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Design of virtual reality modules for multivariable calculus and an examination of student noticing within them

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ABSTRACT

Virtual reality (VR) research in mathematics education has centred largely on geometry content. This paper contributes by describing VR modules developed for another area that heavily involves three-dimensional objects: multivariable calculus. This paper also contributes by describing an empirical study on students' experiences inside the VR modules through student (conceptual) noticing. One finding was that colourful objects had moderate associated conceptual noticing, but that accompanying animation drastically improved the conceptual noticing. Symbolic and textual elements were conceptually noticed much less. While narration was meant to guide the students' noticing, it often did not produce the conceptual noticing intended, though animation again was a key factor. The conceptual noticing was also found to be connected to the students' emerging understandings of the mathematical ideas discussed in the modules. We end by discussing implications for (a) designing VR modules generally, and for (b) learning specific mathematical content within VR.

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1. Introduction

Virtual reality (VR) has created interest from the education community regarding its potential for improved learning (see Kaufmann, 2011). As the technology becomes more affordable, available, and portable, the opportunities for its use are increasing (Martin-Gutiérrez et al., 2017). Research has demonstrated that using VR modules can have positive benefits for student understanding and reasoning (Merchant et al., 2014). While several studies have discussed VR in education more broadly (e.g. see Cheng & Tsai, 2013), there are still relatively few studies within mathematics education that have examined VR. Further, to date, most of this research on VR in mathematics education has focused on geometry content (e.g. Hwang & Hu, 2013; Ibili et al., 2020; Kaufmann et al., 2000; Radu et al., 2015; Song & Lee, 2002; Trien Do & Lee, 2007; Yeh & Nason, 2004). This focus makes sense because geometry often deals with three-dimensional (3-D) objects and shapes that would be easier understood in a 3-D environment. Yet, geometry is not the only mathematical area dealing with significant 3-D

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content. Multivariable calculus also heavily involves many 3-D objects, such as 3-D surfaces, 3-D curves, and 3-D regions. This paper contributes to the existing literature on VR in mathematics education in two ways. First, it describes the research-informed design of VR modules in multivariable calculus, which are focused on the topics of multivariable functions, contour plots, and partial derivatives. Second, it relates an empirical study that examined students' experience within the modules through the construct of *student noticing* (see Section 3 for the research questions).

2. Research-informed design of VR modules for multivariable calculus

In this section, we first review literature from three main areas pertinent to the development of the modules: VR in mathematics education, broader aspects of visual displays in education, and the specific multivariable calculus content that the VR modules deal with. We then explain how this literature is connected to the design of the multivariable calculus modules, and present brief overviews of the modules themselves.

2.1. Literature review

2.1.1. Virtual reality within mathematics education

Virtual reality (VR) is an immersive 3-D environment containing objects, visuals, animations, or other entities, for student exploration. It is closely related to "virtual manipulatives" and "augmented reality," where virtual manipulatives are externally represented on a computer screen and augmented reality superimposes objects onto the person's real-world surroundings (see Bujak et al., 2013; Moyer et al., 2002). These kinds of virtual environments have been shown to be successful in improving students' learning and understanding (Cheng & Tsai, 2013; Merchant et al., 2014; Reimer & Moyer, 2005), by permitting objects to be viewed from different perspectives (Kaufmann, 2011) and supporting spatial reasoning (see also Gutiérrez de Ravé et al., 2016). For instance, Ibili et al. (2020) found that virtual environments helped their grade 5–8 students recognise and create 3-D shapes, and identify their properties. The students were also more successful at calculating volume or area and comparing features of different 3-D shapes. Song and Lee (2002) found that their students using VR were better able to identify conditions for geometric properties, such as information needed to define a unique plane, or conditions for orthogonality between a plane and a line. Yeh and Nason (2004) found that by engaging in VR, their students consolidated geometric language with geometric concepts, such that the VR helped them create semiotic meanings for mathematical terminology or symbols. In addition to gaining conceptual understanding, VR has also been found to increase students' motivation and engagement (Martin-Gutiérrez et al., 2017), and students' affective feelings toward the subject (Hwang & Hu, 2013).

Of course, these types of applications have limitations, too. VR, specifically, does not necessarily provide the chance for "physical" manipulations of objects in the real world (see Bujak et al., 2013). Further, Kerawalla et al. (2006) noted that an augmented reality module used for children's science lessons actually resulted in reduced student engagement. They concluded that it is important to ensure that modules appropriately guide students, so that they can fit within curricular constraints and so that intended learning is maximised.

In mathematics education specifically, despite one early example of VR for beginning algebra (Winn & Bricken, 1992), most VR research has centred on geometry content (Gutiérrez de Ravé et al., 2016; Hwang & Hu, 2013; Ibili et al., 2020; Kaufmann et al., 2000; Radu et al., 2015; Song & Lee, 2002; Trien Do & Lee, 2007; Yeh & Nason, 2004). This makes sense given that VR can allow students to investigate 3-D objects that are not easily described verbally or with two-dimensional (2-D) images (Song & Lee, 2002). Some geometry VR apps include environments where students mostly view geometric objects and examine their relationships (e.g. Kaufmann et al., 2000), while others have students solve problems regarding scaling or positioning and rotating (e.g. Yeh & Nason, 2004).

2.1.2. Design of visuals

We now recap research on the design of visuals in education, such as in textbooks or digital displays. While much of this research is outside of mathematics education, certain results have relevance to the design of a mathematics VR module. First, some of this research has claimed important benefits in addressing aesthetic aspects of visuals (Girod et al., 2010; Jakobsen & Wickman, 2008). These studies claim that having positive aesthetic experiences fosters more enduring learning, where students begin to make easier connections between the concept being learned and contexts outside of the classroom (Girod et al., 2010). However, almost to the contrary, Jaeger and Wiley (2014) showed that purely decorative imagery leads to poorer comprehension. Lee (2010) noted that while there is a significant increase in visuals in science textbooks over time, much of that increase is given to photographs that do not necessarily support student learning. He explains that imagery has shifted some away from "scientific accuracy" to "social connections" to the reader. Taken together, these studies point to a need for a sense of the aesthetic in designing visuals, while at the same time adhering to the key abstract elements that the visuals are meant to represent.

Next, Renkl and Scheiter (2017) explained that one major difficulty students face in making sense of visual displays is poor allocation of attention. The authors suggest that segmenting the information into smaller pieces can be helpful. Prompts for what to examine in a visual were also helpful. One method of doing so is what Goodwin (1994) and Lobato et al. (2013) call "highlighting," where actions are done to make particular features prominent or salient. Other researchers have corroborated the need for scaffolding student usage of visuals (Jaeger & Wiley, 2014; Ross et al., 2017; Van Horne et al., 2016). Jaeger and Wiley (2014) stated that, without guidance, text with diagrams was not any more impactful for learning than pure text. For dynamic imagery, Van Horne et al. (2016) explained that usage should be adopted early in the course and instructors should assist students in identifying what to see.

2.1.3. Multivariable calculus

Research in multivariable calculus education has given particular focus to the teaching and learning of multivariable functions (e.g. $f(x,y)$) and their graphs (Dorko & Weber, 2014; Kabael, 2011; Martinez-Planell & Gonzales, 2012; Trigueros & Martinez-Planell, 2010; Weber & Thompson, 2014; Yerushalmy, 1997). It has also begun to examine the key calculus ideas of partial derivatives (Martinez-Planell et al., 2014, 2015; McGee &

Moore-Russo, 2015; Weber, 2015) and integrals (Jones, 2020; Jones & Dorko, 2015; McGee & Martinez-Planell, 2014).

Trigueros and Martinez-Planell (2010) noted that instructors might assume that generalising single-variable functions and graphs in $\mathbb{R}^2$ to functions and graphs in $\mathbb{R}^3$ is easier than it is. Dorko and Weber (2014) found that previous understandings of domain and range based on inputs and outputs helped students make better sense of domain and range for multivariable functions, $z = f(x,y)$, as opposed to symbol-based interpretations that typically cast y as a dependent variable in the students' minds. Martinez-Planell and Trigueros-Gaisman (2012) explained that graphs in $\mathbb{R}^3$ must be conceptualised in their entirety as objects in order for students to do additional work, such as identifying intersection curves between a graph and a plane or between two graphs. Weber and Thompson (2014) suggested that previously encountered graphs, like the graph of , need to be flexibly manipulated to imagine graphs of multivariable functions like $f(x, y) = y\sin(x)$. Students need to connect different representations, including symbolic function expressions and the visual graphs, such as connecting $f(x, y) = x^2 + y^2$ to a paraboloid (Kabael, 2011; Yerushalmy, 1997).

Another result in the multivariable calculus literature is that many students who successfully complete the course do not have deep understandings of partial derivatives (Martinez-Planell et al., 2014), because of their reliance on memorised facts instead of reasoning about the meaning of a partial derivative. McGee and Moore-Russo (2015) added that it may be crucial for students to explicitly see graphical, numeric, and algebraic representations of partial derivatives, to develop understanding of slope as related to multivariable functions. Further, Martinez-Planell et al. (2015) explain that students need to construct the idea of vertical change versus horizontal changes, in both the x and y directions. This is related to what Weber (2015) calls the "two-change" problem: that there is a potential change in the z coordinate with respect to both the x variable and the y variable. Some students thought that there should only be a single "rate of change value" for a given point, and were confused at the prospect of two (or more) rates of change of z at the same point.

2.2. Connections between literature and design of the VR modules

Because most VR work in mathematics has been done in geometry (e.g. Hwang & Hu, 2013; Ibili et al., 2020; Radu et al., 2015; Song & Lee, 2002), this paper contributes by expanding VR work into multivariable calculus. We focus on VR modules specifically for (a) multivariable functions, (b) contour plots, and (c) partial derivatives, due to the literature base around these topics, as described in the previous subsection. In line with the VR literature, our modules develop conceptual understanding through visual reasoning (see Gutiérrez de Ravé et al., 2016; Kaufmann, 2011; Merchant et al., 2014), by introducing geometric meanings first *before* computation, and then showing symbolic and numeric representations connecting to these meanings (Martinez-Planell et al., 2014; McGee & Moore-Russo, 2015).

Within this conceptual orientation, the modules connect symbolic expressions to their visual counterparts by providing an interface where both were presented and discussed simultaneously (Kabael, 2011; Yerushalmy, 1997). Topics that were reported as challenging in the literature were specifically targeted in these modules. The modules develop generalisations from single-variable functions to multivariable functions through extending domain and range – specifically portraying y as now being an input (Dorko & Weber, 2014) – and connecting to graphical representations (Martinez-Planell & Trigueros-Gaisman, 2013). The modules also examine relationships between 3-D objects (Martinez-Planell & Trigueros-Gaisman, 2012), such as intersections. The modules separate rates of change of z with respect to x and y, to show how the two partial derivative computations are associated with more than one possible "rate" of the function at a given point (Weber, 2015).

The modules make use of orienting narration to help guide the students' attention to the major conceptual ideas (Kerawalla et al., 2006). They were designed with aesthetics in mind through colours, familiar shapes, and animations in order to promote visual reasoning and making connections (Girod et al., 2010; Jakobsen & Wickman, 2008). However, because of the cautions given by Jaeger and Wiley (2014), the aesthetic elements are not merely decorative, but are all geared toward a major focus on clear and accurate information about the actual multivariable calculus concept being examined. Colouring, narration, and some animation was used to highlight specific mathematical elements within the modules (Goodwin, 1994; Lobato et al., 2013).

In line with Renkl and Scheiter (2017), our modules were partitioned into small "pieces", and the students' attention was directed to the key ideas through each piece's narration (Jaeger & Wiley, 2014). Further, the modules we focus on in this paper are part of an entire suite of VR modules meant to be used through a multivariable calculus class, meaning the students adopt the VR early on and the instructor guides the students' early interactions with it (Van Horne et al., 2016).

2.3. The three multivariable calculus VR modules

Having recapped the literature and its relations to the design of the multivariable calculus VR modules, we now give brief descriptions of the actual modules. The modules were developed through a National Science Foundation grant (NSF DUE-1820724), wherein a set of modules were created for an entire multivariable calculus course. The three modules focused on in this paper (multivariable functions, contour plots, and partial derivatives) were a part of this larger set. The set of modules was developed in 2017–2018 by the second and third authors (Long and Becnel). The first author (Jones) was brought in as part of an effort to study students' experiences within the modules. The modules were meant to be experienced by the students outside of class prior to the in-class lessons that would discuss that particular topic. Thus, the modules served as introductions to the topics and were consequently fairly brief and focused on the geometric and conceptual ideas. The ideas from the modules would then be further developed in the next class session.

The modules all had a similar structure. A module was broken down into three or four short "pieces" that were each meant to be experienced in less than five minutes (see Renkl & Scheiter, 2017). Some pieces needed only two to three minutes to experience. Thus, one entire module was meant to take 10-15 min to experience. Every piece opened with some narration intended to provide some guidance for the students in noticing certain ideas (see Kerawalla et al., 2006). The modules then typically had space for the student to explore, with some additional narration or a pop-up question used at times to again help guide the students (see Jaeger & Wiley, 2014). At any time, the viewer could restart the piece, rewind, or skip the narration.

2.3.1. Module 1: multivariable functions

The multivariable functions module begins with a piece in which some narration recaps graphs of familiar one-variable functions (Figure 1a) and explains how these types of functions can be extended to functions of more than one variable (Figure 1b). The relation between the *x* and *y* inputs (as a point in the *x-y* plane) and the output is represented by a yellow vertical line rising from the (*x,y*) point up to the red graph. The students are able to examine different surfaces (Figures 1b and c) in terms of how different input points map to the corresponding output.

The second piece starts with narration about the domain being the set of applicable *x-y* inputs, with an example given of a semi-spherical graph (Figure 2a). The module depicts this domain as a yellow circle with a green interior in the *x-y* plane (Figure 2b). Students can view different surfaces, such as the hemi-sphere in Figure 2a or the logarithmic graph in Figure 2c. They can manipulate the view to essentially project the graph into the *x-y* plane, creating a region corresponding to the 2-D domain, and then identify (*x,y*) points where the function exists or does not exist. The intention of this short piece is for students to expand their meaning of domain from an interval on the *x*-axis to an entire region of (*x,y*) points in $\mathbb{R}^2$.

The third piece narrates how the one-output rule of functions still applies to multivariable functions. Students are able to examine different surfaces (Figures 3a and b) to judge whether *z* represents a function of *x* and *y*. In doing so, an animation in Figure 3a populates the graphs with a "vertical line test" and an animation in Figure 3b shows multiple potential outputs with a single (*x,y*). A pop-up question

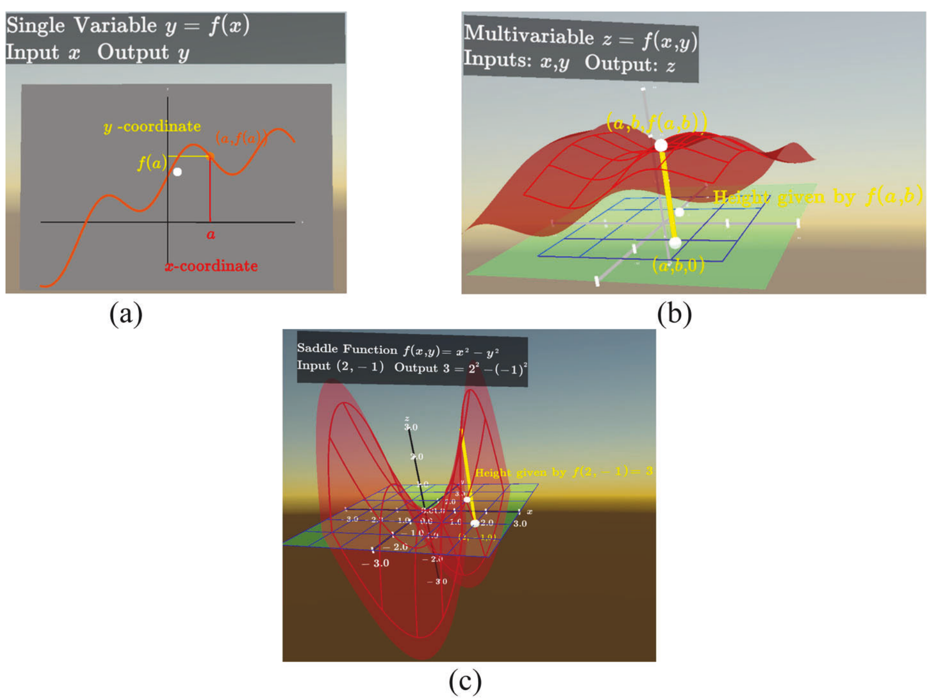


Figure 1. Screenshots from the first piece of the multivariable functions module. Reproduced with permission by the CalcVR Project. Copyright 2021, SFASU.

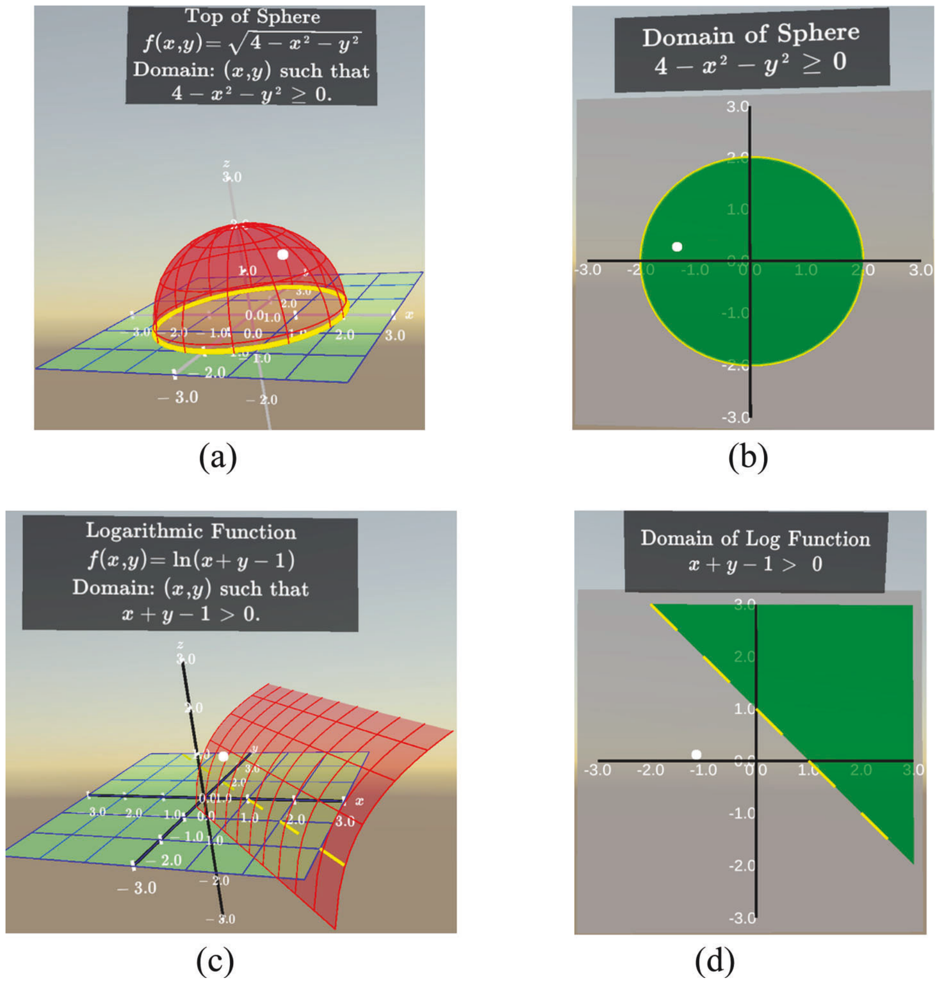
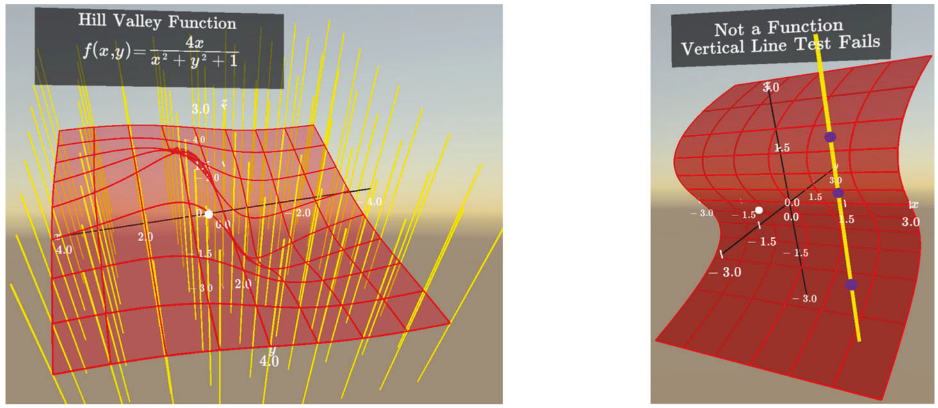


Figure 2. Screenshots from the second piece of the module. Reproduced with permission by the CalcVR Project. Copyright 2021, SFASU.

asks the students whether the surface in Figure 3b could represent x as a function of y and z. The goal of this piece is to help students see that multiple inputs still associate with a single output, but the variable that is considered a function of the others is malleable.

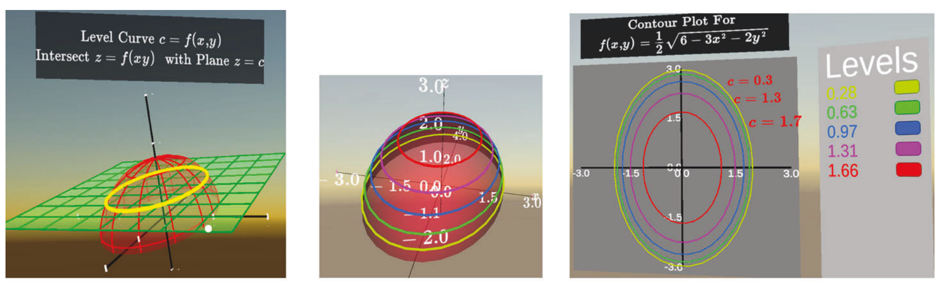
2.3.2. Module 2: contour plots

The contour plot module extends the multivariable functions module by discussing the representation of contour plots. The first piece begins with an image of a weather map and a topographical map and some narration recapping how these are 2-D representations of 3-D structures. A red semi-ellipsoid graph is then presented where students induce an animation of a green horizontal plane “slicing through” the red graph (Figure 4a). A yellow ellipse is traced out the intersection. The student can view different level curves for different heights of the graph (Figure 4b). After doing so, a



(a) (b)

Figure 3. Screenshots from the third piece of the multivariable function module. Reproduced with permission by the CalcVR Project. Copyright 2021, SFASU.



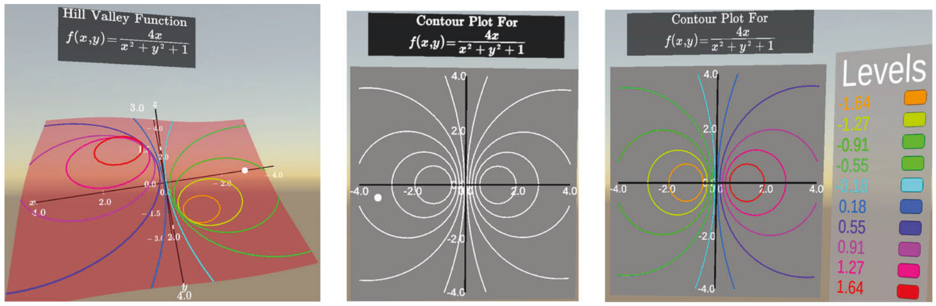
(a) (b) (c)

Figure 4. Screenshots from the first piece of the contour plot module. Reproduced with permission by the CalcVR Project. Copyright 2021, SFASU.

separate panel shows all of the same cross sections placed onto a single *x-y* plane (Figure 4c). The purpose of this piece is to help students make connections between the 2-D contour plot and the 3-D surface it is representing.

The second piece shows a "hill-valley" graph with coloured traces on it (Figure 5a). A contour plot with only grey level curves is shown next to it (Figure 5b), and the students are asked which part of the contour plot corresponds to the "hill" and which part corresponds to the "valley." The students can then switch on a coloured version of the contour map to identify matches between it and the original graph (Figure 5c). The simple goal is for students to understand the necessity of labels in a contour plot.

The final piece begins with narration that level curves can be helpful for non-function surfaces, too. A hyperboloid is shown where students can identify different level curves at different $z = c$ heights (Figure 6a). These level curves are again reproduced in a single *x-y* plane (Figure 6b).



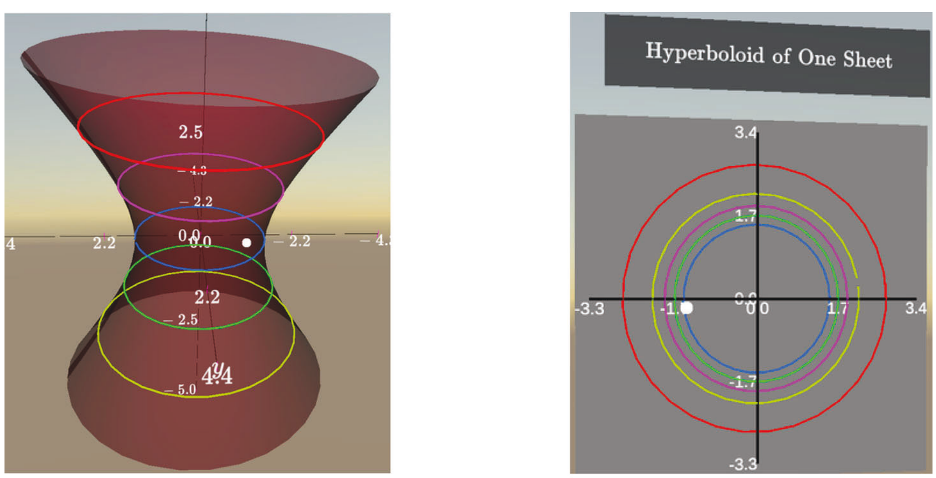
(a) (b) (c)

Figure 5. Screenshots from the second piece of the contour plot module. Reproduced with permission by the CalcVR Project. Copyright 2021, SFASU.

2.3.3. Module 3: partial derivatives

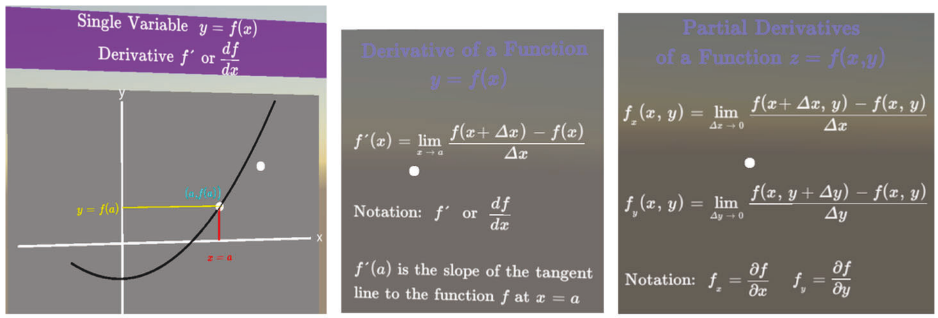
The partial derivative module is more heavily narrated at the beginning, with narration recapping derivatives for single-variable functions (Figure 7a), the limit definition (Figure 7b), and how that definition is extended to partial derivatives (Figure 7c).

The second piece then shows a red graph with "Partial derivative with respect to x" in the textbox. At a point on the graph, the student starts an animation of a blue vertical plane "slicing through" the graph at that point parallel to x-axis (Figure 8a). The student can examine what the intersection of the plane and red graph looks like and a turquoise curve is animated tracing out that intersection (Figure 8b). The same turquoise curve is traced out in an x-z plane to the right (Figure 8c). A yellow tangent line is drawn on both the 3-D and 2-D images (Figure 8b and c). The goal of this piece is for students to see how cross sections permit ideas about single variable derivatives to generalise to multivariable functions. A third piece of the module does a similar thing for the partial derivative with respect to y, using a green plane and a green curve.



(a) (b)

Figure 6. Screenshots from the third piece of the contour plot module. Reproduced with permission by the CalcVR Project. Copyright 2021, SFASU.



(a) (b) (c)

Figure 7. Screenshots from the first piece of the partial derivative module. Reproduced with permission by the CalcVR Project. Copyright 2021, SFASU.

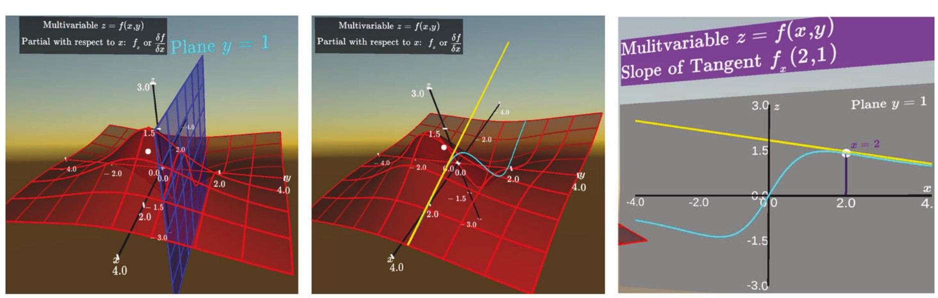
In the fourth and final piece, students can move to a different function and graph, $f(x, y) = x^2 - y^2/2 + 4$. The students can examine either the x partial derivative or y partial derivative. Similar animations show cross-sections of the curve parallel to those axes, as well as the tangent lines (Figures 9a and b). The cross sections and tangent lines are also reproduced on an x-z or y-z plane (Figures 9a and b). The symbolic computation of the partial derivative is shown in a textbox, and students are asked to compare the outcome of the computation to the visible tangent line. The point of this final piece is to help solidify the idea of cross sections and of a derivative being represented by the slope of a line tangent to that cross section.

3. A study on students' experience through the lens of "noticing"

Having described the three modules on multivariable functions, contour plots, and partial derivatives, we now pivot to relate a study that was done on students' experience within the modules, primarily in terms of their *noticing*.

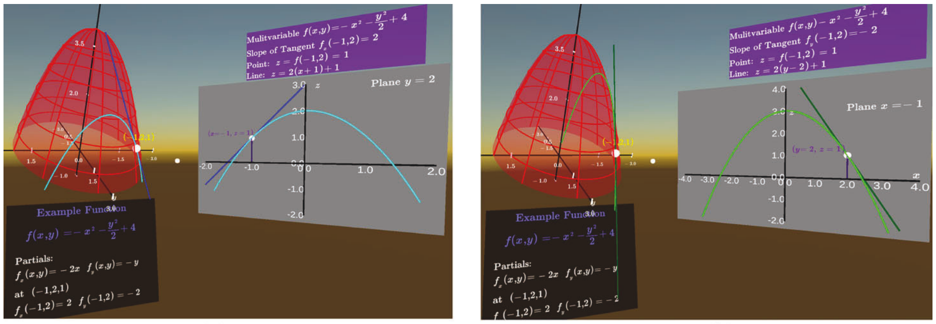
3.1. Theoretical framework: student noticing

In our study, we explored students' experience through the construct of *student noticing*, put forward by Lobato and her colleagues (Lobato et al., 2013; Lobato et al., 2003; Lobato



(a) (b) (c)

Figure 8. Screenshots from the second piece of the partial derivative module. Reproduced with permission by the CalcVR Project. Copyright 2021, SFASU.



(a) (b)

Figure 9. Screenshots from the fourth piece of the partial derivative module. Reproduced with permission by the CalcVR Project. Copyright 2021, SFASU.

et al., 2012). This framework seeks to pay more attention to what *students* see as relevant, as opposed to only what researchers and experts *expect* students to see. Student noticing is important in understanding how students themselves experience a learning environment, and how that relates to their understanding. For example, Lobato et al. (2003) realised that a "slope" task with evenly spaced x intervals led some students to notice only the changes in y, consequently coming to believe that slope essentially is only the change in y. Lobato et al. (2012) then explained that a different task with unevenly spaced x intervals helped students notice both quantities in conceptualising slope. Because VR environments are based on students' personal experience with representations of mathematical ideas, it is critical to understand what they notice (or not) within these environments.

Student noticing is defined as "selecting, interpreting, and working with particular mathematical features or regularities when multiple sources of information compete for students' attention" (Lobato et al., 2012, p. 438). Noticing is based on *centres of focus*, which are the "features, regularities, properties, or conceptual objects to which individual students attend" (Lobato et al., 2012, p. 439). In other words, centres of focus are what students actually notice within an environment.

However, simply seeing something does not necessarily mean it is impacting learning. Important to this study, we make a distinction between *perceptual* noticing of a centre of focus versus *conceptual* noticing of it. While this distinction originates in Lobato et al.'s (2012; 2013) work, we expand on this idea here. We define *perceptual noticing* as simply observing the existence or presence of an object or its characteristics. This type of noticing deals with the basic identification, without necessarily operating on it cognitively to make judgments, assertions, or inferences. In our view, *perceptual noticing* aligns more with the simpler "selecting" part of Lobato et al.'s (2012) definition of noticing. An example might be a student saying, "I see a green horizontal plane sliding into my view." Here, the student acknowledges the presence of the plane and its characteristics "green" and "moving." On the other hand, we define *conceptual noticing* as associating the object with a meaningful conceptual idea, a connection to another idea or object, or an inference. This type of noticing relates to internal cognitive work that attaches some meaning or inference to the perceived object. An example might be a student saying, "That green horizontal plane shows me the cross section of the graph that corresponds to $z = 2$." In this case, the student has operated conceptually on the plane by connecting it to the graph and inferring the intersection as representing output values of 2. In our view, *conceptual noticing* aligns more with the "interpreting and working with" part of Lobato et al.'s (2012) definition. Because the VR modules were designed to enhance learning, we wanted to tease apart perceptual versus conceptual noticing, in order to see what elements instigated the deeper conceptual noticing. Thus, our research questions, stated at the end of this subsection, are focused on conceptual noticing.

One key aspect of the framework is what Lobato et al. (2012) call *features of the tasks*. In this paper we call them "*features of the modules*" to better align the term with the VR modules we examined. *Features* are the parts of the modules that might lead students to notice one thing versus another. For example, if an animation shows an object intersecting with another object, that animation might increase the chances of that intersection being *noticed*. Similarly, if an object is brightly coloured, that feature might more readily call a student's attention to it. Other examples might be if one object appears before another, or if an object moves. These features relate to what Goodwin (1994) and Lobato et al. (2013) call "highlighting," which is making "specific phenomena in a complex perceptual field salient by marking them in some fashion" (Goodwin, 1994, p. 606).

Lobato et al.'s (2012) framework also has the idea of *focusing interactions*, which refer to "discursive practices (conceived broadly to include gesture, diagrams, and talk) that can give rise to particular centres of focus" (p. 440). They clarify that *focusing interactions* relate to "mak[ing] particular information prominent or salient for others" (p. 440). In this way, focusing interactions could be considered a particular type of highlighting, but one that is specific to real-time communication. Because the VR modules examined in this study were to be experienced by students *before* classtime, there is no surrounding classroom discourse. However, the modules do include narration, meaning it is possible for the narration itself to provide some focusing interactions for the student, by making certain items in the modules prominent or salient. Possible examples include verbally referring to an object in the student's field of view or reading a symbolic expression from a textbox.

Having explained the framework we utilised for this study, we can now articulate our research questions using its terminology: (1) How do the features of the modules relate to what was conceptually noticed or not? (2) How does the narration's focusing interactions relate to what was conceptually noticed or not?

3.2. Student participant sample

To explore students' experiences within the multivariable calculus modules, we interviewed students enrolled in a multivariable calculus course that used the VR modules. In our recruitment, we attempted to achieve some purposeful diversity in the sample (Patton, 2002). At the beginning of the semester, a two-question survey was given to the students to obtain simple background data. One question posed a problem and asked the student whether they would rather solve it using a visual/geometric approach or a symbolic/algebraic approach. The other question asked how much their prior calculus classes used technology. We also tried to ensure that we had both female and male student participants. In the end, we chose a group of five students that had some of this diversity, whom we label in this paper Students A, B, C, D, and E. Table 1 summarises the background characteristics of these five students.

3.3. The interview

We used a semi-structured interview protocol for our study (Smith, 1995). The basic interview structure consisted of the student participant and the interviewer viewing together the three modules described in Section 2.4 (each using their own VR headset). After each short piece of the module, the interviewer asked the student to discuss what they had noticed and what they thought the module was conveying. The students were free to talk about whatever they wanted to regarding the modules. The interviewer regularly followed up on student responses to make sure that they explained as much as possible about what they were noticing in the modules. Also, if there were parts of the module that a student did not mention, the interviewer brought them up. Doing so allowed the students to sometimes discuss elements of the module they had in fact noticed (according to their own statements), but had neglected to talk about at the time. However, if it was clear a student had not previously noticed the element, and discussed it only because the interviewer asked about it, that was *not* marked as student noticing. For some of these instances, the student directly stated they had not noticed it before. For a few other instances, it was clear because, after being asked about the element, the student paused for a long time and made statements such as "I'm not sure." However, we erred on the side of *retaining* student explanations and only excluding them if there was good evidence that the student had not done that noticing before being asked.

At the end of a module, the interviewer asked the student to explain what they thought the module was teaching about the concept. They were also asked to describe what they felt they understood about the concept from the module. At the conclusion of the first module (multivariable functions), the interviewer then directed the student to the second module (contour plots). The same procedure described above was employed for this module. After the second module, the interviewer directed the student to the third module (partial derivatives). The same procedure was used for this module as well. The interviews were video recorded and transcribed. Overall, each interview took approximately one hour for the student go through and discuss all three modules.

3.4. Analysis

Analysis was based on the framework by examining the *centres of focus* students noticed, the *features of the modules* that corresponded (or not) to conceptual noticing, and the influence of the narration's *focusing interactions* on conceptual noticing (Lobato et al., 2012; Lobato et al., 2013). The first phase of analysis involved coding (see Bogdan & Biklen, 2007) for centres of focus. To prepare for this first phase, we went through the three modules to list out all of what we called the *potential* centres of focus. These were the specific objects, symbols, shapes, figures, and so forth, within the modules that

Table 1. Background on the interviewed students.

Student	Female/Male	Reasoning preference	Prior technology use
A	M	Algebraic	Moderate
B	F	Geometric	Low
C	F	Geometric	Moderate
D	F	Algebraic	Low
E	M	Algebraic	Extensive

students could pay attention to that had some sort of relevance to the mathematical idea being taught (see Figures 1–9 in Section 2.3). These potential centres of focus became a priori codes. Each student explanation was coded according to the potential centre of focus the student attended during for that explanation. Each explanation was also coded according to whether the noticing was *perceptual* or *conceptual* (see Section 3.1).

The second phase of analysis involved examining the features of the modules as they related to students' conceptual noticing. To prepare for this phase, we took the results from the first phase and identified those that had more instances of conceptual noticing, and those that had less instances of conceptual noticing. For centres of focus with more conceptual noticing, we looked for themes in terms of the modules' features that corresponded to those objects. For those with less conceptual noticing, we also looked for themes in terms of features associated with those less-noticed objects.

The third phase of analysis consisted of examining the narration's possible focusing interactions in terms of conceptual noticing. To do so, we looked specifically at the student noticing that overlapped with the narration. We looked for patterns in terms of which of those centres of focus were more or less conceptually noticed, and what relationship that noticing had with what was contained in the narration.

The final phase of analysis was to take the students' explanations of what the module was trying to teach, and what they understood about the concept, and connect it to the type of noticing that was done. We did this by identifying the objects and connections contained within the explanation of their understanding and then comparing that to the objects they had noticed perceptually and/or conceptually.

3.5. Limitations of methods used in this study

There are some important limitations to this study. First, it is never possible to see inside a student's mind and observe what they are noticing. This study can only report on noticing that students made apparent in some way in their explanations. The interviewer did frequently follow-up and ask students to describe what they were seeing, but there is always the possibility that a student noticed certain parts of the modules without ever saying anything that provided evidence of that noticing. Second, on the other side of the coin, because the interviewer wanted to get the students to talk about what they were experiencing inside of the modules, students may have noticed some elements because they were under an expectation to talk about what they noticed. It is quite possible that some students observing these modules on their own would not have noticed all of the elements they did for the purposes of the interview.

4. Results

4.1. Features associated with more conceptual noticing

In this subsection we explain what features across the three modules were associated with more conceptual noticing. First, and fairly unsurprisingly, some of the most conceptually noticed centres of focus were objects that were brightly coloured or that were familiar shapes. For example, in the multivariable function module, highly conceptually noticed objects were the bright yellow segments depicting the z output for a given

(x,y) input, the way the red grid lines on the graph matched the blue grid lines in the x-y plane, and the bright yellow lines suggesting the vertical line test. In the contour plots module, all of the students conceptually noticed the multiple coloured rings on the various graphs and their connections to the same coloured rings portrayed as a contour map in the x-y plane. In the partial derivatives module, highly conceptually noticed objects were the blue and green planes that "sliced through" the red graph, the coloured intersections between these planes and the red graph, and the yellow tangent line along that intersection curve. Figure 10 shows samples of such brightly coloured objects.

However, it is important to point out that there were some brightly coloured, familiar shapes that were *not* conceptually noticed as much by the students. For example, in the multivariable function module, the bright yellow circle denoting the boundary of the function's domain and the green interior of this circle had less conceptual noticing (only one student) compared to other brightly coloured objects (Figure 11). In the contour plot module, the colourful temperature map and elevation map were not conceptually noticed by any students, and were only perceptually noticed by three. In the partial derivative module, despite being colourful, the various components of the single-variable explanation were noticed less (Figure 11).

What might account for the differences between those brightly coloured objects that were highly conceptually noticed versus those brightly coloured objects that were not? One key feature that seemed to differentiate between them was *animation*. That is, close to two-thirds of the highly conceptually noticed objects had some sort of animation associated with them. For example, the contour plot module contained animations of a level curve being traced out on the graph and a corresponding level curve being traced out in an x-y plane. The partial derivatives module contained animations of the planes cutting through the graph. Animations also traced out the intersections and tangent lines in a dynamic way. On the other hand, the yellow domain circle and green interior that were not conceptually noticed very much had no corresponding animation. This was also true of the temperature/elevation maps and the single-variable graphs. None of these had associated animations. In fact, of the 43 total brightly coloured elements within these modules, only 24 were conceptually noticed by at least half of the students (55.8%). But when looking at just the 15 of these elements that had animation associated with them, all 15 were conceptually noticed by at least half of the students (100%). Thus, while bright colouring may lead to some noticing, animation was a significant feature of the module that increased conceptual noticing.

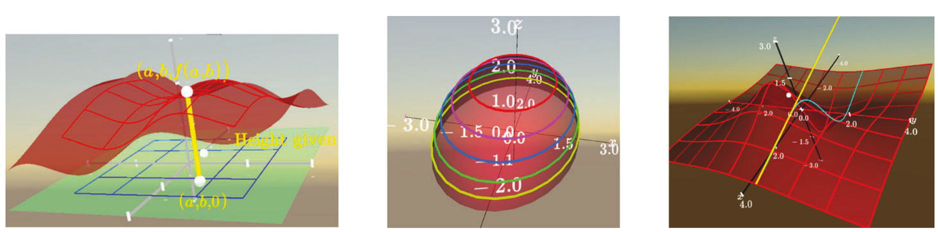


Figure 10. Examples of brightly coloured objects that were highly conceptually noticed.

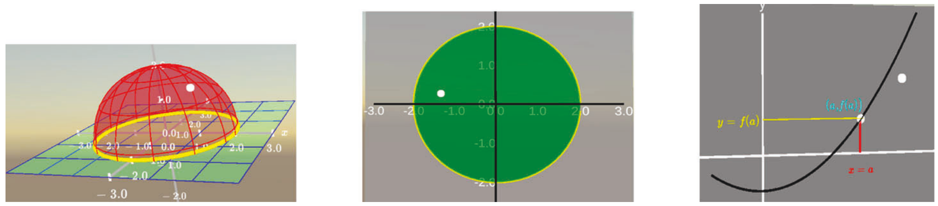


Figure 11. Brightly coloured objects that were *not* conceptually noticed as much.

4.1.1. Connections to student understanding of the mathematics

The result about the link between colour/animation and conceptual noticing has connections to the students' understandings of the mathematical concepts of multivariable functions, contour plots, and partial derivatives. Consider the following sample explanations of how students responded when asked to explain what they understood about each concept after watching that particular module.

> *Student B* [multivariable function module]: A multivariable function is where you have two different variables being put into one function, and you get one answer out … When you put those two values in, you're not going to get multiple answers [i.e. output values]. There's just going to be one answer.

> *Student D* [contour plot module]: I was learning how to, ultimately taking, I feel like, the traces from a three-dimensional figure, taking the traces that are parallel to the x-y plane, and putting those in just a two-dimensional *x-y* graph [contour plot]. But you're still able to represent, through colours, through different colours and a key, what it [the surface] would look like in three dimensions from a two-dimensional graph.

> *Student A* [partial derivative module]: When you're taking a partial derivative … so I guess you're going to make some sort of cut-out [i.e. trace]. If I was going to take the partial derivative of x, then I would fix y … You take that cut-out at y equals 2, you could take a partial derivative at any point on that cut-out … You can take a derivative at any of the x values, and you get that tangent line going across.

Student B's explanation of multivariable functions closely matches with the animated yellow vertical lines that depict several (x,y) points each matching with only one z value (Figure 3). Student D's explanation of contour plots is strongly connected to the animated multi-coloured horizontal cross sections of the semi-ellipsoid (Figure 4). Student A's explanation of partial derivatives explicitly recaps the animated blue plane that cut through the graph and the animated cross section along the plane (Figure 8). Thus, this conceptual noticing had important impacts, not just on their experiences, but on the understanding they were building while participating in the modules.

4.2. Features associated with less conceptual noticing

Of centres of focus within the modules, text and symbolic expressions were noticed at a much lower rate across all three modules. In the multivariable function module, not a single student noticed (even perceptually) the textbox containing "Input: x, Output: y" nor the textbox containing "$f(x, y) = x^2 - y^2$, Output: $3 = 2^2 - (-1)^2$." This was also true, in the contour plot module, for the textboxes containing "Level curve $c = f(x, y)$" and "Intersect $z = f(x, y)$ with $z = c$." In the partial derivative module, the definition "$f'(x) = \lim_{h \to 0} \ldots$" was only perceptually noticed by two students (C and D) and the text "$f'(a)$ is the slope … " was only perceptually noticed by Student D. Similarly, the textboxes "$z = f(x, y)$" and "partial with respect to x: $f_x = \partial f/\partial x$" were only perceptually noticed by Student E. Few textboxes or symbolic expressions had any conceptual noticing associated with them. In fact, of the 19 textual/symbolic elements within these modules, only one was noticed conceptually by three students, only one was noticed conceptually by two, and four were noticed conceptually by just one student. 13 of the textual/symbolic elements were not noticed conceptually by *any* of the students (68.4%). One of the few exceptions was when the definition of a partial derivative was given in the first piece of the partial derivative module. Students B and C gave explanations suggesting conceptual noticing of the definition, as in this excerpt:

> *Student B*: You're going to have two different [partial derivatives]. It's saying, like, you're going to have the $f(x+\Delta x,y)$. So, like, you're still going to, it's not like you're going to completely ignore the y part … The y is still relevant.

Interestingly, despite Students B and C conceptually noticing the limit definitions, this pattern mostly held true even for that first piece of the partial derivative module when the large textbox stating the definitions was literally the only object visible at the time while the narration played (Figure 7 in Section 2.4.3). Students A, D, and E did no conceptual noticing during this part, and mostly just acknowledged the existence of the textbox and its elements during that piece.

4.2.1. Connections to student understanding of the mathematics

This result about text boxes and symbolic expressions also connected to the students' understandings of the mathematical content. Most students did not make use of the information from text boxes or symbolic expressions when explaining their understandings of the concepts. For example, Student D had explained that she understood partial derivatives to represent slopes along 2-D cross sections of a 3-D graph. But, when asked what she understood from the large text box containing the symbolic expressions, she only stated, "All it is, is the partial derivative [long pause] formulas? Equations?" She later added, "It has different notation. Hmm, I'm not sure." Similarly, when Student E was asked about that text box, he simply stated, "I hate limits." When asked about the textual/symbolic elements for the multivariable function module, some of the students made incorrect assertions. For example, Student A suggested that for $f(x, y) = \sqrt{4 - x^2 - y^2}$, if $x = y = 0$, then the resulting $z = 2$ corresponded to the yellow ring in the *x-y* plane around the edge of the graph. These excerpts suggest that the textual/symbolic elements were not significant factors for these students in the understanding they developed about this multivariable calculus content.

4.3. The narration and conceptual noticing

The narration within the modules was intended as a focusing interaction to help guide students' noticing to important objects and ideas (see Section 2.2). However, unexpectedly, sometimes there appeared to be a negative relationship between the narration and the students' conceptual noticing. During some of the narrated portions, there was a decrease in this type of noticing. For example, the narration during the multivariable function's first piece recapped graphs of single-variable functions and how they might be extended to graphs of multivariable functions. Only three students (C, D, E) made any mention of the single-variable graph (Figure 1a), and these were all at the perceptual level. In the second piece, there was less conceptual noticing regarding the red semi-sphere and its associated algebraic function that were described by the narration (Figure 2a). In contour plots, while Students B, C, and D noticed the temperature or elevation graphs, they did so only at the perceptual level. The partial derivatives module opened with narration recapping derivatives of single-variable functions, and how that definition could be extended to partial derivatives with respect to x and y (Figure 7). As described in the previous subsection, three students did no conceptual noticing during this part.

Thus, while the narration had been meant to scaffold conceptual noticing, at times it appeared to have the reverse affect. It might have led the students to notice the *existence* of the objects (perceptual), but it did not always lead them to the conceptual insights we hoped for. We refer to this unintended outcome as "passive noticing" in that the narration may occasionally have led the students to feel something akin to watching a video, wherein they were making fewer connections for themselves.

However, narration did not *always* lead to this "passive" perceptual noticing. In fact, several narrated parts had correspondence with conceptual noticing. For example, in the third piece of the multivariable function module, the narration described how a "vertical line test" also applies to the graphs of multivariable functions. The module displayed a red graph and animated a series of vertical yellow lines rising up through the graph (Figure 3a). All five students conceptually noticed the yellow lines and their relationship to the red graph, such as Student E explaining that each line represented "a coordinate." He elaborated, "It's a point, so when you have an x and y, you're going to have a point. Except now it's a line stemming from that point." He interpreted the intersection of the yellow line and the red graph as an output of z, and concluded, "For it to be a function, it can only have one single output for one input of each variable."

As another example, during the second piece of the partial derivative module, a blue plane was animated slicing through the red graph (Figure 8a). Some narration then explained that the intersection of the blue plane and the graph produced a single curve for which a tangent line could be found, and whose slope was the value of the partial derivative (Figure 8b). The blue plane, the turquoise curve at the intersection, and the yellow tangent line were all highly conceptually noticed objects. To illustrate, consider Student A's statements about what he noticed during this piece.

> *Student A*: You take a cutout of the graph and make that 2-D. And just look at how that [the tangent line] is in one plane.
>
> …
>
> *Student A*: that cutout would obviously change if I were to do, say, $y = -1$.

Here, Student A showed conceptual noticing of the intersecting blue plane, in that it allowed the 3-D graph to reduce to a 2-D curve. He was also aware that different y values could potentially lead to different traces ("cutouts"), giving distinct tangent lines.

What might account for the differences between when the narration was related to "passive" perceptual noticing versus conceptual noticing? As before, one of the major differences between these instances appeared to be *animation*. Those elements that were highly conceptually noticed during narration tended to be animated. Those that were less conceptually noticed during narration tended to not be animated. The recaps of the single variable functions and single variable derivatives had no animation associated with them. The textual and symbolic elements all had no animation associated with them. On the other hand, the "vertical line test" and the creation of the visual representations of the partial derivatives were animated. Thus, as before, animation appeared to play an important role in provoking conceptual noticing.

4.3.1. Connections to student understanding of the mathematics

Similar to what has been explained already, there was a relationship between narration/animation and student understanding. We have already presented examples of how animated elements were incorporated more strongly into the students' developing understandings. The non-animated elements presented during the narration had less uptake when the students described their understanding. For example, after the multivariable function module, no student made mention of symbolic function expressions in their understanding of what multivariable functions are. They described the visual aspects of pairing an (x,y) point to a z output, but their explanations did not contain the idea of algebraic expressions. While that certainly is not to say that the students definitely did not understand algebraic expressions, the point is that it was clearly less central in their emerging understanding. In the contour map module, no student made connections to the real-world examples of topographical maps nor temperature maps. In the partial derivative module, as stated previously, the narrated text box with symbolic expressions had little relevance to what the students described as understanding about partial derivatives. Thus, the lack of conceptual noticing during parts of the narration appeared to influence their understanding of the mathematics.

5. Discussion

This paper contributes to research on VR in mathematics education in two ways. First, this paper described the research-informed design of VR modules in an area that has so far been unattended to in VR research: multivariable calculus. Second, this paper presents an empirical study examining the students' experiences inside of these VR modules, helping to clarify features of the modules that enhanced conceptual noticing and student learning. When these two contributions are taken together, they generate useful implications for both the design of VR modules in mathematics generally, as well as for the learning of multivariable calculus concepts specifically.

5.1. Implications for general design of VR modules in mathematics

The literature on VR has suggested that students may not always know how to allocate their attention (Renkl & Scheiter, 2017), and that scaffolding is needed to direct this attention (Jaeger & Wiley, 2014; Ross et al., 2017; Van Horne et al., 2016). Lobato et al. (2013) and Goodwin (1994) explain the practice of "highlighting" to shape the perceptions of students by making certain objects salient. Our study adds to this literature by providing some specifics about "highlighting" within VR modules. While brightly coloured objects were a somewhat helpful feature in directing attention, it turned out to be hit-or-miss in terms of conceptual noticing. The colourful imagery may certainly have increased the aesthetics of the modules (Girod et al., 2010; Jakobsen & Wickman, 2008), but aesthetics alone is insufficient for robust comprehension (Lee, 2010). Rather, the feature of *animation* was the strongest indicator of conceptual noticing – much stronger than the colourful imagery. Additionally, while one might assume that a well-placed text box with symbolic expressions would help the students see the relationships between the textual/symbolic elements on the visual elements, our study strongly suggested that these kinds of elements were actually quite weak in producing conceptual noticing. The implication is that we need to rethink the role of such textual/symbolic elements in VR environments and how connections to them might be made (we discuss some ideas related to this in the next subsection).

Next, the literature speaks to the need to guide students within the modules to maximise learning (Jaeger & Wiley, 2014; Kerawalla et al., 2006). Our study adds that the way this guidance is provided matters, and that not all guidance is created equal. We assumed that narration would supply the needed scaffolding to direct students' attention and learning. However, the narration was also hit-or-miss in terms of whether it created conceptual noticing, and often seemed to lead to *passive noticing*. As before, adding *animation* to this guidance appeared to be the most productive in actually provoking conceptual noticing. Thus, scaffolding in VR can come in the form of visual manipulation as much as from speech or other forms of directing attention (see also Lobato et al., 2012). In fact, one major conclusion from our study is to promote animation as a key feature within VR. Additionally, to reduce passivity during narration, an important way to provide scaffolding guidance could be to ask immediate questions for the students to answer within the VR module (Lobato et al., 2013). These questions could be accompanied by animation, or the animation may come in afterward as the students have had a chance to think.

Because most VR work in mathematics education has been on geometry content (Gutiérrez de Ravé et al., 2016; Hwang & Hu, 2013; Ibili et al., 2020; Kaufmann et al., 2000; Song & Lee, 2002; Yeh & Nason, 2004), a key implication of our paper is to expand the horizons on what kind of mathematical content might be considered for VR development. So far, the field of mathematics education has not applied VR to other mathematical areas, with the exception of a very early example of algebra VR (Winn & Bricken, 1992). We hope this paper can spur researchers and educators to more broadly consider what mathematical content across the K-16 level (or beyond) could benefit from serious investigations of developing and studying VR modules.

5.2. Implications for learning mathematics concepts

Our study has implications for how the actual mathematics concepts themselves might be learned within VR. One of the broader implications deals with connections between different modes of representation in learning mathematics, such as symbolic, geometric, or numeric (Duval, 2006; Goldin & Shteingold, 2001; Yerushalmy & Shternberg, 2001). Because of the lack of conceptual noticing of textual/symbolic elements in the VR modules, connections between symbolic and visual representations was not nearly as strong in our students as hoped for. Thus, VR modules dealing with mathematics in both symbolic and visual forms needs scaffolding that specifically ensures connections between them (see also King-Sears et al., 2018; Renkl & Scheiter, 2017; Yum et al., 2021). One way to do so would be to simply highlight a symbolic expression and its corresponding visual element simultaneously (see Chen et al., 2007). For example, one could write the inequality $x + y - 1 > 0$ in yellow to match the yellow line in the x-y plane that shows the domain boundary of the function $f(x, y) = \ln(x + y - 1)$. However, because our results suggested that colour alone is less effective than animation, the stronger implication is to try to tie symbols with visuals via animation. As an example, when showing the graph of $f(x, y) = x^2 - y^2$, the module could show $z = x^2$ when $y = 0$, and then animate the parabola shifting up and down as y assumes different values (similar to what is described without VR in Weber & Thompson, 2014). A more ambitious option for animating connections could be to have objects morph into symbols (Bujak et al., 2013). For example, a yellow circle might be animated moving into a textbox and morphing into the symbolic expression $x^2 + y^2 = 4$.

Our study has implications for learning multivariable calculus, specifically. The literature speaks to challenges in generalising functions to $\mathbb{R}^3$ (Trigueros & Martinez-Planell, 2010), including issues in thinking of domains and ranges with y often switching to the role of an input and z now being an output (Dorko & Weber, 2014). The VR modules we present are promising in overcoming these issues, as the students readily interpreted x-y pairs as inputs, matching to a single z output. In the same vein, Martinez-Planell and Trigueros-Gaisman (2012) explained challenges in visualising surfaces and graphs in $\mathbb{R}^3$ in their entirety as a necessary precursor to examining features of those objects, such as intersection curves. The visuals in the VR modules allowed students to examine the objects in this way, permitting a comprehension of how, for example, a horizontal plane slid through a surface to create a level curve. Further, Weber (2015) described an issue students have with conceptualising different rates of change for a multivariable function at a single point. This is likely due to their inability to completely see and manipulate $\mathbb{R}^3$ graphical information (Martinez-Planell & Trigueros-Gaisman, 2012). The VR modules we described show the entire graphical representation of a function, as well as cross sections being dynamically created in different partial derivative directions. All of our students conceptually noticed these elements and were able to describe an understanding that incorporated different possible rates of change. However, because of our students' lack of attention to the textual/symbolic elements, the understanding tended to rely on general "slope," without dissecting it into the quantities *change in z* and *change in x (or y)*. This suggests the need to, again, animate stronger connections between the visual and the symbolic.

Overall, we believe these modules to have important benefits to offer multivariable calculus instructors who wish to better support students' reasoning in multidimensional space. In fact, we hope this paper prompts more creation and testing of other VR modules for multivariable calculus, each potentially addressing specific difficulties or known misconceptions. As different instructors contribute different aspects of visualisation, animation, and manipulation within VR modules for multivariable calculus, undergraduate mathematics education can move closer to addressing many of the issues in learning the content of that course.

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