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We prove the Legendrian Whitney trick, which allows for the removal of intersections between codimension-2 contact submanifolds and Legendrian submanifolds, assuming such a smooth cancellation is possible. We apply this technique to prove the first known existence h–principle for codimension-2 isocontact embeddings, with a prescribed arbitrary contact structure on its domain.

53D10; 53D15, 57R17

1 Introduction

The object of this article will be to show the existence of the Legendrian Whitney trick, which removes smoothly canceling intersections between codimension-2 contact submanifolds and Legendrian submanifolds. We shall also prove the first known existence h–principle for isocontact embeddings. This new h–principle is proven with the Legendrian Whitney trick, and it is as general as possible, as it allows for any codimension (including codimension-2) and any prescribed contact structure on the submanifold.

The smooth Whitney trick is a method for removing points of intersection between two smooth submanifolds; see Kirby [25] and Whitney [36]. This technique rests at the center of differential topology, with direct applications to embeddings problems (see Shapiro [31] and Whitney [36]) and the h–cobordism theorem (see Milnor [29] and Smale [32]). Since its first appearance [36], the Whitney trick has been generalized to include multiple submanifolds (see Mabillard and Wagner [28]), intersections of positive dimension (see Goodwillie [16] and Haefliger [19]) and adapted for 4–dimensional manifolds (see Casson [6] and Freedman [14]). At its core, the Whitney trick states that a smooth cancellation is possible if certain algebraic obstructions vanish, thus measuring the difference between algebraic topology and geometric topology. Here, we study the Legendrian Whitney trick, in the context of contact structures, further crystallizing the difference between smooth topology and contact topology.

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Let $(\mathbb{D}^{2n-1}, \xi_{\mathrm{st}})$ be the standard contact structure in the smooth $(2n-1)$–dimensional disk. In Cartesian coordinates $(x_1, \dots, x_{n-1}, x_n, y_1, \dots, y_{n-1}) \in \mathbb{D}^n \times \mathbb{D}^{n-1} \cong \mathbb{D}^{2n-1}$, the standard contact structure reads $\xi_{\mathrm{st}} = \ker\{dx_n - y_1\,dx_1 - \dots - y_{n-1}\,dx_{n-1}\}$. Let (B, ξ) an arbitrary contact structure in the $(2n+1)$–dimensional standard smooth disk

$$B = \{(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^2 : |x|^2 + |y|^2 + |z|^2 \le 1\}.$$

Consider the two smooth inclusions

$$\mathbb{D}^{2n-1} = \{(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^2 : |x|^2 + |y|^2 \le 1,\ z = 0\} \subseteq B,$$
$$\mathbb{S}^n = \{(x, y, z_1, z_2) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} : |x|^2 + |z_1|^2 = 0.5,\ y = z_2 = 0\} \subseteq B.$$

By definition, a smooth embedding $\mathbb{D}^{2n-1} \to B$ will be said to be smoothly standard if it coincides with the above inclusion $\mathbb{D}^{2n-1} \subseteq B$ near the boundary and it is smoothly isotopic to this same inclusion relative to the boundary. Similarly, a smooth embedding $\mathbb{S}^n \to B$ will be said to be smoothly standard if it is smoothly isotopic to the above inclusion $\mathbb{S}^n \subseteq B$. These local models will be perturbed to achieve transversality as we develop our arguments; see Section 2.1. The main result of the article reads as follows:

Theorem 1 (Legendrian Whitney trick) *Let $\phi\colon (\mathbb{D}^{2n-1}, \partial\mathbb{D}^{2n-1}; \xi_{\mathrm{st}}) \to (B, \partial B; \xi)$ be a proper isocontact embedding, $n \ge 2$ and $\lambda\colon \mathbb{S}^n \to (B, \xi)$ a Legendrian embedding such that ϕ and λ are smoothly standard. Then there exists a compactly supported family of isocontact embeddings $\phi_t\colon (\mathbb{D}^{2n-1}, \xi_{\mathrm{st}}) \to (B, \xi)$ for $t \in [0, 1]$ with $\phi_0 = \phi$ such that $\mathrm{im}(\phi_1) \cap \mathrm{im}(\lambda) = \varnothing$.*

Theorem 1 stands in stark contrast with many known obstructions — see Borman, Eliashberg and Murphy [1], Casals, Murphy and Presas [3], Eliashberg [8] and Vogel [33] — for the removal of intersections, via contact isotopies, between certain subsets in contact manifolds. In particular, pairs of overtwisted disks are known to admit a rigid behavior [33], whereby there exists a smooth isotopy removing their intersections but no such contact isotopy exists. In fact, as shown in [3, Section 8.2], the intersection theory between the overtwisted contact germ and certain Legendrian spheres is strictly richer than the smooth topology. The new insight in Theorem 1 is that in higher dimensions the interaction between Legendrian submanifolds and contact submanifolds is much closer to smooth topology. In the same vein as the smooth Whitney trick, Theorem 1 should lead to a source of applications in higher-dimensional contact topology. We show that it in fact leads to the discovery of a new h–principle in contact topology, as follows.

The study of contact submanifolds in higher dimensions has recently seen many new developments, including the articles of Casals and Etnyre [2], Casals, Murphy and Presas [3], Etnyre and Furukawa [12], Etnyre and Lekili [13], Honda and Huang [23], Lazarev [27] and Pancholi and Pandit [30]. We apply Theorem 1 to completely solve the existence problem for isocontact submanifolds:

Theorem 2 (existence of isocontact submanifolds) *Let (N,ξ_N) and (M,ξ_M) be contact manifolds with $\dim(M)=\dim(N)+2\geq 5$, and $(f^0, F_s^0)\colon (N,\xi_N)\to(M,\xi_M)$ a formal isocontact embedding. Then there exists a family*

$$(f^t, F_s^t)\colon (N,\xi_N)\to(M,\xi_M), \quad t\in[0,1],$$

of formal isocontact embeddings such that $(f^1, F_s^1=df^1)$ is an isocontact embedding.

Theorem 2 shows that any codimension-2 smooth submanifold in a contact manifold (M,ξ_M) can be smoothly isotoped to be a contact submanifold, even with an a priori chosen induced contact structure, as long as the obstructions from algebraic topology vanish. In particular, any almost contact smooth knot $\mathbb{S}^{2n-1}\to(M,\xi_M)$ has a standard contact representative $(\mathbb{S}^{2n-1},\xi_{\mathrm{st}})\to(M,\xi_M)$ in its smooth isotopy class. Paired with M Gromov's h–principle for codimension-k submanifolds for $k\geq 4$, Theorem 2 resolves the existence h–principle for isocontact submanifolds in *any* formal class, ie the existence of codimension-2 contact embeddings with a prescribed contact structure in the domain.[1]

Theorem 1 focuses on the interplay between Legendrian submanifolds and contact submanifolds, the first result of its type. Theorem 2 addresses the *existence* of a contact submanifold in a given smooth class. This existence h–principle regarding isocontact embeddings was actually conjectured and partially solved by the second author and Pandit[2] in [30, Theorem 1], who showed that a given contact structure can be isocontactly embedded assuming the hypothesis that a contact embedding actually exists. Theorem 2 is significantly stronger and resolves this issue entirely: it actually does prove that *existence* always holds and strictly implies the results in the preprint [30]. The analogous statement remains open for symplectic manifolds. Note that, regarding

[1] Being able to prescribe the contact structure in the submanifold is a strong conclusion. Theorem 2 works in full generality, without any assumption on the contact submanifold being (ambiently) Stein fillable or overtwisted, or any other simplifying hypothesis: it holds for *any* contact structure, including all tight nonfillable contact structures.

[2] An alternative argument for their results also appeared in [27, Theorem 4.12].

the *uniqueness* of such a contact submanifold in its isotopy class, the first author and Etnyre [2, Theorem 1.1] recently constructed the first examples of contact submanifolds in all dimensions which are smoothly isotopic but *not* contact isotopic.

The present arguments for Theorems 1 and 2 apply to *higher-dimensional* contact manifolds, and crucially use the dimensional hypothesis. In the remaining case — that of transverse knots in contact 3–manifolds — Theorem 2 holds if we allow ourselves to change the self-linking number (see Geiges [15, Section 3.3]), and thus the conclusion is only valid on the smooth type of the knot, rather than the formal isocontact type of the embedding. This situation is in line with the 3–dimensional Thurston–Bennequin inequality [15, Section 4.6.5] and the difference between Legendrian knots and higher-dimensional Legendrian submanifolds [3].

Organization This article is organized as follows: Section 2 proves Theorem 1 and Section 3 shows Theorem 2 as an application. Regarding notation, given a set C we will denote by $\mathcal{O}p(C)$ an arbitrarily small open neighborhood of it.

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Relation to [23] The results of this article were disseminated during spring and summer 2019, with its arXiv version finally being uploaded August 13, 2019. Simultaneously, and unbeknownst to the authors, K Honda and Y Huang were able to apply their newly developed theory of convex surfaces in higher dimensions to construct *a* contact representative of a smooth submanifold, as presented in [23, Corollary 1.3.5], dating July 13, 2019 on arXiv. Note also that our Theorem 2 is stronger than the statement in [23, Corollary 1.3.5], as it proves that the contact structure on the smooth submanifold can be prescribed, which is not known to be the case if one only applies the convex hypersurface techniques in [23] and Honda [22], as one would still need the results of the second author in [30]. In any case, the two arguments are remarkably different and were developed independently; we view this as a healthy consequence of having active and fruitful research in higher-dimensional contact topology.

2 Legendrian Whitney trick

The classical smooth Whitney trick [25; 36] starts with two oriented submanifolds $S_0, S_1 \subseteq M$ inside a simply connected oriented manifold M, which intersect at finitely many (signed) points. The main assumption is the existence of two oppositely oriented intersection points $p_0, p_1 \in S_0 \cap S_1$. The Whitney trick then consists of canceling these two intersection points by placing them as boundaries of two embedded curves $\psi_0\colon [0,1] \to S_0$ and $\psi_1\colon [0,1] \to S_1$. The crucial part of the argument relies on the existence of an embedded smooth disk $\psi\colon [0,1] \times [0,1] \to M$ which intersects S_0 and S_1 exactly in the images of ψ_i for $i = 0, 1$ only along its boundary, and satisfies $\psi(t,0) = p_0$ and $\psi(t,1) = p_1$. This disk, known as a Whitney disk, is then used to construct a smooth flow that pushes ψ_0 to $\psi_t = \psi(t,-)$ for time t and is supported in an arbitrarily small neighborhood of the disk. The image of S_0 through the flow at time $t = 1+\delta$ for $\delta \in \mathbb{R}^+$ arbitrarily small becomes displaced from S_1 at the points $p_0, p_1 \in S_1$, and no new intersections are introduced.

The Legendrian Whitney trick generalizes the smooth Whitney trick to submanifolds whose generic intersection has dimension bigger than zero. For instance, a Legendrian sphere in $(\mathbb{S}^{2n+1}, \xi_{\mathrm{st}})$ generically intersects a codimension-2 contact submanifold in a smooth submanifold of dimension $n-2$. In this context, the underlying smooth arguments in the contact setting are in line with Haefliger's proof of unknotting in high codimensions [19], and see also Shapiro's theory of deformation cells [31, Section 5].

Let (M,ξ) be a $(2n+1)$–dimensional contact manifold. Consider a Legendrian sphere $S \subseteq (M,\xi)$ and a codimension-2 contact submanifold $(\mathbb{D}, \xi|_{\mathbb{D}}) \subseteq (M,\xi)$. Their intersection $S \cap \mathbb{D} \subseteq (M,\xi)$ is an $(n-2)$–dimensional isotropic submanifold $\Sigma \subseteq \mathbb{D} \cap S$. For the Legendrian Whitney trick, Σ might not be a sphere, and thus not necessarily bound a smooth disk. Instead, the role of the Whitney disk is played by a Legendrian map $\psi\colon W \times [0,1] \to (M,\xi)$ restricting to two isotropic embeddings $\psi_0\colon W \times \{0\} \to S$ and $\psi_1\colon W \times \{1\} \to \mathbb{D}$ such that $\psi_0(\partial W \times \{0\}) = \Sigma = \psi_1(\partial W \times \{1\})$. The image of the Legendrian map ψ is referred to as a Legendrian Whitney bridge, and its interior will lie on the complement $M \setminus (S \cup \mathbb{D})$. It is this Legendrian Whitney bridge that allows us to construct a compactly supported contact isotopy sliding S along W and remove the intersection $\Sigma \subseteq S \cap \mathbb{D}$ without creating a new one.

2.1 Construction of the Legendrian Whitney bridge

Let $(\mathbb{D}, \xi) \subseteq (B, \xi)$ be a standardly embedded contact disk, and $S \subseteq (B, \xi)$ a Legendrian sphere which is also standardly embedded. By genericity, we assume the intersection $\Sigma = \mathbb{D} \cap S$ between $\mathbb{D}$ and S is smoothly transverse. Since S is compact and $\mathbb{D}$ is smoothly standard, there exists an $(n-1)$–dimensional compact manifold W and an embedding $W \to S$ with boundary $\partial W = \Sigma$. Indeed, the submanifold W can be constructed by fixing a $2n$–dimensional smoothly standard disk $\widehat{\mathbb{D}}^{2n}$, satisfying $\mathbb{D} \subseteq \widehat{\mathbb{D}} \subseteq B$, and intersecting the closure of one of the two connected components of $\widehat{\mathbb{D}}^{2n} \setminus \mathbb{D}$ with the Legendrian sphere S.

Let $C_\Sigma = \Sigma \times (-\varepsilon, 0]$ be a collar neighborhood of the boundary $\Sigma \subseteq W$, and denote by $\mathbb{W}$ the quotient of $W \times [0,1]$ by the relation $(w,t) \sim (w,s)$ if and only if $w \in \partial W$ and $s, t \in [0,1]$, ie the quotient $e \colon W \times [0,1] \to \mathbb{W}$ collapses the subset $\partial W \times [0,1]$ to a single copy of the boundary ∂W.

Proposition 3 (Legendrian Whitney bridge) *Let $(\mathbb{D}^{2n-1}, \xi_{\mathrm{st}}) \subseteq (B, \xi)$ be a properly embedded contact disk and $S \subseteq (B, \xi)$ a Legendrian submanifold with both inclusions smoothly standard. For any sufficiently small $\varepsilon \in \mathbb{R}^+$, there exist a compact manifold W, a Legendrian embedding $F \colon \mathbb{W} \to B$ and a smooth quotient map $e \colon W \times [0,1] \to \mathbb{W}$ such that*:

(1) *F is transverse to S and $\mathbb{D}$, $F^{-1}(S) = e(W \times \{0\})$ and $F^{-1}(\mathbb{D}) = e(W \times \{1\})$.*

(2) *$(F \circ e)(\partial W \times [0,1]) = \Sigma = S \cap \mathbb{D}$ and $(F \circ e)|_{\mathring{W} \times [0,1]}$ is a smooth embedding.*

(3) *There exists a Legendrian ribbon $R(\Sigma)$ of $\Sigma \subseteq B$ and a chart $\Sigma \times \mathbb{D}^2(3\varepsilon) \to R(\Sigma) \subset B$ such that the restriction $(F \circ e)|_{C_\Sigma \times [0,1]}$ is mapped to $\Sigma \times \mathbb{D}^2(3\varepsilon)$ and there exist coordinates $(q, s, t) \in C_\Sigma \times [0,1] = \Sigma \times (-\varepsilon, 0] \times [0,1]$ such that*

$$(F \circ e)(q, s, t) = \Psi\big(q, \big(s \cos\big(\tfrac{1}{2} t \pi\big), s \sin\big(\tfrac{1}{2} t \pi\big)\big)\big). \qquad \square$$

Property (3) in Proposition 3 is depicted schematically in Figure 1. It is important to keep in mind that $(q, x, y) \in \Sigma \times \mathbb{D}^2(3\varepsilon)$ are coordinates for the ribbon $R(\Sigma)$. They will be our reference coordinates to study this region and they will remain unchanged for the whole proof.

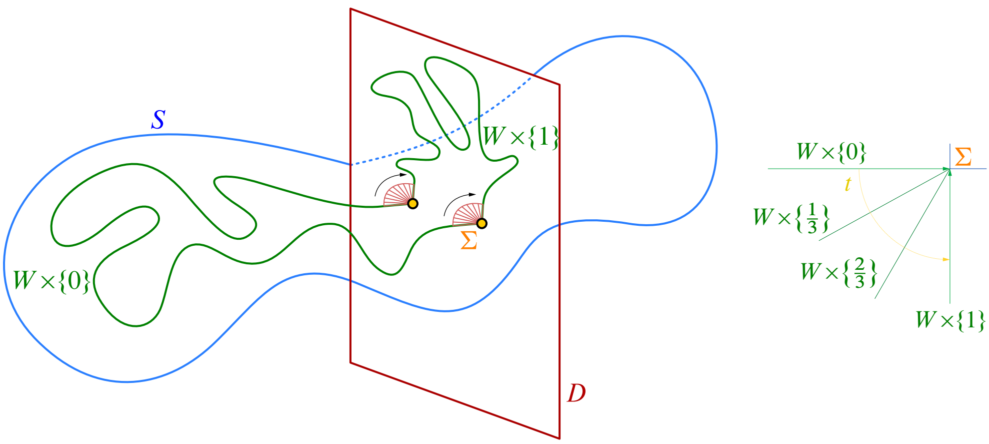

Figure 1: The Legendrian neighborhood $\Sigma\times\mathbb{D}^2$ of the intersection $\Sigma=S\cap\mathbb{D}$ and the image of the Legendrian Whitney bridge, where $\mathbb{W}$ is mapped into the third quadrant. Left: inside the global manifold. Right: the coordinate system.

Remark 4 The map $\psi=F\circ e\colon W\times[0,1]\to B$ will be constructed as a 1–parameter family $\{\psi_t\}_{t\in[0,1]}$ of isotropic embeddings of W such that $\psi_0=(F\circ e)|_{W\times\{0\}}\subseteq S$ and $\psi_1=(F\circ e)|_{W\times\{1\}}\subseteq\mathbb{D}$. However, we first find a family of smooth embeddings $\{\phi_t\}_{t\in[0,1]}$ and then construct the required isotropic family $\{\psi_t\}_{t\in[0,1]}$. Hence, for the proof of Proposition 3, ϕ_t are smooth morphisms and ψ_t are isotropic morphisms.

Regarding notation, a normal bundle of a submanifold $A\subseteq B$ is denoted by ν_A^B. When ν_A^B is an oriented real line bundle, ν_A^B will also denote a choice of nowhere-vanishing vector field trivializing the bundle.

2.2 Proof of Proposition 3

Let us declare $\phi_0=\psi_0$ to be the inclusion $W\subseteq S$. Choose a compatible almost complex structure J_ξ for the symplectic bundle $\xi=(\ker\alpha,d\alpha)$ which extends a complex structure compatible with $(\mathbb{D},\xi|_{\mathbb{D}})$. First, we proceed with the construction of the Legendrian embedding yielding $\mathbb{W}\subseteq B$ near the intersection locus Σ. For that, we choose framings as follows.

Let us trivialize the rank-2 (trivial) normal bundle ν_Σ^S over Σ so that

$$\nu_\Sigma^S=\nu_\Sigma^{\phi_0(W)}\oplus\nu_{\phi_0(W)}^S=\langle v_1\rangle\oplus\langle w_1\rangle,$$

where v_1 and w_1 are fixed but arbitrary nonvanishing sections. Since this rank-2 bundle $\langle v_1, w_1\rangle$ is transverse to $T\mathbb{D}|_\Sigma$, there is a unique vector field v_0, up to positive scalar function, such that $\langle v_0\rangle = \langle\{v_1, w_1, J_\xi w_1\}\rangle \cap T\mathbb{D}|_\Sigma$. In short, $v_0, v_1 \in \Gamma(T\Sigma)$ are vector fields such that:

- v_0 is tangent to $\mathbb{D}$ and transverse to $\phi_0(W)$; v_1 is tangent to $\phi_0(W)$ and transverse to $\mathbb{D}$.
- The subbundle $T\Sigma \oplus \langle v_0, v_1\rangle \subseteq \xi|_\Sigma$ is a Lagrangian space.

Indeed, the second property follows from the fact that $T\Sigma \oplus \langle v_1, w_1\rangle$ is a Lagrangian subspace and we have replaced w_1 by a linear combination of the type

$$v_0 = \lambda_0 w_1 + \lambda_1 J w_1 + \lambda_2 v_1,$$

and such a vector paired with the isotropic subspace $T\Sigma \oplus \langle v_1\rangle$ evaluates to zero for the symplectic structure on ξ_Σ.

For the smooth embedding ϕ_1, we directly invoke the Whitney embedding theorem [35; 36], relative to an open subdomain in which the map is already an embedding. This constructs a smooth embedding $\phi_1\colon W \to \mathbb{D}$ such that $\phi_1(\partial W) = \Sigma$, such that the vector field v_0 is tangent to $\phi_1(W)$ and v_1 is transverse to $\phi_1(W)$. In order to adequately construct the Legendrian ribbon $R(\Sigma)$, let us use the following:

Lemma 5 *Let $e\colon \Sigma \to (M^{2n+1}, \xi)$ be an isotropic embedding with $k < n$ and $\{\tau_{k+1}, \dots, \tau_n\}$ a partial framing for $e^*\xi$ such that $T\Sigma \oplus \langle\{e_*\tau_{k+1}, \dots, e_*\tau_n\}\rangle \subset \xi_{e(S)}$ is a Lagrangian subspace. Then there exists a Legendrian embedding $\tilde{e}\colon \Sigma \times D^{n-k}(1) \to (M, \xi)$ such that $\tilde{e}|_{\Sigma\times\{0\}} = e$ and $(\tilde{e})_*(\partial/\partial x_i)|_{\Sigma\times\{0\}} = \tau_{k+i}$ for $1 \le i \le n-k$.*

Lemma 5 is proven right after we conclude the present proof. Now we apply Lemma 5 to the inclusion $\Sigma = S \cap \mathbb{D}$ and the partial framing $\{\tau_{n-1}, \tau_n\} = \{v_0, v_1\}$, where the vector fields v_0 and v_1 are extended preserving their containment in ξ with the extension of v_1 being transverse to $\phi_1(W)$. This extension yields a family of maps $\{\psi_t\} = \{\phi_t\}$ for $t \in [0, 1]$ in the collar neighborhood $C_\Sigma \times (-\varepsilon, 0]$ that conform to condition (3) in the statement of Proposition 3. Let us now work towards extending $\{\phi_t\}_{[0,1]}$ from the collar $C_\Sigma \times [0, 1]$ to the entire B.

First, let us claim that for $\varepsilon \in \mathbb{R}^+$ small enough, we can construct smooth pushoffs of the two embeddings $\phi_0(W)$ and $\phi_1(W)$ providing a family of embeddings $\{\phi_t\}$ for $t \in [0, \varepsilon) \cup (1-\varepsilon, 1]$ which are each transverse to $\mathbb{D}$ and S, and intersect $\mathbb{D}$ and $\mathcal{S}$

at Σ. In order to prove this, it is enough to find two nonvanishing vector fields τ_0 and τ_1 within ξ which are defined over $\phi_0(W)$ and $\phi_1(W)$, respectively, are normal to them, and extend v_0 and v_1, ie $(\tau_0)|_\Sigma = v_0$ and $(\tau_1)|_\Sigma = v_1$, making sure that they are tangent to the ribbon $R(\Sigma)$. For that, let $\mathbb{D}$ be the connected component of $\widehat{\mathbb{D}} \setminus \mathbb{D}$ which defines W, and note that $\langle v_1 \rangle = \nu_\Sigma^{\phi_0(W)} \simeq (\nu_{\mathbb{D}}^{\Delta})|_\Sigma$. Then we choose τ_1 along $\phi_1(W)$ such that $\langle\{\tau_1\}\rangle = \nu_{\mathbb{D}}^{\Delta}|_{\{\phi_1(W)\}}$. The vector field τ_0 is constructed similarly. Indeed, the vector field $J_\xi \cdot w_1$ is defined along $\phi_0(W)$ and the vector field v_0 that is defined over the collar C_Σ. Since $\langle v_1, w_1 \rangle$ is transverse to $\mathbb{D}$, we have that $\lambda_1 > 0$ and thus the linear interpolation between $J_\xi \cdot w_1$ and v_0 is never zero and transverse to $\phi_0(W)$. Hence, we can choose a cutoff function $h\colon W \to [0,1]$ compactly supported on the collar C_Σ, with $h|_\Sigma = 1$, and declare $\tau_0 = h v_0 + (1-h) J_\xi w_1$.

At this stage of the argument, the Legendrian Whitney bridge $W \times [0,1]$ and its embedding $F \circ e$ are (smoothly) defined by ϕ in the region $A = W \times \big([0,\varepsilon) \cup (1-\varepsilon, 1]\big) \cup C_\Sigma \times [0,1]$. Its extension can be constructed as follows. First, since the contact divisor $\mathbb{D} \subseteq B$ is smoothly standard, $\mathbb{D}$ is contained in a $2n$–dimensional disk $\widehat{\mathbb{D}}$, $\mathbb{D} \subseteq \widehat{\mathbb{D}} \subseteq B$, smoothly isotopic to a linear disk. Recall that Δ is the connected component of $B \setminus \widehat{\mathbb{D}}$ which contains A, out of the two components of $B \setminus \widehat{\mathbb{D}}$, and note that Δ is a smooth $(2n+1)$–dimensional ball. Then we choose an arbitrary smooth map $\widetilde{\phi}\colon W \times [0,1] \to \Delta \subseteq B$ which coincides with ϕ along A, and use Thom's transversality theorem [10] to obtain a C^∞–small perturbation, relative to a neighborhood of A, which does not intersect S. This is possible since $\dim(W \times [0,1]) + \dim S = 2n < 2n+1 = \dim B$. Then, the Whitney embedding theorem applies to perturb the map $\widetilde{\phi}$, relative to A, into a smooth embedding $\phi\colon W \times [0,1] \to \overline{\Delta}$. The image of the restriction of this map ϕ to the open subset $\mathring{W} \times (0,1)$ avoids both S and $\mathbb{D}$. At this stage of the proof, we have constructed a smooth Whitney bridge according to the statement of Proposition 3. The remainder of the proof consists in deforming ϕ to a Legendrian embedding as required.

The Legendrian embedding will be obtained via the h–principle on Legendrian immersions [10, Section 16.1]. Note that this h–principle yields a Legendrian immersion; nevertheless, the Legendrian dimension and codimension in ambient dimension $2n+1$ are n and $n+1$, respectively. Consequently, a generic Legendrian perturbation of a Legendrian immersion yields a Legendrian embedding, as required. In order to apply the h–principle, it now suffices to endow the smooth embedding $\phi\colon W \times [0,1] \to B$ with the structure of a formal isotropic embedding $G\colon W \times [0,1] \to \mathrm{Mon}_{\mathbb{R}}(TW \oplus \mathbb{R}, \phi^*\xi)$. In fact, the bundle $\phi^*\xi$ decomposes as $V \oplus \mathbb{C}$ for a complex bundle V, satisfying that $\phi_0^*\xi = (\phi_0)_* TW \otimes_{\mathbb{R}} \mathbb{C} \oplus \langle \tau_0, J_\xi \tau_0 \rangle$ and $V|_{W \times \{1\}} \oplus \mathbb{C}$, where $V|_{W\times\{1\}} \simeq \xi|_{\mathbb{D}}$.

Indeed, along the collar neighborhood C_Σ, the complexification $(\phi_t)_* TW \otimes_{\mathbb{R}} \mathbb{C}$ is a choice for V, and the linear interpolation $(t\tau_1 + (1-t)\tau_0)|_\Sigma = tv_1 + (1-t)v_0$ is a section trivializing the rank-2 symplectic orthogonal. Then obstruction theory [20] tells us that this trivial subbundle can be extended along $W \times [0,1]$: since ξ is smoothly trivial, choose a smooth map $\widehat{f}\colon W \times [0,1] \to \mathbb{S}^{2n-1}$ extending a fixed section on the boundary, which exists since $\dim W \times [0,1] = n$ and $\mathbb{S}^{2n-1}$ is $(2n-2)$–connected. In conclusion, we have the decomposition $\phi^*\xi = V \oplus \{\widehat{f}, J\widehat{f}\}$ for the pullback of the contact structure.

Let $V_t = V|_{\phi_t(W)}$ and $G_t = G|_{W\times\{t\}}$, and let us require that the formal isotropic embedding (ϕ, G) satisfy the following properties:

(a) $G_t(w)(\{0\}\oplus\mathbb{R}) = \{0\}\oplus\mathbb{C} \subset \xi_{(w,t)}$ for any $w \in W$.

(b) $G(w,t) = d\phi(w,t)$ for $(w,t) \in (W \times [0,\varepsilon) \cup C_\Sigma \times [0,1])$.

(c) $G_t(w)(TW \oplus \{0\}) \subset V_t$ and is a Lagrangian subspace of V_t for any $w \in W$.

The first condition is determined by imposing $G(w,t)(\partial_t) = f(w,t)$, and the second condition merely formalizes that the embedding is already Legendrian, not just formally Legendrian, in certain regions. The third condition can be assumed since $W \times [0,1]$ deformation-retracts to $(W \times [0,\varepsilon]) \cup (C_\Sigma \times [0,1])$. In conclusion, (ϕ, G) is a formal Legendrian immersion of $W \times [0,1]$ and (ϕ_1, G_1) is a formal Legendrian immersion on $\mathbb{D}$, since $V_1 \simeq \xi|_{\mathbb{D}}$. Now, the C^0–dense h–principle for Legendrian immersions and the genericity of the immersion yields a C^0–small perturbation of the pair (ϕ_1, G_1) into a new pair $(\psi_1, F_1' = d\psi_1)$. Then we push the smooth map ψ_1 along the flow of τ_1 to define the maps ψ_t for $t \in (1-\varepsilon, 1]$, which are isotropic embeddings. Finally, the same C^0–dense h–principle for Legendrian immersions relative to the domain for the pair (ϕ, G) produces the required Legendrian Whitney bridge. The embedding $F\colon \mathbb{W} \to B$ defined in the statement is the map ψ extended with the vector fields ∂_s and ∂_t, defined in the two boundaries of $W \times [0,1]$. This concludes the proof of Proposition 3. □

Proof of Lemma 5 View the contact structure $(T^*\Sigma \times \mathbb{C}^{n-k} \times \mathbb{R}, \xi_{\mathrm{st}})$ as a 1–jet space, and extend the map e to an smooth embedding $\hat{e}\colon T^*\Sigma \times \mathbb{C}^{n-k} \times \mathbb{R} \to \mathcal{O}p(\Sigma)$ such that $\hat{e}_*\xi_{\mathrm{st}}$ coincides with ξ over $e(\Sigma)$ and $\partial/\partial x_i$ is mapped to the corresponding framing vector. Since the associated conformal symplectic structures share a Lagrangian subspace, the linear interpolation between them is through conformal symplectic structures. By Moser's stability argument [15], given that the contact structures coincide over S, there is a small neighborhood of Σ that is contactomorphic to a small neighborhood of $\Sigma \subseteq T^*\Sigma \times \mathbb{C}^{n-k} \times \mathbb{R}$. Composing $\hat{e}$ with this contactomorphic yields the required Legendrian embedding. □

2.3 The Legendrian Whitney trick

Let us prove Theorem 1. Apply Proposition 3 to construct a Legendrian Whitney bridge (e, F, W) for the intersection $\Sigma = S \cap \mathbb{D}$. Intuitively, the product structure $W \times [0,1]$ of the domain of the Legendrian Whitney bridge allows us to proceed in the same vein as in the smooth case [31; 36]. In the present case, the central caveat is that the isotopy that displaces the Legendrian sphere S from the contact submanifold $\mathbb{D}$ must also be a contact isotopy. The challenge thus relies in finding a compactly supported contact isotopy which pushes the Legendrian S along the image of the Legendrian Whitney bridge $W \times [0,1]$ in such a manner that the last contactomorphism has removed the intersection Σ.

In the smooth case, this is achieved by integrating a compactly supported extension of the smooth vector field in the direction of $[0,1]$. The core difference between the smooth and the contact situations is the fact that smoothly cutting off a contact vector field might not integrate to a contact isotopy. Hence, in a nutshell, the main contribution of this section is a careful analytical construction of a compactly supported contact vector field, or equivalently a contact Hamiltonian, which produces a contact isotopy with the necessary properties. Let us introduce local coordinates in order to perform these computations.

2.3.1 Coordinate system The Legendrian Whitney bridge $W \times [0,1]$ is the manifold $\mathbb{W} = (W \times [0,1])/\sim$ and thus an open neighborhood of the Legendrian image $F(\mathbb{W}) \subseteq (B^{2n+1}, \xi)$ can be identified with $J^1(\mathbb{W}, \xi_{\mathrm{st}})$. Let $q_n \in [0,1]$ be the coordinate in the second factor of $W \times [0,1]$, and $q_{n-1} \in (-\varepsilon, 0]$ the local coordinate in $\mathcal{O}p(\partial W)$ defined by minus the distance to ∂W. Thus, given the nature of the quotient $W \times [0,1] \to \mathbb{W}$, the pair (q_{n-1}, q_n) can be understood as smooth polar coordinates (defined on the third quadrant) in $\mathcal{O}p(\partial W) \subseteq \mathbb{W}$, parametrizing the 2–dimensional normal slice to ∂W, where $-q_{n-1}$ is the radius and q_n its associated angle.

The intersection $(F \circ e)(W \times \{0\}) = F(\mathbb{W}) \cap S$ of the Whitney bridge with the Legendrian sphere is the image of $W \times \{0\}$, and the intersection $(F \circ e)(W \times \{1\}) = F(\mathbb{W}) \cap \mathbb{D}$ of the Whitney bridge with the contact submanifold is the image of $W \times \{1\}$.

Far from Σ, ie $q_{n-1} \neq 0$, the former can be given by the local[3] equation

$$\{(q_1, \dots, q_{n-1}, q_n) \in \mathbb{W} : q_n = 0\},$$

whereas the latter is defined by $\{(q_1, \dots, q_{n-1}, q_n) \in \mathbb{W} : q_n = 1\}$. Along Σ, given by the vanishing of q_{n-1}, the quotient $W \times [0,1] \to \mathbb{W}$ forces q_n to be ill-defined, and we interpret any restriction on q_n as being satisfied. Thus, the above equation can be understood as meaningful all along $\mathbb{W}$ and we have $W \times \{0\} \cap W \times \{1\} = \Sigma$. This establishes a coordinate system along $\mathbb{W}$, which we can now extend to its neighborhood $J^1(\mathbb{W}, \xi_{\mathrm{st}})$. Indeed, the chosen extension is given by the trivial Legendrian inclusion $\mathbb{W} \subseteq (J^1(\mathbb{W}), \xi_{\mathrm{st}}) \cong (T^*\mathbb{W} \times \mathbb{R}_\tau, \lambda_{\mathbb{W}} - d\tau)$ given by the zero section in $T^*\mathbb{W}$ at $\tau = 0$. Note that this extension is not canonical: there is a choice of a Riemannian metric. We fix a metric such that $\mathbb{D}$ and S are assumed to be orthogonal to the Whitney bridge for the following discussion.

In the model $T^*\mathbb{W} \times \mathbb{R}$, we write $(q, p) \in T^*\mathbb{W}$ for points in the cotangent bundle such that the canonical projection $\pi : T^*\mathbb{W} \to \mathbb{W}$ is given by $\pi(q, p) = q$. The contact submanifold $\mathbb{D}$ is cut out by the equation

$$\mathbb{D} = \{(q, p, \tau) \in T^*\mathbb{W} \times \mathbb{R} : q_n = 1,\ p_n = 0\},$$

where p_n is the conjugate coordinate of q_n. The isotropy and dimension count for $\mathbb{W}$, S and ∂W forces their conjugate momenta to vanish along S and thus we can write

$$S = \{(q, p, \tau) \in T^*\mathbb{W} \times \mathbb{R} : p_1 = p_2 = \dots = p_{n-1} = 0, \tau = q_n = 0\}.$$

Again, these coordinates are understood in the quotient $\mathbb{W}$, and thus the intersection $\Sigma = S \cap \mathbb{D}$ is nonempty, cut out by the equation

$$\Sigma = \{(q, p, \tau) \in T^*\mathbb{W} \times \mathbb{R} : p_1 = p_2 = \dots = p_{n-1} = p_n = 0, \tau = q_{n-1} = q_n = 0\},$$

since the vanishing of the radial coordinate q_{n-1} allows for both equations $\{q_n = 1\}$ and $\{q_n = 0\}$ to be satisfied by the angular coordinate, as in the usual context of smooth polar coordinates. Finally, recall that $(x, y) \in \mathbb{D}^2$ are Cartesian coordinates for the disk spanned by these polar coordinates (q_{n-1}, q_n). Denote their conjugate momenta by (p_x, p_y). Note that, in these Cartesian coordinates, the intersection of the contact submanifold $\mathbb{D}$ with the Legendrian Whitney bridge reads

$$\mathbb{D} \cap \mathcal{O}p(\mathbb{W}) = \{(q, p, \tau) \in T^*\mathbb{W} \times \mathbb{R} : x = 0,\ p_x = 0\}.$$

[3]The coordinates $(q_1, \dots, q_{n-1})$ are only local coordinates in W.

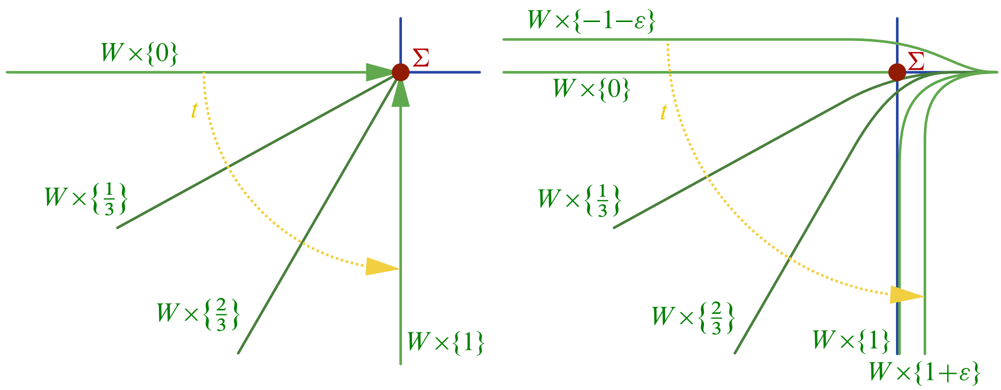

Figure 2: A depiction of $\Sigma\times\mathbb{D}^2 \subseteq B$, and the modification of the Legendrian Whitney bridge $\mathbb{W} = (e, F, W)$ to its extension $\overline{\mathbb{W}} = (\bar{e}, \bar{F}, \overline{W})$, which places the extended collar $\overline{C}_\Sigma\times(-\varepsilon-1, 1+\varepsilon)$ of Σ inside the domain $\Sigma\times\mathbb{D}^2$. In the picture, the angle is extended from $(0,1)$ on the left, and to $(-\varepsilon-1, 1+\varepsilon)$ on the right.

2.3.2 Preparation before sliding The Legendrian Whitney bridge (e, F, W) provided by Proposition 3 can be intuitively depicted as an piece of a smooth open book [15] with binding Σ and page W. The 2–disk factor of the tubular neighborhood $\Sigma\times D^2$ near the binding is parametrized by coordinates (q_{n-1}, q_n), or equivalently (x, y). Before constructing the contact isotopy which removes the intersection $\Sigma = S\cap\mathbb{D}$, let us perform a minor modification placing the binding of this open book off $\Sigma\times\{0\}$, as depicted in Figure 2.

The need for this modification is already present in the smooth Whitney trick, as can be visualized in Figure 3. Indeed, the vector field ∂_{q_n} in the above Whitney bridge (e, F, W) vanishes along Σ, given that the smooth map $F\circ e$ only embeds the quotient $\mathbb{W}$. It is thus necessary to enlarge the collar C_Σ of $\Sigma\subseteq\mathbb{W}$ in order for the extension of the vector field ∂_{q_n} to displace $\mathbb{D}$ from S. This modification is stated in detail in the upcoming Lemma 7.

The notation follows that of Proposition 3, with (s,t) denoted by (q_{n-1}, q_n), $q\in\Sigma$ a point in $\Sigma = \partial W$ and (q, x, y) coordinates in the Legendrian ribbon $R(\Sigma)$ of Σ. Let us also prolong a collar neighborhood of the boundary $\Sigma = \partial W\subseteq W$ from $C_\Sigma = \Sigma\times(-\varepsilon, 0]$ to $\overline{C}_\Sigma = \Sigma\times(-\varepsilon,\varepsilon]$ and use it to define the extension $\overline{W} = W\cup\partial W\times(0,\varepsilon]$. The Legendrian extension $\overline{\mathbb{W}}$ of the Legendrian Whitney bridge $\mathbb{W}$ can be constructed by applying the following:

Lemma 6 *Let $F\colon \Lambda \to (M,\xi)$ be a Legendrian embedding such that Λ is an open smooth manifold which admits a compactification $\widehat{\Lambda}$ such that the boundary $\partial\widehat{\Lambda}$ is a smooth manifold (possibly with corners) and admits a collar neighborhood on the boundary, ie $\mathcal{O}p(\partial\widehat{\Lambda}) \simeq \partial\widehat{\Lambda}\times(\varepsilon,0]$. Then there exist*

(a) *an open manifold $\overline{\Lambda}$ containing Λ, $\Lambda \subseteq \overline{\Lambda}$ such that it admits a compactification whose neighborhood is of the form $\partial\widehat{\Lambda}\times(-\varepsilon,\varepsilon]$,*

(b) *a Legendrian embedding $\overline{F}\colon \overline{\Lambda} \to (M,\xi)$ that coincides with F when restricted to Λ,*

(c) *the image $\overline{F}(\overline{\Lambda})$ lies in an arbitrary small neighborhood of $F(\Lambda)$.*

The proof of this lemma follows the same argument as that of Lemma 5. The embedding F is extended to $\overline{F}\colon \overline{\mathbb{W}} \to B$, by using Lemma 6. Let us remind the reader that $\partial\mathbb{W} = e(W\times\{0\}) \cup e(W\times\{1\})$. In particular, a neighborhood of $e(W\times\{0\})$ inside $\overline{F}(\overline{\mathbb{W}}) \setminus R(\Sigma)$ can be assumed to have coordinates $(w,v) \in e(W\times\{0\})\times(-\delta,\delta)$ such that $q_n = v$. In short, we are extending the domain of definition of the coordinate q_n from the interval $[0,1]$ to the interval $[-\varepsilon, 1+\varepsilon]$. Thus, we have that $q_n \in [-\varepsilon, 1+\varepsilon]$ is a coordinate on $\mathbb{W}$.

This coordinate can be reparametrized into $f(\overline{q}_n) = q_n$, by using a diffeomorphism $f\colon[-\varepsilon-1, 1+\varepsilon] \to [-\varepsilon, 1+\varepsilon]$ satisfying that $q_n = \overline{q}_n$ for $\overline{q}_n \geq 0$. Consequently, the Legendrian S still satisfies the local equation $\{\overline{q}_n = 0\}$. To ease notation, we shall still write q_n for this deformed coordinate $\overline{q}_n$. In the working hypothesis of Proposition 3, we have the following:

Lemma 7 *There exists a smooth map $\bar{e}\colon \overline{W}\times[-\varepsilon-1, 1+\varepsilon] \to \overline{\mathbb{W}}$ such that:*

(a) *The map $\bar{e}$ coincides with e on $(\overline{W}\setminus \overline{C}_\Sigma)\times[0,1]$.*

(b) *The domain $\Sigma\times(-\varepsilon,\varepsilon]\times[-\varepsilon-1, 1+\varepsilon]$ is mapped by $\bar{e}$ into the domain $\partial W\times\mathbb{D}^2(3\varepsilon)$.*

(c) *Let $\overline{\Psi} = \overline{F}\circ\bar{e}$; then $\overline{\Psi}$ on $\Sigma\times\left(-\varepsilon,-\frac{1}{2}\varepsilon\right)\times[-1-\varepsilon, 1+\varepsilon]$ is given by*

$$\overline{\Psi}(q,q_{n-1},q_n) = \begin{cases} (q,q_{n-1},q_n) & \text{if } (q,q_{n-1},q_n)\in\Sigma\times\left(-\varepsilon,-\frac{1}{2}\varepsilon\right)\times[-1-\varepsilon,0], \\ \left(q, q_{n-1}\cos\left(\frac{1}{2}q_n\pi\right), q_{n-1}\sin\left(\frac{1}{2}q_n\pi\right)\right) & \\ & \text{if } (q,q_{n-1},q_n)\in\Sigma\times\left(-\varepsilon,-\frac{1}{2}\varepsilon\right)\times[0,1], \\ (q, q_n-1, q_{n-1}) & \text{if } (q,q_{n-1},q_n)\in\Sigma\times\left(-\varepsilon,-\frac{1}{2}\varepsilon\right)\times(1,1+\varepsilon]. \end{cases}$$

(d) $\bar{\Psi}(q, q_{n-1}, q_n) = (q, q_{n-1} + 2\varepsilon, 0)$ *in the domain* $\Sigma \times \left(\frac{1}{2}\varepsilon, \varepsilon\right) \times [-\varepsilon - 1, 1 + \varepsilon]$.

(e) *In the domain* $\Sigma \times \left(-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon\right) \times [-\varepsilon - 1, 1 + \varepsilon]$, *we have that*

$$\bar{\Psi}(q, q_{n-1}, q_n) = (q, x(q_{n-1}, q_n), y(q_{n-1}, q_n)), \tag{1}$$

where the change of coordinates

$$x \colon \left(-\tfrac{1}{2}\varepsilon, \tfrac{1}{2}\varepsilon\right) \times [-\varepsilon - 1, 1 + \varepsilon] \to \left[-\tfrac{1}{2}\varepsilon, \tfrac{5}{2}\varepsilon\right],$$
$$y \colon \left(-\tfrac{1}{2}\varepsilon, \tfrac{1}{2}\varepsilon\right) \times [-\varepsilon - 1, 1 + \varepsilon] \to \left[-\tfrac{1}{2}\varepsilon, \varepsilon\right]$$

satisfies the inequalities $\partial_{q_{n-1}} x, \partial_{q_n} x > 0$ *on the intersection*

$$\left(\Sigma \times \left(-\tfrac{1}{2}\varepsilon, \tfrac{1}{2}\varepsilon\right) \times [-\varepsilon, 1 + \varepsilon]\right) \cap \mathbb{D}$$

if the x*–coordinate vanishes.*

(f) $e(W \times \{0\}) \subset \bar{e}(\overline{W} \times \{0\})$ *and* $\bar{e}(\overline{W} \times \{0\}) \cap \Sigma \times D^2(3\varepsilon) = \Sigma \times (-3\varepsilon, 3\varepsilon) \times \{0\}$.

Proof The items (a)–(d) and (g) are proven similarly as Proposition 3. The main difference is the addition of the inequalities $\partial_{q_{n-1}} x, \partial_{q_n} x > 0$ in item (e). For that, just note that the condition $\partial q_n / \partial x > 0$ is equivalent to $\partial x / \partial q_n > 0$. Since the initial morphism e satisfies $\partial x / \partial q_{n-1} \geq 0$ and $\partial x / \partial q_n \geq 0$ globally, we can ensure that the extension $\bar{e}$ satisfies this property (e). □

In order to ease notation, we shall refer to this extended Legendrian Whitney bridge still as (e, F, W), understanding that the goal is to place the original intersection Σ in the interior of the bridge. However, the coordinates (x, y) will still refer to the previous embedding, and this is why they satisfy property (e).

2.3.3 The contact isotopy Endowed with the Legendrian Whitney bridge provided by Proposition 3, and duly modified in Lemma 7, we can consider the smooth isotopy given by the flow along the coordinate $q_n \in [-1 - \varepsilon, 1 + \varepsilon]$ defined on $\overline{\mathbb{W}}$, generated by the vector field $\partial_{q_n} \in \Gamma(T\mathbb{W})$, which is nonvanishing away from Σ, and adjusted to remove the intersection at Σ. This is the smooth generalization of the classical Whitney trick, as depicted in Figure 3. This smooth flow, defined along $\mathbb{W}$, lifts to a contact isotopy in the 1–jet bundle $J^1(\mathbb{W}, \xi_{\mathrm{st}})$ [15]. Nevertheless, this is *not* a compactly supported contact isotopy.

The canonical lift of the vector field $\bar{e}_*(\partial_{q_n})$ has contact flow

$$(j^1\phi)^t(w, q_n, p_w, p_n, \tau) = (w, q_n + t, p_w, p_n, \tau),$$

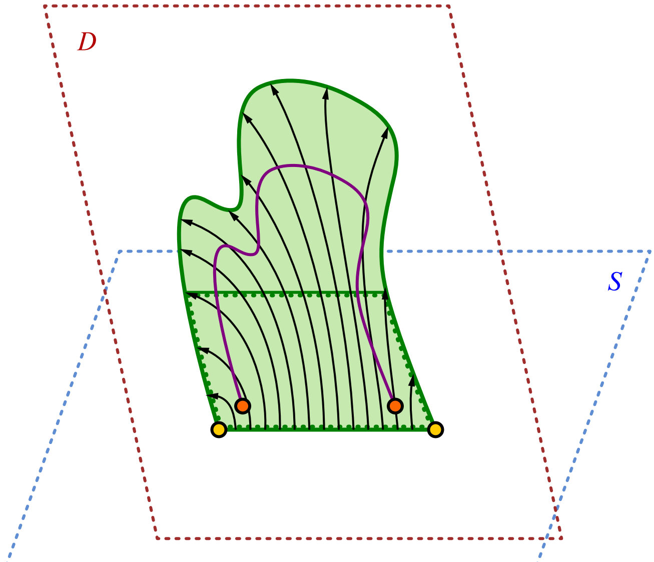

Figure 3: The Legendrian neighborhood $\Sigma \times \mathbb{D}^2$ of the modified intersection $\Sigma = S \cap \mathbb{D}$ (yellow) and the image of the extended Legendrian Whitney bridge (green), where $W \times [0,1]$, the original bridge, is mapped into the third quadrant. The purple curve represents the original boundary of the Legendrian Whitney bridge (e, F, W) and the flowlines of the extended vector field ∂_{q_n} (black) placing the original Σ (orange) in the interior of the extended bridge $\overline{\mathbb{W}}$.

where $w \in W$ belongs to the Legendrian Whitney bridge. Note that this contact flow displaces the Legendrian S from the contact submanifold $\mathbb{D}$ at $t \geq 1 + \varepsilon$. Indeed, the contact flow $(j^1\phi)^t$ preserves the p_n–coordinate and we have the following:

Lemma 8 *Let $z \in \Sigma \times D^2(3\varepsilon)$ be such that the coordinate $p_n(z) \neq 0$ is nonvanishing, $p_{n-1}(z) = 0$ and $\bar{x}(z) = 0$. Then $p_x(z) \neq 0$ and thus $z \notin \mathbb{D}$.*

Proof Let us verify that $p_n \neq 0$ implies $p_x \neq 0$. Indeed, being a change of coordinates, we have

$$p_n\, dq_n + p_{n-1}\, dq_{n-1} = p_x\, dx + p_y\, dy,$$

and, by the hypothesis $p_{n-1} = 0$, we have that

$$p_n\, dq_n + 0\, dq_{n-1} = p_n\, dq_n = p_n \frac{\partial q_n}{\partial \bar{x}}\, dx + p_n \frac{\partial q_n}{\partial \bar{y}}\, dy.$$

Thus we have $p_x = p_n \partial q_n / \partial x$ and, by Lemma 7, we have $\partial q_n / \partial x > 0$. □

From now on, we work with the deformed coordinates on $\overline{\mathbb{W}}$:

- $q_{n-1} \in (-\varepsilon, \varepsilon)$ that is defined on the Legendrian ribbon.
- $q_n \in [-\varepsilon - 1, 1 + \varepsilon]$ that coincides with q_n away from the ribbon and over that domain it is defined by (1).

Lemma 8 shows that

$$p_n((j^1\phi)^t(w, 0, 0, p_n, 0)) \neq 0 \implies p_x((j^1\phi)^t(w, 0, 0, p_n, 0)) \neq 0.$$

This implies that any point $z \equiv (w, 0, 0, p_n, 0) \in S \setminus W_0$ satisfies that $(j^1\phi)^t z$ does not belong to $\mathbb{D}$ for any $t > 0$. This shows that the only intersection of the displaced Legendrian spheres is happening along the image of W_0 through the sliding flow. These intersections, in turn, disappear at time $t = 1 + \varepsilon$. In addition, we will check that this property remains true after applying the cutoffs which achieve the compact support for the contact isotopy. Since the displacement occurs in the q_n directions and the Legendrian Whitney bridge essentially moves along the (q_{n-1}, q_n) directions, the core work for achieving a compactly supported contact isotopy must be in cutting off along the conjugate coordinates (p_{n-1}, p_n) and the Reeb direction τ. Let us start constructing this cutoff.

Let us start in a neighborhood of $W_0 = S \cap \mathbb{W}$, which is one of the two ends of the Whitney bridge. Fix $c \in \mathbb{R}$ with $0 < c < \frac{1}{2}\varepsilon$, and construct a compactly supported cutoff function $\widetilde{G}\colon (-3\varepsilon, 3\varepsilon) \to [0, 1]$ such that:

- $\widetilde{G}(0) = 1$, $\widetilde{G}'(0) = 0$ and $\widetilde{G}'$ is odd.
- For all $p_n \in (-3\varepsilon, 3\varepsilon)$, the associated integral
$$\widetilde{g}(p_n) = \int_0^{p_n} \widetilde{G}'(s) \cdot s \cdot ds$$
satisfies $|\widetilde{g}|_{C_0} \leq 3c$.

In order to construct such $\widetilde{G}$, we can proceed as follows. Consider the function (see Figure 4)

$$\widetilde{G} = \begin{cases} 1 & \text{if } p_n \in [0, c], \\ 1 - 2(p_n - c)/c & \text{if } p_n \in [c, 2c], \\ -1 + (p_n - 2c)/2c & \text{if } p_n \in [2c, 4c], \\ 0 & \text{if } p_n \geq 4c, \end{cases}$$

which we extend to negative values by imposing even symmetry. Then $\widetilde{G}$ satisfies

$$\int_0^{4c} \widetilde{G}'(p_n) \cdot p_n \, dp_n = 0. \tag{2}$$

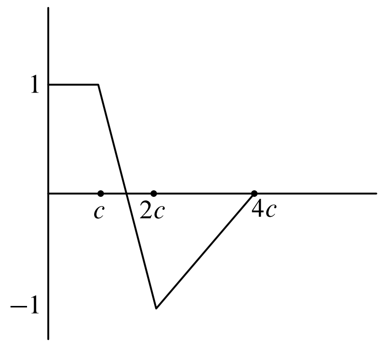

Figure 4: The linearized version $\widetilde{G}$ of the function G

Since $\widetilde{G}'$ is an odd function, $(\widetilde{G}'(p_n)) \cdot p_n$ is even and thus $\widetilde{g}\colon (-3\varepsilon, 3\varepsilon) \to \mathbb{R}$, defined as

$$\widetilde{g}(p_n) = \int_0^{p_n} \widetilde{G}'(s) \cdot s \cdot ds,$$

is an odd function which is also compactly supported in $[-4c, 4c]$ after (2). Since $|\widetilde{g}|_{C_0} \leq 3c$, the above $\widetilde{G}$ satisfies the second property. Finally, smooth out the function $\widetilde{G}$ to a new C^∞ function G, which is C^0–close to it, still has support in the closed interval $[-4c, 4c]$ and satisfies the required properties. In particular, the associated function

$$g(p_n) = \int_{-3\varepsilon}^{p_n} G'(s) \cdot s \cdot ds$$

has compact support in the interval $[-4c, 4c]$, and satisfies $g'(p_n) = G'(p_n) \cdot p_n$ and $|g|_{C^0} \leq 3c$. Note that G can be chosen so that the norm $|g|_{C^0}$ is arbitrary small, by choosing c as small as required.

Now, the flow $\Psi_t(w, q_n, p_w, p_n, \tau) = (w, q_n + tG(p_n), p_w, p_n, \tau - tg(p_n))$ is a strict contact flow defined for $t < 1 + \varepsilon$ on a neighborhood of the intersection $S \cap \mathbb{W}$. It is here assumed that the function g is C^0–small to assure that the map $\tau \to \tau - tg(p_n)$ is well defined, since the τ–coordinate is defined for a small open neighborhood of $\tau = 0$. In the same vein, the inequality $|G|_{C^0} \leq 1$ implies that the function $q_n + tG(p_n)$ lies in the interval $(-\varepsilon - 1, 1 + \varepsilon)$ for $q_n = 0$. So, it is well defined.

In addition, this cutoff flow is the identity for $p_n > 3\varepsilon$ and therefore it glues $\Psi_t(S)$ and S in the transition region $\mathcal{O}p(\{p_n = \pm 3\varepsilon\})$. Thus, this discussion above resolves the contact cutoff regarding the direction associated to the 1–jet of the coordinate q_n. Let us now study the remaining region $q_{n-1} > \varepsilon$, responsible for the direction transversal to the flow in the Whitney trick.

The contact vector field associated to the above constructed flow is given by $X_H = G(p_n)\partial/\partial q_n - g(p_n)\partial/\partial\tau$, whose associated Hamiltonian is $H = G(p_n)\cdot p_n - g(p_n)$. Choose a real number $\delta \in (-\varepsilon, \varepsilon)$ such that for any point $z \in \mathcal{O}p(\mathbb{W})$ with coordinate $q_{n-1}(z) \geq \delta$, we have $x(z) > 0$. Fix a smooth step function $R\colon (-\infty, \varepsilon) \to [0, 1]$ such that

- $R(q_{n-1}) = 1$ for $q_{n-1} \leq \frac{1}{2}(\varepsilon + \delta)$,
- $R(q_{n-1}) = 0$ for $q_{n-1} \geq \frac{1}{4}(3\varepsilon + \delta)$,
- $0 \geq \partial_{q_{n-1}} R \geq -5/(\varepsilon - \delta)$.

and define $\widehat{R} = R \cdot H$, where H is the Hamiltonian above. Then the contact vector field associated to the Hamiltonian $\widehat{R}$ is given by $X_{\widehat{H}} = R\cdot X_H - (\partial_{q_{n-1}} R)\cdot H \cdot \partial_{p_{n-1}}$. Thus the associated contact flow $\widehat{\Psi}_t$ coincides with Ψ_t away from $\mathcal{O}p(\Sigma\times\mathbb{D}^2(3\varepsilon))$. In the neighborhood $\mathcal{O}p(\Sigma\times\mathbb{D}^2(3\varepsilon))$, we obtain

$$\widehat{\Psi}_t(q, q_{n-1}, q_n, p, p_{n-1}, p_n, \tau)$$
$$= \big(q, q_{n-1}, q_n + tG(p_n)R(q_{n-1}), p, p_{n-1} + t\cdot\partial_{q_{n-1}} R\cdot H, p_n, \tau - tg(p_n)R(q_{n-1})\big).$$

Since the upper bound $\|\partial_{q_{n-1}} R\cdot H\| \leq 5/(\varepsilon-\delta)\cdot(c+2c) = 15c/(\varepsilon-\delta)$ can be made arbitrary small by further reducing $c > 0$, the flow of $\widehat{\Psi}_t$ is well defined for time $t < 1+\varepsilon$. As a consequence, we obtain a 1–parametric family of Legendrian submanifolds obtained by flowing the Legendrian S along the contact isotopy $\widehat{\Psi}_t$ for $t \in [0, 1+\varepsilon)$. In addition, the Legendrian $S_{1+\varepsilon}$ is disjoint from the contact submanifold $\mathbb{D}$. Indeed, in the region $S_{1+\varepsilon} \cap (B \setminus \mathcal{O}p(\mathbb{W}))$, the family of Legendrians S_t remains constant and thus the empty intersection with $\mathbb{D}$ persists. In the region $S \cap \mathcal{O}p(W)$ near the Legendrian Whitney bridge, the coordinate p_n is preserved along the contact flow. It thus suffices to show that the Legendrian S_t can only intersect $\mathbb{D}$ in points of the image of the domain $W_0 = S \cap \{p_n = 0\}$ through the flow. This can be discussed in the following two regions:

(a) The complement of the region $\Sigma\times\mathbb{D}^2(3\varepsilon)$. Then $p_x = p_n$ for $x = 0$ and thus, since p_n is preserved, we obtain that the only possible intersections between $\mathbb{D}$ and $\widehat{\Psi}_t(S)$ happen in the image of W_0. The same argument works for the region in $\Sigma\times\mathbb{D}^2(3\varepsilon)$ in the complement of the support of R.

(b) In the remaining region, where the coordinate p_{n-1} is not preserved, we cannot use Lemma 8. Nevertheless, the inequality $x > 0$ is satisfied and preserved by the contact flow in positive time. Thus, since $\mathbb{D}$ is defined by $\{x = p_x = 0\}$, the image of S can only intersect $\mathbb{D}$ along the image of W_0 through the flow.

Consequently, since the coordinate p_n is preserved by the flow, the contact flow coincides with the displacement flow constructed along $\mathbb{W}$, and thus at time $1+\varepsilon$ the intersection between $S_{1+\varepsilon}$ and $\mathbb{D}$ must be empty. □

3 Application to isocontact embeddings

In this section we establish Theorem 2 by using Theorem 1. In line with standard h–principle arguments [4; 5; 10; 17], the isocontact h–principle, Theorem 2, follows from the resolution of the local problem, which is the content of the following:

Theorem 9 *Let (B^{2n+1},ξ) be a contact $(2n+1)$–dimensional disk, $n \geq 2$ and $(\phi, F_s)\colon (\mathbb{D}^{2n-1},\xi_{\mathrm{st}}) \to (B^{2n+1},\xi)$ a proper formal isocontact embedding such that:*

(1) *The embedding is isocontact close to the boundary, ie*

$$(F_s)|_{\mathcal{O}p(\mathbb{D}^{2n+1})} = d\phi|_{\mathcal{O}p(\mathbb{D}^{2n+1})}.$$

(2) *The embedding is smoothly isotopic to the standard embedding.*

Then there exists a family of formally isocontact embeddings $(\phi^t, F_s^t)_{t\in[0,1]}$, relative to the boundary, such that the deformed embedding is isocontact, ie $F_s^1 = d\phi^1$. □

Theorem 9 constructs a solution to the core local model in our existence h–principle. In comparison to the recent work [23], it is crucial to emphasize that the contact embedding ϕ^1 induces the standard contact structure $(\mathbb{D}^{2n-1},\xi_{\mathrm{st}})$, which is the difference between the study of contact embeddings and the existence h–principle for isocontact embeddings.

Theorem 2 follows from Theorem 9, thanks to the h–principle for isocontact embeddings between open contact manifolds [10; 17] applied to the formal isocontact embedding

$$(f_0, F_s)|_{N\setminus \bar{B}_N}\colon N\setminus \bar{B}_N \to M,$$

where B_N is a Darboux ball in (N,ξ_N). Indeed, Theorem 12.3.1 of [10] provides a formal isocontact deformation of $(f_0, F_s)|_{N\setminus \bar{B}_N}$ to a genuinely isocontact embedding, and thus extending the domain of this resulting isocontact embedding from $N\setminus \bar{B}_N$ to N is achieved by applying Theorem 9.

3.1 Intuition for Theorem 9

Let us first consider the 3–dimensional case of a smooth knot $K \subseteq (S^3,\xi_{\mathrm{st}})$ or, in the local context of Theorem 9, a properly embedded smooth arc $K_0 \subseteq (B^3,\xi_{\mathrm{st}})$ within a

Darboux ball (B^3, ξ_{st}) which is positively transverse at the endpoints ∂K_0. The aim is to perform a smooth isotopy K_t for $t \in [0,1]$ relative to the boundary $\mathcal{O}p(B^3)$ such that K_1 is a transverse arc. In the 3–dimensional case this can be done explicitly, as explained in [15, Theorem 3.3.1]. In short, the case where the transverse knot is positively transverse everywhere except at Legendrian subsets can be solved directly by averaging the positivity at the ends. Thus, the only significant reason for K_0 not to be a transverse knot is the existence of interval subsets $I \subseteq K_0$ where the arc is negatively transverse. The solution is to insert a 1–dimensional wrinkle [9], ie the transverse pushoff of a Legendrian stabilization. In conclusion, in the 3–dimensional case the problem translates into a solvable question on plane curves and their slopes, an approach which does not exist in higher dimensions.

The crucial object in the 3–dimensional case is the notion of a transverse stabilization, which is neatly explained in [11, Section 2.8]. As depicted in [11, Figure 21], the transverse stabilization can be understood as a *transverse* bypass. This is where intuition for the higher-dimensional case starts to arise. A Legendrian bypass [22] can be understood as half an overtwisted 2–disk and, even more precisely, the connected sum of a Legendrian arc with the boundary of an overtwisted disk is tantamount to a Legendrian stabilization. Thus, the overtwisted disk can be understood as an object which provides the necessary stabilizations required for the smooth arc to become a transverse (or Legendrian) arc. In fact — and this is crucial — only half of the overtwisted 2–disk is needed.

The coming proof of Theorem 9 is based on the following 3–dimensional geometric heuristic. Instead of searching for half an overtwisted disk in the Darboux ball (B^3, ξ_{st}), which allows for transverse stabilizations, insert an entire overtwisted 2–disk $\mathbb{D}$ away from $\mathcal{O}p(K_0)$. This modifies the standard structure (B^3, ξ_{st}) to an overtwisted structure (B^3, ξ_{ot}), in which the h–principle for isocontact embeddings holds [1; 8]. Thus, the smooth inclusion $K_0 \subseteq (B^3, \xi_{\mathrm{ot}})$ can be made positively transverse, relative to the endpoints. By Gray's isotopy theorem, this can be considered as a smooth isotopy $\{K_t\}_{t\in[0,1]}$ within the fixed contact structure ξ_{ot} such that K_1 is positively transverse. The h–principle does not a priori guarantee that the isotopy $\{K_t\}_{t\in[0,1]}$ only uses half the overtwisted 2–disk $\mathbb{D}$.[4] The argument then relies on proving that any such (formally transverse) isotopy $\{K_t\}_{t\in[0,1]}$ can be made disjoint from half of the overtwisted 2–disk

[4] Although, in this 3–dimensional case, we know we can just use half an overtwisted 2–disk, but this statement is not available in higher dimensions.

$\mathbb{D}$ and that the isotopy $\{K_t\}_{t\in[0,1]}$ can therefore be pushed to exist within (B^3,ξ_{st}), where only half of the overtwisted 2–disk is available.

In the higher-dimensional case, the generalization of the arc $K_0 \subseteq (B^3,\xi_{\mathrm{st}})$ is a standardly embedded $(2n-1)$–dimensional disk $(\mathbb{D},\xi_{\mathrm{st}}) \subseteq (B^{2n+1},\xi_{\mathrm{st}})$. The fact that Theorem 9 is able to conclude an embedding of the standard contact structure $(\mathbb{D},\xi_{\mathrm{st}})$ is a feature of both Theorem 1 and the h–principle [1]. In the 3–dimensional case, there is a unique contact structure induced in a transverse arc, whereas a codimension-2 contact embedding of a smooth disk $\mathbb{D}$ does not necessarily induce the standard contact structure $(\mathbb{D},\xi_{\mathrm{st}})$. In addition, the formal contact type of a smooth embedding in higher dimensions differs from the data of the self-linking number of a transverse knot, which entirely determines the formal transverse type of a smooth knot in (S^3,ξ_{st}). Finally, the proof of Theorem 9 only works in higher dimensions, since it relies on the smooth Whitney trick [25; 36], its Legendrian avatar Theorem 1 and the fact that the space of embedded n–sphere into $\mathbb{R}^{2n+1}$ is connected, which only holds for $n \geq 2$. The following subsection comprises the details of our argument for Theorem 9 in higher dimensions.

3.2 Proof of Theorem 9

First, we shall modify the contact structure (B,ξ) by inserting an overtwisted disk, away from the image $\phi(\mathbb{D})$ of the contact divisor. It is possible to achieve this by choosing a standard Legendrian unknot $\Lambda_0 \subseteq (B,\xi)$, and performing[5] two $(+1)$–surgeries along Λ_0. Let us do so in the framework of adapted contact open books [3; 7]. For that, choose a $(2n+1)$–dimensional Darboux ball $(\mathbb{B},\xi_{\mathrm{st}}) \subseteq (B\setminus\mathbb{D},\xi)$ and $p \in (\mathbb{B},\xi_{\mathrm{st}})$ an interior point. This Darboux ball $(\mathbb{B},\xi_{\mathrm{st}})$ is contactomorphic, relative to its boundary, to the contact connected sum $(\mathbb{B},\xi_{\mathrm{st}})\,\#_p\,(S^{2n+1},\xi_{\mathrm{st}})$ along (a neighborhood of) the point $p \in (\mathbb{B},\xi_{\mathrm{st}})$. The contact structure $(S^{2n+1},\xi_{\mathrm{st}})$ is supported by the adapted open book $(S^{2n+1},\xi_{\mathrm{st}}) = \mathrm{ob}(T^*S^n,\lambda_{\mathrm{st}};\tau_{S^n})$, and the contact connected sum can be performed so that p belongs to the zero section of the Weinstein page $P_0 \cong (T^*S^n,\lambda_{\mathrm{st}})$ at angle $0 \in S^1$. In order to insert an overtwisted disk in this context, change the monodromy $\tau_{S^n} \in \mathrm{Symp}^c(T^*S^n,\lambda_{\mathrm{st}})$ to its inverse $\tau_{S^n}^{-1}$, to obtain an overtwisted contact structure $(S^{2n+1},\xi_{\mathrm{ot}}) = \mathrm{ob}(T^*S^n,\lambda_{\mathrm{st}};\tau_{S^n}^{-1})$, as we proved in [3]. The connected sum $(\mathbb{B},\xi_{\mathrm{st}})\,\#_p\,(S^{2n+1},\xi_{\mathrm{ot}})$ is thus overtwisted and differs from

[5] Alternatively, choose a stabilized Legendrian unknot $\Lambda \subseteq (B\setminus\mathbb{D},\xi)$, formally isotopic to Λ_0, and perform one $(+1)$–surgery.

$(\mathbb{B}, \xi_{\mathrm{st}}) \#_p (S^{2n+1}, \xi_{\mathrm{st}})$ precisely in the gluing prescribed by the monodromy. The $2n$–dimensional overtwisted disk in the adapted open book decomposition $(S^{2n+1}, \xi_{\mathrm{ot}}) = \mathrm{ob}(T^*S^n, \lambda_{\mathrm{st}}; \tau_{S^n}^{-1})$ can be visualized by using 3–dimensional techniques [24; 34] and a thick neighborhood [3, Section 3].

Remark 10 In the case that the symplectic monodromy is an exact symplectomorphism, the action of the monodromy on the contact structure can be concentrated in an arbitrarily small neighborhood of the page P_π at angle $\pi \in S^1$. The symplectic Dehn twist τ_{S^n} is not an exact symplectomorphism, but the difference $\lambda_{\mathrm{st}} - \tau_{S^n}^* \lambda_{\mathrm{st}}$ can be written explicitly [26, Section 1.2] and these two contact connected sums $(\mathbb{B}, \xi_{\mathrm{st}}) \#_p (S^{2n+1}, \xi_{\mathrm{st}})$ and $(\mathbb{B}, \xi_{\mathrm{st}}) \#_p (S^{2n+1}, \xi_{\mathrm{ot}})$ can be assumed to be contactomorphic away from a large enough neighborhood of P_π, diffeomorphic to $\mathcal{O}p(P_\pi)$.

The formal isocontact embedding $(\phi, F_s)\colon (\mathbb{D}, \xi_{\mathrm{st}}) \to (B, \xi)$ can be extended to have the modified overtwisted contact manifold $(B, \xi) \# (S^{2n+1}, \xi_{\mathrm{ot}})$ as a target, since its image lies away from p. Let us apply the h–principle [1] for codimension-2 isocontact embeddings in overtwisted contact manifolds in order to obtain a family $\{(\phi^t, F_s^t)\}_{t \in [0,1]}$ of formally isocontact embeddings which starts at (ϕ, F_s) and finishes at a genuinely isocontact embedding (ϕ^1, F_s^1). Even though the image $(\phi^0, F_s^0) = (\phi, F_s)$ lies away from any overtwisted disk, the image of the embeddings (ϕ^t, F_s^t) might, for some $t \in [0, 1]$, be contained in the $(\mathbb{B}, \xi_{\mathrm{st}}) \#_p (S^{2n+1}, \xi_{\mathrm{ot}})$ region near p. The crucial fact is that, in order to avoid the region $(\mathbb{B}, \xi_{\mathrm{st}}) \#_p (S^{2n+1}, \xi_{\mathrm{ot}})$, it suffices to avoid the Legendrian sphere $S_\pi \subseteq P_\pi \cong (T^*S^n, \lambda_{\mathrm{st}})$ given by the Legendrian lift of the (exact) Lagrangian zero section. This is explained in the following:

Lemma 11 *Let $\varepsilon \in \mathbb{R}^+$, $\theta \in (\pi - \varepsilon, \pi + \varepsilon)$ and*

$$K \subseteq N = (\mathbb{D}_\varepsilon(T^*S^n) \times (\pi - \varepsilon, \pi + \varepsilon), \ker\{\lambda_{\mathrm{st}} + d\theta\})$$

be a compact subset of N such that $K \cap (S^n \times \{\pi\}) = \varnothing$, where $S^n \subseteq T^*S^n$ is the inclusion given by the zero section. Then there exists a compactly supported contact flow $\phi_t \in \mathrm{Cont}(N)$ such that $\phi_1(K) \subseteq \mathcal{O}p(\partial N)$.

Proof The complement of the zero section $S^n \subseteq (\mathbb{D}_\varepsilon(T^*S^n), \lambda_{\mathrm{st}})$ is Liouville symplectomorphic to the symplectization $(\mathbb{S}(T^*S^n) \times \mathbb{R}, e^t \alpha_{\mathrm{st}})$ with $t \in \mathbb{R}$ of the standard unit cotangent bundle $(\mathbb{S}(T^*S^n), \ker \alpha_{\mathrm{st}})$. Thus, the complement of the zero section is modeled on the contactization of the symplectization of $(\mathbb{S}(T^*S^n), \ker \alpha_{\mathrm{st}})$, with contact form $e^t \alpha_{\mathrm{st}} + d\theta$. The vector field $X = \partial_t + \partial_\theta$ is contact and a pseudogradient for the function $t+\theta$. Choose a compactly supported cutoff function $\widehat{H}$ that coincides with its contact Hamiltonian H on a compact set of N whose complement is a small open neighborhood of ∂N. The flow associated to the contact vector field generated by the contact Hamiltonian $\widehat{H}$ satisfies the requirements, as it pushes the complement of the zero section to $\mathcal{O}p(\partial N)$. □

Lemma 11 shows that an embedding whose image lies on the complement of the Legendrian S_π page inside $(\mathbb{B},\xi_{\mathrm{st}})\,\#_p\,(S^{2n+1},\xi_{\mathrm{ot}})$ can be pushed off, with a contact isotopy, away from the region where the contact domains $(\mathbb{B},\xi_{\mathrm{st}})\,\#_p\,(S^{2n+1},\xi_{\mathrm{ot}})$ and $(\mathbb{B},\xi_{\mathrm{st}})\,\#_p\,(S^{2n+1},\xi_{\mathrm{st}})$ differ. Since S_π is a standardly embedded Legendrian sphere in a ball (B,ξ), the Legendrian Whitney trick, stated in Theorem 1 above, can now be used to remove the intersection between the contact disks given by the image of the isocontact embedding (ϕ^1, F_s^1) and the Legendrian sphere S_π. At this stage, the proof could proceed by establishing a parametric version of Theorem 1 which would allow for the removal of intersections between the Legendrian sphere S_π and (ϕ^t, F_s^t) for all $t\in[0,1]$. Instead, given that the maps (ϕ^t, F_s^t) are just formal isocontact embeddings, it suffices to treat this as a differential topology problem, rather than a contact topological problem, as in the following:

Lemma 12 *Let $\phi_t\colon \mathbb{D}^{2n-1}\to B^{2n+1}$ for $t\in[0,1]$ be a family of smoothly standard proper embeddings of the disk such that $n\ge 2$, ϕ_0 and ϕ_1 do not intersect $S_\pi\subseteq B$, a standard n–sphere in the ball, and $\phi_t(\mathcal{O}p(\mathbb{D}^{2n-1}))$ is independent of $t\in[0,1]$. Then there exists a deformation through paths of embeddings $\phi_{t,s}$ to a new 1–parametric family of embeddings $\phi_{t,1}\colon\mathbb{D}\to B$ such that:*

- *$\phi_{t,0}=\phi_t$ and $\phi_{t,s}=\phi_0$ in $\mathcal{O}p(\partial(\mathbb{D}^{2n-1}))$ for all $(t,s)\in[0,1]\times[0,1]$.*
- *For all $s\in[0,1]$, $\phi_{0,s}=\phi_0$ and $\phi_{1,s}=\phi_1$.*
- *The images of $\phi_{t,1}$ do not intersect S_π for all $t\in[0,1]$.*

Proof The diffeomorphism $B^{2n+1}\cong\mathbb{D}^{2n-1}\times\mathbb{D}^2$ implies that $B\setminus\phi_1(\mathbb{D})$ is diffeomorphic to $S^1\times B^{2n}$. Proposition 13, proven below, shows that the space of smoothly embedded n–spheres in $B^{2n+1}\cong\mathbb{D}^{2n-1}$ is connected. By deforming S_π to a sufficiently small sphere with (the endpoint of) an isotopy Ψ_1, the composition $\Psi_1^{-1}\circ\phi_t$ is our required deformation. □

In conclusion, the initial formal isocontact embedding is made genuinely isocontact by inserting an overtwisted disk and using the h–principle [1]. Theorem 1 is used to ensure that the resulting isocontact embedding avoids the Legendrian sphere S_π, ie that it avoids part of the overtwisted disk. Lemma 11 is then used to flow the image of the isocontact embedding away from the region where the overtwisted disk is inserted, and Lemma 12 ensures that the formal isocontact isotopy is also disjoint from S_π. This yields the required isocontact embedding for Theorem 9. □

3.3 Connectedness of the space of n–dimensional embedded spheres in $\mathbb{S}^1 \times \mathbb{D}^{2n}$

Let us conclude the article with Proposition 13, on the connectedness of the space of embeddings into $\mathbb{S}^1 \times B^{2n}$. This is the higher-dimensional analogue of the 3–dimensional arguments in [21] on the topology of the moduli space of unknots. The argument strongly relies on the connectedness of the space of embeddings into B^{2n+1}, as proven by Haefliger [18, Theorem 1].

Proposition 13 *The space of standard embeddings of the n–sphere into $\mathbb{S}^1 \times B^{2n}$ is connected for $n \geq 2$.*

Proof Let $X = \mathbb{S}^1 \times B^{2n}$, and $\pi\colon X \to B^{2n}$ be the projection into the second factor. Let $e\colon S \to X$ be a standard embedding, with components $e = (\theta, m)$, where θ denotes the angle $\theta \in \mathbb{S}^1$, and assume, after genericity and possibly adding a finite number of kinks, that the smooth map $\pi \circ e\colon S \to B^{2n}$ is a generic immersion and the self-intersection points are all paired, ie the signed self-intersection number is zero. Here, a kink is a smooth isotopy of the embedding $e\colon S \to X$ which adds exactly a double point on its image $\pi(e(S))$ of the required sign.

Let us now choose a pair of canceling points $q_0, q_1 \in B^{2n}$ for $\pi(e(S))$ and a Whitney disk $d\colon \mathbb{D}^2 \to B^{2n}$ for the smooth map $\pi \circ e\colon S \to B^{2n}$. The sliding, through a Whitney-move isotopy [25], along this Whitney disk yields a homotopy through immersions L_t of the immersion $L_0 = \pi(e(S))$ into an embedded sphere L_1. This homotopy lies entirely in B^{2n}, and we need to choose a lift for the family of maps $m_t\colon S \to B$, defined by $L_t = m_t(S)$, to a family of embeddings $e_t\colon S \to X$. For instance, choose a lift $\tilde{d}\colon \mathbb{D}^2 \to \mathbb{S}^1 \times B^{2n}$ for the Whitney disk $d\colon \mathbb{D}^2 \to B^{2n}$ such that the interior of $\tilde{d}(\mathbb{D}^2)$ does not intersect S, the lower hemisphere of $\partial(\tilde{d}(\mathbb{D}^2))$ still intersects $e(S)$, and the upper hemisphere of $\partial(\tilde{d}(\mathbb{D}^2))$ does not intersect $e(S)$.

The Whitney disk $\tilde{d}\colon \mathbb{D}^2 \to \mathbb{S}^1 \times B^{2n}$ can now be used to perform a Whitney-move isotopy along it such that both images $e(S)$ and $\pi(e(S))$ through this smooth isotopy remain embedded. Finally, given that the degree of the map $S \to \mathbb{S}^1$ must be zero, we can isotope the smooth sphere S to live within the region $\mathcal{O}p(p) \times B^{2n}$, where $p \in \mathbb{S}^1$ is a point. This yields an embedding of the smooth n–sphere S inside B^{2n+1}. Then Theorem 1 of [18] shows that, for $n \geq 2$, the space of standard embeddings of an n–sphere into B^{2n+1} is connected, concluding the required result. □

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