Homodyne measurement with a Schrödinger cat state as a local oscillator

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Homodyne measurements are a widely used quantum measurement. Using a coherent state of large amplitude as the local oscillator, it can be shown that the quantum homodyne measurement limits to a field quadrature measurement. In this work, we give an example of a general idea: injecting non-classical states as a local oscillator can led to non-classical measurements. Specifically we consider injecting a superposition of coherent states, a Schrödinger cat state, as a local oscillator. We derive the Kraus operators and the positive operator-valued measure (POVM) in this situation and show the POVM is a reflection symmetric quadrature measurement when the coherent state amplitudes are large.

I. INTRODUCTION

Homodyne measurement is a low noise, high sensitivity technique to detect a quadrature of the electromagnetic field. This is achieved by the mixing of a high power, phase stable local oscillator with an input signal and detecting the resulting low frequency components. In practice, homodyne detection can in operate very close to the noise limits imposed by quantum mechanics [1]. Hence homodyne measurements have become a vital component of optical and microwave quantum– optics, communication, and computation.

Balanced homodyne detection is performed by mixing an arbitrary input signal state $|\Psi\rangle$, with a prepared reference state or *local oscillator* (LO) on a 50:50 beam-splitter, see Fig. 1. The two outputs of this beam-splitter are then measured by detectors that produce currents that are proportional to the intensity of the field. The difference between the two currents is the output signal and effectively measures a quadrature of electromagnetic field [2]. The output is considered to be destructively measured, that is all energy contained within the field is fully absorbed in the act of measurement.

In the quantum analysis of homodyne measurements all elements of of the scheme (fields, beamsplitters, and detectors) must be treated as quantum objects. The goal of the analysis is to predict the statistics of the measurement. Many quantum treatments of homodyne detection [3–10] calculate moments of the detectors (or output signal) and show this limits to moments of a quadrature variable. Another approach, taken by Tyc and Sanders [11, 12], is to calculate the Kraus operators and positive operator valued measure (POVM) and show that these limit to the POVM for an ideal quadrature measurement.

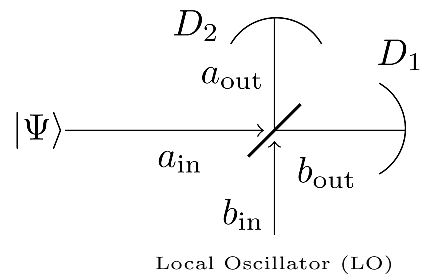

FIG. 1. Balanced homodyne setup. An arbitrary signal state $|\Psi\rangle$ is mixed on a 50:50 beam-splitter with a local oscillator. The two outputs are measured by detectors (D_1, D_2) that produce currents I_i proportional to the intensity of the field. The homodyne measurement result is proportional to the difference of these currents i.e. $I_2 - I_1$. In standard quantum homodyne detection the local oscillator is a large amplitude coherent state. The phase of the coherent state determines the measured quadrature. In this work we consider local oscillators prepared in superpositions of coherent states.

Little consideration to date has been given to states of the LO that are not effectively classical (or non-Gaussian). However there has been related work that has considered variations of standard homodyne measurement. In Refs. [13, 14] Sanders *et al.* considered homodyne detection using a squeezed LO.

Recent work by Thekkadath *et al.* [15] shows that using the setup in Fig. 1 one can project onto an even parity states by using a "reversed" quantum interference argument, a control state and postselecting on an equal number of quanta at the output detectors. The control state determines the even parity state detected and does not necessarily play a role like a local oscillator. Another recent work by Thekkadath *et al.* [16] shows a calculation of the properties of homodyne detection with local oscillators that are coherent states with strengths down to zero, i.e. a weak field local oscillator [17]. This shows that one can smoothly transition from photon counting style

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detections to field quadrature variables. Related issues have been examined by Olivares et al. in Refs. [18, 19]. However in all these cases, the local oscillator here is always a coherent state which is generally considered to have classical properties.

In this work we consider using superpositions of coherent states as local oscillators and derive the corresponding Kraus operators and POVM elements in the strong LO limit. In Sec. II we give an alternative derivation of the results of Tyc and Sanders [11, 12]. We show through application of the algebra of creation and annihilation operators for bosonic fields, that a coherent state LO $|\beta\rangle$, where $\beta = |\beta|e^{i\theta}$, results in the measurement of an arbitrary quadrature projector $|x_\theta\rangle\langle x_\theta|$, in the limit where $|\beta| \to \infty$. In Sec. III we use this alternative derivation to consider a LO that is proportional to $|\beta\rangle \pm |-\beta\rangle$. The resulting POVM is the reflection symmetric measurement of an arbitrary quadrature, i.e. $\propto |x_\theta\rangle\langle x_\theta| + |-x_\theta\rangle\langle -x_\theta|$. In Sec. IV we give numerical examples of the statistics of these measurements in the moderate local oscillator limit. Then in Sec. V we show how the non-classical measurements can be used to prepare a non-classical state of a remote system using only a EPR state. Finally we conclude in Sec. VI.

II. HOMODYNE MEASUREMENT WITH COHERENT STATE LOCAL OSCILLATOR

In this section, we re-derive the Kraus operators and POVM elements for a standard homodyne measurement. Our method is inspired by the work of Tyc and Sanders [11, 12]. However, we use different techniques and variables that are better suited for the later consideration of non-classical local oscillator states.

A. Exact Kraus operator

We will now construct the Kraus operator for Fig. 1 by working backwards from the detectors towards the states.

A departure from the usual treatment of Homodyne measure in the method of Tyc and Sanders is to model the measurement of intensity by ideal photon number resolving detection at detectors 1 and 2. Whereas balanced Homodyne measurement typically involves the light impinging on detectors that respond to intensity. Both treatments consider the photocurrents I produced by the detectors to be proportional to the number operator e.g. $I \propto \langle a^\dagger a\rangle$ and the measurement result is the difference of the photocurrents i.e. $I_1 - I_2 \propto \langle a^\dagger a\rangle - \langle b^\dagger b\rangle$.

Typically the number resolving measurements are modelled by Kraus operators that are projectors onto a Fock state basis, e.g. $\Pi_n = |n\rangle\langle n|$, which might represent a quantum non-demolition detection of photon number. In virtually all cases, optical detectors completely absorb the field and hence the field state after the measurement is mapped to vacuum for any measurement outcomes. We could introduce operators to denote this case $P_n = |0\rangle\langle n|$. However, we are never going to be interested in this conditional state and hence consider this detection to be a "partial projection" $P_n = \langle n|$, which effectively traces out the post-measurement state.

The object that precedes the detectors is a 50:50 beamsplitter and we denote the unitary representing this object as $U_{\rm BS}$. The Heisenberg evolution of the annihilation operators due to $U_{\rm BS}$ is

$$a_{\rm out} := U_{\rm BS}^\dagger\, a_{\rm in}\, U_{\rm BS} = \frac{1}{\sqrt{2}}(a_{\rm in} + b_{\rm in}) \tag{1a}$$

$$b_{\rm out} := U_{\rm BS}^\dagger\, b_{\rm in}\, U_{\rm BS} = \frac{1}{\sqrt{2}}(a_{\rm in} - b_{\rm in}) \tag{1b}$$

where $a_{\rm in}$ is an annihilator operator for the signal mode and $b_{\rm in}$ is an annihilation operator for the local oscillator mode.

The next step is to include the input states. By introducing the input states we may define a Kraus operator $M^{[\beta]}_{n,m}$ that acts on the input Hilbert space of mode a

$$M^{[\beta]}_{n,m}\,|\psi\rangle = (P_n \otimes P_m)U_{\rm BS}(|\text{signal}\rangle \otimes |\text{LO}\rangle), \tag{2}$$

where the left part of the tensor product is the signal mode and the right the local oscillator. The superscript β on is in anticipation of taking a "large local oscillator limit" using a parameter β. By itself the Kraus operator is

$$M^{[\beta]}_{n,m} = \langle n|\,\langle m|\,U_{\rm BS}\,|\text{LO}\rangle \equiv P_n \otimes P_m U_{\rm BS}(I \otimes |\text{LO}\rangle). \tag{3}$$

This equation represents the Kraus operator for measuring the detection event for n and m photons. The Fock basis dual vectors within the Kraus operator $M^{[\beta]}_{n,m}$ of Eq. (3) can be written in terms of creation operators acting on a vacuum tensor product space to give,

$$M^{[\beta]}_{n,m} = \langle 0|\,\langle 0|\,\frac{(a_{\rm out})^n}{\sqrt{n!}}\frac{(b_{\rm out})^m}{\sqrt{m!}}U_{\rm BS}\,|\text{LO}\rangle\,. \tag{4}$$

It is not yet apparent that this is an operator on the input Hilbert space so we perform further manipulations to elucidate this fact.

We transform $a_{\rm out}$ and $b_{\rm out}$ to linear combination of the input operators by inserting identity $U_{\rm BS}U_{\rm BS}^\dagger$ many times between the powers of the annihilation operators and substituting Eq. (1). Doing so gives a Kraus operator

$$\begin{aligned} M^{[\beta]}_{n,m} &= \langle 0|\langle 0|\frac{1}{\sqrt{n!m!}}U_{BS}U_{BS}^\dagger a_{\rm out}^n U_{BS}U_{BS}^\dagger b_{\rm out}^m U_{BS}|\text{LO}\rangle\,, \\ &= \langle 0|\,\langle 0|\,\frac{1}{\sqrt{n!m!}}\left(\frac{a_{\rm in}+b_{\rm in}}{\sqrt{2}}\right)^n\left(\frac{a_{\rm in}-b_{\rm in}}{\sqrt{2}}\right)^m|\text{LO}\rangle\,. \end{aligned} \tag{5}$$

At this point the quantum state of the local oscillator $|\text{LO}\rangle$ is arbitrary and could be replaced with any quantum state. If we now use the fact that in homodyne measurement the LO is a coherent state, i.e. $|\text{LO}\rangle = |\beta\rangle$, on mode $b_{\rm in}$ we arrive at the Kraus operator (with no approximations)

$$M_{n,m}^{[\beta]} = \langle 0| \frac{(a_{\rm in}+\beta)^n (a_{\rm in}-\beta)^m}{2^{(n+m)/2}\sqrt{n!m!}} e^{-|\beta|^2/2}. \tag{6}$$

This expression is now an operator acting *only* on the input signal mode $a_{\rm in}$, and is valid for all values of β including small values. From now on we drop the subscript "in" on the operator to reduce notational clutter i.e. $a_{\rm in} \mapsto a$.

The positive operator valued measure (POVM) corresponding to Eq. (6) is

$$E_{n,m}^{[\beta]} = \left(M_{n,m}^{[\beta]}\right)^\dagger M_{n,m}^{[\beta]}, \tag{7}$$

that this is a valid POVM is evident because $E_{n,m}^{[\beta]} \geq 0$ and $\sum_{n,m} E_{n,m}^{[\beta]} = \mathbb{I}$.

B. Kraus operator approximations

In the Kraus operators and POVM above have not yet seen the emergence of a quadrature-like measurement result. In order to proceed towards our goal of deriving a quadrature measurement from the Kraus operator Eq. (6) we will need to make some assumptions and approximations. Specifically, we will repeatedly take the large amplitude local oscillator (LO), i.e. $|\beta| \to \infty$, and that implies the LO intensity is much larger than the signal i.e. $|\beta|^2 \gg \langle\psi|\,\hat{n}\,|\psi\rangle$.

At this point we will assume $n > m$. This assumption is not actually a significant restriction due to the symmetry between n and m. With this assumption Eq. (6) can be factored as

$$M_{n,m}^{[\beta]} = \langle 0| \left(1+\frac{\hat{a}}{\beta}\right)^{n-m} \left(1-\frac{\hat{a}^2}{\beta^2}\right)^m \frac{e^{-|\beta|^2/4}}{\sqrt{n!}}\left(\frac{\beta}{\sqrt{2}}\right)^n \frac{e^{-|\beta|^2/4}}{\sqrt{m!}}\left(\frac{-\beta}{\sqrt{2}}\right)^m, \tag{8}$$

as $n-m$ is positive.

Following Tyc and Sanders' logic (see appendix A) we change variables representing the measurement output to the difference of the counts

$$x = e^{-i\theta}\tilde{x} = \frac{n-m}{\sqrt{2}|\beta|e^{i\theta}} = \frac{n-m}{\sqrt{2}\beta}, \tag{9}$$

which is essentially an estimator for the quadrature component (hence the use of the variable x). Using this new variable we can write [1]

$$\left(1+\frac{\hat{a}}{\beta}\right)^{n-m} = \left(1+\frac{e^{-i\theta}\hat{a}}{|\beta|}\right)^{\sqrt{2}\tilde{x}|\beta|} \underset{|\beta|\to\infty}{\to} e^{\sqrt{2}e^{-i\theta}\tilde{x}\hat{a}}. \tag{10}$$

[1] Regarding the change of variables. Alternatively one can take $\beta' = \sqrt{2}x\beta$, then substitute into the middle equation to get the standard exponential limit. If $x = 0$ it also works.

In appendix A we show the distribution of outcomes m and n will be peaked around $|\beta|^2/2$, due to the Poisson statistics of the LO overwhelming the signal mode. Thus we replace the random variable m with it's mean: $|\beta|^2/2$. Using this approximation on the second bracketed operator term in Eq. (8) we arrive at

$$\left[1-\frac{e^{-2i\theta}\hat{a}^2}{|\beta|^2}\right]^m \approx \left[1-\frac{e^{-2i\theta}\hat{a}^2}{|\beta|^2}\right]^{\frac{|\beta|^2}{2}} \underset{|\beta|\to\infty}{\to} e^{-\frac{1}{2}e^{-2i\theta}\hat{a}^2} \tag{11}$$

The leading order correction to the above approximation is $O(1/|\beta|^2)$, as shown in appendix B.

So far we have assumed that $n > m$. In appendix C 1 we show that when $m > n$ a cancellation of signs occurs in the expressions equivalent to Eq. (10) and Eq. (11) , which results in the same asymptotic limit. So finally we can approximate the Kraus operator in the strong oscillator limit for all cases as

$$M_{n,m}^{[\beta]} \underset{|\beta|\to\infty}{\approx} \langle 0|\, e^{\sqrt{2}e^{-i\theta}\tilde{x}\hat{a}} e^{-e^{-2i\theta}\hat{a}^2/2} \frac{e^{-|\beta|^2/4}}{\sqrt{n!}}\left(\frac{\beta}{\sqrt{2}}\right)^n \frac{e^{-|\beta|^2/4}}{\sqrt{m!}}\left(\frac{-\beta}{\sqrt{2}}\right)^m. \tag{12}$$

This Kraus operator is the basic result we use for our remaining analysis.

The Kraus operator $M_{n,m}^{[\beta]}$ in the limit of large $|\beta|$ is a good approximation of a quadrature eigenstate projection [20]. To see this, we can write, with the choice of units used here, eigenstates of an arbitrary quadrature $Q(\varphi) = (e^{-i\varphi}a + e^{i\varphi}a^\dagger)/\sqrt{2}$ as

$$|x_\varphi\rangle = \frac{e^{-x^2/2}}{\pi^{1/4}} e^{\sqrt{2}x\chi a^\dagger} e^{-\chi^2 a^{\dagger 2}/2}\,|0\rangle\,, \tag{13}$$

with $\chi = e^{i\varphi}$ and $|0\rangle$ is the vacuum state. (In appendix D we show that $|x_\varphi\rangle$ is an eigenstate of $Q(\varphi)$ using techniques adapted from Ref. [20].) This means the approximation of the Kraus operator can be written as

$$M_{n,m}^{[\beta]} \underset{|\beta|\to\infty}{\approx} \langle x_\theta|\, \pi^{1/4} e^{\tilde{x}^2/2}(-1)^m e^{i\theta(n+m)} \frac{e^{-|\beta|^2/4}}{\sqrt{n!}}\left(\frac{|\beta|}{\sqrt{2}}\right)^n \frac{e^{-|\beta|^2/4}}{\sqrt{m!}}\left(\frac{|\beta|}{\sqrt{2}}\right)^m. \tag{14}$$

with some slight mixing of notation as x is used to represent a quantity proportional to the difference of n and m as defined above. The final two terms are the square root of a Poisson distribution. They can be approximated in the strong local oscillator limit, $|\beta| \to \infty$ using the continuous approximation for the Poisson distribution, that is

$$\left[\frac{e^{-|\beta|^2/2}}{n!}\right]^{\frac{1}{2}} \left[\frac{|\beta|^2}{2}\right]^{\frac{n}{2}} \approx \frac{e^{-(n-|\beta|^2/2)^2/(2|\beta|^2)}}{(\pi|\beta|^2)^{1/4}}\sqrt{dn}. \tag{15}$$

Notice we have introduced the square root of the probability measure of n. This is because the continuous approximation smooths the differences between the values of n which were implicitly 1 in the discrete distribution. This argument on the differential dn applies to the probability, but this expression involves the probability *amplitude* which is why the square root of this measure is required [21]. Using this approximation on the final two terms in Eq. (14) gives two independent normal distributions for n and m with mean and variance $|\beta|^2/2$. However, we wish to resolve the results into scaled sum and differences of n and m. Specifically the exponent of the combined distribution for n and m is

$$-\frac{1}{2|\beta|^2}\left[\left(n-\frac{|\beta|^2}{2}\right)^2+\left(m-\frac{|\beta|^2}{2}\right)^2\right] \\ =-\frac{1}{4|\beta|^2}\left[\left(n+m-|\beta|^2\right)^2+(n-m)^2\right]. \quad (16)$$

This equation shows that the sum of n and m is normal distribution with mean $|\beta|^2$ and variance $|\beta|^2$ and the difference of n and m is a normal distribution with mean zero and variance $|\beta|^2$. The difference of n and m can be scaled to be written in terms of x and the distribution in x will be normal with mean zero and variance $1/2$. The distribution in the scaled x cancels the exponential factor of $e^{\tilde{x}^2/2}$ in Eq. 14 leaving

$$M_{n,m}^{[\beta]}\sqrt{dndm} \underset{|\beta|\to\infty}{\approx} e^{i\theta(n+m)}\left\langle x_\theta\right|(-1)^m \\ \frac{2^{1/4}}{|\beta|^{1/2}}\frac{e^{-(n+m-|\beta|^2)^2/(4|\beta|^2)}}{(2\pi|\beta|^2)^{1/4}}\sqrt{dndm}, \quad (17)$$

where this equation has included the probability measure due to the continuum limit as explained above. Moreover we have pulled out one of the global phase factors $e^{i\theta(n+m)}$ in anticipation of similar factors in the cat state local oscillator examined in Sec. III.

The change of variables can be simplified by introducing w defined as

$$w=\frac{n+m}{\sqrt{2}|\beta|} \quad (18)$$

which will be distributed normally with mean $|\beta|/\sqrt{2}$ and variance $1/2$. To complete the change of variables, the probability measure must be changed, so we need the determinant of the Jacobian

$$dxdw=\left|\frac{\partial(x,w)}{\partial(n,m)}\right|dndm=\frac{1}{|\beta|^2}dndm. \quad (19)$$

With these changes of variables the final approximation for the Kraus operator which can be written as (including changing the m and n subscripts to x and w as there are no more uses of m and n)

$$M_{x,w}^{[\beta]}\sqrt{dwdx} \underset{|\beta|\to\infty}{\approx} e^{i\sqrt{2}|\beta|w\theta}\left\langle x_\theta\right|e^{i\pi\sqrt{2}|\beta|(w-x)} \\ \frac{1}{|\beta|}\frac{e^{-(w-|\beta|/\sqrt{2})^2/2}}{\pi^{1/4}}|\beta|\sqrt{dwdx}. \quad (20)$$

From the original definitions of $\sqrt{2}|\beta|x$ and $\sqrt{2}|\beta|w$ in terms of the discrete variables, these values are actually integers. This means that the first phase factor is either 1 or -1 (though these phase factors disappear in the next step of our derivation). In Sec. III this phase has an important role to play.

The Kraus operator is important for determining the post measurement state, however the measurement statistics are entirely determined by the POVM. The POVM for a scaled difference measurement of x will then be

$$dx\,dw\,E_{x,w}^{[\beta]}=dx\,dw\left(M_{x,w}^{[\beta]}\right)^\dagger M_{x,w}^{[\beta]} \quad (21a)$$

$$=dx\,dw\,\frac{e^{-(w-|\beta|/\sqrt{2})^2}}{\sqrt{\pi}}\left|x_\theta\right\rangle\left\langle x_\theta\right| \quad (21b)$$

which agrees with equation (19) in Ref. [11] when differing notation is taken into account.

In the standard approach to homodyne detection, the w variable, the exact sum of the photons counted, is unobserved. We will integrate over it and take the final large oscillator limit i.e.

$$dx\,E_x \underset{|\beta|\to\infty}{\approx} dx\int_0^\infty dw\,E_{x,w}^{[\beta]}\,, \quad (22)$$

where x is the measurement outcome. Thus POVM with outcome x in homodyne detection of an arbitrary quadrature θ is

$$dx\,E_x \underset{|\beta|\to\infty}{\approx} dx\int_0^\infty dw\frac{e^{-(w-|\beta|)^2}}{\sqrt{\pi}}\left|x_\theta\right\rangle\left\langle x_\theta\right| \\ \underset{|\beta|\to\infty}{\approx}\frac{1}{2}\left[\operatorname{erf}\left(\frac{|\beta|}{\sqrt{2}}\right)+1\right]dx\left|x_\theta\right\rangle\left\langle x_\theta\right| \quad (23a)$$

$$=dx\left|x_\theta\right\rangle\left\langle x_\theta\right|, \quad (23b)$$

which is a well-defined POVM as all elements are positive (projectors) and

$$\int dxE_x=\int dx\left|x_\theta\right\rangle\left\langle x_\theta\right|=\mathbb{I}. \quad (24)$$

III. HOMODYNE MEASUREMENT WITH A CAT STATE LOCAL OSCILLATOR

In this section we derive the Kraus operators and POVM for the case of a local oscillator state that is a superposition of two coherent states, i.e. a cat state. Broadly the derivation here follows the procedures developed Sec. II, i.e. switching to sum and difference variables and taking a large local oscillator limit.

We consider and LO that is a superposition of coherent states of the form

$$\mathcal{N}_\pm(\beta)^{-1}\left(|\beta\rangle\pm|-\beta\rangle\right), \quad (25)$$

where $\mathcal{N}_\pm(\beta)=\sqrt{2(1\pm e^{-2|\beta|^2})}$. Note that as $|\beta|\to\infty$ the normalization limits to $\mathcal{N}_\pm(\beta)\to\sqrt{2}$. The plus superposition consists of only even Fock basis terms as the odd amplitudes follow the sign of the $\pm\beta$ amplitude and cancel to zero. A similar argument follow for the minus superposition but only odd terms survive. Therefore we say the plus superposition is an even parity state the minus superposition an odd parity state.

The *exact* Kraus operator, substituting Eq. (25) into Eq. (5) instead of $|\beta\rangle$, for the oscillator in a superposition of coherent states is

$$M_{n,m}^{[\beta]_\pm} = \langle 0| \frac{e^{-|\beta|^2/2}}{2^{(n+m)/2}\sqrt{n!m!}\mathscr{N}_\pm(\beta)} \left((\hat{a}+\beta)^n(\hat{a}-\beta)^m \pm (\hat{a}-\beta)^n(\hat{a}+\beta)^m\right), \quad (26)$$

where the subscript of $\pm$ on $[\beta]_\pm$ denotes the plus or minus superposition. We assume $n > m$, as we did in deriving Eq. (8), and get

$$M_{n,m}^{[\beta]_\pm} = \langle 0| \frac{e^{-|\beta|^2/2}}{2^{(n+m)/2}\sqrt{n!m!}\mathscr{N}_\pm(\beta)} \left(1-\frac{\hat{a}^2}{\beta^2}\right)^m \left[\beta^n(-\beta)^m\left(1+\frac{\hat{a}}{\beta}\right)^{n-m} \pm (-\beta)^n\beta^m\left(1-\frac{\hat{a}}{\beta}\right)^{n-m}\right]. \quad (27)$$

As shown in appendix C 2, the expression for $m > n$ is the same as this expression but with the sign of the superposition possibly changed depending on the parity of $m-n$. The up shot is: Eq. (27) covers the case of $m > n$ if the information as to which superposition phase applies is incorporated.

It turns out in the large LO limit m is distributed as a Poisson distribution with parameter $\lambda = |\beta|^2/2$ as before, see appendix E for the details. Thus the reasoning around Eq. (11) also applies to Eq. (27). We also perform the change of variables given in Eq. (9) and take the limits given in Eqs. (10) and (11) and make the replacement from the large $|\beta|$ limit i.e. $\mathscr{N}_\pm(\beta) \to \sqrt{2}$.

Using these approximations we have

$$M_{n,m}^{[\beta]_\pm} \underset{|\beta|\to\infty}{\approx} \langle 0|\Big[(-1)^m e^{\sqrt{2}e^{-i\theta}\tilde{x}\hat{a}} e^{-e^{-2i\theta}\hat{a}^2/2} \pm(-1)^n e^{-\sqrt{2}e^{-i\theta}\tilde{x}\hat{a}} e^{-e^{-2i\theta}\hat{a}^2/2}\Big]\frac{1}{\sqrt{2}} \frac{e^{-|\beta|^2/4}}{\sqrt{n!}}\left(\frac{\beta}{\sqrt{2}}\right)^n \frac{e^{-|\beta|^2/4}}{\sqrt{m!}}\left(\frac{\beta}{\sqrt{2}}\right)^m. \quad (28)$$

At this point we recognise an quadrature eigenstate and a rotated quadrature eigenstate, as per Eq. (13), and use that to further simplify the operator to

$$M_{n,m}^{[\beta]_\pm} \underset{|\beta|\to\infty}{\approx} \pi^{1/4} e^{\tilde{x}^2/2} \frac{e^{-|\beta|^2/4}}{\sqrt{n!}}\left(\frac{|\beta|}{\sqrt{2}}\right)^n \frac{e^{-|\beta|^2/4}}{\sqrt{m!}}\left(\frac{|\beta|}{\sqrt{2}}\right)^m \frac{e^{i\theta(n+m)}}{\sqrt{2}}\left[\langle x_\theta|(-1)^m \pm \langle -x_\theta|(-1)^n\right]. \quad (29)$$

Next we approximate the square root of Poisson distributions by normal distributions. The mean and variance of the sum and differences scale in the same way as the previous section giving

$$M_{n,m}^{[\beta]_\pm}\sqrt{dndm} \underset{|\beta|\to\infty}{\approx} \frac{2^{1/4}}{|\beta|^{1/2}} \frac{e^{-(n+m-|\beta|^2)^2/(4|\beta|^2)}}{(2\pi|\beta|^2)^{1/4}}\sqrt{dndm} \frac{e^{i\theta(n+m)}}{\sqrt{2}}\left[\langle x_\theta|(-1)^m \pm \langle -x_\theta|(-1)^n\right], \quad (30)$$

which should be compared to Eq. (17). Notice there is an overall phase of $e^{i\theta(n+m)}$ as there was in Eq. (17). However now we have a relative phase between quadrature eigenstates which is evident in the terms $(-1)^m$ and $(-1)^n$.

Let's pause to consider the implications of these phase factors. For the + cat LO, the Kraus operator is $M_{n,m}^{[\beta]_+} \propto \langle x_\theta|(-1)^m \pm \langle -x_\theta|(-1)^n$ with m and n taking integer values and the phase factors reflect whether m and n are odd or even. Thus the four possible combinations of the m and n dependant phases result in two distinct Kraus operators (upto a global phase), namely

$$M_{n,m}^{[\beta]_+} \propto \begin{cases} \langle x_\theta| + \langle -x_\theta| & n+m \text{ even} \\ \langle x_\theta| - \langle -x_\theta| & n+m \text{ odd} \end{cases}. \quad (31)$$

These measurement operators will project onto states of definite parity. To see this fact we specialize to the position quadrature ($\theta = 0$) and recall the parity operator can be represented as

$$P = (-1)^{a^\dagger a} = \int dx'\, |-x'\rangle\langle x'|\,. \quad (32)$$

One can show that

$$P\,|x_\pm\rangle = (\pm 1)\,|x_\pm\rangle \quad (33)$$

where these eigenstates of the parity operator are

$$|x_\pm\rangle = \frac{1}{\sqrt{2}}(|x\rangle \pm |-x\rangle). \quad (34)$$

A similar argument can be made for a LO using the − cat state but with the signs of the plus and minus on the right-hand side of Eq. (31) exchanged.

Returning to the derivation, we now complete the change of variables using the previously defined variable w, see Eq. (18). From Eq. (30) we can see that w is still distributed normally with mean $|\beta|/\sqrt{2}$ and variance $1/2$. This is because both n and m are approximately Poisson distributed with parameter $\lambda = |\beta|^2/2$ as detailed in appendix E. This gives the Kraus operator

$$M_{x,w}^{[\beta]_\pm}\sqrt{dwdx} \underset{|\beta|\to\infty}{\approx} \frac{e^{i\sqrt{2}|\beta|w\theta}}{\sqrt{2}}\Big[\langle x_\theta|\, e^{i\pi\sqrt{2}|\beta|(x-w)/2} \pm \langle -x_\theta|\, e^{i\pi\sqrt{2}|\beta|(x+w)/2}\Big] \frac{e^{-(w-|\beta|/\sqrt{2})^2/2}}{\pi^{1/4}}\sqrt{dwdx}\,. \quad (35)$$

For the detection process, we care primarily about the POVM which is

$$dw\,dx\,E_{x,w}^{[\beta]\pm} \underset{|\beta|\to\infty}{\approx} dw\,dx\left(M_{x,w}^{[\beta]\pm}\right)^{\dagger} M_{x,w}^{[\beta]\pm}. \tag{36}$$

Substituting our expressions in we find

$$\begin{aligned} dw\,dx\,E_{x,w}^{[\beta]\pm} \underset{|\beta|\to\infty}{\approx} dw\,dx\,\frac{e^{-(w-|\beta|/\sqrt{2})^2}}{\sqrt{\pi}}\frac{1}{2} \\ \Big[\,|x_\theta\rangle\langle x_\theta| \\ \pm |x_\theta\rangle\langle -x_\theta|\,e^{\sqrt{2}i\pi|\beta|w} \\ \pm |-x_\theta\rangle\langle x_\theta|\,e^{-\sqrt{2}i\pi|\beta|w} \\ + |-x_\theta\rangle\langle -x_\theta|\,\Big]. \end{aligned} \tag{37}$$

At this point we can still observe a coherence between the + and − outcomes of the measurement, see the terms with the ± coefficients above. So if knowledge of both the sum and difference variables is retained Eq. (37) is the final result.

However, if we integrate Eq. (37) over w, presuming that it is unobserved like in homodyne detection, gives an expression of the form

$$\begin{aligned} dx\int dw E_{x,w}^{[\beta]\pm} = \\ dx\Big[G(\beta)(|x_\theta\rangle\langle x_\theta| + |-x_\theta\rangle\langle -x_\theta|) \\ \pm I(\beta)(|x_\theta\rangle\langle -x_\theta| + |-x_\theta\rangle\langle x_\theta|)\Big], \end{aligned} \tag{38}$$

where

$$\begin{aligned} I(\beta) &= \int_0^\infty dw\,e^{\pm 2\sqrt{2}i\pi|\beta|w}\,\frac{e^{-(w-|\beta|/\sqrt{2})^2}}{\sqrt{\pi}} \\ &= \frac{1}{2}e^{-2\pi^2|\beta|^2}e^{\pm 2\pi i|\beta|^2}\left[1+\mathrm{erf}\left(\frac{(1\pm 2i\pi)|\beta|}{\sqrt{2}}\right)\right], \\ G(\beta) &= \int_0^\infty dw\,\frac{e^{-(w-|\beta|/\sqrt{2})^2}}{\sqrt{\pi}} = \frac{1}{2}\left[1+\mathrm{erf}\left(\frac{|\beta|}{\sqrt{2}}\right)\right]. \end{aligned} \tag{39}$$

Notice that $I(\beta)$ has an overall envelope of $\exp[-2\pi^2|\beta|^2]$ which very clearly limits to zero as $|\beta|\to\infty$. Thus in the large LO limit these expressions limit to

$$\lim_{|\beta|\to\infty} I(\beta) = 0 \quad \text{and} \quad \lim_{|\beta|\to\infty} G(\beta) = 1. \tag{40}$$

With this the limiting case in the probability is

$$\begin{aligned} dx\,E_x^{\text{cat}} &\underset{|\beta|\to\infty}{=} dx\int dw\,E_{x,w}^{[\beta]\pm} \\ &= \frac{dx}{2}\Big(\,|x_\theta\rangle\langle x_\theta| + |-x_\theta\rangle\langle -x_\theta|\,\Big) \end{aligned} \tag{41}$$

which satisfies the normalisation properties of a standard probability density,

$$\int dx\,E_x^{\text{cat}} = \mathbb{I}. \tag{42}$$

To summarise so far, using an odd or even cat state as a local oscillator we derived the Kraus operators and POVM for a homodyne like measurement. In the limit of large amplitudes in the cat states, the POVM for each measurement outcome is a sum of a quadrature eigenstate e.g. $|x_\theta\rangle$ and it's negative $|-x_\theta\rangle$. This operator projects onto this two-dimensional subspace and hence cannot distinguish between states which equally project onto that subspace. For example, the states $|x_\theta\rangle$, $|-x_\theta\rangle$ and $|c_\theta\rangle = \frac{1}{\sqrt{2}}(|x_\theta\rangle + |-x_\theta\rangle)$ will all give the same probability density. In other words, this measurement is symmetric about reflections through the quadrature origin. For this reason we call it a reflection symmetric measurement.

Note that after integrating out over the sum variable w and taking the large oscillator limit means that the coherence of the cat state in the local oscillator is irrelevant. So one can see such a measurement as a homodyne measurement with randomly chosen classical phase of either 0 or π.

Before the integration was performed, in Eq. (37) the coherence terms contain phases of the form $\sqrt{2}i\pi|\beta|w$. But from the definition of the variable w, $\sqrt{2}|\beta|w$ is an integer. Therefore this phase factor can be treated as ±1 but only with the knowledge of w whereas without this knowledge and the continuum approximation leads to this phase factor tending quickly to zero. The ability to use (or ignore) the information contained in the sum variable seems to be a new feature with the cat state local oscillator and provides a means to engineer a measurement.

There is hence a large number of interconnected concerns when taking these approximations together that may be fragile. In the next section we give numerical computations which try to address these concerns with computations involving finite sized local oscillator states.

IV. EXAMPLE INPUT STATES

The above analysis exposes some general properties of homodyne detection with these types of local oscillators, case studies give rise to some more specific information about the details of the measurement without reliance on numerous approximations. To do this we return to Eq. (30) to calculate the Kraus operators, but we are guided broadly by the properties uncovered in Sec. III.

1. *Vacuum*

For the case of a vacuum input state as the signal, the exact Kraus operator from Eq. (26) acting on a vacuum

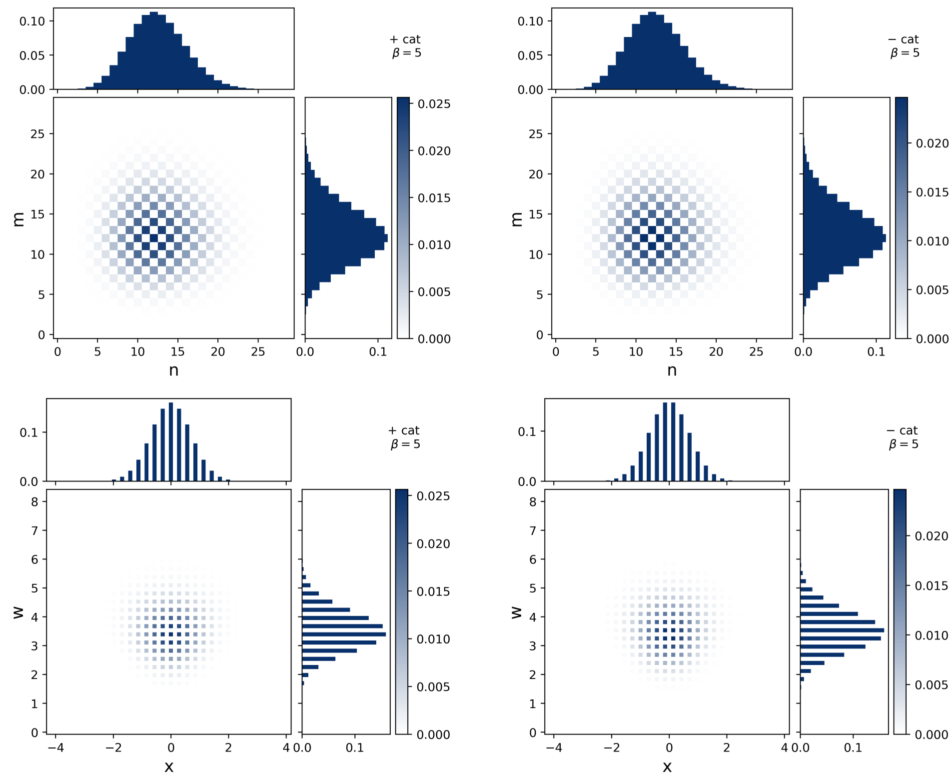


FIG. 2. Click distributions for detecting vacuum (the signal) with a Cat local oscillators where $\beta = 5$ and the expected number of photons in the b mode is $\langle b^\dagger b\rangle = |\beta|^2 = 25$. (Row 1) Original click distributions n and m, marginal distributions are depicted above and right for both the + and − cat LO. The distribution and marginals are centred around $|\beta|^2/2 = 12.5$. (Row 2) Sum $w = (n+m)/(\sqrt{2}\beta)$ and difference $x = (n-m)/(\sqrt{2}\beta)$ variables and the corresponding marginal distributions for the + and − cat LO. As vacuum is even parity state the parity of the LO is evident in the marginal of the difference variable. For example the + cat has support on $x = 0$ (see the marginal x distribution) while the − cat does not.

state gives

$$M^{[\beta]\pm}_{n,m} |0\rangle = \frac{e^{-|\beta|^2/2}}{2^{(n+m)/2}\sqrt{n!m!}\mathcal{N}_\pm(\beta)}\beta^{n+m}\left[(-1)^m \pm (-1)^n\right], \quad (43)$$

where $(1 \pm a/\beta)|0\rangle = |0\rangle$ and $(1 - a^2/\beta^2)|0\rangle = |0\rangle$ have been used to form this expression. The probability of detecting n and m photons is subsequently

$$\Pr(n, m|\beta, \pm, |0\rangle) = \frac{e^{-|\beta|^2/2}}{n!}\left(\frac{|\beta|^2}{2}\right)^n \frac{e^{-|\beta|^2/2}}{m!}\left(\frac{|\beta|^2}{2}\right)^m \left[\frac{(-1)^{m-n} \pm 1}{\mathcal{N}_\pm(\beta)}\right]^2. \quad (44)$$

This equation shows explicitly the underlying Possion distribution envelops with equal Possion variables in both detectors. The signal will be modulated by the alternating interference in the final term in the square brackets. This term comes entirely from the superposition state of the local oscillator.

Now one can change variables into the x and w sum and difference variables for this particular case. However, at this stage, as the continuum approximation has not been made, the expression for this is not much clearer. However, it should be noted that the final term in the square brackets *only* depends on the difference variable x.

In Fig. 2 we plot the probabilities generated by both the "+" cat (i.e. $|\mathrm{LO}\rangle \propto |\beta\rangle + |-\beta\rangle$) and the "−" cat (i.e. $|\mathrm{LO}\rangle \propto |\beta\rangle - |-\beta\rangle$) as a LO when measuring the vacuum. The in the original photodetection variables n, m (top row) the clicks of each variable individually are approximately Possion distributed with half to LO intensity $|\beta|^2/2$ due to the 50:50 beamsplitter. The effects from the superposition of the local oscillator are present in the full distribution of n and m where complete interference gives zero probability from the final term in Eq. (44). This gives rise to the "checker-board" style pattern in the full distribution and the exact terms where the pattern is non-zero depending on the phase of the LO cat-state superposition.

In the sum and difference variables x, w (bottom row) the two marginal distributions exhibit the interference more directly as they essentially look "diagonally" across

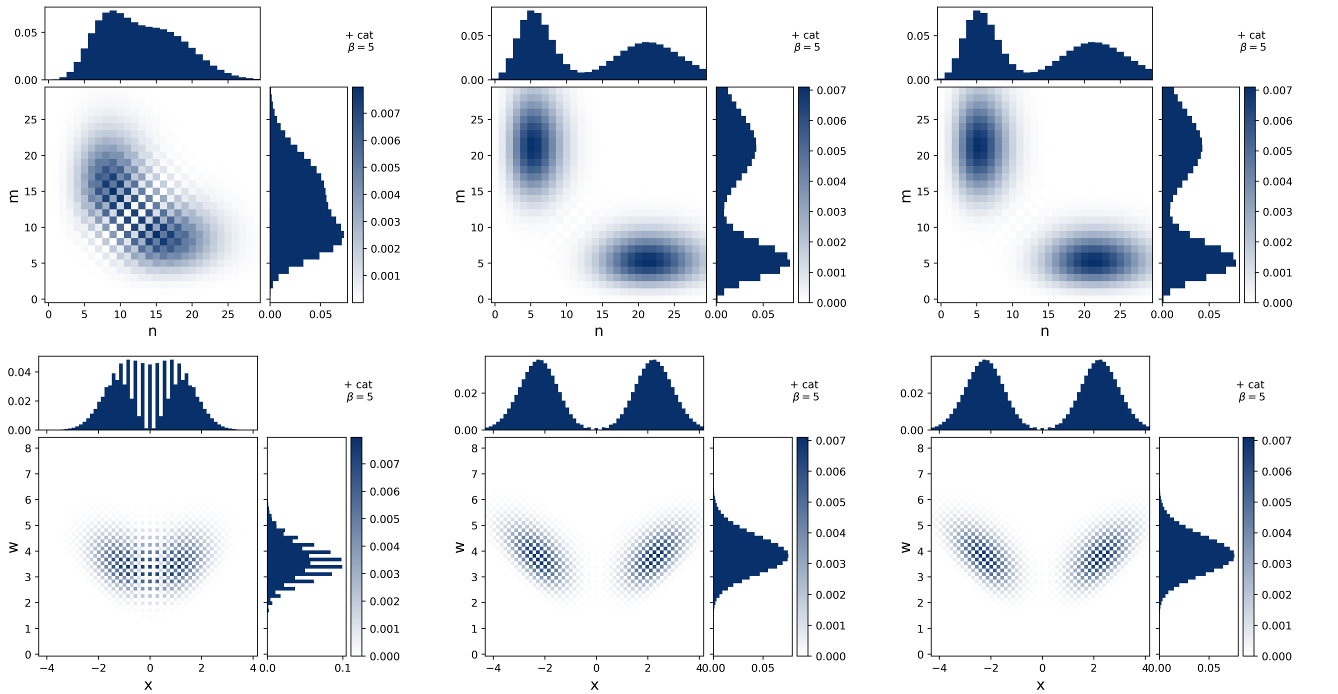


FIG. 3. Click distributions for detecting a coherent state signal with $\alpha = 0.8$ (column 1), $\alpha = 1.6$ (column 2), and $\alpha = -1.6$ (column 3) with a Cat local oscillators where $\beta = 5$ and $\langle n_b \rangle = 25$. (Row 1) Original click distributions and marginal distributions for n and m. (Row 2) Sum $w = (n + m)/(\sqrt{2}\beta)$ and difference $x = (n - m)/(\sqrt{2}\beta)$ variables and corresponding marginal distributions. In contrast to the previous plot the difference variables now contain little probability around $x = 0$. Column 2 and 3 are identical as this measurement can not distinguish between $|\alpha\rangle$ and $|-\alpha\rangle$.

the n, m distribution. The superposition in the local oscillator is evident in these distributions having non-zero probabilities only on odd or even number term depending on the sign of the local oscillator superposition. Note that when changing to x, w variables there exist particular combinations of variables that are permitted for individual x and w but not necessarily when combined together. For example, if $x = 0$ then $m = n$ and hence $m + n$ must be an even number and w is an even multiple of $1/(\sqrt{2}\beta)$. But if $x = 1/(\sqrt{2}\beta)$ (e.g. if $n = 1, m = 0$ or $n = 3, m = 2$, etc), then $n = m + 1$ and $n - m$ is odd. Hence $x = 1/(\sqrt{2}\beta)$ with $w = 1/(\sqrt{2}\beta)$ is not permitted. As, depending on the superposition sign in the LO, particular parities of photon number are suppressed, this leads to stripes in the x and w marginal distributions. These stripes are shifted by $1/(\sqrt{2}\beta)$ between the $+$ and $-$ superpositions in the LO. This shifting occurs as the vacuum state has a definite even parity. Therefore the parity of the sum and difference variables needs to preserve the overall parity relationship between the combined signal and LO.

After the information of the sum variable w is integrated out, the stripes still remain. This is because the parity of the sum and difference variables is determined by the parity of the input states only. This is unlike the measurement in the raw counts m and n where the parity is encoded between the measurement outcomes as well as the parity of the input state. Therefore the transformation to the sum and difference variables gives measurement outcomes that relate the measurements of parity even after integrating out one of the variables. However, the parity information is encoded in outputs that are separated by $1/\sqrt{2}|\beta|$ which is equivalent to single integer changes in the n or m variables. Any process that influences these numbers by a single integer, such as photon loss, would drastically reduce the visibility of this property.

This property of similar marginal distributions shifted by $1/(\sqrt{2}\beta)$ is commonly shared with different possible input signals. For this reason, for other input states, we will only plot the $+$ superposition case of the local oscillator. With the input state being the vacuum, the marginal distributions between n and m or between x and w look very similar. This situation will change as we look a measuring signals from coherent light.

2. *Coherent state*

For the case of a coherent state input to the detector with the superposition local oscillator the amplitude generated from the Kraus operator is

$$M_{n,m}^{[\beta]\pm}|\alpha\rangle = \frac{e^{-|\beta|^2/2}e^{-|\alpha|^2/2}}{2^{(n+m)/2}\sqrt{n!m!}\mathcal{N}_\pm(\beta)} \left[(\alpha+\beta)^n(\alpha-\beta)^m \pm (\alpha-\beta)^n(\alpha+\beta)^m\right]. \quad (45)$$

The probability of detecting n and m photons is subsequently

$$P_{n,m}^{[\beta]\pm}(|\alpha\rangle) = \frac{e^{-|\beta|^2-|\alpha|^2}}{n!m!\mathcal{N}_\pm(\beta)^2} \left|\left(\frac{\alpha+\beta}{\sqrt{2}}\right)^n\left(\frac{\alpha-\beta}{\sqrt{2}}\right)^m \pm \left(\frac{\alpha-\beta}{\sqrt{2}}\right)^n\left(\frac{\alpha+\beta}{\sqrt{2}}\right)^m\right|^2. \quad (46)$$

These probabilities are shown in Fig. 3 for input signal amplitudes of $\alpha = 0.8, 1.6$ and -1.6 and coherent state superpositions for the local oscillators with $\beta = 5$ just like in Fig. 2.

One of the striking things to notice in Fig. 3 are the spikes in the sum and difference variables of the $\alpha = 0.8$ signal, which might be an artefact of using photon number resolving detectors to approximate an intensity measurement. Although, similar spikes are present in the work of Sanders *et al.* [13] which considered a coherent state LO interfering with a cat state signal and in the marginal distribution of a cat state Wigner function. For larger intensity signals e.g. $\alpha = 1.6$ we see the marginal distributions have smoothed out significantly. Importantly in columns 2 and 3 we can see that our measurement does not allow one to distinguish between $|\alpha\rangle$ and $|-\alpha\rangle$. In the large LO limit the distribution of the x variable seems limit to

$$\Pr(x|\alpha) = \mathrm{Tr}[E_x^{\mathrm{cat}}|\alpha\rangle\langle\alpha|] = \frac{1}{2}\big(|\langle\alpha|x_\theta\rangle|^2 + |\langle\alpha|{-x_\theta}\rangle|^2\big), \quad (47)$$

where $\langle\alpha|x_\theta\rangle$ is the inner product between a coherent state and a rotated quadrature eigenstate. This expression also holds when $\alpha = 0$ as in the previous section. Of course this is not the case for the first column where the interference terms are still visible.

3. *Fock state*

For the case of a Fock state input state, i.e. $n|p\rangle = p|p\rangle$, as the signal, equation 26 acting on such a state gives

$$M_{n,m}^{[\beta]\pm}|p\rangle = \frac{e^{-|\beta|^2/2}}{2^{(n+m)/2}\sqrt{n!m!}\mathcal{N}_\pm(\beta)} \langle 0|\left((\hat{a}+\beta)^n(\hat{a}-\beta)^m \pm (\hat{a}-\beta)^n(\hat{a}+\beta)^m\right)|p\rangle. \quad (48)$$

In appendix F we show how to simplify this expression. the resulting closed form helps mainly with numerical computation but offers little insight into the general functional properties. In the large LO limit the distribution of the x variable should limit to

$$\Pr(x|p) = \mathrm{Tr}[E_x^{\mathrm{cat}}|p\rangle\langle p|] = \frac{1}{2}\big(|\langle p|x_\theta\rangle|^2 + |\langle p|{-x_\theta}\rangle|^2\big), \quad (49)$$

where $\langle q|x_\theta\rangle$ is the inner product between a Fock state and a rotated quadrature eigenstate. This in itself will be proportional to the square of a Hermite polynomial $H_p(x)$ as

$$\langle x|p\rangle = \frac{1}{\sqrt{2^p p!}}H_p(x)\langle x|0\rangle. \quad (50)$$

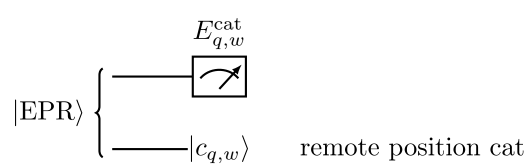
In Fig. 4 the appearance of the square of a Hermite polynomial is evident in the difference variable marginal distributions. Also evident is the effect of parity on the distribution of difference variable x. Columns 1 and 3 have odd parity input states ($|1\rangle, |3\rangle$), while column 2 has and even input parity ($|2\rangle$). Like in the vacuum case the even parity states have support on the difference variable when $x = 0$ and the odd parity states do not.

V. APPLICATION OF CATODYNE TO REMOTE STATE PREPARATION

We now briefly describe an application of our measurement to remotely preparing superposition of position eigenstates or a "Schrödinger Cat state in position". The idea is to have two parties share an EPR state (which is a Gaussian state) and then perform our non-Gaussian "catodyne" measurement, i.e. Eq. (35), on half of the EPR pair. Then conditional on the measurement result q, w the state

$$|c_{q,w}\rangle = \frac{1}{\sqrt{2}}(|q\rangle + (-1)^{f(w)}|-q\rangle) \quad (51)$$

is prepared remotely. In Eq. (51) $|q\rangle$ is an eigenstate of the position operator x i.e. $x|q\rangle = q|q\rangle$, and $f(w)$ is a linear function of the sum variable w. This procedure is summarised in the following circuit.



(52)

The initial state between the remote parties is an ideal EPR state in the position representation

$$|\mathrm{EPR}\rangle = \int dx\, |x\rangle \otimes |x\rangle, \quad (53)$$

which is unnormalizable and unphysical but a limit of states that are routinely made in the optical domain using two mode squeezing.

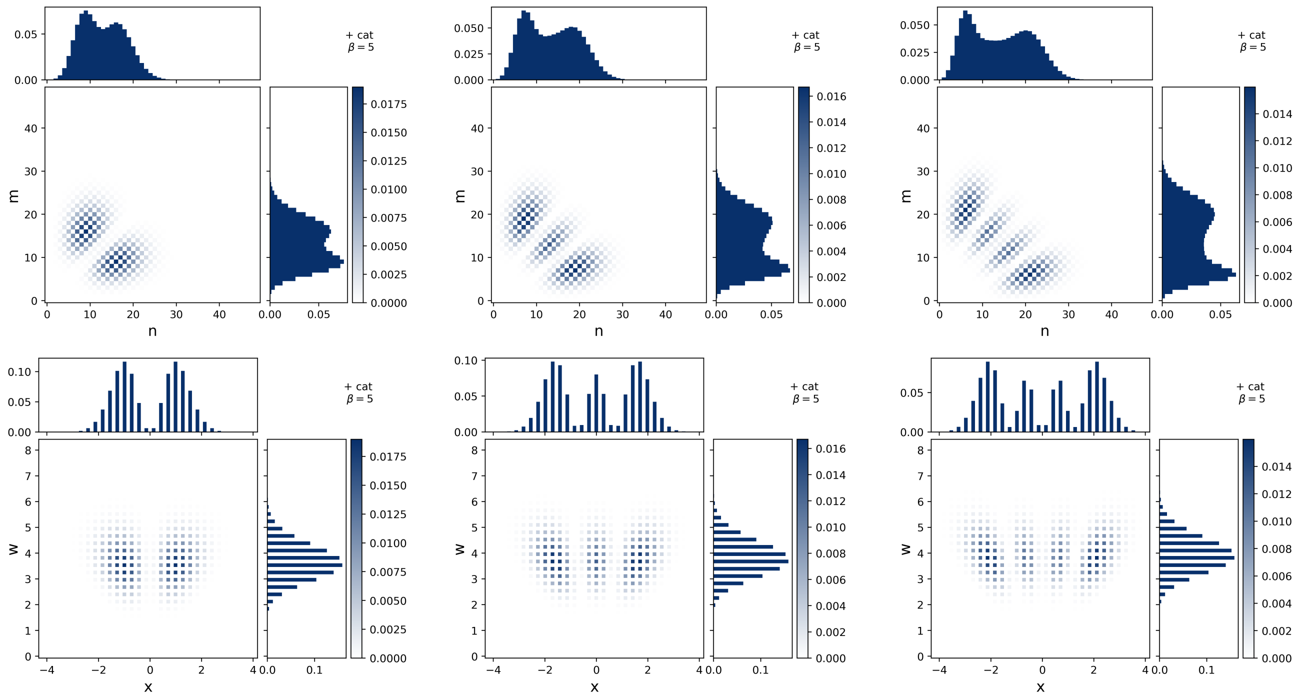


FIG. 4. Click distributions for detecting a Fock state $|p\rangle$ with $p = 1$ (column 1), $p = 2$ (column 2), and $p = 3$ (column 3) with a Cat local oscillators where $\beta = 5$. (Row 1) Original click distributions and marginal distributions for n and m. (Row 2) Sum $w = (n+m)/(\sqrt{2}\beta)$ and difference $x = (n-m)/(\sqrt{2}\beta)$ variables and corresponding marginal distributions.

We will choose to measure one of the systems and allow the other to freely propagate. To compute the state of both systems after the measurement we use the usual measurement update rule

$$|\Phi_{q,w}\rangle = \frac{(M^{[+]}_{q,w} \otimes \mathbb{I})\,|\text{EPR}\rangle}{\sqrt{\langle \text{EPR}|E^{[+]}_{q,w}|\text{EPR}\rangle}}. \tag{54}$$

Since the denominator simply normalize the post measurement state it is instructive to consider the numerator of Eq. (54) alone

$$(M^{[+]}_{q,w} \otimes \mathbb{I})\,|\text{EPR}\rangle \propto (\langle q| \pm \langle -q|) \otimes \mathbb{I} \int dx\, |x\rangle \otimes |x\rangle \ , \tag{55}$$

where we have replaced the w dependence of $M_{q,w}$ with a parity bit $\pm$ so simplify our presentation. Moreover it is clear that the measured mode is absorbed by the detector. Performing the integrals and normalizing the state we arrive at

$$|\Phi_{q,w}\rangle = \frac{1}{\sqrt{2}}(|q\rangle + (-1)^{f(w)}\,|-q\rangle), \tag{56}$$

which is the Schrödinger cat state in position on the remote system.

VI. CONCLUSION

Using a non-classical local oscillator we have constructed a new non-Gaussian measurement. One of the new features that arose, relative to standard homodyne measurements, was the importance of the sum variable. Using the information from the sum variable led to a measurement that had coherence between outcomes (a rank-1 POVM), while integrating out the sum variable led to a loss of coherence (a rank-2 POVM).

Moreover we have shown that these measurements can be used to remotely prepare a non-Gaussian state. While we did not get non-Gaussiannity for free, we injected a LO that was non-Gaussian, the ability to prepare non-Gaussian states via measurement in a teleportation scheme might find applications in quantum computation and communications. The utility of these measurements outside it's usefulness in remote state preparation is unknown. However it is heartening to note that ordinary homodyne (or heterodyne) measurements can be used to measure nonlinear properties such as correlation functions [22]. Thus it is possible that the measurements described herein may find useful and exotic applications.

There are many possible extensions of this work for example one could consider other non-classical states where you can take a large LO limit. Moreover, we have not numerically or analytically studied the convergence of the full measurement operators to the strong local oscillator limit [6, 11]. An important open question is how large does the local oscillator have to be a reasonable approximation to the limiting measurement operators. A second and equally important question is the extension of our single mode analysis to detection of multimode fields [4, 5, 23–26]. Some stepping stones to this end have been made by e.g. Gough *et al.* [27] and Dąbrowska [28] where a quantum trajectory formulation of fields in superpostions of coherent states was derived.

Some of the artefacts we saw in the numerics were due to the fact we considered an idealized situation where we had number resolving detectors. A approximation of an intensity measurement would be to use the finite-efficiency photon number POVM. Given an efficiency $\eta \in [0,1]$ the measurement operator representing report n clicks is

$$E_n^{(\eta)} = \sum_{n=0}^{\infty} \binom{n+m}{n} \eta^n (1-\eta)^m \, |n+m\rangle\langle n+m| \,. \quad (57)$$

Using this operator in the analysis should remove the spurious issues with parity of the w variable we noted in Sec. III. However it is unclear if this will result in a rank 1 or rank 2 POVM.

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Appendix A: Coherent state signals for regular homodyne

In this section we use a coherent state, $|\alpha\rangle$, as a proxy for an arbitrary signal states with $\langle n_{\rm signal}\rangle \ll \langle n_{\rm LO}\rangle$ i.e. $|\alpha|^2 \ll |\beta|^2$. This let's us reason about properties of the click distribution that are due, largely, to the LO.

If the input signal is an coherent state $|\alpha\rangle$ then

$$M_{n,m}^{[\beta]} |\alpha\rangle = \frac{(\alpha+\beta)^n(\alpha-\beta)^m}{2^{(n+m)/2}\sqrt{n!m!}} e^{-(|\beta|^2+|\alpha|^2)/2} |0\rangle = \frac{(\alpha+\beta)^n}{2^{n/2}\sqrt{n!}} e^{-(|\alpha+\beta|)^2/2} \frac{(\alpha-\beta)^m}{2^{m/2}\sqrt{m!}} e^{-(|\alpha-\beta|)^2/2} |0\rangle \quad \text{(A1)}$$

which gives probabilities proportional to separate Poisson distributions

$$P^{[\beta]}(n,m) = \frac{|\alpha+\beta|^{2n}}{2^n n!} e^{-(|\alpha+\beta|)^2} \frac{|\alpha-\beta|^{2m}}{2^m m!} e^{-(|\alpha-\beta|)^2}. \quad \text{(A2)}$$

In the case of a strong local oscillator relative to the input signal, i.e. $\beta \gg \alpha$, then the distributions of n and m are individually peaked nearby a mean value of $|\beta|^2/2$ with standard deviation $\beta/\sqrt{2}$.

The mean difference between n and m (normalised by $\sqrt{2}|\beta|$) can be computed using the statistical moments of the Poisson distribution

$$\frac{\mathrm{E}(n-m)}{\sqrt{2}|\beta|} = \frac{|\alpha+\beta|^2}{2\sqrt{2}|\beta|} - \frac{|\alpha-\beta|^2}{2\sqrt{2}|\beta|} = \frac{e^{-i\theta}\alpha + e^{i\theta}\alpha^*}{\sqrt{2}} \quad \text{(A3)}$$

where $\mathrm{E}(n-m)$ is the expectation of the difference variable and θ is the complex angle of $\beta = |\beta|e^{i\theta}$. This recovers the quadrature component and shows how the signal output is converted into the quadrature signal. The variance is then

$$\frac{\mathrm{Var}(n-m)}{2|\beta|^2} = \frac{|\alpha+\beta|^2}{4|\beta|} + \frac{|\alpha-\beta|^2}{4|\beta|} = \frac{|\alpha|^2+|\beta|^2}{2|\beta|^2} = \frac{1}{2} + \frac{|\alpha|^2}{2|\beta|^2} \quad \text{(A4)}$$

as the variances add. In the strong local oscillator limit the second term tends to zero and this variance approaches the variance of the input coherent state.

Appendix B: Error estimate in approximation

We wish to expand a function f of a random variable X about the mean of X i.e. $\mathrm{E}[X]$. The expansion is [29]

$$\mathrm{E}[f(X)] \approx f\Big(\mathrm{E}[X]\Big) + \frac{1}{2}\mathrm{Var}[X] f''\Big(\mathrm{E}[X]\Big) + \dots \quad \text{(B1)}$$

The function of the random variable m we care about is

$$f(m) = \left(1 - \frac{e^{-2i\theta}\hat{a}^2}{|\beta|^2}\right)^m \tag{B2}$$

Recall that m is the number of clicks in one of the detectors. Because the LO overwhelms the signal, the number of clicks is approximately Poisson distributed and that means that $\mathrm{E}[m] = \mathrm{Var}[m] = \langle m \rangle = |\beta|^2/2$. Further the second derivative of $(1-x)^m$ with respect to m is $(1-x)^m \ln^2(1-x)$. Combined this gives us

$$\mathrm{E}[f(m)] \approx \left(1 - \frac{e^{-2i\theta}\hat{a}^2}{|\beta|^2}\right)^{|\beta|^2/2} + \frac{1}{4}|\beta|^2 \left(1 - \frac{a^2 e^{-2i\theta}}{|\beta|^2}\right)^{|\beta|^2/2} \log^2 \left(1 - \frac{a^2 e^{-2i\theta}}{|\beta|^2}\right) \tag{B3}$$

Now we approximate $\ln^2(1-x) \approx x^2 + O(x^3)$ thus

$$\mathrm{E}[f(m)] \approx \left(1 - \frac{e^{-2i\theta}\hat{a}^2}{|\beta|^2}\right)^{|\beta|^2/2} \left[1 + \frac{1}{4|\beta|^2}\left(a^2 e^{-2i\theta}\right)^2\right], \tag{B4}$$

which shows the leading order correction is $O(1/|\beta|^2)$ as claimed.

So we can be completely comfortable with the approximation we investigate the variance of $f(X)$ as well. The expansion is [29] $\mathrm{Var}\left[f(X)\right] \approx \left(f'(\mathrm{E}\left[X\right])\right)^2 \mathrm{Var}\left[X\right]$. It turns out that it scales as $1/|\beta|^2$ so in the limit $|\beta| \to \infty$ the variance becomes zero.

Appendix C: When $m > n$

1. Coherent state local oscillator

Consider $m > n$ and pull a factor of $(1 - a/\beta)^n$ out of Eq. (6) to arrive at

$$M^{(\beta)}_{n,m} = \langle 0| \left(1 - \frac{\hat{a}}{\beta}\right)^{m-n} \left(1 - \frac{\hat{a}^2}{\beta^2}\right)^n \frac{e^{-|\beta|^2/4}}{\sqrt{n!}} \left(\frac{\beta}{\sqrt{2}}\right)^n \frac{e^{-|\beta|^2/4}}{\sqrt{m!}} \left(\frac{-\beta}{\sqrt{2}}\right)^m. \tag{C1}$$

Using the same definition of x from the main text means that the exponent will have a minus sign which in the $\beta \to \infty$ limit results in the same expression. The second operator term has the exponent changed to n which has the same peak in it's distribution and so again, results in the same asymptotic limit.

2. Superposition state local oscillator

Had we chosen that $m > n$ after equation 26, then this would change equation 27 to become

$$M^{[\beta]\pm}_{n,m} = \langle 0| \frac{e^{-|\beta|^2/2}}{2^{(n+m)/2}\sqrt{n!m!}\mathcal{N}_\pm(\beta)} \left(1 - \frac{\hat{a}^2}{\beta^2}\right)^n \left[\beta^n(-\beta)^m \left(1 + \frac{\hat{a}}{\beta}\right)^{m-n} \pm (-\beta)^n \beta^m \left(1 - \frac{\hat{a}}{\beta}\right)^{m-n}\right]. \tag{C2}$$

If now variables are relabelled to swap n and m

$$M^{[\beta]\pm}_{n,m} = \langle 0| \frac{e^{-|\beta|^2/2}}{2^{(n+m)/2}\sqrt{n!m!}\mathcal{N}_\pm(\beta)} \left(1 - \frac{\hat{a}^2}{\beta^2}\right)^m \left[(-1)^n \beta^{m+n} \left(1 + \frac{\hat{a}}{\beta}\right)^{n-m} \pm (-1)^m \beta^{m+n} \left(1 - \frac{\hat{a}}{\beta}\right)^{n-m}\right]. \tag{C3}$$

This equation, up to a global phase factor, only differs from Eq. (27) in the superposition phase depending on the parity of $n - m$. Therefore, as the analysis presented in the main text incorporates both superposition phases, it actually also covers the case of $m > n$ if the information as to which superposition phase applies is incorporated.

Appendix D: Quadrature Eigenstates

This appendix has two parts. In appendix D 1 we show that

$$|x_\varphi\rangle = \frac{e^{-x^2/2}}{\pi^{1/4}} e^{\sqrt{2}x\chi a^\dagger} e^{-\chi^2 a^{\dagger 2}/2} |0\rangle \,, \tag{D1}$$

with $\chi = e^{-i\theta}$ (for consistency with the LO in the main text in Eq. (9)), is an eigenstate of

$$Q(\varphi) = (e^{-i\varphi}a + e^{i\varphi}a^\dagger)/\sqrt{2} \tag{D2}$$

with eigenvalue x, that is

$$Q(\varphi)\,|x_\varphi\rangle = x\,|x_\varphi\rangle \,. \tag{D3}$$

Then in appendix D 2 we show that

$$\langle x'_\varphi|x_\varphi\rangle = \delta(x - x'). \tag{D4}$$

1. Eigenstates

Our method is inspired by Ref. [20]. We start by considering the operator $Q(\varphi)$ acting on $|x_\varphi\rangle$,

$$Q(\varphi)\,|x_\varphi\rangle = \frac{e^{-x^2/2}}{\pi^{1/4}} \frac{e^{-i\varphi}a + e^{i\varphi}a^\dagger}{\sqrt{2}} e^{-\chi^2 a^{\dagger 2}/2} e^{\sqrt{2}x\chi a^\dagger} |0\rangle \,.$$

Note that two operator exponential commute and so their order does not matter. Left multiply the above equation using a resolution of the the identity

$$I = e^{-\chi^2 a^{\dagger 2}/2} e^{\sqrt{2}x\chi a^\dagger} e^{-\sqrt{2}x\chi a^\dagger} e^{\chi^2 a^{\dagger 2}/2}, \tag{D5}$$

which we will then try to remove the quadrature operator by evaluating the conjugations that surround it. We use of the following commutation relation, $[a, f(a^\dagger)] = \frac{\partial}{\partial a^\dagger} f(a^\dagger)$, for any smooth function f. If $f(a^\dagger) = e^{-\chi^1 a^{\dagger 2}/2}$, then

$$a e^{-\chi^2 a^{\dagger 2}/2} - e^{-\chi^2 a^{\dagger 2}/2} a = -\chi^2 a^\dagger e^{-\chi^2 a^{\dagger 2}/2} \tag{D6}$$

Defining an operator G which is a conjugated version of the quadrature operator using only the first part of the identity resolution, and using the above equation gives,

$$G = e^{\chi^2 a^{\dagger 2}/2} Q(\varphi) e^{-\chi^2 a^{\dagger 2}/2} \tag{D7}$$

$$= e^{\chi^2 a^{\dagger 2}/2} \left(\frac{e^{-i\varphi}a + e^{i\varphi}a^\dagger}{\sqrt{2}} \right) e^{-\chi^2 a^{\dagger 2}/2} \tag{D8}$$

$$= \frac{e^{i\varphi}}{\sqrt{2}} a^\dagger + \frac{e^{-i\varphi}}{\sqrt{2}} (a - \chi^2 a^\dagger) \tag{D9}$$

$$= \frac{a^\dagger}{\sqrt{2}} (e^{i\varphi} - \chi^2 e^{-i\varphi}) + \frac{e^{-i\varphi}}{\sqrt{2}} a. \tag{D10}$$

Next, using the same derivative commutation relation, but with the choice $f(a^\dagger) = e^{\sqrt{2}x\chi a^\dagger}$

$$a e^{\sqrt{2}x\chi a^\dagger} - e^{\sqrt{2}x\chi a^\dagger} a = \sqrt{2}x\chi e^{\sqrt{2}x\chi a^\dagger}, \tag{D11}$$

the G operator can be conjugated again to give the operator H,

$$H = e^{-\sqrt{2}x\chi a^\dagger} G e^{\sqrt{2}x\chi a^\dagger} \tag{D12}$$

$$= e^{-\sqrt{2}x\chi a^\dagger} \left(\frac{a^\dagger}{\sqrt{2}} (e^{i\varphi} - \chi^2 e^{-i\varphi}) + \frac{e^{-i\varphi}}{\sqrt{2}} a \right) e^{\sqrt{2}x\chi a^\dagger}$$

$$= \frac{a^\dagger}{\sqrt{2}} (e^{i\varphi} - \chi^2 e^{-i\varphi}) + \frac{e^{-i\varphi}}{\sqrt{2}} (a + \sqrt{2}x\chi). \tag{D13}$$

Now we set $\theta = -\varphi$ or $\chi = e^{i\theta}$ (which gives consistency with Eq. (13)) which sets the $a^\dagger$ term to zero, and recall that $a\left|0\right\rangle = 0$

$$Q(\varphi)\left|x_\varphi\right\rangle = e^{-\chi^2 a^{\dagger 2}/2} e^{\sqrt{2}x\chi a^\dagger} H\left|0\right\rangle = x e^{-\chi^2 a^{\dagger 2}/2} e^{\sqrt{2}x\chi a^\dagger}\left|0\right\rangle = x\left|x_\varphi\right\rangle, \tag{D14}$$

which shows that the state in D1 is an eigenstate of the quadrature operator. All that remains is to normalise this state to give the desired result.

2. Inner product

In this appendix we compute the normalisation of two squeezed states $|x_\varphi\rangle$ and $|x'_\varphi\rangle$, see Eq. (13). We will show that

$$\langle x'_\varphi|x_\varphi\rangle = \frac{e^{-(x'^2+x^2)/2}}{\sqrt{\pi}}\left\langle 0\right| e^{-\chi^{*2}a^2/2} e^{\sqrt{2}x'\chi^* a} e^{\sqrt{2}x\chi a^\dagger} e^{-\chi^2 a^{\dagger 2}/2}\left|0\right\rangle = \delta(x-x') = \langle x'|x\rangle, \tag{D15}$$

where x and x' are quadrature eigenstates. Note that the normalisation of an x eigenstate is very different to any squeezed state. So we should not expect $\left|\psi\right\rangle = S(r)\left|0\right\rangle$, with length $\sqrt{|\langle\psi|\psi\rangle|^2} = 1$, to have any relationship to $\left|x\right\rangle$ which is unbounded and behaves like a delta function.

To simplify this expression we will insert the identity operator in the coherent state basis twice, i.e.

$$\frac{1}{\pi}\int d^2\alpha\left|\alpha\right\rangle\left\langle\alpha\right| = I. \tag{D16}$$

If we define $\mathcal{N} = e^{-x^2/2}/\pi^{1/4}$ and $\mathcal{N}' = e^{-x'^2/2}/\pi^{1/4}$ and $\mathcal{N}^2 := \mathcal{N}'\mathcal{N}$. Then the inner product becomes

$$\begin{aligned}\langle x'_\varphi|x_\varphi\rangle &= \mathcal{N}^2\left\langle 0\right| e^{-\chi^{*2}a^2/2} e^{\sqrt{2}x'\chi^* a} e^{\sqrt{2}x\chi a^\dagger} e^{-\chi^2 a^{\dagger 2}/2}\left|0\right\rangle \\ &= \frac{\mathcal{N}^2}{\pi^2}\int d^2\alpha d^2\beta\ \left\langle 0\right| e^{-\chi^{*2}a^2/2}\left|\alpha\right\rangle\left\langle\alpha\right| e^{\sqrt{2}x'\chi^* a} e^{\sqrt{2}x\chi a^\dagger}\left|\beta\right\rangle\left\langle\beta\right| e^{-\chi^2 a^{\dagger 2}/2}\left|0\right\rangle.\end{aligned}$$

Next we re-order (in normal order) or commute exponentials of a and $a^\dagger$ i.e.

$$e^{\sqrt{2}x'\chi^* a} e^{\sqrt{2}x\chi a^\dagger} = e^{2xx'} e^{\sqrt{2}x\chi a^\dagger} e^{\sqrt{2}x'\chi^* a}, \tag{D17}$$

where $\chi = e^{-i\theta}$. Doing so and simplifying gives

$$\begin{aligned}\langle x'_\varphi|x_\varphi\rangle &= \frac{\mathcal{N}^2}{\pi^2}\int d^2\alpha d^2\beta\, e^{-|\alpha|^2/2} e^{-\chi^{*2}\alpha^2/2} e^{-|\beta|^2/2} e^{-\chi^2\beta^{*2}/2} e^{2xx'}\left\langle\alpha\right| e^{\sqrt{2}x\chi a^\dagger} e^{\sqrt{2}x'\chi^* a}\left|\beta\right\rangle \\ &= \frac{\mathcal{N}^2}{\pi^2}\int d^2\alpha d^2\beta e^{-|\alpha|^2/2} e^{-\chi^{*2}\alpha^2/2} e^{-|\beta|^2/2} e^{-\chi^2\beta^{*2}/2} e^{2xx'} e^{\sqrt{2}x\chi\alpha^*} e^{\sqrt{2}x'\chi^*\beta} e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2-2\alpha^*\beta)} \\ &= e^{2xx'}\frac{\mathcal{N}^2}{\pi^2}\int d^2\alpha e^{-|\alpha|^2} e^{-\chi^{*2}\alpha^2/2} e^{\sqrt{2}x\chi\alpha^*}\int d^2\beta e^{-|\beta|^2} e^{-\chi^2\beta^{*2}/2} e^{(\sqrt{2}\chi^* x'+\alpha^*)\beta} \\ &= e^{2xx'}\frac{\mathcal{N}^2}{\pi^2}\int d^2\alpha e^{-|\alpha|^2} e^{-\chi^{*2}\alpha^2/2} e^{\sqrt{2}x\chi\alpha^*}\pi e^{-\frac{\chi^2}{2}(\sqrt{2}\chi^* x'+\alpha^*)^2},\end{aligned} \tag{D18}$$

where we used the integral $\int d^2\gamma e^{-|\beta|^2} e^{-a\beta^{*2}} e^{c\beta} = \pi e^{-ac^2}$ to arrive at the last line. Further manipulations give

$$\begin{aligned}\langle x'_\varphi|x_\varphi\rangle &= e^{2xx'} e^{-x'^2}\frac{\mathcal{N}^2}{\pi}\int d^2\alpha e^{-|\alpha|^2} e^{-\chi^{*2}\alpha^2/2} e^{-\chi^2\alpha^{*2}/2} e^{\sqrt{2}(x-x')\chi\alpha^*} \\ &= e^{2xx'} e^{-x'^2}\frac{\mathcal{N}^2}{\pi}\int d^2\alpha e^{-\frac{1}{2}(\chi^*\alpha+\chi\alpha^*)^2} e^{\sqrt{2}(x-x')\chi\alpha^*} \\ &= e^{2xx'} e^{-x'^2} e^{\frac{1}{4}(x-x')^2}\frac{\mathcal{N}^2}{\sqrt{2\pi}}\int d\Im\{\chi^*\alpha\} e^{-i\sqrt{2}(x-x')\Im\{\chi^*\alpha\}} \\ &= e^{2xx'} e^{-x'^2} e^{\frac{1}{4}(x-x')^2}\frac{\mathcal{N}^2}{2\sqrt{\pi}}\int dk e^{-ik(x-x')} \\ &= e^{2xx'} e^{-x'^2} e^{\frac{1}{4}(x-x')^2}\frac{e^{-x'^2/2}e^{-x^2/2}}{\sqrt{\pi}}\sqrt{\pi}\delta(x-x').\end{aligned} \tag{D19}$$

As the last part is $\delta(x-x')$, only the value at $x=x'$ matters. Hence this final expression is equivalent to

$$\langle x'_\varphi | x_\varphi\rangle = \delta(x-x'). \tag{D20}$$

By multiplying the state by the square-root of the inverse of the pre-factor before the delta function, gives the standard normalisation for position eigenstates.

Appendix E: Cat state homodyne

In this section we use coherent states as a proxy for an arbitrary signal states with $\langle n_{\text{signal}}\rangle \ll \langle n_{\text{LO}}\rangle$. This let's us reason about properties of the click distribution that are due, largely, to the LO.

$$M^{[\beta]\pm}_{n,m}|\alpha\rangle = \frac{e^{-(|\beta|^2+|\alpha|^2)/2}}{2^{(n+m)/2}\sqrt{n!m!}\mathcal{N}_\pm(\beta)}\left((\alpha+\beta)^n(\alpha-\beta)^m \pm (\alpha-\beta)^n(\alpha+\beta)^m\right) \tag{E1}$$

$$= \frac{1}{\mathcal{N}_\pm(\beta)}\left(\frac{(\alpha+\beta)^n}{2^{n/2}\sqrt{n!}}e^{-|\alpha+\beta|^2/2}\frac{(\alpha-\beta)^m}{2^{m/2}\sqrt{m!}}e^{-|\alpha-\beta|^2/2} \pm \frac{(\alpha-\beta)^n}{2^{n/2}\sqrt{n!}}e^{-|\alpha-\beta|^2/2}\frac{(\alpha+\beta)^m}{2^{m/2}\sqrt{m!}}e^{-|\alpha+\beta|^2/2}\right) \tag{E2}$$

which is an amplitude. The detection probabilities are

$$\begin{aligned}\Pr(n,m|\pm,\beta,\alpha) = \frac{1}{2(1\pm e^{-2|\beta|^2})}\Bigg[&\frac{|\alpha+\beta|^{2n}}{2^n n!}e^{-|\alpha+\beta|^2}\frac{|\alpha-\beta|^{2m}}{2^m m!}e^{-|\alpha-\beta|^2}\\ &\pm\frac{(\alpha+\beta)^n(\alpha^*-\beta^*)^n}{2^n n!}e^{-|\alpha+\beta|^2}\frac{(\alpha-\beta)^m(\alpha^*+\beta^*)^m}{2^m m!}e^{-|\alpha-\beta|^2}\\ &\pm\frac{(\alpha^*+\beta^*)^n(\alpha-\beta)^n}{2^n n!}e^{-|\alpha+\beta|^2}\frac{(\alpha^*-\beta^*)^m(\alpha+\beta)^m}{2^m m!}e^{-|\alpha-\beta|^2}\\ &+\frac{|\alpha-\beta|^{2n}}{2^n n!}e^{-(|\alpha-\beta|)^2}\frac{|\alpha+\beta|^{2m}}{2^m m!}e^{-(|\alpha+\beta|)^2}\Bigg]\end{aligned} \tag{E3}$$

which gives probabilities proportional to separate Poisson distributions.

1. Marginal click distribution for $\alpha = 0$

Below we assume that β is real, i.e. $\beta = |\beta|$

$$M^{[\beta]\pm}_{n,m}|0\rangle = \frac{e^{-(|\beta|^2)/2}}{2^{(n+m)/2}\sqrt{n!m!}\mathcal{N}_\pm(\beta)}\left((\beta)^n(-\beta)^m \pm (-\beta)^n(\beta)^m\right) = \frac{e^{-(|\beta|^2)/2}}{2^{(n+m)/2}\sqrt{n!m!}\mathcal{N}_\pm(\beta)}|\beta|^n|\beta|^m\left((-1)^m \pm (-1)^n\right) \tag{E4}$$

Thus

$$\Pr(n,m|\pm,\beta,0) = \frac{e^{-|\beta|^2}}{2^{(n+m)}n!m!\mathcal{N}^2_\pm(\beta)}|\beta|^{2n}|\beta|^{2m}2(1\pm(-1)^{n+m}), \tag{E5}$$

where we used $((-1)^m \pm (-1)^n) = 2(1\pm(-1)^{n+m})$. The marginal over n is

$$\Pr(m|\pm,\beta,0) = \sum_{n=0}^{\infty}\Pr(n,m|\pm,\beta,0). \tag{E6}$$

Lets do the + superposition case first

$$\Pr(m|+,\beta,0) = \sum_{n=0}^{\infty} \Pr(n,m|+,\beta,0)$$

$$= \frac{e^{-|\beta|^2/2}}{m!}\left(\frac{|\beta|^2}{2}\right)^m \frac{e^{|\beta|^2}+(-1)^m}{e^{-|\beta|^2}+e^{|\beta|^2}} \tag{E7a}$$

$$= \frac{e^{-|\beta|^2/2}}{m!}\left(\frac{|\beta|^2}{2}\right)^m \underbrace{\frac{(e^{|\beta|^2}+(-1)^m)}{2\cosh|\beta|^2}}_{=1} \tag{E7b}$$

$$= \frac{e^{-|\beta|^2/2}}{m!}\left(\frac{|\beta|^2}{2}\right)^m \tag{E7c}$$

where we have used m is even for the + superposition. *Thus we have a Poisson distribution with mean and variance equal to* $|\beta|^2/2$.

Now we do the minus superposition "−" case

$$\Pr(m|-,\beta,0) = \sum_{n=0}^{\infty} \Pr(n,m|-,\beta,0)$$

$$= \frac{e^{-|\beta|^2}}{m!}\left(\frac{|\beta|^2}{2}\right)^m \frac{e^{|\beta|^2}+(-1)^{m+1}}{-e^{-|\beta|^2}+e^{|\beta|^2}} \tag{E8a}$$

$$= \frac{e^{-|\beta|^2/2}}{m!}\left(\frac{|\beta|^2}{2}\right)^m \underbrace{\frac{e^{-|\beta|^2/2}(e^{|\beta|^2}+(-1)^m)}{2\sinh|\beta|^2}}_{=1}\,. \tag{E8b}$$

The marginals $\Pr(n|\pm,\beta,0)$ look identical with the role of m and n reversed.

Appendix F: Fock state signal

To further evaluate this, consider the fragment

$$\langle 0|\,(\hat{a}+\beta)^n(\hat{a}-\beta)^m\,|p\rangle\,. \tag{F1}$$

The annihilation operator is the only operator within this expression and hence all the operators commute. Therefore the standard binomial expansion can be used,

$$(a+x)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}x^k. \tag{F2}$$

This gives

$$(a+\beta)^n(a-\beta)^m\,|p\rangle = \sum_{k=0}^{m}\sum_{k'=0}^{n}\binom{m}{k}\binom{n}{k'}(-1)^k\beta^{k+k'}\sqrt{\frac{p!}{(p-(m-k)-(n-k'))!}}\,|p-(m-k)-(n-k')\rangle\,. \tag{F3}$$

where any negative values within the ket are equivalent to the zero vector. The expression closed with a $\langle 0|$ is then

$$\langle 0|\,(\hat{a}+\beta)^n(\hat{a}-\beta)^m\,|p\rangle = \sum_{k=0}^{m}\binom{m}{k}\binom{n}{n+m-p-k}(-1)^k\beta^{n+m-p}\sqrt{p!}, \tag{F4}$$

provided $n+m\geq p$ otherwise the expression is zero. This condition occurs because there is no loss considered in the model and hence if p photons are injected into the detector, at the very least they must all be detected. Additional photons can arise from the local oscillator which introduces the terms involving the β. The series has the form of a ordinary hypergeometric function evaluated at -1 and hence can be written

$$\langle 0|\,(\hat{a}+\beta)^n(\hat{a}-\beta)^m\,|p\rangle = \beta^{n+m-p}\sqrt{p!}\begin{cases}\binom{n}{m+n-p}\ {}_2F_1(-m,-m-n+p;1-m+p;-1) & m\leq p\\ \binom{m}{m-p}(-1)^{m-p}\ {}_2F_1(-n,-p;1+m-p;-1) & m>p\end{cases} \tag{F5}$$

again, provided that $n + m \geq p$. The leading two parameters, given this inequality are always negative which corresponds to the finite sum of terms in the defining series. The second terms, given the constraints on each branch, are always greater than or equal to one which ensures no singular points.

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