

GLEASON PARTS AND POINT DERIVATIONS FOR UNIFORM ALGEBRAS WITH DENSE INVERTIBLE GROUP II

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Dedicated to the memory of Andrew Browder

ABSTRACT. Due to the omission of a hypothesis from an elementary lemma in the author's paper "Gleason parts and point derivations for uniform algebras with dense invertible group", some of the proofs presented in that paper are flawed. We prove here that nevertheless, all of the results in that paper, with the exception of the one misstated lemma, are correct. In the process, we strengthen slightly some of the results of that paper.

1. INTRODUCTION

In the author's paper [3], it is shown that there exist compact sets X in $\mathbf{C}^N$ ($N \geq 2$) with nontrivial polynomial or rational hull for which the uniform algebra $P(X)$ or $R(X)$ has a dense set of invertible elements, a large Gleason part, and an abundance of bounded point derivations. This is done by combining new constructions of "hulls with dense invertibles" with a construction of a compact set X such that $R(X)$ has a Gleason part of full measure and a nonzero bounded point derivation at almost every point. (The first example of a "hull with dense invertibles" was constructed by Joel Feinstein and Garth Dales [2].) The purpose of the present paper is to correct two errors in [3] and, in particular, to show that in spite of these errors, all of the results in [3] are correct with the exception of one misstated lemma whose statement is easily corrected.

One error occurs in the proof of [3, Theorem 1.7]. This error is easily corrected. See Section 5.

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The other error at first sight appears even easier to fix. As anyone with a background in complex analysis can see, [3, Lemma 5.1] is false as stated, but becomes true upon the addition of the hypothesis that the set Ω is connected. The trouble is that when the lemma is invoked in [3], the unstated hypothesis that Ω is connected is not considered. The lemma is invoked only in the proofs of Theorems 1.5 and 6.1. In the case of Theorem 1.5, the sets that play the role of Ω when applying the lemma are connected, as the reader can easily verify, so no change is needed in the proof. In the case of Theorem 6.1 though, it is not at all clear that the connectedness hypothesis is satisfied. While it is conceivable that the construction can be carried out so as to insure that the connectedness hypothesis is satisfied, we will instead give a new construction to prove [3, Theorem 6.1]. This new construction, which constitutes the bulk of the present paper, uses the ideas in the earlier (flawed) argument, but also involves considerable additional complications.

Note that the proofs of [3, Theorems 1.2–1.4, and 1.6] relied upon [3, Theorem 6.1], so correcting the proof of [3, Theorem 6.1] is needed to establish the validity of those results also. (The proofs of some of these results (Theorems 1.2–1.4) do not, however, require the full force of [3, Theorem 6.1] and could surely be established by simpler means than the argument we will give for [3, Theorem 6.1].)

The new construction we will give actually yields slightly more than was claimed in [3] in that the set we construct is connected. Specifically, we will establish the following result. Here, and throughout the paper, B denotes the open unit ball $\{z : \|z\| < 1\}$ in $\mathbf{C}^N$, and μ denotes the $2N$ -dimensional Lebesgue measure on $\mathbf{C}^N$.

Theorem 1.1. *For each integer $N \geq 1$, there exists a connected, compact rationally convex set $X \subset B \subset \mathbf{C}^N$ of positive $2N$ -dimensional measure such that the collection of polynomials zero-free on X is dense in $P(\overline{B})$ and such that there is a set $P \subset X$ of full $2N$ -dimensional measure in X such that P is contained in a single Gleason part for $R(X)$ and at every point of P the space of bounded point derivations for $R(X)$ has dimension N . Furthermore, given $\varepsilon > 0$, the set X can be chosen so that $\mu(B \setminus X) < \varepsilon$.*

We remark that as a consequence of this theorem, the set X in [3, Theorem 1.6] can be taken to be connected.

In the next section we recall some standard definitions and notation already used above, and we introduce some other notation we will use. Section 3 is devoted to several lemmas which are used in Section 4 to prove Theorem 1.1. As mentioned earlier, the correction to the

minor error in the proof of [3, Theorem 1.7] is given in the concluding Section 5.

This paper was written while the author was a visitor at the University of Michigan. He would like to thank the Department of Mathematics for its hospitality. He would also like to thank the referees for reading the paper especially carefully and providing helpful feedback.

It is with a mixture of joy and sorrow that I dedicate this paper to the memory of Andrew Browder. Sorrow, of course, that he is no longer with us; joy that I had the privilege of knowing him.

2. PRELIMINARIES

Throughout the paper, N denotes a positive integer. There is no further restriction on the value of N except in two places (the opening sentence and the concluding section of the paper) where it is noted that we impose the condition $N \geq 2$.

For X a compact Hausdorff space, we denote by $C(X)$ the algebra of all continuous complex-valued functions on X with the supremum norm $\|f\|_X = \sup\{|f(x)| : x \in X\}$. A *uniform algebra* on X is a closed subalgebra of $C(X)$ that contains the constant functions and separates the points of X .

For a compact set X in $\mathbf{C}^N$, we denote by $P(X)$ the uniform closure on X of the polynomials in the complex coordinate functions $z_1, \dots, z_N$, and we denote by $R(X)$ the uniform closure of the rational functions holomorphic on (a neighborhood of) X . It is well known that the maximal ideal space of $P(X)$ can be naturally identified with the *polynomial hull* $\widehat{X}$ of X defined by

$$\widehat{X} = \{z \in \mathbf{C}^N : |p(z)| \leq \max_{x \in X} |p(x)| \text{ for all polynomials } p\},$$

and the maximal ideal space of $R(X)$ can be naturally identified with the *rational hull* $h_r(X)$ of X defined by

$$h_r(X) = \{z \in \mathbf{C}^N : p(z) \in p(X) \text{ for all polynomials } p\}.$$

An equivalent formulation of the definition of $h_r(X)$ is that $h_r(X)$ consists precisely of those points $z \in \mathbf{C}^N$ such that every polynomial that vanishes at z also has a zero on X . Another equivalent formulation is that $h_r(X)$ consists precisely of those points $z \in \mathbf{C}^N$ such that every rational function g holomorphic on (a neighborhood of) X is defined at z and satisfies $|g(z)| \leq \max_{x \in X} |g(x)|$.

The open unit disc in the plane will be denoted by D , and the open unit ball in $\mathbf{C}^N$ will be denoted by B . Given a real number $r \geq 0$ and a subset S of $\mathbf{C}^N$, we will denote by rS the set $\{rs : s \in S\}$. For E a

measurable subset of $\mathbf{C}^N = \mathbf{R}^{2N}$, we will denote the $2N$ -dimensional Lebesgue measure of E by $\mu(E)$. The real part of a complex number (or function) z will be denoted by $\Re z$. We will use the notation $\|f\|_S = \sup\{|f(x)| : x \in S\}$ even if S is not compact, and we will adopt the convention that $\|f\|_S = 0$ if S is empty. As in [3], we will use the following notation in which we assume that X is a compact set in $\mathbf{C}^N$ and that Ω is an open set in $\mathbf{C}^N$ each of which contains the origin:

$\mathbf{B}(X)$ denotes the set of rational functions f holomorphic on a neighborhood of X such that $\|f\|_X \leq 1$,

$\mathbf{B}_0(X)$ denotes the set of functions in $\mathbf{B}(X)$ that vanish at the origin,

$\mathbf{B}(\Omega)$ denotes the set of functions f holomorphic on Ω such that $\|f\|_\Omega \leq 1$, and

$\mathbf{B}_0(\Omega)$ denotes the set of functions in $\mathbf{B}(\Omega)$ that vanish at the origin.

Let A be a uniform algebra on a compact space X . The *Gleason parts* for the uniform algebra A are the equivalence classes in the maximal ideal space of A under the equivalence relation $\varphi \sim \psi$ if $\|\varphi - \psi\| < 2$ in the norm on the dual space A^* . (That this really is an equivalence relation is well-known but *not* obvious.) For ϕ a multiplicative linear functional on A , a *point derivation* on A at ϕ is a linear functional ψ on A satisfying the identity

$$\psi(fg) = \psi(f)\phi(g) + \phi(f)\psi(g) \quad \text{for all } f, g \in A.$$

A point derivation is said to be *bounded* if it is bounded (continuous) as a linear functional.

We will make use of the following standard lemma. (For a proof see [1, Lemma 2.6.1].)

Lemma 2.1. *Two multiplicative linear functionals ϕ and ψ on a uniform algebra A lie in the same Gleason part if and only if*

$$\sup\{|\psi(f)| : f \in A, \|f\| \leq 1, \phi(f) = 0\} < 1.$$

We will also need the following elementary observation about bounded point derivations whose easy proof is given in [3, p. 4304].

Lemma 2.2. *Let $X \subset \mathbf{C}^N$ be a compact set. Then the complex vector space of bounded point derivations on $R(X)$ at a particular point $x \in X$ has dimension at most N , and the dimension is exactly N if and only if there is a number $M < \infty$ such that*

$$|\partial f / \partial z_\nu(x)| \leq M \quad \text{for every } f \in \mathbf{B}(X) \text{ and } \nu = 1, \dots, N.$$

3. LEMMAS

In this section we present several lemmas that will be used in the proof of Theorem 1.1.

Lemma 3.1. *Let $X \subset \mathbf{C}^N$ be a compact, rationally convex set. Then each component of X is rationally convex.*

It is also true that each component of a polynomially convex set is polynomially convex, and more generally, that each component of the maximal ideal space of a uniform algebra A is A -convex, but we will not need these facts.

Proof. Let K be a component of X . Since $h_r(K) \subset h_r(X) = X$, it suffices to show that given $x \in X \setminus K$, there is a rational function g holomorphic on a neighborhood of X such that $|g(x)| > \|g\|_K$. Choose a separation X_0, X_1 of X with $K \subset X_0$ and $x \in X_1$. By the Shilov idempotent theorem, there exists a function $f \in R(X)$ such that f is identically 0 on X_0 and identically 1 on X_1 . Now taking for g a rational function holomorphic on a neighborhood of X such that $\|f - g\|_X < 1/4$ completes the proof. $\square$

Lemma 3.2. *Fix $N \geq 1$ and $\varepsilon > 0$. Let $\{p_j\}$ be a countable collection of polynomials on $\mathbf{C}^N$ such that $p_j(0) \neq 0$ for each j . Then there exists a totally disconnected, compact polynomially convex set E with $0 \in E \subset B \subset \mathbf{C}^N$ such that*

- (i) *each p_j is zero-free on E*
- (ii) *$\mu(B \setminus E) < \varepsilon$.*

Proof. The main idea of the proof is the same as the proof of [3, Lemma 4.2].

Obviously it is sufficient to prove the lemma with the conditions $E \subset B$ and $\mu(B \setminus E) < \varepsilon$ replaced by $E \subset \overline{B}$ and $\mu(\overline{B} \setminus E) < \varepsilon$. By multiplying each p_j by a suitable complex number if necessary, we may assume that $\Re p_j(0) \neq 0$ for each j . Furthermore, by enlarging the collection $\{p_j\}$, we may assume that $\{p_j\}$ is dense in $P(\overline{B})$.

For each j , the set $p_j^{-1}(\{z \in \mathbf{C} : \Re z = 0\})$ is a proper real-analytic subvariety of $\mathbf{C}^N$ and hence has $2N$ -dimensional measure zero. Consequently, we can choose $0 < \varepsilon_j < \min\{|\Re p_j(0)|, 1\}$ such that

$$\mu\left(p_j^{-1}(\{z \in \mathbf{C} : |\Re z| < \varepsilon_j\}) \cap \overline{B}\right) < \varepsilon/2^j.$$

Then

$$\mu\left(\bigcup_{j=1}^{\infty} p_j^{-1}(\{z \in \mathbf{C} : |\Re z| < \varepsilon_j\}) \cap \overline{B}\right) < \sum_{j=1}^{\infty} \varepsilon/2^j = \varepsilon.$$

Thus setting $E = \overline{B} \setminus \bigcup_{j=1}^{\infty} p_j^{-1}(\{z \in \mathbf{C} : |\Re z| < \varepsilon_j\})$ we have that $\mu(\overline{B} \setminus E) < \varepsilon$. Obviously $0 \in E \subset \overline{B}$, each p_j is zero-free on E , and E is compact.

For each j , choose a closed disc $\overline{D}_j$ containing $p_j(\overline{B})$. Then

$$E = \bigcap_{j=1}^{\infty} \left(p_j^{-1}(\{z \in \overline{D}_j : |\Re z| \geq \varepsilon_j\}) \cap \overline{B} \right).$$

Each set $\{z \in \overline{D}_j : |\Re z| \geq \varepsilon_j\}$ is polynomially convex since it has connected complement in the plane. Hence each set $p_j^{-1}(\{z \in \overline{D}_j : |\Re z| \geq \varepsilon_j\}) \cap \overline{B}$ is polynomially convex (by the elementary [3, Lemma 3.1]). Consequently, E is polynomially convex.

Let x and y be arbitrary distinct points of E . Because $\{p_j\}$ is dense in $P(\overline{B})$, there is some p_j such that $\Re p_j(x) > 1$ and $\Re p_j(y) < -1$. Because $|\Re p_j| \geq \varepsilon_j$ everywhere on E , it follows that x and y lie in different components of E . Consequently, E is totally disconnected. $\square$

Lemma 3.3. *Fix $N \geq 1$ and $\varepsilon > 0$. Then there exists a compact rationally convex set Y with $0 \in Y \subset B \subset \mathbf{C}^N$ such that*

- (i) *the set of polynomials zero-free on Y is dense in $P(\overline{B})$*
- (ii) $\mu(B \setminus Y) < \varepsilon$
- (iii) *Y is connected.*

In fact, the set Y in the lemma can be taken to be a polynomially convex arc with the property that $P(Y) = C(Y)$ [5, Theorem 1.2], but that is much more difficult to prove and will not be needed here.

Proof. Assume without loss of generality that $\varepsilon < \mu(B)$. Choose a countable collection $\{p_j\}$ of polynomials that is dense in $P(\overline{B})$ and such that $p_j(0) \neq 0$ for each j . By the preceding lemma, there exists a totally disconnected, compact polynomially convex set E such that $0 \in E \subset B$, each p_j is zero-free on E , and $\mu(B \setminus E) < \varepsilon$. Let $\sigma : [0, 1] \rightarrow \mathbf{C}^N$ be a continuous map whose image contains E . (For instance, σ could be a space-filling curve.) Assume without loss of generality that $\sigma(0)$ and $\sigma(1)$ are in the set E , and let $K = \sigma^{-1}(E)$. Let $(a_1, b_1), (a_2, b_2), \dots$ be the disjoint open intervals whose union is $[0, 1] \setminus K$. Define a map $\gamma : [0, 1] \rightarrow \mathbf{C}^N$ by setting $\gamma = \sigma$ on K and taking γ to be affine on each interval $[a_k, b_k]$ for $k = 1, 2, \dots$. The reader can verify that γ is continuous. Consequently, $\gamma([0, 1])$ is a connected, compact subset of $B \subset \mathbf{C}^N$ that contains E .

Set $J = \gamma([0, 1])$. Set $I_k = [a_k, b_k]$ for $k = 1, 2, \dots$. Because each p_j is zero-free on E , there exists $\varepsilon_j > 0$ such that $p_j(E)$ is disjoint from $\{z \in \mathbf{C} : |z| < \varepsilon_j\}$. Each set $p_j(\gamma(I_k))$ has empty interior in the plane, so it follows that the set $p_j(J) \cap \{z \in \mathbf{C} : |z| < \varepsilon_j\}$ is a countable union of nowhere dense sets and thus has empty interior. Thus there exist arbitrarily small complex numbers α such that $p_j + \alpha$ has no zeros on

J . Consequently, conditions (i), (ii), and (iii) all hold with Y replaced by J .

Now set $Y = h_r(J)$. Then conditions (i) and (ii) are immediate, and condition (iii) follows from the connectedness of J by the Shilov idempotent theorem. $\square$

The proof of the next lemma is an easy exercise. However, given that the incorrect statement of the lemma in [3] is what created the need for the present paper, and given the central role the lemma plays in the proof of our main result, we include the proof in full.

Lemma 3.4. *Suppose K is a compact set contained in an open set $\Omega \subset \mathbf{C}^N$ with $0 \in \Omega$. Then there exists an R with $0 < R < 1$ such that $\|f\|_K \leq R$ for all $f \in \mathbf{B}_0(\Omega)$ if and only if K is contained in the component of Ω that contains the origin.*

Proof. Suppose that K is contained in the component of Ω that contains the origin. Assume to get a contradiction that no R as in the statement of the lemma exists. Then for each $n = 1, 2, \dots$, there exists a function $f_n \in \mathbf{B}_0(\Omega)$ such that $\|f_n\|_K \geq 1 - 1/n$. By Montel's theorem, $\mathbf{B}_0(\Omega)$ is a normal family, so some subsequence of (f_n) converges uniformly on compact subsets of Ω to a limit function f . Then f is a holomorphic function on Ω such that $f(0) = 0$, $\|f\|_\Omega \leq 1$, and $\|f\|_K \geq 1$. But then the restriction of f to the component of Ω that contains the origin is a nonconstant holomorphic function on a connected, open set that takes on a maximum in modulus at some point, a contradiction.

Now suppose that K fails to be contained in the component of Ω that contains the origin. Let g be the function on Ω that is identically 0 on the component of Ω that contains the origin and is identically 1 everywhere else on Ω . Then $\|g\|_K = 1$ and $g \in \mathbf{B}_0(\Omega)$. Thus no R as in the statement of the lemma exists. $\square$

The next three lemmas are taken from [3]. (As stated in [3], the second of these lemmas contained the hypothesis that $0 \in K$. That hypothesis is actually irrelevant, and so the statement of the lemma has been modified here.)

Lemma 3.5. [3, Lemma 5.2] *Suppose K is a compact set contained in an open set $\Omega \subset \mathbf{C}^N$. Then there exists an $M < \infty$ such that $\|\partial f / \partial z_\nu\|_K \leq M$ for all $f \in \mathbf{B}(\Omega)$ and $\nu = 1, \dots, N$.*

Lemma 3.6. [3, Lemma 6.2, modified] *Let K be a compact set contained in an open set $\Omega \subset \mathbf{C}^N$ with $0 \in \Omega$, let p be a polynomial on $\mathbf{C}^N$ with no zeros on $K \cup \{0\}$, and let $\varepsilon > 0$ be given. Let $0 < R < 1$,*

and suppose that $\|f\|_K \leq R$ for all $f \in \mathbf{B}_0(\Omega)$. Then there exists an $r > 0$ such that $p^{-1}(r\overline{D}) \cap (K \cup \{0\}) = \emptyset$ and $\|f\|_K \leq R + \varepsilon$ for all $f \in \mathbf{B}_0(\Omega \setminus p^{-1}(r\overline{D}))$.

Lemma 3.7. [3, Lemma 6.3] *Let K be a compact set contained in an open set $\Omega \subset \mathbf{C}^N$, let p be a polynomial on $\mathbf{C}^N$ with no zeros on K , and let $\varepsilon > 0$ be given. Let $0 < M < \infty$, and suppose that $\|\partial f / \partial z_\nu\|_K \leq M$ for all $f \in \mathbf{B}(\Omega)$ and $\nu = 1, \dots, N$. Then there exists an $r > 0$ such that $p^{-1}(r\overline{D}) \cap K = \emptyset$ and $\|\partial f / \partial z_\nu\|_K \leq M + \varepsilon$ for all $f \in \mathbf{B}(\Omega \setminus p^{-1}(r\overline{D}))$ and $\nu = 1, \dots, N$.*

Note that in Lemmas 3.6 and 3.7 the conclusion continues to hold with r replaced by any number r' such that $0 < r' < r$.

Lemma 3.8. *Let K be a compact set contained in an open set $\Omega \subset \mathbf{C}^N$ with $0 \in \Omega$, and let $\varepsilon > 0$ be given. Let $0 < R < 1$, and suppose that $\|f\|_K \leq R$ for all $f \in \mathbf{B}_0(\Omega)$. Let (Ω_n) be an increasing sequence of open sets of $\mathbf{C}^N$ with union Ω . Then there exists an n such that $\Omega_n \supset K \cup \{0\}$ and $\|f\|_K \leq R + \varepsilon$ for all $f \in \mathbf{B}_0(\Omega_n)$.*

Proof. Assume to get a contradiction that no such n exists. Let J be a positive integer large enough that $\Omega_J \supset K \cup \{0\}$. Then for each $n = J, J+1, \dots$, there exists a function $f_n \in \mathbf{B}_0(\Omega_n)$ such that $\|f_n\|_K > R + \varepsilon$. For each $m = J, J+1, \dots$, the set $\{f_n : n \geq m\}$ is a normal family on Ω_m . Thus there is a subsequence of (f_n) that converges uniformly on compact subsets of Ω_J . We may then choose a further subsequence that converges uniformly on compact subsets of Ω_{J+1} . Continuing in this manner taking subsequences of subsequences, and then applying the usual diagonalization argument, we arrive at a subsequence (f_{n_k}) of (f_n) such that for each compact subset L of Ω , the sequence (f_{n_k}) converges uniformly on L . Note that given L , there may be a finite number of terms of the sequence (f_{n_k}) that are not defined on L , but this does not matter. Thus there is a well-defined limit function f that is holomorphic on Ω . The function f is in $\mathbf{B}_0(\Omega)$. Thus by hypothesis, $\|f\|_K \leq R$. But since $f_{n_k} \rightarrow f$ uniformly on K , and $\|f_n\|_K > R + \varepsilon$ for all n , this is a contradiction. $\square$

Lemma 3.9. *Let K be a compact set contained in an open set $\Omega \subset \mathbf{C}^N$, and let $\varepsilon > 0$ be given. Let $0 < M < \infty$, and suppose that $\|\partial f / \partial z_\nu\|_K \leq M$ for all $f \in \mathbf{B}(\Omega)$ and $\nu = 1, \dots, N$. Let (Ω_n) be an increasing sequence of open sets of $\mathbf{C}^N$ with union Ω . Then there exists an n such that $\Omega_n \supset K$ and $\|\partial f / \partial z_\nu\|_K \leq M + \varepsilon$ for all $f \in \mathbf{B}(\Omega_n)$ and $\nu = 1, \dots, N$.*

Proof. This is proven by a normal families argument similar to the one just presented and hence is left to the reader. $\square$

Lemma 3.10. *Given $K \subset U \subset \mathbf{C}^N$ with K a compact set and U a connected open set, there exists a connected open set W with compact closure such that $K \subset W \subset \overline{W} \subset U$.*

Proof. Cover K by a finite number of open balls $B_1, \dots, B_k$ whose closures are contained in U . Let $a_1, \dots, a_k$ be the centers of these balls. For each $j = 2, \dots, k$, choose a path γ_j from a_1 to a_j in U . Let γ_j^* denote the image of the path γ_j . Set $J = \overline{B_1} \cup \dots \cup \overline{B_k} \cup \gamma_2^* \cup \dots \cup \gamma_k^*$. Then J is a compact, connected subset of U that contains K . Cover J by a finite number of open balls $W_1, \dots, W_m$ whose closures are contained in U . Assume without loss of generality that each W_j intersects J . Set $W = W_1 \cup \dots \cup W_m$. Then W is as asserted in the statement of the lemma. $\square$

4. PROOF OF THEOREM 1.1

Recall that given a real number $r \geq 0$ and a subset S of $\mathbf{C}^N$, we denote by rS the set $\{rs : s \in S\}$. Recall also the notations $\mathbf{B}(X)$, $\mathbf{B}_0(X)$, $\mathbf{B}(\Omega)$, and $\mathbf{B}_0(\Omega)$ introduced in Section 2.

Proof of Theorem 1.1. Fix $\varepsilon > 0$. Let Y be the set whose existence is given by Lemma 3.3, and choose a countable collection $\{p_j\}$ of polynomials on $\mathbf{C}^N$ that is dense in $P(\overline{B})$ and such that each p_j is zero-free on Y .

It is to be understood that throughout the proof, all subscripts and superscripts are positive integers. We will choose a strictly decreasing sequence $(s_m)_{m=1}^\infty$ of strictly positive numbers less than 1 and a doubly indexed collection $\{r_j^{(m)} : 1 \leq j \leq m\}$ of strictly positive numbers such that for each j , the sequence $(r_j^{(m)})_{m=j}^\infty$ is strictly increasing and bounded. Given these, we define $\tilde{X}_m$ and $\tilde{\Omega}_m$ by

$$(1) \quad \tilde{X}_m = s_m \overline{B} \setminus \bigcup_{j=1}^m p_j^{-1}(r_j^{(m)} D)$$

and

$$(2) \quad \tilde{\Omega}_m = s_m B \setminus \bigcup_{j=1}^m p_j^{-1}(r_j^{(m)} \overline{D}).$$

The numbers $r_j^{(m)}$ will be chosen small enough that $\tilde{\Omega}_m$, and hence also $\tilde{X}_m$, contains the origin. We denote the component of each of $\tilde{X}_m$ and

$\tilde{\Omega}_m$ that contains the origin respectively by X_m and Ω_m . Note that X_m is compact and Ω_m is open. Note also that

$$\tilde{X}_1 \supset \tilde{\Omega}_1 \supset \tilde{X}_2 \supset \tilde{\Omega}_2 \supset \tilde{X}_3 \supset \tilde{\Omega}_3 \supset \cdots$$

and

$$X_1 \supset \Omega_1 \supset X_2 \supset \Omega_2 \supset X_3 \supset \Omega_3 \supset \cdots.$$

Let

$$X = \bigcap_{m=1}^{\infty} X_m.$$

Letting $s = \inf s_m$ and $r_j = \sup_m r_j^{(m)}$, we have that the intersection $\tilde{X} = \bigcap_{m=1}^{\infty} \tilde{X}_m$ is given by

$$\tilde{X} = s\overline{B} \setminus \bigcup_{j=1}^{\infty} p_j^{-1}(r_j D),$$

and since the intersection of a decreasing sequence of compact connected sets is connected, X is the component of $\tilde{X}$ that contains the origin. We will show that for a suitable choice of the sequence (s_m) and the collection $\{r_j^{(m)}\}$, the set X has the properties stated in the theorem.

The choice of (s_m) and $\{r_j^{(m)}\}$ will involve also choosing a nonstrictly increasing sequence (t_n) of strictly positive numbers and a doubly indexed collection $\{u_j^{(n)} : j = 1, 2, \dots \text{ and } n = 1, 2, \dots\}$ of strictly positive numbers such that for each j , the sequence $(u_j^{(n)})_{n=1}^{\infty}$ is nonstrictly decreasing. The choices will be made in such a way that $t_\alpha < s_\beta$ and $u_j^{(\beta)} > r_j^{(\alpha)}$ for all α, β , and j for which the quantities are defined. We then define $\tilde{K}_n^m$ for all $m \geq n \geq 1$ by

$$(3) \quad \tilde{K}_n^m = t_n \overline{B} \setminus \bigcup_{j=1}^m p_j^{-1}(u_j^{(n)} D).$$

Note that then $\tilde{K}_n^m$ is contained in $\tilde{\Omega}_m$. We set

$$(4) \quad K_n^m = \tilde{K}_n^m \cap \Omega_m.$$

As a component of $\tilde{\Omega}_m$, the set Ω_m is closed in $\tilde{\Omega}_m$, and consequently, K_n^m is compact.

We will also choose sequences of numbers (R_n) and (M_n) such that $0 < R_n < 1$ and $0 < M_n < \infty$ for all n .

We will show that (s_m) , $\{r_j^{(m)}\}$, (t_n) , $\{u_j^{(n)}\}$, (R_n) , and (M_n) can be chosen so that the following conditions hold for all $1 \leq n \leq m$ and

$1 \leq j \leq m$. (We present our required conditions in a slightly odd way so as to set up a proof by induction. Conditions (i)–(vii) and (xii)–(xiii) have already been mentioned above.)

- (i) The numbers $s_m, r_j^{(m)}, t_n$, and $u_j^{(n)}$ are strictly positive.
- (ii) $1 > s_1 > s_2 > \cdots > s_m$
- (iii) $r_j^{(j)} < r_j^{(j+1)} < \cdots < r_j^{(m)}$
- (iv) $t_1 \leq t_2 \leq \cdots \leq t_m$
- (v) $u_j^{(1)} \geq u_j^{(2)} \geq \cdots \geq u_j^{(m)}$
- (vi) $t_m < s_m$ (and hence, $t_\alpha < s_\beta$ for all $1 \leq \alpha \leq m$ and $1 \leq \beta \leq m$)
- (vii) $u_j^{(m)} > r_j^{(m)}$ (and hence $u_j^{(\beta)} > r_j^{(\alpha)}$ for all $j \leq \alpha \leq m$ and $1 \leq \beta \leq m$)
- (viii) $s_m B \supset Y$
- (ix) $p_j^{-1}(r_j^{(m)} \overline{D})$ is disjoint from Y
- (x) $\mu(X_n \setminus K_n^n) \leq (1/n)\mu(B)$
- (xi) $\mu(p_j^{-1}(u_j^{(n)} D) \cap B) \leq (1/2^{j-1})\mu(B)$
- (xii) $0 < R_n < 1$
- (xiii) $0 < M_n < \infty$
- (xiv) For all $f \in \mathbf{B}_0(\Omega_m)$,

$$\|f\|_{K_n^m} \leq R_n + \left(\frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{m-n+1}}\right)(1 - R_n).$$

(When $m = n$, the sum in parentheses is understood to be zero.)

- (xv) For all $f \in \mathbf{B}(\Omega_m)$ and $\nu = 1, \dots, N$,

$$\|\partial f / \partial z_\nu\|_{K_n^m} \leq \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n}}\right)M_n.$$

Assuming for the moment that $(s_m), \{r_j^{(m)}\}, (t_n), \{u_j^{(n)}\}, (R_n)$, and (M_n) have been chosen so that the above conditions are satisfied, we now prove that the set X has the properties stated in the theorem. Obviously X is contained in B . Clearly $\tilde{X}$ is rationally convex, and hence the component X of $\tilde{X}$ is rationally convex by Lemma 3.1. By conditions (viii) and (ix), $\tilde{X} \supset Y$. Therefore, $X \supset Y$ since Y is connected and contains the origin. Consequently, $\mu(B \setminus X) < \varepsilon$. Since each p_j is zero-free on X , the collection of polynomials zero-free on X is dense in $P(\overline{B})$. Finally, for each $n = 1, 2, \dots$, set $K_n = \bigcap_{m=n}^\infty K_n^m$, and set $P = \bigcup_{n=1}^\infty K_n$. Then P is contained in X , and we now show that P is as asserted in the statement of the theorem.

Note that for every n ,

$$X \setminus P \subset X \setminus K_n \subset \left[X \setminus K_n^n \right] \cup \left[\bigcup_{j=n+1}^{\infty} (p_j^{-1}(u_j^{(n)} D) \cap B) \right].$$

Letting $n \rightarrow \infty$, we see that conditions (x) and (xi) imply that $\mu(X \setminus P) = 0$.

Now consider an arbitrary point $z \in P$. Fix n such that $z \in K_n$. Given $g \in \mathbf{B}_0(X)$ and a constant $0 < a < 1$, there is some $m \geq n$ such that $ag \in \mathbf{B}_0(\Omega_m)$. Since $z \in K_n^m$, condition (xiv) gives that

$$|ag(z)| \leq R_n + \left(\frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{m-n+1}}\right)(1 - R_n).$$

Letting $a \rightarrow 1$ shows that this inequality continues to hold with ag replaced by g . Therefore,

$$\sup_{f \in \mathbf{B}_0(X)} |f(z)| \leq R_n + \frac{1}{2}(1 - R_n) < 1.$$

Thus by Lemma 2.1, z lies in the Gleason part of the origin for $R(X)$. Similarly, using condition (xv), we get that

$$\sup_{f \in \mathbf{B}(X)} |(\partial f / \partial z_\nu)(z)| \leq 2M_n < \infty \text{ for all } \nu = 1, \dots, N.$$

Thus by Lemma 2.2, the space of bounded point derivations on $R(X)$ at z has dimension N .

It remains to choose (s_m) , $\{r_j^{(m)}\}$, (t_n) , $\{u_j^{(n)}\}$, (R_n) , and (M_n) . First choose $0 < s_1 < 1$ large enough that $s_1 B \supset Y$, and choose $r_1^{(1)} > 0$ small enough that $p_1^{-1}(r_1^{(1)} \overline{D})$ is disjoint from Y . Then choose $0 < t_1 < s_1$ and $u_1^{(1)} > r_1^{(1)}$. The sets $\tilde{X}_1$, $\tilde{\Omega}_1$, X_1 , Ω_1 , $\tilde{K}_1^1$, and K_1^1 are now defined in accordance with equations (1), (2), (3) and (4), and the remarks that immediately follow them. Lemmas 3.4 and 3.5 yield numbers $0 < R_1 < 1$ and $0 < M_1 < \infty$ such that

$$\|f\|_{K_1^1} \leq R_1 \quad \text{for all } f \in \mathbf{B}_0(\Omega_1)$$

and

$$\|\partial f / \partial z_\nu\|_{K_1^1} \leq M_1 \quad \text{for all } f \in \mathbf{B}(\Omega_1) \text{ and } \nu = 1, \dots, N.$$

Now conditions (i)–(xv) hold for $n = m = j = 1$.

We continue by induction. Suppose that for some $k \geq 1$ we have chosen s_m , $r_j^{(m)}$, t_n , $u_j^{(n)}$, R_n , and M_n for all $1 \leq n \leq m \leq k$ and $1 \leq j \leq m \leq k$ such that conditions (i)–(xv) hold for all these values of n , m , and j . We will refer to conditions (i)–(xv) for these values of n , m , and j as conditions (i)–(xv) of the induction hypothesis.

Choose $u_{k+1}^{(1)} = u_{k+1}^{(2)} = \dots = u_{k+1}^{(k)} > 0$ small enough that for each $n = 1, \dots, k$ we have that $\mu(p_{k+1}^{-1}(u_{k+1}^{(n)}D) \cap B) < (1/2^k)\mu(B)$. Equation (3) now defines sets $\tilde{K}_1^{k+1}, \dots, \tilde{K}_k^{k+1}$. For $n = 1, \dots, k$, set

$$\check{K}_n^{k+1} = \tilde{K}_n^{k+1} \cap \Omega_k,$$

and note that then $\check{K}_n^{k+1} \subset K_n^k$. Note also that each of $\check{K}_1^{k+1}, \dots, \check{K}_k^{k+1}$ is compact. Lemmas 3.6 and 3.7, together with conditions (xiv) and (xv) of the induction hypothesis, yield the existence of $0 < c < u_{k+1}^{(k)}$ such that $0 \notin p_{k+1}^{-1}(c\overline{D})$ and such that for all $n = 1, \dots, k$, we have

$$(5) \quad \|f\|_{\check{K}_n^{k+1}} \leq R_n + \left(\frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{k-n+1}} + \frac{1/2}{2^{(k+1)-n+1}}\right)(1 - R_n)$$

for all $f \in \mathbf{B}_0(\Omega_k \setminus p_{k+1}^{-1}(c\overline{D}))$

and

$$(6) \quad \|\partial f / \partial z_\nu\|_{\check{K}_n^{k+1}} \leq \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{k-n}} + \frac{1/2}{2^{(k+1)-n}}\right)M_n$$

for all $f \in \mathbf{B}(\Omega_k \setminus p_{k+1}^{-1}(c\overline{D}))$ and $\nu = 1, \dots, N$.

Note that by choosing c small enough, we can arrange to have $p_{k+1}^{-1}(c\overline{D})$ disjoint from Y .

Denote the component of $\Omega_k \setminus p_{k+1}^{-1}(c\overline{D})$ containing the origin by U . By the regularity of Lebesgue measure, there exists a compact set K contained in U such that $\mu(U \setminus K) < \frac{1}{2}(1/(k+1))\mu(B)$. By Lemma 3.10, there exists a connected open set W such that $K \subset W \subset \overline{W} \subset U$. Then of course

$$(7) \quad \mu(U \setminus \overline{W}) < \frac{1}{2}(1/(k+1))\mu(B).$$

If we let (α_n) , $(\beta_j^{(n)})_{n=1}^\infty$ for $j = 1, \dots, k$, and (γ_n) be sequences of strictly positive numbers with (α_n) strictly increasing to s_k , with $(\beta_j^{(n)})_{n=1}^\infty$ strictly decreasing to $r_j^{(k)}$, and with (γ_n) strictly decreasing to c , and we set

$$V_n = \alpha_n B \setminus \left[\left(\bigcup_{j=1}^k p_j^{-1}(\beta_j^{(n)}\overline{D}) \right) \cup p_{k+1}^{-1}(\gamma_n\overline{D}) \right],$$

then (V_n) is an increasing sequence of open sets with union $\tilde{\Omega}_k \setminus p_{k+1}^{-1}(c\overline{D})$. Therefore, by Lemmas 3.8 and 3.9, it follows from inequalities (5) and (6) that there exist numbers s_{k+1} and $r_j^{(k+1)}$ for $j = 1, \dots, k+1$ such that $0 < t_k < s_{k+1} < s_k$, such that $r_j^{(k)} < r_j^{(k+1)} < u_j^{(k)}$ for $j = 1, \dots, k$, such that $c < r_{k+1}^{(k+1)} < u_{k+1}^{(k)}$, and such that defining $\tilde{\Omega}_{k+1}$ in accordance with equation (2), we have that $\tilde{\Omega}_{k+1}$ contains each of $\check{K}_1^{k+1}, \dots, \check{K}_k^{k+1}$

and contains the origin and we have, for each $n = 1, \dots, k$, the inequalities

$$(8) \quad \|f\|_{\check{K}_n^{k+1}} \leq R_n + \left(\frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{k-n+1}} + \frac{1}{2^{(k+1)-n+1}}\right)(1 - R_n) \\ \text{for all } f \in \mathbf{B}_0(\tilde{\Omega}_{k+1})$$

and

$$(9) \quad \|\partial f / \partial z_\nu\|_{\check{K}_n^{k+1}} \leq \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{k-n}} + \frac{1}{2^{(k+1)-n}}\right)M_n \\ \text{for all } f \in \mathbf{B}(\tilde{\Omega}_{k+1}) \text{ and } \nu = 1, \dots, N.$$

Then by Lemma 3.4, each of $\check{K}_1^{k+1}, \dots, \check{K}_k^{k+1}$ is contained in the component Ω_{k+1} of $\tilde{\Omega}_{k+1}$ containing the origin, and inequalities (8) and (9) continue to hold with $\tilde{\Omega}_{k+1}$ replaced by Ω_{k+1} . Furthermore, by conditions (viii) and (ix) of the induction hypothesis and the compactness of Y and $\overline{W}$, the numbers s_{k+1} and $r_1^{(k+1)}, \dots, r_{k+1}^{(k+1)}$ can be chosen so that in addition, $s_{k+1}B \supset Y$ and for $j = 1, \dots, k+1$, the set $p_j^{-1}(r_j^{(k+1)}\overline{D})$ is disjoint from Y (and hence $Y \subset \tilde{\Omega}_{k+1}$), and also $\overline{W} \subset \tilde{\Omega}_{k+1}$. Then by the connectedness of Y and $\overline{W}$, we have that $Y \subset \Omega_{k+1}$ and $\overline{W} \subset \Omega_{k+1}$.

With s_{k+1} and $r_1^{(k+1)}, \dots, r_{k+1}^{(k+1)}$ now chosen, $\tilde{X}_{k+1}$ is defined in accordance with equation (1), and X_{k+1} is defined to be the component of $\tilde{X}_{k+1}$ that contains the origin. Note that $\tilde{X}_{k+1} \subset \tilde{\Omega}_k \setminus p_{k+1}^{-1}(c\overline{D})$, and so $X_{k+1} \subset U$. Consequently, $X_{k+1} \setminus \Omega_{k+1} \subset U \setminus \overline{W}$. Therefore, by inequality (7) we have

$$(10) \quad \mu(X_{k+1} \setminus \Omega_{k+1}) \leq \mu(U \setminus \overline{W}) < \frac{1}{2}(1/(k+1))\mu(B).$$

We define the sets $K_1^{k+1}, \dots, K_k^{k+1}$ in accordance with equation (4). Observe that then

$$K_n^{k+1} = \check{K}_n^{k+1} \quad \text{for all } n = 1, \dots, k.$$

Therefore, since inequalities (8) and (9) hold with $\tilde{\Omega}_{k+1}$ replaced by Ω_{k+1} , we have for all $n = 1, \dots, k$ that

$$\|f\|_{K_n^{k+1}} \leq R_n + \left(\frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{k-n+1}} + \frac{1}{2^{(k+1)-n+1}}\right)(1 - R_n) \\ \text{for all } f \in \mathbf{B}_0(\Omega_{k+1})$$

and

$$\|\partial f / \partial z_\nu\|_{K_n^{k+1}} \leq \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{k-n}} + \frac{1}{2^{(k+1)-n}}\right)M_n \\ \text{for all } f \in \mathbf{B}(\Omega_{k+1}) \text{ and } \nu = 1, \dots, N.$$

Finally, choose t_{k+1} and $u_1^{(k+1)}, \dots, u_{k+1}^{(k+1)}$ such that $0 < t_k \leq t_{k+1} < s_{k+1}$ and $u_j^{(k)} \geq u_j^{(k+1)} > r_j^{(k+1)}$ for $j = 1, \dots, k+1$ with t_{k+1} close

enough to s_{k+1} and each $u_j^{(k+1)}$ close enough to $r_j^{(k+1)}$ that defining $\tilde{K}_{k+1}^{k+1}$ in accordance with equation (3) we have $\tilde{K}_{k+1}^{k+1} \subset \tilde{\Omega}_{k+1}$ and

$$\mu(\tilde{\Omega}_{k+1} \setminus \tilde{K}_{k+1}^{k+1}) < \frac{1}{2}(1/(k+1))\mu(B).$$

Then with K_{k+1}^{k+1} defined in accordance with equation (4), we also have

$$\mu(\Omega_{k+1} \setminus K_{k+1}^{k+1}) < \frac{1}{2}(1/(k+1))\mu(B).$$

Combined with inequality (10), this gives that

$$\mu(X_{k+1} \setminus K_{k+1}^{k+1}) < (1/(k+1))\mu(B).$$

Lemmas 3.4 and 3.5 yield numbers $0 < R_{k+1} < 1$ and $0 < M_{k+1} < \infty$ such that

$$\|f\|_{K_{k+1}^{k+1}} \leq R_{k+1} \quad \text{for all } f \in \mathbf{B}_0(\Omega_{k+1})$$

and

$$\|\partial f / \partial z_\nu\|_{K_{k+1}^{k+1}} \leq M_{k+1} \quad \text{for all } f \in \mathbf{B}(\Omega_{k+1}) \text{ and } \nu = 1, \dots, N.$$

Now, as the reader can verify, we have chosen $s_m, r_j^{(m)}, t_n, u_j^{(n)}, R_n,$ and M_n for all $1 \leq n \leq m \leq k+1$ and $1 \leq j \leq m \leq k+1$ in such a way that conditions (i)–(xv) hold for all these values of $n, m,$ and j . Thus the induction can proceed, and the proof of Theorem 1.1 is complete. $\square$

5. CORRECTION TO THE PROOF OF [3, Theorem 1.7]

In the proof of [3, Theorem 1.7] there are compact subsets K and X of $\mathbf{C}^N$ ($N \geq 2$) with $K \subset h_r(X) = \hat{X}$, an injective holomorphic map $\sigma : D \rightarrow \hat{X} \setminus K$, and a polynomial p on $\mathbf{C}^N$ such that $p(\sigma(0)) = 0$ and p has no zeros on K . It is asserted that “because σ is injective, we may assume, by adding to p a small multiple of a suitable first degree polynomial if necessary, that $(\partial(p \circ \sigma)/\partial z)(0) \neq 0$ ”. This is of course false since it could be that $(\partial\sigma/\partial z)(0) = 0$. (For instance, σ could be given by $\sigma(z) = (z^2, z^3)$.) However, all that is needed in the proof of [3, Theorem 1.7] is the ability to assume that p is not identically zero on $\sigma(D)$. This can be achieved by adding to p , if necessary, a small multiple of a polynomial that vanishes at $\sigma(0)$ but is not identically zero on $\sigma(D)$.

A generalization of [3, Theorem 1.7], whose proof runs along the same lines, is given in the author’s paper [4]. There the argument is presented correctly; see the proof of [4, Theorem 5.1].

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