quad (5.9)$$

The symplectic current introduced above, when integrated over some hypersurface Σ , provides a symplectic product on phase space. As discussed in [21, 39], a perturbed Hamiltonian should be constructed by considering the symplectic product of an arbitrary perturbation δg_{ab} with $\delta_\xi \hat{g}_{ab} = \mathcal{L}_\xi \hat{g}_{ab}$:

$$\delta H_\xi = \int_\Sigma \omega(\delta \hat{g}, \mathcal{L}_\xi \hat{g}) = \int_\Sigma d [\delta \mathbf{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta \hat{g})] = \int_{\partial \Sigma} [\delta \mathbf{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta \hat{g})], \quad (5.10)$$

where the second equality follows by equation (5.6) and $\partial \Sigma$ is the boundary of Σ . Thus, the Hamiltonian can then naturally be thought of as an integral over the boundary $\partial \Sigma$.

Now consider the case where the hypersurface Σ extends as a smooth surface to $\mathcal{I}$ in the unphysical spacetime which intersects $\mathcal{I}$ at a cross-section S . We take equation (5.10), rewritten in terms of the unphysical fields which are smooth at $\mathcal{I}$. In general relativity, using the behaviour of the unphysical metric perturbations detailed in section 3.1, the symplectic current has a finite limit to $\mathcal{I}$; as shown in [21]. However, one should not conclude from equation (5.6) that $\delta \mathbf{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta \hat{g})$ has a limit to $\mathcal{I}$; in general relativity it can be shown that $\boldsymbol{\theta}(\delta \hat{g})$ again has a finite limit to $\mathcal{I}$ (see [21]), but $\delta \mathbf{Q}_\xi$ diverges in the limit to $\mathcal{I}$. Note that any procedure to ‘subtract out the diverging part’ is highly non-unique. Fortunately, there is no need to resort to any such ad hoc procedure as we can use some elementary differential geometry to proceed directly as follows.

Lemma 5.1. *Let S' be some sequence of two-spheres in unphysical spacetime which limits (continuously) to a chosen cross-section S of $\mathcal{I}$. Then, the limiting integral*

$$\lim_{S' \rightarrow S} \int_{S'} [\delta \mathbf{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta \hat{g})] \quad (5.11)$$

defined by first integrating $\delta \mathbf{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta \hat{g})$ over the sequence of two-spheres S' and then taking the limit as the sequence tends to $S \subset \mathcal{I}$ exists and is independent of the chosen sequence of two-spheres used in the limit.

Proof. Let S_0 be some two-surface in the unphysical spacetime and let Σ be a smooth three-surface which extends from S_0 and intersects $\mathcal{I}$ at a cross-section S . Note that this surface Σ is a compact three-manifold in the unphysical spacetime. The integral of $\omega(\delta \hat{g}, \mathcal{L}_\xi \hat{g})$ over Σ is necessarily finite since $\omega(\delta \hat{g}, \mathcal{L}_\xi \hat{g})$ is a continuous three-form on Σ (including at the ‘boundary’ at S). Thus integrating equation (5.6) over Σ we obtain⁹

$$\int_\Sigma \omega(\delta \hat{g}, \mathcal{L}_\xi \hat{g}) = \int_{S_0} [\delta \mathbf{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta \hat{g})] - \lim_{S' \rightarrow S} \int_{S'} [\delta \mathbf{Q}_\xi - \xi \cdot \boldsymbol{\theta}(\delta \hat{g})], \quad (5.12)$$

where the last expression on the right-hand side means ‘integrate over some two-sphere $S' \subset \Sigma$, and then take the limit of this two-sphere to the boundary S ’. This limiting procedure is necessary because, although the integral on the left-hand side of equation (5.12) is always finite (as argued above), the two-form integrand on the right-hand side need not have a finite

⁹ We note that on the left-hand side, we have taken Σ to be future-oriented while on the right-hand side, the two-spheres *do not* have the usual outward-facing orientation within Σ , but the opposite. This choice of orientation is more natural, since (as mentioned in remark 2.1), the limit of the orientation as $S' \rightarrow S$ is the future-directed orientation of S within $\mathcal{I}$ as specified by ϵ_2 . This choice of orientation is the opposite of the one that is used by Wald and Zoupas (see footnotes 2, 3 and 8 of [21]), and so some of our equations have the opposite sign.

limit to $\mathcal{S}$ in general. It follows that the limit in the last expression on the right-hand side of equation (5.12) exists. The limit is also independent of the choice of sequence of S' that is used in its definition due to equation (5.6) and the fact that Σ is compact and $\omega(\delta\hat{g}, \mathcal{L}_\xi\hat{g})$ is continuous on Σ . Further, by Stokes' theorem, this expression is the same for *any* choice of the three-surface Σ whose boundary is also S , since $d\omega(\delta\hat{g}, \delta_\xi\hat{g}) = 0$ by equation (5.5) and since $\omega(\delta\hat{g}, \delta_\xi\hat{g})$ extends continuously to $\mathcal{S}$. $\square$

Remark 5.1 (Necessity of ω extending continuously to $\mathcal{S}$). We emphasize that the condition that $\omega(\delta\hat{g}, \delta_\xi\hat{g})$ extend continuously (as a three-form in M) to $\mathcal{S}$ is necessary in the above argument. For instance, if we instead only assume that $\int_\Sigma \omega(\delta\hat{g}, \delta_\xi\hat{g})$ is finite on *every* surface Σ (or that the pullback of $\omega(\delta\hat{g}, \delta_\xi\hat{g})$ to *every* Σ is continuous within Σ), then $d\omega = 0$ does *not* imply that the limit of $\int_{S'} [\delta Q_\xi - \xi \cdot \theta(\delta\hat{g})]$ is well-defined since it can depend on the choice of surface Σ used to define the limiting integral. As this point is often overlooked in the application of Stokes' theorem at $\mathcal{S}$ we provide a simple example on the Euclidean plane below.

On $\mathbb{R}^2$, let (x, y) be the usual Cartesian coordinates. Consider the one-form

$$\omega \equiv \frac{1}{(x^2 + y^2)^{3/2}} [y^2 dx - xy dy]. \quad (5.13)$$

This one-form can be written as the exterior derivative of a 0-form (i.e. a function) as¹⁰

$$\omega = dQ, \quad Q = \frac{x}{(x^2 + y^2)^{1/2}}. \quad (5.14)$$

Note that ω does not extend continuously to the origin $(x, y) = (0, 0)$, but $d\omega = 0$ everywhere else and extends continuously to the origin. Further, it can be checked that the pullback of this ω to any smooth curve Σ through the origin is continuous at the origin *within* this curve. However, we *cannot* use Stokes' theorem to conclude that $\int_\Sigma \omega$ is independent of the choice of curve Σ joining the origin to some other point, since ω is not continuous at the origin as a one-form in $\mathbb{R}^2$. A direct computation shows that for any curve Σ from the origin to $(1, 1)$, we have

$$\int_\Sigma \omega = \frac{1}{\sqrt{2}} - \cos \theta, \quad (5.15)$$

where θ is the angle with the x -axis of the tangent of Σ at the origin. So the integral of ω along any curve through the origin is finite but depends on the curve Σ . Thus, we cannot define the value of Q at the origin by taking such integrals over curves Σ since it depends on the curve used in the limiting procedure. This can be explicitly checked from the expression of Q given above.

Next we show that, in general relativity, the limiting integral of $[\delta Q_\xi - \xi \cdot \theta(\delta\hat{g})]$ is independent of the choice of extension of the BMS symmetry away from $\mathcal{S}$.

¹⁰ The choice of the constant in the function Q is irrelevant for our argument.

Lemma 5.2. *If ξ^a and ξ'^a are equivalent representatives of a BMS symmetry on $\mathcal{I}$ in general relativity then*

$$\lim_{S' \rightarrow S} \int_{S'} [\delta \mathcal{Q}_\xi - \xi \cdot \theta(\delta \hat{g})] = \lim_{S' \rightarrow S} \int_{S'} [\delta \mathcal{Q}_{\xi'} - \xi' \cdot \theta(\delta \hat{g})], \quad (5.16)$$

for all background spacetimes, all perturbations $\delta \hat{g}_{ab}$ and all cross-sections S of $\mathcal{I}$.

Proof. From proposition 4.1, two equivalent representatives of a BMS symmetry are related by $\xi'^a = \xi^a + \Omega^2 W^a$ for some smooth W^a . Hence to prove the desired result we only need to show that the above integral computed for the vector field $\Omega^2 W^a$ vanishes. In general relativity it can be shown that the three-form $\theta(\delta \hat{g})$ is finite at $\mathcal{I}$ [21] and thus $(\Omega^2 W) \cdot \theta(\delta \hat{g}) \equiv 0$. So we only need to compute the Noether charge term which can be written as

$$\mathcal{Q}_{\Omega^2 W} \equiv -\frac{1}{16\pi} \hat{\epsilon}_{abcd} \hat{\nabla}^c (\Omega^2 W^d) = -\frac{1}{16\pi} \epsilon_{abcd} \nabla^c W^d. \quad (5.17)$$

Note that this is manifestly finite at $\mathcal{I}$ and so we can dispense with the limiting procedure used in the integral and evaluate the variation of the above expression directly at $\mathcal{I}$. Using the asymptotic conditions on the metric perturbations (equation (3.8)), we get (we ignore the overall signs and factors)

$$\delta \mathcal{Q}_{\Omega^2 W}|_{\mathcal{I}} \propto \epsilon_{abcd} n^c \gamma^d_e W^e. \quad (5.18)$$

Using the definition of the volume-forms (equation (2.14)) and $n_a l^a \hat{=} -1$, the integral over the cross-section S is then (using $\gamma_{ab} n^b \hat{=} \Omega \gamma_a$)

$$\int_S \delta \mathcal{Q}_{\Omega^2 W} \propto \int_S \epsilon_2 n^a \gamma_{ab} W^b \hat{=} \int_S \epsilon_2 \Omega \gamma_a W^a \hat{=} 0. \quad (5.19)$$

□

The results of lemmas 5.1 and 5.2 prove that the limiting integral of $[\delta \mathcal{Q}_\xi - \xi \cdot \theta(\delta \hat{g})]$ is well-defined on BMS symmetries at $\mathcal{I}$. Thus, from equation (5.12), it would be natural to define a charge associated with the asymptotic symmetry ξ^a at S as a function $Q[\xi; S]$ in the phase space of the theory such that

$$\delta Q[\xi; S] := \lim_{S' \rightarrow S} \int_{S'} [\delta \mathcal{Q}_\xi - \xi \cdot \theta(\delta \hat{g})], \quad (5.20)$$

for all backgrounds $\hat{g}_{ab}$, all perturbations $\delta \hat{g}_{ab}$, and all cross-sections S . However, in general, no such function $Q[\xi; S]$ exists, since the right-hand side is not integrable in phase space; that is, it cannot be written as the variation of some quantity for *all* perturbations. To see this, suppose that the charge defined in equation (5.20) does exist. Then, one must have $(\delta_1 \delta_2 - \delta_2 \delta_1)Q[\xi; S] = 0$ for *all* backgrounds $\hat{g}_{ab}$ and *all* perturbations $\delta_1 \hat{g}_{ab}$, and $\delta_2 \hat{g}_{ab}$ (satisfying the corresponding equations of motion). However, it follows from equations (5.4) and (5.20) and the commutativity of δ_1 and δ_2 that

$$(\delta_1 \delta_2 - \delta_2 \delta_1)Q[\xi, S] = - \int_S \xi \cdot \omega(\delta_1 \hat{g}, \delta_2 \hat{g}). \quad (5.21)$$

Thus, a charge defined by equation (5.20) will exist if the right-hand side of the above equation vanishes. This is the case if ξ^a is tangent to S . However, in general, the right-hand side is non-vanishing, and so one cannot define any charge $Q[\xi; S]$ using equation (5.20).

Remark 5.2 (Subtracting the ‘non-integrable part’). It might be tempting to simply stare at some explicit expression for the right-hand-side of equation (5.20) and then subtract off the ‘non-integrable part’ of the expression to obtain an expression which is manifestly integrable to define the charge at S . But this ‘procedure’ is very ad hoc; for instance suppose one manages to write the right-hand side of equation (5.20) as $\delta A + B$ where δA is a manifestly integrable expression and B is not. However, one can trivially write this in the alternative form $\delta A + B = \delta(A + C) + (B - \delta C)$ where C is some tensorial expression in terms of the available fields. Obviously, $\delta(A + C)$ is integrable while $(B - \delta C)$ is non-integrable for any choice of C . Thus the procedure to ‘subtract off the non-integrable part’ is highly ambiguous—without any additional criteria one cannot know which ‘non-integrable part’ B or $B - \delta C$ should be ‘subtracted off’.

The obstruction to the non-integrability of equation (5.20) was resolved by the rather general prescription of Wald and Zoupas [21]. Their procedure for defining integrable charges associated with asymptotic symmetries can be summarized as follows: let Θ be a symplectic potential for the pullback of the symplectic current to $\mathcal{I}$; that is,

$$\omega_{\leftarrow}(\delta_1 \hat{g}, \delta_2 \hat{g}) = \delta_1 \Theta(\delta_2 \hat{g}) - \delta_2 \Theta(\delta_1 \hat{g}), \quad (5.22)$$

for *all* backgrounds and *all* perturbations, with appropriate asymptotic conditions and equations of motion imposed. Following [21], we require that the choice of Θ satisfies the following properties:

- (a) Θ must be locally and covariantly constructed out of the dynamical fields g_{ab} , the perturbations δg_{ab} , and finitely many of their derivatives, along with any fields in the ‘universal background structure’ present at $\mathcal{I}$ described in section 3;
- (b) Θ must be independent of any arbitrary choices made in specifying the background structure; that is, Θ is conformally invariant and independent of the choice of the auxiliary normal l^a ; and
- (c) If g_{ab} is a stationary background solution, then $\Theta(\hat{g}; \delta \hat{g}) = 0$, for *all* (not necessarily stationary) perturbations δg_{ab} .

The first of these criteria is motivated by the fact that, as laid out in the introduction, the prescription used to define charges should only require the tensor fields that exist on $\mathcal{I}$, and should not depend on additional structure that can be used (for example, some choice of coordinates). The second is required to ensure that the charges that are defined by this prescription are associated with the physical spacetime and do not depend that the particular choices that go into the conformal-completion (such as the conformal factor) or other additional choices like the foliation of $\mathcal{I}$. The final criterion plays an important role in showing that the flux of these charges vanishes for stationary spacetimes as we shall see below.

If such a symplectic potential Θ can be found, define $Q[\xi; S]$ to be a function on the phase space at $\mathcal{I}$ by

$$\delta Q[\xi; S] := \lim_{S' \rightarrow S} \int_{S'} [\delta Q_\xi - \xi \cdot \theta(\delta \hat{g})] + \int_S \xi \cdot \Theta(\delta \hat{g}). \quad (5.23)$$

It can easily be checked (using equations (5.20)–(5.22)) that this expression is integrable in phase space; that is, $(\delta_1 \delta_2 - \delta_2 \delta_1)Q[\xi; S] = 0$. Together with some choice of reference

solution g_0 on which $\mathcal{Q}[\xi; S] = 0$ for all asymptotic symmetries ξ^a and all cross-sections S , equation (5.23) can be integrated in phase space to define the WZ charge $\mathcal{Q}[\xi; S]$ associated with the asymptotic BMS symmetry ξ^a at S . We note the following properties of the WZ charge defined by the above procedure:

- (a) Since Θ is locally and covariantly constructed from the physical metric and the universal structure of $\mathcal{S}$, the charge can similarly be written in terms of quantities that are defined at $\mathcal{S}$. In particular the charge only depends on the BMS symmetry at $\mathcal{S}$ and not on any choice of its extension into the spacetime. While this is clear for the term that is an integral of $\xi \cdot \Theta$, that this is also true for the remaining terms, which are defined by a limit in the unphysical spacetime, follows by lemmas 5.1 and 5.2.
- (b) Moreover, since Θ is chosen to be conformally invariant and independent of the choice of the auxiliary foliation of $\mathcal{S}$ and the choice of the reference solution is to be specified without any particular choice of its conformal completion or the choice of auxiliary foliation, the WZ charge is also conformally invariant and independent of the choice of the auxiliary foliation. Note, however that the WZ charge does depend on the chosen cross-section S on which it is evaluated and hence can depend on the auxiliary normal l_a at S (but does not depend on the choice of l_a away from S).

The flux of the perturbed WZ charge, through a portion $\Delta\mathcal{S}$ of $\mathcal{S}$ whose boundary is given by two cross-sections S_1 and S_2 , is given by (see equations (28) and (29) of [21])

$$\delta\mathcal{F}[\xi; \Delta\mathcal{S}] := \delta\mathcal{Q}[\xi; S_2] - \delta\mathcal{Q}[\xi; S_1] = \int_{\Delta\mathcal{S}} \left[\underset{\leftarrow}{\omega}(\delta\hat{g}, \mathcal{L}_\xi\hat{g}) + d\{\xi \cdot \Theta(\delta\hat{g})\} \right]. \quad (5.24)$$

The last term of this equation can also be written as

$$d[\xi \cdot \Theta(\delta\hat{g})] = \mathcal{L}_\xi \Theta(\delta\hat{g}) = -\underset{\leftarrow}{\omega}(\delta\hat{g}, \mathcal{L}_\xi\hat{g}) + \delta\Theta(\mathcal{L}_\xi\hat{g}), \quad (5.25)$$

where in the first equality, we have used the fact that Θ is a three-form intrinsic to $\mathcal{S}$, and so its exterior derivative is zero, and the second equality follows from the definition of Θ as a symplectic potential for $\underset{\leftarrow}{\omega}$ (equation (5.22)). The flux of the perturbed WZ charge is therefore simply given by

$$\delta\mathcal{F}[\xi; \Delta\mathcal{S}] = \int_{\Delta\mathcal{S}} \delta\Theta(\mathcal{L}_\xi\hat{g}). \quad (5.26)$$

To get the unperturbed charge and flux, we have to choose a reference solution g_0 on which the charges are required to vanish. Since the symplectic potential Θ is required to vanish on stationary backgrounds, we choose the reference solution g_0 to also be stationary. For our concrete case of general relativity, we will pick g_0 to be (any conformal completion of) Minkowski spacetime. Then, the flux of the WZ charge is given by

$$\mathcal{F}[\xi; \Delta\mathcal{S}] = \mathcal{Q}[\xi; S_2] - \mathcal{Q}[\xi; S_1] = \int_{\Delta\mathcal{S}} \Theta(\mathcal{L}_\xi\hat{g}). \quad (5.27)$$

There are two important properties that follow from this expression for the flux. The first, which the flux inherits from Θ , is that, for any stationary background, the flux will vanish. This property captures the fact that, if there is no radiation, there should be no flux and the charges should be conserved quantities. The second is that, if ξ^a is an exact Killing vector field

in the physical spacetime, then the flux will vanish as well. This follows from the vanishing of $\mathcal{L}_\xi \hat{g}_{ab}$ for such vector fields. This property is reminiscent of Noether's theorem: if there is an exact symmetry, then there should be a quantity related to that symmetry (in this case, the charge) that is conserved. In the next two sections, we will find expressions for the WZ flux and the WZ charge in vacuum general relativity. Since the charge and flux calculations will be performed in different ways, we will check these calculations by showing that they agree with equation (5.27).

Finally, we remark that the WZ prescription has certain ambiguities related to the choices of the symplectic potential θ and the choice of Θ . However, it was argued in [21] that these ambiguities do not affect the final result. As the proof is quite involved, we do not present it here, and turn our attention to computing explicit expressions for the WZ flux and charge.

5.1. Wald–Zoupas flux in general relativity

We first consider the flux of the WZ charge. From equation (5.27), it is apparent that determining this flux requires finding Θ . First, a lengthy calculation starting with equation (5.7), then using equation (3.8) for the unphysical metric perturbation δg_{ab} , along with the variation of the vacuum Einstein equations (equation (2.1)), shows that [21]

$$\omega_{\leftarrow}(\delta_1 \hat{g}, \delta_2 \hat{g}) \hat{=} -\frac{1}{32\pi} (\delta_1 S_{ab} \gamma_2^{ab} - \delta_2 S_{ab} \gamma_1^{ab}) \epsilon_3. \quad (5.28)$$

Since $\gamma^{ab} n_b \hat{=} 0$ by equation (3.8), δS_{ab} in this expression can be replaced with $\delta_{\leftarrow} S_{ab}$. Moreover, ρ_{ab} is universal, and so $\delta \rho_{ab} = 0$; as such, equation (2.39) implies that we can replace $\delta_{\leftarrow} S_{ab}$ with δN_{ab} , yielding

$$\begin{aligned} \omega_{\leftarrow}(\delta_1 \hat{g}, \delta_2 \hat{g}) &\hat{=} -\frac{1}{32\pi} (\delta_1 N_{ab} \gamma_2^{ab} - \delta_2 N_{ab} \gamma_1^{ab}) \epsilon_3 \\ &\hat{=} \frac{1}{16\pi} (\delta_1 N^{ab} \delta_1 \sigma_{ab} - \delta_2 N^{ab} \delta_1 \sigma_{ab}) \epsilon_3, \end{aligned} \quad (5.29)$$

where the final equality uses the fact that the News tensor is traceless and equation (3.10).

A symplectic potential for $\omega_{\leftarrow}$ is therefore given by

$$\Theta(\delta \hat{g}) \hat{=} -\frac{1}{32\pi} N^{ab} \gamma_{ab} \epsilon_3 \hat{=} \frac{1}{16\pi} N^{ab} \delta \sigma_{ab} \epsilon_3. \quad (5.30)$$

One must check that this symplectic potential satisfies the requirements above. First, it is constructed from g_{ab} , δg_{ab} , their derivatives (such as S_{ab}), and fields that are part of the universal structure at $\mathcal{I}$ (such as n^a and, less obviously, ρ_{ab} [21]). Moreover, it does not depend on the arbitrary structure that we have provided at $\mathcal{I}$, the choice of conformal factor Ω and auxiliary normal l^a . To see this, first note that the only piece that depends on l^a is $\epsilon_3 = l \cdot \epsilon_4$, but upon taking the pullback, this dependence drops out. To see that it is conformally invariant, use the fact that $\gamma_{ab} = \Omega \delta \hat{g}_{ab}$, and so under a conformal transformation $\Omega \mapsto \omega \Omega$,

$$N^{ab} \mapsto \omega^{-4} N^{ab}, \quad \gamma_{ab} \mapsto \omega \gamma_{ab}, \quad \epsilon_3 \mapsto \omega^3 \epsilon_3, \quad (5.31)$$

and so we find that Θ is conformally invariant. Finally, note that this choice of Θ vanishes on stationary solutions g_{ab} , for any perturbation δg_{ab} , by the argument on pp 53–54 of [27] (which shows that N_{ab} vanishes for stationary vacuum spacetimes).

Using a chosen foliation of $\mathcal{J}$, we now provide a more explicit expression for the WZ flux in terms of fields defined on $\mathcal{J}$. Note that, since the News tensor is traceless, we have that

$$N^{ab}\gamma_{ab}^{(\xi)} \doteq N^{ab} \text{STF } \gamma_{ab}^{(\xi)}. \quad (5.32)$$

As such, from corollary 4.1 we have that our flux is independent of the extension of ξ^a off of $\mathcal{J}$. Plugging in our expression for $\text{STF } \gamma_{ab}^{(\xi)}$ from equation (4.32), we therefore find that¹¹

$$\begin{aligned} \mathcal{F}[\xi; \Delta\mathcal{J}] &\doteq -\frac{1}{16\pi} \int_{\Delta\mathcal{J}} \epsilon_3 N^{ab} \gamma_{ab}^{(\xi)} \doteq -\frac{1}{16\pi} \int_{\Delta\mathcal{J}} \epsilon_3 N^{ab} \\ &\times \left[\frac{1}{2} \beta N_{ab} + (\mathcal{D}_a + \tau_a)(\mathcal{D}_b - \tau_b)\beta + \mathcal{L}_X \sigma_{ab} - \frac{1}{2}(\mathcal{D}_c X^c) \sigma_{ab} \right]. \end{aligned} \quad (5.33)$$

As mentioned in [21] this flux formula is equal to the one obtained by Ashtekar and Streubel [43]; the explicit computation of this equivalence is given in appendix C1.

Note that the first expression is manifestly independent of any choice of foliation of $\mathcal{J}$, a property it inherits from Θ . This is not obviously true for the second, which instead is more useful for explicit computations (such as comparing with expressions in Bondi coordinates). Both, however, are clearly local and covariant, and can be easily shown to be independent of the conformal factor. We reiterate that these properties of the flux are motivated by the desire that these expressions not be dependent on any arbitrary choices that we could make, such as a coordinate system, conformal factor etc. As mentioned below equation (5.27), the flux in equation (5.33) has both the property that it vanishes for stationary backgrounds, as well as when ξ^a is an exact Killing vector field in the physical spacetime. The first of these properties is evident both in the first and second expressions in equation (5.33), as the integrands are both proportional to the News. That the flux vanishes when ξ^a is an exact Killing vector field follows immediately from the fact that $\gamma_{ab}^{(\xi)}$ vanishes for such vector fields.

In the case where ξ^a is a translation, namely $\xi^a = f n^a$, where f obeys equation (4.24), it is the case that

$$\mathcal{F}[fn; \Delta\mathcal{J}] = -\frac{1}{32\pi} \int_{\Delta\mathcal{J}} \epsilon_3 f N^{ab} N_{ab}. \quad (5.34)$$

In the case where f is everywhere positive, ξ^a corresponds to a time-translation in some conformal frame, as mentioned below equation (4.24). In this case, the flux is negative, corresponding to the *loss* of mass/energy during the emission of gravitational waves.

Remark 5.3 (BMS fluxes and Hamiltonians). On any region $\Delta\mathcal{J}$ of null infinity we can consider the integral of the symplectic current $\int_{\Delta\mathcal{J}^{\leftarrow}} \omega(\delta_1 \hat{g}, \delta_2 \hat{g})$ as defining a *symplectic form* on the radiative phase on $\Delta\mathcal{J}$. Then, equation (5.24) implies that

$$\delta\mathcal{F}[\xi; \Delta\mathcal{J}] = \int_{\Delta\mathcal{J}^{\leftarrow}} \omega(\delta \hat{g}, \mathcal{L}_\xi \hat{g}) + \int_{S_2} \xi \cdot \Theta(\delta \hat{g}) - \int_{S_1} \xi \cdot \Theta(\delta \hat{g}). \quad (5.35)$$

Note that if the boundary terms at S_1 and S_2 vanish for all perturbations δg_{ab} and all background solutions g_{ab} then the WZ flux $\mathcal{F}[\xi; \Delta\mathcal{J}]$ would define a *Hamiltonian generator* corresponding

¹¹ The first form of the WZ flux in equation (5.33) was anticipated by Geroch and Winicour long before Wald and Zoupas (see equation (28) of [42])!

to the BMS symmetry ξ^a , as defined by [21, 39]. But these boundary terms do not vanish in general and the WZ flux does not define a Hamiltonian. However, in the case where we consider all of null infinity, instead of some finite portion, and use appropriate boundary conditions at timelike and spatial infinity, then $\mathcal{F}[\xi; \mathcal{S}]$ will be a Hamiltonian generator on the full radiative phase space on $\mathcal{S}$. To see this, let u be a parameter along the null generators of $\mathcal{S}$ such that $n^a \nabla_a u \hat{=} 1$, and impose, as $u \rightarrow \pm\infty$, the following boundary conditions:

$$N_{ab} = O(1/|u|^{1+\epsilon}), \quad \gamma_{ab} = O(1), \quad (5.36)$$

for some $\epsilon > 0$. These conditions ensure that the integral of the symplectic current over all of $\mathcal{S}$, as given in equation (5.29), is finite. Further, equation (4.21a) implies that $\xi^a = O(|u|)$, and so it follows from equations (5.30) and (5.36) that

$$\lim_{u \rightarrow \pm\infty} \xi \cdot \Theta(\delta\hat{g}) = 0. \quad (5.37)$$

Note that the conditions equation (5.36) are preserved by all BMS symmetries, and thus the total flux $\mathcal{F}[\xi; \mathcal{S}]$ defines a Hamiltonian generator for the BMS symmetry ξ^a on the radiative phase space on $\mathcal{S}$.

5.2. Wald–Zoupas charge in general relativity

Having obtained the WZ flux we now wish to find an expression for the WZ charge in terms of fields on $\mathcal{S}$. From equation (5.23), the WZ charge $\mathcal{Q}[\xi; S]$ is determined by

$$\delta\mathcal{Q}[\xi; S] = \lim_{S' \rightarrow S} \int_{S'} [\delta\mathcal{Q}_\xi - \xi \cdot \theta(\delta\hat{g})] + \int_S \xi \cdot \Theta(\delta\hat{g}), \quad (5.38)$$

along with the requirement that $\mathcal{Q}[\xi; S]$ vanish on Minkowski spacetime for all BMS symmetries ξ^a and all cross-sections S . The main difficulty in carrying out this computation directly is that the two-form $\delta\mathcal{Q}_\xi$ does not have a limit to $\mathcal{S}$; as mentioned earlier, it can be shown that $\theta(\delta\hat{g})$ does have a limit to $\mathcal{S}$ in general relativity. To compute the right-hand-side of equation (5.38), one must first choose some family of two-spheres inside the spacetime, evaluate the integral and then take the limit as these two-spheres tend to the chosen cross-section S of $\mathcal{S}$.

To compute this term, we note that by the general arguments in lemmas 5.1 and 5.2,

$$\lim_{S' \rightarrow S} \int_{S'} [\delta\mathcal{Q}_\xi - \xi \cdot \theta(\delta\hat{g})], \quad (5.39)$$

is guaranteed to exist, is independent of the family of two-spheres chosen to take the limit and is independent of how the BMS symmetry is extended into the spacetime away from $\mathcal{S}$. Moreover, it is manifestly independent of the choice of the conformal factor and the choice of the foliation of $\mathcal{S}$. Thus, we can compute equation (5.39) in the choices given by the conformal Bondi–Sachs coordinates in a neighbourhood of $\mathcal{S}$. The detailed construction of these coordinates and the form of the unphysical and physical metrics in these coordinates is given in section 6.1. The conformal Bondi–Sachs coordinates give us a family of null surfaces $\mathcal{N}_u$ labeled by a coordinate u and a family of two-spheres S' labeled by the coordinates Ω and u along each null surface $\mathcal{N}_u$ such that as $\Omega \rightarrow 0$ the two-spheres limit to a cross-section S of $\mathcal{S}$. We use this family of two-spheres to evaluate equation (5.39) and then take the limit $\Omega \rightarrow 0$ along this family.

With this setup, we take the form of the unphysical metric in Bondi–Sachs coordinates (given by equations (6.2) and (6.5)), the corresponding expressions for the unphysical metric perturbations along with the expression for ξ^a derived in equation (6.19). We use this to evaluate equation (5.39) using equations (5.3b) and (5.9), after converting the physical metric and physical metric perturbations to their unphysical counterparts. The resulting expression for the two-form $\delta\mathcal{Q}_\xi$ has a term that diverges as $\Omega \rightarrow 0$, which is given by

$$\varepsilon_2 \Omega^{-1} \left(2X^A \delta U_A^{(2)} \right) = -\varepsilon_2 \Omega^{-1} X^A \mathcal{D}^B \delta C_{AB}. \quad (5.40)$$

The indices A, B are abstract indices for tensor fields on the chosen family of two-spheres, and $\mathcal{D}_A$ is the covariant derivative on these two-spheres. The equality equation (5.40) follows from equation (6.7) which is a consequence of the (linearized) Einstein equation in the Bondi–Sachs coordinates. Now, since X^A is a Lorentz vector field (satisfying equation (6.14) in the conformal Bondi–Sachs coordinates) and δC_{AB} is traceless, this term vanishes when integrated over the two-spheres. As a result, the limit of $\int_{S'} \delta\mathcal{Q}_\xi$ as $S' \rightarrow S$, i.e. $\Omega \rightarrow 0$, is finite. Then, computing the remaining terms in equation (5.39) (which are manifestly finite as $\Omega \rightarrow 0$) we obtain

$$\begin{aligned} \lim_{S' \rightarrow S} \int_{S'} [\delta\mathcal{Q}_\xi - \xi \cdot \boldsymbol{\Theta}(\delta\hat{g})] &= -\frac{1}{8\pi} \delta \int_S \varepsilon_2 \left[\beta (\mathcal{P} + \frac{1}{2} \sigma_{ab} N^{ab}) + X^a \mathcal{J}_a \right. \\ &\quad \left. + X^a \sigma_{ab} \mathcal{D}_c \sigma^{bc} - \frac{1}{4} \sigma_{ab} \sigma^{ab} \mathcal{D}_c X^c \right] \\ &\quad - \frac{1}{16\pi} \int_S \varepsilon_2 \beta N^{ab} \delta \sigma_{ab}. \end{aligned} \quad (5.41)$$

where we have used equation (6.10) to convert the metric components in the conformal Bondi–Sachs coordinates to covariant quantities defined on $\mathcal{S}$. We can immediately see this expression is non-integrable due to the last term¹².

Using equations (2.14), (5.30) and (6.19), we have

$$\int_S \xi \cdot \boldsymbol{\Theta}(\delta\hat{g}) = \frac{1}{16\pi} \int_S \varepsilon_2 \beta N^{ab} \delta \sigma_{ab}. \quad (5.42)$$

Thus we have the perturbed WZ charge

$$\begin{aligned} \delta\mathcal{Q}[\xi; S] &= \lim_{S' \rightarrow S} \int_{S'} [\delta\mathcal{Q}_\xi - \xi \cdot \boldsymbol{\Theta}(\delta\hat{g})] + \int_S \xi \cdot \boldsymbol{\Theta}(\delta\hat{g}) \\ &= -\frac{1}{8\pi} \delta \int_S \varepsilon_2 \left[\beta (\mathcal{P} + \frac{1}{2} \sigma_{ab} N^{ab}) + X^a \mathcal{J}_a + X^a \sigma_{ab} \mathcal{D}_c \sigma^{bc} - \frac{1}{4} \sigma_{ab} \sigma^{ab} \mathcal{D}_c X^c \right]. \end{aligned} \quad (5.43)$$

Therefore, the expression for the WZ charge is given by

¹² As emphasized in remark 5.2 above, this *should not* be taken to mean that the last line above is the ‘non-integrable part’ of the expression which should be ‘simply subtracted away’ without specifying any additional criteria.

$$\begin{aligned} \mathcal{Q}[\xi; S] \hat{=} & -\frac{1}{8\pi} \int_S \epsilon_2 \left[\beta(\mathcal{P} + \frac{1}{2} \sigma^{ab} N_{ab}) + X^a \mathcal{J}_a + X^a \sigma_{ab} (\mathcal{D}_c - \tau_c) \sigma^{bc} \right. \\ & \left. - \frac{1}{4} \sigma_{ab} \sigma^{ab} (\mathcal{D}_c - 2\tau_c) X^c \right]. \end{aligned} \quad (5.44)$$

Note that in equation (5.44) we have added terms which depend on τ_a ; these terms cancel amongst each other using the identity equation (D.3). These additional terms make each term in the integrand of conformal weight -2 (which can be verified using the conformal weights given in equations (2.29)–(2.31) and (4.11)) which, along with the conformal weight $+2$ of ϵ_2 , makes the charge conformally-invariant. Further, since these τ_a terms cancel the charge is independent of the choice of the foliation and only depends on the chosen cross-section S . We will verify below that the flux of this charge across any region $\Delta\mathcal{S}$ is given by equation (5.33). Finally, since the flux vanishes in Minkowski spacetime we can compute the charge on any cross-section of $\mathcal{S}$. Using a shear-free cross-section we see that the charge also vanishes on any cross-section of $\mathcal{S}$ in Minkowski spacetime. Thus, the charge formula in equation (5.44) satisfies all the properties required by the WZ charge.

For a supertranslation symmetry with $\beta \hat{=} f$ satisfying $\mathcal{L}_n f \hat{=} 0$ it is straightforward to verify that equation (5.44) reproduces Geroch's supermomentum [21, 27]. Further, if at some chosen cross-section S we pick $\beta|_S \hat{=} 0$ then the resulting formula is equal to the linkage charge defined by Geroch and Winicour [21, 42]; this was proven in [21] and we show this more explicitly in appendix C2. Now, while on any fixed cross-section S we can decompose a general BMS symmetry into a supertranslation part and a part tangent to S , this decomposition is not preserved along $\mathcal{S}$ (see also remark 4.2). Thus, in general the WZ charge is *not* the sum of Geroch's supermomentum with the Geroch–Winicour charge. For an exact Killing vector field ξ^a in the physical spacetime, the WZ charge agrees with the charge that is given by the Komar formula. We discuss in more detail appendix C2. To see this, note that, for exact Killing vector fields, the linkage charge agrees with the Komar formula for Killing vector fields which asymptotically become Lorentz vector fields at S , and agrees up to a factor of two for Killing vector fields that become translations at S . The WZ charge, similarly, equals the linkage charge for Lorentz vector fields at S , and up to a factor of two for translations; as such, the WZ charge and the Komar formula agree for exact Killing vector fields. Note that the Komar formula is (up to constant factors) the integral of the Noether charge in equation (5.9).

By the general arguments following equation (5.24), the change in the charge equation (5.44) between two cross-sections is given by the flux formula equation (5.33). However, showing this explicitly is a non-trivial computation, which we detail in the remainder of this section.

Let S_2 and S_1 be any two cross-sections of $\mathcal{S}$, with S_2 to the future of S_1 , and let $\Delta\mathcal{S}$ be the portion of $\mathcal{S}$ bounded by these cross-sections. Then, the change in the charge is given by

$$\begin{aligned} \mathcal{F}[\xi; \Delta\mathcal{S}] \hat{=} \mathcal{Q}[\xi; S_2] - \mathcal{Q}[\xi; S_1] \hat{=} & -\frac{1}{8\pi} \int_{\Delta\mathcal{S}} \epsilon_3 \mathcal{L}_n \left[\beta(\mathcal{P} + \frac{1}{2} \sigma^{ab} N_{ab}) + X^a \mathcal{J}_a \right. \\ & \left. + X^a \sigma_{ab} \mathcal{D}_c \sigma^{bc} - \frac{1}{4} \sigma_{ab} \sigma^{ab} \mathcal{D}_c X^c \right]. \end{aligned} \quad (5.45)$$

Note that we have dropped the τ_a terms from the expression since they do not contribute to the charge as explained above.

Now we simplify equation (5.45) term-by-term starting with the first and second terms. Using $\mathcal{J}_a n^a \hat{=} 0$, as well as equations (2.25b), (2.25d), (2.37), (4.21a) and (4.21b) and the

integration-by-parts formula equation (2.22), we see that the first two terms in equation (5.45) contribute

$$-\frac{1}{8\pi} \int_{\Delta\mathcal{S}} \epsilon_3 \left[N^{ab} \left(\frac{1}{4} \beta N_{ab} + \frac{1}{2} (\mathcal{D}_a + \tau_a)(\mathcal{D}_b - \tau_b)\beta + (\mathcal{D}_a + \tau_a)(X^c \sigma_{bc}) \right. \right. \\ \left. \left. + \frac{1}{4} \sigma_{ab}(\mathcal{D}_c - \tau_c)X^c \right) - \frac{1}{2} \epsilon_a{}^b X^a (\mathcal{D}_b + 3\tau_b) \mathcal{P}^* \right], \quad (5.46)$$

to the flux. Now consider the contribution of the third term in equation (5.45):

$$\mathcal{L}_n(X^a \sigma_{ab} \mathcal{D}_c \sigma^{bc}) \hat{=} \frac{1}{2} X^a N_{ab} \mathcal{D}_c \sigma^{bc} + X^a \sigma_{ab} \mathcal{L}_n \mathcal{D}_c \sigma^{bc} \quad (5.47)$$

where we have used equations (4.21b) and (2.32). For the last term above we have

$$\begin{aligned} \mathcal{L}_n \mathcal{D}_c \sigma^{bc} &\hat{=} \mathcal{L}_n(Q^a{}_c Q^b{}_d) \nabla_a \sigma^{cd} + Q^a{}_c Q^b{}_d \mathcal{L}_n \nabla_a \sigma^{cd} \\ &\hat{=} (Q^a{}_c n^b \tau_d + Q^b{}_d n^a \tau_c) \nabla_a \sigma^{cd} + \frac{1}{2} \mathcal{D}_a N^{ab} \\ &\hat{=} (Q^a{}_c n^b \tau_d) \nabla_a \sigma^{cd} + \frac{1}{2} (\mathcal{D}_a + \tau_a) N^{ab}, \end{aligned} \quad (5.48)$$

where we have made liberal use of the Bondi condition (equation (2.6)) along with equations (2.11) and (2.18), and commuted the $\mathcal{L}_n$ past the ∇_a using $\sigma^{ab} n_a \hat{=} Q_{ab} n^a \hat{=} 0$. Using the above expression in equation (5.47) the first term vanishes by $\sigma_{ab} n^b \hat{=} 0$ and we get

$$\mathcal{L}_n(X^a \sigma_{ab} \mathcal{D}_c \sigma^{bc}) \hat{=} \frac{1}{2} X^a N_{ab} \mathcal{D}_c \sigma^{bc} + \frac{1}{2} (\mathcal{D}_c + \tau_c)(X^a \sigma_{ab} N^{bc}) - \frac{1}{2} N_{bc} \mathcal{D}^b (X_a \sigma^{ac}). \quad (5.49)$$

Note that the second term on the right-hand side above drops out of the flux formula using equation (2.22).

Consider now the fourth term in equation (5.45). Since $\alpha_{(\xi)} \hat{=} \frac{1}{2} \mathcal{D}_a X^a$ and $\mathcal{L}_n \alpha_{(\xi)} \hat{=} 0$ (equation (4.7)), using equation (2.32) we get

$$-\frac{1}{4} \mathcal{L}_n(\sigma_{ab} \sigma^{ab} \mathcal{D}_c X^c) \hat{=} -\frac{1}{4} \sigma^{ab} N_{ab} \mathcal{D}_c X^c. \quad (5.50)$$

Putting together equations (5.46), (5.49) and (5.50), we see that the flux is given by

$$\begin{aligned} \mathcal{F}[\xi; \Delta\mathcal{S}] &\hat{=} -\frac{1}{8\pi} \int_{\Delta\mathcal{S}} \epsilon_3 \left[N^{ab} \left(\frac{1}{4} \beta N_{ab} + \frac{1}{2} (\mathcal{D}_a + \tau_a)(\mathcal{D}_b - \tau_b)\beta \right. \right. \\ &\quad \left. \left. + \frac{1}{4} (\mathcal{D}_a + 2\tau_a)(X^c \sigma_{bc}) - \frac{1}{2} \sigma_{ab} \tau_c X^c + \frac{1}{2} X^a \mathcal{D}_c \sigma^{bc} \right) - \frac{1}{2} \epsilon_a{}^b X^a (\mathcal{D}_b + 3\tau_b) \mathcal{P}^* \right]. \end{aligned} \quad (5.51)$$

Next we simplify the last term using the identity equation (2.38) for $\mathcal{P}^*$. The term arising from this identity which involves derivatives of the shear reads $\epsilon_3 \epsilon^{de} \epsilon_a{}^b X^a (\mathcal{D}_b + 3\tau_b) [\mathcal{D}_d (\mathcal{D}_c - \tau_c) \sigma_e{}^c]$. This term can be shown to vanish upon integrating over the cross-sections as follows. Note that this term is conformally-invariant (which can be seen using equation (2.38) and the conformal weights given in equations (2.28), (2.29), (2.31) and (4.11)). Therefore, we can evaluate this term in the Bondi frame where the metric on the cross-sections is chosen to be the unit

round metric and $\tau_a \hat{=} 0$. This allows us to make use of spherical harmonics— σ_{ab} is a symmetric and trace-free tensor and thus is supported on $\ell \geq 2$ tensor spherical harmonics while X^a is supported on $\ell = 1$ vector spherical harmonics (see remark 4.1). Using the orthogonality of the spherical harmonics, the integral of this term over the cross-sections vanishes. The remaining term arising from equation (2.38) contains the News tensor which, after an integration-by-parts using equation (2.22), becomes

$$-\frac{1}{4}\varepsilon^{ab}\varepsilon_e{}^d N_{ac}\sigma_b{}^c(\mathcal{D}_d - 2\tau_d)X^e \hat{=} \frac{1}{2}N_{ab}\sigma_b{}^c\mathcal{D}^{[a}X^{c]} - N_{ab}\sigma_b{}^cX^k\tau^a{}_k. \quad (5.52)$$

Replacing the above for the last term in equation (5.51) we get

$$\begin{aligned} \mathcal{F}[\xi; \Delta\mathcal{J}] \hat{=} & -\frac{1}{8\pi}\int_{\Delta\mathcal{J}}\varepsilon_3 N^{ab}\left[\frac{1}{4}\beta N_{ab} + \frac{1}{2}(\mathcal{D}_a + \tau_a)(\mathcal{D}_b - \tau_b)\beta\right. \\ & \left. + \frac{1}{2}(\mathcal{D}_a + 2\tau_a)(X^c\sigma_{bc}) - \frac{1}{2}\sigma_{ab}\tau_c X^c + \frac{1}{2}X_a\mathcal{D}^c\sigma_{bc} + \frac{1}{2}\sigma_b{}^c\mathcal{D}_{[a}X_{c]} - \sigma_b{}^cX_{[c}\tau_{a]} \right]. \end{aligned} \quad (5.53)$$

The last term on the first line and first term on the second line in the expression above can together be written as

$$N^{ab}(X^c\sigma_{bc}\tau_a - \frac{1}{2}\sigma_{ab}\tau_c X^c) \hat{=} N^{ab}\sigma_b{}^c(X_c\tau_a - \frac{1}{2}Q_{ca}X_d\tau^d). \quad (5.54)$$

Moreover, it follows from equation (D.3) that $N_{ab}\sigma_b{}^c \hat{=} \text{pure trace term} + N_{b[a}\sigma_{c]}{}^b$ and so equation (5.54) becomes $N^{ab}\sigma_b{}^cX_{[c}\tau_{a]}$. This exactly cancels the last term in equation (5.53), and therefore the flux formula simplifies to

$$\begin{aligned} \mathcal{F}[\xi; \Delta\mathcal{J}] \hat{=} & -\frac{1}{8\pi}\int_{\Delta\mathcal{J}}\varepsilon_3 N^{ab}\left[\frac{1}{4}\beta N_{ab} + \frac{1}{2}(\mathcal{D}_a + \tau_a)(\mathcal{D}_b - \tau_b)\beta\right. \\ & \left. + \frac{1}{2}\mathcal{D}_a(X^c\sigma_{bc}) + \frac{1}{2}X_a\mathcal{D}^c\sigma_{bc} + \frac{1}{2}\sigma_b{}^c\mathcal{D}_{[a}X_{c]} \right]. \end{aligned} \quad (5.55)$$

The second line above can be simplified using equation (D.5) to get

$$\mathcal{F}[\xi; \Delta\mathcal{J}] \hat{=} -\frac{1}{16\pi}\int_{\Delta\mathcal{J}}\varepsilon_3 N^{ab}\left[\frac{1}{2}\beta N_{ab} + (\mathcal{D}_a + \tau_a)(\mathcal{D}_b - \tau_b)\beta + \mathcal{L}_X\sigma_{ab} - \frac{1}{2}(\mathcal{D}_c X^c)\sigma_{ab}\right] \quad (5.56)$$

which matches the WZ flux derived in equation (5.33).

6. Expressions in some coordinate systems

Our entire preceding analysis was completely covariant, without referring to any particular coordinate systems. In this section, we consider two examples of coordinate systems for the unphysical spacetime in a neighbourhood of null infinity. These coordinates can be used to also obtain asymptotic coordinates for the physical spacetime. We will find the asymptotic form of the metric, both physical and unphysical, and derive the expressions for the BMS symmetries and their charges in these coordinates.

As described in section 3, one can construct a geometrically defined coordinate system (Ω, u, x^A) at $\mathcal{I}$ where x^A are coordinates on the cross-sections of $\mathcal{I}$, u satisfies $n^a \nabla_a u \hat{=} 1$ and the conformal factor Ω is chosen so that the Bondi condition is satisfied and so that the induced metric on the cross-sections is the unit round metric s_{AB} on $\mathbb{S}^2$. Then for any asymptotically flat spacetime the line element of the unphysical metric at $\mathcal{I}$ is

$$ds^2 \hat{=} 2 d\Omega du + s_{AB} dx^A dx^B. \quad (6.1)$$

These coordinates can be extended away from $\mathcal{I}$ in different ways, and these give rise to the different coordinates that are often used in the analysis of asymptotic symmetries and their associated charges. Two commonly used coordinates are the Bondi–Sachs coordinates and conformal Gaussian null coordinates, and we will focus on these in the remainder of this section. We emphasize that the form of the (physical or unphysical) metric in these coordinates follows directly from the construction of the coordinate systems and the covariant definition of asymptotic flatness without any additional assumptions. We also show that in any such coordinate system one can obtain the BMS symmetries as coordinate transformations which preserve the asymptotic form of the metric.

6.1. Bondi–Sachs coordinates

One way to extend the coordinates described above away from $\mathcal{I}$ is as follows. Let S_u be cross-sections of $\mathcal{I}$ with $u = \text{constant}$, and consider a family of null surfaces $\mathcal{N}_u$ which intersect $\mathcal{I}$ transversely in the cross-sections S_u . These surfaces $\mathcal{N}_u$ foliate a neighbourhood of $\mathcal{I}$ by null surfaces. We first extend the coordinate u away from $\mathcal{I}$ so that it is constant along each null surface $\mathcal{N}_u$. Then $l_a := -\nabla_a u$ is the null normal to each $\mathcal{N}_u$ with $l^a l_a = 0$ in addition to $l^a n_a \hat{=} -1$ as above. Then we extend the angular coordinates x^A on each cross-section S_u by parallel transport i.e. $l^a \nabla_a x^A = 0$.

Fixing the induced metric on cross sections of $\mathcal{I}$ to be the unit round sphere metric fixes the conformal factor, Ω , on $\mathcal{I}$. To extend Ω away from $\mathcal{I}$, we use the freedom in the conformal factor off of $\mathcal{I}$ to demand that the two-spheres at constant u and Ω have the same area element as the unit sphere, that is, if h_{AB} is the two-metric on the surfaces of constant u and Ω then we demand that $\det h = \det s$ in the x^A -coordinates. This fixes Ω uniquely away from $\mathcal{I}$. Thus we have set up the conformal Bondi–Sachs coordinate system (Ω, u, θ^A) in a neighborhood of $\mathcal{I}$.

The most general form of the unphysical metric in conformal Bondi–Sachs coordinates is given by¹³

$$ds^2 \hat{=} -W e^{2B} du^2 + 2e^{2B} d\Omega du + h_{AB}(dx^A - U^A du)(dx^B - U^B du), \quad (6.2)$$

where the metric components $g_{\Omega\Omega}$ and $g_{\Omega A}$ vanish everywhere due to the conditions $l^a l_a = l^a \nabla_a x^A = 0$, and W , B , h_{AB} , and U^A are smooth functions of the coordinates (Ω, u, x^A) . Since the metric at $\mathcal{I}$ is given by equation (6.1) we also have

$$W = O(\Omega), \quad B = O(\Omega), \quad U^A = O(\Omega), \quad h_{AB} = s_{AB} + O(\Omega). \quad (6.3)$$

Further, evaluating the Bondi condition $\nabla_a n_b \hat{=} 0$ (equation (2.6)) gives

$$W = O(\Omega^2), \quad B = O(\Omega^2), \quad U^A = O(\Omega^2). \quad (6.4)$$

¹³ Note that the function we denote by B is conventionally denoted by β , but we use a different symbol to avoid conflict with the n^a -component of a BMS vector field.

We therefore consider the following expansion of the metric components

$$\begin{aligned} W &= \Omega^2 W^{(2)} - 2\Omega^3 M + O(\Omega^4), & U^A &= \Omega^2 U^{(2)A} + 2\Omega^3 L^A + O(\Omega^4), \\ B &= \Omega^2 B^{(2)} + O(\Omega^3), & h_{AB} &= s_{AB} + \Omega C_{AB} + \Omega^2 d_{AB} + O(\Omega^3). \end{aligned} \quad (6.5)$$

Next, imposing $\det h = \det s$, we get

$$s^{AB} C_{AB} = 0, \quad s^{AB} d_{AB} = \frac{1}{2} C^{AB} C_{AB}. \quad (6.6)$$

We then impose the Einstein equation, equation (2.1), order by order in Ω . At $O(\Omega^0)$, it gives

$$W^{(2)} = 1, \quad B^{(2)} = -\frac{1}{32} C^{AB} C_{AB}, \quad U_A^{(2)} = -\frac{1}{2} \mathcal{D}^B C_{AB}, \quad (6.7)$$

while at $O(\Omega)$ we get

$$\partial_u \text{STF } d_{AB} = \mathcal{D}^A \text{STF } d_{AB} = 0. \quad (6.8)$$

Since d_{AB} is a smooth tensor on a two-sphere this implies that $\text{STF } d_{AB} = 0$ and thus (from equation (6.6))

$$d_{AB} = \frac{1}{4} C^{CD} C_{CD} s_{AB}. \quad (6.9)$$

The metric component C_{AB} is related to the shear and the News tensor while M and L_A are related to the Weyl tensor components (equation (2.24)) through

$$\begin{aligned} \sigma_{AB} &= -\frac{1}{2} C_{AB}, & N_{AB} &= -\partial_u C_{AB}, \\ \mathcal{P} &= -2M + \frac{1}{4} C_{AB} N^{AB}, \\ \mathcal{J}_A &= 3L_A - \frac{3}{32} \mathcal{D}_A (C_{BC} C^{BC}) - \frac{3}{4} C_A{}^B \mathcal{D}^C C_{BC}. \end{aligned} \quad (6.10)$$

Imposing the Einstein equation to higher order in Ω either relates the higher order metric components to the lower order ones or gives evolution equations along u —for instance, one gets equations for $\partial_u M$ and $\partial_u L_A$ which are equivalent to equation (2.25) using equation (6.10). We will not need the explicit form of these higher order equations in our analysis.

The conformal Bondi–Sachs coordinates defined above can be used to define the *physical Bondi–Sachs coordinates* (r, u, x^A) which are often used in the asymptotic analysis near $\mathcal{I}$.¹⁴ Define the physical ‘radial coordinate’ $r := \Omega^{-1}$, and using (r, u, x^A) as coordinates, the physical metric $\hat{g}_{ab} = \Omega^{-2} g_{ab} = r^2 g_{ab}$ has the line element

$$d\hat{s}^2 \equiv -U e^{2B} du^2 - 2e^{2B} du dr + r^2 h_{AB} (dx^A - U^A du)(dx^B - U^B du), \quad (6.11)$$

¹⁴ We present these equations in the two-sphere covariant form appearing in [44], and in a more modern form in [22, 45], instead of the original notation of [1].

where, from equations (6.2), (6.5), (6.6) and (6.9), we have the asymptotic expansions

$$\begin{aligned}
 U &= r^2 W = 1 - \frac{2}{r} M + O(1/r^2), \\
 B &= -\frac{1}{32r^2} C^{AB} C_{AB} + O(1/r^3), \\
 U^A &= -\frac{1}{2r^2} \mathcal{D}_B C^{AB} + \frac{2}{r^3} L^A + O(1/r^4), \\
 h_{AB} &= s_{AB} + \frac{1}{r} C_{AB} + \frac{1}{4r^2} s_{AB} C^{CD} C_{CD} + O(1/r^3).
 \end{aligned} \tag{6.12}$$

Note that since Ω , and hence r , is chosen so that $\det h = \det s$, the area (in the physical metric) of the two-spheres of constant u and r is precisely $4\pi r^2$. Thus r is a ‘radial coordinate’ along the outgoing null surfaces $\mathcal{N}_u$ as constructed by Bondi and van der Burg [1]. We emphasize that the Bondi–Sachs asymptotic form of the physical metric (equations (6.11) and (6.12)) directly follows from the covariant definition of asymptotic flatness in a particular choice of coordinate system and *does not* involve any additional assumptions.

Having chosen the conformal Bondi–Sachs coordinate system, the asymptotic BMS symmetries are coordinate transformations which preserve the Bondi–Sachs form of the unphysical metric. We shall only consider the infinitesimal coordinate transformations, i.e. we take (Ω, u, x^A) and (Ω', u', x'^A) to be two conformal Bondi–Sachs coordinates, as constructed above, related by an infinitesimal coordinate transformation parametrized by a vector field ξ^a . We next obtain an expression for this vector field in the coordinate system (Ω, u, x^A) .

Since in both coordinates (Ω, u, x^A) and (Ω', u', x'^A) , null infinity $\mathcal{I}$ lies at $\Omega = \Omega' = 0$, the component ξ^Ω vanishes at $\Omega = 0$, i.e. ξ^a must be tangent to $\mathcal{I}$. Thus, ξ^a can be written in the coordinate system (Ω, u, x^A) as

$$\xi^a \equiv \beta \partial_u + X^A \partial_A + \Omega Z^a \partial_a + \Omega^2 W^a \partial_a + O(\Omega^3), \tag{6.13}$$

where each of β, X^A, Z^a and W^a are functions of (u, x^A) .

Next we note that, since we are using the conformal factor itself as a coordinate, an infinitesimal change in the coordinate system is accompanied by an infinitesimal change of the conformal factor parametrized by the component ξ^Ω . Thus, when changing the coordinate system, the unphysical metric changes infinitesimally by $\mathcal{L}_\xi g_{ab} - 2\Omega^{-1} \xi^\Omega g_{ab}$; note that this is finite at $\mathcal{I}$ since ξ^Ω vanishes there. We now require that this change preserve the Bondi–Sachs form of the metric obtained in equation (6.2) along with equation (6.5) and the equations below.

As discussed above, the metric on $\mathcal{I}$, given by equation (6.1), is universal and so we require that at $O(\Omega^0)$, $\mathcal{L}_\xi g_{ab} - 2\Omega^{-1} \xi^\Omega g_{ab} \hat{=} 0$. This gives us the following conditions [46]: X^A is constant along u and satisfies the conformal Killing equation on the cross-sections, that is,

$$\mathcal{D}_{(A} X_{B)} = \frac{1}{2} q_{AB} \mathcal{D}_C X^C, \quad \partial_u X^A = 0, \tag{6.14}$$

while the components of Z^a satisfy

$$Z^\Omega = \partial_u \beta = \frac{1}{2} \mathcal{D}_A X^A, \quad Z^u = 0, \quad Z_A = -\mathcal{D}_A \beta. \tag{6.15}$$

The first condition in equation (6.15) allows us to solve for the u -dependence of β to get

$$\beta = f + \frac{1}{2}(u - u_0)\mathcal{D}_A X^A, \quad (6.16)$$

where f is an arbitrary function of x^A which denotes the value of β at some choice of cross-section with $u = u_0$.

At $O(\Omega)$, requiring that the form of the unphysical metric be preserved, we further obtain

$$W^u = 0, \quad W_A = \frac{1}{2}C_{AB}\mathcal{D}^B\beta, \quad (6.17)$$

and the requirement that the metric component C_{AB} remain trace-free with respect to the unit round sphere metric gives us

$$W^\Omega = \frac{1}{2}\mathcal{D}_A Z^A = -\frac{1}{2}\mathcal{D}^2\beta. \quad (6.18)$$

At this order one also obtains the transformation of C_{AB} under a BMS symmetry which, upon using equation (6.10), coincides with equation (4.33) (see also equation (2.18b) of [22]).

Putting all of this together, we obtain

$$\xi^a \equiv \beta\partial_u + X^A\partial_A + \Omega(-\mathcal{D}^A\beta\partial_A + \frac{1}{2}\mathcal{D}_A X^A\partial_\Omega) + \frac{1}{2}\Omega^2(-\mathcal{D}^2\beta\partial_\Omega + C^{AB}\mathcal{D}_B\beta\partial_A) + O(\Omega^3). \quad (6.19)$$

Up to $O(\Omega)$, this expression agrees with the covariant expression derived in section 4. The form of the $O(\Omega^2)$ terms here is fixed by the choice of the conformal Bondi–Sachs coordinates and that also matches equation (2.16) of [22]. One could continue this computation to higher orders in Ω , which gives expressions for higher order terms of the components of ξ^a (which appear in equations (3.5)–(3.7) of [46]) and also the transformation laws for the various metric components, but we will not require these expressions.

Using the relations equation (6.10) in equation (5.44) the WZ charge on *any* cross-section S of $\mathcal{I}$ can be written in terms of the metric components in the Bondi–Sachs form of the metric to get

$$\mathcal{Q}[\xi, S] \hat{=} -\frac{1}{8\pi}\int_S \epsilon_2 \left[-2M\beta + X^A \left(3L_A - \frac{1}{32}\mathcal{D}_A(C_{BC}C^{BC}) - \frac{1}{2}C_A{}^B\mathcal{D}^C C_{BC} \right) \right]. \quad (6.20)$$

The above charge expression matches the charge expression given by Flanagan and Nichols in equation (3.5) of [22], even though their expression was calculated on a cross section of $\mathcal{I}$ where the News vanishes. A similar expression was also obtained by Barnich and Troessaert [24] with non-vanishing News but they do not find the integrable charge.

From the above expression we also see that the function M determines the Bondi mass at any cross-section of $\mathcal{I}$ and can be called the *mass aspect*. In the Bondi–Sachs coordinates it coincides with the (constant) mass parameter of Kerr spacetimes. Similarly, in the usual choice of Bondi–Sachs coordinates in Kerr spacetime the angular momentum parameter a appears in the metric component L_A [25]. Note however, on cross-sections with shear, there are other terms containing the C_{AB} in the charge formulae. In any case, since there is no preferred Lorentz subalgebra in the BMS symmetries (see remark 4.2), there is no preferred notion of ‘angular momentum aspect’—so identifying any particular metric component with ‘the angular momentum’ is a moot point.

6.2. Conformal Gaußian null coordinates

Instead of extending the coordinates (Ω, u, x^A) away from $\mathcal{S}$ along null hypersurfaces, we can extend them into the (unphysical) spacetime along affine null geodesics, transverse to $\mathcal{S}$. We recall this construction below which leads to the *conformal Gaußian null coordinates* in a neighborhood of $\mathcal{S}$.

We fix the conformal factor to leading order away from $\mathcal{S}$ as follows. Consider the expansion of l^a at $\mathcal{S}$ defined by

$$\vartheta := Q^{ab} \nabla_a l_b \hat{=} \frac{1}{2} Q^{ab} \mathcal{L}_l g_{ab}. \quad (6.21)$$

We can set this expansion to vanish by suitably choosing the conformal factor away from $\mathcal{S}$ as follows. Note that the conformal factor on $\mathcal{S}$ has already been fixed so that the induced metric on cross-sections of $\mathcal{S}$ is the unit-sphere metric. So consider a new conformal factor $\tilde{\Omega} = \omega \Omega$ (with $\omega \hat{=} 1$) so that $\tilde{g}_{ab} = \omega^2 g_{ab}$. Then, the expansion of the new auxiliary normal $\tilde{l}^a \hat{=} l^a$ can be computed to be (the behaviour of the auxiliary normal away from $\mathcal{S}$ is not relevant here)

$$\tilde{\vartheta} \hat{=} \frac{1}{2} \tilde{Q}^{ab} \mathcal{L}_{\tilde{l}} \tilde{g}_{ab} \hat{=} \vartheta + 2 \mathcal{L}_l \omega. \quad (6.22)$$

Then, choosing ω to be any solution of $\vartheta + 2 \mathcal{L}_l \omega = 0$ with $\omega \hat{=} 1$ we can set $\tilde{\vartheta} \hat{=} 0$. In the rest of this section we work the choice of conformal factor where the auxiliary normal is expansion-free at $\mathcal{S}$ —we drop the ‘tilde’ from the notation.

Having made this choice, we then extend the vector field $l^a \hat{=} -\partial/\partial\Omega$ away from $\mathcal{S}$ such that it is the generator of affine null geodesics so that $l^b \nabla_b l^a = 0$. We can further use the remaining freedom in the conformal factor Ω so that Ω is the affine parameter along these null geodesics generated by l^a . To summarize, we can always choose the conformal factor Ω and extend the auxiliary normal away from $\mathcal{S}$ so that in a neighbourhood of $\mathcal{S}$ we have

$$l^a = -\frac{\partial}{\partial\Omega}, \quad \vartheta \hat{=} 0, \quad l^a l_a = 0, \quad l^b \nabla_b l^a = 0. \quad (6.23)$$

Finally, we extend (u, x^A) into the spacetime by parallel-transport along l^a , that is, we require

$$l^a \nabla_a u = l^a \nabla_a x^A = 0. \quad (6.24)$$

This construction gives us the conformal Gaußian null coordinates in a neighbourhood of $\mathcal{S}$.

The most general form of the unphysical metric in these coordinates is given by

$$ds^2 = 2 du(d\Omega - \alpha du - \beta_A dx^A) + h_{AB} dx^A dx^B. \quad (6.25)$$

To see why this is the most general form, note that $g_{\Omega\Omega} = 0$ by $l^a l_a = 0$ and $g_{\Omega A} = 0$ by $l^a \nabla_a x^A = 0$. Then, $l^b \nabla_b l^a = 0$ gives $\frac{\partial g_{u\Omega}}{\partial\Omega} = 0$ which implies $g_{u\Omega} = 1$ by the condition $l^a n_a \hat{=} -1$. Further,

$$\alpha = O(\Omega), \quad \beta_A = O(\Omega), \quad h_{AB} = s_{AB} + O(\Omega), \quad (6.26)$$

since the metric at $\mathcal{S}$ is given by equation (6.1). Imposing the Bondi condition $\nabla_a n_b \hat{=} 0$ leads to the following conditions

$$\alpha = O(\Omega^2), \quad \beta_A = O(\Omega^2). \quad (6.27)$$

We therefore consider the asymptotic expansions

$$\begin{aligned}\alpha &= \Omega^2 \alpha^{(2)} + \Omega^3 \alpha^{(3)} + O(\Omega^4), & \beta_A &= \Omega^2 \beta_A^{(2)} + \Omega^3 \beta_A^{(3)} + O(\Omega^4) \\ h_{AB} &= s_{AB} + \Omega C_{AB} + \Omega^2 h_{AB}^{(2)} + O(\Omega^3).\end{aligned}\quad (6.28)$$

The condition that the expansion of l^a vanishes on $\mathcal{I}$ gives us

$$s^{AB} C_{AB} = 0. \quad (6.29)$$

We then impose the Einstein equation (2.1). At $O(\Omega^0)$, this gives the conditions

$$\alpha^{(2)} = \frac{1}{2}, \quad \beta_A^{(2)} = -\frac{1}{2} \mathcal{D}^B C_{AB}, \quad s^{AB} h_{AB}^{(2)} = \frac{1}{4} C^{AB} C_{AB}, \quad (6.30)$$

while at $O(\Omega)$, it implies

$$\mathcal{D}^B h_{AB}^{(2)} = \frac{1}{8} \mathcal{D}_A (C_{BC} C^{BC}), \quad \partial_u h_{AB}^{(2)} = \frac{1}{8} s_{AB} \partial_u (C_{CD} C^{CD}). \quad (6.31)$$

As in the Bondi–Sachs case, this implies that $h_{AB}^{(2)}$ is pure trace.

The metric coefficients C_{AB} , $\alpha^{(3)}$ and $\beta_A^{(3)}$ are directly related to the shear and the Weyl tensor components as follows

$$\sigma_{AB} = -\frac{1}{2} C_{AB}, \quad \mathcal{P} = 2\alpha^{(3)}, \quad \mathcal{J}_A = \frac{3}{2} \beta_A^{(3)}. \quad (6.32)$$

One could continue computing the Einstein equations to higher orders but we will not need to do so.

To write an expression for the physical metric define $\lambda := \Omega^{-1}$ so that in the coordinates (λ, u, x^A) the physical metric $\hat{g}_{ab} = \Omega^{-2} g_{ab} = \lambda^2 g_{ab}$ has the components

$$\begin{aligned}\hat{g}_{\lambda\lambda} &= 0, & \hat{g}_{A\lambda} &= 0, & \hat{g}_{u\lambda} &= -1, \\ \hat{g}_{uu} &= -1 - \frac{1}{\lambda} \mathcal{P} + O(1/\lambda^2), \\ \hat{g}_{uA} &= \frac{1}{2} \mathcal{D}^B C_{AB} - \frac{2}{3\lambda} \mathcal{J}_A + O(1/\lambda^2), \\ \hat{g}_{AB} &= \lambda^2 s_{AB} + \lambda C_{AB} + \frac{1}{8} s_{AB} C_{CD} C^{CD} + O(1/\lambda).\end{aligned}\quad (6.33)$$

Note that the vector field

$$\hat{l}^a \equiv \frac{\partial}{\partial \lambda} = -\Omega^2 \frac{\partial}{\partial \Omega} = \Omega^2 l^a, \quad (6.34)$$

generates outgoing null geodesics which are affinely parametrized with respect to the physical metric $\hat{g}_{ab}$ with the affine parameter being λ . Equation (6.33) is consistent with equations (5) and (58) of [13] put together (with the additional condition that our C_{AB} is tracefree since we picked l^a to be expansion-free; see remark 6.1). This also gives the physical metric in the coordinates used in the *affine-null* form used in [47, 48], as well as the Newman–Unti coordinates [49].

We can also derive the form of the BMS vector fields by considering infinitesimal coordinate transformations between two conformal Gaussian null coordinate systems and demanding that

the conditions on the unphysical metric derived above be preserved. Since the computation proceeds exactly as in the case of Bondi–Sachs coordinates detailed above, we skip the details. The end result is that the BMS vector field in conformal Gaussian null coordinates takes the same form as equation (6.19) above—the difference in the form of the BMS vector fields written in conformal Gaussian null coordinates and conformal Bondi–Sachs coordinates only appears at $O(\Omega^3)$ and higher. Note that this form is different than the one obtained by [13] since our coordinates differ slightly from theirs as explained in remark 6.1 below. The WZ charge (equation (5.44)) can be straightforwardly written in these coordinates using equation (6.32).

Remark 6.1 (Comparison of different conformal Gaussian null coordinates).

The conformal Gaussian null coordinates constructed in this section are closely related to, but not the same as, the ones used in [13, 50, 51]. Note that these references use the freedom in the conformal factor to set the metric coefficient $\alpha^{(2)} = 1/2$ (as in equation (6.30)) and then the Einstein equations imply that the expansion ϑ of the auxiliary normal l^a is constant along n^a on $\mathcal{S}$, i.e. $\mathcal{L}_n \vartheta \doteq 0$. In contrast, we used the conformal freedom to set $\vartheta \doteq 0$ and then $\alpha^{(2)} = 1/2$ follows from the Einstein equation.

7. Discussion

The aim of this paper was to obtain, using the WZ prescription, manifestly covariant expressions for the charges and fluxes corresponding to the BMS symmetries at null infinity in asymptotically flat spacetimes in vacuum general relativity. While (special cases of) these expressions have appeared in various places in the literature, they are usually written in a way that obscures their covariant and conformally-invariant nature, for example, by restricting to specific coordinate frames near null infinity. The expressions we obtained are manifestly covariant, conformally-invariant and do not rely on a preferred choice of foliation of null infinity or on any arbitrary extension of the BMS symmetries away from null infinity. We also recast our charge expression in two specific choices of coordinate frames near null infinity and showed, for example, that in Bondi–Sachs coordinates, it reduces to the expression obtained by Flanagan and Nichols, even on cross-sections of $\mathcal{S}$ where the News tensor is non-vanishing. In appendices C1–C3, we compare the WZ charge and flux formulae to some other expressions that have appeared in the literature, including the Ashtekar–Streubel flux formula, the Komar and linkage formulae and Penrose’s twistor charge formula. In particular, we explicitly show that the flux of the WZ charge matches the flux expression of Ashtekar and Streubel.

While our analysis was limited to null infinity in asymptotically flat spacetimes in vacuum general relativity, the WZ prescription is, in fact, much more general and can be used to obtain local and covariant charges for arbitrary diffeomorphism covariant Lagrangian theories of gravity including gravity coupled to electromagnetism [37] and Brans–Dicke theory [52]. The WZ prescription has also been applied to the context of symmetries and charges associated with finite null surfaces and horizons [53, 54] and spatial infinity [55] in asymptotically flat spacetimes in vacuum general relativity. We hope that the explicit computations presented here will be useful for similar analyses in other contexts, for instance, in spacetimes with compact extra dimensions [56].

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Data availability statement

No new data were created or analysed in this study.

Appendix A. General choices of conformal factor

In the body of the paper, we worked in a conformal frame where

$$\Phi := \frac{1}{4} \nabla_a n^a \hat{=} 0, \quad (\text{A.1})$$

which, as a result of the Einstein equation (equation (2.1)), implies the conditions equations (2.6) and (2.7). This choice, however, was made purely for convenience and is not essential to the results of this paper. In this appendix, we state some of our main results in general conformal frames where $\Phi \neq 0$ and therefore the Bondi condition does not hold. In this context, one is allowed more general conformal transformations of the form

$$\Omega \mapsto \omega \Omega, \quad \text{where } \mathcal{L}_n \omega \neq 0. \quad (\text{A.2})$$

Using the fact that S_{ab} is smooth at $\mathcal{I}$, equation (2.1) implies that in general

$$\nabla_a n_b \hat{=} \Phi g_{ab}, \quad \lim_{\rightarrow \mathcal{I}} \Omega^{-1} n_a n^a = 2\Phi, \quad (\text{A.3})$$

which generalize equations (2.6) and (2.7). In these conformal frames, the pullback of the unphysical metric to $\mathcal{I}$, q_{ab} , satisfies

$$\mathcal{L}_n q_{ab} \hat{=} 2\Phi q_{ab}. \quad (\text{A.4})$$

It is important to note that Φ is universal in the sense of section 3, that is, it is independent of the physical spacetime under consideration and can, without any loss of generality, be picked to be the same for the conformal completion of any asymptotically-flat physical spacetime. Hence, $\delta\Phi = 0$ on phase space.

In these general conformal frames, evolution equations for components of the Weyl tensor on $\mathcal{I}$ given in equation (2.25) also get generalized and may be found in equation (3.3) of [4] (see also equation (B.13) for their expressions in the GHP formalism). Moreover, the definition of the News tensor is generalized to

$$N_{ab} := 2Q_a^c Q_b^d (\mathcal{L}_n - \Phi) \sigma_{cd}, \quad (\text{A.5})$$

(given by equation (B.15) in the GHP formalism) where σ_{ab} is (still) defined by equation (2.16) and has conformal weight one. In addition, equation (2.36) is generalized to

$$N_{ab} \hat{=} \text{STF} [S_{ab} - 2\Phi \sigma_{ab} + 2(\mathcal{D}_a + \tau_a) \tau_b], \quad (\text{A.6})$$

(which corresponds to equation (B.17) in the GHP formalism). Note that these expressions for the News tensor are defined in any conformal frame and are conformally invariant under all conformal transformations, including those of the form equation (A.2), whereas, for example,

the expression due to Geroch, discussed in equation (2.39), is only defined in frames where the Bondi condition holds.

In general conformal frames where $\Phi \neq 0$, a BMS symmetry is given by $\xi^a \triangleq \beta n^a + X^a$ where $X^a l_a \triangleq 0$. Here, β, X^a satisfy the relations (which are derived using the same method as for equation (4.21))

$$\begin{aligned} (\mathcal{L}_n - \Phi)\beta &\triangleq \frac{1}{2}(\mathcal{D}_a - 2\tau_a)X^a, & \mathcal{L}_n X^a &\triangleq n^a X_b \tau^b, \\ \text{STF } \mathcal{D}_a X_b &\triangleq 0, & \alpha_{(\xi)} &\triangleq \Phi\beta + \frac{1}{2}\mathcal{D}_a X^a, \end{aligned} \quad (\text{A.7})$$

which correspond to equation (B.22) in the GHP formalism. It is worth emphasizing that this is *not* a different symmetry algebra than the one considered in the body of the paper. The observation that, for example, in these conformal frames a pure supertranslation, $\xi^a \triangleq f n^a$, is no longer constrained to satisfy $\mathcal{L}_n f \triangleq 0$ and that f can have a non-trivial functional dependence on the normal direction along $\mathcal{S}$, should not be taken to mean that symmetry algebra has become bigger. The point is simply that supertranslations are associated with conformally-weighted functions which have different-looking functional forms in different conformal frames.

With these considerations all of our computations can be carried out in a similar manner; one only has to keep track of the chosen Φ through the calculations. For any BMS symmetry with β and X^a subject to equation (A.7) and N_{ab} given by equation (A.5), the expressions for the WZ flux (equation (5.56)) and charge (equation (5.44)), remain unchanged in these conformal frames and are given by

$$\mathcal{F}[\xi; \Delta\mathcal{S}] \triangleq -\frac{1}{16\pi} \int_{\Delta\mathcal{S}} \varepsilon_3 N^{ab} \left[\frac{1}{2}\beta N_{ab} + (\mathcal{D}_a + \tau_a)(\mathcal{D}_b - \tau_b)\beta + \mathcal{L}_X \sigma_{ab} - \frac{1}{2}(\mathcal{D}_c X^c)\sigma_{ab} \right], \quad (\text{A.8})$$

and

$$\mathcal{Q}[\xi; \mathcal{S}] \triangleq -\frac{1}{8\pi} \int_{\mathcal{S}} \varepsilon_2 \left[\beta(\mathcal{P} + \frac{1}{2}\sigma^{ab}N_{ab}) + X^a \mathcal{J}_a + X^a \sigma_{ab}(\mathcal{D}_c - \tau_c)\sigma^{bc} - \frac{1}{4}\sigma_{ab}\sigma^{ab}(\mathcal{D}_c - 2\tau_c)X^c \right], \quad (\text{A.9})$$

respectively. These expressions are invariant under all conformal transformations including those of the form equation (A.2).

Appendix B. The GHP formalism at $\mathcal{S}$

In this appendix, we include a brief review of the GHP formalism (see [4, 57, 58] for details) applied at null infinity. We also present some of the expressions written in the body of the paper, including those for the WZ charge and flux, in this formalism.

In order to be consistent with our conventions for the signature and Riemann curvature of the metric, we will use the sign conventions of [59].¹⁵ We pick a null tetrad $(n^a, l^a, m^a, \bar{m}^a)$ which is normalised such that $n_a l^a = -m_a \bar{m}^a = -1$, with all other inner products vanishing. The metric then takes the form $g_{ab} = -2n_{(a} l_{b)} + 2m_{(a} \bar{m}_{b)}$. It is easy to see that the form of the

¹⁵ Note however that l^a in the notation of [59] corresponds to our n^a while their k^a corresponds to our l^a .

metric is invariant under the following transformations,

$$l^a \mapsto \lambda \bar{\lambda} l^a, \quad n^a \mapsto (\lambda \bar{\lambda})^{-1} n^a, \quad m^a \mapsto \lambda (\bar{\lambda})^{-1} m^a, \quad (\text{B.1})$$

for any complex scalar λ . These are called *GHP transformations*. A scalar field ξ , associated with a choice of null tetrad, is said to have *GHP weight* (p, q) if, under the GHP transformations in equation (B.1), ξ transforms as

$$\xi \mapsto \lambda^p \bar{\lambda}^q \xi. \quad (\text{B.2})$$

We will indicate the GHP weight of such fields by writing $\xi \triangleq (p, q)$.

We now adapt the choice of the null tetrad to $\mathcal{S}$. Note that the conformal factor Ω , and therefore $\nabla^a \Omega$, are independent of the choice of null tetrad. To use the GHP formalism consistently with the transformations equation (B.1), we therefore need to define (on $\mathcal{S}$) a scalar field A such that

$$A n^a \triangleq \nabla^a \Omega \quad \text{where } A \triangleq (1, 1). \quad (\text{B.3})$$

When converting to tensor expressions used in the body of the paper, we set $A = 1$ at the end of the calculation. As in section 2.1, we choose l^a to be the auxiliary null normal to a foliation of $\mathcal{S}$. The tetrads $(m^a, \bar{m}^a)$ are then a *complex* orthonormal basis for the cross-sections of $\mathcal{S}$ in the chosen foliation, and are related to Q_{ab} and ε_{ab} by

$$Q_{ab} \triangleq 2m_{(a}\bar{m}_{b)}, \quad \varepsilon_{ab} \triangleq 2im_{[a}\bar{m}_{b]}. \quad (\text{B.4})$$

The definition of the GHP spin coefficients in our conventions can be found in equation (7.2) of [59]; see also footnote 12 above. This definition differs by a sign from the ones used in [57, 58]. At $\mathcal{S}$, using $\nabla_a \nabla_b \Omega|_{\mathcal{S}} = \Phi g_{ab}$ (from equation (A.3)) and equation (B.3) we obtain

$$\begin{aligned} \kappa' &\triangleq \sigma' \triangleq \tau' \triangleq 0, \\ \rho' &\triangleq -A^{-1}\Phi, \\ \epsilon &\triangleq -\frac{1}{2}(n^a l^b \nabla_b l_a - \bar{m}^a l^b \nabla_b m_a), \\ \beta &\triangleq \frac{1}{2}(m^a \nabla_a \ln A + \bar{m}^a m^b \nabla_b m_a), \\ \epsilon' &\triangleq \frac{1}{2}(-\mathcal{L}_n \ln A + A^{-1}\Phi + m^a n^b \nabla_b \bar{m}_a), \\ \beta' &\triangleq \frac{1}{2}(-\bar{m}^a \nabla_a \ln A + m^a \bar{m}^b \nabla_b \bar{m}_a), \end{aligned} \quad (\text{B.5})$$

while the spin coefficients $\kappa, \rho, \sigma, \tau$ are arbitrary. Note that when the conformal factor is chosen to satisfy the Bondi condition we have $\Phi \triangleq 0$ and thus $\rho' \triangleq 0$. Also, equation (B.5) implies that

$$(\rho' + \rho')A \triangleq \delta A \triangleq 0, \quad (\text{B.6})$$

which is equation (9.8.26) of [60].

Further, since l_a is chosen to be normal to the cross-sections of $\mathcal{S}$, we have $\rho = \bar{\rho}$ (i.e. the twist of l_a vanishes), and the expansion of l_a is given by

$$\vartheta \triangleq -2 \operatorname{Re} \rho \triangleq -2\rho. \quad (\text{B.7})$$

The spin coefficient κ depends on the choice of extension of l_a off of $\mathcal{S}$, while σ and τ are related to the tensors σ_{ab} and τ_a defined in equations (2.16) and (2.17) by

$$\begin{aligned}\sigma_{ab} &\hat{=} -\sigma \bar{m}_a \bar{m}_b - \bar{\sigma} m_a m_b, & \sigma &\hat{=} -\sigma_{ab} m^a m^b, \\ \tau_a &\hat{=} -\tau \bar{m}_a - \bar{\tau} m_a, & \tau &\hat{=} -\tau_a m^a.\end{aligned}\quad (\text{B.8})$$

Note that the spin coefficients $\epsilon, \beta, \epsilon', \beta'$ are not GHP-weighted quantities, and in fact their GHP transformations contain derivatives of λ . These are used to define the GHP *derivative operators* ($\flat, \flat', \eth, \eth'$) (see equation (7.41) of [59]) which map GHP-weighted scalars to (other) GHP-weighted scalars.

Recall that, under a change of conformal factor $\Omega \mapsto \omega \Omega$, the transformations of the metric and null tetrad on $\mathcal{S}$ are given by

$$g_{ab} \mapsto \omega^2 g_{ab} \quad (n^a, l^a, m^a, \bar{m}^a) \mapsto \omega^{-1} (n^a, l^a, m^a, \bar{m}^a), \quad (\text{B.9})$$

where ω has GHP-weight $(0, 0)$. Under this transformation, a scalar field ξ associated with the choice of null tetrad will have an additional conformal weight w if it transforms as $\xi \mapsto \omega^w \xi$. We will indicate the combined *conformal-GHP weight* of such scalars by writing $\xi \hat{=} (p, q; w)$.

Under the conformal transformation in equation (B.9), the GHP derivatives of a conformally-weighted GHP scalar ξ will, in general, pick up derivatives of ω . However, using the fact that the spin coefficients ρ' and τ are not conformally-weighted and transform as (see also equations (2.5) and (2.30))

$$\rho' \mapsto \omega^{-1} (\rho' - \flat' \ln \omega), \quad \tau \mapsto \omega^{-1} (\tau - \eth \ln \omega), \quad (\text{B.10})$$

we can ‘correct’ the conformal behaviour of the GHP derivatives by adding suitable combinations of ρ' and τ to define the *conformal-GHP derivatives*. For our purposes we only need the conformal-GHP derivatives tangential to $\mathcal{S}$, which are given by (see equation (5.6.36) of [61], where we set $w_0 = w_1 = -1/2$ as in equation (5.6.26) (iii) of [61] for compatibility with the conformal transformations of the null tetrad given in equation (B.9))

$$\begin{aligned}\left[\flat' + (w + \tfrac{1}{2}(p + q)) \rho' \right] \xi &\hat{=} (p - 1, q - 1; w - 1), \\ \left[\eth + (w - \tfrac{1}{2}(p - q)) \tau \right] \xi &\hat{=} (p + 1, q - 1; w - 1), \\ \left[\eth' + (w + \tfrac{1}{2}(p - q)) \bar{\tau} \right] \xi &\hat{=} (p - 1, q + 1; w - 1),\end{aligned}\quad (\text{B.11})$$

where $\xi \hat{=} (p, q; w)$ is any conformally-weighted GHP scalar. We note that the spin coefficient τ' is, in our formalism, conformally-invariant on $\mathcal{S}$, contrary to its transformation given in equation (5.6.27) (iii) of [61]. This difference arises because, under $\Omega \mapsto \omega \Omega$, the normal n_a , in our definition, transforms away from $\mathcal{S}$ as $n_a \mapsto \omega n_a + \Omega A^{-1} \nabla_a \omega$. For this reason, we have defined the conformal-GHP derivative corresponding to $\eth'$ with $\bar{\tau}$ instead of τ' . This definition corresponds to the conformal operators $\frac{1}{2}(\eth_{\mathcal{S}} + \eth'_{\mathcal{S}})$ and $\frac{1}{2}(\eth_{\mathcal{S}} + \eth'_{\mathcal{S}})$ that were introduced in equation (5.6.36) of [61]. The conformal-GHP derivatives are useful for translating expressions from a Bondi frame to arbitrary choices of conformal factor and foliation, as follows: in the Bondi frame, where $\rho' = \tau = 0$, the conformal-GHP derivatives equation (B.11) are equivalent to the usual GHP derivatives. Thus, to generalise any expression in the Bondi frame to arbitrary conformal frames, we replace the usual GHP derivatives in that expression with the conformal-GHP derivatives, taking into account the appropriate weights. We will summarize the essential relations below; see appendix A of [4] for details of this procedure.

By the peeling theorem (see theorem 9.6.41 of [60]) the usual GHP components Ψ_i of the Weyl tensor vanish at $\mathcal{I}$ and the components of $\Omega^{-1}C_{abcd}$ are smooth at $\mathcal{I}$. We denote the components of $\Omega^{-1}C_{abcd}$ in the chosen tetrad at $\mathcal{I}$, which we call the (unphysical) *Weyl scalars*, by

$$\begin{aligned}\psi_4 &:= \mathcal{R}_{ab}\bar{m}^a\bar{m}^b && \doteq (-4, 0; -3), \\ \psi_3 &:= \mathcal{S}_a\bar{m}^a && \doteq (-2, 0; -3), \\ \psi_2 &:= \frac{1}{2}(\mathcal{P} + i\mathcal{P}^*) && \doteq (0, 0; -3), \\ \psi_1 &:= \mathcal{J}_am^a && \doteq (2, 0; -3), \\ \psi_0 &:= \mathcal{I}_{ab}m^am^b && \doteq (4, 0; -3).\end{aligned}\tag{B.12}$$

The relations between the spin coefficients, $\Psi_i = \Omega\psi_i$, and components of the Ricci tensor in the GHP formalism can be found in [58].

Equation (2.3a) implies that the Weyl scalars ψ_i in the unphysical spacetime satisfy the same Bianchi identity at $\mathcal{I}$ as the Weyl scalars $\hat{\Psi}_i$ defined by the physical Weyl tensor, with all the physical Ricci tensor components set to zero by the vacuum Einstein equations (see equations (9.10.1) and (9.10.2) of [60]). Taking the prime transform (defined by equation (2.7) of [58]) of the Bianchi identities equations (2.33)–(2.36) of [58], with their Ψ_i replaced by ψ_i and the Ricci tensor terms set to zero, we have

$$(\mathfrak{p}' - 4\rho')\psi_3 \doteq (\delta - \tau)\psi_4, \tag{B.13a}$$

$$(\mathfrak{p}' - 3\rho')\psi_2 \doteq (\delta - 2\tau)\psi_3 + \sigma\psi_4, \tag{B.13b}$$

$$(\mathfrak{p}' - 2\rho')\psi_1 \doteq (\delta - 3\tau)\psi_2 + 2\sigma\psi_3, \tag{B.13c}$$

$$(\mathfrak{p}' - \rho')\psi_0 \doteq (\delta - 4\tau)\psi_1 + 3\sigma\psi_2. \tag{B.13d}$$

If the conformal factor satisfies the Bondi condition (i.e. $\rho' = 0$), these reduce to equations (9.10.4)–(9.10.7) of [60]. Using equations (B.5) and (B.12) and setting $A = 1$, we see that equation (B.13) is equivalent to equation (2.25).

The spin coefficient σ has conformal-GHP weight $(3, -1; -1)$. The *complex News function* N is defined by

$$\bar{N} := -\mathfrak{p}'\sigma \doteq (2, -2; -2). \tag{B.14}$$

The *conformally-invariant* News tensor N_{ab} is given by

$$N_{ab} \doteq 2Nm_am_b + 2\bar{N}\bar{m}_a\bar{m}_b, \quad 2\bar{N} \doteq N_{ab}m^am^b. \tag{B.15}$$

Using equations (B.8), (B.14), (B.15) and (B.17), we reproduce the definition of the News tensor in equation (2.32). Note that equation (B.14) differs from the definition of the News function in equation (9.8.73) of [60] by terms involving the spin coefficient τ , which vanishes in the Bondi frame. In arbitrary conformal frames we choose equation (B.14) as the definition of the News function, since the corresponding News tensor in equation (B.15) is conformally-invariant.

The News function is related to the Weyl scalars by

$$A\psi_4 \triangleq (\mathfrak{p}' - 2\rho')N, \quad A\psi_3 \triangleq \delta N. \quad (\text{B.16})$$

Rewriting the above in terms of tensors using equations (B.12) and (B.15) and setting $\rho' \triangleq 0$ and $A = 1$, we get equation (2.37). Moreover, using equation (2.25) of [58], along with equations (B.5) and (B.14), we have

$$\overline{N} \triangleq \Phi_{02} - \rho'\sigma - (\delta - \tau)\tau, \quad (\text{B.17})$$

where, in our conventions, $\Phi_{02} := \frac{1}{2}S_{ab}m^am^b$ [59]; this corresponds to equation (2.36) when $\rho' \triangleq 0$, or equation (A.6) when the Bondi condition is not imposed.

Using $\Psi_i = \Omega\psi_i$ with equations (B.5) and (B.17) in equation (2.34) of [58], we also get

$$A\text{Im } \psi_2 \triangleq \text{Im}[(\delta' - \overline{\tau})(\delta' + \overline{\tau})\sigma + \sigma N], \quad (\text{B.18})$$

which corresponds to equation (2.38). Similarly, from equation (2.21) of [58] we have

$$\Phi_{01} \triangleq \delta\rho - \delta'\sigma, \quad (\text{B.19})$$

where $\Phi_{01} = \frac{1}{2}S_{ab}l^am^b$; using equation (B.7), this corresponds to equation (C.5).

For any vector ξ^a tangent to $\mathcal{S}$, we have the tetrad components

$$\xi^a \triangleq \beta\nabla^a\Omega + X^a \triangleq (A\beta)n^a + Xm^a + \overline{X}\overline{m}^a, \quad (\text{B.20})$$

where the conformal-GHP weights of β and X are:

$$A\beta \triangleq (1, 1; 1), \quad X \triangleq (-1, 1; 1). \quad (\text{B.21})$$

The conditions for ξ^a to be a BMS symmetry are given by

$$\begin{aligned} (\mathfrak{p}' + 2\rho')(A\beta) &\triangleq \text{Re}[(\delta + 2\tau)X], \\ (\mathfrak{p}' + 2\rho')X &\triangleq 0, \quad \delta'X \triangleq 0. \end{aligned} \quad (\text{B.22})$$

For a supertranslation ($\beta = f, X = 0$) to be a translation we have the additional condition

$$(\delta - \tau)(\delta + \tau)(Af) \triangleq 0. \quad (\text{B.23})$$

Using equation (B.5) and setting $A = 1$, we reproduce equations (4.24) and (A.7); when $\rho' = 0$, the former reduces to equation (4.21).

Finally, the WZ charge in equation (5.44), in GHP notation, takes the form

$$\mathcal{Q}[\xi; S] \triangleq -\frac{1}{4\pi}\text{Re} \int_S \varepsilon_2 \left[\beta(\psi_2 - A^{-1}\sigma N) + A^{-1}X\psi_1 + A^{-2}X\sigma(\delta + \tau)\overline{\sigma} - \frac{1}{2}A^{-2}\sigma\overline{\sigma}(\delta + 2\tau)X \right], \quad (\text{B.24})$$

while the WZ flux is given by

$$\mathcal{F}[\xi; \Delta \mathcal{J}] \triangleq -\frac{1}{4\pi} \text{Re} \int_{\Delta \mathcal{J}} \epsilon_3 N \left[\frac{1}{2} \beta \bar{N} + (\delta - \tau)(\delta + \tau) \beta \right. \\ \left. - 2A^{-1}(\text{Re}[X \delta] \sigma + \sigma \delta X) + A^{-1} \sigma \text{Re}[\delta X] \right]. \quad (\text{B.25})$$

Appendix C. Comparison to other charge formulae

In this appendix we compare the WZ prescription to other formulations of the charge and fluxes associated with the BMS symmetries. In particular, we consider the Hamiltonian formulation of Ashtekar and Streubel, the Noether/Komar charge formula and their linkage versions and also the charges defined by considering twistors at null infinity.

C1. Ashtekar–Streubel flux and charge

In this appendix, we compare the expression for the WZ flux given in equation (5.55) to the flux given by Ashtekar and Streubel [43]. Rewriting the expression for the flux given in equation (4.14) of [43] in our conventions, we obtain

$$\mathcal{F}^{(\text{AS})}[\xi; \Delta \mathcal{J}] \triangleq -\frac{1}{16\pi} \int_{\Delta \mathcal{J}} \epsilon_3 N^{ab} (\mathcal{L}_\xi D_a - D_a \mathcal{L}_\xi) l_b, \quad (\text{C.1})$$

where D_a is the derivative operator induced on $\mathcal{J}$ by the unphysical derivative operator ∇_a as defined on pp 46 of [27]. The derivative operator D_a satisfies

$$D_a n^b \triangleq 0, \quad D_a q_{bc} \triangleq 0, \quad D_a v_b \triangleq (\delta_a^c + n_a l^c)(\delta_b^d + n_d l^b) \nabla_c v_d, \quad (\text{C.2})$$

where v_a is any covector field in the unphysical spacetime and v_a is its pullback to $\mathcal{J}$.

To compare the flux equation (C.1) to the WZ flux we need to compute the quantity $\text{STF}[(\mathcal{L}_\xi D_a - D_a \mathcal{L}_\xi) l_b]$ for a BMS symmetry ξ^a . First, note that $(\mathcal{L}_\xi D_a - D_a \mathcal{L}_\xi) l_b$ can be expressed in terms of the Riemann tensor of D_a which can, in turn, be written in terms of S_{ab} using equation (3.5) of [43]. This gives

$$(\mathcal{L}_\xi D_a - D_a \mathcal{L}_\xi) l_b \triangleq \xi^c (q_{b[c} S_{a]}^d + S_{b[c} \delta_{a]}^d) l_d - l_c D_a D_b \xi^c. \quad (\text{C.3})$$

For a BMS symmetry $\xi^a \triangleq \beta n^a + X^a$, we get

$$\text{STF}[(\mathcal{L}_\xi D_a - D_a \mathcal{L}_\xi) l_b] \triangleq \text{STF} \left[\frac{1}{2} X_b S_a^c l_c + \frac{1}{2} S_{ab} \beta + D_a D_b \beta - l_c D_a D_b X^c \right]. \quad (\text{C.4})$$

Next, we use the following relation (which follows from equation (B.19) in the GHP notation)

$$\frac{1}{2} Q_b^c S_{ac} l^a \triangleq \mathcal{D}^a \sigma_{ab} - \frac{1}{2} \mathcal{D}_b \vartheta, \quad (\text{C.5})$$

where $\vartheta \triangleq Q^{ab} \nabla_a l_b$ is the expansion of l^a . Therefore, the first term in equation (C.4) gives

$$\text{STF} \left[\frac{1}{2} X_b S_a^c l_c \right] \triangleq \text{STF} \left[X_b \mathcal{D}^c \sigma_{ac} - \frac{1}{2} X_b \mathcal{D}_a \vartheta \right]. \quad (\text{C.6})$$

Using equation (2.36), the second term in equation (C.4) is

$$\text{STF}[\frac{1}{2}S_{ab}\beta] \hat{=} \beta \text{STF}[\frac{1}{2}N_{ab} - (\mathcal{D}_a\tau_b + \tau_a\tau_b)]. \quad (\text{C.7})$$

The third term in equation (C.4) can be expressed as

$$\begin{aligned} \text{STF} D_a D_b \beta &\hat{=} \text{STF}(\mathcal{D}_a \mathcal{D}_b \beta - \mathcal{L}_n \beta \nabla_a l_b) \\ &= \text{STF} \left[\mathcal{D}_a \mathcal{D}_b \beta - \frac{1}{2}(\mathcal{D}_c - 2\tau_c)X^c \sigma_{ab} \right], \end{aligned} \quad (\text{C.8})$$

where we have used $\mathcal{L}_n \beta \hat{=} \frac{1}{2}(\mathcal{D}_a - 2\tau_a)X^a$ (equation (4.21a)). For the fourth term in equation (C.4), using $l_a X^a \hat{=} 0$ we have $-l_c D_a D_b X^c = D_a(X^c D_b l_c) + D_b X^c D_a l_c$. We substitute $D_a l_b \hat{=} \sigma_{ab} + \frac{1}{2}Q_{ab}\vartheta - l_a \tau_b$ and take the STF of the resulting expression. Using $\text{STF} \mathcal{D}_a X_b \hat{=} 0$ (equation (4.21c)), we get

$$\text{STF}[-l_c D_a D_b X^c] \hat{=} \text{STF}[\mathcal{D}_a(X^c \sigma_{bc}) + \frac{1}{2}X_b \mathcal{D}_a \vartheta - X^c \tau_c \sigma_{ab} + \sigma_a^c \mathcal{D}_b X_c]. \quad (\text{C.9})$$

Putting together equations (C.6)–(C.9) we get

$$\begin{aligned} \text{STF}[(\mathcal{L}_\xi D_a - D_a \mathcal{L}_\xi) l_b] &\hat{=} \text{STF} \left[\frac{1}{2}\beta N_{ab} + (\mathcal{D}_a + \tau_a)(\mathcal{D}_b - \tau_b)\beta \right. \\ &\quad \left. + X_b \mathcal{D}^c \sigma_{ac} - \frac{1}{2}\mathcal{D}_c X^c \sigma_{ab} + \mathcal{D}_a(X^c \sigma_{bc}) + \sigma_a^c \mathcal{D}_b X_c \right]. \end{aligned} \quad (\text{C.10})$$

Finally, we simplify the second line above using proposition D.1. As a result, we find

$$\mathcal{F}^{(\text{AS})}[\xi; \Delta \mathcal{J}] \hat{=} -\frac{1}{16\pi} \int_{\Delta, \mathcal{J}} \epsilon_3 N^{ab} \left[\frac{1}{2}\beta N_{ab} + (\mathcal{D}_a + \tau_a)(\mathcal{D}_b - \tau_b)\beta + \mathcal{L}_X \sigma_{ab} - \frac{1}{2}\sigma_{ab} \mathcal{D}_c X^c \right]. \quad (\text{C.11})$$

This is the same as the WZ flux formula given in equation (5.56). A charge expression whose flux is given by the Ashtekar–Streubel formula, for the case where $\tau_a = \vartheta = 0$, was also considered in [62]. Our charge expression, when adapted to $\tau_a = \vartheta = 0$, differs from the one in [62] which we believe to be a result of sign errors in some of the terms given in [62].

C2. Komar formulae and linkage charges

In this section we review the Komar formulae for conserved charges associated with Killing vector fields and the linkage charges which generalize these to asymptotic BMS symmetries. We show that for translations and Lorentz symmetries these reproduce the WZ charge formula (up to proportionality constants), however for supertranslations the linkage charge does not coincide with the WZ charge.

For physical spacetimes with a Killing vector field a prescription for a corresponding conserved quantity was given by Komar [63]. For stationary spacetimes (with a timelike Killing field $\hat{t}^a$) and axisymmetric spacetimes (with an axial Killing field $\hat{\phi}^a$) this gives a prescription for the Komar mass $\mathcal{M}^{(\text{K})}$ and Komar angular momentum $\mathcal{J}^{(\text{K})}$, respectively, defined as

$$\mathcal{M}^{(\text{K})} := -\frac{1}{8\pi} \int_S \hat{\epsilon}_{abcd} \hat{\nabla}^c \hat{t}^d, \quad \mathcal{J}^{(\text{K})} := \frac{1}{16\pi} \int_S \hat{\epsilon}_{abcd} \hat{\nabla}^c \hat{\phi}^d. \quad (\text{C.12})$$

Using the vacuum Einstein equation and the Killing equation it can be checked that these expressions are identically conserved and so can be evaluated on any two-sphere S in the spacetime. The relative difference in the sign arises because $\hat{t}^a$ is timelike while $\hat{\phi}^a$ is spacelike with the choice dictated by standard conventions for fields in Minkowski spacetime. The relative ‘factor of two’ between the two formulae is essentially put in ‘by hand’ to match the standard notions of mass and angular momentum in the case of the Kerr spacetime.

Note that the integrands in equation (C.12) are proportional to the Noether charge two-form (equation (5.9)) evaluated for the respective Killing fields. Thus, for asymptotic BMS symmetries we could define the asymptotic charges using the Noether charge to get

$$\mathcal{Q}^{(N)}[\xi; S] = -\frac{1}{16\pi} \lim_{S' \rightarrow S} \int_{S'} \hat{\varepsilon}_{abcd} \hat{\nabla}^c \xi^d = -\frac{1}{16\pi} \lim_{S' \rightarrow S} \int_{S'} \varepsilon_{abcd} \nabla^c (\Omega^{-2} \xi^d). \quad (\text{C.13})$$

It can be shown that the limit of the integrals defined above is finite and is independent of the manner in which S' limits to S (this is similar to the proof of lemma 5.1; see [42]). However, a major drawback of using the Noether charge as the charge associated with a BMS symmetry is that equation (C.13) depends on how the BMS symmetry is extended away from $\mathcal{S}$: if ξ'^a and ξ^a are equivalent representatives of a BMS symmetry, so that $\xi'^a = \xi^a + \Omega^2 W^a$ (from proposition 4.1) then we have

$$\mathcal{Q}^{(N)}[\xi'; S] - \mathcal{Q}^{(N)}[\xi; S] = -\frac{1}{16\pi} \int_S \varepsilon_{abcd} \nabla^c W^d. \quad (\text{C.14})$$

It is clear that one can obtain *any* value for the Noether charge on S associated with the BMS symmetry simply by using different extensions of the symmetry into the spacetime. Note, if ξ^a is an exact Killing vector field of the physical spacetime then ξ^a is determined everywhere by its value on $\mathcal{S}$ by the Killing equation in the physical spacetime, or equivalently the conformal Killing equation in the unphysical spacetime and this ambiguity does not arise. Thus, for a Killing vector field the Noether charge (i.e. the Komar formula) at S is unambiguously defined. But for asymptotic BMS symmetries which need not be exact Killing fields the Noether charge cannot be unambiguously defined.

A way of avoiding this dependence on the extension of the BMS vector fields would be to suitably modify the Noether charge formula (so that it reduces to the Noether charge for a Killing field) and/or prescribe a choice of extension for the BMS vector field. Such a prescription is given by the *linkage charge* which we summarize next.

- (a) One formulation of the linkage charge is given as follows [44, 64]: consider a *fixed* cross-section S of $\mathcal{S}$, and let $\mathcal{N}$ be an outgoing null hypersurface which intersects $\mathcal{S}$ in S . Let l^a be a null geodesic vector field (defined on $\mathcal{N}$) which is tangent to $\mathcal{N}$ and coincides with the auxiliary normal at S . Given a BMS vector field ξ^a on S we extend it along the null surface $\mathcal{N}$ by the condition

$$\left(\gamma_{ab}^{(\xi)} - \frac{1}{2} \gamma^{(\xi)} g_{ab} \right) l^b \Big|_{\mathcal{N}} = 0, \quad (\text{C.15})$$

where $\gamma_{ab}^{(\xi)}$ is the unphysical metric perturbation generated by ξ^a (see equations (4.2) and (4.5)) and $\gamma^{(\xi)}$ is its trace. Then the linkage charge at S is defined by

$$\mathcal{Q}^{(L)}[\xi; S] := -\frac{1}{16\pi} \lim_{S' \rightarrow S} \int_{S'} [\varepsilon_{abcd} \nabla^c (\Omega^{-2} \xi^d) + \Omega^{-1} \gamma^{(\xi)} \varepsilon_{ab}], \quad (\text{C.16})$$

where the integral is evaluated on two-sphere cross-sections S' of the null surface $\mathcal{N}$, with area-element $\varepsilon_{ab} := \varepsilon_{abcd} n^c l^d (l^e n_e)^{-1}$, and then the limit is taken to S along $\mathcal{N}$.

- (b) Another formulation of the linkage charge, which does not require fixing a cross-section and a null surface was given by Geroch and Winicour [42]. In this formulation the BMS vector field ξ^a on $\mathcal{I}$ is extended away from $\mathcal{I}$ by the condition

$$\gamma^{(\xi)} = 0, \quad (\text{C.17})$$

and the linkage charge is defined by

$$\mathcal{Q}^{(L)}[\xi; S] := -\frac{1}{16\pi} \lim_{S' \rightarrow S} \int_{S'} \varepsilon_{abcd} \nabla^c (\Omega^{-2} \xi^d), \quad (\text{C.18})$$

where now the integral is evaluated on *any* family of two-spheres S' in M and then the limit is taken as S' limits to S .

It was shown in [42] that both formulations of the linkage charge give the same charge on *any* choice of cross-section S (when the respective conditions in each formulation are satisfied). The advantages and disadvantages of both formulations are also discussed in [42]. An explicit expression for the linkage charge in terms of fields on $\mathcal{I}$ was computed by Winicour [64]. After integration-by-parts on the cross-section S , the expression for the linkage charge given in equation (5.6) of [64] (in the GHP notation) can be written as¹⁶

$$\begin{aligned} \mathcal{Q}^{(L)}[\xi; S] \doteq \frac{1}{4\pi} \text{Re} \int_S \varepsilon_2 \left[\frac{1}{2} \beta (\psi_2 - A^{-1} \sigma N) - \frac{1}{2} \beta A^{-1} (\delta - \tau) (\delta + \tau) \bar{\sigma} \right. \\ \left. + A^{-1} X \psi_1 + A^{-2} X \sigma (\delta + \tau) \bar{\sigma} - \frac{1}{2} A^{-2} \sigma \bar{\sigma} (\delta + 2\tau) X \right]. \end{aligned} \quad (\text{C.19})$$

The linkage charge expression is conformally-invariant and thus only depends on the physical spacetime and *not* on the choice of the conformal-completion. If we pick $\beta|_S = 0$, i.e. ξ^a is a Lorentz vector field at S , the linkage charge coincides with the WZ charge equation (B.24); this was argued by Wald and Zoupas [21]. For a BMS translation (with $\beta = f$ satisfying equation (B.23) and $X = 0$) note that the third term in the first line of equation (C.19) vanishes upon integration-by-parts (also using equation (B.6)) and we get *half* of the WZ charge. Thus, the linkage charges also have a ‘factor of two’ discrepancy between the mass and angular momentum similar to the Komar formulae equation (C.12). Note that the WZ charge gives the correct relative factor between mass and angular momentum without any need to fix these factors in an ad hoc manner. Since exact Killing vector fields in the physical spacetime can be shown to be part of some Poincaré algebra at $\mathcal{I}$ (see remark 4.2), we conclude that the WZ charge of an exact Killing vector field agrees with the Komar formula.

However for supertranslations which are not translations the third term in equation (C.19) does not vanish when the shear σ on the cross-section S is non-zero, and the linkage charge

¹⁶Note that the function $\bar{N}$ defined in equation (3.30) of [64] is in fact $\bar{\Phi}_{02}$ in our notation and we have used equation (B.17) to convert to our notation. Similarly, the quantity P defined in equation (3.31) of [64] vanishes at $\mathcal{I}$ due to the Bondi condition. We have also restored factors of A (defined in equation (B.3)) to make the expression invariant under GHP transformations (equation (B.1)).

does not coincide with the WZ charge or the Geroch supermomentum [27]. Thus, the linkage charge for supertranslations is non-zero even in Minkowski spacetime when the cross-section is not shear-free. This also implies that the flux of the linkage charge between arbitrary cross-sections does not vanish in Minkowski spacetime; though the flux does vanish when computed between two shear-free cross-sections. A detailed comparison of the linkage flux and the Ashtekar–Streubel flux (which equals the WZ flux, see appendix C1), with similar conclusions, was given in [65].

C3. Twistor charge

Another expression for a charge associated with the BMS symmetries was given by Penrose [66] motivated by twistor theory (see also [23, 60, 67]). To compare to these references we will work with the GHP formalism, set $\rho' \hat{=} 0$ (i.e. the Bondi condition is satisfied) and $\tau \hat{=} 0$ (equivalent to a choice of foliation where $\tau_a \hat{=} 0$).

With these choices being made, we choose a cross-section S of $\mathcal{S}$ and define two-surface twistors (ω^0, ω^1) on S satisfying¹⁷

$$\begin{aligned} \delta' \omega^0 &\hat{=} 0, \quad \delta \omega^1 \hat{=} \sigma \omega^0, \\ \omega^0 &\hat{=} (-1, 0; \tfrac{1}{2}), \quad \omega^1 \hat{=} (1, 0; \tfrac{1}{2}), \end{aligned} \quad (\text{C.20})$$

Given two solutions (ω^0, ω^1) and $(\tilde{\omega}^0, \tilde{\omega}^1)$ of equation (C.20) we can form the (complex) vector field

$$\xi_{\mathbb{C}}^a = (A\beta)n^a + X m^a, \quad \beta = -i(\omega^0 \tilde{\omega}^1 + \omega^1 \tilde{\omega}^0), \quad A^{-1}X = -2i\omega^0 \tilde{\omega}^0, \quad (\text{C.21})$$

where A is as defined in equation (B.3). Then equation (C.20) implies

$$\delta' X \hat{=} 0, \quad (\text{C.22a})$$

$$\delta^2(A\beta) \hat{=} \tfrac{1}{2}\sigma \delta X + \delta(\sigma X). \quad (\text{C.22b})$$

The first equation says that $X m^a$ generates a (complex) Lorentz transformation on S , while the second is a restriction to a (complex) Poincaré subalgebra at S . The ‘complexification’ is necessary since equation (C.22b) in general does not have any real solutions for β . A fixed choice of cross-section S is necessary since equation (C.22b) is *not* preserved along the null generators of $\mathcal{S}$ unless $\beta = 0$ or $\bar{N} = 0$, i.e. the News vanishes. For this choice of subalgebra the (complex) twistor charge at S proposed by Penrose [66] is

$$\mathcal{Q}^{(\text{T})}[\xi_{\mathbb{C}}; S] \hat{=} -\frac{i}{4\pi} \int_S \epsilon_2 [(\omega^0 \tilde{\omega}^1 + \omega^1 \tilde{\omega}^0)\psi_2 + \omega^0 \tilde{\omega}^0 \psi_1 + \omega^1 \tilde{\omega}^1 \psi_3]. \quad (\text{C.23})$$

Using $A\psi_3 = \delta N$ (equation (B.16)), and integrating by parts on S we can write this charge in terms of the vector field $\xi_{\mathbb{C}}^a$, defined in equation (C.21), as

$$\mathcal{Q}^{(\text{T})}[\xi_{\mathbb{C}}; S] \hat{=} \frac{1}{8\pi} \int_S \epsilon_2 [2\beta(\psi_2 - A^{-1}\sigma N) + A^{-1}X\psi_1]. \quad (\text{C.24})$$

¹⁷ The pair (ω^0, ω^1) are the components of a two-spinor defined at S in a spin frame compatible with the GHP tetrad frame at S ; see [67].

Using equation (C.22) and integrating-by-parts on S , equation (C.24) can be written as (see also equation (11) of [23])

$$\begin{aligned} \mathcal{Q}^{(T)}[\xi_{\mathbb{C}}; S] \triangleq \frac{1}{8\pi} \int_S \epsilon_2 \left[2\beta(\psi_2 - A^{-1}\sigma N - iA^{-1}\text{Im } \delta' \delta' \sigma) \right. \\ \left. + A^{-1}X\psi_1 + A^{-2}X\sigma \delta \bar{\sigma} - \frac{1}{2}A^{-2}\sigma \bar{\sigma} \delta X \right]. \end{aligned} \quad (\text{C.25})$$

To compare this expression to the WZ charge we can use equation (B.18) (with $\tau \triangleq 0$) to get

$$\mathcal{Q}^{(T)}[\xi_{\mathbb{C}}; S] \triangleq \frac{1}{8\pi} \int_S \epsilon_2 \left[2\beta \text{Re}(\psi_2 - A^{-1}\sigma N) + A^{-1}X\psi_1 + A^{-2}X\sigma \delta \bar{\sigma} - \frac{1}{2}A^{-2}\sigma \bar{\sigma} \delta X \right]. \quad (\text{C.26})$$

Comparing to equation (B.24) we see that the twistor charge is the same as the WZ charge for the complex vector field $\xi_{\mathbb{C}}^a = \beta n^a + X m^a$ in the Poincaré subalgebra at S chosen by equation (C.22). The flux of the twistor charge was shown to be equivalent to the (complex) Ashtekar–Streubel flux in [67].

Appendix D. Symmetric and tracefree tensors in two dimensions

This appendix collects some useful identities for symmetric, tracefree tensors in two dimensions. In the main body of the paper these are applied to tensors on a cross-section S of $\mathcal{I}$, but they hold for any two-dimensional orientable manifold S with a Riemannian metric Q_{ab} ; as such, for simplicity, we drop the hats on all equalities.

The results of this appendix primarily rely on the following relation between the two-dimensional Riemannian metric Q_{ab} and the corresponding area-element ε_{ab} , (see equation (B.2.12) of Wald [26]):

$$\varepsilon^{ab} \varepsilon_{cd} = 2Q^{[a} Q^{b]}_{[c} Q^{d]}_{d].} \quad (\text{D.1})$$

For a tensor field A_{ab} , let $\text{STF} A_{ab}$ be the symmetric tracefree part as defined in equation (2.13). Then, we can decompose A_{ab} into an antisymmetric part, a pure trace part, and a symmetric, tracefree part:

$$A_{ab} = A_{[ab]} + \frac{1}{2}A^c{}_c Q_{ab} + \text{STF} A_{ab}. \quad (\text{D.2})$$

Most of the tensors on S which arise in our main analysis are symmetric and tracefree, so we collect the identities satisfied by such tensors in the following proposition:

Proposition D.1. *Let A_{ab} and B_{ab} be symmetric and tracefree tensors on S ; then the following identities hold:*

$$\text{STF}(A_{ac} B^c{}_b) = 0, \quad (\text{D.3})$$

$$\mathcal{D}_{[a} A_{b]c} = Q_{c[a} \mathcal{D}^d A_{b]d}, \quad (\text{D.4})$$

where $\mathcal{D}$ is the covariant derivative compatible with Q_{ab} , and

$$\text{STF} \left(\mathcal{L}_X A_{ab} - \frac{1}{2} A_{ab} \mathcal{D}_c X^c \right) = \text{STF} \left[X_a \mathcal{D}^c A_{bc} + \mathcal{D}_a (X^c A_{bc}) + A_a{}^c \mathcal{D}_{[b} X_{c]} \right], \quad (\text{D.5})$$

for any vector field X^a on S .

Proof. For any tensor A_{ab} on S , define a notion of left and right dual, respectively, by

$$(*A)_{ab} := \varepsilon_{ca} A^c{}_b, \quad (A^*)_{ab} := A_a{}^c \varepsilon_{cb}. \quad (\text{D.6})$$

It follows from equation (D.1) that

$$(**A)_{ab} = -A_{ab}, \quad (A^{**})_{ab} = -A_{ab}, \quad (*A^*)_{ab} = A_c{}^c Q_{ab} - A_{ba}. \quad (\text{D.7})$$

Using equation (D.7), we therefore can characterize A_{ab} by how its left and right duals are related:

$$(A^*)_{ab} = \begin{cases} (*A)_{ab} & \iff A_{ab} = \text{STF } A_{ab} \\ -(^*A)_{ab} & \iff \text{STF } A_{ab} = 0. \end{cases} \quad (\text{D.8})$$

Now, let A_{ab} and B_{ab} be two symmetric, tracefree tensors, and consider the left and right duals of $U_{ab} = A_{ac} B^c{}_b$. Using equations (D.6) and (D.8) we obtain

$$\begin{aligned} (U^*)_{ab} &= A_{ac} (B^*)^c{}_b = A_{ac} (^*B)^c{}_b = A_{ac} \varepsilon^{dc} B_{db} \\ &= -(A^*)_{ac} B^c{}_b = -(^*A)_{ac} B^c{}_b = -(^*U)_{ab}. \end{aligned} \quad (\text{D.9})$$

Then, using equation (D.8) we get that $\text{STF}(A_{ac} B^c{}_b) = 0$.

To show equation (D.4) we start with the fact that

$$\varepsilon^{cd} \varepsilon^{ab} \mathcal{D}_a A_{bc} = 2 \mathcal{D}_a (Q^{a[c} Q^{d]b} A_{bc}) = \mathcal{D}_c A^{dc}, \quad (\text{D.10})$$

where the first equality uses equation (D.1) and the compatibility of $\mathcal{D}_a$ and ε^{ab} , and the second follows from the fact that $Q^{ab} A_{ab} = 0$. As such, the trace of equation (D.1) implies that

$$\varepsilon^{ab} \mathcal{D}_a A_{bc} = \varepsilon_{cd} \mathcal{D}_e A^{de}. \quad (\text{D.11})$$

The result equation (D.4) then follows by equation (D.1).

Finally, consider the Lie derivative of A_{ab} with respect to some vector field X^a on S . We have that

$$\begin{aligned} &\text{STF} \left(\mathcal{L}_X A_{ab} - \frac{1}{2} A_{ab} \mathcal{D}_c X^c \right) \\ &= \text{STF} \left(X^c \mathcal{D}_c A_{ab} + 2 A_a{}^c \mathcal{D}_b X_c - \frac{1}{2} A_{ab} \mathcal{D}_c X^c \right) \\ &= \text{STF} \left[2 X^c \mathcal{D}_{[c} A_{a]b} + \mathcal{D}_a (X^c A_{bc}) + A_a{}^c \mathcal{D}_b X_c - \frac{1}{2} A_{ab} \mathcal{D}_c X^c \right], \end{aligned} \quad (\text{D.12})$$

where the first equation follows by the definition of the Lie derivative and the second from the Leibniz rule. Using equation (D.4) for the first term in this expression and rewriting the last two terms as

$$\text{STF} [A_a{}^c \mathcal{D}_{[b} X_{c]} + A_a{}^c \text{STF } \mathcal{D}_b X_c] = \text{STF} [A_a{}^c \mathcal{D}_{[b} X_{c]}], \quad (\text{D.13})$$

and using equation (D.3), we find equation (D.5). $\square$

Finally, we show that divergence-free, symmetric, and trace-free tensors A_{ab} on S must vanish. Unlike the above proposition, which holds for any two-dimensional manifold S , this is a consequence of the fact that any cross-section S of $\mathcal{I}$ has the topology of $\mathbb{S}^2$. Our proof is based on one given in lemma 5 of [27] (for similar discussion, see appendix A.4 of [62] or appendix C of [68]):

Proposition D.2. *Let A_{ab} be a symmetric, trace-free tensor on S that satisfies $\mathcal{D}_a A^{ab} = 0$. Then A_{ab} vanishes.*

Proof. Consider any conformal Killing vector ξ^a on S , and define

$$v_a := A_{ab} \xi^b. \quad (\text{D.14})$$

First note that, since A_{ab} is symmetric and trace-free and ξ^a is a conformal Killing field on S , v_a is divergence-free:

$$\mathcal{D}_a v^a = A^{ab} \mathcal{D}_a \xi_b = 0. \quad (\text{D.15})$$

Next, we show that v_a is curl-free. To do so, we use equation (D.1) to show that

$$\varepsilon_{ab} \varepsilon^{ac} A_c{}^b = Q_{ab} A^{ab} = 0, \quad Q_{ab} \varepsilon^{ac} A_c{}^b = \varepsilon_{ab} A^{ab} = 0, \quad (\text{D.16})$$

so that $\varepsilon^{ac} A_c{}^b$ is also a symmetric, trace-free tensor. Then, proposition D.1 implies that

$$\mathcal{D}_a (\varepsilon^{ac} A_c{}^b) = \varepsilon^{ac} Q_a{}^b \mathcal{D}_d A_{cd} = 0, \quad (\text{D.17})$$

and so

$$\varepsilon^{ab} \mathcal{D}_a v_b = \mathcal{D}_a (\varepsilon^{ac} A_c{}^b \xi_b) = \varepsilon^{ac} A_c{}^b \mathcal{D}_a \xi_b = 0. \quad (\text{D.18})$$

As such, v_a is both divergence- and curl-free, and so it must vanish (as S has the topology of $\mathbb{S}^2$): that is,

$$A_{ab} \xi^b = 0, \quad (\text{D.19})$$

for any conformal Killing vector ξ^a on S . However, at each point, the set of conformal Killing vectors on S spans the tangent space, and so we find that $A_{ab} = 0$. $\square$

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