



On maximum packings of λ -fold complete 3-uniform hypergraphs with triple-hyperstars of size 4

Amber Armstrong^a, Ryan C. Bunge^b, William Duncan^b, Saad I. El-Zanati^b,
Kristin Koe^b, Rachel Stutzman^c

^aSt. Paul Academy and Summit School, St. Paul, MN

^bIllinois State University, Normal, IL 61790-4520

^cGreenville University, Greenville, IL

rcbunge@ilstu.edu, saad@ilstu.edu

Abstract

A symmetric triple-hyperstar is a connected, 3-uniform hypergraph where, for some edge $\{a, b, c\}$, vertices a , b , and c all have degree $k > 1$ and all other edges contain exactly 2 vertices of degree 1. Let H denote the symmetric triple-hyperstar with 4 edges and, for positive integers λ and v , let ${}^\lambda K_v^{(3)}$ denote the λ -fold complete 3-uniform hypergraph on v vertices. We find maximum packings of ${}^\lambda K_v^{(3)}$ with copies of H .

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1. Introduction

A hypergraph H consists of a finite, nonempty set V of vertices and a finite collection $E = \{e_1, e_2, \dots, e_m\}$ of nonempty subsets of V called *hyperedges* or simply *edges*. For a given hypergraph H , we use $V(H)$ and $E(H)$ to denote the vertex set and the edge set (or multiset) of H , respectively. We call $|V(H)|$ and $|E(H)|$ the *order* and *size* of H , respectively. A hypergraph H is *simple* if no edge appears more than once in $E(H)$. If for each $e \in E(H)$ we have $|e| = t$, then H is said to be *t-uniform*. Thus *t-uniform* hypergraphs are generalizations of the concept

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of a graph (where $t = 2$). Graphs with repeated edges are often called *multigraphs*. If H is a simple hypergraph and λ is a positive integer, then λ -fold H , denoted ${}^\lambda H$, is the multi-hypergraph obtained from H by repeating each edge exactly λ times. The hypergraph with vertex set V and edge set the set of all t -element subsets of V is called the *complete t -uniform hypergraph on V* and is denoted by $K_V^{(t)}$. If $v = |V|$, then ${}^\lambda K_v^{(t)}$ is called the *λ -fold complete t -uniform hypergraph of order v* and is used to denote any hypergraph isomorphic to ${}^\lambda K_V^{(t)}$. When $t = 2$, we will use ${}^\lambda K_v$ in place of ${}^\lambda K_v^{(2)}$. Similarly, if $\lambda = 1$, then we will use $K_v^{(t)}$ in place of ${}^1 K_v^{(t)}$. If H' is a subhypergraph of H , then $H \setminus H'$ denotes the hypergraph obtained from H by deleting the edges of H' . We may refer to $H \setminus H'$ as the hypergraph H with a *hole* H' . The vertices in H' may be referred to as the vertices in the hole.

A commonly studied problem in combinatorics concerns decompositions of graphs or multigraphs into edge-disjoint subgraphs. A *decomposition* of a multigraph K is a set $\Delta = \{G_1, G_2, \dots, G_s\}$ of subgraphs of K such that $\{E(G_1), E(G_2), \dots, E(G_s)\}$ is a partition of $E(K)$. If each element of Δ is isomorphic to a fixed graph G , then Δ is called a *G -decomposition* of K . If exactly one element $L \in \Delta$ is not isomorphic to G , then Δ is called a *G -packing* of K with *leave* L . Such a G -packing is *maximum* if no other possible G -packing of K has a leave of a smaller size than that of L . Clearly, if $|E(L)| < |E(G)|$, then the G -packing is maximum. Moreover, a G -decomposition of K can be viewed as a maximum G -packing with an empty leave.

A G -decomposition of ${}^\lambda K_v$ is also known as a *G -design of order v and index λ* . A K_k -design of order v and index λ is usually known as a $2-(v, k, \lambda)$ *design* or as a *balanced incomplete block design of index λ* or a (v, k, λ) -*BIBD*. The problem of determining all v for which there exists a G -design of order v is of special interest (see [1] for a survey).

The notion of decompositions of graphs naturally extends to hypergraphs. A *decomposition* of a hypergraph K is a set $\Delta = \{H_1, H_2, \dots, H_s\}$ of subhypergraphs of K such that $\{E(H_1), E(H_2), \dots, E(H_s)\}$ is a partition of $E(K)$. Any element of Δ isomorphic to a fixed hypergraph H is called an *H -block*. If all elements of Δ are H -blocks, then Δ is called an *H -decomposition* of K . If exactly one element $L \in \Delta$ is not an H -block, then Δ is called an *H -packing* of K with *leave* L , where we again define such a packing to be *maximum* if L has the fewest edges possible. An H -decomposition of ${}^\lambda K_v^{(t)}$ is called an *H -design of order v and index λ* . The problem of determining all v for which there exists an H -design of order v and index λ is called the *λ -fold spectrum problem for H -designs*.

A $K_k^{(t)}$ -design of order v and index λ is a generalization of $2-(v, k, \lambda)$ designs and is known as a $t-(v, k, \lambda)$ *design* or simply as a *t -design*. A summary of results on t -designs appears in [16]. A $t-(v, k, 1)$ design is also known as a *Steiner system* and is denoted by $S(t, v, k)$ (see [9] for a summary of results on Steiner systems). Keevash [15] has recently shown that for all t and k the obvious necessary conditions for the existence of an $S(t, k, v)$ -design are sufficient for sufficiently large values of v . Similar results were obtained by Glock, Kühn, Lo, and Osthus [10, 11] and extended to include the corresponding asymptotic results for H -designs of order v for all uniform hypergraphs H . These results for t -uniform hypergraphs mirror the celebrated results of Wilson [24] for graphs. Although these asymptotic results assure the existence of H -designs for sufficiently large values of v for any uniform hypergraph H , the spectrum problem has been settled for very few hypergraphs of uniformity larger than 2.

In the study of graph decompositions, a fair amount of the focus has been on G -decompositions of K_v where G is a graph with a relatively small number of edges (see [1] and [6] for known results). Some authors have investigated the corresponding problem for 3-uniform hypergraphs. For example, in [4], the 1-fold spectrum problem is settled for all 3-uniform hypergraphs on 4 or fewer vertices. More recently, the 1-fold spectrum problem was settled in [5] for all 3-uniform hypergraphs with at most 6 vertices and at most 3 edges. In [5], they also settle the 1-fold spectrum problem for the 3-uniform hypergraph of order 6 whose edges form the lines of the Pasch configuration. Authors have also considered H -designs where H is a 3-uniform hypergraph whose edge set is defined by the faces of a regular polyhedron. Let T , O , and I denote the tetrahedron, the octahedron, and the icosahedron hypergraphs, respectively. The hypergraph T is the same as $K_4^{(3)}$, and its spectrum was settled in 1960 by Hanani [12]. In another paper [13], Hanani settled the spectrum problem for O -designs and gave necessary conditions for the existence of I -designs. The 1-fold spectrum problem is also settled for a type of 3-uniform hyperstars which is part of a larger class of hypergraphs known as delta-systems. For a positive integer m , let $S_m^{(3)}$ denote the 3-uniform hypergraph of size m that consists of one vertex of degree m and $2m$ vertices of degree one. Necessary and sufficient conditions for the existence of $S_m^{(3)}$ -decompositions of $K_v^{(3)}$ are given in [22] for $m \in \{4, 5, 6\}$ and settled in [19] for any m . Some results on maximum $S_m^{(3)}$ -packings of $K_v^{(3)}$ are given in [20]. Perhaps the best known general result on decompositions of complete t -uniform hypergraphs is Baranyai's result [3] on the existence of 1-factorizations of $K_{mt}^{(t)}$ for all positive integers m . There are, however, several articles on decompositions of complete t -uniform hypergraphs (see [2] and [21]) and of t -uniform t -partite hypergraphs (see [17] and [23]) into variations on the concept of a Hamilton cycle. There are also several results on decompositions of 3-uniform hypergraphs into structures known as Berge cycles with a given number of edges (see for example [14] and [18]). We note however that the Berge cycles in these decompositions are not required to be isomorphic.

In this paper we are interested in maximum H -packings of ${}^\lambda K_v^{(3)}$, where H is a 3-uniform symmetric triple-hyperstar with 4 edges. A *triple-hyperstar* is a connected 3-uniform hypergraph where, for some edge $\{a, b, c\}$, vertices a , b , and c all have degree greater than 1 and all other edges contain exactly two vertices of degree 1. That is, if the degrees of vertices a , b , and c in the triple-hyperstar are $m_1 + 1$, $m_2 + 1$, and $m_3 + 1$, respectively, then the removal of edge $\{a, b, c\}$ would result in the hypergraph consisting of three components, namely $S_{m_1}^{(3)}$, $S_{m_2}^{(3)}$, and $S_{m_3}^{(3)}$. We call such a triple-hyperstar *symmetric* if $m_1 = m_2 = m_3 = m$. Thus a symmetric triple-hyperstar has $6m + 3$ vertices and $3m + 1$ edges. We are interested in the case $m = 1$.

Let $H[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9]$ denote the symmetric triple-hyperstar H with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ and edge set $\{\{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}, \{v_2, v_6, v_7\}, \{v_3, v_8, v_9\}\}$ as seen in Figure 1. Here we show that for all $v \geq 9$ and $\lambda \geq 1$, there exists a maximum H -packing of ${}^\lambda K_v^{(3)}$ where the leave has fewer than 4 edges.

1.1. Additional Notation and Terminology

Let $\mathbb{Z}_n$ denote the group of integers modulo n . We next define some notation for certain types of 3-uniform hypergraphs.

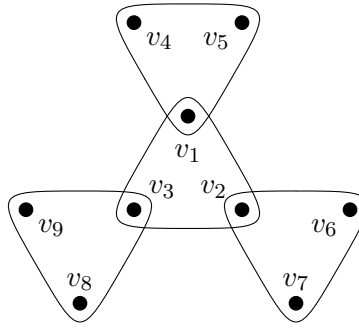


Figure 1. The symmetric triple-hyperstar H of size 4, denoted by $H[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9]$.

Let U_1, U_2, U_3 be pairwise disjoint sets. The hypergraph with vertex set $U_1 \cup U_2 \cup U_3$ and edge set consisting of all 3-element sets having exactly one vertex in each of U_1, U_2, U_3 is denoted by $K_{U_1, U_2, U_3}^{(3)}$. The hypergraph with vertex set $U_1 \cup U_2$ and edge set consisting of all 3-element sets having at most 2 vertices in each of U_1, U_2 is denoted by $L_{U_1, U_2}^{(3)}$. If $|U_i| = u_i$ for $i \in \{1, 2, 3\}$, we may use $K_{u_1, u_2, u_3}^{(3)}$ or $L_{u_1, u_2}^{(3)}$ to denote any hypergraph that is isomorphic to $K_{U_1, U_2, U_3}^{(3)}$ or $L_{U_1, U_2}^{(3)}$, respectively.

2. Main Results

2.1. Decompositions and Packings of Simple Hypergraphs

We begin by giving necessary conditions for the existence of an H -decomposition of $K_v^{(3)}$. An obvious necessary condition is that 4 must divide the number of edges in $K_v^{(3)}$, and thus we must have $v \equiv 0, 1, 2, 4$, or $6 \pmod{8}$. Since $K_1^{(3)}$ and $K_2^{(3)}$ contain no edges, it is vacuously true that H decomposes $K_1^{(3)}$ and $K_2^{(3)}$. Also, since H has order 9, there is no H -decomposition of $K_4^{(3)}$, $K_6^{(3)}$, or $K_8^{(3)}$. Hence, we have the following.

Lemma 1. *There exists an H -decomposition of $K_v^{(3)}$ only if $v \equiv 0, 1, 2, 4$, or $6 \pmod{8}$ and $v \notin \{4, 6, 8\}$.*

We intend to prove that the above conditions are sufficient by showing how to construct H -decompositions of $K_v^{(3)}$ for all $v \equiv 0, 1, 2, 4$, or $6 \pmod{8}$ with $v \geq 9$. Our constructions are dependent on the many small examples given in the Appendix. We begin by proving a lemma that is fundamental to our constructions.

Lemma 2. *Let n, x , and r be nonnegative integers such that $nx + r \geq 3$. There exists a decomposition of $K_{nx+r}^{(3)}$ that is comprised of isomorphic copies of each of the following under the given conditions:*

- $K_r^{(3)}$ if $x = 0$,
- $K_{n+r}^{(3)}$ if $x \geq 1$,

- $K_{n+r}^{(3)} \setminus K_r^{(3)}$ if $x \geq 2$,
- $K_{r,n,n}^{(3)} \cup L_{n,n}^{(3)}$ if $x \geq 2$,
- $K_{n,n,n}^{(3)}$ if $x \geq 3$.

Furthermore, if $x \geq 1$ and $r \geq 3$, then the decomposition contains exactly one isomorphic copy of $K_{n+r}^{(3)}$.

Proof. If $x \in \{0, 1\}$, the decomposition is trivial. Similarly, if $n = 0$, then $r \geq 3$, and the result is trivial because $K_r^{(3)} = K_{n+r}^{(3)} = K_{nx+r}^{(3)}$ while $K_{n+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,n,n}^{(3)} \cup L_{n,n}^{(3)}$, and $K_{n,n,n}^{(3)}$ are all empty (i.e., contain no edges). For the remainder of the proof, we assume that $x \geq 2$ and $n \geq 1$.

Let $V_0, V_1, \dots, V_x$ be pairwise disjoint sets of vertices with $|V_0| = r$ and $|V_1| = |V_2| = \dots = |V_x| = n$. Then, the decomposition of $K_{nx+r}^{(3)}$ results from the fact that the complete 3-uniform hypergraph on the vertex set $V_0 \cup V_1 \cup \dots \cup V_x$, which is $nx + r$ vertices, can be viewed as the (edge-disjoint) union

$$K_{V_1 \cup V_0}^{(3)} \cup \bigcup_{2 \leq i \leq x} \left(K_{V_i \cup V_0}^{(3)} \setminus K_{V_0}^{(3)} \right) \cup \bigcup_{1 \leq i < j \leq x} \left(K_{V_0, V_i, V_j}^{(3)} \cup L_{V_i, V_j}^{(3)} \right) \cup \bigcup_{1 \leq i < j < k \leq x} \left(K_{V_i, V_j, V_k}^{(3)} \right).$$

In addition, if $r \geq 3$, the single isomorphic copy of $K_{n+r}^{(3)}$ in the decomposition is $K_{V_1 \cup V_0}^{(3)}$. $\square$

We now give our main results.

Theorem 3. *There exists an H -decomposition of $K_v^{(3)}$ if and only if $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod{8}$ and $v \notin \{4, 6, 8\}$.*

Proof. The necessary conditions for the existence of an H -decomposition of $K_v^{(3)}$ are established in Lemma 1. Thus we need only to establish their sufficiency. Let $v = 8x + r$ where $x \geq 1$ and $r \in \{1, 2, 4, 6, 8\}$. By Lemma 2 it suffices to find H -decompositions of $K_{8+r}^{(3)}$, $K_{8+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,8,8}^{(3)} \cup L_{8,8}^{(3)}$, and $K_{8,8,8}^{(3)}$. We note that if $r \in \{1, 2\}$ then $K_{8+r}^{(3)} \setminus K_r^{(3)}$ is isomorphic to $K_{8+r}^{(3)}$. Also, $K_{3,8,8}^{(3)}$ decomposes $K_{6,8,8}^{(3)}$, and $K_{4,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$. Thus, it suffices to find H -decompositions of $K_9^{(3)}$, $K_{10}^{(3)}$, $K_{12}^{(3)}$, $K_{14}^{(3)}$, $K_{16}^{(3)}$, $K_{12}^{(3)} \setminus K_4^{(3)}$, $K_{14}^{(3)} \setminus K_6^{(3)}$, $K_{16}^{(3)} \setminus K_8^{(3)}$, $K_{1,8,8}^{(3)} \cup L_{8,8}^{(3)}$, $K_{2,8,8}^{(3)} \cup L_{8,8}^{(3)}$, $K_{3,8,8}^{(3)}$, $K_{4,8,8}^{(3)}$, and $L_{8,8}^{(3)}$, which are each shown to exist within Examples 1–16. $\square$

Theorem 4. *If $v \geq 9$ is an integer, then there exists a maximum H -packing of $K_v^{(3)}$ where the leave has fewer than four edges.*

Proof. If $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod{8}$, then the result follows from the H -decomposition result in Theorem 3, which translates to a maximum H -packing with an empty leave. Hence, we need only consider when $v \equiv 3, 5, \text{ or } 7 \pmod{8}$. Let $v = 8x + r$ where $x \geq 1$ and $r \in \{3, 5, 7\}$. By Lemma 2 it suffices to find

- a maximum H -packing of $K_{8+r}^{(3)}$ with a leave consisting of fewer than four edges and
- H -decompositions of $K_{8+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,8,8}^{(3)} \cup L_{8,8}^{(3)}$, and $K_{8,8,8}^{(3)}$.

We note that an H -decomposition of $K_{11}^{(3)} \setminus K_3^{(3)}$ is a subset of an H -packing of $K_{11}^{(3)}$ with a leave consisting of the single edge in the hole, which is necessarily then a maximum H -packing of $K_{11}^{(3)}$. Also, $K_{7,8,8}^{(3)}$ is decomposable into copies of $K_{3,8,8}^{(3)}$ and $K_{4,8,8}^{(3)}$, and $K_{4,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$. Thus, it suffices to find maximum H -packings (with leaves of fewer than four edges) of $K_{11}^{(3)}$, $K_{13}^{(3)}$, and $K_{15}^{(3)}$, which are each shown to exist in Examples 17–19, and H -decompositions of $K_{13}^{(3)} \setminus K_5^{(3)}$, $K_{15}^{(3)} \setminus K_7^{(3)}$, $K_{3,8,8}^{(3)}$, $K_{4,8,8}^{(3)}$, $K_{5,8,8}^{(3)}$, and $L_{8,8}^{(3)}$, which are each shown to exist within Examples 6–15. $\square$

2.2. Results for any Positive Index

We show here the necessary conditions for an H -decomposition of λ -fold $K_v^{(3)}$ for any positive integer λ . This will inform our choice on which combinations of λ and v we search for decompositions of $\lambda K_v^{(3)}$ versus finding maximum packings.

Lemma 5. *Let $v \geq 9$ be an integer. There exists an H -decomposition of λ -fold $K_v^{(3)}$ only if the following hold:*

- if $\gcd(\lambda, 4) = 1$, then $v \equiv 0, 1, 2, 4$, or $6 \pmod{8}$;
- if $\gcd(\lambda, 4) = 2$, then $v \equiv 0, 1$, or $2 \pmod{4}$;
- if $\gcd(\lambda, 4) = 4$, then $v \geq 9$.

Proof. Suppose there exists an H -decomposition of $\lambda K_v^{(3)}$. Since $|E(H)| = 4$, we must have $4 \mid \lambda \binom{v}{3} = \lambda v(v-1)(v-2)/6$, and thus $8 \mid \lambda v(v-1)(v-2)$. First, if $\gcd(\lambda, 4) = 1$, then $8 \mid v(v-1)(v-2)$, and thus $v \equiv 0, 1, 2, 4$, or $6 \pmod{8}$. Second, if $\gcd(\lambda, 4) = 2$, then $4 \mid v(v-1)(v-2)$, and thus $v \equiv 0, 1$, or $2 \pmod{4}$. Finally, if $\gcd(\lambda, 4) = 4$, then $2 \mid v(v-1)(v-2)$, which is true for any $v \geq 9$. $\square$

Next, we settle the decomposition and maximum packing results for some small values of λ .

Theorem 6. *Let $v \geq 9$ be an integer. There exists an H -decomposition of 2-fold $K_v^{(3)}$ if $v \equiv 0, 1$, or $2 \pmod{4}$.*

Proof. If $v \equiv 0, 1, 2, 4$, or $6 \pmod{8}$, then the result follows from 2 copies of an H -decomposition of $K_v^{(3)}$, which exists by Theorem 3. Hence, we need only consider when $v \equiv 5 \pmod{8}$.

First, we consider when $v = 13$. Let $v_1, v_2, \dots, v_9 \in V(K_v^{(3)})$. By Example 18, there exist both a maximum H -packing, say Δ_1 , of $K_{13}^{(3)}$ with a leave consisting of two edges that share a single vertex and a maximum H -packing, say Δ_2 , of $K_{13}^{(3)}$ with a leave consisting of two vertex-disjoint edges. Let L_1 and L_2 be the leaves of Δ_1 and Δ_2 , respectively. Without loss of generality, we may assume that

$$\begin{aligned} E(L_1) &= \{\{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}\}, \\ E(L_2) &= \{\{v_2, v_6, v_7\}, \{v_3, v_8, v_9\}\}. \end{aligned}$$

Now, let L' be the hypergraph with edge set $E(L_1) \cup E(L_2)$. Hence, L' is isomorphic to H , and the (multi-)set

$$(\Delta_1 \setminus \{L_1\}) \cup (\Delta_2 \setminus \{L_2\}) \cup \{L'\}$$

is a collection of H -blocks such that each edge of $K_{13}^{(3)}$ is represented exactly twice. Therefore, we have an H -decomposition of ${}^2K_{13}^{(3)}$.

Now, let $v = 8x + 5$ where $x \geq 2$. By Lemma 2 it suffices to find H -decompositions of (2-fold) $K_{13}^{(3)}$, $K_{13}^{(3)} \setminus K_5^{(3)}$, $K_{5,8,8}^{(3)} \cup L_{8,8}^{(3)}$, and $K_{8,8,8}^{(3)}$. We note that $K_{4,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$, and we already have that H decomposes ${}^2K_{13}^{(3)}$. Thus, we need only additionally find H -decompositions of $K_{13}^{(3)} \setminus K_5^{(3)}$, $K_{5,8,8}^{(3)}$, $L_{8,8}^{(3)}$, and $K_{4,8,8}^{(3)}$, which exist by Examples 13, 11, 6, and 10, respectively. $\square$

Theorem 7. *If $v \geq 9$ is an integer, then there exists a maximum H -packing of 2-fold $K_v^{(3)}$ where the leave has no edges or two vertex-disjoint edges.*

Proof. If $v \equiv 0, 1$, or $2 \pmod{4}$, then the result follows from the H -decomposition result in Theorem 6, which translates to a maximum H -packing with an empty leave. Hence, we need only consider when $v \equiv 3 \pmod{4}$.

First, we consider when $v = 11$. Let Δ_1 and Δ_2 be maximum H -packings of $K_{11}^{(3)}$ with leaves L_1 and L_2 , respectively, which exist by Example 17. Now, let L' be the hypergraph with edge (multi-)set $E(L_1) \cup E(L_2)$. Hence, L' consists of two edges. In fact, we further note that L' can be any hypergraph with two edges, including ${}^2K_3^{(3)}$. Hence, the (multi-)set

$$(\Delta_1 \setminus \{L_1\}) \cup (\Delta_2 \setminus \{L_2\}) \cup \{L'\}$$

is a maximum H -packing of ${}^2K_{11}^{(3)}$ with a leave, L' , consisting of two (possibly vertex-disjoint) edges.

Second, we consider when $v = 15$. Let $v_1, v_2, \dots, v_9 \in V(K_v^{(3)})$. By Example 19, there exist maximum H -packings of $K_{15}^{(3)}$ where the leaves consist of three disjoint edges. Let Δ_1 and Δ_2 be such H -packings of $K_{15}^{(3)}$ with leaves L_1 and L_2 , respectively. Without loss of generality, we may assume that

$$\begin{aligned} E(L_1) &= \{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}, \{v_7, v_8, v_9\}\}, \\ E(L_2) &= \{\{v_1, v_4, v_5\}, \{v_2, v_6, v_7\}, \{v_3, v_8, v_9\}\}. \end{aligned}$$

Now, let L' be the hypergraph with edge set $E(L_1) \cup E(L_2)$. We note that L' is decomposable into copies of $K_3^{(3)}$ and H . That is, if we let L'' be the hypergraph with edge set $\{\{v_4, v_5, v_6\}, \{v_7, v_8, v_9\}\}$, then $L' \setminus L''$ is isomorphic to H , and the (multi-)set

$$(\Delta_1 \setminus \{L_1\}) \cup (\Delta_2 \setminus \{L_2\}) \cup \{L' \setminus L'', L''\}$$

is a maximum H -packing of ${}^2K_{15}^{(3)}$ with a leave, L'' , consisting of two (disjoint) edges.

Now, let $v = 8x + r$ where $x \geq 2$ and $r \in \{3, 7\}$. By Lemma 2 it suffices to find

- a maximum H -packing of (2-fold) $K_{8+r}^{(3)}$ with a leave consisting of fewer than four edges and
- H -decompositions of $K_{8+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,8,8}^{(3)} \cup L_{8,8}^{(3)}$, and $K_{8,8,8}^{(3)}$.

We already have the maximum H -packing results. Also, we note that $K_{7,8,8}^{(3)}$ is decomposable into copies of $K_{3,8,8}^{(3)}$ and $K_{4,8,8}^{(3)}$, and $K_{4,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$. Thus, we need only additionally find H -decompositions of $K_{11}^{(3)} \setminus K_3^{(3)}$, $K_{15}^{(3)} \setminus K_7^{(3)}$, $K_{3,8,8}^{(3)}$, $K_{4,8,8}^{(3)}$, and $L_{8,8}^{(3)}$, which exist by Examples 17, 15, 9, 10, and 6, respectively. $\square$

Theorem 8. *If $v \geq 9$ is an integer, then there exists a maximum H -packing of 3-fold $K_v^{(3)}$ where the leave has fewer than four edges.*

Proof. If $v \equiv 0, 1, 2, 4$, or $6 \pmod{8}$, then the result follows from the H -decomposition result in Theorem 3, which translates to a maximum H -packing with an empty leave. Hence, we need only consider when $v \equiv 3, 5$, or $7 \pmod{8}$.

First, we consider when $v = 11$. Let Δ_1 be a maximum H -packing of $K_{11}^{(3)}$ with leave L_1 consisting of a single edge, which exists by Example 17, and let Δ_2 be a maximum H -packing of ${}^2K_{11}^{(3)}$ with leave L_2 consisting of two edges, which exists by Theorem 7. Now, let L' be the hypergraph with edge (multi-)set $E(L_1) \cup E(L_2)$. Hence, L' consists of three edges. In fact, we further note that L' can be any hypergraph with three edges, including ${}^3K_3^{(3)}$. Hence, the (multi-)set

$$(\Delta_1 \setminus \{L_1\}) \cup (\Delta_2 \setminus \{L_2\}) \cup \{L'\}$$

is a maximum H -packing of ${}^3K_{11}^{(3)}$ with a leave, L' , consisting of three edges.

Second, we consider when $v = 13$. Let Δ_1 be a maximum H -packing of $K_{13}^{(3)}$ with leave L_1 consisting of two edges, which exists by Example 18, and let Δ_2 be an H -decomposition of ${}^2K_{13}^{(3)}$, which exists by Theorem 6. Hence, the (multi-)set $\Delta_1 \cup \Delta_2$ is a maximum H -packing of ${}^3K_{13}^{(3)}$ with a leave, L_1 , consisting of two edges.

Third, we consider when $v = 15$. Let $v_1, v_2, \dots, v_9 \in V(K_v^{(3)})$, let Δ_1 be a maximum H -packing of $K_{15}^{(3)}$ with leave L_1 consisting of a three vertex-disjoint edges, which exists by Example 19, and let Δ_2 be a maximum H -packing of ${}^2K_{15}^{(3)}$ with leave L_2 consisting of two vertex-disjoint edges, which exists by Theorem 7. Without loss of generality, we may assume that

$$\begin{aligned} E(L_1) &= \{\{v_1, v_4, v_5\}, \{v_2, v_6, v_7\}, \{v_3, v_8, v_9\}\}, \\ E(L_2) &= \{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}\}. \end{aligned}$$

Now, let L' be the hypergraph with edge set $E(L_1) \cup E(L_2)$. We note that L' is decomposable into copies of $K_3^{(3)}$ and H . That is, if we let L'' be the hypergraph with the single edge $\{v_4, v_5, v_6\}$, then $L' \setminus L''$ is isomorphic to H , and the (multi-)set

$$(\Delta_1 \setminus \{L_1\}) \cup (\Delta_2 \setminus \{L_2\}) \cup \{L' \setminus L'', L''\}$$

is a maximum H -packing of ${}^3K_{15}^{(3)}$ with a leave, L'' , consisting of one edges.

Now, let $v = 8x + r$ where $x \geq 2$ and $r \in \{3, 5, 7\}$. By Lemma 2 it suffices to find

- a maximum H -packing of (3-fold) $K_{8+r}^{(3)}$ with a leave consisting of fewer than four edges and
- H -decompositions of $K_{8+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,8,8}^{(3)} \cup L_{8,8}^{(3)}$, and $K_{8,8,8}^{(3)}$.

We already have the maximum H -packing results. Also, we note that $K_{7,8,8}^{(3)}$ is decomposable into copies of $K_{3,8,8}^{(3)}$ and $K_{4,8,8}^{(3)}$, and $K_{4,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$. Thus, we need only additionally find H -decompositions of $K_{11}^{(3)} \setminus K_3^{(3)}$, $K_{13}^{(3)} \setminus K_5^{(3)}$, $K_{15}^{(3)} \setminus K_7^{(3)}$, $K_{3,8,8}^{(3)}$, $K_{5,8,8}^{(3)}$, $K_{4,8,8}^{(3)}$, and $L_{8,8}^{(3)}$, which exist by Examples 17, 13, 15, 9, 11, 10, and 6, respectively. $\square$

Theorem 9. *Let $v \geq 9$ be an integer. There exists an H -decomposition of 4-fold $K_v^{(3)}$.*

Proof. If $v \equiv 0, 1$, or $2 \pmod{4}$, then the result follows from 2 copies of an H -decomposition of ${}^2K_v^{(3)}$, which exists by Theorem 6. Hence, we need only consider when $v \equiv 3 \pmod{4}$. Let $v_1, v_2, \dots, v_9 \in V(K_v^{(3)})$.

First, we consider when $v = 11$. For $i \in \{1, 2, 3, 4\}$, let Δ_i be a maximum H -packing of $K_{11}^{(3)}$ with leave L_i consisting of a single edge, which exists by Example 17. Without loss of generality, we may assume that

$$\begin{aligned} E(L_1) &= \{\{v_1, v_2, v_3\}\}, & E(L_2) &= \{\{v_1, v_4, v_5\}\}, \\ E(L_3) &= \{\{v_2, v_6, v_7\}\}, & E(L_4) &= \{\{v_3, v_8, v_9\}\}. \end{aligned}$$

Now, let L' be the hypergraph with edge set $E(L_1) \cup E(L_2) \cup E(L_3) \cup E(L_4)$. Hence, L' is isomorphic to H , and the (multi-)set

$$L' \cup \bigcup_{i=1}^4 (\Delta_i \setminus \{L_i\})$$

is a collection of H -blocks such that each edge of $K_{11}^{(3)}$ is represented exactly four times. Therefore, we have an H -decomposition of ${}^4K_{11}^{(3)}$.

Second, we consider when $v = 15$. Let Δ_1 be a maximum H -packing of $K_{15}^{(3)}$ with leave L_1 consisting of a three vertex-disjoint edges, which exists by Example 19, and let Δ_2 be a maximum H -packing of ${}^3K_{15}^{(3)}$ with leave L_2 consisting of a single edge, which exists by Theorem 8. Without loss of generality, we may assume that

$$\begin{aligned} E(L_1) &= \{\{v_1, v_4, v_5\}, \{v_2, v_6, v_7\}, \{v_3, v_8, v_9\}\}, \\ E(L_2) &= \{\{v_1, v_2, v_3\}\}. \end{aligned}$$

Now, let L' be the hypergraph with edge set $E(L_1) \cup E(L_2)$. Hence, L' is isomorphic to H , and the (multi-)set

$$(\Delta_1 \setminus \{L_1\}) \cup (\Delta_2 \setminus \{L_2\}) \cup \{L'\}$$

is a collection of H -blocks such that each edge of $K_{15}^{(3)}$ is represented exactly four times. Therefore, we have an H -decomposition of ${}^4K_{15}^{(3)}$.

Now, let $v = 8x + r$ where $x \geq 2$ and $r \in \{3, 7\}$. By Lemma 2 it suffices to find H -decompositions of (4-fold) $K_{8+r}^{(3)}$, $K_{8+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,8,8}^{(3)} \cup L_{8,8}^{(3)}$, and $K_{8,8,8}^{(3)}$. We note that $K_{7,8,8}^{(3)}$ is decomposable into copies of $K_{3,8,8}^{(3)}$ and $K_{4,8,8}^{(3)}$. Also, $K_{4,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$, and we already have that H decomposes ${}^4K_{11}^{(3)}$ and ${}^4K_{15}^{(3)}$. Thus, we need only additionally find H -decompositions of $K_{11}^{(3)} \setminus K_3^{(3)}$, $K_{15}^{(3)} \setminus K_7^{(3)}$, $K_{3,8,8}^{(3)}$, $K_{4,8,8}^{(3)}$, and $L_{8,8}^{(3)}$, which exist by Examples 17, 15, 9, 10, and 6, respectively. $\square$

Finally, we show that the necessary conditions for the existence of an H -decomposition of λ -fold $K_v^{(3)}$ are sufficient.

Theorem 10. *Let λ and v be positive integers with $v \geq 9$. There exists an H -decomposition of λ -fold K_v if and only if the following hold:*

- if $\gcd(\lambda, 4) = 1$, then $v \equiv 0, 1, 2, 4$, or $6 \pmod{8}$;
- if $\gcd(\lambda, 4) = 2$, then $v \equiv 0, 1$, or $2 \pmod{4}$;
- if $\gcd(\lambda, 4) = 4$, then $v \geq 9$.

Proof. The necessary conditions are established in Lemma 5. For sufficiency, we consider the following cases.

Case 1. $\lambda \equiv 0 \pmod{4}$

Let $\lambda = 4t$ for some positive integer t . Then the result follows from t copies of an H -decomposition of ${}^4K_v^{(3)}$, which exists by Theorem 9.

Case 2. $\lambda \equiv 1$ or $3 \pmod{4}$

Since $\gcd(\lambda, 4) = 1$, we have that $v \equiv 0, 1, 2, 4$, or $6 \pmod{8}$. Let $\lambda = 4t + r$ for some integers $t \geq 0$ and $r \in \{1, 3\}$. Then the result follows from t copies of an H -decomposition of ${}^4K_v^{(3)}$, which exists by Theorem 9, and r copies of an H -decomposition of $K_v^{(3)}$, which exists by Theorem 3.

Case 3. $\lambda \equiv 2 \pmod{4}$

Since $\gcd(\lambda, 4) = 2$, we have that $v \equiv 0, 1$, or $2 \pmod{4}$. Let $\lambda = 4t + 2$ for some nonnegative integer t . Then the result follows from t copies of an H -decomposition of ${}^4K_v^{(3)}$, which exists by Theorem 9, and 1 copy of an H -decomposition of ${}^2K_v^{(3)}$, which exists by Theorem 6. $\square$

Theorem 11. *If $v \geq 9$ is an integer, then there exists a maximum H -packing of λ -fold $K_v^{(3)}$ where the leave has fewer than four edges.*

Proof. If $1 \leq \lambda \leq 3$, then the result follows from Theorems 4, 7, and 8. If $\lambda = 4$, then the result follows from the H -decomposition result in Theorem 9, which translates to a maximum H -packing with an empty leave. For the remainder of the proof, we assume that $\lambda \geq 5$. Let $\lambda = 4t + r$ for some integers $t \geq 1$ and $r \in \{1, 4\}$. Then the result follows from t copies of an H -decomposition of ${}^4K_v^{(3)}$, which exists by Theorem 9, and 1 copy of a maximum H -packing of r -fold $K_v^{(3)}$. $\square$

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Appendix: Some Small Examples

We give several examples of H -decompositions and H -packings that are used in proving our main result.

Decomposition Examples

Example 1. Let $V\left(K_9^{(3)}\right) = \mathbb{Z}_7 \cup \{\infty_1, \infty_2\}$ and let

$$B = \{H[0, 1, 4, 5, 6, \infty_1, 3, \infty_2, 2], H[\infty_1, \infty_2, 0, 3, 6, 1, 2, 4, 5], H[0, 2, 5, \infty_2, 4, \infty_1, 1, 6, 3]\}.$$

Then an H -decomposition of $K_9^{(3)}$ consists of the orbits of the H -blocks in B under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, 2\}$, and $j \mapsto j + 1 \pmod{7}$.

Example 2. Let $V(K_{10}^{(3)}) = \mathbb{Z}_{10}$ and let

$$B = \{H[0, 2, 4, 8, 9, 3, 6, 5, 1], H[0, 2, 7, 1, 6, 5, 8, 9, 3], H[0, 1, 5, 7, 9, 2, 4, 8, 3]\}.$$

Then an H -decomposition of $K_{10}^{(3)}$ consists of the orbits of the H -blocks in B under the action of the map $j \mapsto j + 1 \pmod{10}$.

Example 3. Let $V(K_{12}^{(3)}) = \mathbb{Z}_{11} \cup \{\infty\}$ and let

$$B = \{H[0, 1, 3, 8, 10, 2, 5, 6, 7], H[0, 1, 5, \infty, 6, 2, 8, 10, 3], H[0, 6, 9, 2, 5, 10, 3, \infty, 8], \\ H[\infty, 0, 3, 8, 10, 2, 4, 6, 9], H[0, 1, 2, \infty, 7, 5, 10, 3, 8]\}.$$

Then an H -decomposition of $K_{12}^{(3)}$ consists of the orbits of the H -blocks in B under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{11}$.

Example 4. Let $V(K_{14}^{(3)}) = \mathbb{Z}_{13} \cup \{\infty\}$ and let

$$B = \{H[0, 1, 3, 10, 12, 2, 5, 6, 7], H[0, 1, 5, 7, 12, 2, 10, 6, 11], H[\infty, 4, 6, 0, 1, 2, 3, 5, 12], \\ H[\infty, 4, 8, 0, 3, 7, 12, 11, 1], H[\infty, 6, 11, 12, 5, 8, 10, 2, 7], H[0, 2, 7, 6, 10, 4, 11, 12, 1], \\ H[0, 2, 5, 8, 11, 6, 12, 3, 9]\}.$$

Then an H -decomposition of $K_{14}^{(3)}$ consists of the orbits of the H -blocks in B under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{13}$.

Example 5. Let $V(K_{16}^{(3)}) = \mathbb{Z}_{15} \cup \{\infty\}$ and let

$$B_1 = \{H[0, 1, 3, 5, 6, 2, 14, 4, 9], H[0, 2, 5, 3, 11, 4, 14, 8, 12], H[0, 1, 4, \infty, 7, 2, 13, 8, 12], \\ H[0, 2, 6, 3, 9, 4, 13, \infty, 11], H[0, 2, 8, 7, 14, 4, 11, \infty, 10], H[0, 1, 7, 4, 9, 2, 10, \infty, 13], \\ H[0, 1, 5, 3, 6, 2, 12, \infty, 8], H[0, 2, 7, \infty, 1, 4, 12, 9, 11], H[0, 3, 8, 4, 10, 6, 13, \infty, 12]\}, \\ B_2 = \{H[0, 5, 10, 1, 2, 6, 7, 11, 12], H[1, 6, 11, 2, 3, 7, 8, 12, 13], H[2, 7, 12, 3, 4, 8, 9, 13, 14], \\ H[3, 8, 13, 4, 5, 9, 10, 14, 0], H[4, 9, 14, 5, 6, 10, 11, 0, 1]\}.$$

Then an H -decomposition of $K_{16}^{(3)}$ consists of the orbits of the H -blocks in B_1 under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{15}$ along with the H -blocks in B_2 .

Example 6. Let $V(L_{8,8}^{(3)}) = \mathbb{Z}_{16}$ with vertex partition $\{\{0, 2, 4, 6, 8, 10, 12, 14\}, \{1, 3, 5, 7, 9, 11, 13, 15\}\}$ and let

$$B = \{H[0, 1, 2, 7, 9, 4, 14, 8, 13], H[0, 1, 3, 5, 14, 2, 15, 8, 10], H[0, 4, 5, 7, 15, 8, 3, 12, 13], \\ H[0, 6, 13, 3, 7, 12, 15, 1, 4], H[0, 1, 7, 9, 15, 3, 14, 2, 4], H[0, 1, 6, 10, 15, 8, 12, 13, 2], \\ H[0, 1, 4, 3, 11, 2, 14, 12, 15]\}.$$

Then an H -decomposition of $L_{8,8}^{(3)}$ consists of the orbits of the H -blocks in B under the action of the map $j \mapsto j + 1 \pmod{16}$.

Example 7. Let $V\left(L_{8,8}^{(3)} \cup K_{1,8,8}^{(3)}\right) = \mathbb{Z}_{16} \cup \{\infty\}$ with vertex partition $\{\{\infty\}, \{0, 2, 4, 6, 8, 10, 12, 14\}, \{1, 3, 5, 7, 9, 11, 13, 15\}\}$ and let

$$B = \{H[0, 1, 3, 5, 14, 2, 15, 8, 10], H[0, 4, 5, 7, 15, 8, 3, 12, 13], H[0, 6, 13, 3, 7, 12, 15, 1, 4], \\ H[0, 1, 7, 9, 15, 3, 14, 2, 4], H[0, 1, 6, 10, 15, 8, 12, 13, 2], H[0, 1, 4, 3, 11, 2, 14, 12, 15], \\ H[\infty, 0, 9, 10, 13, 7, 14, 15, 4], H[\infty, 0, 11, 3, 4, 1, 2, 5, 8]\}.$$

Then an H -decomposition of $L_{8,8}^{(3)} \cup K_{1,8,8}^{(3)}$ consists of the orbits of the H -blocks in B under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{16}$.

Example 8. Let $V\left(L_{8,8}^{(3)} \cup K_{2,8,8}^{(3)}\right) = \mathbb{Z}_{16} \cup \{\infty_1, \infty_2\}$ with vertex partition $\{\{\infty_1, \infty_2\}, \{0, 2, 4, 6, 8, 10, 12, 14\}, \{1, 3, 5, 7, 9, 11, 13, 15\}\}$ and let

$$B = \{H[0, 1, 3, 5, 14, 2, 15, 8, 10], H[0, 4, 5, 7, 15, 8, 3, 12, 13], H[\infty_1, 0, 15, 3, 10, 1, 4, 11, 14], \\ H[0, 1, 7, 9, 15, 3, 14, 2, 4], H[0, 1, 6, 10, 15, 8, 12, 13, 2], H[\infty_1, 0, 3, 9, 14, 5, 13, 8, 11], \\ H[0, 6, 13, 3, 7, 12, 15, 1, 4], H[\infty_2, 0, 9, 5, 6, 14, 15, 11, 2], H[\infty_2, 0, 13, 5, 10, 3, 6, 2, 7]\}.$$

Then an H -decomposition of $L_{8,8}^{(3)} \cup K_{2,8,8}^{(3)}$ consists of the orbits of the H -blocks in B under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, 2\}$, and $j \mapsto j + 1 \pmod{16}$.

Example 9. Let $V\left(K_{3,8,8}^{(3)}\right) = \mathbb{Z}_{16} \cup \{\infty_1, \infty_2, \infty_3\}$ with vertex partition $\{\{\infty_1, \infty_2, \infty_3\}, \{0, 2, 4, 6, 8, 10, 12, 14\}, \{1, 3, 5, 7, 9, 11, 13, 15\}\}$ and let

$$B = \{H[\infty_1, 0, 1, 2, 5, \infty_2, 7, \infty_3, 6], H[\infty_2, 0, 1, 2, 5, \infty_3, 7, \infty_1, 6], \\ H[\infty_3, 0, 1, 2, 5, \infty_1, 7, \infty_2, 6]\}.$$

Then an H -decomposition of $K_{3,8,8}^{(3)}$ consists of the orbits of the H -blocks in B under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, 2, 3\}$, and $j \mapsto j + 1 \pmod{16}$.

Example 10. Let $V\left(K_{4,8,8}^{(3)}\right) = \mathbb{Z}_{16} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ with vertex partition $\{\{\infty_1, \infty_2, \infty_3, \infty_4\}, \{0, 2, 4, 6, 8, 10, 12, 14\}, \{1, 3, 5, 7, 9, 11, 13, 15\}\}$ and let

$$B = \{H[\infty_1, 0, 1, 2, 5, \infty_2, 7, \infty_3, 6], H[\infty_2, 0, 1, 2, 5, \infty_3, 7, \infty_4, 6], \\ H[\infty_3, 0, 1, 2, 5, \infty_4, 7, \infty_1, 6], H[\infty_4, 0, 1, 2, 5, \infty_1, 7, \infty_2, 6]\}.$$

Then an H -decomposition of $K_{4,8,8}^{(3)}$ consists of the orbits of the H -blocks in B under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, \dots, 4\}$, and $j \mapsto j + 1 \pmod{16}$.

Example 11. Let $V\left(K_{5,8,8}^{(3)}\right) = \mathbb{Z}_{16} \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ with vertex partition $\{\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}, \{0, 2, 4, 6, 8, 10, 12, 14\}, \{1, 3, 5, 7, 9, 11, 13, 15\}\}$ and let

$$B = \{H[\infty_1, 0, 1, 2, 5, \infty_2, 7, \infty_3, 6], H[\infty_2, 0, 1, 2, 5, \infty_3, 7, \infty_4, 6], \\ H[\infty_3, 0, 1, 2, 5, \infty_4, 7, \infty_5, 6], H[\infty_4, 0, 1, 2, 5, \infty_5, 7, \infty_1, 6], \\ H[\infty_5, 0, 1, 2, 5, \infty_1, 7, \infty_2, 6]\}.$$

Then an H -decomposition of $K_{5,8,8}^{(3)}$ consists of the orbits of the H -blocks in B under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, \dots, 5\}$, and $j \mapsto j + 1 \pmod{16}$.

Example 12. Let $V(K_{12}^{(3)} \setminus K_4^{(3)}) = \mathbb{Z}_{12}$ with $0, 3, 6, 9$ being the vertices in the hole and let

$$\begin{aligned} B_1 = \{ & H[0, 3, 7, 2, 5, 6, 11, 9, 4], H[0, 2, 6, 1, 11, 4, 10, 8, 9], \\ & H[0, 1, 6, 7, 11, 2, 8, 10, 5], H[0, 1, 4, 8, 11, 3, 5, 9, 2] \}, \\ B_2 = \{ & H[7, 8, 10, 1, 4, 2, 5, 0, 9], H[1, 2, 4, 7, 10, 8, 11, 3, 6], H[8, 9, 11, 0, 4, 6, 7, 2, 5], \\ & H[1, 10, 11, 5, 9, 4, 7, 0, 2], H[2, 3, 5, 6, 10, 0, 1, 8, 11], H[4, 5, 7, 1, 10, 6, 8, 11, 3] \}. \end{aligned}$$

Then an H -decomposition of $K_{12}^{(3)} \setminus K_4^{(3)}$ consists of the orbits of the H -blocks in B_1 under the action of the map $j \mapsto j + 1 \pmod{12}$ along with the H -blocks in B_2 .

Example 13. Let $V(K_{13}^{(3)} \setminus K_5^{(3)}) = \mathbb{Z}_8 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ with $\infty_1, \infty_2, \infty_3, \infty_4, \infty_5$ being the vertices in the hole and let

$$\begin{aligned} B_1 = \{ & H[\infty_3, \infty_5, 0, 1, 3, \infty_2, 7, \infty_1, \infty_4], H[\infty_4, \infty_5, 0, \infty_3, 6, \infty_1, 7, 2, 3], \\ & H[\infty_2, \infty_4, 0, 5, 7, 1, 4, \infty_5, 3], H[\infty_4, 0, 2, 4, 5, \infty_1, 7, \infty_5, 3] \}, \\ B_2 = \{ & H[\infty_1, \infty_3, 0, 3, 5, 1, 2, 4, 6], H[\infty_1, \infty_3, 1, 4, 6, 2, 3, 5, 7], \\ & H[\infty_1, \infty_3, 2, 5, 7, 3, 4, 6, 0], H[\infty_1, \infty_3, 3, 6, 0, 4, 5, 7, 1], \\ & H[\infty_1, \infty_3, 4, 7, 1, 5, 6, 0, 2], H[\infty_1, \infty_3, 5, 0, 2, 6, 7, 1, 3], \\ & H[\infty_1, \infty_3, 6, 1, 3, 7, 0, 2, 4], H[2, 4, 7, \infty_2, \infty_3, \infty_1, 1, 5, 6], \\ & H[3, 5, 0, \infty_2, \infty_3, \infty_1, 2, 6, 7], H[4, 6, 1, \infty_2, \infty_3, \infty_1, 3, 7, 0], \\ & H[5, 7, 2, \infty_2, \infty_3, \infty_1, 4, 0, 1], H[6, 0, 3, \infty_2, \infty_3, \infty_1, 5, 1, 2], \\ & H[7, 1, 4, \infty_2, \infty_3, \infty_1, 6, 2, 3], H[0, 2, 5, \infty_2, \infty_3, \infty_1, 7, 3, 4], \\ & H[2, 3, 7, \infty_3, 5, \infty_1, \infty_2, 1, 6], H[3, 4, 0, \infty_3, 6, \infty_1, \infty_2, 2, 7], \\ & H[4, 5, 1, \infty_3, 7, \infty_1, \infty_2, 3, 0], H[5, 6, 2, \infty_3, 0, \infty_1, \infty_2, 4, 1], \\ & H[6, 7, 3, \infty_3, 1, \infty_1, \infty_2, 5, 2], H[7, 0, 4, \infty_3, 2, \infty_1, \infty_2, 6, 3], \\ & H[0, 1, 5, \infty_3, 3, \infty_1, \infty_2, 7, 4], H[\infty_2, \infty_3, 1, 0, 4, \infty_1, 7, 2, 6], \\ & H[\infty_2, 1, 5, \infty_1, 2, \infty_3, 0, 6, 4], H[1, 3, 6, \infty_3, 4, 5, 7, \infty_2, 2], \\ & H[\infty_1, 0, 3, 2, 4, 5, 6, \infty_2, 7], H[\infty_2, 0, 1, 3, 6, \infty_1, 4, \infty_3, 5], \\ & H[\infty_2, 1, 2, 4, 7, \infty_1, 5, \infty_3, 6], H[\infty_2, 2, 3, 5, 0, \infty_1, 6, \infty_3, 7], \\ & H[\infty_2, 3, 4, 6, 1, \infty_1, 7, \infty_3, 0], H[\infty_2, 4, 5, 7, 2, \infty_4, 0, \infty_5, 1], \\ & H[\infty_2, 5, 6, 0, 3, \infty_4, 1, \infty_5, 2], H[\infty_2, 6, 7, 1, 4, \infty_4, 2, \infty_5, 3], \\ & H[\infty_2, 7, 0, 2, 5, \infty_4, 3, \infty_5, 4], H[\infty_5, 0, 2, 1, 3, 4, 5, 6, 7], \\ & H[\infty_5, 3, 5, 2, 4, 7, 0, 6, 1], H[\infty_5, 4, 6, 5, 7, 0, 1, 2, 3], H[\infty_5, 1, 7, 0, 6, 2, 5, 3, 4] \}. \end{aligned}$$

Then an H -decomposition of $K_{13}^{(3)} \setminus K_5^{(3)}$ consists of the orbits of the H -blocks in B_1 under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, \dots, 5\}$, and $j \mapsto j + 1 \pmod{8}$ along with the H -blocks in B_2 .

Example 14. Let $V(K_{14}^{(3)} \setminus K_6^{(3)}) = \mathbb{Z}_{12} \cup \{\infty_1, \infty_2\}$ with $0, 3, 6, 9, \infty_1, \infty_2$ being the vertices in the hole and let

$$\begin{aligned} B_1 = & \{H[0, 1, 5, 7, 11, 2, 10, 6, 9], H[\infty_1, 0, 1, 6, 8, 10, 11, \infty_2, 2], H[\infty_2, 0, 4, 1, 3, 9, 11, \infty_1, 8], \\ & H[0, 1, 6, \infty_2, 7, 2, 8, \infty_1, 11], H[0, 2, 5, 7, 10, 4, 8, 9, 11], H[0, 2, 4, 3, 8, 5, 9, 6, 11]\}, \\ B_2 = & \{H[\infty_1, 2, 8, 5, 11, 1, 4, 7, 10], H[\infty_2, 5, 11, 2, 8, 4, 7, 1, 10], H[0, 1, 3, 4, 8, \infty_1, 7, 2, 5], \\ & H[3, 4, 6, 7, 11, \infty_1, 10, 5, 8], H[6, 7, 9, 2, 10, \infty_2, 1, 8, 11], H[0, 9, 10, 2, 11, 1, 5, \infty_2, 4], \\ & H[\infty_1, \infty_2, 1, 2, 5, 8, 11, 4, 7], H[\infty_1, \infty_2, 2, 1, 4, 7, 10, 5, 8], \\ & H[\infty_1, \infty_2, 4, 5, 8, 2, 11, 7, 10], H[\infty_1, \infty_2, 5, 4, 7, 1, 10, 8, 11], \\ & H[\infty_1, \infty_2, 7, 8, 11, 2, 5, 1, 10], H[\infty_1, \infty_2, 8, 7, 10, 1, 4, 2, 11], \\ & H[\infty_1, \infty_2, 10, 2, 11, 5, 8, 1, 4], H[\infty_1, \infty_2, 11, 1, 10, 4, 7, 2, 5]\}. \end{aligned}$$

Then an H -decomposition of $K_{14}^{(3)} \setminus K_6^{(3)}$ consists of the orbits of the H -blocks in B_1 under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, 2\}$, and $j \mapsto j + 1 \pmod{12}$ along with the H -blocks in B_2 .

Example 15. Let $V(K_{15}^{(3)} \setminus K_7^{(3)}) = \mathbb{Z}_8 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7\}$ with $\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7$ being the vertices in the hole and let

$$\begin{aligned} B_1 = & \{H[\infty_1, \infty_4, 0, \infty_5, 5, 3, 4, \infty_6, \infty_7], H[\infty_2, \infty_4, 0, \infty_5, 5, 2, 4, \infty_7, 3], \\ & H[\infty_3, \infty_4, 0, \infty_5, 5, \infty_7, 4, \infty_6, 3], H[\infty_4, \infty_5, 0, 4, 7, \infty_6, 1, \infty_3, 2], \\ & H[\infty_3, \infty_6, 0, \infty_7, 6, 3, 4, \infty_1, 1], H[\infty_5, \infty_7, 0, 5, 6, \infty_1, 4, \infty_6, 2], \\ & H[\infty_2, \infty_6, 0, \infty_7, 6, \infty_4, 2, \infty_5, 3], H[\infty_7, 3, 5, 0, 1, \infty_1, \infty_6, \infty_2, 7]\}, \\ B_2 = & \{H[\infty_1, \infty_3, 0, 3, 5, 1, 2, 4, 6], H[\infty_1, \infty_3, 1, 4, 6, 2, 3, 5, 7], \\ & H[\infty_1, \infty_3, 2, 5, 7, 3, 4, 6, 0], H[\infty_1, \infty_3, 3, 6, 0, 4, 5, 7, 1], \\ & H[\infty_1, \infty_3, 4, 7, 1, 5, 6, 0, 2], H[\infty_1, \infty_3, 5, 0, 2, 6, 7, 1, 3], \\ & H[\infty_1, \infty_3, 6, 1, 3, 7, 0, 2, 4], H[2, 4, 7, \infty_2, \infty_3, \infty_1, 1, 5, 6], \\ & H[3, 5, 0, \infty_2, \infty_3, \infty_1, 2, 6, 7], H[4, 6, 1, \infty_2, \infty_3, \infty_1, 3, 7, 0], \\ & H[5, 7, 2, \infty_2, \infty_3, \infty_1, 4, 0, 1], H[6, 0, 3, \infty_2, \infty_3, \infty_1, 5, 1, 2], \\ & H[7, 1, 4, \infty_2, \infty_3, \infty_1, 6, 2, 3], H[0, 2, 5, \infty_2, \infty_3, \infty_1, 7, 3, 4], \\ & H[2, 3, 7, \infty_3, 5, \infty_1, \infty_2, 1, 6], H[3, 4, 0, \infty_3, 6, \infty_1, \infty_2, 2, 7], \\ & H[4, 5, 1, \infty_3, 7, \infty_1, \infty_2, 3, 0], H[5, 6, 2, \infty_3, 0, \infty_1, \infty_2, 4, 1], \\ & H[6, 7, 3, \infty_3, 1, \infty_1, \infty_2, 5, 2], H[7, 0, 4, \infty_3, 2, \infty_1, \infty_2, 6, 3], \\ & H[0, 1, 5, \infty_3, 3, \infty_1, \infty_2, 7, 4], H[\infty_2, \infty_3, 1, 0, 4, \infty_1, 7, 2, 6], \\ & H[\infty_2, 1, 5, \infty_1, 2, \infty_3, 0, 6, 4], H[1, 3, 6, \infty_3, 4, 5, 7, \infty_2, 2], \\ & H[\infty_1, 0, 3, 2, 4, 5, 6, \infty_2, 7], H[\infty_2, 0, 1, 3, 6, \infty_1, 4, \infty_3, 5], \\ & H[\infty_2, 1, 2, 4, 7, \infty_1, 5, \infty_3, 6], H[\infty_2, 2, 3, 5, 0, \infty_1, 6, \infty_3, 7], \end{aligned}$$

$$\begin{aligned}
& H[\infty_2, 3, 4, 6, 1, \infty_1, 7, \infty_3, 0], H[\infty_2, 4, 5, 7, 2, \infty_4, 0, \infty_5, 1], \\
& H[\infty_2, 5, 6, 0, 3, \infty_4, 1, \infty_5, 2], H[\infty_2, 6, 7, 1, 4, \infty_4, 2, \infty_5, 3], \\
& H[\infty_2, 7, 0, 2, 5, \infty_4, 3, \infty_5, 4], H[\infty_6, 0, 4, 2, 6, 5, 7, 1, 3], H[\infty_6, 1, 5, 3, 7, 6, 0, 2, 4], \\
& H[\infty_7, 2, 6, 0, 4, 7, 1, 3, 5], H[\infty_7, 3, 7, 1, 5, 0, 2, 4, 6], H[\infty_5, 0, 2, 1, 3, 4, 5, 6, 7], \\
& H[\infty_5, 3, 5, 2, 4, 7, 0, 6, 1], H[\infty_5, 4, 6, 5, 7, 0, 1, 2, 3], H[\infty_5, 1, 7, 0, 6, 2, 5, 3, 4] \}.
\end{aligned}$$

Then an H -decomposition of $K_{15}^{(3)} \setminus K_7^{(3)}$ consists of the orbits of the H -blocks in B_1 under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, \dots, 7\}$, and $j \mapsto j + 1 \pmod{8}$ along with the H -blocks in B_2 .

Example 16. Let $V(K_{16}^{(3)} \setminus K_8^{(3)}) = \mathbb{Z}_8 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7, \infty_8\}$ with $\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7, \infty_8$ being the vertices in the hole and let

$$\begin{aligned}
B_1 = \{ & H[\infty_1, \infty_2, 0, \infty_3, 1, \infty_5, 2, \infty_4, \infty_8], H[\infty_2, \infty_3, 0, \infty_4, 1, \infty_6, 2, \infty_1, \infty_5], \\
& H[\infty_3, \infty_4, 0, \infty_5, 1, \infty_7, 2, \infty_1, \infty_8], H[\infty_4, \infty_5, 0, \infty_6, 1, \infty_8, 2, \infty_3, \infty_7], \\
& H[\infty_5, \infty_6, 0, \infty_7, 1, \infty_1, 2, \infty_2, \infty_8], H[\infty_6, \infty_7, 0, \infty_8, 1, \infty_2, 2, \infty_1, \infty_4], \\
& H[\infty_7, \infty_8, 0, \infty_1, 1, \infty_3, 2, \infty_2, \infty_6], H[\infty_5, 0, 7, 2, 5, \infty_6, 3, \infty_8, 4], \\
& H[\infty_6, 0, 1, 2, 4, \infty_1, 6, \infty_7, 3], H[\infty_7, 0, 1, 3, 6, \infty_3, 5, \infty_4, 4], \\
& H[\infty_8, 0, 1, 2, 4, \infty_4, 6, \infty_3, 3], H[0, 1, 4, \infty_1, 3, 2, 7, \infty_2, 6], H[0, 2, 4, \infty_2, 5, 3, 7, 6, 1] \}, \\
B_2 = \{ & H[0, 1, 2, \infty_1, 4, \infty_5, 3, \infty_2, 6], H[1, 2, 3, \infty_1, 5, \infty_5, 4, \infty_2, 7], \\
& H[2, 3, 4, \infty_1, 6, \infty_5, 5, \infty_2, 0], H[3, 4, 5, \infty_1, 7, \infty_5, 6, \infty_2, 1], \\
& H[4, 5, 6, \infty_3, 0, \infty_5, 7, \infty_4, 2], H[5, 6, 7, \infty_3, 1, \infty_5, 0, \infty_4, 3], \\
& H[6, 7, 0, \infty_3, 2, \infty_5, 1, \infty_4, 4], H[7, 0, 1, \infty_3, 3, \infty_5, 2, \infty_4, 5], \\
& H[0, 1, 3, \infty_5, 4, \infty_6, 5, \infty_1, 2], H[1, 2, 4, \infty_5, 5, \infty_6, 6, \infty_1, 3], \\
& H[2, 3, 5, \infty_5, 6, \infty_6, 7, \infty_1, 4], H[3, 4, 6, \infty_5, 7, \infty_6, 0, \infty_1, 5], \\
& H[4, 5, 7, \infty_7, 0, \infty_8, 1, \infty_1, 6], H[5, 6, 0, \infty_7, 1, \infty_8, 2, \infty_1, 7], \\
& H[6, 7, 1, \infty_7, 2, \infty_8, 3, \infty_1, 0], H[7, 0, 2, \infty_7, 3, \infty_8, 4, \infty_1, 1], \\
& H[\infty_2, 0, 1, 5, 6, \infty_3, 7, \infty_4, 2], H[\infty_3, 0, 1, 5, 6, \infty_4, 7, \infty_2, 2], \\
& H[\infty_4, 0, 1, 5, 6, \infty_2, 7, \infty_3, 2], H[\infty_2, 3, 4, 6, 7, \infty_3, 2, \infty_4, 5], \\
& H[\infty_3, 3, 4, 6, 7, \infty_4, 2, \infty_2, 5], H[\infty_4, 3, 4, 6, 7, \infty_2, 2, \infty_3, 5] \}.
\end{aligned}$$

Then an H -decomposition of $K_{16}^{(3)} \setminus K_8^{(3)}$ consists of the orbits of the H -blocks in B_1 under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, \dots, 8\}$, and $j \mapsto j + 1 \pmod{8}$ along with the H -blocks in B_2 .

Maximum Packing Examples

Example 17. Let $V(K_{11}^{(3)}) = \mathbb{Z}_{10} \cup \{\infty\}$ and let

$$B_1 = \{H[0, 2, 7, 1, 4, \infty, 9, 3, 6], H[0, 3, 6, 1, 5, \infty, 9, 7, 2], H[0, 2, 5, 1, 3, \infty, 4, 7, 8]\},$$

$$B_2 = \{H[\infty, 0, 1, 8, 9, 2, 6, 5, 7], H[\infty, 1, 2, 0, 9, 3, 7, 6, 8], H[\infty, 2, 3, 5, 6, 4, 8, 7, 9], \\ H[\infty, 3, 4, 6, 7, 5, 9, 8, 0], H[\infty, 4, 5, 7, 8, 6, 0, 9, 1], H[3, 5, 7, 2, 4, \infty, 0, 8, 9], \\ H[4, 6, 8, 3, 5, \infty, 1, 0, 2], H[5, 7, 9, 4, 6, \infty, 2, 0, 1], H[0, 6, 8, 2, 4, 5, 7, \infty, 3], \\ H[1, 7, 9, 2, 3, 6, 8, \infty, 4], H[0, 1, 2, 8, 9, 3, 5, 4, 6]\}.$$

Then a maximum H -packing of $K_{11}^{(3)}$ consists of the orbits of the H -blocks in B_1 under the action of the map $\infty \mapsto \infty$ and $j \mapsto j+1 \pmod{10}$ along with the H -blocks in B_2 and a leave consisting of the edge $\{1, 3, 9\}$.

Example 18. Let $V(K_{13}^{(3)}) = \mathbb{Z}_{13}$ and let

$$B_1 = \{H[0, 3, 7, 6, 10, 5, 11, 9, 1], H[0, 2, 11, 1, 7, 5, 12, 3, 8], H[0, 3, 5, 8, 10, 7, 1, 9, 11], \\ H[0, 1, 5, 8, 12, 2, 7, 10, 11], H[0, 1, 3, 10, 12, 2, 5, 6, 7]\}, \\ B_2 = \{H[0, 4, 8, 1, 12, 5, 6, 9, 10], H[1, 5, 9, 2, 3, 6, 7, 10, 11], H[2, 6, 10, 3, 4, 7, 8, 11, 12], \\ H[3, 4, 5, 7, 11, 8, 12, 10, 1], H[7, 8, 9, 11, 2, 12, 3, 0, 4], H[11, 12, 0, 2, 6, 3, 7, 5, 9]\}.$$

Then a maximum H -packing of $K_{13}^{(3)}$ consists of the orbits of the H -blocks in B_1 under the action of the map $j \mapsto j+1 \pmod{13}$ along with the H -blocks in B_2 and a leave consisting of the edges $\{0, 1, 2\}$ and $\{1, 6, 10\}$, which share a single vertex. Additionally, let

$$B'_2 = (B_2 \setminus \{H[2, 6, 10, 3, 4, 7, 8, 11, 12]\}) \cup \{H[2, 6, 10, 0, 1, 7, 8, 11, 12]\}.$$

Then a maximum H -packing of $K_{13}^{(3)}$ consists of the orbits of the H -blocks in B_1 under the action of the map $j \mapsto j+1 \pmod{13}$ along with the H -blocks in B'_2 and a leave consisting of the edges $\{1, 6, 10\}$ and $\{2, 3, 4\}$, which are vertex-disjoint.

Example 19. Let $V(K_{15}^{(3)}) = \mathbb{Z}_{15}$ and let

$$B_1 = \{H[0, 4, 9, 6, 11, 7, 14, 12, 2], H[0, 4, 8, 3, 6, 7, 13, 10, 12], H[0, 1, 3, 12, 14, 2, 5, 6, 7], \\ H[0, 1, 6, 9, 14, 2, 12, 7, 11], H[0, 2, 8, 7, 13, 4, 12, 1, 3], H[0, 3, 7, 8, 12, 5, 14, 9, 13], \\ H[0, 2, 12, 7, 8, 10, 1, 3, 11]\}, \\ B_2 = \{H[0, 5, 10, 1, 2, 6, 7, 11, 12], H[1, 6, 11, 2, 3, 7, 8, 12, 13], H[2, 7, 12, 3, 4, 8, 9, 13, 14], \\ H[3, 8, 13, 4, 5, 9, 10, 14, 0], H[4, 9, 14, 5, 6, 10, 11, 0, 1], H[0, 2, 5, 13, 3, 12, 14, 7, 10], \\ H[4, 6, 9, 14, 1, 8, 11, 12, 7], H[8, 10, 13, 3, 5, 12, 0, 11, 1]\}.$$

Then a maximum H -packing of $K_{15}^{(3)}$ consists of the orbits of the H -blocks in B_1 under the action of the map $j \mapsto j+1 \pmod{15}$ along with the H -blocks in B_2 and a leave consisting of the edges $\{1, 3, 6\}$, $\{2, 4, 7\}$, and $\{9, 11, 14\}$, which are vertex-disjoint.