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ORIGINAL RESEARCH ARTICLE

# Transport equations with inflow boundary conditions

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**Abstract**

We provide bounds in a Sobolev-space framework for transport equations with nontrivial inflow and outflow. We give, for the first time, bounds on the gradient of the solution with the type of inflow boundary conditions that occur in Poiseuille flow. Following ground-breaking work of the late Charles Amick, we name a generalization of this type of flow domain in his honor. We prove gradient bounds in Lebesgue spaces for general Amick domains which are crucial for proving well posedness of the grade-two fluid model. We include a complete review of transport equations with inflow boundary conditions, providing novel proofs in most cases. To illustrate the theory, we review and extend an example of Bernard that clarifies the singularities of solutions of transport equations with nonzero inflow boundary conditions.

**Mathematics Subject Classification** 35M35 · 65M60

## 1 Introduction

Transport equations [11] are central to many models, including compressible Navier–Stokes equations [16], non-Newtonian fluids [9], conservation laws [8], and in many other areas. Many models of non-Newtonian fluids [10, 12, 13, 18–20, 26] may be viewed as a coupling of the Navier–Stokes equations with a transport equation. Smooth solutions of transport equations have been of interest recently [2–6, 21]. It is not difficult to show that $H^1$ solutions exist if there is no inflow boundary [12], but inflow boundary conditions yield more complex behavior.

Despite the extensive research to date on transport with nonzero inflow boundary conditions, the critical case of Poiseuille flow has not been addressed. The simplest version of this case is Poiseuille flow in a channel or pipe, which we both address and extend to a class of more general geometries. The primary result is Theorem 5.1, which establishes the existence and unicity of the solution with nontrivial inflow conditions in what we will refer to as Amick [1] domains. These domains, as depicted in Fig. 1, and which we define precisely

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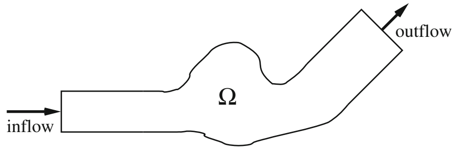



**Fig. 1** Domain for Amick's theorem

in Sect. 5, feature inflow and outflow tubes where the flow is asymptotically Poiseuille-like. Such domains are essential for simulating experiments involving non-Newtonian fluid flow. In particular, many rheometers [15, 17, 23] involve Amick domains, and thus our results allow, for the first time, analysis of algorithms for simulating such devices. The gradient bounds derived here are critical for analyzing the grade-two fluid model [22].

To place the results of Sect. 5 in context, we review and extend some previous results, giving simpler derivations in most cases. In particular, Theorem 3.1 is the fundamental result in Lebesgue spaces for transport problems, and we extend the range of applicable Lebesgue exponents and present a new proof by modifying the techniques introduced in [14]. The proof of the results of Sect. 5 are actually quite simple by comparison, so our approach offers an alternative entry point to understanding these estimates.

Surprisingly, quite singular solutions arise even in standard flow problems, such as Poiseuille flow. This is one of the simplest flow problems, whose solution is given by a quadratic polynomial for Newtonian fluids. We will show that for Poiseuille flow, the gradient of the transport solution fails to be square integrable, but we give bounds in $W_q^1$ for $q < 2$. (We provide definitions for standard Lebesgue and Sobolev spaces used here in Appendix A.) This shows that the results of Theorem 5.1 are sharp. An example of Bernard [5] further shows that the gradient of the solution can fail to be integrable for certain types of inflow conditions. We rederive this example and extend it both analytically and computationally.

## 2 Problem definition

We consider transport in a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$, with advection velocity $\mathbf{u} \in H^1(\Omega)$. See Appendix A for our notation for Lebesgue and Sobolev spaces. Define the inflow boundary $\Gamma_-$ and the outflow boundary $\Gamma_+$ by

$$\Gamma_\pm = \left\{ \mathbf{x} \in \partial\Omega \;\middle|\; \pm\mathbf{u}(\mathbf{x}) \cdot \mathbf{n} > 0 \right\}, \tag{1}$$

where $\mathbf{n}$ denotes the outward normal to $\partial\Omega$.

Consider the transport problem to find $\mathbf{w} \in L^q(\Omega)^m$ such that

$$\mathcal{C}\mathbf{w} + \mathbf{u} \cdot \nabla\mathbf{w} = \mathbf{f} \text{ in } \Omega, \quad \mathbf{w} = \mathbf{w}_0 \text{ on } \Gamma_-, \tag{2}$$

where $\mathcal{C}(\mathbf{x}) \in \mathcal{L}_m$ for $\mathbf{x} \in \Omega$, $\mathbf{f} \in L^q(\Omega)^m$, $m \geq 1$. Here, $\mathcal{L}_m$ is the space of linear operators on $\mathbb{R}^m$, which is of course isomorphic to $\mathbb{R}^m \times \mathbb{R}^m$ via matrix representation.

When $\mathbf{u} \in H^1(\Omega)^d$, $\Gamma_-$ is a measurable subset of $\partial\Omega$. But to define the meaning of the transport inflow boundary condition, Bernard [4] makes further assumptions on $\mathbf{u}$, as follows. Assume that we can write

$$\partial\Omega = \overline{\Gamma_-} \cup \overline{\Gamma_1}, \quad \Gamma_- \cap \Gamma_1 = \emptyset, \quad \overline{\Gamma_-} \cap \overline{\Gamma_1} = \cup_{k=1}^{\kappa} K_k, \text{ with each } K_k \text{ being Lipschitz.} \tag{3}$$

Thus $\Gamma_1$ is the complement of the inflow boundary $\Gamma_-$ in $\partial\Omega$ and the sets $K_k$ are the (finite number of) $d-2$ dimensional connected components of $\overline{\Gamma_-}\cap\overline{\Gamma_1}$. Regarding the requirement that each $K_k$ is Lipschitz, see [4,(1.7)]. In two dimensions ($d=2$), each $K_k$ is a point.

We will assume that $\mathbf{u}$ corresponds to incompressible flow, that is,

$$\nabla\cdot\mathbf{u}=0 \quad \text{in } \Omega.$$

We can make the space of solutions more precise by defining

$$X_{\mathbf{u}}=\left\{\mathbf{w}\in L^2(\Omega)^m \;\middle|\; \mathbf{u}\cdot\nabla\mathbf{w}\in L^2(\Omega)^m\right\},$$

with the norm

$$\|\,\mathbf{w}\,\|_{X_{\mathbf{u}}}=\sqrt{\|\,\mathbf{w}\,\|^2_{L^2(\Omega)}+\|\,\mathbf{u}\cdot\nabla\mathbf{w}\,\|^2_{L^2(\Omega)}}\,.$$

It is proved in [10] that $X_{\mathbf{u}}$ is well defined for $\mathbf{u}\in H^1(\Omega)^d$, and that smooth functions are dense. Under the condition (3), it is shown in [4] that the restriction of $\mathbf{w}\in X_{\mathbf{u}}$ to $\Gamma_-$ is well defined. We will be interested in more regular solutions in $L^q$ for $q>2$, but we will not need to modify $X_u$ in any way. Thus we can rely on the results of [10] without further modifications. Since we are restricting our attention to bounded domains, we have $\mathbf{w}\in L^2$ if $\mathbf{w}\in L^q$ for $q>2$. We now give some examples of interest.

## 2.1 Oldroyd models

Oldroyd rheological models [12] have a transport equation linking the fluid stress $\mathbf{T}$ with fluid velocity $\mathbf{u}$. In the Oldroyd equations, the operator $\mathcal{C}$ takes the form

$$\mathcal{C}\mathbf{T}=\lambda_1(\mathbf{R}\circ\mathbf{T}+\mathbf{T}\circ\mathbf{R}^t)-\mu_1(\mathbf{E}\circ\mathbf{T}+\mathbf{T}\circ\mathbf{E}),$$

where the multiplication operator $\circ$ is matrix multiplication and $\lambda_1$ and $\mu_1$ are real numbers. Here $m=d^2$, and $\mathbf{R}$ and $\mathbf{E}$ are the matrix-valued functions defined by

$$\mathbf{R}=\tfrac{1}{2}(\nabla\mathbf{u}^t-\nabla\mathbf{u}),\qquad \mathbf{E}=\tfrac{1}{2}(\nabla\mathbf{u}^t+\nabla\mathbf{u}).$$

## 2.2 Derivatives

Many existing results [2–4, 6, 10, 12, 13] show that the transport equation (2) is well posed on Lebesgue spaces. But there is significant interest in regularity of the solutions. A transport equation for the gradient of the solution can be obtained by taking the gradient of the transport equation. Consider the simplified model ($m=1$)

$$w+\mathbf{u}\cdot\nabla w=f,\qquad w=0 \text{ on } \Gamma_-. \tag{4}$$

The transport equation for $\mathbf{w}=\nabla w$ is

$$(\mathcal{I}+\nabla\mathbf{u})\mathbf{w}+\mathbf{u}\cdot\nabla\mathbf{w}=\nabla f. \tag{5}$$

Thus $\mathcal{C}=\mathcal{I}+\nabla\mathbf{u}$, with $m=d$. The relevant boundary conditions will be discussed in Sect. 4.3.

## 3 Bounds in Lebesgue spaces

We begin with bounds in Lebesgue spaces as we need the results as building blocks for the bounds in Sobolev spaces derived subsequently. The main theorem extends the result [4, Theorem 3.3] for $q = 2$ to arbitrary Lebesgue exponent $q \geq 2$. This improvement will be crucial later for bounds on derivatives.

**Theorem 3.1** *Suppose that $\Omega$ has a Lipschitz boundary, $2 \leq q \leq \infty$, $\mathbf{u} \in H^1(\Omega)^d$, the Bernard condition (3) holds, $\mathbf{f} \in L^q(\Omega)^m$, and $\mathcal{C}$ is bounded and positive definite, that is,*

$$c_1|\boldsymbol{\xi}|^2 \geq \boldsymbol{\xi}^t \mathcal{C} \boldsymbol{\xi} \geq c_0|\boldsymbol{\xi}|^2 \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^m. \tag{6}$$

*Suppose further that $\mathbf{w}_0$ can be extended to all of $\Omega$ so that $\mathbf{w}_0 \in X_{\mathbf{u}} \cap L^q(\Omega)^m$ and $\mathbf{u} \cdot \nabla \mathbf{w}_0 \in L^q(\Omega)^m$. Then (2) has a unique solution $\mathbf{w} \in X_{\mathbf{u}}$ satisfying*

$$\| \mathbf{w} \|_{L^q(\Omega)} \leq \frac{1}{c_0} \| \mathbf{f} \|_{L^q(\Omega)} + \left(1 + \frac{c_1}{c_0}\right) \| \mathbf{w}_0 \|_{L^q(\Omega)} + \frac{1}{c_0} \| \mathbf{u} \cdot \nabla \mathbf{w}_0 \|_{L^q(\Omega)}. \tag{7}$$

In the case that $q \geq 2$, existence and uniqueness follow from [4, Theorem 3.3] for $q = 2$. Thus the main challenge is to establish the bound (7) for $q > 2$. The proof follows closely [14] with the main difference being the inflow boundary condition that has been added here.

### 3.1 Sufficient to prove it for $\mathbf{w}_0 = \mathbf{0}$

We first show that it is sufficient to consider the case $\mathbf{w}_0 = \mathbf{0}$. So suppose for the moment that we have the bound (7) in the case that $\mathbf{w}|_{\Gamma_-} = \mathbf{0}$:

$$\| \mathbf{w} \|_{L^q(\Omega)} \leq \frac{1}{c_0} \| \mathbf{f} \|_{L^q(\Omega)}.$$

Now consider a general $\mathbf{w}_0 \in X_{\mathbf{u}}$. Let $\mathbf{z}$ solve

$$\mathcal{C}\mathbf{z} + \mathbf{u} \cdot \nabla \mathbf{z} = \hat{f} \text{ in } \Omega, \quad \mathbf{z}|_{\Gamma_-} = \mathbf{0}, \qquad \hat{f} = f - \mathcal{C}\mathbf{w}_0 - \mathbf{u} \cdot \nabla \mathbf{w}_0.$$

Define $\mathbf{w} = \mathbf{z} + \mathbf{w}_0$, which satisfies (2). Then

$$\| \mathbf{w} \|_{L^q(\Omega)} \leq \| \mathbf{z} \|_{L^q(\Omega)} + \| \mathbf{w}_0 \|_{L^q(\Omega)} \leq \frac{1}{c_0} \| \mathbf{f} - \mathcal{C}\mathbf{w}_0 - \mathbf{u} \cdot \nabla \mathbf{w}_0 \|_{L^q(\Omega)} + \| \mathbf{w}_0 \|_{L^q(\Omega)},$$

which implies (7). Thus all we must do is establish the bound (7) in the case $\mathbf{w}_0 = \mathbf{0}$.

We now break down the proof into two cases depending on $q$.

### 3.2 Proof for $\mathbf{w}_0 = \mathbf{0}$ and $2 \leq q < \infty$

Following [14], we regularize the transport problem to obtain a standard diffusion-advection problem [25, chapter 15]:

$$-\epsilon \Delta \mathbf{w}^\epsilon + \mathcal{C}\mathbf{w}^\epsilon + \mathbf{u} \cdot \nabla \mathbf{w}^\epsilon = \mathbf{f} \text{ in } \Omega, \quad \mathbf{w}^\epsilon = \mathbf{0} \text{ on } \Gamma_-, \quad \frac{\partial \mathbf{w}^\epsilon}{\partial n} = \mathbf{0} \text{ on } \partial\Omega \backslash \Gamma_-, \tag{8}$$

where $\epsilon > 0$. Define the space $W = \left\{ \mathbf{w} \in H^1(\Omega)^m \;\middle|\; \mathbf{w} = \mathbf{0} \text{ on } \Gamma_- \right\}$. Multiplying (8) by a suitable test function $\boldsymbol{\sigma} \in W$, integrating and integrating by parts gives

$$a_\epsilon(\mathbf{w}^\epsilon, \boldsymbol{\sigma}) = \int_\Omega \mathbf{f} \cdot \boldsymbol{\sigma} \, d\mathbf{x} \tag{9}$$

where

$$a_\epsilon(\boldsymbol{\tau},\boldsymbol{\sigma}) = \epsilon \int_\Omega \sum_{i=1}^m \nabla\tau_i \cdot \nabla\sigma_i \, d\mathbf{x} + \int_\Omega (\mathcal{C}\boldsymbol{\tau}) \cdot \boldsymbol{\sigma} \, d\mathbf{x} + \int_\Omega (\mathbf{u} \cdot \nabla\boldsymbol{\tau}) \cdot \boldsymbol{\sigma} \, d\mathbf{x}. \tag{10}$$

In (10), we have used vector dot-product notation, but we have written the first integral explicitly in terms of components, since $m$ is not connected to the dimension $d$ of the domain. Thus $(\mathcal{C}\boldsymbol{\tau}) \cdot \boldsymbol{\sigma}$ is a dot-product of $m$ vectors, whereas $\mathbf{u} \cdot \nabla$ is a dot-product of $d$ vectors.

Now we address the last term in (10). Note that

$$\nabla\cdot \left(\mathbf{u}|\boldsymbol{\sigma}|^2\right) = 2(\mathbf{u} \cdot \nabla\boldsymbol{\sigma}) \cdot \boldsymbol{\sigma}$$

because $\nabla\cdot \mathbf{u} = 0$. For $\boldsymbol{\tau} = \boldsymbol{\sigma} \in W$, the last term in (10) with $\boldsymbol{\tau} = \boldsymbol{\sigma}$ takes the form

$$\int_\Omega (\mathbf{u} \cdot \nabla\boldsymbol{\sigma}) \cdot \boldsymbol{\sigma} \, d\mathbf{x} = \int_\Omega \tfrac{1}{2}\nabla\cdot \left(\mathbf{u}|\boldsymbol{\sigma}|^2\right) d\mathbf{x} = \oint_{\Gamma_+} |\boldsymbol{\sigma}|^2(\mathbf{u} \cdot \mathbf{n}) \, ds + \oint_{\Gamma_-} |\boldsymbol{\sigma}|^2(\mathbf{u} \cdot \mathbf{n}) \, ds \tag{11}$$

since $\nabla\cdot \mathbf{u} = 0$ and $\mathbf{u} \cdot \mathbf{n} = 0$ on the rest of $\partial\Omega$. Technically, we must be sure that all quantities in this equality are well defined. The smoothness of $\mathbf{u}$ insures all quantities in (11) are integrable. Note that

$$\oint_{\Gamma_+} |\boldsymbol{\sigma}|^2(\mathbf{u} \cdot \mathbf{n}) \, ds \geq 0, \qquad \oint_{\Gamma_-} |\boldsymbol{\sigma}|^2(\mathbf{u} \cdot \mathbf{n}) \, ds \leq 0,$$

due to the definition (1) of the inflow and outflow boundaries. Thus we conclude that

$$\int_\Omega (\mathbf{u} \cdot \nabla\boldsymbol{\sigma}) \cdot \boldsymbol{\sigma} \, d\mathbf{x} \geq - \oint_{\Gamma_-} |\boldsymbol{\sigma}|^2|\mathbf{u} \cdot \mathbf{n}| \, ds. \tag{12}$$

We have written $\mathbf{u} \cdot \mathbf{n} = -|\mathbf{u} \cdot \mathbf{n}|$ on $\Gamma_-$ just to clarify the sign of this term.

If we have $\boldsymbol{\sigma} = \mathbf{0}$ on $\Gamma_-$, then (10) and (12) imply

$$a_\epsilon(\boldsymbol{\sigma},\boldsymbol{\sigma}) \geq \epsilon \int_\Omega |\nabla\boldsymbol{\sigma}|^2 \, d\mathbf{x} + c_0 \int_\Omega |\boldsymbol{\sigma}|^2 \, d\mathbf{x}, \tag{13}$$

in view of (6). Here we have used the short-hand notation $|\nabla\boldsymbol{\sigma}|^2 = \sum_{i=1}^m |\nabla\sigma_i|^2$ whereas $|\boldsymbol{\sigma}|^2 = \sum_{i=1}^m \sigma_i^2$.

Following [14], we use the Yoshida regularization and define

$$\boldsymbol{\sigma}_a = \phi(a|\mathbf{w}^\epsilon|^{q-2})\mathbf{w}^\epsilon,$$

where $a > 0$ and $\phi$ is defined by $\phi(t)\left(1 + t\phi(t)^{q-2}\right) = 1$, $t \geq 0$. Note that $\phi(0) = 1$. Then for $t = a|\mathbf{w}^\epsilon|^{q-2}$ we get

$$\boldsymbol{\sigma}_a + a|\boldsymbol{\sigma}_a|^{q-2}\boldsymbol{\sigma}_a = \mathbf{w}^\epsilon. \tag{14}$$

The function $\phi$ satisfies [14] $\phi'(0) = -1$ and $\phi(t) \approx t^{-1/(q-1)}$ as $t \to \infty$. Thus as $\mathbf{w}^\epsilon(\mathbf{x}) \to \infty$, $\boldsymbol{\sigma}_a(\mathbf{x}) \to 0$. Similarly, it is shown in [14] that

$$|\nabla\boldsymbol{\sigma}_a| \leq |\nabla\mathbf{w}^\epsilon| \quad \text{a. e. in } \Omega.$$

Then

$$|a_\epsilon(\mathbf{w}^\epsilon, |\boldsymbol{\sigma}_a|^{q-2}\boldsymbol{\sigma}_a)| = \left| \int_\Omega \mathbf{f} \cdot \boldsymbol{\sigma}_a|\boldsymbol{\sigma}_a|^{q-2} \, d\mathbf{x} \right| \leq \| \mathbf{f} \|_{L^q(\Omega)} \| \boldsymbol{\sigma}_a \|_{L^q(\Omega)}^{q-1}.$$

Therefore (13) and (14) imply

$$a_\epsilon(\boldsymbol{\sigma}_a, |\boldsymbol{\sigma}_a|^{q-2}\boldsymbol{\sigma}_a) = a_\epsilon(\mathbf{w}^\epsilon, |\boldsymbol{\sigma}_a|^{q-2}\boldsymbol{\sigma}_a) - a\, a_\epsilon(|\boldsymbol{\sigma}_a|^{q-2}\boldsymbol{\sigma}_a, |\boldsymbol{\sigma}_a|^{q-2}\boldsymbol{\sigma}_a)$$
$$\leq \|\, \mathbf{f}\, \|_{L^q(\Omega)} \|\, \boldsymbol{\sigma}_a\, \|_{L^q(\Omega)}^{q-1}.$$

But it is also shown in [14] that

$$\nabla \boldsymbol{\sigma}_a : \nabla(|\boldsymbol{\sigma}_a|^{q-2}\boldsymbol{\sigma}_a) \geq 0 \quad \text{a. e. in } \Omega.$$

From the non-negativity of the first term in the definition (10) of the form $a_\epsilon(\cdot, \cdot)$, we have

$$\int_\Omega |\boldsymbol{\sigma}_a|^{q-2}\boldsymbol{\sigma}_a^t(\mathcal{C}\boldsymbol{\sigma}_a)\, d\mathbf{x} + \int_\Omega (\mathbf{u} \cdot \nabla\boldsymbol{\sigma}_a) \cdot |\boldsymbol{\sigma}_a|^{q-2}\boldsymbol{\sigma}_a\, d\mathbf{x} \leq \|\, \mathbf{f}\, \|_{L^q(\Omega)} \|\, \boldsymbol{\sigma}_a\, \|_{L^q(\Omega)}^{q-1}. \tag{15}$$

As in the derivation (11) and (12), we use $\nabla\cdot \big(\mathbf{u}|\boldsymbol{\sigma}|^q\big) = q(\mathbf{u} \cdot \nabla\boldsymbol{\sigma}) \cdot \boldsymbol{\sigma}$ and find

$$\begin{aligned}\int_\Omega (\mathbf{u} \cdot \nabla\boldsymbol{\sigma}_a) \cdot |\boldsymbol{\sigma}_a|^{q-2}\boldsymbol{\sigma}_a\, d\mathbf{x} &= \frac{1}{q}\int_\Omega \mathbf{u} \cdot \nabla|\boldsymbol{\sigma}_a|^q\, dx = \frac{1}{q}\int_\Omega \nabla\cdot(\mathbf{u}|\boldsymbol{\sigma}_a|^q)\, dx \\ &= \oint_{\Gamma_+} |\boldsymbol{\sigma}_a|^q(\mathbf{u}\cdot\mathbf{n})\, ds + \oint_{\Gamma_-} |\boldsymbol{\sigma}_a|^q(\mathbf{u}\cdot\mathbf{n})\, ds \geq 0,\end{aligned} \tag{16}$$

since $\boldsymbol{\sigma}_a = \mathbf{0}$ on $\Gamma_-$ (recall that $\mathbf{w}^\epsilon = \mathbf{0}$ on $\Gamma_-$). Applying (6) and (15), we find

$$c_0\int_\Omega |\boldsymbol{\sigma}_a|^q\, d\mathbf{x} \leq \int_\Omega |\boldsymbol{\sigma}_a|^{q-2}\boldsymbol{\sigma}_a^t(\mathcal{C}\boldsymbol{\sigma}_a)\, d\mathbf{x} \leq \|\, \mathbf{f}\, \|_{L^q(\Omega)} \|\, \boldsymbol{\sigma}_a\, \|_{L^q(\Omega)}^{q-1}. \tag{17}$$

Dividing by $\|\, \boldsymbol{\sigma}_a\, \|_{L^q(\Omega)}^{q-1}$, we have shown that

$$c_0\|\, \boldsymbol{\sigma}_a\, \|_{L^q(\Omega)} \leq \|\, \mathbf{f}\, \|_{L^q(\Omega)}$$

for any $a > 0$. Letting $a \to 0$ forces $\phi(a\ \cdot) \to 1$, and thus $\sigma_a \to \mathbf{w}^\epsilon$, and as shown in [14, Theorem 3]

$$c_0\|\, \mathbf{w}^\epsilon\, \|_{L^q(\Omega)} \leq \|\, \mathbf{f}\, \|_{L^q(\Omega)}.$$

Letting $\epsilon \to 0$ shows (as in [14]) that

$$c_0\|\, \mathbf{w}\, \|_{L^q(\Omega)} \leq \|\, \mathbf{f}\, \|_{L^q(\Omega)} \tag{18}$$

as claimed. □

### 3.3 Proof for $\mathbf{w}_0 = 0$ and $q = \infty$

If $\mathbf{f} \in L^\infty(\Omega)^m$, then we can apply (18) for all finite $q$ to get

$$c_0\|\, \mathbf{w}\, \|_{L^q(\Omega)} \leq \|\, \mathbf{f}\, \|_{L^q(\Omega)} \leq \|\, \mathbf{f}\, \|_{L^\infty(\Omega)}|\Omega|^{1/q}.$$

Since $|\Omega|^{1/q} \to 1$ as $q \to \infty$, we conclude that $\mathbf{w} \in L^\infty(\Omega)^m$ and

$$\|\, \mathbf{w}\, \|_{L^\infty(\Omega)} = \lim_{q\to\infty} \|\, \mathbf{w}\, \|_{L^q(\Omega)} \leq \frac{1}{c_0}\|\, \mathbf{f}\, \|_{L^\infty(\Omega)}.$$

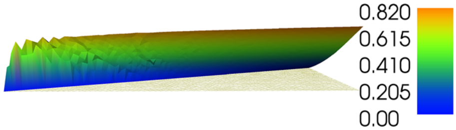
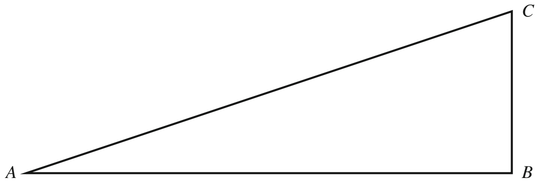



**Fig. 2** Bernard's example on a mesh with $M = 50$ used for the meshing program `mshr`. The impact of this choice of mesh parameter input can be seen in the figure since the complete triangulation can be seen under the computed surface

**Fig. 3** Bernard's triangle $\Omega$




## 4 Derivative estimates

A hallmark of the $L^q$ estimates is that there is no nonlinear dependence on the size of $\mathbf{u}$. This is not the case for estimates of derivatives. We begin with an example that provides an upper bound on our expectations for derivative estimates (Fig. 2).

### 4.1 Bernard's example

There is an example in [5] with a polynomial (and thus smooth) $\mathbf{u}$ on a triangular domain $\Omega$ for which $f \in H^1(\Omega)$ but $w \notin H^1(\Omega)$ $(m = 1)$. The vertices of the triangle are $A = (-2, \frac{1}{3})$, $B = (0, \frac{1}{3})$, and $C = (0, 1)$, as depicted in Fig. 3. The solution $w$ is of the form

$$w(x, y) = 1 - e^{\frac{1}{\alpha(x,y)} - \frac{1}{y}}, \qquad \alpha(x, y) = p(x, y)^{1/3} + \tfrac{1}{3}, \qquad p(x, y) = \tfrac{1}{3}\big(xy^2 + y - \tfrac{1}{9}\big). \tag{19}$$

Differentiating, we find

$$\nabla w(x, y) = e^{\frac{1}{\alpha(x,y)} - \frac{1}{y}}\Big(\frac{\nabla\alpha}{\alpha^2} - (0, y^{-2})\Big), \ \nabla\alpha = \tfrac{1}{3}p^{-2/3}\nabla p, \ \nabla p(x, y) = \tfrac{1}{3}\big(y^2, 2xy + 1\big).$$

The singularity in $w$ is visible if we consider

$$p|_{\overline{AB}} = \tfrac{1}{3}\big(\tfrac{1}{9}x + \tfrac{1}{3} - \tfrac{1}{9}\big) = \frac{1}{27}(x + 2),$$

which implies that

$$\alpha|_{\overline{AB}} = \tfrac{1}{3}\big((x + 2)^{1/3} + 1\big) \implies (\alpha_x)|_{\overline{AB}} = \tfrac{1}{9}(x + 2)^{-2/3}. \tag{20}$$

But we can also identity the restriction of $p$ to $\overline{AC}$. The line $\overline{AC}$ can be parameterized by $y = \frac{1}{3}x + 1$. Thus $p|_{\overline{AB}}$ satisfies

$$\begin{aligned} p(x, \tfrac{1}{3}x + 1) &= \tfrac{1}{3}\big(x(\tfrac{1}{3}x + 1)^2 + \tfrac{1}{3}x + 1 - \tfrac{1}{9}\big) = \tfrac{1}{3}\big(\tfrac{1}{9}x^3 + \tfrac{2}{3}x^2 + (1 + \tfrac{1}{3})x + \tfrac{8}{9}\big) \\ &= \frac{1}{27}\big(x^3 + 6x^2 + 12x + 8\big) = \frac{1}{27}(x + 2)^3 = \Big(\frac{x + 2}{3}\Big)^3. \end{aligned} \tag{21}$$

**Table 1** $L^2$ error for solving (25) using piecewise linears on (a) the triangular domain (left), and (b) quadrilateral domain (right, $A' = (-2, \frac{1}{4})$)

| (a) Triangle | | | (b) Quadrilateral | | |
|---|---|---|---|---|---|
| $M$ | Ratio | Error | $M$ | Ratio | Error |
| 50 | 7.23 | 1.14e−02 | 50 | 6.28 | 1.19e−02 |
| 100 | 7.93 | 6.19e−03 | 100 | 7.65 | 7.90e−03 |
| 200 | 8.49 | 3.56e−03 | 200 | 7.62 | 4.17e−03 |
| 400 | 9.38 | 2.19e−03 | 400 | 8.48 | 2.60e−03 |

$M$ is the mesh parameter used by `mshr`, *ratio* is the ratio of the $H^1$ and $L^2$ norms of the discrete solution, and *error* is the $L^2$ norm error between the computed solution and the exact solution in (19). The $L^2$ norm of the solution is (triangle) 0.37 and (quadrilateral) 0.44

We introduce the notation $F_x$ to mean the partial derivative of a general function $F$ of two variables with respect to the $x$-variable. We will use this for $\alpha$ and $p$. Thus

$$(\alpha_x)|_{\overline{AC}} = 3(x+2)^{-2} p_x = (x+2)^{-2}(y^2)|_{\overline{AC}} = (x+2)^{-2}(\tfrac{1}{3}x+1)^2 = \frac{(x+3)^2}{9(x+2)^2}. \tag{22}$$

This is an even stronger singularity at $A$ than seen on $\overline{AB}$. Thus $\alpha_x$ (and hence $w_x$) is not even integrable on $\Omega$.

The minimum of $p$ over $\Omega$ is 0, and so $\alpha \geq \frac{1}{3}$ on $\Omega$. Thus $w$ is smooth in the interior of $\Omega$, with the only singularity at the boundary point $A$. The velocity $\mathbf{u}$ in [5] is given by

$$\mathbf{u}(x, y) = (2xy + 1, -y^2). \tag{23}$$

Thus $\mathbf{u} \cdot \nabla p = 0$. Similarly $\mathbf{u} \cdot \nabla \alpha = 0$. Therefore

$$\mathbf{u} \cdot \nabla w = -e^{\frac{1}{\alpha(x,y)} - \frac{1}{y}} \mathbf{u}(x, y) \cdot (0, y^{-2}) = e^{\frac{1}{\alpha(x,y)} - \frac{1}{y}} = 1 - w. \tag{24}$$

Thus

$$w + \mathbf{u} \cdot \nabla w = 1 \text{ in } \Omega, \qquad w|_{\overline{AC}} = 0. \tag{25}$$

But $w_x \notin L^p(\Omega)$ for any $p \geq 1$.

In Fig. 2, we show the result of a standard Galerkin method for solving (25) using piecewise linears, with no special upwinding or other techniques used to stabilize the numerical method. We obtain a smooth solution away from the singularity, but near the singularity it is not surprising to see some wiggles. On the other hand, the solution accuracy is quite good, as indicated in Table 1, especially given the lack of regularity of the solution. As expected, the $H^1$ norm of the computed solution grows as the mesh is refined.

The boundary condition holds because

$$\alpha|_{\overline{AC}} = y.$$

This follows since on the line $\overline{AC}$, $x = 3(y-1)$. Thus

$$3p|_{\overline{AC}} = 3(y-1)y^2 + y - \tfrac{1}{9} = 3y^3 - 3y^2 + y - \tfrac{1}{9} = 3(y - \tfrac{1}{3})^3 \implies p^{1/3}|_{\overline{AC}} = y - \tfrac{1}{3}.$$

We can identify the inflow and outflow regions of the boundary. A simple calculation gives $(\mathbf{u} \cdot \mathbf{n})|_{BC} = 1$. Similarly, $(\mathbf{u} \cdot \mathbf{n})|_{AB} = 1/9$. We note that on $\overline{AC}$, $\mathbf{n} = \frac{1}{\sqrt{10}}(-1, 3)$. Recall

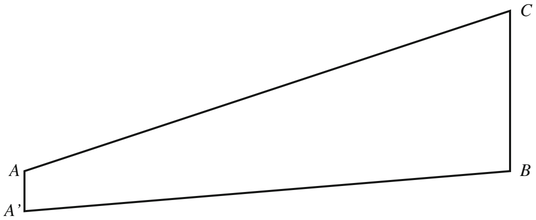



**Fig. 4** A quadrilateral related to Bernard's triangle

that $\overline{AC}$ can be parameterized by $y = \frac{1}{3}x + 1$. Thus

$$\begin{aligned} -\sqrt{10}\,\mathbf{u}\cdot\mathbf{n}|_{\overline{AC}} &= 2x(\tfrac{1}{3}x+1)+1+3(\tfrac{1}{3}x+1)^2 = \tfrac{2}{3}x^2+2x+1+\tfrac{1}{3}x^2+2x+3 \\ &= x^2+4x+4=(x+2)^2. \end{aligned} \tag{26}$$

Thus $\mathbf{u}\cdot\mathbf{n}$ vanishes to second order at $A$. For the purposes of comparing with Theorem 4.1, we observe that

$$\nabla\mathbf{u}(A) = \begin{pmatrix} 2/3 & -4 \\ 0 & -2/3 \end{pmatrix}. \tag{27}$$

## 4.2 Bernard's quadrilateral

The fact that the singularity in Bernard's example occurs at the point $A$ does not depend on the small angle there. Indeed, one can add another point $A'$, slightly below $A$, to get a quadrilateral $\Omega'$, as shown in Fig. 4. The general properties of the example still hold. In particular, the inflow boundary remains $\overline{AC}$, as indicated by the flow patterns in Fig. 5. More precisely, we have

$$\mathbf{u}\cdot\mathbf{n}|_{\overline{AA'}} = 4y-1>0$$

as long as $y > \frac{1}{4}$.

If we pick $A' = (-2, \frac{1}{4})$, then the line $\overline{A'B}$ is parameterized by

$$y = \frac{1}{24}(x+8).$$

Therefore

$$\mathbf{n}_{\overline{A'B}} = \frac{1}{\sqrt{577}}(1,-24).$$

Recalling (23), we have

$$\mathbf{u}|_{\overline{A'B}} = \Big(\frac{2x(x+8)}{24}+1, -\frac{1}{576}(x+8)^2\Big) = \frac{1}{24}\big(2x^2+16x+24, -\frac{1}{24}(x+8)^2\big).$$

Therefore

$$(\mathbf{n}\cdot\mathbf{u})|_{\overline{A'B}} = \frac{1}{24\sqrt{577}}\big(2x^2+16x+24+(x+8)^2\big) = \frac{1}{24\sqrt{577}}\big(3x^2+32x+88\big).$$

It is elementary to show that

$$3x^2+32x+88 \ge \frac{8}{3} \quad \text{for all } x.$$

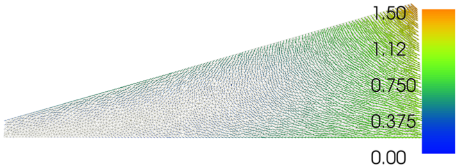



**Fig. 5** Plot of the vector field $\mathbf{u}$ defined in (23) on $\Omega'$

Thus $\overline{A'B}$ is an outflow region of the quadrilateral boundary for $\mathbf{u}$ given in (23).

In Fig. 6, we show the result of a standard Galerkin method for solving (25) on the Bernard quadrilateral using piecewise linears, with no special upwinding or other techniques used to stabilize the numerical method. We again obtain a smooth solution away from the singularity, but near the singularity it is again not surprising to see some wiggles. On the other hand, the solution accuracy is quite good, as indicated in Table 1, especially given the lack of regularity of the solution. As expected, the $H^1$ norm of the computed solution grows as the mesh is refined, but it is smaller than for the triangle case.

### 4.3 Gradient bounds for restricted *f*

A version of the following corollary of Theorem 3.1 was given in [5, Theorem 2.1]. For completeness, we sketch the proof since this result is central to the subsequent development in this paper.

**Theorem 4.1** *In addition to the assumptions in Theorem 3.1, suppose* $\mathbf{u} \in W^1_\infty(\Omega)^d$, $f \in W^1_q(\Omega)$, *and there exists* $c_0 > 0$ *such that*

$$\boldsymbol{\xi}^t(I - \nabla\mathbf{u}(\mathbf{x}))\boldsymbol{\xi} \geq c_0|\boldsymbol{\xi}|^2 \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^d,\ \mathbf{x} \in \Omega. \tag{28}$$

*Then* (4) *has a unique solution* $w \in H^1(\Omega)^d$ *satisfying*

$$\|\nabla w\|_{L^q(\Omega)} \leq \frac{1}{c_0}\|\nabla f\|_{L^q(\Omega)}, \tag{29}$$

*provided that that* $f = 0$ *on* $\Gamma_-$.

***Proof*** We summarize the proof of [5, Theorem 2.1]. Since $w = 0$ on $\Gamma_-$, the transport equation implies that $\mathbf{u} \cdot \nabla w = 0$ on $\Gamma_-$. Since $\mathbf{u} \cdot \mathbf{n} < 0$ in the interior of $\Gamma_-$, we conclude that $\mathbf{n} \cdot \nabla w = 0$ on $\Gamma_-$. Similarly, the tangential derivatives also vanish, we we conclude that $\nabla w = 0$ on $\Gamma_-$.

We apply Theorem 3.1 to $\mathbf{w} = \nabla w$, with $m = d$. The equation for $\mathbf{w}$ is (5). Thus we conclude that

$$\|\nabla w\|_{L^q(\Omega)} = \|\mathbf{w}\|_{L^q(\Omega)} \leq \frac{1}{c_0}\|\nabla f\|_{L^q(\Omega)},$$

in the case that $f = 0$ on $\Gamma_-$. □

The condition (28) holds if $\|\nabla\mathbf{u}\|_{L^\infty(\Omega)}$ is sufficiently small. But the condition (28) fails for $\nabla\mathbf{u}(\mathbf{x})$ for $\mathbf{x}$ near $A$ with the velocity $\mathbf{u}$ in Sect. 4.1, as indicated by (27). Note that if $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, then (29) holds for all $f \in W^1_q(\Omega)$, since $\Gamma_-$ is the empty set in this case.

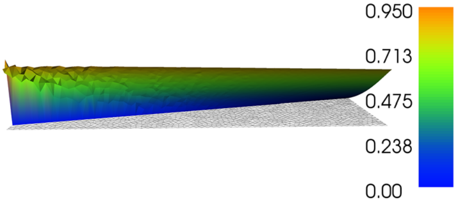



**Fig. 6** Bernard's example on a mesh with $M = 50$ on the quadrilateral domain $\Omega'$

### 4.4 General *f*

Theorem 4.1 can be used [5] to prove results when $f$ does not vanish on $\Gamma_-$, but with further restrictions. The idea is to pick $w_0 \in H^2(\Omega)$ such that $w_0 = 0$ on $\Gamma_-$ and

$$\frac{\partial w_0}{\partial n} = \frac{f}{\mathbf{u} \cdot \mathbf{n}}$$

on $\Gamma_-$. This construction satisfies

$$(w_0 + \mathbf{u} \cdot \nabla w_0)|_{\Gamma_-} = f|_{\Gamma_-}.$$

Then Theorem 4.1 is applied to $w - w_0$. If $\mathbf{u} \cdot \mathbf{n}$ is bounded below on $\Gamma_-$, then $w_0$ can be chosen so that $\| w_0 \|_{H^2(\Omega)} \leq C \| f \|_{H^1(\Omega)}$. See the proof of [5, Theorem 2.2] for details.

## 5 Amick's theorem

Amick's theorem [1] establishes existence of flows in domains which have inflow and outflow tubes, as shown in Fig. 1, where the flow is asymptotically Poiseuille-like. We call such domains Amick domains, and we assume that the imposed boundary conditions are compatible Poiseuille flows, that is, the input and output flux is the same. For such domains in three dimensions, the sets $K_k$ in (3) are circles. For simplicity, we limit the dimension to $d = 2$ or $d = 3$.

**Definition 5.1** Let $\Omega$ be a Lipschitz domain and recall the definition (1) of the inflow and outflow boundaries, where we now suppose that $\mathbf{u} \in W^1_\infty(\Omega)$. We say that $\Omega$ is an Amick domain (for $\mathbf{u}$) if there are neighborhoods $\Omega_\pm$ of $\Gamma_\pm$ in $\Omega$ which coincide with a channel (for $d = 2$) or a pipe (for $d = 3$) and $\mathbf{u}$ tends to Poiseuille flow on $\Gamma_\pm$, with the input and output flux being the same.

In particular, the definition implies that $\mathbf{u}$ is tangential (or zero) on $\partial\Omega$ outside of $\Gamma_\pm$. The definition of the Amick domain is really for a pair, $\Omega$ and $\mathbf{u}$. We have indicated that by the parenthetical (for $\mathbf{u}$) after "domain" in the second sentence of Definition 5.1. The vector field $\mathbf{u}$ is arbitrary except for the requirement to be Poiseuille-like at the inlet and outlet. For example, it could be defined by solving a Stokes-like equation with Poiseuille flow boundary conditions [22].

Figure 7 shows the horizontal component of the solution of the Stokes equations with Poiseuille flow boundary conditions in an expanded channel, an example of an Amick domain. We see that the Poiseuille (quadratic) flow is adopted fairly quickly for Reynolds number zero. For larger Reynolds numbers, longer input and output channel buffers would have to be used to guarantee asymptotic similarity, e.g., the same flow profile in and near the expanding region.

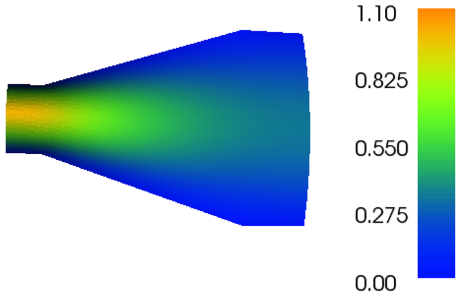



**Fig. 7** Horizontal component of the solution of the Stokes equations in an expanded channel, an example of an Amick domain

Amick's theorem provides the basis for the choice of boundary conditions in both physical and computational experiments. It says that if we make the input and output pipes sufficiently long, we can be assured that Poiseuille flow will be an appropriate boundary condition. For this reason, we focus first on transport in channels and pipes with Poiseuille flow profiles.

### 5.1 Poiseuille flow

Suppose that the $x$-axis is the centerline for the input or output section of $\Omega$. Then Poiseuille flow takes the form $\mathbf{u}(x, y) = (u(y), 0)$ in two dimensions or $\mathbf{u}(x, y, z) = (u(y, z), 0, 0)$ in three dimensions. We begin with the two-dimensional case, and we write the input section as

$$\big\{(x, y) \,\big|\, 0 \le x \le X,\ |y| \le 1\big\}$$

after scaling the coordinates appropriately.

Theorem 3.1 in [5] gives the most general result known for $H^1$ estimates for transport equations with inhomogeneous boundary conditions. But it requires $\mathbf{u}(\mathbf{x}) \ne \mathbf{0}$ at boundary points of $\Gamma_- = \big\{(0, y) \,\big|\, |y| \le 1\big\}$. In standard Poiseuille flow, $\mathbf{u}(x, y) = (u(y), 0)$, where $u$ vanishes at $y = \pm 1$, and this condition does not hold. On the other hand,

$$u(y) = 1 - y^2 \tag{30}$$

vanishes only to first order at the boundary points of $\Gamma_-$, unlike the example in Sect. 4.1 where the flow velocity vanishes to second order at the boundary point $A$ of $\Gamma_-$. Thus it is of interest to look in detail at the solution in this case.

For Poiseuille flow, the transport equation takes the form

$$w + u w_x = f, \qquad w(0, y) = 0. \tag{31}$$

It suffices to assume that $f$ is independent of $x$, because if we analyze the case $f = f(y)$ we can apply Theorem 4.1 to $f(x, y) - f(0, y)$. We claim that (31) is solved by

$$w(x, y) = f(y)\big(1 - e^{-x/u(y)}\big). \tag{32}$$

Checking, we have

$$w_x(x, y) = \frac{f(y)}{u(y)} e^{-x/u(y)}. \tag{33}$$

Thus

$$w(x, y) + u(y) w_x(x, y) = f(y)\big(1 - e^{-x/u(y)}\big) + f(y) e^{-x/u(y)} = f(y),$$

as claimed. Differentiating (32) with respect to $y$ gives

$$w_y(x,y) = f'(y)\big(1 - e^{-x/u(y)}\big) - x\frac{u'(y)}{u(y)^2} f(y) e^{-x/u(y)}. \tag{34}$$

Therefore

$$\nabla w(x,y) = \Big(0,\ f'(y)\big(1 - e^{-x/u(y)}\big)\Big) + e^{-x/u(y)}\frac{f(y)}{u(y)}\Big(1, -x\frac{u'(y)}{u(y)}\Big). \tag{35}$$

Let $\tau = x/u(y)$. Then

$$\Big|\frac{e^{-x/u(y)}}{u(y)}\Big| = \Big|\frac{e^{-\tau}}{u(y)}\Big| = \Big|\frac{\tau e^{-\tau}}{x}\Big|. \tag{36}$$

Since $e^{-\tau} \le 1$ and $\tau e^{-\tau} \le e^{-1} \le 1$, for all $\tau \ge 0$, we have

$$\Big|\frac{e^{-x/u(y)}}{u(y)}\Big| \le \min\{|u(y)|^{-1}, x^{-1}\}. \tag{37}$$

Before proceeding, we introduce a small technical lemma.

**Lemma 5.1** *Suppose that* $a, b > 0$. *Then*

$$\min\{1/a, 1/b\} \le \frac{\sqrt{2}}{\sqrt{a^2+b^2}}. \tag{38}$$

***Proof*** Without loss of generality, assume that $a \le b$. Then $\min\{1/a, 1/b\} = 1/b = 1/\max\{a,b\}$. Similarly $\sqrt{a^2+b^2} \le \sqrt{b^2+b^2} = \sqrt{2}b = \sqrt{2}\max\{a,b\}$. Putting the above two observations together we have

$$(\min\{1/a, 1/b\})^{-1} = \max\{a,b\} \ge \frac{1}{\sqrt{2}}\sqrt{a^2+b^2}.$$

Inverting this gives the required bound. □

Applying the lemma to (37), we get

$$\Big|\frac{e^{-x/u(y)}}{u(y)}\Big| \le \frac{\sqrt{2}}{\sqrt{x^2+u(y)^2}}. \tag{39}$$

We conclude from (35) and (39) that the only singularities in $\nabla w$ occur at the boundary points of $\Gamma_-$. From the formula (30) for $u$, we conclude that

$$\Big|\frac{e^{-x/u(y)}}{u(y)}\Big| \le \frac{\sqrt{2}}{\sqrt{x^2+(1-y^2)^2}}. \tag{40}$$

Similarly,

$$\Big|e^{-x/u(y)}\frac{xu'(y)}{u(y)^2}\Big| = |u'(y)|\Big|\frac{e^{-\tau}\tau}{u(y)}\Big| = |u'(y)|\Big|\frac{e^{-\tau}\tau^2}{x}\Big|. \tag{41}$$

Note that $\tau^2 e^{-\tau} \le 4e^{-4} < e^{-1}$. Since $\sqrt{2}e^{-1} < 1$, we conclude from Lemma 5.1 that

$$\Big|e^{-x/u(y)}\frac{f(y)xu'(y)}{u(y)^2}\Big| \le |f(y)u'(y)|\big(x^2+(1-y^2)^2\big)^{-1/2}. \tag{42}$$

Thus

$$|\nabla w(x,y)| \le |f'(y)| + \sqrt{2}|f(y)|(1+|u'(y)|)\big(x^2+(1-y^2)^2\big)^{-1/2}. \tag{43}$$

Define $\Omega_\gamma = \{(x,y)\in\Omega \,\big|\, x<\gamma\}$. Note that

$$\int_{\Omega_\gamma} \big(x^2+(1-y^2)^2\big)^{-p/2}\,dx\,dy < \infty \tag{44}$$

for any $p<2$. Thus $\nabla w \in L^p$ for $p<2$ if $f$ is bounded. More generally, suppose that $1<s<2$, and define

$$t=\frac{2+s}{2-s},\quad t'=\frac{2+s}{2s} \quad\Longrightarrow\quad \frac{1}{t}+\frac{1}{t'}=1.$$

Hölder's inequality implies that

$$\begin{aligned}\int_{\Omega_\gamma} |f(y)|^s\big(x^2+(1-y^2)^2\big)^{-s/2}\,dxdy \le& \left(\int_{\Omega_\gamma}|f(y)|^{st}\,dxdy\right)^{1/t}\\ &\left(\int_{\Omega_\gamma}|x^2+(1-y^2)^2|^{-st'/2}\,dxdy\right)^{1/t'}.\end{aligned} \tag{45}$$

Define $r=st$ and

$$c_s=\sqrt{2}\left(\int_{\Omega_\gamma}|x^2+(1-y^2)^2|^{-st'/2}\,dxdy\right)^{1/st'}. \tag{46}$$

Note that

$$st'=\frac{2+s}{2}<2.$$

Thus (43), (45) and (46) combine to give

$$\|\nabla w\|_{L^s(\Omega_\gamma)} \le c_s\|f\|_{L^r(\Omega)} + \|\nabla f\|_{L^r(\Omega)}. \tag{47}$$

Poincaré's inequality and (47) imply that

$$\|w\|_{W^1_s(\Omega_\gamma)} \le c_P\|f\|_{W^1_r(\Omega)}. \tag{48}$$

Therefore, we claim the following result.

**Theorem 5.1** *Suppose that the assumptions of Theorem 4.1 hold on* $\mathbf{u}$*, and that* $s<2$*. Pick* $r=s(2+s)/(2-s)$ *and assume that* $f\in L^r(\Omega)$ *and* $\nabla f\in L^s(\Omega)^d$*. Suppose that either (a)* $\mathbf{u}\cdot\mathbf{n}=0$ *on* $\partial\Omega$ *or (b)* $\Omega$ *is an Amick domain with Poiseuille-like boundary conditions on the inflow and outflow boundaries. In the case (b), assume that* $f$ *is radially symmetric in the inflow and outflow pipes of the Amick domain. Then there is a constant* $c_s$ *depending on* $s$ *and* $\Omega$ *such that the unique solution* $w$ *of* (4) *satisfies* $w\in W^1_s(\Omega)^d$ *and*

$$\|\nabla w\|_{L^s(\Omega)} \le c_s\Big(\|f\|_{W^1_s(\Omega)} + \big(1+\|\mathbf{u}\|_{W^1_\infty(\Omega_\gamma)}\big)\|f\|_{W^1_r(\Omega)}\Big). \tag{49}$$

Note that in case (a), this follows from Theorem 4.1 since $\Gamma_-$ is the empty set. Note that Theorem 5.1 easily extends to Amick multidomains, in which there are multiple Poiseuille input and output channels or tubes. The constant $c_s$ in (49) is not the same as the one in (46), but they are related as indicated in the proof of Theorem 5.1.

### 5.2 Proof of Theorem 5.1 in 2D

We are nearly done with the proof of Theorem 5.1 in 2D, but we need to show how we can subtract the solution (32) and use Theorem 4.1. Let $\tilde{w}$ denote the solution of (4) and write

$$v = \tilde{w} - w\hat{\chi}, \quad \hat{\chi}(x, y) = (1 - 2x + x^2)\chi_{[0,1]}(x),$$

where $\chi_{[0,1]}$ is the characteristic function of the unit interval. Here we are assuming that the inflow domain is a channel of length $\gamma \geq 1$. If not, make the cut-off more abrupt. Note that $\| \hat{\chi} \|_{L^\infty(\Omega)} = 1$ and $\nabla\hat{\chi} = (-2 + 2x, 0)\chi_{[0,1]}$. Then

$$v + \mathbf{u} \cdot \nabla v = f - \big(w\hat{\chi} + u(w\hat{\chi})_x\big) = f(1 - \hat{\chi}) + uw\hat{\chi}_x = f(1 - \hat{\chi}) + uw(2 - 2x)\chi_{[0,1]}.$$

Applying Theorem 4.1, we find

$$\begin{aligned}
\| \nabla v \|_{L^q(\Omega)} &\leq \frac{1}{c_0} \| \nabla\big(f(1 - \hat{\chi}) + uw\hat{\chi}_x\big) \|_{L^q(\Omega)} \\
&\leq \frac{1}{c_0}\Big(\| \nabla f \|_{L^q(\Omega)} + 2\| f \|_{L^q(\Omega)} + \| \nabla(uw) \|_{L^q(\Omega_\gamma)} + 2\| uw \|_{L^q(\Omega_\gamma)}\Big) \\
&\leq C\big(\| f \|_{W^1_q(\Omega)} + \| \mathbf{u} \|_{W^1_\infty(\Omega_\gamma)} \| w \|_{W^1_q(\Omega_\gamma)}\big).
\end{aligned} \tag{50}$$

Note that

$$\| \nabla(\hat{\chi}w) \|_{L^q(\Omega)} \leq C\| w \|_{W^1_q(\Omega_\gamma)}. \tag{51}$$

From the definition of $v$ and the combination of (50) and (51), we find

$$\begin{aligned}
\| \nabla\tilde{w} \|_{L^q(\Omega)} &\leq \| \nabla v \|_{L^q(\Omega)} + \| \nabla(\hat{\chi}w) \|_{L^q(\Omega)} \\
&\leq C\Big(\| f \|_{W^1_q(\Omega)} + \big(1 + \| \mathbf{u} \|_{W^1_\infty(\Omega_\gamma)}\big)\| w \|_{W^1_q(\Omega_\gamma)}\Big),
\end{aligned} \tag{52}$$

for a possibly larger constant $C$. Combining with (48) gives

$$\| \nabla\tilde{w} \|_{L^q(\Omega)} \leq C\Big(\| f \|_{W^1_q(\Omega)} + \big(1 + \| \mathbf{u} \|_{W^1_\infty(\Omega_\gamma)}\big)\| f \|_{W^1_r(\Omega)}\Big), \tag{53}$$

which completes the proof.

### 5.3 Sharpness

Other bounds could be proved, especially ones with weights involving the distance to the boundary of the inflow boundary, $\partial\Gamma_-$. However, for standard Lebesgue spaces, we can show the restriction on $q$ is sharp as follows.

Taking $f \equiv 1$, we find

$$w_x(x, y) = \frac{e^{-x/u(y)}}{u(y)}, \qquad w_y(x, y) = -\frac{xe^{-x/u(y)}u'(y)}{u(y)^2},$$

which implies that (take $\tau = x/u(y)$)

$$\int_0^X |w_x(x, y)|^2\, dx = \int_0^X \frac{e^{-2x/u(y)}}{u(y)^2}\, dx = \frac{1}{u(y)} \int_0^{X/u(y)} e^{-2\tau}\, d\tau.$$

Thus as $u(y) \to 0$,

$$\int_0^X |w_x(x, y)|^2\, dx \approx \frac{1}{u(y)} \int_0^\infty e^{-2\tau}\, d\tau = \frac{1}{2u(y)}.$$

Thus

$$\int_{-1+\delta}^{1-\delta}\int_0^X |w_x(x,y)|^2\,dx \to \infty$$

as $\delta \to 0$. Thus we conclude that $\nabla w \notin L^2$.

### 5.4 Special *f*

Suppose that $f(y) = f(0,y) \le C_f u(y)$ for some positive constant $C_f$. Then (35) implies that

$$|\nabla w(x,y)| \le |f'(y)|\big(1-e^{-\tau}\big) + C_f e^{-\tau}\big(1+\tau|u'(y)|\big), \tag{54}$$

where $\tau = x/u(y)$. More generally, let $g(y) = f(y)/u(y)$. Then

$$\begin{aligned} |\nabla w(x,y)| &\le |f'(y)|\big(1-e^{-\tau}\big) + |g(y)|e^{-\tau}\big(1+\tau|u'(y)|\big) \\ &\le |f'(y)| + |g(y)|\big(1+|u'(y)|\big). \end{aligned} \tag{55}$$

These observations are similar to what is observed in Sect. 4.4.

### 5.5 Proof of Theorem 5.1 in 3D

We modify the notation in Sect. 5.1.

Suppose that the $x$-axis is the centerline for the input or output section of $\Omega$. Then Poiseuille flow takes the form $\mathbf{u}(x,y,z) = (u(r),0,0)$ in three dimensions, where $r = \sqrt{y^2+z^2}$. We write the input section as

$$\big\{(x,y,z)\;\big|\;0 \le x \le X,\;\; y^2+z^2 \le 1\big\}$$

after scaling the coordinates appropriately.

For Poiseuille flow, the transport equation takes the form

$$w + uw_x = f, \qquad w(0,y,z) = 0. \tag{56}$$

It suffices to assume that $f$ is independent of $x$, because if we analyze the case $f = f(y)$ we can apply Theorem 4.1 to $f(x,y) - f(0,y)$. As before, (56) is solved by

$$w(x,y,z) = f(r)\big(1-e^{-x/u(r)}\big), \qquad r = \sqrt{y^2+z^2}. \tag{57}$$

Differentiating (57) with respect to $r$ gives

$$w_r(x,r) = f'(r)\big(1-e^{-x/u(r)}\big) - x\frac{u'(r)}{u(r)^2} f(r)e^{-x/u(r)}. \tag{58}$$

Let $\tau = x/u(r)$. Then

$$\left|e^{-x/u(r)}\frac{f(r)}{u(r)}\right| = \left|e^{-\tau}\frac{f(r)}{u(r)}\right| = \left|\tau e^{-\tau}\frac{f(r)}{x}\right|. \tag{59}$$

Since $e^{-\tau} \le 1$ and $\tau e^{-\tau} \le e^{-1}$, applying (38) as before, we have

$$\left|e^{-x/u(r)}\frac{f(r)}{u(r)}\right| \le |f(r)|\min\{|u(r)|^{-1}, x^{-1}\} \le \sqrt{2}|f(r)|\big(x^2+u(r)^2\big)^{-1/2}. \tag{60}$$

Similarly,

$$\left|e^{-x/u(r)}\frac{f(r)xu'(r)}{u(r)^2}\right| = \left|e^{-\tau}\tau\frac{f(r)u'(r)}{u(r)}\right| = \left|e^{-\tau}\tau^2\frac{f(r)u'(r)}{x}\right|. \tag{61}$$

Thus

$$\left|e^{-x/u(r)}\frac{f(r)xu'(r)}{u(r)^2}\right| \le \sqrt{2}e^{-1}|f(r)u'(r)|\big(x^2+u(r)^2\big)^{-1/2}. \tag{62}$$

Therefore

$$|\nabla w(x,r)| \le |f'(r)| + \sqrt{2}|f(r)|(1+|u'(r)|)\big(x^2+u(r)^2\big)^{-1/2}. \tag{63}$$

Thus $\nabla w \in L^p$ for $p < 2$.

The cut-off argument is the same. Therefore, we have proved Theorem 5.1 in three dimensions.

## 6 Negative norms

To obtain bounds in negative norms, we need to consider the adjoint transport equation

$$\mathcal{C}\boldsymbol{\phi} - \mathbf{u}\cdot\nabla\boldsymbol{\phi} = \boldsymbol{\psi} \text{ in } \Omega, \qquad \boldsymbol{\phi} = \mathbf{0} \text{ in } \Gamma_+, \tag{64}$$

where we recall that $\Gamma_+$ is defined in (1). In particular, we will be interested in the case that $\boldsymbol{\psi} \in W_q^1(\Omega)^d$ and the resulting solution $\boldsymbol{\phi} \in W_q^1(\Omega)^d$. The equations (2) and (64) are essentially the same: we just switch flow directions $\mathbf{u} \to -\mathbf{u}$.

Suppose that $\mathbf{w} \in X_{\mathbf{u}}$ with $\mathbf{w} = 0$ on $\Gamma_-$. Then $\mathbf{u}\cdot\nabla\mathbf{w} \in L^2(\Omega)^d$. Let $\boldsymbol{\phi} \in H^1(\Omega)^d$ with $\boldsymbol{\phi} = 0$ on $\Gamma_+$. Then $(\mathbf{w}\cdot\boldsymbol{\phi})(\mathbf{u}\cdot\mathbf{n}) = 0$ on $\partial\Omega$. We have

$$\nabla\cdot\big(\mathbf{u}(\mathbf{w}\cdot\boldsymbol{\phi})\big) = (\mathbf{u}\cdot\nabla\mathbf{w})\cdot\boldsymbol{\phi} + (\mathbf{u}\cdot\nabla\boldsymbol{\phi})\cdot\mathbf{w} \tag{65}$$

in $\Omega$ since $\nabla\cdot\mathbf{u} = 0$ in $\Omega$. This holds as an expression in $L^1(\Omega)$. Thus the divergence theorem implies that

$$(\mathbf{u}\cdot\nabla\mathbf{w}, \boldsymbol{\phi}) = -(\mathbf{w}, \mathbf{u}\cdot\nabla\boldsymbol{\phi}). \tag{66}$$

Suppose that $\mathbf{w}$ solves (2) with $\mathbf{w} = 0$ on $\Gamma_-$. Then (66) implies

$$(\mathbf{w}, \mathcal{C}\boldsymbol{\phi} - \mathbf{u}\cdot\nabla\boldsymbol{\phi}) = (\mathbf{f}, \boldsymbol{\phi}).$$

Let $\boldsymbol{\psi} \in C_0^\infty(\Omega)^d$ be any test function. Apply (18) and (29) to the solution of (64) with data $\boldsymbol{\psi}$, namely the function $\boldsymbol{\phi}$ satisfying

$$\mathcal{C}\boldsymbol{\phi} - \mathbf{u}\cdot\nabla\boldsymbol{\phi} = \boldsymbol{\psi} \text{ in } \Omega, \qquad \boldsymbol{\phi} = 0 \text{ on } \Gamma_+, \qquad \|\,\boldsymbol{\phi}\,\|_{W_q^1(\Omega)} \le c_q\|\,\boldsymbol{\psi}\,\|_{W_q^1(\Omega)},$$

where $c_q = 2^{(q-1)/q}/c_0$, cf. (69). Then

$$\frac{(\mathbf{w}, \boldsymbol{\psi})}{\|\,\boldsymbol{\psi}\,\|_{W_q^1(\Omega)}} = \frac{(\mathbf{f}, \boldsymbol{\phi})}{\|\,\boldsymbol{\psi}\,\|_{W_q^1(\Omega)}} \le \frac{c_q(\mathbf{f}, \boldsymbol{\phi})}{\|\,\boldsymbol{\phi}\,\|_{W_q^1(\Omega)}}.$$

Thus we conclude that

$$\|\,\mathbf{w}\,\|_{(W_q^1)'(\Omega)} \le c_q \sup_{\mathbf{0}\neq\boldsymbol{\phi}\in W_{q+}^1(\Omega)^d} \frac{(\mathbf{f}, \boldsymbol{\phi})}{\|\,\boldsymbol{\phi}\,\|_{W_q^1(\Omega)}}, \tag{67}$$

where $W^1_{q+}(\Omega) = \left\{ v \in W^1_q(\Omega) \;\middle|\; v = 0 \text{ on } \Gamma_+ \right\}$.

Thus we have proved the following.

**Theorem 6.1** *Under the conditions of Theorem 5.1, there is a constant $c_q$ such that*

$$\| \mathbf{w} \|_{(W^1_q)'(\Omega)} \le c_q \sup_{\mathbf{0} \ne \boldsymbol{\phi} \in W^1_{q+}(\Omega)^d} \frac{(\mathbf{f}, \boldsymbol{\phi})}{\| \boldsymbol{\phi} \|_{W^1_q(\Omega)}}. \tag{68}$$

## 7 Conclusions and questions

We have shown that transport equations with inflow boundary conditions may be posed in Sobolev spaces. We showed that for simple problems like Poiseuille flow there is a singularity at the boundary of the inflow section of $\partial\Omega$ that arises naturally, due to the fact that $\mathbf{u}$ vanishes there. We extended this observation to prove a general result on Amick domains. We reviewed an example of Bernard that shows that the gradient of the solution of the transport equation is not even integrable if $\mathbf{u}$ vanishes to second order there.

Since the proof of bounds in Sect. 4.3 do not require the assumption of Bernard (3), it is an interesting question whether this assumption is essential for Theorem 3.1.

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## A Spaces

Here we collect the notation used for various Sobolev spaces and norms. We denote by $L^p(\Omega)$ the Lebesgue spaces [7] of $p$-th power integrable functions, with norm

$$\| f \|_{L^p(\Omega)} = \left( \int_\Omega |f(\mathbf{x})|^p \, dx \right)^{1/p}.$$

Note that we can easily apply the same notation to vector or tensor valued $f$. We think of tensors of any arity as vectors of the appropriate length, and we think of $|f(\mathbf{x})|$ as the Euclidean length of this vector. For tensors of arity 2 (i.e., matrices) this is the same as the Frobenius norm. We will write the spaces for such tensor-valued functions as $L^p(\Omega)^m$ for the appropriate $m$ (e.g., $m = d^2$ for arity 2). Similarly, we denote by $L^\infty(\Omega)$ the Lebesgue space of essentially bounded functions, with

$$\| f \|_{L^\infty(\Omega)} = \sup \left\{ |f(\mathbf{x})| \;\middle|\; \text{a.e. } \mathbf{x} \in \Omega \right\}.$$

Correspondingly, we define Sobolev spaces and norms of order $m$ by

$$\| f \|_{W^m_p(\Omega)} = \left( \sum_{|\alpha| \le m} \| D^\alpha f \|^p_{L^p(\Omega)} \right)^{1/p},$$

where $D^\alpha$ is the weak derivative $\partial^\alpha / \partial \mathbf{x}^{|\alpha|}$ [7]. More precisely, the spaces $W^m_p(\Omega)$ are defined as the subspaces of $L^p(\Omega)$ for which the corresponding norm is finite. The case $p = 2$ is denoted by $H$:

$$H^m(\Omega) = W^m_2(\Omega).$$

Correspondingly, we define

$$\| \nabla^m f \|_{L^p(\Omega)} = \left( \sum_{|\alpha|=m} \| D^\alpha f \|^p_{L^p(\Omega)} \right)^{1/p}.$$

Note that

$$\| f \|_{W^1_p(\Omega)} \le \| f \|_{L^p(\Omega)} + \| \nabla f \|_{L^p(\Omega)} \le 2^{(p-1)/p} \| f \|_{W^1_p(\Omega)}. \tag{69}$$

We will briefly use the space $H^1_0(\Omega)$ of $f \in H^1(\Omega)$ such that $f = 0$ on $\partial\Omega$. The dual space $H^{-1}(\Omega)^d$ is the set of Schwartz distributions [24] for which the dual norm

$$\| \mathbf{u} \|_{H^{-1}(\Omega)} = \sup_{\mathbf{0} \neq \boldsymbol{\phi} \in H^1_0(\Omega)^d} \frac{\langle \mathbf{u} \cdot \boldsymbol{\phi} \rangle}{\| \boldsymbol{\phi} \|_{H^1(\Omega)}}$$

is finite.

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