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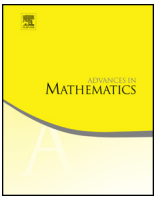

Chiral principal series categories I: Finite dimensional calculations

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ABSTRACT

This paper begins a series studying D-modules on the Feigin-Frenkel semi-infinite flag variety from the perspective of the Beilinson-Drinfeld factorization (or chiral) theory.

Here we calculate Whittaker-twisted cohomology groups of Zastava spaces, which are certain finite-dimensional subvarieties of the affine Grassmannian. We show that such cohomology groups realize the nilradical of a Borel subalgebra for the Langlands dual group in a precise sense, following earlier work of Feigin-Finkelberg-Kuznetsov-Mirkovic and Braverman-Gaitsgory. Moreover, we compare this geometric realization of the Langlands dual group to the standard one provided by (factorizable) geometric Satake.

Contents

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1. Introduction

1.1. Semi-infinite flag variety

This paper begins a series concerning D-modules on the *semi-infinite flag variety* of Feigin-Frenkel.

Let G be a split reductive group over k a field of characteristic zero. Let B be a Borel with radical N and reductive quotient $B/N = T$.

Let X be a smooth curve. We let $x \in X$ be a fixed k-point. Let $O_x = k[[t_x]]$ and $K_x = k((t_x))$ be the rings of Taylor and Laurent series based at x. Let $\mathcal{D}_x$ and $\overset{o}{\mathcal{D}}_x$ denote the spectra of these rings.

Informally, the semi-infinite flag variety should be a quotient $\mathrm{Fl}_x^{\frac{\infty}{2}} := G(K_x)/N(K_x)T(O_x)$, but this quotient is by an infinite-dimensional group and therefore leaves the realm of usual algebraic geometry.

Still, we will explain in future work [37] how to make precise sense of D-modules on $\mathrm{Fl}_x^{\frac{\infty}{2}}$, but we ask the reader to take on faith for this introduction that such a category makes sense.[1] This category will not play any explicit role in the present paper, and will be carefully discussed in [37]; however, it plays an important motivational role in this introduction.

1.2. Why semi-infinite flags?

The desire for a theory of sheaves on the semi-infinite flag variety stretches back to the early days of geometric representation theory: see [16], [18], [17], [13], and [3]. Among these works, there are diverse goals and perspectives, showing the rich representation theoretic nature of $\mathrm{Fl}_x^{\frac{\infty}{2}}$.

- [16] explains that the analogy between Wakimoto modules for an affine Kac-Moody algebra $\hat{\mathfrak{g}}_{\kappa,x}$ and Verma modules for the finite-dimensional algebra $\mathfrak{g}$ should be understood through the Beilinson-Bernstein localization picture, with $\mathrm{Fl}_x^{\frac{\infty}{2}}$ playing the role of the finite-dimensional G/B.
- [18], [17] and [3] relate the semi-infinite flag variety to representations of Lusztig's small quantum group, following Finkelberg, Feigin-Frenkel and Lusztig.

[1] In fact, the definition is short: one takes the category $D^!(G(K_x))$ from [8] (cf. also [36]) and imposes the coinvariant condition with respect to the group indscheme $N(K_x)T(O_x)$.

- As noted in [3], $D(\mathrm{Fl}_x^{\frac{\infty}{2}}) = D(G(K_x)/N(K_x)T(O_x))$ plays the role of the *universal unramified principal series representation* of $G(K_x)$ in the categorical setting of local geometric Langlands (see [20] and [8] for some modern discussion of this framework and its ambitions).

However, these references (except [16], which is not rigorous on these points) use ad hoc *finite-dimensional models* for the semi-infinite flag variety.

Remark 1.2.1. One of our principal motivations in this work and its sequels is to study $D(\mathrm{Fl}_x^{\frac{\infty}{2}})$ from the perspective of the geometric Langlands program, and then to use local to global methods to apply this to the study of geometric Eisenstein series in the global unramified geometric Langlands program. But this present work is also closely[2] connected to the above, earlier work, as we hope to explore further in the future.

1.3. The present series of papers will introduce the whole category $D(\mathrm{Fl}_x^{\frac{\infty}{2}})$ and study some interesting parts of its representation theory: e.g., we will explain how to compute Exts between certain objects in terms of the Langlands dual group.

Studying the whole of $D(\mathrm{Fl}_x^{\frac{\infty}{2}})$ was neglected by previous works (presumably) due to the technical, infinite-dimensional nature of its construction.

1.4. The role of the present paper

Whatever the definition of $D(\mathrm{Fl}_x^{\frac{\infty}{2}})$ is, it is not obvious how to compute directly with it. The primary problem is that we do not have such a good theory of perverse sheaves in the infinite type setting: the usual theory [7] of middle extensions — which is so crucial in connecting combinatorics (e.g., Langlands duality) and geometry — does not exist for embeddings of infinite codimension.

Therefore, to study $D(\mathrm{Fl}_x^{\frac{\infty}{2}})$, it is necessary to reduce our computations to finite-dimensional ones. This paper performs those computations. The application to semi-infinite flags is postponed to [37]; here the category $D(\mathrm{Fl}_x^{\frac{\infty}{2}})$ plays only a motivational role. (That said, the author finds the calculations, which concern twisted cohomologies of Zastava spaces, to be interesting in their own right.)

1.5. In the remainder of the introduction, we will discuss problems close to those to be considered in [37], and discuss the contents of the present paper and their connection to the above problems. This is by way of motivating the contents of the present work.

[2] But non-trivially, due to the ad hoc definitions in earlier works.

1.6. $D(\mathrm{Fl}_x^{\frac{\infty}{2}})$ in terms of affine flags

One can show[3] that $D(\mathrm{Fl}_x^{\frac{\infty}{2}})$ is equivalent to the category $D(\mathrm{Fl}_{G,x}^{\mathrm{aff}})$ of D-modules on the *affine flag variety* $\mathrm{Fl}_{G,x}^{\mathrm{aff}} = G(K_x)/I$, where $I \subseteq G(O_x)$ is the Iwahori subgroup, in a $G(K_x)$-equivariant way.[4]

At first pass, this means that essentially[5] every question in local geometric Langlands about $D(\mathrm{Fl}_x^{\frac{\infty}{2}})$ has either been answered in the exhaustive works of Bezrukavnikov and collaborators (especially [1], [2], and [9]), or else is completely out of reach (e.g., some conjectures from [20]).

Thus, it would appear that there is nothing new to say about $D(\mathrm{Fl}_x^{\frac{\infty}{2}})$.

1.7. The factorization perspective

However, there is a significant difference between the affine and semi-infinite flag varieties: the latter *factorizes* in the sense of Beilinson-Drinfeld [6].

We refer to the introduction to [35] for an introduction to factorization. Modulo the non-existence of $\mathrm{Fl}_x^{\frac{\infty}{2}}$, let us recall that this essentially means that for each finite set I, we have a *semi-infinite flag variety* $\mathrm{Fl}_{X^I}^{\frac{\infty}{2}}$ over X^I whose fiber at a point $(x_i)_{i\in I} \in X^I$ is the product $\prod_{\{x_i\}_{i\in I}} \mathrm{Fl}_{x_i}^{\frac{\infty}{2}}$. Here $\{x_i\}_{i\in I}$ is the unordered set in which we have forgotten the multiplicities with which points appear.

However, it is well-known that the Iwahori subgroup (unlike $G(O_x)$) does not factorize.[6]

Remark 1.7.1. The methods of the Bezrukavnikov school do not readily adapt to studying $\mathrm{Fl}_x^{\frac{\infty}{2}}$ factorizably: they heavily rely on the ind-finite type and ind-proper nature of $\mathrm{Fl}_G^{\mathrm{aff}}$, which are not manifested in the factorization setting.

1.8. As discussed in the introduction to [35], there are several reasons to care about factorization structures.

- Most imminently (from the perspective of Remark 1.2.1), the theory of chiral homology (cf. [6]) provides a way of constructing *global* invariants from factorizable local

[3] This result will appear in [37].

[4] This is compatible with the analogy with p-adic representation theory: cf. [14].

[5] This is not completely true: for the study of Kac-Moody algebras, the semi-infinite flag variety has an interesting global sections functor. It differs from the global sections functor of the affine flag variety in as much as Wakimoto modules differ from Verma modules.

[6] It is instructive to try and fail to define a factorization version of the Iwahori subgroup that lives over X^2: a point should be a pair of points x_1, x_2 in X, G-bundle on X with a trivialization away from x_1 and x_2, and with a reduction to the Borel B at the points x_1 and x_2. However, for this to define a scheme, we need to ask for a reduction to B at the *divisor-theoretic union* of the points x_1 and x_2. Therefore, over a point x in the diagonal $X \subseteq X^2$, we are asking for a reduction to B on the first infinitesimal neighborhood of x, which defines a subgroup of $G(O_x)$ smaller than the Iwahori group.

ones. Therefore, identifying spectral and geometric factorization categories allows us to compare globally defined invariants as well.

- Factorization structures also play a key, if sometimes subtle, role in the purely local theory. Let us mention one manifestation of this: the localization theory [21] (at critical level) for $\mathrm{Fl}_G^{\mathrm{aff}}$ has to do with the structure of the Kac-Moody algebra $\widehat{\mathfrak{g}}_{crit}$ as a bare Lie algebra. A factorizable localization theory for $\mathrm{Fl}_x^{\frac{\infty}{2}}$ would connect to the vertex algebra structure on its vacuum representation.
- In [18], [17], [3], and [10], sheaves on $\mathrm{Fl}_x^{\frac{\infty}{2}}$ are defined using factorization structures. We anticipate the eventual comparison between our category $D(\mathrm{Fl}_x^{\frac{\infty}{2}})$ and the previous ones to pass through the factorization structure of $\mathrm{Fl}_x^{\frac{\infty}{2}}$.

1.9. Main conjecture

Our main conjecture is about Langlands duality for certain factorization categories: the geometric side concerns some D-modules on the semi-infinite flag variety, and the spectral side concerns coherent sheaves on certain spaces of local systems.

See below for a more evocative description of the two sides.

1.10. Let B^- be a Borel opposite to B, and let N^- denote its unipotent radical.

Recall that for any category $\mathcal{C}$ acted on by $G(K_x)$ in the sense of [8], we can form its *Whittaker subcategory*, $\mathsf{Whit}(\mathcal{C}) \subseteq \mathcal{C}$ consisting of objects equivariant against a non-degenerate character of $N^-(K)$.

Moreover, up to certain twists (which we ignore in this introduction: see §2.8 for their definitions), this makes sense factorizably.

For each finite set I, there is therefore a category $\mathsf{Whit}_{X^I}^{\frac{\infty}{2}}$ of Whittaker equivariant D-modules on $\mathrm{Fl}_{X^I}^{\frac{\infty}{2}}$, and the assignment $I \mapsto \mathsf{Whit}_{X^I}^{\frac{\infty}{2}}$ defines a *factorization category* in the sense of [35]. This forms the geometric side of our conjecture.

1.11. For a point $x \in X$ and an affine algebraic group Γ, let $\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$ denote the prestack of de Rham local systems with structure group Γ on $\overset{o}{\mathcal{D}}_x$.

Formally: we have the indscheme Conn_Γ of $\mathrm{Lie}(\Gamma)$-valued 1-forms (i.e., connection forms) on the punctured disc, which is equipped with the usual gauge action of $\Gamma(K_x)$. We form the quotient and denote this by $\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$.

Remark 1.11.1. $\mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$ is not an algebraic stack of any kind because we quotient by the loop group $\Gamma(K_x)$, an indscheme of ind-infinite type. It might be considered as a kind of *semi-infinite Artin stack*, the theory of which has unfortunately not been developed.

The assignment $x \mapsto \mathrm{LocSys}_\Gamma(\overset{o}{\mathcal{D}}_x)$ factorizes in an obvious way.

1.12. Recall that for a finite type (derived) scheme (or stack) Z, [27] has defined a DG category $\mathsf{IndCoh}(Z)$ of *ind-coherent sheaves* on Z.[7]

We would like to take as the spectral side of our equivalence the factorization category:

$$x \mapsto \mathsf{IndCoh}\big(\mathrm{LocSys}_{\check{B}}(\overset{o}{\mathcal{D}}_x) \underset{\mathrm{LocSys}_{\check{T}}(\overset{o}{\mathcal{D}}_x)}{\times} \mathrm{LocSys}_{\check{T}}(\mathcal{D}_x)\big).$$

Here and everywhere, we use e.g. $\check{G}$ to refer to the reductive group Langlands dual to G, and $\check{B} \subseteq \check{G}$ to refer to the corresponding Borel subgroup, etc. (cf. §1.41).

However, note that IndCoh has not been defined in this setting: the spaces of local systems on the punctured disc are defined as the quotient of an indscheme of ind-infinite type by a group of ind-infinite type.

We ignore this problem in what follows, describing a substitute in §1.15 below.

1.13. We now formulate the following conjecture:

Main Conjecture. *There is an equivalence of factorization categories:*

$$\mathsf{Whit}^{\frac{\infty}{2}} \xrightarrow{\simeq} \Big(x \mapsto \mathsf{IndCoh}\big(\mathrm{LocSys}_{\check{B}}(\overset{o}{\mathcal{D}}_x) \underset{\mathrm{LocSys}_{\check{T}}(\overset{o}{\mathcal{D}}_x)}{\times} \mathrm{LocSys}_{\check{T}}(\mathcal{D}_x)\big)\Big). \tag{1.13.1}$$

Remark 1.13.1. Identifying D-modules on the affine flag variety and on the semi-infinite flag variety, one can show that fiberwise, this conjecture recovers the main result of [1]. However, as noted in Remark 1.7.1, the methods of [1] are not amenable to the factorizable setting.

1.14. What is contained in this paper?

In [18], Finkelberg and Mirkovic argue that their *Zastava* spaces provide finite-dimensional models for the geometry of the semi-infinite flag variety.

In essence, we are using this model in the present paper: we compute some twisted cohomology groups of Zastava spaces, and these computations will provide the main input for our later study [37] of semi-infinite flag varieties.

In §1.15-1.21, we describe a certain factorization algebra $\Upsilon_{\check{\mathfrak{n}}}$ and its role in the main conjecture (from §1.13). In §1.22-1.27, we recall some tactile aspects of the geometry of Zastava spaces. Finally, in §1.28-1.37, we formulate the main results of this text: these realize $\Upsilon_{\check{\mathfrak{n}}}$ (and its modules) as twisted cohomology groups on Zastava space.

[7] For the reader unfamiliar with the theory of [27], we recall that this sheaf theoretic framework is very close to the more familiar QCoh, but is the natural setting for Grothendieck's functor $f^!$ of exceptional inverse image (as opposed to the functor f^*, which is adapted to QCoh).

Remark 1.14.1. Some of the descriptions below may go a bit quickly for a reader who is a non-expert in this area. We hope that for such a reader, the material that follows helps to supplement what it is written more slowly in the body of the text.

1.15. The factorization algebra $\Upsilon_{\check{\mathfrak{n}}}$

To describe the main results of this paper, we need to describe how we model the spectral side of the main conjecture i.e., the category of ind-coherent sheaves on the appropriate space of local systems.

We will do this using the *graded factorization algebra* $\Upsilon_{\check{\mathfrak{n}}}$, introduced in [12].

After preliminary remarks about what graded factorization algebras are in §1.16, we introduce $\Upsilon_{\check{\mathfrak{n}}}$ in §1.17. Finally, in §1.20-1.21, we explain why factorization modules for $\Upsilon_{\check{\mathfrak{n}}}$ are related to the spectral side of the main conjecture.

1.16. Let $\check{\Lambda}^{pos} \subseteq \check{\Lambda} := \{\text{cocharacters of } G\}$ denote the $\mathbb{Z}^{\geqslant 0}$-span of the simple coroots (relative to B).

Let $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$ denote the space of $\check{\Lambda}^{pos}$-valued divisors on X. I.e., its $\overline{k}$-points are written:

$$\sum \check{\lambda}_i \cdot x_i \tag{1.16.1}$$

for some finite set $\{x_i\} \subseteq X(\overline{k})$, $\check{\lambda}_i \in \check{\Lambda}^{pos}$. For G of semi-simple rank 1, this space is the union of the symmetric powers of X, while for general G, its connected components are products of symmetric powers of X.

For $\check{\lambda} \in \check{\Lambda}^{pos}$, we let $\mathrm{Div}_{\mathrm{eff}}^{\check{\lambda}}$ denote the connected component of $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$ of divisors of total degree $\check{\lambda}$ (i.e., in the above we have $\sum \check{\lambda}_i = \check{\lambda}$).

A *($\check{\Lambda}^{pos}$)-graded factorization algebra* is the datum of D-modules:

$$\mathcal{A}^{\check{\lambda}} \in D(\mathrm{Div}_{\mathrm{eff}}^{\check{\lambda}}), \qquad \check{\lambda} \in \check{\Lambda}^{pos}$$

plus symmetric and associative[8] isomorphisms:

$$\mathcal{A}^{\check{\lambda}+\check{\mu}}|_{[\mathrm{Div}_{\mathrm{eff}}^{\check{\lambda}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\mu}}]_{disj}} \simeq \mathcal{A}^{\check{\lambda}} \boxtimes \mathcal{A}^{\check{\mu}}|_{[\mathrm{Div}_{\mathrm{eff}}^{\check{\lambda}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\mu}}]_{disj}}.$$

Here:

$$[\mathrm{Div}_{\mathrm{eff}}^{\check{\lambda}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\mu}}]_{disj} \subseteq \mathrm{Div}_{\mathrm{eff}}^{\check{\lambda}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\mu}}$$

denotes the open locus of pairs of (colored) divisors with disjoint supports, which we consider mapping to $\mathrm{Div}_{\mathrm{eff}}^{\check{\lambda}+\check{\mu}}$ through the map of addition of divisors (which is étale on this locus).

[8] These terms should be understood in the homotopical sense, including higher compatibilities. We do not emphasize these points in the introduction.

Remark 1.16.1. The theory of graded factorization algebras closely imitates the theory of factorization algebras from [6], with the above $\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}$ replacing the *Ran space* from [6].

1.17. The $\check{\Lambda}^{pos}$-graded Lie algebra $\check{\mathfrak{n}}$ defines a *Lie-$*$ algebra*[9]:

$$\check{\mathfrak{n}}_X := \oplus_{\check{\alpha} \text{ a coroot of } G} \ \check{\mathfrak{n}}^{\check{\alpha}} \otimes \Delta^{\check{\alpha}}_{*,dR}(k_X) \in D(\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}).$$

In this notation, for a finite type scheme S, k_S denotes its (D-module version of the) constant sheaf; $\check{\mathfrak{n}}^{\check{\alpha}}$ denotes the corresponding graded component of $\check{\mathfrak{n}}$; and $\Delta^{\check{\alpha}} : X \to \mathrm{Div}^{\check{\alpha}}_{\mathrm{eff}}$ denotes the diagonal embedding.

As in [6], we may form the *chiral enveloping algebra* of $\check{\mathfrak{n}}_X$: we let $\Upsilon_{\check{\mathfrak{n}}}$ denote the corresponding factorization algebra. For the reader unfamiliar with [6], we remind that $\Upsilon_{\check{\mathfrak{n}}}$ is associated to $\check{\mathfrak{n}}_X$ as a sort of Chevalley complex; in particular, the $*$-fiber of $\Upsilon_{\check{\mathfrak{n}}}$ at a point (1.16.1) is:

$$\underset{i}{\otimes} C_\bullet(\check{\mathfrak{n}})^{\check{\lambda}_i}$$

where $C_\bullet$ denotes the (homological) Chevalley complex of a Lie algebra (i.e., the complex computing Lie algebra homology).

1.18. Next, we recall that in the general setup of §1.16, to a graded factorization algebra $\mathcal{A}$ and a closed point $x \in X$, we can associate a DG category $\mathcal{A}$–$\mathsf{mod}^{\mathrm{fact}}_x$ of its ($\check{\Lambda}$-graded) factorization modules at $x \in X$.

First, let $\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}}$ denote the indscheme of $\check{\Lambda}$-valued divisors on X that are $\check{\Lambda}^{pos}$-valued on $X \setminus x$. So $\overline{k}$-points of this space are sums:

$$\check{\mu} \cdot x + \sum \check{\lambda}_i \cdot x_i$$

where $\check{\mu} \in \check{\Lambda}$ and $\check{\lambda}_i \in \check{\Lambda}^{pos}$ and $\{x_i\} \subseteq X(\overline{k})$ finite. (To see the indscheme structure, bound how small $\check{\mu}$ can be.)

Then a *factorization module* for $\mathcal{A}$ is a D-module $M \in D(\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}})$ equipped with an isomorphism:

$$\mathrm{add}^!(M)|_{[\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}} \times \mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}}]_{disj}} \simeq \mathcal{A} \boxtimes M|_{[\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}} \times \mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}}]_{disj}}$$

which is associative with respect to the factorization structure on $\mathcal{A}$, where add is the map:

[9] Here the structure of Lie-$*$ algebra is defined in [6] (see also [19], [35] for derived versions). For the reader's sake, we simply note that this datum encodes the natural structure on $\check{\mathfrak{n}}_X$ inherited from the Lie bracket on $\check{\mathfrak{n}}$.

$$\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}} \times \mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}} \to \mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}}$$

of addition of divisors.

Factorization modules form a DG category in the obvious way.

Remark 1.18.1. In what follows, we will need *unital* versions of the above, i.e., *unital* factorization algebras and unital modules. This is a technical requirement, and for the sake of brevity we do not spell it out here, referring to [6] or [35] for details. However, this is the reason that notations of the form $\mathcal{A}$–$\mathsf{mod}^{\mathrm{fact}}_{un,x}$ appear below instead of $\mathcal{A}$–$\mathsf{mod}^{\mathrm{fact}}_{x}$. However, we remark that whatever these unital structures are, chiral envelopes always carry them, and in particular $\Upsilon_{\check{\mathfrak{n}}}$ does.

Remark 1.18.2. Note that the affine Grassmannian $\mathrm{Gr}_{T,x} = T(K_x)/T(O_x)$ with structure group T embeds into $\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}}$ as the locus of divisors supported at the point x. We remind that the reduced scheme underlying $\mathrm{Gr}_{T,x}$ is the discrete scheme $\check{\Lambda}$.

1.19. The following provides the connection between $\Upsilon_{\check{\mathfrak{n}}}$ and the main conjecture.

Principle.

(1) There is a canonical equivalence:

$$\Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,x} \simeq \mathsf{IndCoh}\Big(\mathrm{LocSys}_{\check{B}}(\overset{o}{\mathcal{D}}_x)^{\wedge}_{\mathrm{LocSys}_{\check{T}}(\mathcal{D}_x)} \underset{\mathrm{LocSys}_{\check{T}}(\overset{o}{\mathcal{D}}_x)}{\times} \mathrm{LocSys}_{\check{T}}(\mathcal{D}_x)\Big) \quad (1.19.1)$$

where $\mathrm{LocSys}_{\check{B}}(\overset{o}{\mathcal{D}}_x)^{\wedge}_{\mathrm{LocSys}_{\check{T}}(\mathcal{D}_x)}$ *indicates the formal completion of* $\mathrm{LocSys}_{\check{B}}(\overset{o}{\mathcal{D}}_x)$ *with respect to the map from* $\mathrm{LocSys}_{\check{T}}(\mathcal{D}_x)$.[10]

(2) Under this equivalence, the functor[11]*:*

$$\Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,x} \xrightarrow{\mathrm{Oblv}} D(\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}}) \xrightarrow{!-\textit{restriction}} D(\mathrm{Gr}_{T,x}) \simeq \mathsf{Rep}(\check{T})$$
$$\simeq \mathsf{QCoh}(\mathrm{LocSys}_{\check{T}}(\mathcal{D}_x))$$

corresponds to the functor of !*-restriction along the map:*

$$\mathrm{LocSys}_{\check{T}}(\mathcal{D}_x) \to \mathrm{LocSys}_{\check{B}}(\overset{o}{\mathcal{D}}_x)^{\wedge} \underset{\mathrm{LocSys}_{\check{T}}(\overset{o}{\mathcal{D}}_x)}{\times} \mathrm{LocSys}_{\check{T}}(\mathcal{D}_x).$$

[10] For a fixed k-point $x \in X$, $\mathrm{LocSys}_{\check{B}}(\overset{o}{\mathcal{D}}_x)^{\wedge}_{\mathrm{LocSys}_{\check{T}}(\mathcal{D}_x)}$ is isomorphic to $\check{\mathfrak{b}}^{\wedge}_0/\check{N}^{\wedge}\cdot\check{T}$, so the whole fiber product is isomorphic to $\check{\mathfrak{n}}^{\wedge}_0/\check{N}^{\wedge}\check{T}$. Here $\check{N}^{\wedge}$ is the formal group for $\check{N}$, i.e., the formal completion at the identity.

[11] Here and throughout the text, for an algebraic group Γ, $\mathsf{Rep}(\Gamma)$ denotes the *derived* (i.e., DG) category of its representations, i.e., $\mathsf{QCoh}(\mathbb{B}\Gamma)$.

(3) The above two facts generalize to the factorization setting, where x is replaced by several points allowed to move and collide.

Remark 1.19.1. We categorize the above assertion as a principle and not a theorem because the right hand side of (1.19.1) is not defined factorizably (we remind that this is because IndCoh is only defined in finite type situations, while LocSys leaves this world). Therefore, the reader might take it simply as a definition.

For the reader familiar with derived deformation theory (as in [31], [27]) and [6], we will explain heuristically in §1.20-1.21 why we take this principle as given. However, the reader who is not familiar with these subjects may safely skip this material, as it plays only a motivational role for us.

Remark 1.19.2. We note that (heuristically) ind-coherent sheaves on (1.19.1) should be a full subcategory of $\mathsf{IndCoh}\big(\mathrm{LocSys}_{\check{B}}(\overset{o}{\mathcal{D}}_x) \underset{\mathrm{LocSys}_{\check{T}}(\overset{o}{\mathcal{D}}_x)}{\times} \mathrm{LocSys}_{\check{T}}(\mathcal{D}_x)\big)$.[12]

In [37], we will use the computations of the present paper to construct a functor:

$$\mathsf{Whit}^{\frac{\infty}{2}}_x \to \mathsf{IndCoh}\big(\mathrm{LocSys}_{\check{B}}(\overset{o}{\mathcal{D}}_x)^{\wedge} \underset{\mathrm{LocSys}_{\check{T}}(\overset{o}{\mathcal{D}}_x)}{\times} \mathrm{LocSys}_{\check{T}}(\mathcal{D}_x)\big) := \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,x}$$

and identify a full subcategory of $\mathsf{Whit}^{\frac{\infty}{2}}_x$ on which this functor is an equivalence. Moreover, this equivalence is factorizable, and therefore gives the main conjecture (from §1.13) when restricted to these full subcategories.

1.20. As stated above, the reader may safely skip §1.20-1.21, which are included to justify the principle of §1.19.

We briefly recall Lurie's approach to deformation theory [31].

Suppose that $\mathcal{X}$ is a "nice enough" stack and $x \in \mathcal{X}$ is a k-point, with the formal completion of $\mathcal{X}$ at x denoted by $\mathcal{X}^{\wedge}_x$. Then the fiber $T_{\mathcal{X},x}[-1]$ of the shifted tangent complex of $\mathcal{X}$ at x identifies with the Lie algebra of the (derived) automorphism group (alias: inertia) $\mathrm{Aut}_x(\mathcal{X}) := x \times_{\mathcal{X}} x$ of $\mathcal{X}$ at x, and there is an identification of the DG category $\mathsf{IndCoh}(\mathcal{X}^{\wedge}_x)$ of ind-coherent sheaves on the formal completion of $\mathcal{X}$ at x with $T_{\mathcal{X},x}[-1]$-modules.

1.21. At the trivial local system, the fiber of the shifted tangent complex of $\mathrm{LocSys}_{\check{N}}(\overset{o}{\mathcal{D}}_x)$ is the (derived) Lie algebra $H^*_{dR}(\overset{o}{\mathcal{D}}_x, \check{\mathfrak{n}} \otimes k_{\overset{o}{\mathcal{D}}_x})$. The philosophy of [6] indicates that modules for this Lie algebra should be equivalent to factorization modules for the chiral envelope of the Lie-$*$ algebra $\check{\mathfrak{n}} \otimes k_X$ on X.

[12] This combines the facts that ind-coherent sheaves on a formal completion are a full subcategory of ind-coherent sheaves of the whole space, and the fact that ind-coherent sheaves on the classifying stack of the formal group of a unipotent group are a full subcategory of ind-coherent sheaves on the classifying stack of the group.

The $\check{\Lambda}$-graded variant of this—that is, the version in the setting of §1.16 in which symmetric powers of the curve replace the Ran space from [6]—provides the principle of §1.19.

Remark 1.21.1. To conclude this line of reasoning, we note that the objection of Remark 1.19.1 really only applies factorizably. For a fixed point $x \in X$, the relevant spaces of local systems *are* locally of finite type. In effect, the above outlines the proof of assertions (1) and (2) for such a single point.

1.22. Zastava spaces

Next, we describe the most salient features of Zastava spaces. We remark that this geometry is reviewed in detail in §2.

1.23. There are two Zastava spaces, $\overset{o}{\mathcal{Z}}$ and $\mathcal{Z}$, each fibered over $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$: the relationship is that $\overset{o}{\mathcal{Z}}$ embeds into $\mathcal{Z}$ as an open, and for this reason, we sometimes refer to $\mathcal{Z}$ as Zastava space and $\overset{o}{\mathcal{Z}}$ as *open* Zastava space.

For the purposes of this introduction, we content ourselves with a description of the fibers of the maps:

$$\begin{array}{ccc} \overset{o}{\mathcal{Z}} & \hookrightarrow & \mathcal{Z} \\ & \searrow^{\overset{o}{\pi}} & \downarrow{\scriptstyle \pi} \\ & & \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \end{array}.$$

To give this description, we will first recall the so-called *central fibers* of the Zastava spaces.

1.24. Recall that e.g. $\mathrm{Gr}_{G,x}$ denotes the affine Grassmannian $G(K_x)/G(O_x)$ of G at x.

For $x \in X$ a geometric point and $\check{\lambda} \in \check{\Lambda}^{pos}$, define the *central fiber* $\overset{o}{\mathfrak{Z}}{}_x^{\check{\lambda}}$ as the intersection:

$$\begin{aligned} \mathrm{Gr}_{N^-,x} \cap \mathrm{Gr}_{B,x}^{\check{\lambda}} &= \Big(N^-(K_x)G(O_x) \bigcap N(K_x)\check{\lambda}(t_x)G(O_x)\Big)/G(O_x) \subseteq G(K_x)/G(O_x) \\ &= \mathrm{Gr}_{G,x} \end{aligned}$$

where t_x is any uniformizer at x. Here we recall that $\mathrm{Gr}_{N^-,x} = N^-(K_x)G(O_x)/G(O_x)$ and $\mathrm{Gr}_{B,x}^{\check{\lambda}} = N(K_x)\check{\lambda}(t_x)G(O_x)/G(O_x)$ embed into $\mathrm{Gr}_{G,x}$ as ind-locally closed subschemes (of infinite dimension and codimension).[13]

[13] The requirement that $\check{\lambda} \in \check{\Lambda}^{pos}$ is included so that this intersection is non-empty.

A small miracle: the intersections $\overset{o}{\mathfrak{Z}}{}^{\check\lambda}_x$ are finite type, and equidimensional of dimension $(\rho, \check\lambda)$.

Example 1.24.1. For $\check\lambda = \check\alpha$ a simple coroot, one has $\overset{o}{\mathfrak{Z}}{}^{\check\alpha}_x \simeq \mathbb{A}^1 \setminus \{0\}$.

1.25. Let $\overline{\mathrm{Gr}}{}^{\check\lambda}_{B,x}$ denote the closure of $\mathrm{Gr}^{\check\lambda}_{B,x}$ in $\mathrm{Gr}_{G,x}$.[14] We remind that $\overline{\mathrm{Gr}}{}^{\check\lambda}_{B,x}$ has an (infinite) stratification by the ind-locally closed subschemes $\mathrm{Gr}^{\check\lambda-\check\mu}_{B,x}$ for $\check\mu \in \check\Lambda^{pos}$.

We then define $\mathfrak{Z}^{\check\lambda}_x$ as the corresponding intersection:

$$\mathrm{Gr}_{N^-,x} \cap \overline{\mathrm{Gr}}{}^{\check\lambda}_{B,x} \subseteq \mathrm{Gr}_{G,x}\,.$$

Again, this intersection is finite-dimensional, and equidimensional of dimension $(\rho, \check\lambda)$.

Example 1.25.1. For $\check\lambda = \check\alpha$ a simple coroot, one has $\mathfrak{Z}^{\check\alpha}_x \simeq \mathbb{A}^1$.

1.26. Now, for a k-point (1.16.1) of $\mathrm{Div}^{\check\lambda}_{\mathrm{eff}}$ (for $\check\lambda := \sum \check\lambda_i$), the corresponding fiber of $\overset{o}{\mathcal{Z}}{}^{\check\lambda}$ along $\overset{o}{\pi}$ is:

$$\prod \overset{o}{\mathfrak{Z}}{}^{\check\lambda_i}_{x_i} \tag{1.26.1}$$

and similarly for $\mathcal{Z}$.

Again, $\overset{o}{\mathcal{Z}}{}^{\check\lambda}$ and $\mathcal{Z}^{\check\lambda}$ are equidimensional of dimension $(2\rho, \check\lambda)$, and moreover, $\overset{o}{\mathcal{Z}}{}^{\check\lambda}$ is actually smooth.

1.27. Finally, there is a canonical map $\mathrm{can} : \mathcal{Z} \to \mathbb{G}_a$, which is constructed (fiberwise) as follows.

First, define the map $N^-(K_x) \to \mathbb{G}_a$ by:

$$\begin{aligned} N^-(K_x) &\to (N^-/[N^-,N^-])(K_x) \\ &\simeq \prod_{\text{simple roots}} K_x \xrightarrow{\text{sum over coordinates}} K_x \xrightarrow{f \mapsto \mathrm{Res}(f\cdot dt_x)} \mathbb{G}_a \end{aligned}$$

where Res denotes the residue map and t_x is a coordinate in K_x.

Remark 1.27.1. The twists we mentioned in §1.10 are included so that we do not have to choose a coordinate t_x, but rather have a canonical residue map to $\mathbb{G}_a$. But we continue to ignore these twists, reminding simply that they are spelled out in §2.8.

[14] As a moduli problem, $\overline{\mathrm{Gr}}{}^{\check\lambda}_{B,x}$ can be defined analogously to Drinfeld's compactification of $\mathrm{Bun}^{\check\lambda}_B$.

It is clear that this map factors uniquely through the projection $N^-(K_x) \to \mathrm{Gr}_{N^-}$.

We now map (1.26.1) by embedding into the product of Gr_{N^-,x_i} and summing the corresponding maps to $\mathbb{G}_a$ over the points x_i.

In what follows, we let $\psi_{\mathcal{Z}} \in D(\mathcal{Z})$ (resp. $\psi_{\overset{o}{\mathcal{Z}}} \in D(\overset{o}{\mathcal{Z}})$) denote the !-pullback of the Artin-Shreier (i.e., exponential) D-module on $\mathbb{G}_a$ (normalized to be in the same cohomological degree as the dualizing D-module of $\mathbb{G}_a$).

Remark 1.27.2. The above map $N^-(K_x) \to \mathbb{G}_a$ is referred to as the *Whittaker character*, and we refer to sheaves constructed out of the Artin-Shreier sheaf (e.g., $\psi_{\overset{o}{\mathcal{Z}}}$, $\psi_{\mathcal{Z}}$) as *Whittaker sheaves*.

1.28. Formulation of the main results of this paper

Here is a rough overview of the main results of this paper, to be expanded upon below:

Roughly, the first main result of this paper, Theorem 4.6.1, identifies $\Upsilon_{\check{\mathfrak{n}}}$ with certain Whittaker cohomology groups on Zastava space; see [12] for more details. This theorem, following [12] and [17], provides passage from the group G to the dual group $\check{G}$ (via $\Upsilon_{\check{\mathfrak{n}}}$) which is *different* from geometric Satake.

The second main result, Theorem 7.9.1 (see also Theorem 5.14.1) compares Theorem 4.6.1 with the geometric Satake equivalence.

1.29. We now give a more precise description of the above theorems.

Our first main result is the following.

Theorem *(Theorem 4.6.1).* $\overset{o}{\pi}_{*,dR}(\psi_{\overset{o}{\mathcal{Z}}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}}) \in D(\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}})$ *is concentrated in cohomological degree zero, and identifies canonically with* $\Upsilon_{\check{\mathfrak{n}}}$. *Here* IC *indicates the intersection cohomology sheaf*[15] *(by smoothness of the* $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$*, this just effects cohomological shifts on the connected component of* $\overset{o}{\mathcal{Z}}$*).*

Moreover, the factorization structure on Zastava spaces induces a factorization algebra structure on $\overset{o}{\pi}_{*,dR}(\psi_{\overset{o}{\mathcal{Z}}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}})$*, and the above equivalence upgrades to an equivalence of factorization algebras.*

In words: the ($\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}$-parametrized) cohomology of Zastava spaces twisted by the Whittaker sheaf is $\Upsilon_{\check{\mathfrak{n}}}$.

Remark 1.29.1. We draw the reader's attention to §1.35 below for a closely related result, but which is less imminently related to the theme of semi-infinite flags.

[15] Since $\overset{o}{\mathcal{Z}}$ is a union of the varieties $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$, we define this IC sheaf as the direct sum of the IC sheaves of the connected components.

1.30. Polar Zastava space

To formulate Theorem 5.14.1, we introduce a certain indscheme $\overset{o}{\mathcal{Z}}{}^{\infty\cdot x}$ with a map $\overset{o}{\pi}{}^{\infty\cdot x}: \overset{o}{\mathcal{Z}}{}^{\infty\cdot x} \to \mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}}$, where the geometry is certainly analogous to $\overset{o}{\pi}: \overset{o}{\mathcal{Z}} \to \mathrm{Div}^{\check{\Lambda}}_{\mathrm{eff}}$. (Here we remind that $\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}}$ parametrizes $\check{\Lambda}$-valued divisors on X that are $\check{\Lambda}^{pos}$-valued on $X \setminus x$.)

As with $\overset{o}{\mathcal{Z}}$, for this introduction we only describe the fibers of the map $\overset{o}{\pi}{}^{\infty\cdot x}$. Namely, at a point[16] $\check{\mu}\cdot x + \sum \check{\lambda}_i \cdot x_i$ of $\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}}$, the fiber is:

$$\mathrm{Gr}^{\check{\mu}}_{B,x} \times \prod_i \overset{o}{\mathfrak{Z}}{}^{\check{\lambda}_i}_{x_i}.$$

We refer the reader to §5 for more details on the definition.

1.31. We will explain in §5 how geometric Satake produces a functor $\mathsf{Rep}(\check{G}) \to D(\overset{o}{\mathcal{Z}}{}^{\infty\cdot x})$.

Though this functor is not so complicated, giving its definition here would require further digressions, so we ask the reader to take this point on faith. Instead, for the purposes of an overview, we refer to §1.33, where we explain what is going on when we restrict to divisors supported at the point x, and certainly we refer to §5 where a detailed construction of this functor is given.

Example 1.31.1. The above functor sends the trivial representation to the $*$-extension of $\psi_{\overset{o}{\mathcal{Z}}}$ under the natural embedding $\overset{o}{\mathcal{Z}} \hookrightarrow \overset{o}{\mathcal{Z}}{}^{\infty\cdot x}$.

We now obtain a functor:

$$\mathsf{Rep}(\check{G}) \to D(\overset{o}{\mathcal{Z}}{}^{\infty\cdot x}) \xrightarrow{\overset{o}{\pi}{}^{\infty\cdot x}_{*,dR}} D(\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}}).$$

For geometric reasons explained in §5, Theorem 4.6.1 allows us to upgrade this construction to a functor:

$$\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x} : \mathsf{Rep}(\check{G}) \to \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,x}.$$

We now have the following compatibility between geometric Satake and Theorem 4.6.1.

Theorem *(Theorem 5.14.1). The functor* $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x}$ *is canonically identified with the functor* $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},x}$*, which by definition is the functor:*

[16] We remind that this means that $\check{\mu} \in \check{\Lambda}$ and $\check{\lambda}_i \in \check{\Lambda}^{pos}$.

$$\mathsf{Rep}(\check{G}) \xrightarrow{\text{Res}} \mathsf{Rep}(\check{B}) \xrightarrow{\text{Res}} \check{\mathfrak{n}}\text{–}\mathsf{mod}(\mathsf{Rep}(\check{T})) \xrightarrow{\text{ind}^{ch}} \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\text{fact}}_{un,x}.$$

Here ind^{ch} *is the* chiral induction *functor from Lie-* modules for* $\check{\mathfrak{n}} \otimes k_X$ *to factorization modules for* $\Upsilon_{\check{\mathfrak{n}}}$.

Remark 1.31.2. Here we remind the reader that *chiral induction* is introduced (abelian categorically) in [6] §3.7.15. Like the chiral enveloping algebra operation used to define $\Upsilon_{\check{\mathfrak{n}}}$, chiral induction is again a kind of homological Chevalley complex.

Example 1.31.3. For the trivial representation, Example 1.31.1 reduces Theorem 5.14.1 to Theorem 4.6.1. Here, the claim is that $\text{Chev}^{\text{geom}}_{\check{\mathfrak{n}},x}$ of the trivial representation is the D-module on $D(\text{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\text{eff}})$ obtained by pushforward from $\Upsilon_{\check{\mathfrak{n}}}$ along $\text{Div}^{\check{\Lambda}^{pos}}_{\text{eff}} \hookrightarrow \text{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\text{eff}}$, i.e., the so-called *vacuum representation* of $\Upsilon_{\check{\mathfrak{n}}}$ (at x).

1.32. Our last main result is the following, which we leave vague here.

Theorem *(Theorem 7.9.1). A generalization of Theorem 5.14.1 holds when we work factorizably in the variable* x*, i.e., working at several points at once, allowing them to move and to collide.*

Somewhat more precisely, we define in §6 a DG category $\mathsf{Rep}(\check{G})_{X^I}$ over X^I_{dR} (i.e., with a $D(X^I)$-module category structure) encoding the symmetric monoidal structure on $\mathsf{Rep}(\check{G})_{X^I}$.[17] Most of §6 is devoted to giving preliminary technical constructions that allow us to formulate Theorem 7.9.1.

1.33. Interpretation in terms of $\text{Fl}^{\frac{\infty}{2}}_x$

We now indicate briefly what e.g. Theorem 5.14.1 has to do with $\text{Fl}^{\frac{\infty}{2}}_x$. This section has nothing to do with the contents of the paper, and therefore can be skipped; we include it only to make contact with our earlier motivation.

Fix a closed point $x \in X$, and consider the *spherical Whittaker category* $\mathsf{Whit}^{sph}_x \subseteq D(\text{Gr}_{G,x})$, which by definition is the Whittaker category (in the sense of §1.10) of $D(\text{Gr}_{G,x})$. There is a canonical object in this category (supported on $\text{Gr}_{N^-,x} \subseteq \text{Gr}_{G,x}$), and one can show (cf. Theorem 6.36.1) that the resulting functor:

$$\mathsf{Rep}(\check{G}) \xrightarrow{\text{geometric Satake}} \mathsf{Sph}_{G,x} \to \mathsf{Whit}^{sph}_x$$

is an equivalence, where $\mathsf{Sph}_{G,x} \coloneqq D(\text{Gr}_{G,x})^{G(O_x)}$ is the spherical Hecke category, and the latter functor is convolution with this preferred object of Whit^{sph}_x.

[17] The construction of $\mathsf{Rep}(\check{G})_{X^I}$ is a categorification of the construction of [6] that associated a factorization algebra with a usual commutative algebra.

Let $i^{\frac{\infty}{2},!}: D(\mathrm{Fl}_x^{\frac{\infty}{2}}) \to D(\mathrm{Gr}_{T,x})$ denote the functor encoding !-restriction along:

$$i^{\frac{\infty}{2}}: \mathrm{Gr}_{T,x} = B(K_x)/N(K_x)T(O_x) \hookrightarrow G(K_x)/N(K_x)T(O_x) = \mathrm{Fl}_x^{\frac{\infty}{2}}.$$

Consider the problem of computing the composite functor:

$$\begin{aligned}\mathsf{Rep}(\check{G}) \simeq \mathsf{Whit}_x^{sph} &\xrightarrow{\text{pullback}} \mathsf{Whit}(G(K_x)/B(O_x)) \xrightarrow{\text{pushforward}} \mathsf{Whit}_x^{\frac{\infty}{2}} \xrightarrow{i^{\frac{\infty}{2},!}} D(\mathrm{Gr}_{T,x}) \\ &\simeq \mathsf{Rep}(\check{T}).\end{aligned}$$

By base-change, this amounts to computing pullback-pushforward of Whittaker[18] sheaves along the correspondence:

$$\begin{array}{ccccc} & & G(K_x)/B(O_x) \underset{\mathrm{Fl}_x^{\frac{\infty}{2}}}{\times} \mathrm{Gr}_{T,x} & = & \mathrm{Gr}_{B,x} \\ & \swarrow & & & \searrow \\ \mathrm{Gr}_{G,x} & & & & \mathrm{Gr}_{T,x}. \end{array}$$

One can see this is exactly the picture obtained by restricting the problem of Theorem 5.14.1 to $\mathrm{Gr}_{T,x} \subseteq \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x}$, and therefore we obtain an answer in terms of factorization $\Upsilon_{\check{\mathfrak{n}}}$-modules. Namely, this result says that the resulting functor:

$$\mathsf{Rep}(\check{G}) \to \mathsf{Rep}(\check{T})$$

is computed as Lie algebra homology along $\check{\mathfrak{n}}$.

Remark 1.33.1. The point of upgrading Theorem 5.14.1 to Theorem 7.9.1 is to allow a picture of this sort which is factorizable in terms of the point x, i.e., in which we replace the point $x \in X$ by a variable point in X^I for some finite set I.

1.34. Methods

We now remark one what goes into the proofs of the above theorems.

1.35. Our key computational tool is the following result.

Theorem *(Limiting case of the Casselman-Shalika formula, Theorem 3.4.1). The pushforward $\pi^{\check{\lambda}}_{*,dR}(\psi_{\mathcal{Z}} \overset{!}{\otimes} \mathrm{IC}_{\mathcal{Z}}) \in D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}})$ is the (one-dimensional) skyscraper sheaf at the zero divisor (concentrated in cohomological degree zero).*

[18] It is crucial here that our character be with respect to N^-, not N.

In particular, the restriction of this pushforward to each $\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}$ *with* $0 \neq \check{\lambda} \in \check{\Lambda}^{pos}$ *vanishes.*

We prove this using reasonably standard methods (cf. [13]) for studying sheaves on Zastava spaces.

1.36. Our other major tool is the study of $\Upsilon_{\check{\mathfrak{n}}}$ given in [12], where $\Upsilon_{\check{\mathfrak{n}}}$ is connected to the *untwisted* cohomologies of Zastava spaces (in a less derived framework than in Theorem 4.6.1).

1.37. Finally, we remark that the proofs of Theorems 3.4.1, 4.6.1, and 5.14.1 are elementary: they use only standard perverse sheaf theory, and do not require the use of DG categories or non-holonomic D-modules. (In particular, these theorems work in the ℓ-adic setting, with the usual Artin-Shreier sheaf replacing the exponential sheaf.) The reader uncomfortable with higher category theory should run into no difficulties here by replacing the words *DG category* by *triangulated category* essentially everywhere (one exception: it is important that the definition of $\Upsilon_{\check{\mathfrak{n}}}$–$\mathsf{mod}^{\mathrm{fact}}_{un,x}$ be understood higher categorically).

However, Theorem 7.9.1 is not elementary in this sense. This is the essential reason for the length of §6: we are trying to construct an isomorphism of combinatorial nature in a higher categorical setting, and this is essentially impossible except in particularly fortuitous circumstances. We show in §6, §7 and Appendix B that the theory of *ULA sheaves* provides a suitable method for this particular problem.

1.38. Structure of the paper

§2 is a mostly self-contained review of the geometry of Zastava spaces. In §3 and §4, we prove the limiting case of the Casselman-Shalika formula (Theorem 3.4.1) and use it to realize $\Upsilon_{\check{\mathfrak{n}}}$ in the geometry of Zastava spaces (Theorem 4.6.1). Then in §5, we give our first comparison (Theorem 5.14.1) between geometric Satake and the above construction of $\Upsilon_{\check{\mathfrak{n}}}$.

The remainder of the paper is dedicated to a generalization (Theorem 7.9.1) involving the fusion structure from the geometric Satake theorem. In §6, we introduce prerequisite ideas and discuss the factorizable geometric Satake theorem; in particular, Theorem 6.36.1 proves a version of the factorizable Casselman-Shalika equivalence of [22], which is a folklore result in the subject. In §7, we use this language to formulate a comparison between geometric Satake and our construction of $\Upsilon_{\check{\mathfrak{n}}}$ using the factorizable structures on both sides.

There are two appendices. Appendix A proves a technical categorical lemma from §6. Appendix B introduces a general categorical language based on the theory of universally locally acyclic (ULA) sheaves, and which is suitable for general use in §6. The ULA methods are essential for §6-7.

1.39. Conventions

For the remainder of this introduction, we establish the conventions for the remainder of the text.

1.40. We fix a field k of characteristic zero throughout the paper. All schemes, etc, are understood to be defined over k.

1.41. Lie theory

We fix the following notations from Lie theory.

Let G be a split reductive group over k, let B be a Borel subgroup of G with unipotent radical N and let T be the Cartan B/N. Let B^- be a Borel opposite to B, i.e., $B^- \cap B \xrightarrow{\simeq} T$. Let N^- denote the unipotent radical of B^-.

Let $\check{G}$ denote the corresponding Langlands dual group with corresponding Borel $\check{B}$, who in turn has unipotent radical $\check{N}$ and torus $\check{T} = \check{B}/\check{N}$, and similarly for $\check{B}^-$ and $\check{N}^-$.

Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{n}$, $\mathfrak{t}$, $\mathfrak{b}^-$, $\mathfrak{n}^-$, $\check{\mathfrak{g}}$, $\check{\mathfrak{b}}$, $\check{\mathfrak{n}}$, $\check{\mathfrak{t}}$, $\check{\mathfrak{b}}^-$ and $\check{\mathfrak{n}}^-$ denote the corresponding Lie algebras.

Let Λ denote the lattice of weights of T and let $\check{\Lambda}$ denote the lattice of coweights. We let Λ^+ (resp. $\check{\Lambda}^+$) denote the dominant weights (resp. coweights), and let $\check{\Lambda}^{pos}$ denote the $\mathbb{Z}^{\geqslant 0}$-span of the simple coroots.

Let $\mathcal{I}_G$ be the set of vertices in the Dynkin diagram of G. We recall that $\mathcal{I}_G$ is canonically identified with the set of simple positive roots and coroots of G. For $i \in \mathcal{I}_G$, we let $\alpha_i \in \Lambda$ (resp. $\check{\alpha}_i \in \check{\Lambda}$) denote the corresponding root (resp. coroot).

Moreover, we fix a choice of Chevalley generators $\{f_i\}_{i \in \mathcal{I}_G}$ of $\mathfrak{n}^-$.

Finally, we use the notation $\rho \in \Lambda$ for the half-sum of the positive roots of $\mathfrak{g}$, and similarly for $\check{\rho} \in \check{\Lambda}$.

1.42. For an algebraic group Γ, let $\mathbb{B}\Gamma$ denote the classifying stack $\mathrm{Spec}(k)/\Gamma$ for Γ.

1.43. Let X be a smooth projective curve.

We let Bun_G denote the moduli stack of G-bundles on X. Recall that Bun_G is a smooth Artin stack locally of finite type (though not quasi-compact).

Similarly, we let Bun_B, Bun_N, and Bun_T denote the corresponding moduli stacks of bundles on X. However, we note that we will abuse notation in dealing specifically with bundles of structure group N^-: we will systematically incorporate a twist discussed in detail in §2.8.

1.44. Categorical remarks

The ultimate result in this paper, Theorem 7.9.1, is about computing a certain factorization functor between factorization (DG) categories. This means that we need to work in a higher categorical framework (cf. [30], [32]) at this point.

We will impose some notations and conventions below regarding this framework. With that said, the reader may read up to §5 essentially without ever worrying about higher categories.

1.45. We impose the convention that essentially everything is assumed derived. We will make this more clear below, but first, we note the only exception: schemes can be understood as classical schemes throughout the body of the paper, since we deal only with D-modules on them.

1.46. We find it convenient to assume higher category theory as the basic assumption in our language. That is, we will understand "category" and "1-category" to mean "$(\infty, 1)$-category," "colimit" to (necessarily) mean "homotopy colimit," "groupoid" to mean "∞-groupoid" (aliases: homotopy type, space, etc.), and so on. We use the phrase "set" interchangeably with "discrete groupoid," i.e., a groupoid whose higher homotopy groups at any basepoint vanish.

When we need to refer to the more traditional notion of category, we use the term $(1, 1)$-category.

As an example: we let Gpd denote the category (i.e., ∞-category) of groupoids (i.e., ∞-groupoids).

1.47. DG categories

By DG category, we mean an (accessible) stable (∞-)category enriched over k-vector spaces.

We denote the category of DG categories under k-linear exact functors by DGCat and the category of cocomplete[19] DG categories under continuous[20] k-linear functors by DGCat_{cont}.

We consider DGCat_{cont} as equipped with the symmetric monoidal structure $\otimes$ from [32] §6.3. For $\mathcal{C}, \mathcal{D} \in \mathsf{DGCat}_{cont}$ and for $\mathcal{F} \in \mathcal{C}$ and $\mathcal{G} \in \mathcal{D}$, we let $\mathcal{F} \boxtimes \mathcal{G}$ denote the induced object of $\mathcal{C} \otimes \mathcal{D}$, since this notation is compatible with geometric settings.

For $\mathcal{C}$ an algebra in DGCat_{cont}, we let $\mathcal{C}$–mod denote $\mathcal{C}$–$\mathsf{mod}(\mathsf{DGCat}_{cont})$: no other interpretations of $\mathcal{C}$-module category will be considered, and moreover, $\mathcal{C}$ should systematically be regarded as an algebra in DGCat_{cont}.

For $\mathcal{C}$ a DG category equipped with a t-structure, we let $\mathcal{C}^{\geqslant 0}$ denote the subcategory of *coconnective* objects, and $\mathcal{C}^{\leqslant 0}$ the subcategory of *connective* objects (i.e., the notation is the standard notation for the convention of *cohomological* grading). We let $\mathcal{C}^{\heartsuit}$ denote the heart of the t-structure.

[19] We actually mean *presentable*, which differs from cocomplete by a set-theoretic condition that will always be satisfied for us throughout this text.

[20] There is some disagreement in the literature of the meaning of this word. By *continuous functor*, we mean a functor commuting with filtered colimits. Similarly, by a *cocomplete* category, we mean one admitting all colimits.

We let Vect denote the DG category of k-vector spaces: this DG category has a t-structure with heart $\mathsf{Vect}^{\heartsuit}$ the abelian category of k-vector spaces.

We use the material of the short note [24] freely, taking for granted the reader's comfort with the ideas of [24].

1.48. For a scheme S locally of finite type, we let $D(S)$ denote its DG category of D-modules. For a map $f : S \to T$, we let $f^! : D(T) \to D(S)$ and $f_{*,dR} : D(S) \to D(T)$ denote the corresponding functors.

We always equip $D(S)$ with the *perverse* t-structure,[21] i.e., the one for which IC_S lies in the heart of the t-structure. In particular, if S is smooth of dimension d, then the dualizing sheaf ω_S lies in degree $-d$ and the constant sheaf k_S lies in degree d. We sometimes refer to objects in the heart of this t-structure as *perverse sheaves* (especially if the object is holonomic), hoping this will not cause any confusion (since we do not assume $k = \mathbb{C}$, we are in no position to apply the Riemann-Hilbert correspondence).

1.49. Finally, we use the notation Oblv throughout for various forgetful functors.

1.50. Acknowledgments

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2. Review of Zastava spaces

2.1. In this section, we review the geometry of Zastava spaces, introduced in [18] and [13].

Note that this section plays a purely expository role; our only hope is that by emphasizing the role of *local Zastava stacks*, some of the basic geometry becomes more transparent than other treatments.

[21] Alias: the *right* (as opposed to left) t-structure. cf. [6] and [27].

2.2. Remarks on G

For simplicity, we assume throughout this section that G has a simply-connected derived group.

However, [3] §4.1 (cf. also [40] §7) explains how to remove this hypothesis, and the basic geometry of Zastava spaces and Drinfeld compactifications remains exactly the same. The reader may therefore either assume G has simply-connected derived group for the rest of this text, or may refer to [40] for how to remove this hypothesis (we note that this applies just as well for citations to [11], [12], and [13]).

2.3. The basic affine space

Recall that the map:

$$G/N \to \overline{G/N} := \mathrm{Spec}(H^0(\Gamma(G/N, \mathcal{O}_{G/N}))) = \mathrm{Spec}(\mathrm{Fun}(G)^N)$$

is an open embedding. We call G/N the *basic affine space* and $\overline{G/N}$ the *affine closure of the basic affine space*.

The following result is direct from the Peter-Weyl theorem.

Lemma 2.3.1. *For S an affine test scheme,*[22] *a map $\varphi : S \to \overline{G/N}$ with $\varphi^{-1}(G/N)$ dense in S is equivalent to a* Drinfeld structure *on the trivial G-bundle $G \times S \to S$, i.e., a sequence of maps for $\lambda \in \Lambda^+$:*

$$\sigma^\lambda : \ell^\lambda \underset{k}{\otimes} \mathcal{O}_S \to V^\lambda \underset{k}{\otimes} \mathcal{O}_S$$

that are monomorphisms of quasi-coherent sheaves and satisfy the Plücker relations.

Remark 2.3.2. By *dense*, we mean scheme-theoretically, not topologically (e.g., for Noetherian S, the difference here is only apparent in the presence of associated points).

Example 2.3.3. For $G = \mathrm{SL}_2$, $\overline{G/N}$ identifies equivariantly with $\mathbb{A}^2$. The corresponding map $\mathrm{SL}_2 \to \mathbb{A}^2$ here is given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, c) \in \mathbb{A}^2.$$

2.4.

Let $\overline{T}$ be the closure of $T = B/N \subseteq G/N$ in $\overline{G/N}$.

[22] It is important here that S is a *classical* scheme, i.e., not DG.

Lemma 2.4.1.

(1) $\overline{T}$ *is the toric variety* $\mathrm{Spec}(k[\Lambda^+])$ *(here* $k[\Lambda^+]$ *is the monoid algebra defined by the monoid* Λ^+*). Here the map* $T = \mathrm{Spec}(k[\Lambda]) \to \overline{T}$ *corresponds to the embedding* $\Lambda^+ \to \Lambda$ *and the map* $\mathrm{Fun}(G)^N \to k[\Lambda^+]$ *realizes the latter as* N*-coinvariants of the former.*

(2) The action of T *on* $\overline{G/N}$ *extends to an action of the monoid* $\overline{T}$ *on* $\overline{G/N}$ *(where the coalgebra structure on* $\mathrm{Fun}(\overline{T}) = k[\Lambda^+]$ *is the canonical one, that is, defined by the diagonal map for the monoid* Λ^+*).*

Here (1) follows again from the Peter-Weyl theorem and (2) follows similarly, noting that $V^\lambda \otimes \ell^{\lambda,\vee} \subseteq \mathrm{Fun}(G)^N = \mathrm{Fun}(\overline{G/N})$ has Λ-grading (relative to the right action of T on $\overline{G/N}$) equal to $\lambda \in \Lambda^+$.

2.5. Note that (after the choice of opposite Borel) $\overline{T}$ is canonically a retract of $\overline{G/N}$, i.e., the embedding $\overline{T} \hookrightarrow \overline{G/N}$ admits a canonical splitting:

$$\overline{G/N} \to \overline{T}. \tag{2.5.1}$$

Indeed, the retract corresponds to the map $k[\Lambda^+] \to \mathrm{Fun}(G)^N$ sending λ to the canonical element in:

$$\ell^\lambda \otimes \ell^{\lambda,\vee} \subseteq V^\lambda \otimes V^{\lambda,\vee} \subseteq \mathrm{Fun}(G)$$

(note that the embedding $\ell^{\lambda,\vee} \hookrightarrow V^{\lambda,\vee}$ uses the opposite Borel).

By construction, this map factors as $\overline{G/N} \to N^- \backslash (\overline{G/N}) \to \overline{T}$.

Let T act on $\overline{G/N}$ through the action induced by the adjoint action of T on G. Choosing a regular dominant coweight $\lambda_0 \in \check{\Lambda}^+$ we obtain a $\mathbb{G}_m$-action on $\overline{G/N}$ that contracts[23] onto $\overline{T}$. The induced map $\overline{G/N} \to \overline{T}$ coincides with the one constructed above.

Warning 2.5.1. The induced map $G/N \to \overline{T}$ does *not* factor through T. The inverse image in $\overline{G/N}$ of $T \subseteq \overline{T}$ is the open Bruhat cell $B^- N/N$.

2.6. Define the stack $\overline{\mathbb{B}B}$ as $G\backslash\overline{G/N}/T$. Note that $\overline{\mathbb{B}B}$ has canonical maps to $\mathbb{B}G$ and $\mathbb{B}T$.

[23] We recall that a *contracting* $\mathbb{G}_m$*-action* on an algebraic stack $\mathcal{Y}$ is an action of the multiplicative monoid $\mathbb{A}^1$ on $\mathcal{Y}$. For schemes, this is a property of the underlying $\mathbb{G}_m$ action, but for stacks it is not. Therefore, by the phrase "that contracts," we rather mean that it canonically admits the structure of contracting $\mathbb{G}_m$-action. See [15] for further discussion of these points.

2.7. Local Zastava stacks

Let $\overset{o}{\zeta}$ denote the stack $B^- \backslash G/B = \mathbb{B}B^- \times_{\mathbb{B}G} \mathbb{B}B$ and let ζ denote the stack $B^- \backslash (\overline{G/N})/T = \mathbb{B}B^- \times_{\mathbb{B}G} \overline{\mathbb{B}B}$. We have the sequence of open embeddings:

$$\mathbb{B}T \hookrightarrow \overset{o}{\zeta} \hookrightarrow \zeta$$

where $\mathbb{B}T$ embeds as the open Bruhat cell.

The map $\mathbb{B}T \hookrightarrow \zeta$ factors as:

$$\mathbb{B}T = T\backslash(T/T) \hookrightarrow T\backslash(\overline{T}/T) = \mathbb{B}T \times \overline{T}/T \hookrightarrow \zeta. \tag{2.7.1}$$

One immediately verifies that the retraction $\overline{G/N} \to \overline{T}$ of (2.5.1) is $B^- \times T$-equivariant, where B^- acts on the left on $\overline{G/N}$ and T acts on the right, and the action on $\overline{T}$ is similar but is induced by the $T \times T$-action and the homomorphism $B^- \times T \to T \times T$. Therefore, we obtain a canonical map:

$$\zeta = B^-\backslash\overline{G/N}/T \to B^-\backslash\overline{T}/T \to T\backslash\overline{T}/T.$$

Moreover, up to the choice of λ_0 from [3] this retraction realizes $\mathbb{B}T \times \overline{T}/T$ as a *deformation retract* of ζ.

We will identify $T\backslash\overline{T}/T$ with $\mathbb{B}T \times \overline{T}/T$ in what follows by writing the former as $T\backslash(\overline{T}/T)$ and noting that T acts trivially here on $\overline{T}/T$.

In particular, we obtain a canonical map:

$$\zeta \to \overline{T}/T. \tag{2.7.2}$$

By Lemma 2.4.1 (2) we have an action of the monoid stack $\overline{T}/T$ on ζ. The morphism $\zeta \xrightarrow{r} \mathbb{B}T \times \overline{T}/T \xrightarrow{p_2} \overline{T}/T$ is $\overline{T}/T$-equivariant.

Lemma 2.7.1. *A map $\varphi : S \to \overline{T}/T$ with $\varphi^{-1}(\mathrm{Spec}(k))$ dense (where $\mathrm{Spec}(k)$ is realized as the open point T/T) is canonically equivalent to a $\check{\Lambda}^{neg} := -\check{\Lambda}^{pos}$-valued Cartier divisor on S.*

First, we recall the following standard result.

Lemma 2.7.2. *A map $S \to \mathbb{G}_m\backslash\mathbb{A}^1$ with inverse image of the open point dense is equivalent to the data of an effective Cartier divisor on S.*

Proof. Tautologically, a map $S \to \mathbb{G}_m\backslash\mathbb{A}^1$ is equivalent to a line bundle $\mathcal{L}$ on S with a section $s \in \Gamma(S, \mathcal{L})$.

We need to check that the morphism $\mathcal{O}_S \xrightarrow{s} \mathcal{L}$ is a monomorphism of quasi-coherent sheaves under the density hypothesis. This is a local statement, so we can trivialize $\mathcal{L}$.

Now s is a function f whose locus of non-vanishing is dense, and it is easy to see that this is equivalent to f being a non-zero divisor. □

Proof of Lemma 2.7.1. Let $G' \subseteq G$ denote the derived subgroup $[G,G]$ of G and let $T' = T \cap G'$ and $N' = N \cap G'$. Then with $\overline{T}'$ defined as the closure of T' in the affine closure of G'/N', the induced map:

$$\overline{T}'/T' \to \overline{T}/T$$

is an isomorphism, reducing to the case $G = G'$.

Because the derived group (assumed to be equal to G now) is assumed simply-connected, we have canonical fundamental weights $\{\vartheta_i\}_{i\in\mathcal{I}_G}$, $\vartheta_i \in \Lambda^+$. The map $\prod_{i\in\mathcal{I}_G} \vartheta_i : T \to \prod_{i\in\mathcal{I}_G} \mathbb{G}_m$ extends to a map $\overline{T} \to \prod_{i\in\mathcal{I}_G} \mathbb{A}^1$ inducing an isomorphism:

$$\overline{T}/T \xrightarrow{\simeq} (\mathbb{A}^1/\mathbb{G}_m)^{\mathcal{I}_G}.$$

Because we use the right action of T on $\overline{T}$, the functions on $\overline{T}$ are graded negatively, and therefore we obtain the desired result. □

2.8. Twists

Fix an irreducible smooth projective curve X. We digress for a minute to normalize certain twists.

Let Ω_X denote the sheaf of differentials on X. For an integer n, we will sometimes use the notation Ω_X^n for $\Omega_X^{\otimes n}$, there being no risk for confusion with n-forms because X is a curve.

We fix $\Omega_X^{\frac{1}{2}}$ a square root of Ω_X. This choice extends the definition of Ω_X^n to $n \in \frac{1}{2}\mathbb{Z}$. We obtain the T-bundle:

$$\mathcal{P}_T^{can} := \check{\rho}(\Omega_X^{-1}) := 2\check{\rho}(\Omega_X^{-\frac{1}{2}}). \tag{2.8.1}$$

We use the following notation:

$$\mathrm{Bun}_{N^-} := \mathrm{Bun}_{B^-} \underset{\mathrm{Bun}_T}{\times} \{\mathcal{P}_T^{can}\}$$

$$\mathrm{Bun}_{\mathbb{G}_a^-} := \mathrm{Bun}_{\mathbb{G}_m \ltimes \mathbb{G}_a} \underset{\mathrm{Bun}_{\mathbb{G}_m}}{\times} \{\Omega_X^{-\frac{1}{2}}\}.$$

Here $\mathbb{G}_m \ltimes \mathbb{G}_a$ is the opposite Borel of PGL_2.

Note that $\mathrm{Bun}_{\mathbb{G}_a^-}$ classifies extensions of $\Omega_X^{-\frac{1}{2}}$ by $\Omega_X^{\frac{1}{2}}$ and therefore there is a canonical map:

$$\mathrm{can}_{\mathbb{G}_a^-} : \mathrm{Bun}_{\mathbb{G}_a^-} \to H^1(X, \Omega_X) = \mathbb{G}_a.$$

The choice of Chevalley generators $\{f_i\}_{i\in\mathcal{I}_G}$ of $\mathfrak{n}^-$ defines a map:

$$B^-/[N^-,N^-]\to\prod_{i\in\mathcal{I}_G}(\mathbb{G}_m\ltimes\mathbb{G}_a).$$

By definition of $\mathcal{P}_T^{can}$, this induces a map:

$$\prod_{i\in\mathcal{I}_G}\mathfrak{r}_i:\mathrm{Bun}_{N^-}\to\prod_{i\in\mathcal{I}_G}\mathrm{Bun}_{\mathbb{G}_a^-}.$$

We form the sequence:

$$\mathrm{Bun}_{N^-}\to\prod_{i\in\mathcal{I}_G}\mathrm{Bun}_{\mathbb{G}_a^-}\xrightarrow{\prod_{i\in\mathcal{I}_G}\mathrm{can}_{\mathbb{G}_a^-}}\prod_{i\in\mathcal{I}_G}\mathbb{G}_a\to\mathbb{G}_a$$

and denote the composition by:

$$\mathrm{can}:\mathrm{Bun}_{N^-}\to\mathbb{G}_a. \tag{2.8.2}$$

Remark 2.8.1. The above notation is obviously somewhat abusive, since the notation Bun_{N^-} is often used without this twist. For our purposes, we prefer this convention because it simplifies the notation.

2.9. For a pointed stack $(\mathcal{Y},y\in\mathcal{Y}(k))$ and a test scheme S, we say that $X\times S\to\mathcal{Y}$ is *non-degenerate* if there exists $U\subseteq X\times S$ universally schematically dense relative to S in the sense of [28] Exp. XVIII, and such that the induced map $U\to\mathcal{Y}$ admits a factorization as $U\to\mathrm{Spec}(k)\xrightarrow{y}\mathcal{Y}$ (so this is a property for a map, not a structure). We let $\mathrm{Maps}_{non-degen.}(X,\mathcal{Y})$ denote the open substack of $\mathrm{Maps}(X,\mathcal{Y})$ consisting of non-degenerate maps $X\to\mathcal{Y}$.

We consider $\overset{o}{\zeta}$ and ζ as pointed stacks via the embedding from $\mathbb{B}T$ and its canonical base-point. We consider $\overline{T}/T$ as a pointed stack in the obvious way.

2.10. Zastava spaces

Observe that there is a canonical map:

$$\zeta\to\mathbb{B}T \tag{2.10.1}$$

given as the composition:

$$\zeta=\mathbb{B}B^-\underset{\mathbb{B}G}{\times}\overline{\mathbb{B}B}\to\mathbb{B}B^-\to\mathbb{B}T.$$

Let $\mathcal{Z}$ be the stack of $\mathcal{P}_T^{can}$-twisted non-degenerate maps $X\to\zeta$, i.e., the fiber product:

$$\mathrm{Maps}_{non-degen.}(X,\zeta)\underset{\mathrm{Bun}_T}{\times}\{\mathcal{P}_T^{can}\}$$

where the map $\mathrm{Maps}_{non-degen.}(X,\zeta) \to \mathrm{Bun}_T$ is given by (2.10.1).

Let $\overset{o}{\mathcal{Z}} \subseteq \mathcal{Z}$ be the open substack of $\mathcal{P}_T^{can}$-twisted non-degenerate maps $X \to \overset{o}{\zeta}$. Note that $\mathcal{Z}$ and $\overset{o}{\mathcal{Z}}$ lie in $\mathsf{Sch} \subseteq \mathsf{PreStk}$. We call $\mathcal{Z}$ the *Zastava space* and $\overset{o}{\mathcal{Z}}$ the *open Zastava space*. We let $\jmath : \overset{o}{\mathcal{Z}} \to \mathcal{Z}$ denote the corresponding open embedding.

We have a Cartesian square where all maps are open embeddings:

$$\begin{array}{ccc} \overset{o}{\mathcal{Z}} & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{N^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_B & \longrightarrow & \mathrm{Bun}_{N^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_B \end{array}$$

The vertical arrows realize the source as the subscheme of the target where the two reductions are generically transverse.

2.11. Let $\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}} = \mathrm{Maps}_{non-degen.}(X, \overline{T}/T)$ denote the scheme of $\check{\Lambda}^{pos}$-divisors on X (we include the subscript "eff" for emphasis that we are not taking $\check{\Lambda}$-valued divisors).

We have the canonical map:

$$\deg : \pi_0(\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}) \to \check{\Lambda}^{pos}.$$

For $\check{\lambda} \in \check{\Lambda}^{pos}$ let $\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}$ denote the corresponding connected component of $\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}$.

Remark 2.11.1. Writing $\check{\lambda} = \sum_{i \in \mathcal{I}_G} n_i \check{\alpha}_i$ as a sum of simple coroots, we see that $\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}$ is a product $\prod_{i \in \mathcal{I}_G} \mathrm{Sym}^{n_i} X$ of the corresponding symmetric powers of the curve.

Recall that we have the canonical map $r : \zeta \to \mathbb{B}T \times \overline{T}/T$. For any non-degenerate map $X \times S \to \zeta$, Warning 2.5.1 implies that the induced map to $\overline{T}/T$ (given by composing r with the second projection) is non-degenerate as well.

Therefore we obtain the map:

$$\pi : \mathcal{Z} \to \mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}.$$

We let $\overset{o}{\pi}$ denote the restriction of π to $\overset{o}{\mathcal{Z}}$. It is well-known that the morphism π is affine.

Let $\mathcal{Z}^{\check{\lambda}}$ (resp. $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$) denote the fiber of $\mathcal{Z}$ (resp. $\overset{o}{\mathcal{Z}}$) over $\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}$. We let $\pi^{\check{\lambda}}$ (resp. $\overset{o}{\pi}{}^{\check{\lambda}}$) denote the restriction of π to $\mathcal{Z}^{\check{\lambda}}$ (resp. $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$). We let $\jmath^{\check{\lambda}} : \overset{o}{\mathcal{Z}}{}^{\check{\lambda}} \to \mathcal{Z}^{\check{\lambda}}$ denote the restriction of the open embedding $\jmath$.

Note that π admits a canonical section $\mathfrak{s} : \mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}} \to \mathcal{Z}$, whose restriction to each $\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}$ we denote by $\mathfrak{s}^{\check{\lambda}}$. Note that up to a choice of regular dominant coweight, the situation is given by contraction.

Each $\mathcal{Z}^{\check{\lambda}}$ is of finite type (and therefore the same holds for $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$). It is known (cf. [13] Corollary 3.8) that $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$ is a smooth variety.

For $\check{\lambda} = 0$, we have $\overset{o}{\mathcal{Z}}{}^{0} = \mathcal{Z}^{0} = \mathrm{Div}^{0}_{\mathrm{eff}} = \mathrm{Spec}(k)$.

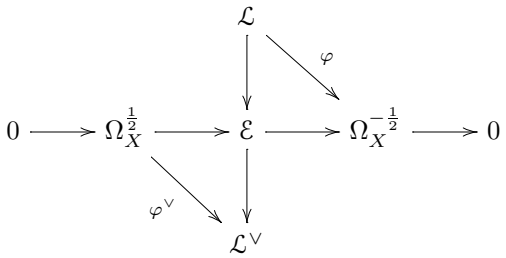
We have a canonical (up to choice of Chevalley generators) map $\mathcal{Z} \to \mathbb{G}_a$ defined as the composition $\mathcal{Z} \to \mathrm{Bun}_{N^-} \xrightarrow{\mathrm{can}} \mathbb{G}_a$. For $\check{\alpha}_i$ a positive simple coroot the induced map:

$$\mathcal{Z}^{\check{\alpha}_i} \to \mathrm{Div}^{\check{\alpha}_i}_{\mathrm{eff}} \times \mathbb{G}_a = X \times \mathbb{G}_a \tag{2.11.1}$$

is an isomorphism that identifies $\overset{o}{\mathcal{Z}}{}^{\check{\alpha}_i}$ with $X \times \mathbb{G}_m$.

The dimension of $\mathcal{Z}^{\check{\lambda}}$ and $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$ is $(2\rho, \check{\lambda}) = (\rho, \check{\lambda}) + \dim \mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}$ (this follows e.g. from the factorization property discussed in §2.12 below and then by the realization discussed in §2.13 of the central fiber as an intersection of semi-infinite orbits in the Grassmannian, that are known by [13] §6 to be equidimensional with dimension $(\rho, \check{\lambda})$).

Example 2.11.2. Let us explain in more detail the case of $G = \mathrm{SL}_2$. In this case, tensoring with the bundle $\Omega_X^{\frac{1}{2}}$ identifies $\mathcal{Z}$ with the moduli of commutative diagrams:



in which the composition $\mathcal{L} \to \mathcal{L}^\vee$ is zero and the morphism φ is non-zero. The open subscheme $\overset{o}{\mathcal{Z}}$ is the locus where the induced map $\mathrm{Coker}(\mathcal{L} \to \mathcal{E}) \to \mathcal{L}^\vee$ is an isomorphism. The associated divisor of such a datum is defined by the injection $\mathcal{L} \hookrightarrow \Omega_X^{-\frac{1}{2}}$.

Over a point $x \in X$, we have an identification of the fiber $\overset{o}{\mathcal{Z}}{}^{1}_{x}$ of $\overset{o}{\mathcal{Z}}{}^{1}$ over $x \in X$ (considering $1 \in \mathbb{Z} = \check{\Lambda}_{\mathrm{SL}_2}$ as the unique positive simple coroot) with $\mathbb{G}_m$. Up to the twist by our square root $\Omega_X^{\frac{1}{2}}$, the point $1 \in \mathbb{G}_m$ corresponds to a canonical extension of $\mathcal{O}_X$ by Ω_X associated to the point x, that can be constructed explicitly using the *Atiyah sequence* of the line bundle $\mathcal{O}_X(x)$.

Recall that for a vector bundle $\mathcal{E}$, the Atiyah sequence (cf. [4]) is a canonical short exact sequence:

$$0 \to \mathrm{End}(\mathcal{E}) \to \mathrm{At}(\mathcal{E}) \to T_X \to 0$$

whose splittings correspond to connections on $\mathcal{E}$. For a line bundle $\mathcal{L}$, we obtain a canonical extension $\mathrm{At}(\mathcal{L}) \otimes \Omega^1_X$ of $\mathcal{O}_X$ by Ω^1_X. Taking $\mathcal{L} = \mathcal{O}_X(x)$, we obtain the extension underlying the canonical point of $\overset{o}{\mathcal{Z}}{}^{1}_{x}$.

Note that we have a canonical map $\mathcal{L} = \mathcal{O}_X(x) \to \mathrm{At}(\mathcal{O}_X(x)) \otimes \Omega^1_X$ that may be thought of as a splitting of the Atiyah sequence with a pole of order 1, and this splitting corresponds to the obvious connection on $\mathcal{O}_X(x)$ with a pole of order 1. This defines the corresponding point of $\overset{o}{\mathcal{Z}}{}^1$ completely.

2.12. Factorization

Now we recall the crucial factorization property of $\mathcal{Z}$.

Let $\mathrm{add} : \mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}} \times \mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}} \to \mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}$ denote the addition map for the commutative monoid structure defined by addition of divisors. For $\check{\lambda}$ and $\check{\mu}$ fixed, we let $\mathrm{add}^{\check{\lambda},\check{\mu}}$ denote the induced map $\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}} \times \mathrm{Div}^{\check{\mu}}_{\mathrm{eff}} \to \mathrm{Div}^{\check{\lambda}+\check{\mu}}_{\mathrm{eff}}$.

Define:

$$[\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}} \times \mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}]_{disj} \subseteq \mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}} \times \mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}$$

as the moduli of pairs of disjoint $\check{\Lambda}^{pos}$-divisors. Note that the restriction of add to this locus is étale.

Then we have canonical *factorization* isomorphisms:

$$\mathcal{Z} \underset{\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}}{\times} [\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}} \times \mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}]_{disj} \xrightarrow{\simeq} (\mathcal{Z} \times \mathcal{Z}) \underset{\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}} \times \mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}}{\times} [\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}} \times \mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}]_{disj}$$

that are associative in the natural sense.

The morphisms π and $\mathfrak{s}$ are compatible with the factorization structure.

2.13. The central fiber

By definition, the *central fiber* $\mathfrak{Z}^{\check{\lambda}}$ of the Zastava space $\mathcal{Z}^{\check{\lambda}}$ is the fiber product:

$$\mathfrak{Z}^{\check{\lambda}} := \mathcal{Z}^{\check{\lambda}} \underset{\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}}{\times} X$$

where $X \to \mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}$ is the closed "diagonal" embedding, i.e., it is the closed subscheme where the divisor is concentrated at a single point. We let $\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}}$ denote the open in $\mathfrak{Z}^{\check{\lambda}}$ corresponding to $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}} \hookrightarrow \mathcal{Z}^{\check{\lambda}}$. Similarly, we let $\mathfrak{Z} \subseteq \mathcal{Z}$ be the closed corresponding to the union of the $\mathfrak{Z}^{\check{\lambda}}$.

We let $\beta^{\check{\lambda}}$ (resp. $\gamma^{\check{\lambda}}$) denote the closed embedding $\mathfrak{Z}^{\check{\lambda}} \hookrightarrow \mathcal{Z}^{\check{\lambda}}$ (resp. $\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}} \hookrightarrow \overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$).

2.14. Twisted affine Grassmannian

Let $\mathcal{P}^{can}_G, \mathcal{P}^{can}_B$ and $\mathcal{P}^{can}_{B^-}$ be the torsors induced by the T-torsor $\mathcal{P}^{can}_T$ under the embeddings of T into each of these groups.

We let $\mathrm{Gr}_{G,X}$ denote the $\mathcal{P}_G^{can}$*-twisted Beilinson-Drinfeld affine Grassmannian* classifying a point $x \in X$, a G-bundle $\mathcal{P}_G$ on X, and an isomorphism $\mathcal{P}_G^{can}|_{X\setminus x} \simeq \mathcal{P}_G|_{X\setminus x}$. More precisely, the S-points are:

$$S \mapsto \left\{ \begin{array}{l} x : S \to X,\ \mathcal{P}_G \text{ a } G\text{-bundle on } X \times S, \\ \alpha \text{ an isomorphism } \mathcal{P}_G|_{X\times S\setminus \Gamma_x} \simeq \mathcal{P}_G^{can}|_{X\times S\setminus \Gamma_x} \end{array} \right\}.$$

Similarly for $\mathrm{Gr}_{B,X}$, etc. We define $\mathrm{Gr}_{N^-,X} := \mathrm{Gr}_{B^-,X} \times_{\mathrm{Gr}_{T,X}} X$, where the map $X \to \mathrm{Gr}_{T,X}$ being the tautological section.

Let $\overline{\mathrm{Gr}}_{B,X}$ denote the *union of closures of semi-infinite orbits*, i.e., the indscheme:

$$\overline{\mathrm{Gr}}_{B,X} : S \mapsto \left\{ \begin{array}{l} x : S \to X,\ \varphi : X \times S \to G \setminus (\overline{G/N})/T, \\ \alpha \text{ a factorization of } \varphi|_{(X\times S)\setminus \Gamma_x} \text{ through the} \\ \text{canonical map } \mathrm{Spec}(k) \to G \setminus (\overline{G/N})/T. \end{array} \right\}.$$

Here Γ_x denotes the graph of the map x.

2.15. In the above notation, we have a canonical isomorphism:

$$\mathfrak{Z} \xrightarrow{\simeq} \mathrm{Gr}_{N^-,X} \underset{\mathrm{Gr}_{G,X}}{\times} \overline{\mathrm{Gr}}_{B,X}.$$

Indeed, this is immediate from the definitions.

Note that $\mathrm{Gr}_{B,X}$ has a canonical map to $\mathrm{Gr}_{T,X} = \coprod_{\check\lambda\in\check\Lambda} \mathrm{Gr}_{T,X}^{\check\lambda}$. Letting $\mathrm{Gr}_{B,X}^{\check\lambda}$ be the fiber over the corresponding connected component of $\mathrm{Gr}_{T,X}$, we obtain:

$$\mathfrak{Z}^{\check\lambda} \xrightarrow{\simeq} \mathrm{Gr}_{N^-,X} \underset{\mathrm{Gr}_{G,X}}{\times} \overline{\mathrm{Gr}}_{B,X}^{\check\lambda}.$$

2.16. By §2.7, we have an action of $\mathrm{Div}_{\mathrm{eff}}^{\check\Lambda^{pos}}$ on $\mathcal{Z}$ so that the morphism π is $\mathrm{Div}_{\mathrm{eff}}^{\check\Lambda^{pos}}$-equivariant. We let $\mathrm{act}_{\mathcal{Z}}$ denote the action map $\mathrm{Div}_{\mathrm{eff}}^{\check\Lambda^{pos}} \times \mathcal{Z} \to \mathcal{Z}$. We abuse notation in denoting the induced map $\mathrm{Div}_{\mathrm{eff}}^{\check\Lambda^{pos}} \times \overset{o}{\mathcal{Z}} \to \mathcal{Z}$ by $\mathrm{act}_{\overset{o}{\mathcal{Z}}}$ (that does not define an action on $\overset{o}{\mathcal{Z}}$, i.e., this map does not factor through $\overset{o}{\mathcal{Z}}$).

For $\check\lambda \in \check\Lambda$ acting on $\mathcal{Z}^{\check\lambda}$ defines the map:

$$\mathrm{act}_{\mathcal{Z}}^{\check\lambda} : \mathrm{Div}_{\mathrm{eff}}^{\check\Lambda^{pos}} \times \mathcal{Z}^{\check\lambda} \to \mathcal{Z}.$$

For $\check\eta \in \check\Lambda^{pos}$ we use the notation $\mathrm{act}_{\mathcal{Z}}^{\check\eta,\check\lambda}$ for the induced map:

$$\mathrm{act}_{\mathcal{Z}}^{\check\eta,\check\lambda} : \mathrm{Div}_{\mathrm{eff}}^{\check\eta} \times \mathcal{Z}^{\check\lambda} \to \mathcal{Z}^{\check\lambda+\check\eta}.$$

Similarly, we have the maps $\mathrm{act}_{\overset{o}{\mathcal{Z}}}^{\check\lambda}$ and $\mathrm{act}_{\overset{o}{\mathcal{Z}}}^{\check\lambda,\check\eta}$.

The following lemma is well-known (see e.g. [13]).

Lemma 2.16.1. *For every $\check{\lambda},\check{\eta} \in \check{\Lambda}^{pos}$, the map $\mathrm{act}^{\check{\lambda},\check{\eta}}_{\mathcal{Z}}$ (resp. $\mathrm{act}^{\check{\lambda},\check{\eta}}_{\overset{o}{\mathcal{Z}}}$) is finite (resp. a locally closed embedding). For fixed $\check{\lambda}$ the set of locally closed subschemes of $\mathcal{Z}^{\check{\lambda}}$:*

$$\{\mathrm{act}^{\check{\eta},\check{\mu}}_{\overset{o}{\mathcal{Z}}}(\mathrm{Div}^{\check{\eta}}_{\mathrm{eff}} \times \overset{o}{\mathcal{Z}}{}^{\check{\mu}})\}_{\substack{\check{\mu}+\check{\eta}=\check{\lambda} \\ \check{\mu},\check{\eta}\in\check{\Lambda}^{pos}}}$$

forms a stratification.

2.17. Intersection cohomology of Zastava

For $\check{\lambda} \in \check{\Lambda}^{pos}$ we now review the description from [13] of the fibers of the intersection cohomology D-module $\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}}$ along the strata described above, i.e., the D-modules:

$$\mathrm{act}^{\check{\eta},\check{\mu},!}_{\overset{o}{\mathcal{Z}}}(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}}) \in D(\mathrm{Div}^{\check{\eta}}_{\mathrm{eff}} \times \overset{o}{\mathcal{Z}}{}^{\check{\mu}}), \quad \check{\eta},\check{\mu} \in \check{\Lambda}^{pos}, \check{\mu}+\check{\eta}=\check{\lambda}.$$

Theorem 2.17.1.

(1) With notation as above, the regular holonomic D-module:

$$\mathrm{act}^{\check{\eta},\check{\mu},!}_{\overset{o}{\mathcal{Z}}}(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}}) \in D(\mathrm{Div}^{\check{\eta}}_{\mathrm{eff}} \times \overset{o}{\mathcal{Z}}{}^{\check{\mu}}) \tag{2.17.1}$$

is concentrated in constructible cohomological degree $-\dim \overset{o}{\mathcal{Z}}{}^{\check{\mu}}$.

(2) For $x \in X$ a point, the further $*$-restriction of (2.17.1) to $\overset{o}{\mathfrak{Z}}{}^{\check{\mu}}_x$ is a lisse sheaf in constructible degree $-\dim \overset{o}{\mathcal{Z}}{}^{\check{\mu}}$ isomorphic to:

$$U(\check{\mathfrak{n}})(\check{\eta}) \otimes k_{\overset{o}{\mathfrak{Z}}{}^{\check{\mu}}_x}[\dim \overset{o}{\mathcal{Z}}{}^{\check{\mu}}]$$

where $U(\check{\mathfrak{n}})(\check{\eta})$ indicates the $\check{\eta}$-weight space.

(3) The $!$-restriction of (2.17.1) to $\overset{o}{\mathfrak{Z}}{}^{\check{\mu}}_x$ is isomorphic to:

$$\mathrm{Sym}(\check{\mathfrak{n}}[-2])(\check{\eta}) \otimes \omega_{\overset{o}{\mathfrak{Z}}{}^{\check{\mu}}_x}[-\dim \overset{o}{\mathcal{Z}}{}^{\check{\mu}}] \tag{2.17.2}$$

for $\mathrm{Sym}(\check{\mathfrak{n}}[-2])(\check{\eta})$ the $\check{\eta}$-weight space of the (DG) symmetric algebra $\mathrm{Sym}(\check{\mathfrak{n}}[-2]) \in \mathsf{Rep}(\check{T})$.

Remark 2.17.2. Recall from the above that $\overset{o}{\mathcal{Z}}{}^{\check{\mu}}$ is equidimensional with $\dim \mathcal{Z}^{\check{\mu}} = 2(\rho,\check{\mu})$.

Remark 2.17.3. This theorem is a combination of Theorem 4.5 and Lemma 4.3 of [13] using the inductive procedure of [13].

2.18. Locality

For X a smooth (possibly affine) curve with choice of $\Omega_X^{\frac{1}{2}}$, we obtain an identical geometric picture. One can either realize this by restriction from a compactification, or by reinterpreting e.g. the map $\mathcal{Z} \to \mathbb{G}_a$ through residues instead of through global cohomology.

3. Limiting case of the Casselman-Shalika formula

3.1. The goal for this section is to prove Theorem 3.4.1, on the vanishing of the IC-Whittaker cohomology groups of Zastava spaces. This vanishing will play a central role in the remainder of the paper.

Remark 3.1.1. The method of proof is essentially by a reduction to the geometric Casselman-Shalika formula of [22].

Remark 3.1.2. We are grateful to Dennis Gaitsgory for suggesting this result to us.

3.2. Artin-Schreier sheaves

We define the !-Artin-Schreier D-module $\psi \in D(\mathbb{G}_a)$ to be the exponential local system normalized cohomologically so that $\psi[-1] \in D(\mathbb{G}_a)^{\heartsuit}$. Note that ψ is multiplicative with respect to !-pullback.

3.3. For $\check{\lambda} \in \check{\Lambda}^{pos}$, let $\psi_{\mathcal{Z}^{\check{\lambda}}} \in D(\mathcal{Z}^{\check{\lambda}})$ denote the !-pullback of the Artin-Schreier D-module ψ along the composition:

$$\mathcal{Z}^{\check{\lambda}} \to \mathrm{Bun}_{N^-} \xrightarrow{\mathrm{can}} \mathbb{G}_a.$$

Note that $\psi_{\mathcal{Z}^{\check{\lambda}}} \overset{!}{\otimes} \mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}} \in D(\mathcal{Z}^{\check{\lambda}})^{\heartsuit}$.

We then also define:

$$\psi_{\overset{\circ}{\mathcal{Z}}{}^{\check{\lambda}}} = \jmath^{\check{\lambda},!}(\psi_{\mathcal{Z}^{\check{\lambda}}}).$$

3.4. The main result of this section is the following:

Theorem 3.4.1. *If $\check{\lambda} \neq 0$, then:*

$$\pi^{\check{\lambda}}_{*,dR}(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}} \overset{!}{\otimes} \psi_{\mathcal{Z}^{\check{\lambda}}}) = 0.$$

The proof will be given in §3.6 below.

This theorem is étale local on X, and therefore we may assume that we have $X = \mathbb{A}^1$. In particular, we have a fixed trivialization of $\Omega_X^{\frac{1}{2}}$.

3.5. Central fibers via affine Schubert varieties

In the proof of Theorem 3.4.1 we will use Proposition 3.5.1 below. We note that it is well-known, though we do not know a published reference.

Throughout §3.5, we work only with reduced schemes and indschemes, so all symbols refer to the reduced indscheme underlying the corresponding indscheme. Note that this restriction does not affect D-modules on the corresponding spaces.

Let $T(K)_X$ denote the group indscheme over X of meromorphic jets into T (so the fiber of $T(K)_X$ at $x \in X$ is the loop group $T(K_x)$). Because we have chosen an identification $X \simeq \mathbb{A}^1$, we have a canonical homomorphism:

$$\begin{gathered} \mathrm{Gr}_{T,X} \simeq \mathbb{A}^1 \times \check{\Lambda} \to T(K)_X \simeq \mathbb{A}^1 \times T(K) \\ (x, \check{\lambda}) \mapsto (x, \check{\lambda}(t)) \end{gathered}$$

where t is the uniformizer of $\mathbb{A}^1$ (of course, the formula $\mathrm{Gr}_{T,X} \simeq \mathbb{A}^1 \times \check{\Lambda}$ is only valid at the reduced level). This induces an action of the X-group indscheme $\mathrm{Gr}_{T,X}$ on $\mathrm{Gr}_{B,X}$, $\mathrm{Gr}_{G,X}$ and $\mathrm{Gr}_{N^-,X} = \mathrm{Gr}^0_{B^-,X}$.

Using this action, we obtain a canonical isomorphism:

$$\mathfrak{Z}^{\check{\lambda}} = \mathrm{Gr}^0_{B^-,X} \underset{\mathrm{Gr}_{G,X}}{\times} \overline{\mathrm{Gr}}^{\check{\lambda}}_{B,X} \xrightarrow{\simeq} \mathrm{Gr}^{\check{\eta}}_{B^-,X} \underset{\mathrm{Gr}_{G,X}}{\times} \overline{\mathrm{Gr}}^{\check{\lambda}+\check{\eta}}_{B,X}$$

of X-schemes for every $\check{\eta} \in \check{\Lambda}$.

Below, we let $\mathrm{Gr}^{\check{\lambda}}_{G,x}$ denote the $G(O_x)$-orbit through the point $\check{\lambda}(t) \in \mathrm{Gr}_{G,x}$, and similarly for $\mathrm{Gr}^{\check{\lambda}}_{G,X}$.

Proposition 3.5.1. *For $\check{\eta}$ deep enough*[24] *in the dominant chamber we have:*

$$\mathrm{Gr}^{\check{\eta}}_{B^-,X} \underset{\mathrm{Gr}_{G,X}}{\times} \overline{\mathrm{Gr}}^{\check{\lambda}+\check{\eta}}_{B,X} = \mathrm{Gr}^{\check{\eta}}_{B^-,X} \underset{\mathrm{Gr}_{G,X}}{\times} \overline{\mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,X}}.$$

This equality also identifies:

$$\mathrm{Gr}^{\check{\eta}}_{B^-,X} \underset{\mathrm{Gr}_{G,X}}{\times} \mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{B,X} = \mathrm{Gr}^{\check{\eta}}_{B^-,X} \underset{\mathrm{Gr}_{G,X}}{\times} \mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,X}.$$

Proof. It suffices to verify the result fiberwise and therefore we fix $x = 0 \in X = \mathbb{A}^1$ (this is really just a notational convenience here). We let $\mathfrak{Z}^{\check{\lambda}}_x$ (resp. $\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}}_x$) denote the fiber of $\mathfrak{Z}^{\check{\lambda}}$ (resp. $\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}}$) at x. Let $t \in K_x$ be a coordinate at x.

Because there are only finitely many $0 \leqslant \check{\mu} \leqslant \check{\lambda}$ and because each $\overset{o}{\mathfrak{Z}}{}^{\check{\mu}}_x$ is finite type, for $\check{\eta}$ deep enough in the dominant chamber we have:

[24] The implicit bound depends on $\check{\lambda}$.

$$\overset{o}{\mathfrak{Z}}{}^{\check{\mu}}_x = \mathrm{Gr}_{N^-,x} \cap \mathrm{Ad}_{-\check{\eta}(t)}(N(O_x)) \cdot \check{\mu}(t)$$

($\check{\mu}(t)$ being regarded as a point in $\mathrm{Gr}_{G,x}$ here and the intersection symbol is short-hand for fiber product over $\mathrm{Gr}_{G,x}$) for all $0 \leqslant \check{\mu} \leqslant \check{\lambda}$. Choosing $\check{\eta}$ possibly larger, we can also assume that $\check{\eta} + \check{\mu}$ is dominant for all $0 \leqslant \check{\mu} \leqslant \check{\lambda}$. Then we claim that such a choice $\check{\eta}$ suffices for the purposes of the proposition.

Observe that for each $0 \leqslant \check{\mu} \leqslant \check{\lambda}$ we have:

$$\mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \mathrm{Gr}^{\check{\mu}+\check{\eta}}_{B,x} = \check{\eta}(t) \cdot \overset{o}{\mathfrak{Z}}{}^{\check{\mu}}_x \subseteq \mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \Big(N(O_x) \cdot (\check{\mu} + \check{\eta})(t) \Big) \subseteq \mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \mathrm{Gr}^{\check{\mu}+\check{\eta}}_{G,x}\,.$$

Recall (cf. [34]) that $\overline{\mathrm{Gr}}^{\check{\lambda}+\check{\eta}}_{B,x}$ is a union of strata:

$$\overline{\mathrm{Gr}}^{\check{\mu}+\check{\eta}}_{B,x}\,,\ \check{\mu} \leqslant \check{\lambda}$$

while for $\check{\mu}$:

$$\mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \mathrm{Gr}^{\check{\mu}+\check{\eta}}_{B,x} = \emptyset$$

unless $\check{\mu} \geqslant 0$. Therefore, $\mathrm{Gr}^{\check{\eta}}_{B^-,x}$ intersects $\overline{\mathrm{Gr}}^{\check{\mu}}_{B,x}$ only in the strata $\mathrm{Gr}^{\check{\mu}+\check{\eta}}_{B,x}$ for $0 \leqslant \check{\mu} \leqslant \check{\lambda}$.

The above analysis therefore shows that:

$$\mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \overline{\mathrm{Gr}}^{\check{\lambda}+\check{\eta}}_{B,x} \subseteq \mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \overline{\mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,x}}.$$

Now observe that $B(O_x) \cdot (\check{\lambda} + \check{\eta})(t)$ is open in $\mathrm{Gr}^{\check{\lambda}}$. Therefore, we have:

$$\overline{\mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,x}} \subseteq \overline{\mathrm{Gr}}^{\check{\lambda}+\check{\eta}}_{B}$$

giving the opposite inclusion above.

It remains to show that the equality identifies $\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}}_x$ in the desired way. We have already shown that:

$$\mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{B,x} \subseteq \mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,x},$$

so it remains to prove the opposite inclusion. Suppose that y is a geometric point of the right hand side. Then, by the Iwasawa decomposition, $y \in \mathrm{Gr}^{\check{\mu}+\check{\eta}}_{B,x}$ for some (unique) $\check{\mu} \in \check{\Lambda}$ and we wish to show that $\check{\mu} = \check{\lambda}$.

Because:

$$y \in \mathrm{Gr}^{\check{\mu}+\check{\eta}}_{B,x} \cap \overline{\mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,x}} \neq \emptyset$$

we have $\check{\mu} \leqslant \check{\lambda}$. We also have:

$$y \in \mathrm{Gr}^{\check{\mu}+\check{\eta}}_{B,x} \cap \mathrm{Gr}^{\check{\eta}}_{B^-,x} \neq \emptyset$$

which implies $\check{\mu} \geqslant 0$. Therefore, by construction of η we have:

$$y \in \mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \mathrm{Gr}^{\check{\mu}+\check{\eta}}_{B,x} \subseteq \mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \mathrm{Gr}^{\check{\mu}+\check{\eta}}_{G,x} \subseteq \mathrm{Gr}^{\check{\mu}+\check{\eta}}_{G,x}$$

but $\mathrm{Gr}^{\check{\mu}+\check{\eta}}_{G,x} \cap \mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,x} = \emptyset$ if $\check{\mu} \neq \check{\lambda}$ (because $\check{\mu}+\check{\eta}$ and $\check{\lambda}+\check{\eta}$ are assumed dominant) and therefore we must have $\check{\mu} = \check{\lambda}$ as desired. □

We continue to use the notation introduced in the proof of Proposition 3.5.1.

Recall that $\beta^{\check{\lambda}}$ (resp. $\gamma^{\check{\lambda}}$) denotes the closed embedding $\mathfrak{Z}^{\check{\lambda}} \hookrightarrow \mathcal{Z}^{\check{\lambda}}$ (resp. $\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}} \hookrightarrow \overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$). For $x \in X$, let $\beta^{\check{\lambda}}_x$ (resp. $\gamma^{\check{\lambda}}_x$) denote the closed embedding $\mathfrak{Z}^{\check{\lambda}}_x \hookrightarrow \mathcal{Z}^{\check{\lambda}}$ (resp. $\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}}_x \hookrightarrow \overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$).

Corollary 3.5.2. *For every $x \in X$, the cohomology:*

$$H^*_{dR}\Big(\mathfrak{Z}^{\check{\lambda}}_x, \beta^{\check{\lambda},!}_x(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}} \overset{!}{\otimes} \psi_{\mathcal{Z}^{\check{\lambda}}})\Big) \tag{3.5.1}$$

is concentrated in non-negative cohomological degrees. For $0 \neq \check{\lambda}$, it is concentrated in strictly positive cohomological degrees.

Remark 3.5.3. It follows a posteriori from Theorem 3.4.1 that the whole cohomology vanishes for $0 \neq \check{\lambda}$.

Proof. First, we claim that when either:

- $i < 0$, or:
- $i = 0$ and $\check{\lambda} \neq 0$

we have:

$$H^i_{dR}\Big(\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}}_x, \gamma^{\check{\lambda},!}_x(\mathrm{IC}_{\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}} \overset{!}{\otimes} \psi_{\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}})\Big) = 0. \tag{3.5.2}$$

Indeed, from the smoothness of $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$, we see that $\gamma^{\check{\lambda},!}_x(\mathrm{IC}_{\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}}) = \omega_{\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}}_x}[-\dim \overset{o}{\mathcal{Z}}{}^{\check{\lambda}}]$, so this object is concentrated in perverse cohomological degrees:

$$\geqslant -\dim(\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}}_x) + \dim(\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}) = \dim(\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}}_x).$$

The same bound applies when we twist by the local system $\psi_{\mathcal{Z}^{\check{\lambda}}}$. This gives the desired vanishing in negative degrees, as de Rham cohomology on $\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}}_x$ has amplitude $\geqslant -\dim(\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}}_x)$ with respect to the perverse t-structure.

Moreover, from Proposition 3.5.1 and the Casselman-Shalika formula ([22] Theorem 1), we deduce that, for $\check{\lambda} \neq 0$, the restriction of our rank one local system to every irreducible component of $\overset{o}{\mathfrak{Z}}{}^{\check{\lambda}}_x$ is moreover *non-constant*. This gives (3.5.2) for in the case $i = 0$ and $\check{\lambda} \neq 0$.

To complete the argument, note that by Theorem 2.17.1 (3), for $0 \leqslant \check{\mu} \leqslant \check{\lambda}$, the !-restriction of $\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}}$ to $\overset{o}{\mathfrak{Z}}{}^{\check{\mu}}_x$ lies in perverse cohomological degrees $\geqslant (\rho, \check{\mu})$, with strict inequality for $\check{\mu} \neq \check{\lambda}$.

By lisseness of $\psi_{\mathcal{Z}^{\check{\lambda}}}$, we deduce that for $0 \leqslant \check{\mu} < \check{\lambda}$, $\beta_x^{\check{\lambda},!}(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}} \overset{!}{\otimes} \psi_{\mathcal{Z}^{\check{\lambda}}})$ has !-restriction to $\overset{o}{\mathfrak{Z}}{}^{\check{\mu}}_x$ in perverse cohomological degrees strictly greater than $(\rho, \check{\mu}) = \dim(\overset{o}{\mathfrak{Z}}{}^{\check{\mu}}_x)$. Therefore, the non-positive de Rham cohomologies of these restrictions vanish.

Therefore, only the open stratum can contribute to the non-positive cohomology. We now obtain the result by (3.5.2). □

Corollary 3.5.4. *For $0 \neq \check{\lambda} \in \check{\Lambda}^{pos}$, we have:*

$$\chi\left(H^*_{dR}\left(\mathfrak{Z}^{\check{\lambda}}_x, \beta_x^{\check{\lambda},!}(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}} \overset{!}{\otimes} \psi_{\mathcal{Z}^{\check{\lambda}}})\right)\right) = 0.$$

Proof. For $\check{\eta} \in \check{\Lambda}$ sufficiently deep in the dominant chamber (in particular, satisfying the conclusion of Proposition 3.5.1), we will show:

$$[\beta_x^{\check{\lambda},!}(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}})] = [\iota^!(\mathrm{IC}_{\overline{\mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,x}}})] \in K_0(D^b_{hol}(\mathfrak{Z}^{\check{\lambda}}_x)) \tag{3.5.3}$$

in the Grothendieck group of complexes of (coherent and) holonomic D-modules on $\mathfrak{Z}^{\check{\lambda}}_x$. Here the map ι is defined as:

$$\mathfrak{Z}^{\check{\lambda}}_x \xrightarrow{\simeq} \mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \overline{\mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,x}} \to \overline{\mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,x}}.$$

It suffices to show that for each $0 \leqslant \check{\mu} \leqslant \check{\lambda}$, the !-restrictions of these classes coincide in the Grothendieck group of:

$$\mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \mathrm{Gr}^{\check{\mu}+\check{\eta}}_{G,x}.$$

Indeed, these locally closed subvarieties form a stratification.

First, note that the !-restriction of $\mathrm{IC}_{\overline{\mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,x}}}$ to $\mathrm{Gr}^{\check{\lambda}+\check{\mu}}_{G,x}$ has constant cohomologies (by $G(O_x)$-equivariance). Moreover, by [33] Theorems 11c and 6.1, the corresponding class in the Grothendieck group is the dimension of the weight component:

$$\dim\left(V^{-w_0(\check{\lambda}+\check{\eta})}(-\check{\mu}-\check{\eta})\right) \cdot [\omega_{\mathrm{Gr}^{\check{\lambda}+\check{\mu}}_{G,x}}].$$

Further !-restricting to $\mathrm{Gr}^{\check{\eta}}_{B^-,x} \cap \mathrm{Gr}^{\check{\mu}+\check{\eta}}_{G,x}$, we obtain that the right hand side of our equation is given by:

$$\dim V^{-w_0(\check{\lambda}+\check{\eta})}(-\check{\mu}-\check{\eta})\cdot[\omega_{\mathrm{Gr}^{\check{\eta}}_{B^-,x}\cap\mathrm{Gr}^{\check{\mu}+\check{\eta}}_{G,x}}].$$

By having $U(\check{\mathfrak{n}})$ act on a lowest weight vector of $V^{-w_0(\check{\lambda}+\check{\eta})}$, we observe that for $\check{\eta}$ large enough, we have[25]:

$$V^{-w_0(\check{\lambda}+\check{\eta})}(-\check{\mu}-\check{\eta})\simeq U(\check{\mathfrak{n}})(\check{\lambda}-\check{\mu})\simeq \mathrm{Sym}(\check{\mathfrak{n}})(\check{\lambda}-\check{\mu}).$$

The similar identification for the left hand side of (3.5.3) follows from the choice of $\check{\eta}$ (so that $\mathrm{Gr}^{\check{\eta}}_{B^-,x}\cap\mathrm{Gr}^{\check{\mu}+\check{\eta}}_{G,x}$ identifies with $\overset{o}{\mathfrak{Z}}{}^{\check{\mu}}_x$) and Theorem 2.17.1 (3).

Appealing to (3.5.3), we see that in order to deduce the corollary, it suffices to prove that:

$$\chi\left(H^*_{dR}\left(\mathfrak{Z}^{\check{\lambda}}_x,\iota^!(\mathrm{IC}_{\overline{\mathrm{Gr}^{\check{\lambda}+\check{\eta}}_{G,x}}})\overset{!}{\otimes}\beta^{\check{\lambda},!}_x(\psi_{\mathcal{Z}^{\check{\lambda}}})\right)\right)=0.$$

Even better: by the geometric Casselman-Shalika formula [22], this cohomology itself vanishes. □

3.6. Now we give the proof of Theorem 3.4.1.

Proof of Theorem 3.4.1. We proceed by induction on $(\rho,\check{\lambda})$, so we assume the result holds for all $0<\check{\mu}<\check{\lambda}$. By factorization and induction, we see that $\mathcal{F}:=\pi^{\check{\lambda}}_{*,dR}(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}}\overset{!}{\otimes}\psi_{\mathcal{Z}^{\check{\lambda}}})$ is concentrated on the main diagonal $X\subseteq\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}$.

The $(*=!$-)restriction of $\mathcal{F}$ to X is the $*$-pushforward along $\mathfrak{Z}^{\check{\lambda}}\to X$ of $\beta^{\check{\lambda},!}(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}}\overset{!}{\otimes}\psi_{\mathcal{Z}^{\check{\lambda}}})$. Moreover, since $\mathfrak{Z}^{\check{\lambda}}\to X$ is a Zariski-locally trivial fibration, the cohomologies of $\mathcal{F}$ on X are lisse and the fiber at $x\in X$ is:

$$H^*_{dR}\left(\mathfrak{Z}^{\check{\lambda}}_x,\beta^{\check{\lambda},!}_x(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}}\overset{!}{\otimes}\psi_{\mathcal{Z}^{\check{\lambda}}})\right).$$

Because $\pi^{\check{\lambda}}$ is affine and $\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}}\overset{!}{\otimes}\psi_{\mathcal{Z}^{\check{\lambda}}}$ is a perverse sheaf, $\mathcal{F}$ lies in perverse degrees $\leqslant 0$. Moreover, by Corollary 3.5.2, its !-fibers are concentrated in strictly positive degrees. Since $\mathcal{F}$ is lisse along X, this implies that $\mathcal{F}$ is actually perverse. Now Corollary 3.5.4 provides the vanishing of the Euler characteristics of the fibers of $\mathcal{F}$, giving the result. □

4. Identification of the Chevalley complex

4.1. The goal for this section is to identify the Chevalley complex in the cohomology of Zastava space with coefficients in the Whittaker sheaf: this is the content of Theorem 4.6.1.

[25] We include the second isomorphism for the reader's convenience, in comparing with the statement of Theorem 2.17.1. We are not concerned here with a canonical choice of such isomorphism; for our purposes, it is enough that these are two finite-dimensional vector spaces of the same dimensions.

The argument combines Theorem 3.4.1 with results from [12].

Remark 4.1.1. Theorem 4.6.1 is one of the central results of this text: as explained in the introduction, it provides a connection between Whittaker sheaves on the semi-infinite flag variety and the factorization algebra $\Upsilon_{\check{\mathfrak{n}}}$, and therefore relates to the main conjecture of the introduction.

4.2. We will use the language of *graded factorization algebras*.

The definition should encode the following: a $\mathbb{Z}^{\geqslant 0}$-graded factorization algebra is a system $\mathcal{A}_n \in D(\mathrm{Sym}^n X)$ such that we have, for every pair m, n we have isomorphisms:

$$\Big(\mathcal{A}_m \boxtimes \mathcal{A}_n\Big)|_{[\mathrm{Sym}^m X \times \mathrm{Sym}^n X]_{disj}} \xrightarrow{\simeq} \Big(\mathcal{A}_{m+n}\Big)|_{[\mathrm{Sym}^m X \times \mathrm{Sym}^n X]_{disj}}$$

satisfying (higher) associativity and commutativity. Note that the addition map $\mathrm{Sym}^m X \times \mathrm{Sym}^n X \to \mathrm{Sym}^{m+n} X$ is étale when restricted to the disjoint locus, and therefore the restriction notation above is unambiguous.

Formally, the scheme $\mathrm{Sym}\, X = \coprod_n \mathrm{Sym}^n X$ is naturally a commutative algebra under correspondences, where the multiplication is induced by the maps:

$$\begin{array}{ccccc} & & [\mathrm{Sym}^n X \times \mathrm{Sym}^m X]_{disj} & & \\ & \swarrow & & \searrow & \\ \mathrm{Sym}^n X \times \mathrm{Sym}^m X & & & & \mathrm{Sym}^{m+n} X. \end{array}$$

Therefore, as in [35] §6, we can apply the formalism of [35] §5 to obtain the desired theory.

Remark 4.2.1. We will only be working with graded factorization algebras in the heart of the t-structure, and therefore the language may be worked out by hand as in [6], i.e., without needing to appeal to [35].

Similarly, we have the notion of $\check{\Lambda}^{pos}$-graded factorization algebra: it is a collection of D-modules on the schemes $\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}$ with similar identifications as above.

4.3. Recall that [12] has introduced a certain $\check{\Lambda}^{pos}$-graded commutative factorization algebra, i.e., a commutative factorization D-module on $\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}$. This algebra incarnates the homological Chevalley complex of $\check{\mathfrak{n}}$. In [12], this algebra is denoted by $\Upsilon(\check{\mathfrak{n}}_X)$: we use the notation $\Upsilon_{\check{\mathfrak{n}}}$ instead. We denote the component of $\Upsilon_{\check{\mathfrak{n}}}$ on $\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}$ by $\Upsilon^{\check{\lambda}}_{\check{\mathfrak{n}}}$. Recall from [12] that each $\Upsilon^{\check{\lambda}}_{\check{\mathfrak{n}}}$ lies in $D(\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}})^{\heartsuit}$.[26]

[26] We explicitly note that in this section we exclusively use the usual (perverse) t-structure.

Remark 4.3.1. To remind the reader of the relation between $\Upsilon_{\check{\mathfrak{n}}}$ and the homological Chevalley complex $C_\bullet(\check{\mathfrak{n}})$ of $\check{\mathfrak{n}}$, we recall that the $*$-fiber of $\Upsilon_{\check{\mathfrak{n}}}$ at a $\check{\Lambda}^{pos}$-colored divisor $\sum_{i=1}^n \check{\lambda}_i \cdot x_i$ (here $\check{\lambda}_i \in \check{\Lambda}^{pos}$ and the $x_i \in X$ are distinct closed points) is canonically identified with:

$$\overset{n}{\underset{i=1}{\otimes}} C_\bullet(\check{\mathfrak{n}})^{\check{\lambda}_i}$$

where $C_\bullet(\check{\mathfrak{n}})^{\check{\lambda}_i}$ denotes the $\check{\lambda}_i$-graded piece of the complex.

Remark 4.3.2. The $\check{\Lambda}^{pos}$-graded vector space:

$$\check{\mathfrak{n}} = \underset{\check{\alpha} \text{ a positive coroot}}{\oplus} \check{\mathfrak{n}}^{\check{\alpha}}$$

gives rise to the D-module:

$$\check{\mathfrak{n}}_X \coloneqq \underset{\check{\alpha} \text{ a positive coroot}}{\oplus} \Delta^{\check{\alpha}}_{*,dR}(\check{\mathfrak{n}}^{\check{\alpha}} \otimes k_X) \in D(\mathrm{Div}^{\check{\Lambda}}_{\mathrm{eff}})$$

where for $\check{\lambda} \in \check{\Lambda}$, $\Delta^{\check{\lambda}} : X \to \mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}$ is the diagonal embedding. The Lie algebra structure on $\check{\mathfrak{n}}$ gives a Lie-$*$ structure on $\check{\mathfrak{n}}_X$.

Then $\Upsilon_{\check{\mathfrak{n}}}$ is tautologically given as the factorization algebra associated to the chiral enveloping algebra of this Lie-$*$ algebra.

Remark 4.3.3. We emphasize the miracle mentioned above and crucially exploited in [12] (and below): although $C_\bullet(\check{\mathfrak{n}})$ is a cocommutative (DG) coalgebra that is very much non-classical, its D-module avatar *does* lie in the heart of the t-structure. Of course, this is no contradiction, since the $*$-fibers of a perverse sheaf need only live in degrees $\leqslant 0$.

4.4. Observe that $\jmath_{*,dR}(\mathrm{IC}_{\overset{\circ}{\mathcal{Z}}})$ naturally factorizes on $\mathcal{Z}$. Therefore, $\mathfrak{s}^{*,dR}\jmath_{*,dR}(\mathrm{IC}_{\overset{\circ}{\mathcal{Z}}})$ is naturally a factorization D-module in $D(\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}})$.

The following key identification is essentially proved in [12], but we include a proof with detailed references to [12] for completeness.

Theorem 4.4.1. *There is a canonical identification:*

$$H^0(\mathfrak{s}^{*,dR}\jmath_{*,dR}(\mathrm{IC}_{\overset{\circ}{\mathcal{Z}}})) \xrightarrow{\simeq} \Upsilon_{\check{\mathfrak{n}}}$$

of $\check{\Lambda}^{pos}$-graded factorization algebras.

Remark 4.4.2. To orient the reader on cohomological shifts, we note that for $\check{\lambda} \in \check{\Lambda}^{pos}$ fixed, $\mathrm{IC}_{\overset{\circ}{\mathcal{Z}}^{\check{\lambda}}}$ is concentrated in degree 0 and therefore the above H^0 is the maximal cohomology group of the complex $\mathfrak{s}^{*,dR}\jmath_{*,dR}(\mathrm{IC}_{\overset{\circ}{\mathcal{Z}}^{\check{\lambda}}})$.

Proof of Theorem 4.4.1. Let $j : \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\mathrm{simple}} \hookrightarrow \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$ denote the open consisting of *simple* divisors, i.e., its geometric points are divisors of the form $\sum_{i=1}^{n} \check{\alpha}_i \cdot x_i$ for $\check{\alpha}_i$ a positive simple coroot and the points $\{x_i\}$ pairwise distinct. For each $\check{\lambda} \in \check{\Lambda}^{pos}$, we let $j^{\check{\lambda}} : \mathrm{Div}_{\mathrm{eff}}^{\check{\lambda},\mathrm{simple}} \to \mathrm{Div}_{\mathrm{eff}}^{\check{\lambda}}$ denote the corresponding open embedding. Note that j and each embedding $j^{\check{\lambda}}$ is affine.

Observe that $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\mathrm{simple}}$ has a factorization structure induced by that of $\mathrm{Div}_{\mathrm{eff}}$. The restriction of $\Upsilon_{\check{\mathfrak{n}}}$ to $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\mathrm{simple}}$ identifies canonically with the exterior product over $i \in \mathcal{I}_G$ of the corresponding sign (rank 1) local systems under the identification:

$$\mathrm{Div}_{\mathrm{eff}}^{\check{\lambda},\mathrm{simple}} \simeq \prod_{i \in \mathcal{I}_G} \mathrm{Sym}^{n_i,\mathrm{simple}} X$$

where $\check{\lambda} = \sum_{i \in \mathcal{I}_G} n_i \check{\alpha}_i$ and on the right the subscript *simple* means *simple effective divisor* in the same sense as above. Moreover, these identifications are compatible with the factorization structure in the natural sense.

Let $\overset{o}{\mathcal{Z}}{}^{\mathrm{simple}}$ and $\overset{o}{\mathcal{Z}}{}^{\check{\lambda},\mathrm{simple}}$ denote the corresponding opens in $\overset{o}{\mathcal{Z}}$ and $\overset{o}{\mathcal{Z}}{}^{\check{\lambda}}$ obtained by fiber product. Let $\mathfrak{s}^{\mathrm{simple}}$ and $\mathfrak{s}^{\check{\lambda},\mathrm{simple}}$ denote the corresponding restrictions of $\mathfrak{s}$ and $\mathfrak{s}^{\check{\lambda}}$.

Then $\overset{o}{\mathcal{Z}}{}^{\check{\lambda},\mathrm{simple}} \xrightarrow{\simeq} \mathrm{Div}_{\mathrm{eff}}^{\check{\lambda},\mathrm{simple}} \times \mathbb{G}_m^{(\rho,\check{\lambda})}$ as a $\mathrm{Div}_{\mathrm{eff}}^{\check{\lambda},\mathrm{simple}}$-scheme by (2.11.1), and these identifications are compatible with factorization.

Therefore, we deduce an isomorphism:

$$H^0(\mathfrak{s}^{\mathrm{simple},*,dR} \jmath_{*,dR}(\mathrm{IC}_{\overset{o}{\mathcal{Z}}{}^{\mathrm{simple}}})) \xrightarrow{\simeq} j^!(\Upsilon_{\check{\mathfrak{n}}})$$

of factorization D-modules on $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\mathrm{simple}}$ (note that the sign local system appears on the left by the Koszul rule of signs).

Therefore, we obtain a diagram:

$$\begin{array}{ccc} j_! H^0(\mathfrak{s}^{\mathrm{simple},*,dR} \jmath_*(\mathrm{IC}_{\overset{o}{\mathcal{Z}}{}^{\mathrm{simple}}})) & \xrightarrow{\simeq} & j_! j^!(\Upsilon_{\check{\mathfrak{n}}}) \\ \downarrow & & \downarrow \\ H^0(\mathfrak{s}^{*,dR} \jmath_{*,dR}(\mathrm{IC}_{\overset{o}{\mathcal{Z}}})) & & \Upsilon_{\check{\mathfrak{n}}} \end{array} \tag{4.4.1}$$

Note that the top horizontal arrow is a map of factorization algebras on $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$.

By (the Verdier duals to) [12] Lemma 4.8 and Proposition 4.9, the vertical maps in (4.4.1) are epimorphisms in the abelian category $D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}})^{\heartsuit}$. Moreover, by the analysis in [12] §4.10, there is a (necessarily unique) isomorphism:

$$H^0(\mathfrak{s}^{*,dR} \jmath_{*,dR}(\mathrm{IC}_{\overset{o}{\mathcal{Z}}})) \xrightarrow{\simeq} \Upsilon_{\check{\mathfrak{n}}}$$

completing the square (4.4.1). By uniqueness, this isomorphism is necessarily an isomorphism of factorizable D-modules. □

4.5. Observe that the D-module $\psi_{\overset{o}{\mathcal{Z}}}$ canonically factorizes on $\overset{o}{\mathcal{Z}}$. Therefore, $\jmath_{*,dR}(\psi_{\overset{o}{\mathcal{Z}}})$ factorizes in $D(\mathcal{Z})$.

By Theorem 4.4.1, we have for each $\check{\lambda} \in \check{\Lambda}^{pos}$ we have a map:

$$\jmath^{\check{\lambda}}_{*,dR}(\mathrm{IC}_{\overset{o}{\mathcal{Z}}^{\check{\lambda}}}) \to \mathfrak{s}^{\check{\lambda}}_{*,dR} H^0\Big(\mathfrak{s}^{\check{\lambda},*,dR}\jmath^{\check{\lambda}}_{*,dR}(\mathrm{IC}_{\overset{o}{\mathcal{Z}}^{\check{\lambda}}})\Big) = \mathfrak{s}_{*,dR}(\Upsilon^{\check{\lambda}}_{\check{\mathfrak{n}}}). \tag{4.5.1}$$

These maps are compatible with factorization as we vary $\check{\lambda}$.

Lemma 4.5.1. *The map* (4.5.1) *is an epimorphism in the abelian category* $D(\mathcal{Z}^{\check{\lambda}})^{\heartsuit}$.

Proof. This is a general phenomenon: for any closed embedding $i : D \to Z$ of finite type schemes and any holonomic $\mathcal{F} \in D(Z)^{\heartsuit}$, the map $\mathcal{F} \to i_{*,dR}H^0(i^{*,dR}(\mathcal{F}))$ is an epimorphism. □

4.6. Applying $\psi_{\mathcal{Z}} \overset{!}{\otimes} -$ to (4.5.1) and using the canonical identifications $\mathfrak{s}^{\check{\lambda},!,dR}(\psi_{\mathcal{Z}^{\check{\lambda}}}) \xrightarrow{\simeq} \omega_{\mathrm{Div}^{\check{\lambda}}_{\mathrm{eff}}}$, we obtain maps:

$$\eta^{\check{\lambda}} : \jmath^{\check{\lambda}}_{*,dR}(\psi_{\overset{o}{\mathcal{Z}}^{\check{\lambda}}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}^{\check{\lambda}}}) \to \mathfrak{s}_{*,dR}(\Upsilon^{\check{\lambda}}_{\check{\mathfrak{n}}}).$$

Because everything above is compatible with factorization as we vary $\check{\lambda}$, the maps $\eta^{\check{\lambda}}$ are as well.

We let $\eta : \jmath_{*,dR}(\psi_{\overset{o}{\mathcal{Z}}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}}) \to \mathfrak{s}_{*,dR}(\Upsilon_{\check{\mathfrak{n}}})$ denote the induced map of factorizable D-modules on $\mathcal{Z}$.

Theorem 4.6.1. *The map:*

$$\overset{o}{\pi}_{*,dR}(\psi_{\overset{o}{\mathcal{Z}}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}}) = \pi_{*,dR}\jmath_{*,dR}(\psi_{\overset{o}{\mathcal{Z}}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}}) \xrightarrow{\pi_{*,dR}(\eta)} \pi_{*,dR}\mathfrak{s}_{*,dR}(\Upsilon_{\check{\mathfrak{n}}}) = \Upsilon_{\check{\mathfrak{n}}} \tag{4.6.1}$$

is an equivalence of factorization D*-modules on* $\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}$.

Remark 4.6.2. In particular, the theorem asserts that $\overset{o}{\pi}_{*,dR}(\psi_{\overset{o}{\mathcal{Z}}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}})$ is concentrated in cohomological degree 0.

Proof of Theorem 4.6.1. It suffices to show for fixed $\check{\lambda} \in \check{\Lambda}^{pos}$ that $\pi^{\check{\lambda}}_{*,dR}(\eta^{\check{\lambda}})$ is an equivalence.

Recall from [12] Corollary 4.5 that we have an equality:

$$[\jmath^{\check{\lambda}}_{*,dR}(\mathrm{IC}_{\overset{o}{\mathcal{Z}}^{\check{\lambda}}})] = \sum_{\substack{\check{\mu},\check{\eta}\in\check{\Lambda}^{pos}\\ \check{\mu}+\check{\eta}=\check{\lambda}}} [\mathrm{act}^{\check{\eta},\check{\mu}}_{\mathcal{Z},*,dR}(\Upsilon^{\check{\eta}}_{\check{\mathfrak{n}}} \boxtimes \mathrm{IC}_{\mathcal{Z}^{\check{\mu}}})] \in K_0(D^b_{hol}(\mathcal{Z}^{\check{\lambda}})), \tag{4.6.2}$$

in the Grothendieck group of (coherent and) holonomic D-modules. Therefore, because $\psi_{\mathcal{Z}}$ is lisse, we obtain a similar equality:

$$[\jmath^{\check\lambda}_{*,dR}(\psi_{\overset{\circ}{\mathcal{Z}}{}^{\check\lambda}} \overset{!}{\otimes} \mathrm{IC}_{\overset{\circ}{\mathcal{Z}}{}^{\check\lambda}})] = \sum_{\substack{\check\mu,\check\eta\in\check\Lambda^{pos}\\ \check\mu+\check\eta=\check\lambda}} [\mathrm{act}^{\check\eta,\check\mu}_{\mathcal{Z},*,dR}\left(\Upsilon^{\check\eta}_{\check{\mathfrak{n}}} \boxtimes (\psi_{\mathcal{Z}^{\check\mu}} \overset{!}{\otimes} \mathrm{IC}_{\mathcal{Z}^{\check\mu}})\right)] \tag{4.6.3}$$

by the projection formula.

For every decomposition $\check\mu+\check\eta=\check\lambda$, we have:

$$\pi^{\check\lambda}_{*,dR}\,\mathrm{act}^{\check\eta,\check\mu}_{\mathcal{Z},*,dR}\left(\Upsilon^{\check\eta}_{\check{\mathfrak{n}}} \boxtimes (\psi_{\mathcal{Z}^{\check\mu}} \overset{!}{\otimes} \mathrm{IC}_{\mathcal{Z}^{\check\mu}})\right) = \mathrm{add}^{\check\eta,\check\mu}_{*,dR}\left(\Upsilon^{\check\eta}_{\check{\mathfrak{n}}} \boxtimes \pi^{\check\mu}_{*,dR}(\psi_{\mathcal{Z}^{\check\mu}} \overset{!}{\otimes} \mathrm{IC}_{\mathcal{Z}^{\check\mu}})\right).$$

By Theorem 3.4.1, this term vanishes for $\check\mu\neq 0$.

Therefore, we see that the left hand side of (4.6.1) is concentrated in degree 0, and that it agrees in the Grothendieck group with the right hand side.

Moreover, by affineness of $\pi^{\check\lambda}$, the functor $\pi^{\check\lambda}_{*,dR}$ is right exact. Therefore, by Lemma 4.5.1, the map $\pi^{\check\lambda}_{*,dR}(\eta^{\check\lambda})$ is an epimorphism in the heart of the t-structure; since the source and target agree in the Grothendieck group, we obtain that our map is an isomorphism. □

5. Hecke functors: Zastava calculation over a point

5.1. Next, we compare Theorem 4.6.1 with the geometric Satake equivalence.

More precisely, given a representation V of the dual group $\check G$, there are two ways to associate a factorization $\Upsilon_{\check{\mathfrak{n}}}$-module: one is through its Chevalley complex $C_\bullet(\check{\mathfrak{n}}, V)$, and the other is through a geometric procedure explained below, relying on geometric Satake and Theorem 4.6.1. In what follows, we refer to these two operations as the *spectral* and *geometric* Chevalley functors respectively.

The main result of this section, Theorem 5.14.1, identifies the two functors.

Notation 5.1.1. We fix a k-point $x\in X$ in what follows.

5.2. Polar Drinfeld structures

Suppose X is proper for the moment.

Recall the ind-algebraic stack $\overline{\mathrm{Bun}}^{\infty\cdot x}_{N^-}$ from [22]: it parametrizes $\mathcal{P}_G$ a G-bundle on X and non-zero maps[27]:

$$\Omega_X^{\otimes(\check\rho,\lambda)} \to V^\lambda_{\mathcal{P}_G}(\infty\cdot x)$$

defined for each dominant weight λ and satisfying the Plucker relations.

[27] Here if $(\check\rho,\lambda)$ is half integral, we appeal to our choice of $\Omega_X^{\frac{1}{2}}$.

Example 5.2.1. Let $G = SL_2$. Then $\overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x}$ classifies the datum of an SL_2-bundle $\mathcal{E}$ and a non-zero map $\Omega_X^{\frac{1}{2}} \to \mathcal{E}(\infty \cdot x)$.[28]

Example 5.2.2. For $G = \mathbb{G}_m$, $\overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x}$ is the affine Grassmannian for T at x.

5.3. Hecke action

The key feature of $\overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x}$ is that Hecke functors at x act on $D(\overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x})$. More precisely, the action of the Hecke groupoid on Bun_G lifts in the obvious way to an action on $\overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x}$.

For definiteness, we introduce the following notation. Let $\mathcal{H}_G^x$ denote the *Hecke stack* at x, parametrizing pairs of G-bundles on X identified away from x. Let h_1 and h_2 denote the two projections $\mathcal{H}_G^x \to \mathrm{Bun}_G$.

Define the *Drinfeld-Hecke* stack $\mathcal{H}_{G,\mathrm{Drin}}^x$ as the fiber product:

$$\mathcal{H}_G^x \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x}$$

where we use the map $h_1 : \mathcal{H}_G^x \to \mathrm{Bun}_G$ in order to form this fiber product. We abuse notation in using the same notation for the two projections $\mathcal{H}_{G,\mathrm{Drin}}^x \to \overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x}$.

Example 5.3.1. Let $G = SL_2$. Then $\mathcal{H}_{G,\mathrm{Drin}}^x$ parametrizes a pair of SL_2-bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ identified away from x and a non-zero map $\Omega_X^{\frac{1}{2}} \to \mathcal{E}_1(\infty \cdot x)$. The two projections h_1 and h_2 correspond to the maps to $\overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x}$ sending a datum as above to:

$$\left(\mathcal{E}_1, \Omega_X^{\frac{1}{2}} \to \mathcal{E}_1(\infty \cdot x)\right)$$
$$\left(\mathcal{E}_2, \Omega_X^{\frac{1}{2}} \to \mathcal{E}_1(\infty \cdot x) \xrightarrow{\simeq} \mathcal{E}_2(\infty \cdot x)\right)$$

respectively.

We have the usual procedure for producing D-modules on $\mathcal{H}_G^x$ from objects of $\mathsf{Sph}_{G,x} := D(\mathrm{Gr}_{G,x})^{G(O_x)}$. These give Hecke functors acting on $D(\mathrm{Bun}_G)$, considering $\mathcal{H}_G^x$ as a correspondence from Bun_G to itself. We normalize our Hecke functors so that we !-pullback along h_1 and $*$-pushforward along h_2. The same discussion applies for $D(\overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x})$.

We use $\star$ to denote the action by convolution of $\mathsf{Sph}_{G,x}$ on these categories.

[28] Here we are slightly abusing notation in letting $\mathcal{E}$ denote the rank two vector bundle underlying our SL_2-bundle.

5.4. Polar Zastava space

We define:

$$\overset{o}{\mathcal{Z}}{}^{\infty\cdot x} \subseteq \mathrm{Bun}_B \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x}$$

as the open characterized by the usual condition of generic transversality. We remark that $\overset{o}{\mathcal{Z}}{}^{\infty\cdot x}$ is an indscheme.

Note that $\overset{o}{\mathcal{Z}} \subseteq \overset{o}{\mathcal{Z}}{}^{\infty\cdot x}$ is the fiber of $\overset{o}{\mathcal{Z}}{}^{\infty\cdot x}$ along $\mathrm{Bun}_{N^-} \subseteq \overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x}$.

Remark 5.4.1. As in the case of usual Zastava, note that $\overset{o}{\mathcal{Z}}{}^{\infty\cdot x}$ is of *local* nature with respect to X: i.e., the definition makes sense for any smooth curve, and is étale local on the curve. Therefore, we typically remove our requirement that X is proper in what follows.

5.5. Let $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x}$ be the indscheme parametrizing $\check{\Lambda}$-valued divisors on X that are $\check{\Lambda}^{pos}$-valued away from x.

As for usual Zastava space, we have the map:

$$\overset{o}{\mathcal{Z}}{}^{\infty\cdot x} \xrightarrow{\overset{o}{\pi}{}^{\infty\cdot x}} \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x}.$$

Remark 5.5.1. There is a canonical map $\deg : \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x} \to \check{\Lambda}$ (considering the target as a discrete k-scheme) of taking the total degree of a divisor.

5.6. Factorization patterns

Note that $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x}$ is a *unital factorization module space* for $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$. This means that e.g. we have a correspondence:

$$\begin{array}{ccc} & \mathcal{H} & \\ \swarrow & & \searrow \\ \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}} \times \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x} & & \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x}. \end{array}$$

For this action, the left leg of the correspondence is the open embedding encoding disjointness of pairs of divisors, while the right leg is given by addition. (For the sake of clarity, let us note that the only reasonable notion of the support of a divisor in $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x}$ requires that x always lie in the support.)

Therefore, as in §4.2, we can talk about unital factorization modules in $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x}$ for a unital graded factorization algebra $\mathcal{A} \in D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}})$. We denote this category by $\mathcal{A}\text{–}\mathsf{mod}_{un,x}^{\mathrm{fact}}$.

Remark 5.6.1. The factorization action of $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$ on $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x}$ is *commutative* in the sense of [35] §7. Indeed, it comes from the obvious action of the monoid $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$ on $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x}$.

Remark 5.6.2. We emphasize that there is no Ran space appearing here: all the geometry occurs on finite-dimensional spaces of divisors.

5.7. There is a similar picture to the above for Zastava. More precisely, $\overset{o}{\mathcal{Z}}{}^{\infty\cdot x}$ is a unital factorization module space for $\overset{o}{\mathcal{Z}}$ in a way compatible with the structure maps to and from the spaces of divisors.

Therefore, for a unital factorization algebra $\mathcal{B}$ on $\overset{o}{\mathcal{Z}}$, we can form the category $\mathcal{B}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(\overset{o}{\mathcal{Z}}{}^{\infty\cdot x})$. Moreover, for $\mathcal{M} \in \mathcal{B}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(\overset{o}{\mathcal{Z}}{}^{\infty\cdot x})$, $\overset{o}{\pi}{}^{\infty\cdot x}_{*,dR}(\mathcal{M})$ is tautologically an object of $\overset{o}{\pi}_{*,dR}(\mathcal{B})\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,x}$. We denote the corresponding functor by:

$$\overset{o}{\pi}{}^{\infty\cdot x}_{*,dR} : \mathcal{B}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(\overset{o}{\mathcal{Z}}{}^{\infty\cdot x}) \to \overset{o}{\pi}_{*,dR}(\mathcal{B})\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,x}.$$

5.8. Construction of the geometric Chevalley functor

We now define a functor:

$$\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x} : \mathsf{Rep}(\check{G}) \to \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,x}$$

using the factorization pattern for Zastava space.

Remark 5.8.1. Following our conventions, $\mathsf{Rep}(\check{G})$ denotes the DG category of representations of $\check{G}$.

Remark 5.8.2. We will give a global interpretation of the induced functor to $D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x})$ in §5.12; this phrasing may be easier to understand at first pass.

5.9. First, observe that there is a natural "compactification" $\mathcal{Z}^{\infty\cdot x}$ of $\overset{o}{\mathcal{Z}}{}^{\infty\cdot x}$: for X proper, it is the appropriate open locus in:

$$\mathcal{Z}^{\infty\cdot x} \subseteq \overline{\mathrm{Bun}}{}^{\infty\cdot x}_{B} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}{}^{\infty\cdot x}_{N^-}.$$

Here $\overline{\mathrm{Bun}}{}^{\infty\cdot x}_{B}$ is defined analogously to $\overline{\mathrm{Bun}}{}^{\infty\cdot x}_{N^-}$; we remark that it has a structure map to Bun_T with fibers the variants of $\overline{\mathrm{Bun}}{}^{\infty\cdot x}_{N^-}$ for other bundles. Again, $\mathcal{Z}^{\infty\cdot x}$ is of local nature on the curve X.

The advantage of $\mathcal{Z}^{\infty\cdot x}$ is that there is a Hecke action here, so $\mathsf{Sph}_{G,x}$ acts on $D(\mathcal{Z}^{\infty\cdot x})$. Note that !-pullback from $\overline{\mathrm{Bun}}{}^{\infty\cdot x}_{N^-}$ commutes with Hecke functors.

There is again a canonical map to $\mathrm{Div}^{\check{\Lambda}^{pos}_{\mathrm{eff}},\infty\cdot x}$, and the factorization pattern of §5.7 carries over in this setting as well, that is, $\mathcal{Z}^{\infty\cdot x}$ is a unital factorization module space for $\mathcal{Z}$.

Moreover, this factorization scheme is compatible with the Hecke action: more precisely, the action by correspondences of $G(O_x)\backslash \mathrm{Gr}_{G,x}$ on $\mathcal{Z}^{\infty\cdot x}$ upgrades to an action on $\mathcal{Z}^{\infty\cdot x}$ considered as a unital factorization module space for $\mathcal{Z}$. Concretely, this implies that for any unital factorization algebra $\mathcal{B}$ on $\mathcal{Z}$, $\mathcal{B}$–$\mathsf{mod}^{\mathrm{fact}}_{un}(\mathcal{Z}^{\infty\cdot x})$ carries a canonical action of $\mathsf{Rep}(\check{G})$ by convolution, extending the action of $\mathsf{Rep}(\check{G})$ on $D(\mathcal{Z}^{\infty\cdot x})$.

5.10. Define $\mathcal{Y} \subseteq \mathcal{Z}^{\infty\cdot x}$ as the preimage of $\mathrm{Bun}_{N^-} \subseteq \overline{\mathrm{Bun}}^{\infty\cdot x}_{N^-}$ in $\mathcal{Z}^{\infty\cdot x}$. Again, $\mathcal{Y}$ is of local nature on X.

Remark 5.10.1. The notation $\overset{o}{\mathcal{Z}}{}^{\infty\cdot x}$ would be just as appropriate for $\mathcal{Y}$ as for the space we have denoted in this way: both are polar versions of $\overset{o}{\mathcal{Z}}$, but for $\overset{o}{\mathcal{Z}}{}^{\infty\cdot x}$ we allow poles for the N^--bundle, while for $\mathcal{Y}$ we allow poles for the B-bundle.

There is a canonical map $\mathcal{Y} \to \mathbb{G}_a$ which e.g. for X proper comes from the canonical map $\mathrm{Bun}_{N^-} \to \mathbb{G}_a$. We can !-pullback the exponential D-module ψ on $\mathbb{G}_a$ (normalized as always to be in perverse degree -1): we denote the resulting D-module by $\psi_{\mathcal{Y}} \in D(\mathcal{Y})$.

We then cohomologically renormalize: define $\psi^{\mathrm{IC}}_{\mathcal{Y}}$ by:

$$\psi^{\mathrm{IC}}_{\mathcal{Y}} := \psi_{\mathcal{Y}}[-(2\rho, \deg)].$$

Here we recall that we have a degree map $\mathcal{Y} \subseteq \mathcal{Z}^{\infty\cdot x} \to \check{\Lambda}$, so pairing with 2ρ, we obtain an integer valued function on $\mathcal{Y}$: we are shifting accordingly.

Remark 5.10.2. The reason for this shift is the normalization of Theorem 4.6.1: this shift is implicit there in the notation $\overset{!}{\otimes}\, \mathrm{IC}_{\overset{o}{\mathcal{Z}}}$. This is also the reason for our notation $\psi^{\mathrm{IC}}_{\mathcal{Y}}$.

5.11. Recall that $\jmath$ denotes the embedding $\overset{o}{\mathcal{Z}} \hookrightarrow \mathcal{Z}$. We let $\jmath^{\infty\cdot x}$ denote the map $\overset{o}{\mathcal{Z}}{}^{\infty\cdot x} \hookrightarrow \mathcal{Z}^{\infty\cdot x}$.

Let:

$$\mathrm{Sat}^{\heartsuit}_x : \mathsf{Rep}(\check{G})^{\heartsuit} \xrightarrow{\simeq} \mathsf{Sph}^{\heartsuit}_{G,x}$$

denote the geometric Satake equivalence. Then let:

$$\mathrm{Sat}^{naive}_x : \mathsf{Rep}(\check{G}) \to \mathsf{Sph}_{G,x}$$

denote the induced functor.

We then define $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x}$ as the following composition:

$$\mathsf{Rep}(\check{G}) \xrightarrow{\mathrm{Sat}^{naive}_x} \mathsf{Sph}_{G,x} \xrightarrow{-\star\psi^{\mathrm{IC}}_{\mathcal{Y}}}$$

$$\jmath_{*,dR}(\psi_{\overset{o}{\mathcal{Z}}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}})\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(\mathcal{Z}^{\infty\cdot x}) \xrightarrow{\jmath^{\infty\cdot x,!}} (\psi_{\overset{o}{\mathcal{Z}}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}})\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(\overset{o}{\mathcal{Z}}{}^{\infty\cdot x}) \xrightarrow{\overset{o}{\pi}{}^{\infty\cdot x}_{*,dR}} \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,x}. \tag{5.11.1}$$

Here in the last step, we have appealed to the identification:

$$\overset{o}{\pi}{}_{*,dR}(\psi_{\overset{o}{\mathcal{Z}}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}})) = \Upsilon_{\check{\mathfrak{n}}}$$

of Theorem 4.6.1. We also abuse notation in not distinguishing between $\psi^{\mathrm{IC}}_{\mathcal{Y}}$ and its $*$-pushforward to $\mathcal{Z}^{\infty\cdot x}$.

Remark 5.11.1. The above construction has many steps, so let us spell out each in a little more detail.

First, $\omega_{\mathcal{Y}}$ is canonically factorization module for $\omega_{\overset{o}{\mathcal{Z}}}$ by functoriality. The same applies for $\psi_{\mathcal{Y}}$ and $\psi_{\overset{o}{\mathcal{Z}}}$, or the shifted versions $\psi^{\mathrm{IC}}_{\mathcal{Y}}$ and $\psi_{\overset{o}{\mathcal{Z}}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}}$.

We can then $*$-extend to obtain that $\psi^{\mathrm{IC}}_{\mathcal{Y}}$ (or rather, its $*$-extension to $\mathcal{Z}^{\infty\cdot x}$) is canonically a factorization module for $\jmath_{*,dR}(\psi_{\overset{o}{\mathcal{Z}}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}})$.

As in §5.9, there is an action of $\mathsf{Rep}(\check{G})$ on $\jmath_{*,dR}(\psi_{\overset{o}{\mathcal{Z}}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}})$. So for any $V \in \mathsf{Rep}(\check{G})$, we can form

$$V \star \psi^{\mathrm{IC}}_{\mathcal{Y}} \in \jmath_{*,dR}(\psi_{\overset{o}{\mathcal{Z}}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}})\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(\mathcal{Z}^{\infty\cdot x}).$$

Next, we !-restrict to $\overset{o}{\mathcal{Z}}{}^{\infty x}$ to obtain a factorization module

$$\jmath^{\infty\cdot x,!}(V \star \psi^{\mathrm{IC}}_{\mathcal{Y}}) \in \psi_{\overset{o}{\mathcal{Z}}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}})\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(\overset{o}{\mathcal{Z}}{}^{\infty\cdot x}).$$

Finally, we apply the construction of §5.7 and Theorem 4.6.1 to observe that

$$\overset{o}{\pi}{}^{\infty\cdot x}_{*,dR}\jmath^{\infty\cdot x,!}(V \star \psi^{\mathrm{IC}}_{\mathcal{Y}})$$

is a factorization module for $\Upsilon_{\check{\mathfrak{n}}}$, completing the construction.

5.12. Global interpretation

As promised in Remark 5.8.2, we will now give a description of the functor $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x}$ in the case X is proper.

Since X is proper, we can speak about Bun_{N^-} and its relatives. Let $\mathcal{W}hit \in D(\mathrm{Bun}_{N^-})$ denote the canonical Whittaker sheaf, i.e., the !-pullback of the exponential sheaf on $\mathbb{G}_a$ (normalized as always to be in perverse degree -1). We then have the functor:

$$\mathsf{Rep}(\check{G}) \to D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x})$$

given as the composition:

$$\mathsf{Rep}(\check{G}) \xrightarrow{\mathrm{Sat}_x^{naive}} \mathsf{Sph}_{G,x} \xrightarrow{-\star \mathcal{W}hit} D(\overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x}) \to D(\overset{o}{\mathcal{Z}}{}^{\infty\cdot x}) \xrightarrow{\overset{o}{\pi}{}^{\infty\cdot x}_{*,dR}} D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x}).$$

That is, we apply geometric Satake, convolve with the $(* = !\text{-})$pushforward of $\mathcal{W}hit$ to $D(\overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x})$, !-pullback to $\overset{o}{\mathcal{Z}}{}^{\infty\cdot x}$, and then $*$-pushforward along $\overset{o}{\pi}{}^{\infty\cdot x}$.

Since !-pullback from $\overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x}$ to $\mathcal{Z}^{\infty\cdot x}$ commutes with Hecke functors, up to the cohomological shifts by degrees, this functor computes the object of $D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x})$ underlying the factorization $\Upsilon_{\check{\mathfrak{n}}}$-module coming from $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x}$.

5.13. Spectral Chevalley functor

We need some remarks on factorization modules for $\Upsilon_{\check{\mathfrak{n}}}$:

Recall from Remark 4.3.2 that $\Upsilon_{\check{\mathfrak{n}}}$ is defined as the chiral enveloping algebra of the graded Lie-$*$ algebra $\check{\mathfrak{n}}_X \in D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}})$. By Remark 5.6.1, we may speak of Lie-$*$ modules for $\check{\mathfrak{n}}_X$ on $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x}$: the definition follows [35] §7.19. Let $\check{\mathfrak{n}}_X$–mod_x denote the DG category of Lie-$*$ modules for $\check{\mathfrak{n}}_X$ supported on $\mathrm{Gr}_{T,x} \subseteq \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x}$ (this embedding is as divisors supported at x). We have a tautological equivalence:

$$\check{\mathfrak{n}}_X\text{–}\mathsf{mod}_x \simeq \check{\mathfrak{n}}\text{–}\mathsf{mod}(\mathsf{Rep}(\check{T})) \tag{5.13.1}$$

coming from identifying $\mathsf{Rep}(\check{T})$ with the DG category of $\check{\Lambda}$-graded vector spaces. Note that the right hand side of this equation is just the category of $\check{\Lambda}$-graded $\check{\mathfrak{n}}$-representations.

Moreover, by [35] §7.19, we have an induction functor $\mathrm{ind}^{ch} : \check{\mathfrak{n}}_X\text{–}\mathsf{mod}_x \to \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,x}$.

We then define $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},x} : \mathsf{Rep}(\check{G}) \to \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,x}$ as the composition:

$$\mathsf{Rep}(\check{G}) \xrightarrow{\mathrm{Oblv}} \mathsf{Rep}(\check{B}) \xrightarrow{\mathrm{Oblv}} \check{\mathfrak{n}}\text{–}\mathsf{mod}(\mathsf{Rep}(\check{T})) \overset{(5.13.1)}{\simeq} \check{\mathfrak{n}}_X\text{–}\mathsf{mod}_x \xrightarrow{\mathrm{ind}^{ch}} \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,x}. \tag{5.13.2}$$

5.14. Formulation of the main result

We can now give the main result of this section.

Theorem 5.14.1. *There exists a canonical isomorphism between the functors* $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},x}$ *and* $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x}$.

The proof will be given in §5.16 below after some preliminary remarks.

Remark 5.14.2. As stated, the result is a bit flimsy: we only claim that there is an identification of functors. The purpose of §7 is essentially to strengthen this identification so that it preserves structure encoding something about the symmetric monoidal structure of $\mathsf{Rep}(\check{G})$.

5.15. Equalizing the Hecke action

Suppose temporarily that X is a smooth proper curve. One then has the following relationship between Hecke functors acting on $\overline{\mathrm{Bun}}_B^{\infty\cdot x}$ and Hecke functors acting on $\overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x}$.

Let α (resp. β) denote the projection $\mathcal{Z}^{\infty\cdot x} \to \overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x}$ (resp. $\mathcal{Z}^{\infty\cdot x} \to \overline{\mathrm{Bun}}_B^{\infty\cdot x}$). Recall that $\alpha^!$ and $\beta^!$ commute with the actions of $\mathsf{Sph}_{G,x}$.

Let $\pi^{\infty\cdot x}$ denote the canonical map $\mathcal{Z}^{\infty\cdot x} \to \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x}$.

Lemma 5.15.1. *For $\mathcal{F} \in D(\overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x})$, $\mathcal{G} \in D(\overline{\mathrm{Bun}}_B^{\infty\cdot x})$, and $\mathcal{S} \in \mathsf{Sph}_{G,x}$, there is a canonical identification:*

$$\pi^{\infty\cdot x}_{*,dR}\Big(\alpha^!(\mathcal{S}\star\mathcal{F}) \overset{!}{\otimes} \beta^!(\mathcal{G})\Big) \simeq \pi^{\infty\cdot x}_{*,dR}\Big(\alpha^!(\mathcal{F}) \overset{!}{\otimes} \beta^!(\mathcal{S}\star\mathcal{G})\Big).$$

Proof. By base-change, each of these functors is constructed using a kernel on some correspondence between $\overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x} \times G(O_x)\backslash \mathrm{Gr}_{G,x} \times \overline{\mathrm{Bun}}_B^{\infty\cdot x}$ and $\mathrm{Div}_{\mathrm{eff}}^{\infty\cdot x}$.

In both cases, one finds that this correspondence is just the Hecke groupoid (at x) for Zastava, mapping via h_1 to $\overline{\mathrm{Bun}}_{N^-}^{\infty\cdot x}$ and via h_2 to $\overline{\mathrm{Bun}}_B^{\infty\cdot x}$, with the kernel being defined by $\mathcal{S}$. □

5.16. We now give the proof of Theorem 5.14.1.

Proof of Theorem 5.14.1. As $\mathsf{Rep}(\check{G})$ is semi-simple, we are reduced to showing this result for $V = V^{\check{\lambda}}$ an irreducible highest weight representation with highest weight $\check{\lambda} \in \check{\Lambda}^+$.

Step 1. First, we introduce some notation that we will use below.

The commutative monoid structure on $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x}$ (given by addition of divisors) induces a symmetric monoidal structure on $D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x})$, which we de note by $\star$ here. Clearly $D(\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}})$ is a symmetric monoidal subcategory.

In addition, $\mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos}}$ canonically acts on $\overline{\mathrm{Bun}}_B^{\infty\cdot x}$ and $\overset{o}{\mathcal{Z}}{}^{\infty\cdot x}$. We again denote the corresponding action on D-module categories by $\star$.

Finally, for $\check{\mu} \in \check{\Lambda}$, we let $\delta_{\check{\mu},x}$ be the skyscraper D-module at the corresponding point $\check{\mu}\cdot x \in \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x}$ (which obviously lies in $\mathrm{Gr}_{T,x} \subseteq \mathrm{Div}_{\mathrm{eff}}^{\check{\Lambda}^{pos},\infty\cdot x}$.

Step 2. Next, we describe the strategy of the argument, which is similar to the proof of Theorem 4.4.1.

Let $j : U \hookrightarrow \mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}}$ be the locally closed subscheme parametrizing divisors of the form:

$$w_0(\check{\lambda}) \cdot x + \sum \check{\alpha}_i \cdot x_i$$

where $x_i \in X$ are pairwise disjoint and distinct from x (this is the analogue of the open $\mathrm{Div}^{\check{\Lambda}^{pos},\mathrm{simple}}_{\mathrm{eff}} \subseteq \mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}$ which appeared in the proof of Theorem 4.4.1).

We have an easy commutative diagram:

$$\begin{array}{ccc} j_! j^! \,\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x}(V^{\check{\lambda}}) & \xrightarrow{\simeq} & j_! j^! \,\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},x}(V^{\check{\lambda}}) = j_! j^!(\delta_{w_0(\check{\lambda}),x} \star \Upsilon_{\check{\mathfrak{n}}}) \\ \downarrow & & \downarrow \\ \mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x}(V^{\check{\lambda}}) & & \mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},x}(V^{\check{\lambda}}). \end{array} \tag{5.16.1}$$

One easily sees that the right vertical map is an epimorphism (this is [12] Lemma 9.2).

It suffices to show that the left vertical map in (5.16.1) is an epimorphism, and that there exists a (necessarily unique) isomorphism in the bottom row of the diagram (5.16.1).

This statement is local on X, and therefore we can (and do) assume that X is proper in what follows.

Step 3. We now recall some the chiral PBW theorem in our setting. We refer to [35] §7.19 for proofs, (which closely follows [6] and [19] on these points).

For $\mathcal{F} \in D(\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}))$, we let $\mathrm{Sym}^{n,\star}(\mathcal{F}) \in D(\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}}))$ denote its nth symmetric power with respect to $\star$.

Then the chiral PBW theorem implies that $\Upsilon_{\check{\mathfrak{n}}}$ carries a canonical filtration indexed by $\mathbb{Z}^{\geqslant 0}$ with $\mathrm{gr}_n\, \Upsilon_{\check{\mathfrak{n}}} \simeq \mathrm{Sym}^{n,\star}(\check{\mathfrak{n}}_X) \in D(\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}})$.

More generally, for any $\mathcal{F} \in \check{\mathfrak{n}}_X\text{–}\mathsf{mod}_x$, $\mathsf{Ind}^{ch}(\mathcal{F})$ carries a canonical $\mathbb{Z}^{\geqslant 0}$-filtration (as an object of $D(\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}}))$ with $\mathrm{gr}_n\, \mathsf{Ind}^{ch}(\mathcal{F}) \simeq \mathrm{Sym}^{n,\star}(\check{\mathfrak{n}}_X) \star \mathcal{F}$.

In particular, forgetting factorization module structures, $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},x}(V)$ carries a canonical increasing filtration with:

$$\mathrm{gr}_n\, \mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},x}(V) \simeq \underset{\check{\mu} \in \check{\Lambda}}{\oplus} V(\check{\mu}) \otimes \delta_{\check{\mu},x} \star \mathrm{Sym}^{n,\star}(\check{\mathfrak{n}}_X).$$

Step 4. We now claim that $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x}(V^{\check{\lambda}})$ lies in the heart of the t-structure, and that $[\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x}(V^{\check{\lambda}})] = [\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},x}(V^{\check{\lambda}})]$ in the Grothendieck group.[29]

[29] Properly, we mean the following. Note that $\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}}$ has connected components indexed by $\check{\Lambda}$. We mean that for every $\check{\eta} \in \check{\Lambda}$, the restriction of these two D-modules to the corresponding connected component are (coherent and) holonomic D-modules, and the two coincide in the Grothendieck group of this component.

By Lemma 5.15.1, for every representation V of $\check{G}$ we have:

$$\begin{aligned}\pi^{\infty\cdot x}_{*,dR}\big(\alpha^!(\mathrm{Sat}_x(V)\star \mathcal{W}hit)\overset{!}{\otimes}\beta^!(\mathrm{IC}_{\mathrm{Bun}_B})\big)\simeq\\ \pi^{\infty\cdot x}_{*,dR}\big(\alpha^!(\mathcal{W}hit)\overset{!}{\otimes}\beta^!(\mathrm{Sat}_x(V)\star \mathrm{IC}_{\mathrm{Bun}_B})\big).\end{aligned} \tag{5.16.2}$$

Here $\mathrm{IC}_{\mathrm{Bun}_B}$ indicates the $*$-extension of this D-module to $\overline{\mathrm{Bun}}_B^{\infty\cdot x}$.

By definition, $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x}(V)$ is the left hand side of (5.16.2).

Now recall from [12] §8.7 that $\mathrm{Sat}_x(V)\star \mathrm{IC}_{\mathrm{Bun}_B}$ carries a canonical (finite for finite-dimensional V) filtration indexed by coweights with:

$$\mathrm{gr}_{\check{\mu}}(\mathrm{Sat}_x(V)\star \mathrm{IC}_{\mathrm{Bun}_B})\simeq V(\check{\mu})\otimes\delta_{\check{\mu},x}\star \mathrm{IC}_{\mathrm{Bun}_B}\in D(\overline{\mathrm{Bun}}_B^{\infty\cdot x}).$$

We find that $\beta^!(\mathrm{Sat}_x(V)\star \mathrm{IC}_{\mathrm{Bun}_B})$ carries a similar filtration with

$$\mathrm{gr}_{\check{\mu}}(\beta^!(\mathrm{Sat}_x(V)\star \mathrm{IC}_{\mathrm{Bun}_B}))\simeq V(\check{\mu})\otimes\delta_{\check{\mu},x}\star \mathrm{IC}_{\overset{o}{\mathcal{Z}}}$$

where, as above, $\mathrm{IC}_{\overset{o}{\mathcal{Z}}}\in D(\overset{o}{\mathcal{Z}}^{\infty\cdot x})$ is shorthand for the $*$-extension of this D-module.

By Theorem 4.6.1, we conclude that $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x}(V)$ carries a filtration with:

$$\mathrm{gr}_{\check{\mu}}(\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x}(V))\simeq V(\check{\mu})\otimes\delta_{\check{\mu},x}\star\Upsilon_{\check{\mathfrak{n}}}.$$

This immediately implies that $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x}(V)$ is concentrated in degree 0.

Finally, the identification with $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},x}(V)$ in the Grothendieck group follows from Step 3.

Step 5. We will use (a slight variant of) the following construction.[30]

Suppose that Y is a variety and $\mathcal{F}\in D(Y\times\mathbb{A}^1)^{\mathbb{G}_m}$ is $\mathbb{G}_m$-equivariant for the action of $\mathbb{G}_m$ by homotheties on the second factor, and that $\mathcal{F}$ is concentrated in negative (perverse) cohomological degrees.

For $c\in k$, let i_c denote the embedding $Y\times\{c\}\hookrightarrow Y\times\mathbb{A}^1$.

Then, for each $k\in\mathbb{Z}$, the theory of vanishing cycles furnishes specialization maps:

$$H^k(i_1^!(\mathcal{F}))\to H^k(i_0^!(\mathcal{F}))\in D(Y)^{\heartsuit} \tag{5.16.3}$$

that are functorial in $\mathcal{F}$, and which is an epimorphism for $k=0$. Indeed, these maps arise from the boundary map in the triangle[31]:

$$i_0^!(\mathcal{F})\to\Phi^{un}(\mathcal{F})\xrightarrow{\mathrm{var}}\Psi^{un}(\mathcal{F})\xrightarrow{+1}$$

[30] As Dennis Gaitsgory pointed out to us, one can argue somewhat more directly, by combining Lemma 5.15.1 with Theorem 8.11 from [12] (and the limiting case of the Casselman-Shalika formula, Theorem 3.4.1).

[31] Our nearby and vanishing cycles functors are normalized to preserve perversity.

when we use $\mathbb{G}_m$-equivariance to identify $\Psi^{un}(\mathcal{F})$ with $\mathcal{F}_1[1]$. The t-exactness of Φ^{un} and the assumption that $\mathcal{F}$ is in degrees < 0 shows that (5.16.3) is an epimorphism for $k = 0$:

$$\ldots \to H^{-1}(\Psi^{un}(\mathcal{F})) = H^0(i_1^!(\mathcal{F})) \to H^0(i_0^!(\mathcal{F}_0)) \to H^0(\Phi^{un}(\mathcal{F})) = 0$$

Step 6. We now apply the previous discussion to see that:

$$\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x}(V) \simeq \mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},x}(V)$$

as objects of $D(\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}})$ for $V \in \mathsf{Rep}(\check{G})^{\heartsuit}$ finite-dimensional.

Forget for the moment that we chose Chevalley generators $\{f_i\}$ and let W denote the vector space $(\mathfrak{n}^-/[\mathfrak{n}^-,\mathfrak{n}^-])^*$. Note that T acts on W through its adjoint action on $\mathfrak{n}^-$. Let $\overset{o}{W} \subseteq W$ denote the open subscheme corresponding to non-degenerate characters.

Then we have a canonical map:

$$\mathcal{Y} \times W \to \mathbb{G}_a$$

by imitating the construction of the map $\mathrm{can} : \mathcal{Z} \to \mathbb{G}_a$ of (2.8.2). Note that this map is T-equivariant for the diagonal action on the source and the trivial action on the target.[32]

Let $\mathcal{W} \in D(\mathcal{Z}^{\infty\cdot x} \times W)^T$ denote the result of !-pulling back of the exponential D-module on $\mathbb{G}_a$ to $\mathcal{Y} \times W$ and then $*$-extending. We then define:

$$\widetilde{\mathcal{W}} := (\overset{o}{\pi}{}^{\infty\cdot x} \times \mathrm{id}_W)_{*,dR}\,(\jmath^{\infty\cdot x} \times \mathrm{id}_W)^!\Big(\mathrm{Sat}^{naive}_x(V) \star \mathcal{W}[-(2\rho,\deg)]\Big)$$
$$\in D(\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}} \times W)^T.$$

Here the T-equivariance now refers to the T-action coming from the *trivial* action on $\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}}$. The notation for the cohomological shift is as in §5.10.

By T-equivariance, the cohomologies of our $\widetilde{\mathcal{W}}$ are constant along the open stratum $\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}} \times \overset{o}{W}$.

Moreover, note that $\widetilde{\mathcal{W}}$ is concentrated in cohomological degrees $\leqslant -\mathrm{rank}(G) = -\dim(W)$: this again follows from Lemma 5.15.1, §8.7 of [12], and ind-affineness of $\overset{o}{\pi}{}^{\infty\cdot x}$.

Therefore, !-restricting to the line through our given non-degenerate character, Step 5 gives us the specialization map:

$$H^0(\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x}(V)) \to H^0\big(\overset{o}{\pi}{}^{\infty\cdot x}_{*,dR}\jmath^{\infty\cdot x,!}(\mathrm{Sat}^{naive}_x(V) \star \omega_{\mathcal{Y}}[-(2\rho,\deg)])\big)$$
$$\in D(\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff}})^{\heartsuit}.$$

[32] We use the canonical T-action on $\mathcal{Z}$, coming from the action of T on Bun_{N^-} induced by its adjoint action on N^-.

By Step 5, this specialization map is an epimorphism.

However, the Zastava space version of Theorem 8.8 from [12] (which is implicit in [12] and easy to deduce from there) implies that the right hand term coincides with $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},x}(V)$, and therefore this map is an isomorphism by our Grothendieck group calculation.

Step 7. Finally, we observe that the isomorphism $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x}(V) \simeq \mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},x}(V)$ constructed above fits into the diagram (5.16.1). Indeed, this follows from functoriality of the specialization and the proof of Theorem 8.8 from [12], namely, that the construction of [12] is characterized by Lemma 9.2 from [12].

Therefore, we do actually obtain an isomorphism of factorization modules, giving the desired result. □

6. Around factorizable Satake

6.1. Our goal in §7 is to prove a generalization of Theorem 5.14.1 in which we treat several points $\{x_1, \ldots, x_n\} \subseteq X$, allowing these points to move and collide (in the sense of the Ran space formalism). This section plays a supplementary and technical role for this purpose.

6.2. Generalizing the geometric side of Theorem 5.14.1 is an old idea: one should use the Beilinson-Drinfeld affine Grassmannian Gr_{G,X^I} and the corresponding factorizable version of the Satake category.

Therefore, we need a geometric Satake theorem over powers of the curve. This has been treated in [23], but the treatment of [23] is inconvenient for us, relying too much on specific aspects of perverse sheaves that do not generalize to non-holonomic D-modules.

6.3. The goal for this section is to give a treatment of factorizable geometric Satake for D-modules.

However, most of the work here actually goes into treating formal properties of the spectral side of this equivalence. Here we have DG categories $\mathsf{Rep}(\check{G})_{X^I}$ which provide factorizable versions of the category $\mathsf{Rep}(\check{G})$ appearing in the Satake theory.

These categories arise from a general construction, taking $\mathcal{C}$ a symmetric monoidal object of DGCat_{cont} (so we assume the tensor product commutes with colimits in each variable), and producing $\mathcal{C}_{X^I} \in D(X^I)\text{–}\mathsf{mod}$. As we will see, this construction is especially well-behaved for $\mathcal{C}$ rigid monoidal (as for $\mathcal{C} = \mathsf{Rep}(\check{G})$).

6.4. Structure of this section

We treat the construction and general properties of the categories $\mathcal{C}_{X^I}$ in §6.5-6.18, especially treating the case where $\mathcal{C}$ is rigid. We specialize to the case where $\mathcal{C}$ is representations of an affine algebraic group in §6.19.

We then discuss the (naive) factorizable Satake theorem from §6.28 until the end of this section.

6.5. Let $\mathcal{C} \in \mathsf{ComAlg}(\mathsf{DGCat}_{cont})$ be a symmetric monoidal DG category. We denote the monoidal operation in $\mathcal{C}$ by $\otimes$.

6.6. Factorization

Recall from [35] §7 that we have an operation attaching to each finite set I a $D(X^I)$-module category $\mathcal{C}_{X^I}$.[33]

We will give an essentially self-contained treatment of this construction below, but first give examples to give the reader a feeling for the construction.

Example 6.6.1. For $I = *$, we have $\mathcal{C}_X = \mathcal{C} \otimes D(X)$.

Example 6.6.2. Let $I = \{1, 2\}$. Let j denote the open embedding $U = X \times X \backslash X \hookrightarrow X \times X$.

Then we have a fiber square:

$$\begin{array}{ccc} \mathcal{C}_{X^2} & \longrightarrow & \mathcal{C} \otimes D(X^2) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{C}} \otimes j^!} \\ \big(\mathcal{C} \otimes \mathcal{C}\big) \otimes D(U) & \xrightarrow{(-\otimes-)\otimes \mathrm{id}_{D(X^2)}} & \mathcal{C} \otimes D(U). \end{array}$$

We emphasize that $(- \otimes -)$ indicates the tensor product morphism $\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$.

Example 6.6.3. If Γ is an affine algebraic group and we take $\mathcal{C} = \mathsf{Rep}(\Gamma)$, then the above says that $\mathsf{Rep}(\Gamma)_{X^2}$ parametrizes a representation of Γ over X^2_{dR} with the structure of a $\Gamma \times \Gamma$-representation on the complement to the diagonal, compatible under the diagonal embedding $\Gamma \hookrightarrow \Gamma \times \Gamma$.

6.7. For the general construction of $\mathcal{C}_{X^I}$, we need the following combinatorics.

First, for any surjection $p : I \twoheadrightarrow J$ of finite sets, let $U(p)$ denote the open subscheme of points $(x_i)_{i \in I}$ with $x_i \neq x_{i'}$ whenever $p(i) \neq p(i')$.

Example 6.7.1. For $p : I \to *$, we have $U(p) = X^I$. For $p : I \xrightarrow{\mathrm{id}} I$, $U(p)$ is the locus X^I_{disj} of pairwise disjoint points in X^I.

We let $\mathcal{S}_I$ denote the $(1,1)$-category indexing data $I \xrightarrow{p} J \xrightarrow{q} K$, where we allow morphisms of diagrams that are contravariant in J and covariant in K, and surjective termwise.

[33] In [35], we use the notation $\Gamma(X^I_{dR}, \mathsf{Loc}_{X^I_{dR}}(\mathcal{C}))$ in place of $\mathcal{C}_{X^I}$.

6.8. For every $\Sigma = (I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K)$ in $\mathcal{S}_I$, define $\mathcal{C}_\Sigma \in D(X^I)$–$\mathsf{mod}$ as:

$$\mathcal{C}_\Sigma = D(U(p)) \otimes \mathcal{C}^{\otimes K}.$$

For $\Sigma_1 \to \Sigma_2 \in \mathcal{S}_I$, we have a canonical map $\mathcal{C}_{\Sigma_1} \to \mathcal{C}_{\Sigma_2} \in D(X^I)$–$\mathsf{mod}$ constructed as follows. If the morphism $\Sigma_1 \to \Sigma_2$ is induced by the diagram:

$$\begin{array}{ccccc} I & \xrightarrow{p_1} & J_1 & \xrightarrow{q_1} & K_1 \\ \| & & \uparrow & & \downarrow \alpha \\ I & \xrightarrow{p_2} & J_2 & \xrightarrow{q_2} & K_2 \end{array}$$

then our functor is given as the tensor product of:

$$\mathcal{C}^{\otimes K_1} \to \mathcal{C}^{\otimes K_2}$$

$$\boxtimes_{k \in K_1} \mathcal{F}_k \mapsto \boxtimes_{k' \in K_2} \left(\otimes_{k'' \in \alpha^{-1}(k')} \mathcal{F}_{k''} \right)$$

and the D-module restriction along the map $U(p_2) \to U(p_1)$.

It is easy to upgrade this description to the homotopical level to define a functor:

$$\mathcal{S}_I \to D(X^I)\text{–}\mathsf{mod}.$$

We define $\mathcal{C}_{X^I}$ as the limit of this functor.

Example 6.8.1. It is immediate to see that this description recovers our earlier formulae for $I = *$ and $I = \{1, 2\}$.

Remark 6.8.2. This construction unwinds to say the following: we have an object $\mathcal{F} \in \mathcal{C} \otimes D(X^I)$ such that for every $p : I \twoheadrightarrow J$, its restriction to $\mathcal{C} \otimes D(U(p))$ has been lifted to an object of $\mathcal{C}^{\otimes J} \otimes D(U(p))$.

Example 6.8.3. For $\mathcal{C} = \mathsf{Rep}(\Gamma)$ with Γ an affine algebraic group, this construction is a derived version of the construction of [23] §2.5.

Remark 6.8.4. Obviously each $\mathcal{C}_{X^I}$ is a commutative algebra in $D(X^I)$–mod. Indeed, each term $\mathcal{C}_\Sigma = D(U(p)) \otimes \mathcal{C}^{\otimes K}$ is so, and the structure functors are canonically symmetric monoidal. We have an obvious symmetric monoidal functor:

$$\mathrm{Loc} = \mathrm{Loc}_{X^I} : \mathcal{C}^{\otimes I} \to \mathcal{C}_{X^I}$$

for each I, with these functors being compatible under diagonal maps.

6.9. Factorization

It follows from [35] §7 that the assignment $I \mapsto \mathcal{C}_{X^I}$ defines a commutative unital chiral category on X_{dR}. For the sake of completeness, the salient pieces of structure here are twofold:

(1) For every pair of finite sets I_1 and I_2, we have a symmetric monoidal map:

$$\mathcal{C}_{X^{I_1}} \otimes \mathcal{C}_{X^{I_2}} \to \mathcal{C}_{X^{I_1 \amalg I_2}}$$

of $D(X^{I_1 \amalg I_2})$-module categories that is an equivalence after tensoring with $D([X^{I_1} \times X^{I_2}]_{disj})$.

(2) For every $I_1 \twoheadrightarrow I_2$, an identification:

$$\mathcal{C}_{X^{I_1}} \underset{D(X^{I_1})}{\otimes} D(X^{I_2}) \simeq \mathcal{C}_{X^{I_2}}.$$

These should satisfy the obvious compatibilities, which we do not spell out here because in the homotopical setting they are a bit difficult to say: we refer to [35] §7 for a precise formulation.

We will construct these maps in §6.10 and 6.11.

6.10. First, suppose $I = I_1 \coprod I_2$.

Define a functor $\mathcal{S}_I \to \mathcal{S}_{I_1}$ as follows. We send $I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K$ to $I_1 \twoheadrightarrow \mathrm{Image}(p|_{I_1}) \twoheadrightarrow \mathrm{Image}(q \circ p|_{I_1})$. It is easy to see that this actually defines a functor. We have a similar functor $\mathcal{S}_I \to \mathcal{S}_{I_2}$, so we obtain $\mathcal{S}_I \to \mathcal{S}_{I_1} \times \mathcal{S}_{I_2}$.

Given $I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K$ as above, let e.g. $I_1 \overset{p_1}{\twoheadrightarrow} J_1 \overset{q_1}{\twoheadrightarrow} K_1$ denote the corresponding object of $\mathcal{S}_{I_1}$.

We have a canonical map:

$$U(p) \hookrightarrow U(p_1) \times U(q_1) \subseteq X^{I_1} \times X^{I_2} = X^I.$$

We also have a canonical map $\mathcal{C}^{\otimes K_1} \otimes \mathcal{C}^{\otimes K_2} \to \mathcal{C}^{\otimes K}$ induced by tensor product and the obvious map $K_1 \coprod K_2 \to K$. Together, we obtain maps:

$$(D(U(p_1)) \otimes \mathcal{C}^{\otimes K_1}) \otimes (D(U(p_2)) \otimes \mathcal{C}^{\otimes K_2}) \to D(U(p)) \otimes \mathcal{C}^K$$

that in passage to the limit define

$$\mathcal{C}_{X^{I_1}} \otimes \mathcal{C}_{X^{I_2}} \to \mathcal{C}_{X^I}.$$

That this map is an equivalence over the disjoint locus follows from a cofinality argument.

6.11. Next, suppose for $f : I_1 \twoheadrightarrow I_2$ is given. We obtain $\mathcal{S}_{I_2} \to \mathcal{S}_{I_1}$ by restriction. Moreover, for any given $I_2 \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K \in \mathcal{S}_{I_2}$, we have the functorial identifications:

$$D(U(p)) \otimes \mathcal{C}^{\otimes K} \simeq (D(U(p \circ f)) \underset{D(X^{I_1})}{\otimes} D(X^{I_2})) \otimes \mathcal{C}^{\otimes K})$$

that give a map:

$$\mathcal{C}_{X^{I_1}} \underset{D(X^{I_1})}{\otimes} D(X^{I_2}) \to \mathcal{C}_{X^{I_2}}.$$

An easy cofinality argument shows that this map is an equivalence.

6.12. A variant

We now discuss a variant of the preceding material a categorical level down.

6.13. First, if A is a commutative algebra in Vect, then there is an assignment $I \mapsto A_{X^I} \in D(X^I)$ defining a commutative factorization algebra. Indeed, it is given by the same procedure as before—we have:

$$A_{X^I} \coloneqq \lim_{(I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K) \in \mathcal{S}_I} j_{p,*,dR}(A^{\otimes K} \otimes \omega_{U(p)}) \in D(X^I). \tag{6.13.1}$$

The structure maps are as before.

6.14. More generally, when $\mathcal{C}$ is as before and $A \in \mathcal{C}$ is a commutative algebra, we can attach a (commutative) factorization algebra $I \mapsto A_{X^I} \in \mathcal{C}_{X^I}$.

We will need this construction in this generality below. However, the above formula does not make sense, since there is no way to make sense of $j_{p,*,dR}(\omega_{U(p)}) \otimes A^{\otimes K}$ as an object of $\mathcal{C}_{X^I}$. So we need the following additional remarks:

We do have A_{X^I} defined as an object of $D(X^I) \otimes \mathcal{C}$ by the above formula. Moreover, as in §6.10, for every $p : I \twoheadrightarrow J$ we have canonical multiplication maps:

$$\underset{j \in J}{\boxtimes} A_{X^{I_j}} \to A_{X^I} \in D(X^I) \otimes \mathcal{C}$$

where I_j is the fiber of I at $j \in J$, and where our exterior product should be understood as a mix of the tensor product for $\mathcal{C}$ and the exterior product of D-modules. This map is an equivalence over $U(p)$.

This says that for every p as above, the restriction of A_{X^I} to $U(p)$ has a canonical structure as an object of $D(U(p)) \otimes \mathcal{C}^{\otimes J}$, lifting its structure of an object of $D(U(p)) \otimes \mathcal{C}$. Moreover, this is compatible with further restrictions in the natural sense. This is exactly the data needed to upgrade A_{X^I} to an object of $\mathcal{C}_{X^I}$ (which we denote by the same name).

6.15. ULA objects

For the remainder of the section, assume that $\mathcal{C}$ is compactly generated and *rigid*: recall that rigidity means that this means that the unit $\mathbb{1}_{\mathcal{C}}$ is compact and every $V \in \mathcal{C}$ compact admits a dual.

Under this rigidity assumption, we discuss ULA aspects of the categories $\mathcal{C}_{X^I}$: we refer the reader to Appendix B for the terminology here, which we assume for the remainder of this section.

6.16. Recall that $\mathsf{QCoh}(X^I, \mathcal{C}_{X^I})$ denotes the object of $\mathsf{QCoh}(X^I)$–mod obtained from $\mathcal{C}_{X^I} \in D(X^I)$–$\mathsf{mod}$ by induction along the (symmetric monoidal) forgetful functor $D(X^I) \to \mathsf{QCoh}(X^I)$.

Proposition 6.16.1. *For $\mathcal{F} \in \mathcal{C}^{\otimes I}$ compact, $\mathrm{Loc}_{X^I}(\mathcal{F}) \in \mathcal{C}_{X^I}$ is ULA.*

We will deduce this from the following lemma.

Let $\mathbb{1}_{\mathcal{C}_{X^I}} = \mathrm{Loc}_{X^I}(\mathbb{1}_{\mathcal{C}})$ denote the unit for the ($D(X^I)$-linear) symmetric monoidal structure on $\mathcal{C}_{X^I}$.

Lemma 6.16.2. *$\mathbb{1}_{\mathcal{C}_{X^I}}$ is ULA.*

Proof. By 1-affineness (see [25]) of X_{dR} and X, the induction functor:

$$D(X)\text{–}\mathsf{mod} \to \mathsf{QCoh}(X)\text{–}\mathsf{mod}$$

commutes with limits.[34]

It follows that $\mathsf{QCoh}(X^I, \mathcal{C}_{X^I})$ is computed by a similar limit as defines $\mathcal{C}_{X^I}$, but with $\mathsf{QCoh}(U(p))$ replacing $D(U(p))$ everywhere.

Since this limit is finite and since each of the terms corresponding to $\mathrm{Oblv}(\mathbb{1}_{\mathcal{C}_{X^I}}) \in \mathsf{QCoh}(X^I, \mathcal{C}_{X^I})$ is compact, we obtain the claim. □

Proof of Proposition 6.16.1. Since the functor $\mathcal{C}^{\otimes I} \to \mathcal{C}_{X^I}$ is symmetric monoidal and since each compact object in $\mathcal{C}^{\otimes I}$ admits a dual by assumption, we immediately obtain the result from Lemma 6.16.2. □

Remark 6.16.3. Proposition 6.16.1 fails for more general $\mathcal{C}$: the tensor product $\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ typically fails to preserve compact objects, which implies that Loc_{X^2} does not preserve compact objects.

[34] To fill in the details: 1-affineness means that $\mathsf{QCoh}(X)$–mod (resp. $D(X)$–mod) canonically identifies with $\mathsf{ShvCat}(X)$ (resp. $\mathsf{ShvCat}(X_{dR})$), where we refer to [25] for the definition of ShvCat. Our induction functor then corresponds to pullback for sheaves of categories. By [25] §1.1.5, pullback for sheaves of categories always commutes with limits.

6.17. We now deduce the following result about the categories $\mathcal{C}_{X^I}$ (for the terminology, see Definition B.6.1).

Theorem 6.17.1. *$\mathcal{C}_{X^I}$ is ULA over X^I.*

We will use the following lemma, which is implicit but not quite stated in [25].

Lemma 6.17.2. *Let S be a (possibly DG) scheme (almost) of finite type, and let $i : T \hookrightarrow S$ be a closed subscheme with complement $j : U \hookrightarrow S$. For $\mathcal{D} \in \mathsf{QCoh}(S)\text{–}\mathsf{mod}$, the composite functor:*

$$\mathrm{Ker}(j^* : \mathcal{D} \to \mathcal{D}_U) \hookrightarrow \mathcal{D} \to \mathcal{D} \underset{\mathsf{QCoh}(S)}{\otimes} \mathsf{QCoh}(S^\wedge_T) \tag{6.17.1}$$

is an equivalence, where $S^\wedge_T$ is the formal completion of S along T.

Proof. By [25] Proposition 4.1.5, the restriction functor:

$$\mathsf{QCoh}(S^\wedge_T)\text{–}\mathsf{mod} \to \mathsf{QCoh}(S)\text{–}\mathsf{mod}$$

is fully-faithful with essential image being those module categories on which objects of $\mathsf{QCoh}(U) \subseteq \mathsf{QCoh}(S)$ act by zero. But the endofunctor $\mathrm{Ker}(j^*)$ of $\mathsf{QCoh}(S)\text{–}\mathsf{mod}$ is a localization functor for the same subcategory, giving the claim. □

Proof of Theorem 6.17.1. Suppose $\mathcal{G} \in \mathsf{QCoh}(X^I, \mathcal{C}_{X^I})$ is some object with:

$$\mathrm{Hom}_{\mathsf{QCoh}(X^I, \mathcal{C}_{X^I})}(\mathcal{P} \otimes \mathrm{Oblv}\, \mathrm{Loc}_{X^I}(\mathcal{F}), \mathcal{G}) = 0$$

for all $\mathcal{P} \in \mathsf{QCoh}(X^I)$ perfect and all $\mathcal{F} \in \mathcal{C}^{\otimes I}$ compact. Then by Proposition 6.16.1, it suffices to show that $\mathcal{G} = 0$.

Fix $p : I \twoheadrightarrow J$. We will show by decreasing induction on $|J|$ that the restriction of $\mathcal{G}$ to $U(p)$ is zero.

We have the closed embedding $X^J_{disj} \hookrightarrow U(p)$ with complement being the union:

$$U(p) \setminus (X^J_{disj}) = \bigcup_{I \overset{q}{\twoheadrightarrow} J' \overset{q'}{\underset{\neq}{\twoheadrightarrow}} J, q'q = p} U(q).$$

In particular, the inductive hypothesis implies that the restriction of $\mathcal{G}$ to this complement is zero.

Let $\mathcal{X}$ denote the formal completion of X^J_{disj} in $U(p)$ and let $i_p : \mathcal{X} \hookrightarrow U(p)$ denote the embedding. By Lemma 6.17.2, it suffices to show that:

$$i_p^*(\mathcal{G}) = 0 \in \mathsf{QCoh}(\mathcal{X}, \mathcal{C}_{X^I}) \coloneqq \mathsf{QCoh}(X^I, \mathcal{C}_{X^I}) \underset{\mathsf{QCoh}(X^I)}{\otimes} \mathsf{QCoh}(\mathcal{X}).$$

The map $\mathcal{X} \to X^I_{dR}$ factors through $X^J_{disj,dR}$ (embedded via p), so by factorization we have:

$$\mathsf{QCoh}(X^I, \mathcal{C}_{X^I}) \underset{\mathsf{QCoh}(X^I)}{\otimes} \mathsf{QCoh}(\mathcal{X}) = \mathcal{C}_{X^I} \underset{D(X^I)}{\otimes} \mathsf{QCoh}(\mathcal{X}) \simeq \mathcal{C}^{\otimes J} \otimes \mathsf{QCoh}(\mathcal{X}).$$

This identification is compatible with the functors Loc in the following way. Let $\overset{p}{\otimes} : \mathcal{C}^{\otimes I} \to \mathcal{C}^{\otimes J}$ denote the map induced by the tensor structure on $\mathcal{C}$. We then have a commutative diagram:

$$\begin{array}{ccccc} \mathcal{C}^{\otimes I} & \xrightarrow{\mathrm{Loc}_{X^I}} & \mathcal{C}_{X^I} & \longrightarrow & \mathsf{QCoh}(X^I, \mathcal{C}_{X^I}) \\ \downarrow{\scriptstyle \overset{p}{\otimes}} & & & & \downarrow{\scriptstyle i_p^*} \\ \mathcal{C}^{\otimes J} & & \xrightarrow{\mathrm{id} \otimes \mathcal{O}_{\mathcal{X}}} & & \mathcal{C}^{\otimes J} \otimes \mathsf{QCoh}(\mathcal{X}), \end{array}$$

by construction.

Since $\mathsf{QCoh}(\mathcal{X})$ is compactly generated by objects of the form $i_p^*(\mathcal{P})$ with $\mathcal{P} \in \mathsf{QCoh}(U(p))$ perfect (and with set-theoretic support in X^J_{disj}), we reduce to the following:

Each $\mathcal{F} \in \mathcal{C}^{\otimes J}$ compact then defines a continuous functor $F_{\mathcal{F}} : \mathcal{C}^{\otimes J} \otimes \mathsf{QCoh}(\mathcal{X}) \to \mathsf{QCoh}(\mathcal{X})$, and our claim amounts to showing that an object in $\mathcal{C}^{\otimes J} \otimes \mathsf{QCoh}(\mathcal{X})$ is zero if and only if each functor $F_{\mathcal{F}}$ annihilates it, but this is obvious e.g. from the theory of dualizable categories. □

6.18. Dualizability

Next, we record the following technical result.

Lemma 6.18.1. *For every $\mathcal{D} \in D(X^I)$–mod, the canonical map:*

$$\begin{aligned} \mathcal{C}_{X^I} \underset{D(X^I)}{\otimes} \mathcal{D} = & \lim_{(I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K) \in \mathcal{S}_I} \left(\mathcal{C}^{\otimes K} \otimes D(U(p)) \right) \underset{D(X^I)}{\otimes} \mathcal{D} \\ & \to \lim_{(I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K) \in \mathcal{S}_I} \left(\mathcal{C}^{\otimes K} \otimes D(U(p)) \underset{D(X^I)}{\otimes} \mathcal{D} \right) \end{aligned}$$

is an equivalence.

This proof is digressive, so we postpone the proof to Appendix A, assuming it for the remainder of this section.

We obtain the following consequence.[35]

[35] We remark that this result is strictly weaker than the above, and more direct to prove.

Corollary 6.18.2. *$\mathcal{C}_{X^I}$ is dualizable and self-dual as a $D(X^I)$-module category.*

Remark 6.18.3. In fact, one can avoid the full strength of Lemma 6.18.1 for our purposes: we include it because it gives an aesthetically nicer treatment, and because it appears to be an important technical result that should be included for the sake of completeness.

With that said, we apply it below only for $\mathcal{D} = \mathsf{Sph}_{G,X^I}$, and here it is easier: it follows from the dualizability of Sph_{G,X^I} as a $D(X^I)$-module category, which is much more straightforward.

6.19. Let Γ be an affine algebraic group. We now specialize the above to the case $\mathcal{C} = \mathsf{Rep}(\Gamma)$.

6.20. Induction

Our main tool in treating $\mathsf{Rep}(\Gamma)_{X^I}$ is the good behavior of the *induction* functor $\mathrm{Av}^w_{X^I,*} : D(X^I) \to \mathsf{Rep}(\Gamma)_{X^I}$ introduced below.

6.21. The symmetric monoidal forgetful functor $\mathrm{Oblv} : \mathsf{Rep}(\Gamma) \to \mathsf{Vect}$ induces a conservative functor $\mathrm{Oblv}_{X^I} : \mathsf{Rep}(\Gamma)_{X^I} \to D(X^I)$ compatible with $D(X^I)$-linear symmetric monoidal structures.

We abuse notation in also letting Oblv_{X^I} denote the $\mathsf{QCoh}(X^I)$-linear functor:

$$\mathrm{Oblv}_{X^I} : \mathsf{QCoh}(X^I, \mathsf{Rep}(\Gamma)_{X^I}) \to \mathsf{QCoh}(X^I)$$

promising the reader to always take caution to make clear which functor we mean in the sequel.

6.22. Applying the discussion of §6.14, we obtain $\mathcal{O}_{\Gamma,X^I} \in \mathsf{Rep}(\Gamma)_{X^I}$ factorizable corresponding to the regular representation $\mathcal{O}_\Gamma \in \mathsf{Rep}(\Gamma)$ of Γ (so we are not distinguishing between the sheaf $\mathcal{O}_\Gamma$ and its global sections in this notation).[36]

Proposition 6.22.1.

(1) The functor $\mathrm{Oblv}_{X^I} : \mathsf{Rep}(\Gamma)_{X^I} \to D(X^I)$ admits a $D(X^I)$-linear right adjoint[37] *$\mathrm{Av}^w_{X^I,*} : D(X^I) \to \mathsf{Rep}(\Gamma)_{X^I}$ compatible with factorization.*[38]

(2) The functor $\mathrm{Av}^w_{X^I,*}$ maps ω_{X^I} to the factorization algebra $\mathcal{O}_{\Gamma,X^I}$ introduced above.

[36] The D-module $\mathrm{Oblv}_{X^I}(\mathcal{O}_{\Gamma,X^I}) \in D(X^I)$ (or its shift cohomologically up by $|I|$, depending on one's conventions) appears in [6] as factorization algebra associated with the constant D_X-scheme $\Gamma \times X^I \to X^I$.

[37] The superscript w stands for *weak*, and is included for compatibility with [20] §20.

[38] More generally, the proof below shows that the analogous statement holds more generally for any symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D} \in \mathsf{DGCat}_{cont}$ with $\mathcal{C}$ rigid, where this is generalizing the forgetful functor $\mathrm{Oblv} : \mathsf{Rep}(\Gamma) \to \mathsf{Vect}$.

Proof. By Proposition B.7.1 and Theorem 6.17.1, it suffices to show that Oblv_{X^I} maps the ULA generators $\mathrm{Loc}_{X^I}(V)$ of $\mathsf{Rep}(\Gamma)_{X^I}$ to ULA objects of $D(X^I)$, which is obvious.

For the second part, note that the counit map $\mathcal{O}_\Gamma \to k \in \mathsf{ComAlg}(\mathsf{Vect})$ induces a map $\mathrm{Oblv}_{X^I}\mathcal{O}_{\Gamma,X^I} \to \omega_{X^I} \in D(X^I)$ factorizably, and therefore induces factorizable maps:

$$\mathcal{O}_{\Gamma,X^I} \to \mathrm{Av}^w_{X^I,*}(\omega_{X^I}).$$

By factorization, it is enough to show that this map is an equivalence for $I = *$, where it is clear. □

6.23. Coalgebras

We now realize the categories $\mathsf{Rep}(\Gamma)_{X^I}$ in more explicit terms.

Lemma 6.23.1. *The functor* Oblv_{X^I} *is comonadic, i.e., satisfies the conditions of the comonadic Barr-Beck theorem.*

In fact, we will prove the following strengthening:

Lemma 6.23.2. *For any* $\mathcal{D} \in D(X^I)$–mod*, the forgetful functor:*

$$\mathrm{Oblv}_{X^I} \otimes \mathrm{id}_{\mathcal{D}} : \mathsf{Rep}(\Gamma)_{X^I} \underset{D(X^I)}{\otimes} \mathcal{D} \to \mathcal{D}$$

is comonadic.

Proof. Using Lemma 6.18.1, we deduce that $\mathrm{Oblv}_{X^I} \otimes \mathrm{id}_{\mathcal{D}}$ arises by passage to the limit over $\mathcal{S}_I$ from the compatible system of functors:

$$\mathsf{Rep}(\Gamma)^{\otimes K} \otimes D(U(p)) \underset{D(X^I)}{\otimes} \mathcal{D} \to D(U(p)) \underset{D(X^I)}{\otimes} \mathcal{D}.$$

Therefore, it suffices to show that each of these functors is conservative and commutes with Oblv-split totalizations.

But by [25] Theorem 2.2.2 and Lemma 5.5.4, the functor $\mathsf{Rep}(\Gamma^n) \otimes \mathcal{E} \to \mathcal{E}$ is comonadic for any $\mathcal{E} \in \mathsf{DGCat}_{cont}$. This obviously gives the claim. □

6.24. t-structures

It turns out that the categories $\mathsf{Rep}(\Gamma)_{X^I}$ admit particularly favorable t-structures.

Proposition 6.24.1. *There is a unique t-structure on* $\mathsf{Rep}(\Gamma)_{X^I}$ *(resp.* $\mathsf{QCoh}(X^I, \mathsf{Rep}(\Gamma)_{X^I})$*) such that* Oblv_{X^I} *is t-exact. This t-structure is left and right complete.*

Proof. We first treat the quasi-coherent case.

For every $(I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K) \in \mathcal{S}_I$, the category:

$$\mathsf{Rep}(\Gamma)^{\otimes J} \otimes \mathsf{QCoh}(U(p)) = \mathsf{QCoh}(\mathbb{B}\Gamma^J \times U(p))$$

admits a canonical t-structure, since it is quasi-coherent sheaves on an algebraic stack. This t-structure is left and right complete, and the forgetful functor to $\mathsf{QCoh}(U(p))$ is obviously t-exact. Moreover, the structure functors corresponding to maps in $\mathcal{S}_I$ are t-exact, and therefore we obtain a t-structure with the desired properties on the limit, which is $\mathsf{QCoh}(X^I, \mathsf{Rep}(\Gamma)_{X^I})$.

We now deduce the D-module version. We have the adjoint functors[39]:

$$\mathsf{QCoh}(X^I, \mathsf{Rep}(\Gamma)_{X^I}) \underset{\text{Oblv}}{\overset{\text{ind}}{\rightleftarrows}} \mathsf{Rep}(\Gamma)_{X^I}.$$

Since the monad Oblv ind is t-exact on $\mathsf{QCoh}(X^I, \mathsf{Rep}(\Gamma)_{X^I})$ and since Oblv is conservative, it follows that $\mathsf{Rep}(\Gamma)_{X^I}$ admits a unique t-structure such that the functor[40] $\text{Oblv}[\dim(X^I)] = \text{Oblv}[|I|] : \mathsf{Rep}(\Gamma)_{X^I} \to \mathsf{QCoh}(X^I, \mathsf{Rep}(\Gamma)_{X^I})$ is t-exact. Since this functor is continuous and commutes with limits (being a right adjoint), this t-structure on $\mathsf{Rep}(\Gamma)_{X^I}$ is left and right complete.

It remains to see that $\text{Oblv}_{X^I} : \mathsf{Rep}(\Gamma)_{X^I} \to D(X^I)$ is t-exact. This is immediate: we see that the t-structure we have constructed is the unique one for which the composition $\mathsf{Rep}(\Gamma)_{X^I} \xrightarrow{\text{Oblv}[|I|]} \mathsf{QCoh}(\mathsf{Rep}(\Gamma)_{X^I}) \xrightarrow{\text{Oblv}_{X^I}} \mathsf{QCoh}(X^I)$ is t-exact, and this composition coincides with $\mathsf{Rep}(\Gamma)_{X^I} \xrightarrow{\text{Oblv}_{X^I}} D(X^I) \xrightarrow{\text{Oblv}[|I|]} \mathsf{QCoh}(X^I)$. We obtain the claim, since the standard t-structure on $D(X^I)$ is the unique one for which $\text{Oblv}_{X^I}[|I|] : D(X^I) \to \mathsf{QCoh}(X^I)$ is t-exact. □

Proposition 6.24.2. *The functor* $\text{Av}^w_{X^I,*} : D(X^I) \to \mathsf{Rep}(\Gamma)_{X^I}$ *is t-exact for the t-structure of Proposition 6.24.1, and similarly for the corresponding quasi-coherent functor* $\mathsf{QCoh}(X^I) \to \mathsf{QCoh}(X^I, \mathsf{Rep}(\Gamma)_{X^I})$.

We will use the following result of [6]. We include a proof for completeness.

Lemma 6.24.3. *Let $A \in \mathsf{Vect}^\heartsuit$ be a classical (unital) commutative algebra and let $I \mapsto A_{X^I} \in D(X^I)$ be the corresponding factorization algebra. Then $A_{X^I}[-|I|] \in D(X^I)^\heartsuit$.*

Proof. We can assume $|I| > 1$, since otherwise the result is clear.

[39] Apologies are due to the reader for using the different functors Oblv and Oblv_{X^I} in almost the same breath.

[40] We use a cohomological shift here since for S smooth, $\text{Oblv} : D(S) \to \mathsf{QCoh}(S)$ only t-exact up to shift by the dimension, since $\text{Oblv}(\omega_S) = \mathcal{O}_S$. This is because we are working with the so-called left forgetful functor, not the right one.

Choose $i, j \in I$ distinct. Let $I \twoheadrightarrow \overline{I}$ be the set obtained by contracting i and j onto a single element (so $|\overline{I}| = |I| - 1$).

The map $I \twoheadrightarrow \overline{I}$ defines a diagonal closed embedding $\Delta : X^{\overline{I}} \to X^I$. Let $j : U \hookrightarrow X^I$ denote the complement, which here is affine.

Since $\Delta^!(A_{X^I}) = A_{X^{\overline{I}}}$, the result follows inductively if we show that the map $j_{*,dR}j^!(A_{X^I}) \to \Delta_{*,dR}\Delta^!(A_{X^I})[1]$ is surjective after taking cohomology in degree $-|I|$.

Writing $I = \{i\} \coprod \overline{I}$ using the evident splitting, we obtain the following commutative diagram from unitality of A and from the commutative factorization structure:

$$\begin{array}{ccccc}
\omega_X \boxtimes A_{X^{\overline{I}}} & \longrightarrow & j_{*,dR}j^!(\omega_X \boxtimes A_{X^{\overline{I}}}) & \longrightarrow & \Delta_{*,dR}(A_{X^{\overline{I}}})[1] \\
\downarrow & & \downarrow & & \downarrow \\
A_X \boxtimes A_{X^{\overline{I}}} & & j_{*,dR}j^!(A_X \boxtimes A_{X^{\overline{I}}}) & & \simeq \\
\downarrow & & \downarrow & & \downarrow \\
A_{X^I} & \longrightarrow & j_{*,dR}j^!(A_{X^I}) & \longrightarrow & \Delta_{*,dR}\Delta^!(A_{X^I})[1]
\end{array}$$

The top line is obviously (by induction) a short exact sequence in the $|I|$-shifted heart of the t-structure. Since the right vertical map is an isomorphism, this implies the claim. □

Proof of Proposition 6.24.2. E.g., in the quasi-coherent setting: it suffices to show that $\mathrm{Av}^w_{X^I,*} \circ \mathrm{Oblv}_{X^I}$ is t-exact. This composition is given by tensoring with $\mathcal{O}_{\Gamma,X^I} \in D(X^I)$ by construction, which we have just seen is in the heart of the t-structure (since $\mathrm{Oblv} : D(X^I) \to \mathsf{QCoh}(X^I)$ is t-exact only after a shift by $|I|$).

It follows that this functor is right t-exact, since it is given by tensoring with something in the heart. But it is also left t-exact, since it is right adjoint to the t-exact functor Oblv_{X^I}. □

Corollary 6.24.4. $\mathsf{Rep}(\Gamma)_{X^I}$ *is the derived category of the heart of this t-structure.*

Proof. At the level of bounded below derived categories, this is a formal consequence of the corresponding fact for $D(X^I)$ and the fact that Oblv_{X^I} and $\mathrm{Av}^w_{X^I,*}$ are t-exact.

To treat unbounded derived categories, it suffices to show that the derived category of $\mathsf{Rep}(\Gamma)^{\heartsuit}_{X^I}$ is left complete, but this is clear: the category has finite homological dimension. □

6.25. Constructibility

We now show how to recover $\mathsf{Rep}(\Gamma)_{X^I}$ from a holonomic version.

This material is not necessary for our purposes, but we include it for completeness. The reader may safely skip straight to §6.28.

6.26. Let $D_{hol}(X^I) \subseteq D(X^I)$ denote the ind-completion of the subcategory of $D(X^I)$ formed by compact objects (i.e., coherent D-modules) that are holonomic in the usual sense. We emphasize that we allow infinite direct sums of holonomic objects to be counted as such.

Definition 6.26.1. Define the *holonomic subcategory* $\mathsf{Rep}(\Gamma)_{X^I,hol}$ of $\mathsf{Rep}(\Gamma)_{X^I}$ to consist of those objects that map into $D_{hol}(X^I)$ under the forgetful functor.

Remark 6.26.2. We have:

$$\begin{aligned}\mathsf{Rep}(\Gamma)_{X^I,hol} \simeq \lim_{(I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K)} \mathsf{Rep}(\Gamma)^{\otimes K} \otimes D_{hol}(U(p)) \subseteq \\ \lim_{(I \overset{p}{\twoheadrightarrow} J \overset{q}{\twoheadrightarrow} K)} \mathsf{Rep}(\Gamma)^{\otimes K} \otimes D(U(p)) =: \mathsf{Rep}(\Gamma)_{X^I}.\end{aligned} \tag{6.26.1}$$

Indeed, the key point is that $\mathsf{Rep}(\Gamma)^{\otimes K} \otimes D_{hol}(U(p)) \to \mathsf{Rep}(\Gamma)^{\otimes K} \otimes D(U(p))$ is actually fully-faithful, and this follows from the general fact that tensoring a fully-faithful functor (here $D_{hol}(U(p)) \hookrightarrow D(U(p))$ with a dualizable category (here $\mathsf{Rep}(\Gamma)^{\otimes K}$) gives a fully-faithful functor.

Since e.g. for each $p : I \twoheadrightarrow J$, $D_{hol}(X^J)$ is dualizable as a $D_{hol}(X^I)$-module category (for the same reason as for the non-holonomic categories), we deduce that $\mathsf{Rep}(\Gamma)_{X^I,hol}$ satisfies the same factorization patterns at $\mathsf{Rep}(\Gamma)_{X^I}$, but with holonomic D-module categories being used everywhere. Indeed, the arguments we gave were basically formal cofinality arguments, and therefore apply verbatim.

6.27. We have the following technical result.

Proposition 6.27.1. *The functor:*

$$\mathsf{Rep}(\Gamma)_{X^I,hol} \underset{D_{hol}(X^I)}{\otimes} D(X^I) \to \mathsf{Rep}(\Gamma)_{X^I}$$

is an equivalence.

Remark 6.27.2. In light of (6.26.1), this amounts to commuting a limit with a tensor product. However, we are not sure how to use this perspective to give a direct argument, since $D(X^I)$ is (almost surely) not dualizable as a $D_{hol}(X^I)$-module category.

Proof of Proposition 6.27.1. The idea is to appeal to use Proposition B.8.1.

Step 1. Let $V \in \mathsf{Rep}(\Gamma)^{\otimes I}$ be given. We claim that $\mathrm{Loc}_{X^I}(V)$ lies in $\mathsf{Rep}(\Gamma)_{X^I,hol}$ and that the induced object of $\mathsf{Rep}(\Gamma)_{X^I,hol} \underset{D_{hol}(X^I)}{\otimes} D(X^I)$ is ULA in this category (considered as a $D(X^I)$-module category in the obvious way) if V is compact.

Indeed, that $\mathrm{Loc}_{X^I}(V)$ is holonomic follows since $\mathrm{Oblv}_{X^I}(V)$ is lisse. The ULA condition then follows from Proposition B.5.1 and Remark B.5.2.

Step 2. Next, we claim that $\mathsf{Rep}(\Gamma)_{X^I,hol}$ is generated as a $D_{hol}(X^I)$-module category by the objects $\mathrm{Loc}_{X^I}(V)$, $V \in \mathsf{Rep}(\Gamma)^{\otimes I}$, i.e., the minimal $D_{hol}(X^I)$-module subcategory of $\mathsf{Rep}(\Gamma)_{X^I,hol}$ containing the $\mathrm{Loc}_{X^I}(V)$ is the whole category.

Indeed, this follows as in the proof of Theorem 6.17.1.

Step 3. We now claim that $\mathsf{Rep}(\Gamma)_{X^I,hol} \underset{D_{hol}(X^I)}{\otimes} D(X^I)$ is ULA as a $D(X^I)$-module category.

We have to show that $\mathsf{Rep}(\Gamma)_{X^I} \underset{D_{hol}(X^I)}{\otimes} \mathsf{QCoh}(X^I)$ is generated as a $\mathsf{QCoh}(X^I)$-module category by objects coming from $\mathrm{Loc}_{X^I}(V)$. But this is clear from Step 2.

Step 4. Finally, we apply Proposition B.8.1 to obtain the result:

Our functor sends a set of ULA generators to ULA objects. And moreover, by Remark 6.26.2, this functor is an equivalence after tensoring with $D(X^J_{disj})$ for each $p : I \twoheadrightarrow J$, giving the result. □

Remark 6.27.3. Taking (6.26.1) as a definition of $\mathcal{C}_{X^I,hol}$ for general rigid $\mathcal{C}$, the above argument shows that the analogue of Proposition 6.27.1 is true in this generality.

6.28. The naive Satake functor

We now specialize the above to $\Gamma = \check{G}$.

6.29. Digression: more on twists

We will work with Grassmannians and loop groups twisted by $\mathcal{P}^{can}_T$ as in §2.14.

To define Gr_{H,X^I} for $H \in \{T, B, N^-, G\}$, one exactly follows §2.14.

Similarly, we have a group scheme (resp. group indscheme) $H(O)_{X^I}$ (resp. $H(K)_{X^I}$) over X^I for H as above, and $H(K)_{X^I}$ acts canonically on Gr_{G,X^I} (as H is a subgroup of G by assumption). Trivializing $\mathcal{P}^{can}_T$ locally on X, the picture becomes the usual picture for factorizable versions of the arc and loop groups: cf. [6] and [29] for example.

6.30. Let Sph_{G,X^I} denote the spherical Hecke category $D(\mathrm{Gr}_{G,X^I})^{G(O)_{X^I}}$. The assignment $I \mapsto \mathsf{Sph}_{G,X^I}$ defines a factorization monoidal category.

Our goal for the remainder of this section is to construct and study certain monoidal functors:

$$\mathrm{Sat}^{naive}_{X^I} : \mathsf{Rep}(\check{G})_{X^I} \to \mathsf{Sph}_{G,X^I}$$

compatible with factorization.

Remark 6.30.1. We follow Gaitsgory in calling this functor *naive* because it is an equivalence only on the hearts of the t-structures (indeed, it is not an equivalence on Exts between unit objects, since equivariant cohomology appears in the right hand side but not the left).

6.31. The following results provide toy models for constructing the functors $\mathrm{Sat}^{naive}_{X^I}$.

Lemma 6.31.1. *For $\mathcal{D} \in \mathsf{DGCat}_{cont}$, the map:*

$$\{F : \mathsf{Rep}(\Gamma) \to \mathcal{D} \in \mathsf{DGCat}_{cont}\} \to \mathcal{O}_\Gamma\text{–}\mathsf{comod}(\mathcal{D})$$
$$F \mapsto F(\mathcal{O}_\Gamma)$$

is an equivalence.

Proof. Since $\mathsf{Rep}(\Gamma)$ is self-dual and since $\mathsf{Rep}(\Gamma) \otimes \mathcal{D} \xrightarrow{\mathrm{Oblv}} \mathsf{Vect} \otimes \mathcal{D} = \mathcal{D}$ is comonadic (cf. the proof of Lemma 6.23.1), we obtain the claim. □

Lemma 6.31.2. *For $\mathcal{D} \in \mathsf{Alg}(\mathsf{DGCat}_{cont})$ a monoidal (in the cocomplete sense) DG category, the map:*

$$\{F : \mathsf{Rep}(\Gamma) \to \mathcal{D} \text{ continuous and lax monoidal}\} \to \mathsf{Alg}(\mathcal{O}_\Gamma\text{–}\mathsf{comod}(\mathcal{D}))$$
$$F \mapsto F(\mathcal{O}_\Gamma)$$

is an equivalence. Here $\mathcal{O}_\Gamma$–$\mathsf{comod}(\mathcal{D})$ is equipped with the obvious monoidal structure, induced from that of $\mathcal{D}$.

Remark 6.31.3. Here is a heuristic for Lemma 6.31.2:

Given $A \in \mathcal{O}_\Gamma$–$\mathsf{comod}(\mathcal{D})$, the corresponding functor $\mathsf{Rep}(\Gamma) \to \mathcal{D}$ is given by the formula $V \mapsto (V \otimes A)^\Gamma$ (where the invariants here are of course derived). If A is moreover equipped with a Γ-equivariant algebra structure, we obtain the canonical maps:

$$(V \otimes A)^\Gamma \otimes (W \otimes A)^\Gamma \to (V \otimes A \otimes W \otimes A)^\Gamma = (V \otimes W \otimes A \otimes A)^\Gamma \to (V \otimes W \otimes A)^\Gamma$$

as desired, where the last map comes from the multiplication on A.

Proof of Lemma 6.31.2. This follows e.g. from the identification of the monoidal structure of $\mathsf{Rep}(\Gamma) \otimes \mathcal{D}$ with the Day convolution structure on the functor category $\mathsf{Hom}_{\mathsf{DGCat}_{cont}}(\mathsf{Rep}(\Gamma), \mathcal{D})$, identifying the two via self-duality of $\mathsf{Rep}(\Gamma)$. □

6.32. We will use the following more sophisticated version of the above lemmas.

Lemma 6.32.1. *For $\mathcal{D} \in D(X^I)$–mod, the functor:*

$$\{F : \mathsf{Rep}(\Gamma)_{X^I} \to \mathcal{D} \in D(X^I)\text{–mod}\} \to \mathsf{Rep}(\Gamma)_{X^I} \underset{D(X^I)}{\otimes} \mathcal{D} \overset{Lem.\ 6.23.2}{=} \mathcal{O}_{\Gamma,X^I}\text{–comod}(\mathcal{D})$$

$$F \mapsto F(\mathcal{O}_{\Gamma,X^I})$$

is an equivalence. Giving a lax monoidal structure in the left hand side amounts to giving an algebra structure on the right hand side.

Proof. By Lemma 6.18.1, $D(X^I)$-linear functors $\mathsf{Rep}(\Gamma)_{X^I} \to \mathcal{D}$ are equivalent to objects of $\mathsf{Rep}(\Gamma)_{X^I} \otimes_{D(X^I)} \mathcal{D}$.

The result then follows from Lemma 6.23.2 and Lemma 6.31.2. □

6.33. Construction of the functor

By Lemma 6.32.1, to construct $\mathrm{Sat}^{naive}_{X^I}$ as a lax monoidal functor, we need to specify an object of $\mathsf{Rep}(\check{G})_{X^I} \otimes_{D(X^I)} \mathsf{Sph}_{G,X^I}$ with an algebra structure.

Such objects $\mathcal{H}^{ch}_{X^I} \in \mathsf{Rep}(\check{G})_{X^I} \otimes_{D(X^I)} \mathsf{Sph}_{G,X^I}$ are defined factorizably in Appendix B of [23] (they go by the name *chiral Hecke algebra* and were probably first constructed by Beilinson).[41] For each I, $\mathcal{H}^{ch}_{X^I}$ is concentrated in cohomological degree $-|I|$.

Example 6.33.1. For $I = *$, $\mathcal{H}^{ch}_X$ comes from the regular representation of $\check{G}$ under geometric Satake.

Remark 6.33.2. We emphasize that the general construction (and the data required to define the output) is purely abelian categorical, and comes from the usual construction of the geometric Satake equivalence.

Lemma 6.33.3. *The lax monoidal functors* $\mathrm{Sat}^{naive}_{X^I}$ *are actually monoidal.*

Proof. We need to check that some maps between some objects of Sph_{G,X^I} are isomorphisms. It suffices to do this after restriction to strata on X^I, and by factorization, we reduce to the case $I = *$ where it follows from usual geometric Satake and the construction of the chiral Hecke algebra. □

6.34. We have the following important fact:

Proposition 6.34.1. $\mathrm{Sat}^{naive}_{X^I}$ *is t-exact.*

We begin with the following.

[41] In the notation of [23], we have $\mathcal{H}^{ch}_{X^d} = \mathcal{R}^d_X[d] = {}^f\mathcal{R}^d_X[d]$.

Lemma 6.34.2. *The functor* $\mathsf{Sph}_{G,X^I} \to \mathsf{Sph}_{G,X^I}$ *defined by convolution with* $\mathcal{H}^{ch}_{X^I}$ *is* t*-exact.*

Proof. Recall that for each I and J, there is the *exterior convolution* functor:

$$\mathsf{Sph}_{G,X^I} \otimes \mathsf{Sph}_{G,X^J} \to \mathsf{Sph}_{G,X^{I \amalg J}}$$

which is a morphism of $D(X^{I \amalg J})$-module categories.[42] The relation to usual convolution is that for $J = I$, convolution is obtained by applying exterior convolution and then !-restricting to the diagonal.

The usual semi-smallness argument shows that exterior convolution is t-exact. Therefore, since $\mathcal{H}^{ch}_{X^I}$ lies in degree $-|I|$, we deduce from the above that convolution with $\mathcal{H}^{ch}_{X^I}$ has cohomological amplitude $[-|I|, 0]$: in particular, it is right t-exact.

It remains to see that this convolution functor is left t-exact. For a given partition $p : I \twoheadrightarrow J$, let $i_p : X^J_{disj} \to X^I$ denote the embedding of the corresponding stratum of X^I. The !-restriction of $\mathcal{H}^{ch}_{X^I}$ to X^J_{disj} is concentrated in cohomological degree $-|J|$, and is the object corresponding to the regular representation under geometric Satake. It follows that the functor of convolution with $i_{p,*,dR} i_p^! (\mathcal{H}^{ch}_{X^I})$ is left t-exact from the exactness of convolution in the Satake category for a point. We now obtain the claim by dévissage. □

Proof of Proposition 6.34.1. First, we claim that our functor is left t-exact.

We can write $\mathrm{Sat}^{naive}_{X^I}$ as a composition of tensoring $\mathcal{F}$ with the delta D-module on the unit of Gr_{G,X^I}, convolving with $\mathcal{H}^{ch}_{X^I}$, and then taking invariants with respect to the diagonal actions for the $\check{G}^J$. The first step is obviously t-exact, and the second step is t-exact by Lemma 6.34.2; the third step is obviously left t-exact.

It remains to show that it is right t-exact.

First, let $V \in \mathsf{Rep}(\check{G}^I)^\heartsuit = \mathsf{Rep}(\check{G})^{\otimes I, \heartsuit}$. We claim that convolution with $\mathrm{Sat}^{naive}_{X^I}(\mathrm{Loc}_{X^I}(V))$ is t-exact (as an endofunctor of Sph_{G,X^I}).

It suffices to show this for V finite-dimensional, and then duality of V and monoidality of $\mathrm{Sat}^{naive}_{X^I}$ reduces us to showing exactness in either direction: we show that this convolution functor is left t-exact. This then follows by the same stratification argument as in the proof of Lemma 6.34.2.

In particular, convolving with the unit, we see that $\mathrm{Sat}^{naive}_{X^I}(\mathrm{Loc}_{X^I}(V))$ is concentrated in cohomological degree $-|I|$, and more generally, $\mathrm{Sat}^{naive}_{X^I} \circ \mathrm{Loc}_{X^I}$ is t-exact up to this same cohomological shift.

For simplicity, we localize on X to assume X is affine. Then by Theorem 6.17.1, $\mathsf{Rep}(\check{G})^{\leqslant 0}_{X^I}$ is generated under colimits by objects of the form $\mathrm{ind}\,\mathrm{Oblv}(\mathrm{Loc}_{X^I}(V))$ for $V \in \mathsf{Rep}(I^I)^{\leqslant |I|}$: indeed, this follows from the observation that $\mathrm{ind}\,\mathrm{Oblv}$ is t-exact,

[42] We emphasize that I and J play an asymmetric role in the definition, i.e., the definition depends on an *ordered* pair of finite sets, not just a pair of finite sets.

which is true since after applying Oblv again, it is given by tensoring with the ind-vector bundle of differential operators on X^I. The same reasoning shows that ind Oblv is t-exact on Sph_{G,X^I}, giving the result. □

6.35. The naive Satake theorem

We will not need the following result, but include a proof for completeness. Since we are not going to use it, we permit ourselves to provide substandard detail.

Theorem 6.35.1. *The functor* $\mathrm{Sat}^{naive}_{X^I}$ *induces an equivalence between the hearts of the t-structures:*

$$\mathrm{Sat}^{\heartsuit}_{X^I} : \mathsf{Rep}(\check{G})^{\heartsuit}_{X^I} \xrightarrow{\simeq} \mathsf{Sph}^{\heartsuit}_{G,X^I}.$$

We will give an argument in §6.37.

Remark 6.35.2. In the setting of perverse sheaves, Theorem 6.35.1 is proved in [23] Appendix B. We provide a different argument from [23] that more easily deals with the problem of non-holonomic D-modules.

6.36. Spherical Whittaker sheaves

Our argument for Satake will appeal to the following. Let $\mathsf{Whit}^{sph}_{X^I}$ denote the category of Whittaker D-modules on Gr_{G,X^I}, i.e., D-modules equivariant against $N^-(K)_{X^I}$ equipped with its standard character (we use the $\check{\rho}(\omega_X)$-twist here).

We have a canonical functor $\mathsf{Sph}_{G,X^I} \to \mathsf{Whit}^{sph}_{X^I}$ given by convolution with the *unit* object $\mathrm{unit}_{\mathsf{Whit}^{sph}_{X^I}} \in \mathsf{Whit}^{sph}_{X^I}$, i.e., the canonical object cleanly extended from Gr_{N^-,X^I} (i.e., the $*$ and !-extensions coincide here).

Theorem 6.36.1 *(Frenkel-Gaitsgory-Vilonen, Gaitsgory, Beraldo). The composite functor:*

$$\mathsf{Rep}(\check{G})_{X^I} \xrightarrow{\mathrm{Sat}^{naive}_{X^I}} \mathsf{Sph}_{G,X^I} \to \mathsf{Whit}^{sph}_{X^I}$$

is an equivalence.

Proof. We will appeal to Proposition B.8.1.

It is easy to see that the unit object of $\mathsf{Whit}^{sph}_{X^I}$ is ULA: this follows from the usual cleanness argument. We then formally deduce from dualizability of ULA objects in $\mathsf{Rep}(\check{G})_{X^I}$ and monoidality of $\mathrm{Sat}^{naive}_{X^I}$ that the above functor sends ULA objects to ULA objects.

Then since these sheaves of categories are locally constant along strata (by factorizability), we obtain the claim by noting that this functor is an equivalence over a point,

as follows from [22] and the comparison of local and global[43] definitions of spherical Whittaker categories, as has been done e.g. in the unpublished work [26]. □

We also use the following fact about Whittaker categories.

Lemma 6.36.2. *The object* $\mathrm{Av}_{G(O)_{X^I},*}(\mathrm{unit}_{\mathsf{Whit}^{sph}_{X^I}}) \in \mathsf{Sph}_{G,X^I}$ *lies in cohomological degrees* $\geqslant -|I|$. *The adjunction map:*

$$\mathrm{unit}_{\mathsf{Sph}_{G,X^I}} \to \mathrm{Av}_{G(O)_{X^I},*}(\mathrm{unit}_{\mathsf{Whit}^{sph}_{X^I}})$$

is an equivalence on cohomology in degree $-|I|$. *(Here* $\mathrm{unit}_{\mathsf{Sph}_{G,X^I}}$ *is the delta D-module on* X^I $*$*-pushed forward to* Gr_{G,X^I} *using the tautological section).*

Proof. The corresponding fact over a point is obvious: the fact that $\mathsf{Sph}_{G,x} \to \mathsf{Whit}^{sph}_x$ is t-exact on hearts of t-structures implies that its right adjoint is left t-exact, so applying the above averaging to the unit, one obtains an object in degrees $\geqslant 0$. The adjunction map is an equivalence on 0th cohomology because $\mathsf{Sph}_{G,x} \to \mathsf{Whit}^{sph}_x$ is an equivalence on hearts of t-structures.

We then deduce that from factorization that for each $p : I \twoheadrightarrow J$, the !-restriction of:

$$\mathrm{Coker}(\mathrm{unit}_{\mathsf{Sph}_{G,X^I}} \to \mathrm{Av}_{G(O)_{X^I},*}(\mathrm{unit}_{\mathsf{Whit}^{sph}_{X^I}}))$$

to the corresponding stratum X^J_{disj} defined by p is concentrated in cohomological degrees $> -|J|$, which immediately gives the claim. □

6.37. We now deduce factorizable Satake.

Proof of Theorem 6.35.1. We have an adjunction $\mathsf{Sph}_{G,X^I} \rightleftarrows \mathsf{Whit}^{sph}_{X^I}$ where the left adjoint is convolution with the unit and the right adjoint is $*$-averaging with respect to $G(O)_{X^I}$.

From Theorem 6.36.1, we obtain the adjunction:

$$\mathsf{Sph}_{G,X^I} \underset{\mathrm{Sat}^{naive}_{X^I}}{\rightleftarrows} \mathsf{Rep}(\check{G})_{X^I}.$$

Since $\mathrm{Sat}^{naive}_{X^I}$ is t-exact, we obtain a corresponding adjunction between the hearts of the t-structure. Lemma 6.36.2 implies that the left adjoint is fully-faithful at the abelian categorical level, and the right adjoint $\mathrm{Sat}^{naive,\heartsuit}_{X^I}$ is conservative by Theorem 6.36.1, so we obtain the claim. □

[43] I.e., using Drinfeld's compactifications as in [22].

7. Hecke functors: Zastava with moving points

7.1. As in §5, the main result of this section, Theorem 7.9.1, will compare geometrically and spectrally defined *Chevalley* functors. However, in this section, we work over powers of the curve: we are giving a compatibility now between Theorem 4.6.1 and the factorizable Satake theorem of §6.

7.2. Structure of this section

In §7.3-7.9, we give *moving points* analogues of the constructions of §5 and formulate our main theorem.

The remainder of the section is dedicated to deducing this theorem from Theorem 7.9.1.

There are two main difficulties in proving the main theorem: working over powers of the curve presents difficulties, and the fact that we are giving a combinatorial (i.e., involving Langlands duality) comparison of functors in the derived setting.

The former we treat by exploiting ULA objects: cf. Appendix B and §6. These at once exhibit good functoriality properties and provide a method for passing from information over the disjoint locus X^I_{disj} to the whole of X^I.

We treat the homotopical difficulties by exploiting a useful t-structure on factorization $\Upsilon_{\check{\mathfrak{n}}}$-modules, cf. Proposition 7.11.1.

7.3. Define the indscheme $\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I}$ over X^I as parametrizing an I-tuple $x=(x_i)$ of points of X and a $\check{\Lambda}$-valued divisor on X that is $\check{\Lambda}^{pos}$-valued on $X\setminus\{x_i\}$.

Warning 7.3.1. The notation $\infty\cdot x$ in the superscript belies that x is a dynamic variable: it is used to denote our I-tuple of points in X. We maintain this convention in what follows, keeping the subscript X^I to indicate that we work over powers of the curve now.

Remark 7.3.2. We again have a degree map $\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I}\to\check{\Lambda}$.

Let $\Upsilon_{\check{\mathfrak{n}}}$–$\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$ denote the DG category of unital factorization modules for $\Upsilon_{\check{\mathfrak{n}}}$ on $\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I}$.

The two functors we will compare will go from $\mathsf{Rep}(\check{G})_{X^I}$ to $\Upsilon_{\check{\mathfrak{n}}}$–$\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$.

7.4. Geometric Chevalley functor

To construct the geometric Chevalley functor, we imitate much of the geometry that appeared in §5.2-5.14.

7.5. For starters, define $\overline{\mathrm{Bun}}^{\infty\cdot x}_{N^-,X^I}\to X^I$ as parametrizing $x=(x_i)_{i\in I}\in X^I$, a G-bundle $\mathcal{P}_G$ on X, and non-zero maps:

$$\Omega_X^{\otimes(\check{\rho},\lambda)} \to V^{\lambda}_{\mathcal{P}_G}(\infty \cdot x)$$

defined for each dominant weight λ and satisfying the Plucker relations, in the notation of §5.2. Here the notation of twisting by $\mathcal{O}_X(\infty \cdot x)$ makes sense in S-points: for $x = (x_i)_{i\in I} : S \to X^I$, we take the sum of the Cartier divisors on $X \times S$ associated with the graphs of the maps x_i to define $\mathcal{O}_{X\times S}(x)$.

7.6. We can imitate the other constructions in the same fashion, giving the ind-scheme $\overset{o}{\mathcal{Z}}{}^{\infty\cdot x}_{X^I}$ (resp. $\mathcal{Z}^{\infty\cdot x}$) over X^I and the map $\overset{o}{\pi}{}^{\infty\cdot x}_{X^I} : \overset{o}{\mathcal{Z}}{}^{\infty\cdot x}_{X^I} \to \mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I}$ (resp. $\pi^{\infty\cdot x}_{X^I} : \mathcal{Z}^{\infty\cdot x}_{X^I} \to \mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I}$).

Let $\mathcal{Y}_{X^I}$ be the inverse image of $X^I \times \mathrm{Bun}_{N^-} \subseteq \overline{\mathrm{Bun}}^{\infty\cdot x}_{N^-,X^I}$. We have a distinguished object $\psi_{\mathcal{Y}_{X^I}} \in D(\mathcal{Y}_{X^I})$, obtained by !-pullback from $\omega_{X^I} \boxtimes \mathcal{W}hit \in D(X^I \times \mathrm{Bun}_{N^-})$, and an object $\psi^{\mathrm{IC}}_{\mathcal{Y}_{X^I}}$ obtained from $\psi_{\mathcal{Y}_{X^I}}$ by cohomologically shifting according to degrees, just as in §5.10.

We also have a $D(X^I)$-linear action of Sph_{G,X^I} on $D(\mathcal{Z}^{\infty\cdot x}_{X^I})$.

We obtain a $D(X^I)$-linear functor:

$$\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I} : \mathsf{Rep}(\check{G})_{X^I} \to \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$$

imitating our earlier functor $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x}$. Indeed, we use the naive Satake functor, convolve with (the $*$-pushforward of) $\psi^{\mathrm{IC}}_{\mathcal{Y}_{X^I}}$, !-restrict to $\overset{o}{\mathcal{Z}}{}^{\infty\cdot x}_{X^I}$, and then $*$-pushforward to $\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I}$, exactly as in §5.11. That is, our functor is the suitable composition:

$$\mathsf{Rep}(\check{G})_{X^I} \xrightarrow{\mathrm{Sat}^{naive}_{X^I}} \mathsf{Sph}_{G,X^I} \xrightarrow{-\star\psi_{\mathcal{Y}^{\mathrm{IC}}_{X^I}}} \jmath_{*,dR}(\psi_{\overset{o}{\mathcal{Z}}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}})\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(\mathcal{Z}^{\infty\cdot x}_{X^I}) \to$$

$$(\psi_{\overset{o}{\mathcal{Z}}} \overset{!}{\otimes} \mathrm{IC}_{\overset{o}{\mathcal{Z}}})\text{–}\mathsf{mod}^{\mathrm{fact}}_{un}(\overset{o}{\mathcal{Z}}{}^{\infty\cdot x}_{X^I}) \xrightarrow{\overset{o}{\pi}{}^{\infty\cdot x}_{*,dR}} \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}.$$

7.7. Spectral Chevalley functor

To construct $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},X^I}$, we will use the following.

Lemma 7.7.1. *The category $\check{\mathfrak{n}}_X$–mod_{X^I} of Lie-$*$ modules on X^I for $\check{\mathfrak{n}}_X \in \mathsf{Rep}(\check{T})_X$ is canonically identified with the category $\check{\mathfrak{n}}$–$\mathsf{mod}(\mathsf{Rep}(\check{T}))_{X^I}$, i.e., the $D(X^I)$-module category associated with the symmetric monoidal DG category $\check{\mathfrak{n}}$–$\mathsf{mod}(\mathsf{Rep}(\check{T}))$ by the procedure of §6.6.*

Proof. Let $\Gamma \subseteq X \times X^I$ be the union of the graphs of the projections $X^I \to X$. Let α (resp. β) denote the projection from Γ to X (resp. X^I).

Since β is proper, one finds that:

$$\beta_{*,dR}\alpha^! : D(X) \to D(X^I)$$

is colax symmetric monoidal, and in particular maps Lie coalgebras for $(D(X), \overset{!}{\otimes})$ to Lie coalgebras for $(D(X^I), \overset{!}{\otimes})$.

Moreover, if $L \in D(X)$ is a Lie-$*$ algebra and compact as a D-module, then its Verdier dual $\mathbb{D}L$ is a Lie coalgebra in $(D(X), \overset{!}{\otimes})$, and L-modules on X^I are equivalent to $\beta_{*,dR}\alpha^!(\mathbb{D}L)$-comodules. We have an obvious translation of this for the *graded* case, where e.g. $D(\mathrm{Gr}_{T,X^I})$ replaces $D(X^I)$. (See [39] Proposition 4.5.2 for a non-derived version of this; essentially the same argument works in general.)

One then easily finds that for $V \in \mathsf{Vect}$, one has:

$$\beta_{*,dR}\alpha^!(V \otimes \omega_X) = \lim_{(I \twoheadrightarrow J \twoheadrightarrow K) \in \mathcal{S}_I} j_{p,*,dR}(V^{\oplus K} \otimes \omega_{U(p)})$$

where the notation is as in §6. We remark that this limit is a "logarithm" of the one appearing in (6.13.1): we use the addition maps $V^{\oplus K} \to V^{\oplus K'}$ for $K \twoheadrightarrow K'$ to give the structure maps in the limit, i.e., the canonical structure of commutative algebra on V in $(\mathsf{Vect}, \oplus)$.

Moreover, this identification is compatible with Lie cobrackets, so that the Lie coalgebra $(\check{\mathfrak{n}}^\vee) \otimes \omega_X$ maps to the Lie coalgebra:

$$\beta_{*,dR}\alpha^!(\check{\mathfrak{n}}^\vee \otimes \omega_X) = \lim_{(I \twoheadrightarrow J \twoheadrightarrow K) \in \mathcal{S}_I} j_{p,*,dR}((\check{\mathfrak{n}}^\vee)^{\oplus K} \otimes \omega_{U(p)}).$$

This immediately gives the claim. □

Remark 7.7.2. We identify $\check{\mathfrak{n}}_X$–mod_{X^I} and $\check{\mathfrak{n}}$–$\mathsf{mod}(\mathsf{Rep}(\check{T}))_{X^I}$ in what follows. We emphasize that although the $\check{\Lambda}$-grading does not appear explicitly in the notation, it is implicit in the fact that $\check{\mathfrak{n}}_X$ is always considered as $\check{\Lambda}$-graded.

We obtain the restriction functor:

$$\mathsf{Rep}(\check{B})_{X^I} \to \check{\mathfrak{n}}_X\text{–}\mathsf{mod}_{X^I}.$$

Using the chiral induction functor $\mathrm{ind}^{ch} : \check{\mathfrak{n}}_X\text{–}\mathsf{mod}_{X^I} \to \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$ and the restriction functor from $\check{G}$ to $\check{B}$, we obtain:

$$\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},X^I} : \mathsf{Rep}(\check{G})_{X^I} \to \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$$

as desired.

7.8. For convenience, we record the following consequence of Lemma 7.7.1. The reader may skip this section.

Recall from [35] §6.12 and §8.14 that the *external fusion* construction defines a lax unital factorization category structure on the assignment:

$$I \mapsto \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}.$$

Corollary 7.8.1. *The lax factorization structure is a true factorization structure. I.e., for every $I, J \in \mathsf{Set}_{<\infty}$, the external fusion functor:*

$$[\Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I} \otimes \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^J}] \underset{D(X^{I \amalg J})}{\otimes} D([X^I \times X^J]_{disj})$$
$$\to \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^{I \amalg J}} \underset{D(X^{I \amalg J})}{\otimes} D([X^I \times X^J]_{disj})$$

is an equivalence.

Proof. The corresponding result for Lie-$*$ modules over $\check{\mathfrak{n}}_X$ follows from Lemma 7.7.1. Using the adjoint functors $(\mathrm{ind}^{ch}, \mathrm{Oblv}^{ch})$, we see that factorization modules for $\Upsilon_{\check{\mathfrak{n}}}$ are modules for a monad on Lie-$*$ modules, and the two monads obviously match up e.g. by the chiral PBW theorem. □

7.9. Formulation of the main theorem

Observe that formation of each of the functors $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},X^I}$ and $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}$ are compatible with factorization as we vary the finite set I (here we use the *external fusion* construction $\Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$).

Theorem 7.9.1. *The factorization functors $I \mapsto \mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},X^I}$ and $I \mapsto \mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}$ are canonically isomorphic as factorization functors.*

The proof of Theorem 7.9.1 will occupy the remainder of this section.

Remark 7.9.2. Here is the idea of the argument: since both functors factorize, we know the result over strata of X^I by Theorem 5.14.1. We glue these isomorphisms over all of X^I by analyzing ULA objects.

Remark 7.9.3. This theorem is somewhat loose as stated, as it does not specify how they are isomorphic. This is because the construction of the isomorphism is somewhat difficult, due in part to the difficulty of constructing anything at all in the higher categorical setting.

However, we remark that for G simply-connected, we will see that such an isomorphism of factorization functors is uniquely characterized as such. Similarly, for G a torus, it is easy to write down such an isomorphism by hand (just as it is easy to write down the (naive) geometric Satake by hand in this case). This should be taken to indicate the existence of a canonical isomorphism in general. We refer to Remark 7.10.2 and §7.22 for further discussion of this point.

7.10. First, we observe the following.

Lemma 7.10.1. $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},X^I}$ *and* $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}$ *are canonically isomorphic for $I = *$.*

Proof. We are comparing two $D(X)$-linear functors:

$$\mathsf{Rep}(\check{G})_X = \mathsf{Rep}(\check{G}) \otimes D(X) \to \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X}$$

or equivalently, two continuous functors:

$$\mathsf{Rep}(\check{G}) \to \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X}.$$

By lisseness along X, we obtain the result from Theorem 5.14.1 (alternatively: the methods of Theorem 5.14.1 work when the point x is allowed to vary, giving the result). □

Remark 7.10.2. In what follows, we will see that the isomorphism of Theorem 7.9.1 is uniquely pinned down by a choice of isomorphism over X, i.e., an isomorphism as in Lemma 7.10.1. Indeed, this will follow from Proposition 7.18.2. Note that we have constructed such an isomorphism explicitly in the proof of Theorem 5.14.1, and therefore this completely pins down Theorem 7.9.1.

7.11. Digression: a t-structure on factorization modules

We now construct a convenient t-structure for $\Upsilon_{\check{\mathfrak{n}}}$-modules.

Proposition 7.11.1.

(1) There is a (necessarily unique) t-structure on $\Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$ such that the forgetful functor:

$$\mathrm{Oblv}^{ch} : \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I} \to D(\mathrm{Div}^{\infty\cdot x}_{\mathrm{eff},X^I}) \tag{7.11.1}$$

is t-exact.

(2) With respect to this t-structure, the chiral induction functor:

$$\mathrm{ind}^{ch} : \check{\mathfrak{n}}_X\text{–}\mathsf{mod}_{X^I} \to \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$$

is t-exact with respect to the t-structure on the left hand side coming from Proposition 7.7.1.

(3) This t-structure is left and right complete.

Proof. Note that we have a commutative diagram:

$$\begin{array}{ccc} \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I} & \longrightarrow & \check{\mathfrak{n}}_X\text{–}\mathsf{mod}_{X^I} \\ \downarrow{\scriptstyle \mathrm{Oblv}^{ch}} & & \downarrow \\ D(\mathrm{Div}^{pos,\infty\cdot x}_{\mathrm{eff},X^I}) & \xrightarrow{i^!} & D(\mathrm{Gr}_{T,X^I}) \end{array} \tag{7.11.2}$$

where we use i to denote the map $\mathrm{Gr}_{T,X^I} \to \mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I}$.

Define $(\Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I})^{\leqslant 0}$ as the subcategory generated under colimits by $\check{\mathfrak{n}}_X\text{–}\mathsf{mod}^{\leqslant 0}_{X^I}$ by ind^{ch}. This defines a t-structure in the usual way. Note that an object lies in $(\Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I})^{>0}$ if and only if its image under Oblv^{ch} lies in $\check{\mathfrak{n}}_X\text{–}\mathsf{mod}^{>0}_{X^I}$.

The main observation is that the composition $\mathrm{Oblv}^{ch} \circ \mathrm{ind}^{ch}$ is t-exact:

The PBW theorem for factorization modules [35] §7.19 says that for $M \in \check{\mathfrak{n}}_X\text{–}\mathsf{mod}_{X^I}$, $\mathrm{ind}^{ch}(M)$ has a filtration as an object of $D(\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I})$ with subquotients given by symmetric group invariants of the $*$-pushforward of:

$$\underbrace{\check{\mathfrak{n}}_X[1] \boxtimes \ldots \boxtimes \check{\mathfrak{n}}_X[1]}_{n \text{ times}} \boxtimes M \in D\big((\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}})^n \times \mathrm{Gr}_{T,X^I}\big)$$

along the addition map to $\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I}$. Formation of this exterior product is obviously t-exact, the $*$-pushforward operation is t-exact by finiteness, and the S_n-invariants are t-exact as we work in characteristic 0. So we obtain our claim.

Then from the commutative diagram (7.11.2), we see that $\mathrm{Oblv}^{ch} \circ \mathrm{ind}^{ch}$ is left t-exact. This immediately implies the t-exactness of ind^{ch}.

It remains to show that Oblv^{ch} is t-exact. By the above computation of $\mathrm{Oblv}^{ch}\,\mathrm{ind}^{ch}$, it is right t-exact.

Suppose $M \in \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$ with $i^!\,\mathrm{Oblv}^{ch}(M) \in D(\mathrm{Gr}_{T,X^I})^{>0}$. By factorization and since $\Upsilon_{\check{\mathfrak{n}}} \in D(\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}})^{\heartsuit}$, we deduce that $\mathrm{Oblv}^{ch}(M)$ is in degree > 0. By the commutative diagram (7.11.2), this hypothesis is equivalent to assuming that $M \in (\Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I})^{>0}$, so we deduce our left t-exactness.

Finally, that this t-structure is left and right complete follows immediately from (1). □

Corollary 7.11.2. *The functor* $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},X^I} : \mathsf{Rep}(\check{G})_{X^I} \to \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$ *is t-exact.*

7.12. ULA objects

Next, we discuss the behavior of ULA objects under the Chevalley functors.

In the discussion that follows, we use the term *ULA* as an abbreviation for *ULA over* X^I.

7.13. We begin with a technical remark on the spectral side.

Proposition 7.13.1.

(1) The functor $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},X^I}$ *maps ULA objects in* $\mathsf{Rep}(\check{G})_{X^I}$ *to ULA objects in* $\Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$.

(2) For every $V \in \mathsf{Rep}(\check{G})^{\otimes I}$, *the object* $\mathrm{Oblv}^{ch}\,\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},X^I}(\mathrm{Loc}_{X^I}(V)) \in D(\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I})$ *underlying* $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},X^I}(\mathrm{Loc}_{X^I}(V))$ *is ind-ULA.*

More precisely, if V is compact, then for every $\check{\lambda} \in \check{\Lambda}$, the restriction of this D-module to the locus of divisors of total degree $\check{\lambda}$ is ULA.[44]

Proof. The functor $\mathsf{Rep}(\check{B})_{X^I} \to \check{\mathfrak{n}}_X\text{–}\mathsf{mod}_{X^I}$ preserves ULA objects by the same argument as in Proposition 6.16.1, and then the first part follows from $D(X^I)$-linearity of the adjoint functors $\check{\mathfrak{n}}_X\text{–}\mathsf{mod}_{X^I} \underset{}{\overset{\mathrm{ind}^{ch}}{\rightleftarrows}} \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$.

For the second part, we claim more generally that $\mathrm{Oblv}^{ch}\,\mathrm{ind}^{ch}$ maps ULA objects in $\mathsf{Rep}(\check{B})_{X^I}$ to objects in $D(\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I})$ whose restriction to each degree is ULA.

To this end, we immediately reduce to the case of one-dimensional representations of $\check{B}^I$, since every compact object of $\mathsf{Rep}(\check{B})^{\otimes I}$ admits a finite filtration with such objects as the subquotients.

In the case of the trivial representation of $\check{B}$, the corresponding object is the *vacuum* representation, which in this setting is obtained by $*$-pushforward from $\omega_{X^I} \boxtimes \Upsilon_{\check{\mathfrak{n}}}$ along the obvious map:

$$X^I \times \mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}} \to \mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I}.$$

Since this map is a closed embedding, we obtain the claim since $\omega_{X^I} \boxtimes \Upsilon_{\check{\mathfrak{n}}}$ obviously has the corresponding property.

The general case of a 1-dimensional representation differs from this situation by a translation on $\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I}$, giving the claim here as well. □

7.14. Next, we make the following observation on the geometric side.

Proposition 7.14.1.

(1) For every $V \in \mathsf{Rep}(\check{G})^{\otimes I}$, $\mathrm{Oblv}^{ch}\,\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}(\mathrm{Loc}_{X^I}(V)) \in D(\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I})$ is ind-ULA.

More precisely, for V compact and $\check{\lambda} \in \check{\Lambda}$, the restriction of $\mathrm{Oblv}^{ch} \times \mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}(\mathrm{Loc}_{X^I}(V))$ to the locus of divisors of total degree $\check{\lambda}$ is ULA.

(2) For $V \in \mathsf{Rep}(\check{G})^{\otimes I,\heartsuit}$, $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}(\mathrm{Loc}_{X^I}(V)) \in \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$ lies in cohomological degree $-|I|$.

Proof. As in Proposition 7.13.1 suffices to show that for $V \in \mathsf{Rep}(\check{G})^{\otimes I,\heartsuit}$ compact, $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}(\mathrm{Loc}_{X^I}(V))$ admits a filtration by $\check{\Lambda}^I$ with $\check{\mu}$-subquotient $\mathrm{ind}^{ch}(\ell^{\check{\mu}}) \otimes V(\check{\mu})$, where $V(\check{\mu})$ is the $\check{\mu}$-weight space of V and $\ell^{\check{\mu}} \in \mathsf{Rep}(\check{B})^{\otimes I,\heartsuit}$ is the corresponding one dimensional representation.

[44] Note that this claim is wrong if we do not restrict to components, since ULA objects are compact.

This follows exactly as in Step 4 of the proof of Theorem 5.14.1: the weight space of $V \in \mathsf{Rep}(\check{G})^{\otimes I}$ appears as a semi-infinite integral à la Mirkovic-Vilonen by the appropriate moving points version of Lemma 5.15.1. □

7.15. We now deduce the following key result, comparing $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}$ and $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},X^I}$ on ULA objects.

Proposition 7.15.1. *The two functors:*

$$\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I} \circ \mathrm{Loc}_{X^I} : \mathsf{Rep}(\check{G})^{\otimes I} \to \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$$
$$\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},X^I} \circ \mathrm{Loc}_{X^I} : \mathsf{Rep}(\check{G})^{\otimes I} \to \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$$

are isomorphic.

More precisely, there exists a unique such isomorphism extending the isomorphism between these functors over X^I_{disj} coming from Lemma 7.10.1 and factorization.

Proof. It suffices to produce an isomorphism between the restrictions of $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}$ and $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},X^I}$ to the category of compact objects in the heart of $\mathsf{Rep}(\check{G})^{\otimes I,\heartsuit}$.

Suppose $V \in \mathsf{Rep}(\check{G})^{\otimes I,\heartsuit}$ is compact. By [38] IV.2.8,[45] ULAness of $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}(\mathrm{Loc}_{X^I}(V))$ and perversity (up to shift) imply that as a D-module, $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}(\mathrm{Loc}_{X^I}(V))$ is concentrated in one degree, and as such, it is middle extended from this disjoint locus. The same conclusion holds for $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},X^I}(\mathrm{Loc}_{X^I}(V))$ for the same reason.

Since the isomorphism above over $\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I} \times_{X^I} X^I_{disj}$ is compatible with factorization module structures, we deduce that the factorization module structures on $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}(\mathrm{Loc}_{X^I}(V))$ and $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},X^I}(\mathrm{Loc}_{X^I}(V))$ are compatible with the middle extension construction, and we obtain that these two are isomorphic as factorization modules for $\Upsilon_{\check{\mathfrak{n}}}$. □

Corollary 7.15.2. *The functor* $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}$ *is t-exact.*

Proof. For simplicity, we localize to assume that X is affine.

First, we claim that $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}$ is right t-exact.

Indeed, as in the proof of Proposition 6.34.1, $\mathsf{Rep}(\check{G})^{\leqslant 0}_{X^I}$ is generated under colimits by objects of the form $\mathrm{ind}\,\mathrm{Oblv}(\mathrm{Loc}_{X^I}(V)) = D_{X^I} \overset{!}{\otimes} \mathrm{Loc}_{X^I}(V)$ for $V \in \mathsf{Rep}(\check{G}^I)^{\leqslant |I|} = \mathsf{Rep}(\check{G})^{\otimes I,\leqslant |I|}$.

[45] Note that [38] only formulates its claim for complements to *smooth* Cartier divisors, since this reference only defines the ULA condition in this case. However, the claim from [38] is still true in this generality, as one sees by combining Beilinson's theory [5] and Corollary B.5.3.

The functor $D_{X^I} \overset{!}{\otimes} -$ is t-exact on $D(\mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I})$ (since after applying forgetful functors, it is given by tensoring with the ind-vector bundle that is the pullback of differential operators on X^I), and since:

$$\begin{aligned}&\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}(D_{X^I} \overset{!}{\otimes} \mathrm{Loc}_{X^I}(V))\\ &\quad = D_{X^I} \overset{!}{\otimes} \mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}(\mathrm{Loc}_{X^I}(V)) \overset{\text{Prop. 7.15.1}}{=} D_{X^I} \overset{!}{\otimes} \mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},X^I}(\mathrm{Loc}_{X^I}(V))\end{aligned}$$

we obtain the result from Corollary 7.11.2.

For left t-exactness: let $p : I \twoheadrightarrow J$ be given, and let i_p denote the corresponding locally closed embedding $X^J_{disj} \to X^I$. Note that the functors $i_p^! \mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}$ are left t-exact by factorization. Therefore, since $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}$ is filtered by the functors $i_{p,*,dR} i_p^! \mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}$, we obtain the claim. □

Warning 7.15.3. It is *not* clear at this point that the isomorphisms of Proposition 7.15.1 are compatible with restrictions to diagonals. Here we note that, as in the proof of [38], this question reduces to the abelian category, and here it becomes a concrete, yes-or-no question. The problem is that the isomorphism of Proposition 7.15.1 was based on middle extending from $X^I_{disj} \subseteq X^I$, and for $X^J \hookrightarrow X^I$, X^I_{disj} and X^J_{disj} do not speak to one another. We will deal with this problem in §7.22.

7.16. Factoring through $\mathsf{Rep}(\check{B})_{X^I}$

Next, we construct a functor:

$$'\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I} : \mathsf{Rep}(\check{G})_{X^I} \to \mathsf{Rep}(\check{B})_{X^I}$$

so that the composition:

$$\mathsf{Rep}(\check{G})_{X^I} \xrightarrow{'\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}} \mathsf{Rep}(\check{B})_{X^I} \to \check{\mathfrak{n}}_X\text{–}\mathsf{mod}_{X^I} \xrightarrow{\mathrm{ind}^{ch}} \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$$

identifies with $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}$.

Lemma 7.16.1. *The t-exact functor:*

$$\mathsf{Rep}(\check{B})_{X^I} \to \check{\mathfrak{n}}_X\text{–}\mathsf{mod}_{X^I} \xrightarrow{\mathrm{ind}^{ch}} \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$$

is fully-faithful on the hearts of the t-structures.

Proof. The functor $\mathsf{Rep}(\check{B})_{X^I} \to \check{\mathfrak{n}}_X\text{–}\mathsf{mod}_{X^I}$ is obviously fully-faithful (even at the derived level), as is clear by writing both categories as limits and using the fully-faithfulness of the functors $\mathsf{Rep}(\check{B})^{\otimes J} \to \check{\mathfrak{n}}\text{–}\mathsf{mod}(\mathsf{Rep}(\check{T})^{\otimes J}$.

So it remains to show that $\mathrm{ind}^{ch} : \check{\mathfrak{n}}_X\text{–}\mathsf{mod}_{X^I} \to \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$ is fully-faithful at the abelian categorical level.

This follows from the chiral PBW theorem, as in the proof of Proposition 7.11.1. Indeed, let Oblv^{ch} denote the right adjoint to ind^{ch}. Then for $M \in \check{\mathfrak{n}}_X\text{–}\mathsf{mod}^{\heartsuit}_{X^I}$, $\mathrm{Oblv}^{ch}\,\mathrm{ind}^{ch}(M)$ is filtered as a D-module with associated graded terms:

$$i^!\,\mathrm{add}_{n,*,dR}\Big(\underbrace{\check{\mathfrak{n}}_X[1]\boxtimes\ldots\boxtimes\check{\mathfrak{n}}_X[1]}_{n\text{ times}}\boxtimes i_{*,dR}(M)\Big)^{S_n} \in D(\mathrm{Gr}_{T,X^I}) \tag{7.16.1}$$

where add_n is the addition map:

$$(\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}})^n \times \mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I} \to \mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I}$$

and i is the embedding $\mathrm{Gr}_{T,X^I} \hookrightarrow \mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I}$. It suffices to show that H^0 of this term vanishes for $n \neq 0$.

Observe that we have a fiber square:

$$\begin{array}{ccc}
\underbrace{\mathrm{Gr}^{\mathrm{eff}}_{T,X^I}\underset{X^I}{\times}\ldots\underset{X^I}{\times}\mathrm{Gr}^{\mathrm{eff}}_{T,X^I}}_{n\text{ times}}\underset{X^I}{\times}\mathrm{Gr}_{T,X^I} & \longrightarrow & (\mathrm{Div}^{\check{\Lambda}^{pos}}_{\mathrm{eff}})^n \times \mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I} \\
\downarrow & & \downarrow{\scriptstyle \mathrm{add}_n} \\
\mathrm{Gr}_{T,X^I} & \xrightarrow{\ i\ } & \mathrm{Div}^{\check{\Lambda}^{pos},\infty\cdot x}_{\mathrm{eff},X^I}
\end{array}$$

where $\mathrm{Gr}^{\mathrm{eff}}_{T,X^I}$ is the locus of points in $X^I \times \mathrm{Div}_{\mathrm{eff}}$ of pairs $((x_i)_{i\in I}, D)$ so that D is zero when restricted to $X \setminus \{x_i\}$ (so the reduced fiber of $\mathrm{Gr}^{\mathrm{eff}}_{T,X^I}$ over a point $x \in X \xrightarrow{\Delta} X^I$ is the discrete scheme $\check{\Lambda}^{pos}$).

Let $\Gamma \subseteq X \times X^I$ be the incidence divisor, as in the proof of Lemma 7.7.1. For $\check{\lambda}$ given, we have a canonical map $\beta^{\check{\lambda}} : \Gamma \to \mathrm{Gr}^{\mathrm{eff}}_{T,X^I}$ over X^I, sending $(x,(x_i)^n_{i=1}) \in \Gamma$ to the divisor $\check{\lambda}\cdot x$. More generally, for every datum $(\check{\lambda}_r)^n_{r=1}$ with $\check{\lambda}_r \in \check{\Lambda}^{pos}$, we obtain a map:

$$\beta^{(\check{\lambda}_r)^n_{r=1}} : \Gamma \underset{X^I}{\times}\ldots\underset{X^I}{\times}\Gamma \to \mathrm{Gr}^{\mathrm{eff}}_{T,X^I}\underset{X^I}{\times}\ldots\underset{X^I}{\times}\mathrm{Gr}^{\mathrm{eff}}_{T,X^I}\,.$$

By base-change, the !-restriction of $\check{\mathfrak{n}}_X[1]\boxtimes\ldots\boxtimes\check{\mathfrak{n}}_X[1]\boxtimes i_{*,dR}(M)$ to $\mathrm{Gr}^{\mathrm{eff}}_{T,X^I}\underset{X^I}{\times}\ldots\underset{X^I}{\times}\mathrm{Gr}^{\mathrm{eff}}_{T,X^I}\underset{X^I}{\times}\mathrm{Gr}_{T,X^I}$ is the direct sum of terms:

$$\beta^{(\check{\alpha}_r)^n_{r=1}}_{*,dR}\Big(p_1^!\phi^!(\check{\mathfrak{n}}^{\check{\alpha}_1}\overset{!}{\otimes}k_X[1])\overset{!}{\otimes}\ldots\overset{!}{\otimes}p_n^!\phi^!(\check{\mathfrak{n}}^{\check{\alpha}_n}\overset{!}{\otimes}k_X[1])\overset{!}{\otimes}p_{n+1}^!(M)\Big),$$

where the p_i are the projections and φ is the map $\Gamma \to X$, and where the sum runs over all n-tuples $(\check{\alpha}_r)^n_{r=1}$ of positive coroots. Since $k_X[1] = \omega_X[-1]$, these terms are concentrated in cohomological degree $\geqslant n$, which gives the claim. □

Proposition 7.16.2. *The functor* $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}|_{\mathsf{Rep}(\check{G})^{\heartsuit}_{X^I}}$ *factors through* $\mathsf{Rep}(\check{B})^{\heartsuit}_{X^I} \subseteq \Upsilon_{\check{\mathfrak{n}}}\text{-}\mathsf{mod}^{\mathrm{fact}}_{un,X^I}$.

Proof. Since $\mathsf{Rep}(\check{G})^{\leqslant 0}_{X^I}$ is generated under colimits by objects of the form $\mathrm{ind} \times \mathrm{Oblv}(\mathrm{Loc}_{X^I}(V)) = D_{X^I} \overset{!}{\otimes} \mathrm{Loc}_{X^I}(V)$ for $V \in \mathsf{Rep}(\check{G}^I)^{\leqslant |I|} = \mathsf{Rep}(\check{G})^{\otimes I, \leqslant |I|}$, $\mathsf{Rep}(\check{G})^{\heartsuit}_{X^I}$ is generated under (for emphasis: possibly non-filtered) colimits by the top cohomologies of such objects, i.e., by objects of the form $D_{X^I} \overset{!}{\otimes} \mathrm{Loc}_{X^I}(V)$ for $V \in \mathsf{Rep}(\check{G})^{\otimes I}$ concentrated in degree $|I|$.

But we have seen that such objects map into $\mathsf{Rep}(\check{B})^{\heartsuit}_{X^I}$, giving the claim. □

We now obtain the desired functor ${}'\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}$ from Corollary 6.24.4, i.e., from the fact that $\mathsf{Rep}(\check{G})_{X^I}$ is the derived category of its heart. These functors factorize as one varies I.

7.17. Kernels

By Lemma 6.32.1, the functor ${}'\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}$ is defined by a kernel:

$$\mathcal{K}^{\mathrm{geom}}_{X^I} \in \mathsf{Rep}(\check{G} \times \check{B})_{X^I}.$$

Recall that the object of $\mathsf{Rep}(\check{B})_{X^I}$ underlying $\mathcal{K}^{\mathrm{geom}}_{X^I}$ is ${}'\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}(\mathcal{O}_{\check{G},X^I})$.

We recall that one recovers the functor ${}'\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}$ from the kernel $\mathcal{K}^{\mathrm{geom}}_{X^I}$ by the following construction. First, for $\mathcal{F} \in \mathsf{Rep}(\check{G})_{X^I}$, observe that $\mathcal{F} \overset{!}{\otimes} \mathcal{K}_{X^I} \in \mathsf{Rep}(\check{G} \times \check{G} \times \check{B})_{X^I}$. Then for each $p : I \twoheadrightarrow J$, we restrict to $U(p)$ and form invariants with respect to $\check{G}^J$, which acts diagonally through the embedding:

$$\check{G}^J \overset{\Delta_{\check{G}^J} \times 1}{\hookrightarrow} (\check{G} \times \check{G} \times \check{B})^J.$$

Let $\mathcal{K}^{\mathrm{spec}}_{X^I} \in \mathsf{Rep}(\check{G} \times \check{B})_{X^I}$ denote the kernel defining the tautological functor $\mathsf{Rep}(\check{G}) \to \mathsf{Rep}(\check{B})$, i.e., for each $p : I \twoheadrightarrow J$, $\mathcal{K}^{\mathrm{spec}}_{X^I}|_{U(p)}$ is given by the regular representation $\mathcal{O}_{\check{G}^J}$ considered as a $(\check{G}^J, \check{B}^J)$-bimodule by restriction from its $(\check{G}^J, \check{G}^J)$-bimodule structure (i.e., forgetting $\mathcal{K}^{\mathrm{spec}}_{X^I}$ down to $\mathsf{Rep}(\check{G})_{X^I}$, we recover $\mathcal{O}_{\check{G},X^I}$ from §6).

7.18. We have the following preliminary observations about these kernels.

Lemma 7.18.1. $\mathcal{K}^{\mathrm{geom}}_{X^I}$ *and* $\mathcal{K}^{\mathrm{spec}}_{X^I}$ *are concentrated in cohomological degree* $-|I|$ *in* $\mathsf{Rep}(\check{G} \times \check{B})_{X^I}$.

Proof. For $\mathcal{K}^{\mathrm{spec}}_{X^I}$, this follows from Lemma 6.24.3.

By construction, we recover $\mathcal{K}^{\mathrm{geom}}_{X^I}$ as an object of $\mathsf{Rep}(\check{B})_{X^I}$ by evaluating ${}'\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}$ on $\mathcal{O}_{\check{G},X^I}$. Since this object is concentrated in degree $-|I|$ by Lemma 6.24.3, we obtain the claim from t-exactness of $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}$. □

Proposition 7.18.2. *The group*[46] *of automorphisms of* $\mathcal{K}^{\mathrm{spec}}_{X^I}$ *restricting to the identity automorphism on* X^I_{disj} *is trivial.*

Proof. Note that the underlying object of Gpd underlying this group is a set by Lemma 7.18.1.

Then automorphisms of $\mathcal{K}^{\mathrm{spec}}_{X^I}$ inject into automorphisms of $\mathcal{O}_{\check{G},X^I} \in \mathsf{Rep}(\check{G})_{X^I}$, so it suffices to verify the claim here.

By adjunction, we have[47]:

$$\mathrm{Hom}_{\mathsf{Rep}(\check{G})_{X^I}}(\mathcal{O}_{\check{G},X^I}, \mathcal{O}_{\check{G},X^I}) = \mathrm{Hom}_{D(X^I)}(\mathcal{O}_{\check{G},X^I}, \omega_{X^I}).$$

Therefore, it suffices to show that:

$$\mathrm{Hom}_{D(X^I)}(\mathcal{O}_{\check{G},X^I}, \omega_{X^I}) \to \mathrm{Hom}_{D(X^I_{disj})}(j^!(\mathcal{O}_{\check{G},X^I}), \omega_{X^I_{disj}}) \tag{7.18.1}$$

is an injection, where j denotes the open embedding $X^I_{disj} \hookrightarrow X^I$.

Note that $j^!(\mathcal{O}_{\check{G},X^I}) \simeq j^!(\mathrm{Loc}_{X^I}(\mathcal{O}_{\check{G}}))$ is obviously ind-lisse, so $j_!$ is defined on it. Let i denote the closed embedding of the union of all diagonal divisors into X^I, so j is the complementary open embedding. We then have the long exact sequence:

$$0 \to \mathrm{Hom}(i_{*,dR}i^{*,dR}(\mathcal{O}_{\check{G},X^I}), \omega_{X^I}) \to \mathrm{Hom}(\mathcal{O}_{\check{G},X^I}, \omega_{X^I}) \to \mathrm{Hom}(j_!j^!(\mathcal{O}_{\check{G},X^I}), \omega_{X^I}) = \mathrm{Hom}(j^!(\mathcal{O}_{\check{G},X^I}), \omega_{X^I_{disj}}) \to \dots.$$

We can compute the first term as:

$$\mathrm{Hom}(i_{*,dR}i^{*,dR}(\mathcal{O}_{\check{G},X^I}), \omega_{X^I}) = \mathrm{Hom}(i^{*,dR}(\mathcal{O}_{\check{G},X^I}), i^!(\omega_{X^I}))$$

which we then see vanishes, since $i^{*,dR}(\mathcal{O}_{\check{G},X^I})$ is obviously concentrated in cohomological degrees $\leqslant -|I|$ (since $\mathcal{O}_{\check{G},X^I}$ is in degree $-|I|$), while $i^!(\omega_{X^I})$ is the dualizing sheaf of a variety of dimension $|I|-1$, and therefore is concentrated in cohomological degrees $\geqslant -|I|+1$. □

Remark 7.18.3. Note that by factorization and by the $|I| = 1$ case, we have an isomorphism between $\mathcal{K}^{\mathrm{geom}}_{X^I}$ and $\mathcal{K}^{\mathrm{spec}}_{X^I}$ over the disjoint locus. We deduce from Proposition 7.18.2 that there is at most one isomorphism extending this given isomorphism,

[46] Here by group, we mean a group object of Gpd.

[47] We emphasize here that Hom means the groupoid of maps, not the whole chain complex of maps. In particular, these Homs are actually sets, not more general groupoids.

or equivalently, there is at most one isomorphism between ${}'\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}$ and the functor of restriction of representations that extends the known isomorphism over the disjoint locus.

7.19. Commutative structure

The following discussion will play an important role in the sequel.

By factorization:

$$I \mapsto \mathcal{K}^{\mathrm{geom}}_{X^I} := {}'\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}(\mathcal{O}_{\check{G},X^I}) \in \mathsf{Rep}(\check{G}\times\check{B})_{X^I}$$

is a factorization algebra in a commutative factorization category.

Lemma 7.19.1. *$I \mapsto \mathcal{K}^{\mathrm{geom}}_{X^I}$ is a commutative factorization algebra.*

Remark 7.19.2. Since each term $\mathcal{K}^{\mathrm{geom}}_{X^I}$ is concentrated in cohomological degree $-|I|$, this factorization algebra is *classical*, i.e., of the kind considered in [6]. In particular, its commutativity is a property, not a structure.

Proof of Lemma 7.19.1. Let Ξ denote the functor:

$$\Xi : \mathsf{Rep}(\check{G}\times\check{B})_X \otimes \mathsf{Rep}(\check{G}\times\check{B})_X \to \mathsf{Rep}(\check{G}\times\check{B})_{X^2}.$$

By [6] §3.4, we only need to show that there is a map:

$$\Xi(\mathcal{K}^{\mathrm{geom}}_X \boxtimes \mathcal{K}^{\mathrm{geom}}_X) \to \mathcal{K}^{\mathrm{geom}}_{X^2} \in \mathsf{Rep}(\check{G}\times \mathsf{Rep}(\check{B})_{X^2} \tag{7.19.1}$$

extending the factorization isomorphism on $X^2 \setminus X$.

Let i denote diagonal embedding $X \hookrightarrow X^2$ and let j denote the complementary open embedding $X^2_{disj} \hookrightarrow X^2$.

Since $i^!(\mathcal{K}^{\mathrm{geom}}_{X^2}) = \mathcal{K}^{\mathrm{geom}}_X$ is in cohomological degree -1, we have a short exact sequence:

$$0 \to \mathcal{K}^{\mathrm{geom}}_{X^2} \to j_{*,dR}j^!(\mathcal{K}^{\mathrm{geom}}_{X^2}) \to i_{*,dR}(\mathcal{K}^{\mathrm{geom}}_X)[1] \to 0$$

in the shifted heart of the t-structure.

Therefore, the obstruction to a map (7.19.1) is the existence of a non-zero map:

$$\mathcal{K}^{\mathrm{geom}}_X \boxtimes \mathcal{K}^{\mathrm{geom}}_X \to i_{*,dR}(\mathcal{K}^{\mathrm{geom}}_X)[1].$$

We know (from the $I = *$ case of §7.10) that $\mathcal{K}^{\mathrm{geom}}_X = \mathrm{Loc}_X(\mathcal{O}_{\check{G}})$, so $\mathcal{K}^{\mathrm{geom}}_X \boxtimes \mathcal{K}_X$ is similarly localized. It follows that $i^{*,dR}(\mathcal{K}^{\mathrm{geom}}_X \boxtimes \mathcal{K}^{\mathrm{geom}}_X)$ is concentrated in cohomological degree -3, while $\mathcal{K}^{\mathrm{geom}}_X[1]$ is concentrated in cohomological degree -2, giving the claim. □

7.20. Lemma 7.19.1 endows $\mathcal{K}^{\mathrm{geom}}_X$ with the structure of commutative algebra object of $\mathsf{Rep}(\check{G}\times\check{B})_X$. Moreover, the Beilinson-Drinfeld theory [6] §3.4 implies that $\mathcal{K}^{\mathrm{geom}}_{X^I}$ can be recovered from $\mathcal{K}^{\mathrm{geom}}_X$ equipped with its commutative algebra structure.

Using Lemma 6.32.1, it follows that the factorization functor ${}'\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}}}$ is induced from a symmetric monoidal functor $F_1 : \mathsf{Rep}(\check{G}) \xrightarrow{\simeq} \mathsf{Rep}(\check{B})$.

Moreover, since $\mathcal{K}^{\mathrm{geom}}_X$ is isomorphic to $\mathcal{O}_{\check{G},X}$ in $\mathsf{Rep}(\check{G}\times\check{B})_X$, the underlying functor of this symmetric monoidal functor F_1 is isomorphic to the forgetful functor. It follows that F_1 factors through the full subcategory $\mathsf{Rep}(\check{G}) \subseteq \mathsf{Rep}(\check{B})$. In what follows, we denote the resulting symmetric monoidal functor $\mathsf{Rep}(\check{G}) \to \mathsf{Rep}(\check{G})$ by F.

Therefore, we obtain a symmetric monoidal functor $F : \mathsf{Rep}(\check{G}) \to \mathsf{Rep}(\check{G})$ inducing the factorization functor ${}'\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}}}$ by composing F with the restriction functor to $\mathsf{Rep}(\check{B})$ and applying the functoriality of the construction $\mathcal{C} \mapsto (I \mapsto \mathcal{C}_{X^I})$ from §6. Moreover, F is isomorphic to $\mathrm{id}_{\mathsf{Rep}(\check{G})}$ as a functor, so in particular, is a symmetric monoidal equivalence.

7.21. We claim that F is equivalent as a symmetric monoidal functor to the identity functor. Indeed, this follows from the next lemma.

Lemma 7.21.1. *Let $\widetilde{F} : \mathsf{Rep}(\check{G}) \to \mathsf{Rep}(\check{G})$ be a symmetric monoidal equivalence such that for every $\check{\lambda} \in \check{\Lambda}^+$, $\widetilde{F}(V^{\check{\lambda}})$ is equivalent to $V^{\check{\lambda}}$ in $\mathsf{Rep}(\check{G})$. Then $\widetilde{F}$ is (non-canonically) equivalent as a symmetric monoidal functor to the identity functor.*

Proof. By[48] the Tannakian formalism, $\widetilde{F}$ is given by restriction along an isomorphism $\varphi : \check{G} \xrightarrow{\simeq} \check{G}$. We need to show that φ is an inner automorphism. We now obtain the result, since the outer automorphism group of a reductive group is the automorphism group of its based root datum and since our assumption implies that the corresponding isomorphism is the identity on $\check{\Lambda}^+$, and therefore the identity for all of $\check{\Lambda}$. □

7.22. Trivializing the central gerbe

The above shows that there exists an isomorphism of the factorization functors $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}}}$ and $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}}}$.

However, the above technique is not strong enough yet to produce a particular isomorphism. Indeed, the isomorphism of Lemma 7.21.1 is non-canonical: the center $Z(\check{G})$ acts by automorphisms on the symmetric monoidal functor $\mathrm{id}_{\mathsf{Rep}(\check{G})}$.

Unwinding the above constructions, we see that factorizable isomorphisms between $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}}}$ and $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}}}$ form a trivial $Z(\check{G})$-gerbe.

[48] In fact, if k is not algebraically closed, there may be a non-trivial $\check{G}$-torsor on $\mathrm{Spec}(k)$ obstructing this argument. But one can a priori see that this torsor is trivial using the observations of §7.25. (Alternatively, one may assume k is algebraically closed in what follows, and then observe that the ultimate isomorphism we produce is manifestly Galois equivariant.)

In order to trivialize this gerbe, it suffices (by Proposition 7.18.2, cf. Remark 7.18.3) to show the following.

Proposition 7.22.1. *There exists a (necessarily unique) isomorphism of factorization functors* $\mathrm{Chev}_{\check{\mathfrak{n}}}^{\mathrm{geom}} \simeq \mathrm{Chev}_{\check{\mathfrak{n}}}^{\mathrm{spec}}$ *whose restriction to* X *is the one given by Lemma 7.10.1.*

Remark 7.22.2. Even when $Z(\check{G}) = *$, this assertion is not obvious: cf. Warning 7.15.3. Essentially, the difficulty is that the identity functor of $\mathsf{Rep}(\check{G})$ admits many automorphisms that are not tensor automorphisms.

7.23. We will deduce the above proposition using the following setup.

Lemma 7.23.1. *Suppose that we are given a symmetric monoidal functor* $F : \mathsf{Rep}(\check{G}) \to \mathsf{Rep}(\check{G})$ *such that* F *is (abstractly) isomorphic to the identity as a tensor functor, and such that we are given a fixed isomorphism:*

$$\alpha : \mathrm{Res}_{\check{T}}^{\check{G}} \circ F \simeq \mathrm{Res}_{\check{T}}^{\check{G}}$$

of symmetric monoidal functors $\mathsf{Rep}(\check{G}) \to \mathsf{Rep}(\check{T})$ *(*Res *indicates the restriction functor here).*

Then there exists an isomorphism of symmetric monoidal functors between F *and the identity functor on* $\mathsf{Rep}(\check{G})$ *inducing* α *if and only if, for every* $V \in \mathsf{Rep}(\check{G})^{\heartsuit}$ *irreducible, there exists an isomorphism* $\beta_V : F(V) \xrightarrow{\simeq} V \in \mathsf{Rep}(\check{G})$ *inducing the map:*

$$\alpha(V) : \mathrm{Res}_{\check{T}}^{\check{G}} F(V) \simeq \mathrm{Res}_{\check{T}}^{\check{G}}(V) \in \mathsf{Rep}(\check{T})$$

upon application of $\mathrm{Res}_{\check{T}}^{\check{G}}$.

Moreover, a symmetric monoidal isomorphism between F *and the identity compatible with* α *is unique if it exists. At the level of objects, it is given by the maps* β_V.

Remark 7.23.2. In words: an isomorphism α as above may not be compatible with any tensor isomorphism between F and the identity. Indeed, consider the case where $\check{G}$ is adjoint, so that a tensor isomorphism between F and the identity is unique if it exists, while there are many choices for α as above. However, if this isomorphism exists, it is unique. Moreover, the lemma provides an objectwise criterion to test whether or not such an isomorphism exists.

Proof. Choose some isomorphism γ between F and the identity functor (of symmetric monoidal functors). From α, we obtain a symmetric monoidal automorphism of $\mathrm{Res}_{\check{T}}^{\check{G}}$. By Tannakian theory, this is given by the action of some $t \in \check{T}(k)$.

Since the symmetric monoidal automorphism group of the identity functor of $\mathsf{Rep}(\check{G})$ is the center of this group, it suffices to show that t lies in the center of $\check{G}$. (Moreover, we immediately deduce the uniqueness from this observation).

To this end, it suffices to show that t acts by a scalar on every irreducible representation on $\check{G}$. But by Schur's lemma, this follows from our hypothesis. □

7.24. We now indicate how to apply Lemma 7.23.1 in our setup.

7.25. First, we give factorizable identifications of the composite functors:

$$\mathsf{Rep}(\check{G})_{X^I} \xrightarrow{'\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}} \mathsf{Rep}(\check{B})_{X^I} \to \mathsf{Rep}(\check{T})_{X^I}$$

with the functors induced from $\mathrm{Res}^{\check{G}}_{\check{T}}$.

Indeed, we have done this implicitly already in the proof of Proposition 7.14.1: one rewrites the functors $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},X^I}$ using (the appropriate generalization of) Lemma 5.15.1, and then uses the (factorizable[49] form of the) Mirkovic-Vilonen identification of restriction as cohomology along semi-infinite orbits.

7.26. Now suppose that $V \in \mathsf{Rep}(\check{G})^{\heartsuit}$ is irreducible.

Then for $x \in X$, Theorem 5.14.1 produces a certain isomorphism between $\mathrm{Chev}^{\mathrm{geom}}_{\check{\mathfrak{n}},x}(V)$ and $\mathrm{Chev}^{\mathrm{spec}}_{\check{\mathfrak{n}},x}(V)$ in $\mathsf{Rep}(\check{B})^{\heartsuit} \subseteq \Upsilon_{\check{\mathfrak{n}}}\text{–}\mathsf{mod}^{\mathrm{fact},\heartsuit}_{un,x}$.

To check that the conditions of Lemma 7.23.1 are satisfied, it suffices to show that this isomorphism induces the isomorphism of §7.25 when we map to $\mathsf{Rep}(\check{T})$.

Fort this, recall that the isomorphism of Theorem 5.14.1 was constructed using a related isomorphism from [12] Theorem 8.8. The isomorphism of [12] has the property above, as is noted in [12]. Since the construction in Theorem 5.14.1 for reducing to the setting of [12] is compatible with further restriction to $\mathsf{Rep}(\check{T})$, we obtain the claim.

Appendix A. Proof of Lemma 6.18.1

A.1. Suppose that we have a diagram $i \mapsto \mathcal{C}_i \in \mathsf{DGCat}_{cont}$ of categories with each $\mathcal{C}_i$ dualizable with dual $\mathcal{C}_i^\vee$ in the sense of [24].

In this case, we can form the dual diagram $i \mapsto \mathcal{C}_i^\vee$.

We can ask: when is $\mathcal{C} := \lim_{i \in \mathcal{I}^{op}} \mathcal{C}_i$ dualizable with dual $\mathrm{colim}_{i \in \mathcal{I}} \mathcal{C}_i^\vee$? More precisely, there is a canonical Vect valued pairing between the limit and colimit here, and we can ask when it realizes the two categories as mutually dual.

As in [24], we recall that this occurs if and only if $\mathrm{colim}_{i \in \mathcal{I}} \mathcal{C}_i^\vee$ is dualizable, which occurs if and only if, for every $\mathcal{D} \in \mathsf{DGCat}_{cont}$, the canonical map:

$$\Big(\lim_{i \in \mathcal{I}^{op}} \mathcal{C}_i\Big) \otimes \mathcal{D} \to \lim_{i \in \mathcal{I}^{op}} \big(\mathcal{C}_i \otimes \mathcal{D}\big)$$

is an equivalence.

[49] This generalization is straightforward given the Mirkovic-Vilonen theory and the methods of this section and §6.

This section gives a criterion, Lemma A.2.1, in which this occurs, and which we will use to deduce Lemma 6.18.1 in §A.3

A.2. A dualizability condition

Suppose we have a diagram:

$$\begin{array}{ccc} & & \mathcal{C}_2 \\ & & \downarrow{\scriptstyle \psi} \\ \mathcal{C}_1 & \xrightarrow{F} & \mathcal{C}_3 \end{array}$$

of dualizable categories. Let $\mathcal{C}$ denote the fiber product of this diagram.

The main result of this section is the following.

Lemma A.2.1. *Suppose that ψ and F have right adjoints φ and G respectively. Suppose in addition that G is fully-faithful.*

Then if each $\mathcal{C}_i$ is dualizable, $\mathcal{C}$ is dualizable as well. Moreover, for each $\mathcal{D} \in \mathsf{DGCat}_{cont}$, the canonical map:

$$\mathcal{C} \otimes \mathcal{D} \to \mathcal{C}_1 \otimes \mathcal{D} \underset{\mathcal{C}_3 \otimes \mathcal{D}}{\times} \mathcal{C}_2 \otimes \mathcal{D} \tag{A.2.1}$$

is an equivalence.

The proof of this lemma is given in §A.7.

A.3. Proof of Lemma 6.18.1

We now explain how to deduce Lemma 6.18.1.

Proof that Lemma A.2.1 implies Lemma 6.18.1. Fix I a finite set. We proceed by induction on $|I|$, the case $|I| = 1$ being obvious.

Recall that we have $\mathcal{C} \in \mathsf{DGCat}_{cont}$ rigid and symmetric monoidal, and X a smooth curve.

By 1-affineness of X^I_{dR} and X^I (cf. [25]), we easily reduce to checking the corresponding fact in the quasi-coherent setting. Note that by rigidity of $\mathsf{QCoh}(X^I)$, dualizability questions in $\mathsf{QCoh}(X^I)$–mod are equivalent to dualizability questions in DGCat_{cont}.

Let $U \subseteq X^I$ be the complement of the diagonally embedded $X \hookrightarrow X^I$. We can then express $\mathcal{C}_{X^I}$ as a fiber product:

$$\begin{array}{ccc} \mathsf{QCoh}(X^I, \mathcal{C}_{X^I}) & \longrightarrow & \mathsf{QCoh}(X^I, \mathcal{C}_{X^I}) \underset{\mathsf{QCoh}(X^I)}{\otimes} \mathsf{QCoh}(U) \\ \downarrow & & \downarrow \\ \mathsf{QCoh}(X^I) \otimes \mathcal{C} & \longrightarrow & \mathsf{QCoh}(U) \otimes \mathcal{C}. \end{array}$$

The two structure functors involved in defining this pullback admit continuous right adjoints, and the right adjoint to the bottom functor is fully-faithful. Moreover, the bottom two terms are obviously dualizable. Therefore, by Lemma A.2.1, it suffices to see that formation of the limit involved in defining the top right term commutes with tensor products over $\mathsf{QCoh}(U)$.

Note that U is covered by the open subsets $U(p)$ for $p : I \twoheadrightarrow J$ with $|J| > 1$. By Zariski descent for sheaves of categories, it suffices to check the commutation of tensor products and limits after restriction to each $U(p)$. But this follows from factorization and induction, using the same cofinality result as in §6.10. □

A.4. The remainder of this section is devoted to the proof of Lemma A.2.1.

A.5. Gluing

Define the *glued* category Glue to consist of the triples $(\mathcal{F}, \mathcal{G}, \eta)$ where $\mathcal{F} \in \mathcal{C}_1$, $\mathcal{G} \in \mathcal{C}_2$, and η is a morphism $\eta : \psi(\mathcal{G}) \to F(\mathcal{F}) \in \mathcal{C}_3$.

Note that the limit $\mathcal{C} := \mathcal{C}_1 \times_{\mathcal{C}_3} \mathcal{C}_2$ is a full subcategory of Glue.

Lemma A.5.1. *The functor $\mathcal{C} \hookrightarrow \mathsf{Glue}$ admits a continuous right adjoint.*

Proof. We construct this right adjoint explicitly:

For $(\mathcal{F}, \mathcal{G}, \eta)$ as above, define $\widetilde{\mathcal{F}} \in \mathcal{C}_1$ as the fiber product:

$$\begin{array}{ccc} \widetilde{\mathcal{F}} & \longrightarrow & G\psi(\mathcal{G}) \\ \downarrow & & \downarrow{\scriptstyle G(\eta)} \\ \mathcal{F} & \longrightarrow & GF(\mathcal{F}). \end{array}$$

Since G is fully-faithful, the map $\varepsilon : F(\widetilde{\mathcal{F}}) \to FG\psi(\mathcal{G}) \simeq \psi(\mathcal{G})$ is an isomorphism, and therefore $(\widetilde{\mathcal{F}}, \mathcal{G}, \varepsilon)$ defines an object of $\mathcal{C}$. It is easy to see that the resulting functor is the desired right adjoint. □

A.6. Let $\mathcal{D} \in \mathsf{DGCat}_{cont}$ be given.

Define $\mathsf{Glue}_{\mathcal{D}}$ as with Glue, but instead use the diagram:

$$\begin{array}{ccc} & & \mathcal{C}_2 \otimes \mathcal{D} \\ & & \downarrow{\scriptstyle \psi \otimes \mathrm{id}_{\mathcal{D}}} \\ \mathcal{C}_1 \otimes \mathcal{D} & \xrightarrow{F \otimes \mathrm{id}_{\mathcal{D}}} & \mathcal{C}_3 \otimes \mathcal{D} \end{array}$$

Lemma A.6.1. *The canonical functor:*

$$\mathsf{Glue} \otimes \mathcal{D} \to \mathsf{Glue}_{\mathcal{D}}$$

is an equivalence.

Proof. First, we give a description of functors $\mathsf{Glue} \to \mathcal{E} \in \mathsf{DGCat}_{cont}$ for a test object $\mathcal{E}$:

We claim that such a functor is equivalent to the datum of a pair $\xi_0 : \mathcal{C}_1 \to \mathcal{E}$ and $\xi_1 : \mathcal{C}_2 \to \mathcal{E}$ of continuous functors, plus a natural transformation:

$$\xi_1 \varphi F \to \xi_0$$

of functors $\mathcal{C}_1 \to \mathcal{E}$.

Indeed, given a functor $\Xi : \mathsf{Glue} \to \mathcal{E}$ as above, we obtain such a datum as follows: for $\mathcal{F} \in \mathcal{C}_1$, we let $\xi_0(\mathcal{F}) := \Xi(\mathcal{F}[-1], 0, 0)$, for $\mathcal{G} \in \mathcal{C}_2$ we let $\xi_1(\mathcal{G}) := \Xi(0, \mathcal{G}, 0)$ (here we write objects of Glue as triples as above). The natural transformation comes from the boundary morphism for the exact triangle Glue:

$$(\mathcal{F}, 0, 0) \to (\mathcal{F}, \varphi F(\mathcal{F}), \eta_{\mathcal{F}}) \to (0, \varphi F(\mathcal{F}), 0) \xrightarrow{+1}$$

where $\eta_{\mathcal{F}}$ is the adjunction map $\psi \varphi F(\mathcal{F}) \to F(\mathcal{F})$. It is straightforward to see that this construction is an equivalence.

This universal property then makes the above property clear. □

A.7. We now deduce the lemma.

Proof of Lemma A.2.1. We need to see that for every $\mathcal{D} \in \mathsf{DGCat}_{cont}$, the map (A.2.1) is an equivalence.

First, observe that each of these categories is a full subcategory of $\mathsf{Glue}_{\mathcal{D}}$. Indeed, for the left hand side of (A.2.1), this follows from Lemma A.5.1, and for the right hand side, this follows from Lemma A.6.1. Moreover, this is compatible with the above functor by construction.

Let L denote the right adjoint to $i : \mathcal{C} \hookrightarrow \mathsf{Glue}$, and let $L_{\mathcal{D}}$ denote the right adjoint to the embedding:

$$i_{\mathcal{D}} : \mathcal{C}_1 \otimes \mathcal{D} \underset{\mathcal{C}_3 \otimes \mathcal{D}}{\times} \mathcal{C}_2 \otimes \mathcal{D} \hookrightarrow \mathsf{Glue}_{\mathcal{D}}.$$

We need to show that:

$$(i \circ L) \otimes \mathrm{id}_{\mathcal{D}} = i_{\mathcal{D}} \circ L_{\mathcal{D}}$$

as endofunctors of $\mathsf{Glue}_{\mathcal{D}}$, since the image of the left hand side is the left hand side of (A.2.1), and the image of the right hand side is the right hand side of (A.2.1).

But writing $\mathsf{Glue}_{\mathcal{D}}$ as $\mathsf{Glue} \otimes \mathcal{D}$, this becomes clear. □

Appendix B. Universal local acyclicity

B.1. Notation

Let S be a scheme of finite type and let $\mathcal{C}$ be a $D(S)$-module category in DGCat_{cont}. Let $\mathsf{QCoh}(S, \mathcal{C})$ denote the category $\mathcal{C} \otimes_{D(S)} \mathsf{QCoh}(S)$.

Remark B.1.1. Everything in this section works with S a general DG scheme almost of finite type. The reader comfortable with derived algebraic geometry may therefore happily understand *scheme* in the derived sense everywhere here.

B.2. The adjoint functors[50]:

$$\mathsf{QCoh}(S) \underset{\text{Oblv}}{\overset{\text{ind}}{\rightleftarrows}} D(S)$$

induce adjoint functors:

$$\mathsf{QCoh}(S, \mathcal{C}) \underset{\text{Oblv}}{\overset{\text{ind}}{\rightleftarrows}} \mathcal{C}.$$

Lemma B.2.1. *The functor* $\mathrm{Oblv} : \mathcal{C} \to \mathsf{QCoh}(S, \mathcal{C})$ *is conservative.*

Proof. This is shown in [27] in the case $\mathcal{C} = D(S)$.

In the general case, it suffices to show that $\mathrm{ind} : \mathsf{QCoh}(S, \mathcal{C}) \to \mathcal{C}$ generates the target under colimits. It suffices to show that the functor:

$$\mathsf{QCoh}(S) \otimes \mathcal{C} \to D(S) \otimes \mathcal{C} \to D(S) \underset{D(S)}{\otimes} \mathcal{C}$$

generates, as it factors through ind. But the first term generates by the [27] result, and the second term obviously generates. □

[50] Throughout this section, we use only the "left" forgetful and induction functors from [27].

B.3. Universal local acyclicity

We have the following notion.

Definition B.3.1. $\mathcal{F} \in \mathcal{C}$ is *universally locally acyclic (ULA)* over S if $\mathrm{Oblv}(\mathcal{F}) \in \mathsf{QCoh}(S, \mathcal{C})$ is compact.

Notation B.3.2. We let $\mathcal{C}^{ULA} \subseteq \mathcal{C}$ denote the full (non-cocomplete) subcategory of ULA objects.

B.4. We have the following basic consequences of the definition.

Proposition B.4.1. *For every $\mathcal{F} \in \mathcal{C}^{ULA}$ and for every compact $\mathcal{G} \in D(S)$, $\mathcal{G} \overset{!}{\otimes} \mathcal{F}$ is compact in $\mathcal{C}$.*

Proof. Since $\mathrm{ind} : \mathsf{QCoh}(S) \to D(S)$ generates the target, objects of the form $\mathrm{ind}(\mathcal{P}) \in D(S)$ for $\mathcal{P} \in \mathsf{QCoh}(S)$ perfect generate the compact objects in the target under finite colimits and direct summands.

Therefore, it suffices to see that $\mathrm{ind}(\mathcal{P}) \overset{!}{\otimes} \mathcal{F}$ is compact for every perfect $\mathcal{P} \in \mathsf{QCoh}(S)$.

To this end, it suffices to show:

$$\mathrm{ind}(\mathcal{P} \otimes \mathrm{Oblv}(\mathcal{F})) \xrightarrow{\simeq} \mathrm{ind}(\mathcal{P}) \overset{!}{\otimes} \mathcal{F} \tag{B.4.1}$$

since the left hand side is obviously compact by the ULA condition on $\mathcal{F}$. We have an obvious map from the left hand side to the right hand side. To show it is an isomorphism, we localize to assume S is affine, and then by continuity this allows us to check the claim when $\mathcal{P} = \mathcal{O}_S$. Then the claim follows because ind and Oblv are $D(S)$-linear functors. □

Corollary B.4.2. *Any $\mathcal{F} \in \mathcal{C}^{ULA}$ is compact in $\mathcal{C}$.*

Example B.4.3. Suppose that S is smooth and $\mathcal{C} = D(S)$. Then $\mathcal{F}$ is ULA if and only if $\mathcal{F}$ is compact with lisse cohomologies. Indeed, if $\mathcal{F}$ is ULA, the cohomologies of $\mathrm{Oblv}(\mathcal{F}) \in \mathsf{QCoh}(S)$ are coherent sheaves and therefore the cohomologies of $\mathcal{F}$ are lisse.

Proposition B.4.4. *Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a morphism in $D(S)$–mod with a $D(S)$-linear right adjoint G. Then F maps ULA objects to ULA objects.*

Proof. We have the commutative diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow{\scriptstyle \mathrm{Oblv}} & & \downarrow{\scriptstyle \mathrm{Oblv}} \\ \mathsf{QCoh}(S, \mathcal{C}) & \longrightarrow & \mathsf{QCoh}(S, \mathcal{D}) \end{array}$$

and the functor $\mathsf{QCoh}(S, \mathcal{C}) \to \mathsf{QCoh}(S, \mathcal{D})$ preserves compact objects by assumption on F. $\square$

B.5. Reformulations

For $\mathcal{F} \in \mathcal{C}$, let $\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, -) : \mathcal{C} \to D(S)$ denote the (possibly non-continuous) functor right adjoint to $D(S) \to \mathcal{C}$ given by tensoring with $\mathcal{F}$.

Proposition B.5.1. *For $\mathcal{F} \in \mathcal{C}$, the following conditions are equivalent.*

(1) $\mathcal{F}$ is ULA.
(2) $\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, -) : \mathcal{C} \to D(S)$ is continuous and $D(S)$-linear.
(3) For every $\mathcal{M} \in D(S)$–mod and every $M \in \mathcal{M}$ compact, the induced object:

$$\mathcal{F} \underset{D(S)}{\boxtimes} M \in \mathcal{C} \underset{D(S)}{\otimes} \mathcal{M}$$

is compact.

Proof. First, we show (1) implies (2).

By Proposition B.4.1, the functor $D(S) \to \mathcal{C}$ of tensoring with $\mathcal{F}$ preserves compact objects, so its right adjoint is continuous. We need to show that $\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, -)$ is $D(S)$-linear.

Observe first that $\mathrm{Oblv}\, \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, -)$ computes[51] $\underline{\mathrm{Hom}}_{\mathsf{QCoh}(S,\mathcal{C})}(\mathrm{Oblv}(\mathcal{F}), \mathrm{Oblv}(-)) : \mathcal{C} \to \mathsf{QCoh}(S)$. Indeed, both are right adjoints to $(- \overset{!}{\otimes} \mathcal{F}) \circ \mathrm{ind} = \mathrm{ind} \circ (- \otimes \mathrm{Oblv}(\mathcal{F}))$, where we have identified these functors by (B.4.1).

Then observe that:

$$\underline{\mathrm{Hom}}_{\mathsf{QCoh}(S,\mathcal{C})}(\mathrm{Oblv}(\mathcal{F}), -) : \mathsf{QCoh}(S, \mathcal{C}) \to \mathsf{QCoh}(S)$$

is a morphism of $\mathsf{QCoh}(S)$-module categories: this follows from rigidity of $\mathsf{QCoh}(S)$. This now easily gives the claim since Oblv is conservative.

Next, we show that (2) implies (3).

Let $\mathcal{M}$ and $M \in \mathcal{M}$ be as given. The composite functor:

$$\mathsf{Vect} \xrightarrow{- \otimes M} \mathcal{M} = D(S) \underset{D(S)}{\otimes} \mathcal{M} \xrightarrow{(- \overset{!}{\otimes} \mathcal{F}) \otimes \mathrm{id}_{\mathcal{M}}} \mathcal{C} \underset{D(S)}{\otimes} \mathcal{M}$$

obviously sends $k \in \mathsf{Vect}$ to $\mathcal{F} \underset{D(S)}{\boxtimes} \mathcal{M}$. But this composite functor also obviously admits a continuous right adjoint: the first functor does because M is compact, and the second functor does because $D(S) \to \mathcal{C}$ admits a $D(S)$-linear right adjoint by assumption.

[51] The notation indicates internal Hom for $\mathsf{QCoh}(S, \mathcal{C})$ considered as a $\mathsf{QCoh}(S)$-module category.

It remains to show that (3) implies (1), but this is tautological: take $\mathcal{M} = \mathsf{QCoh}(S)$. □

Remark B.5.2. Note that conditions (2) and (3) make sense for any algebra $\mathcal{A} \in \mathsf{DGCat}_{cont}$ replacing $D(S)$ and any $\mathcal{F} \in \mathcal{C}$ a right $\mathcal{A}$-module category in DGCat_{cont}. That (2) implies (3) holds in this generality follows by the same argument.

Here is a sample application of this perspective.

Corollary B.5.3. *For $\mathcal{G} \in D(U)$ holonomic and $\mathcal{F} \in \mathcal{C}^{ULA}$, $j_!(\mathcal{G} \overset{!}{\otimes} j^!(\mathcal{F})) \in \mathcal{C}$ is defined, and the natural map:*

$$j_!(\mathcal{G} \overset{!}{\otimes} j^!(\mathcal{F})) \to j_!(\mathcal{G}) \overset{!}{\otimes} \mathcal{F}$$

is an isomorphism. In particular, $j_!(\mathcal{F})$ is defined.

Proof. We begin by showing that there is an isomorphism:

$$j^!(\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, -)) \simeq \underline{\mathrm{Hom}}_{\mathcal{C}_U}(j^!(\mathcal{F}), j^!(-))$$

as functors $\mathcal{C} \to D(U)$. Indeed, we have:

$$j_{*,dR} j^!(\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, -)) = j_{*,dR}(\omega_U) \overset{!}{\otimes} \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, -) = \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, j_{*,dR}(\omega_U) \overset{!}{\otimes} (-))$$

and the right hand side obviously identifies with $j_{*,dR}\underline{\mathrm{Hom}}_{\mathcal{C}_U}(j^!(\mathcal{F}), j^!(-))$.

Now for any $\widetilde{\mathcal{F}} \in \mathcal{C}$, we see:

$$\mathrm{Hom}_{\mathcal{C}}(j_!(\mathcal{G}) \overset{!}{\otimes} \mathcal{F}, \widetilde{\mathcal{F}}) = \mathrm{Hom}_{D(S)}(j_!(\mathcal{G}), \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \widetilde{\mathcal{F}})) = \mathrm{Hom}_{D(U)}(\mathcal{G}, j^!\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \widetilde{\mathcal{F}})) =$$

$$\mathrm{Hom}_{D(U)}(\mathcal{G}, \underline{\mathrm{Hom}}_{\mathcal{C}}(j^!(\mathcal{F}), j^!(\widetilde{\mathcal{F}}))) = \mathrm{Hom}_{\mathcal{C}_U}(\mathcal{G} \overset{!}{\otimes} j^!(\mathcal{F}), j^!(\widetilde{\mathcal{F}}))$$

as desired. □

B.6. We now discuss a ULA condition for $D(S)$-module categories themselves.

Definition B.6.1. $\mathcal{C}$ as above is *ULA* over S if $\mathsf{QCoh}(S, \mathcal{C})$ is compactly generated by objects of the form $\mathcal{P} \otimes \mathrm{Oblv}(\mathcal{F})$ with $\mathcal{F} \in \mathcal{C}^{ULA}$ and $\mathcal{P} \in \mathsf{QCoh}(S)$ perfect.

Example B.6.2. $D(S)$ is ULA. Indeed, ω_S is ULA with $\mathrm{Oblv}(\omega_S) = \mathcal{O}_S$.

Lemma B.6.3. *If $\mathcal{C}$ is ULA, then $\mathcal{C}$ is compactly generated.*

Proof. Immediate from conservativity of Oblv. □

B.7. In this setting, we have the following converse to Proposition B.4.4.

Proposition B.7.1. *For $\mathcal{C}$ ULA, a $D(S)$-linear functor $F : \mathcal{C} \to \mathcal{D}$ admits a $D(S)$-linear right adjoint if and only if F preserves ULA objects.*

Proof. We have already seen one direction in Proposition B.4.4. For the converse, suppose F preserves ULA objects.

Since $\mathcal{C}$ is compactly generated and F preserves compact objects, F admits a continuous right adjoint G.

We will check linearity using Proposition B.5.1:

Suppose that $\mathcal{F} \in D(S)$. We want to show that the natural transformation:

$$\mathcal{F} \overset{!}{\otimes} G(-) \to G(\mathcal{F} \overset{!}{\otimes} -)$$

of functors $\mathcal{D} \to \mathcal{C}$ is an equivalence.

It is easy to see that it is enough to show that for any $\mathcal{G} \in \mathcal{C}^{ULA}$, the natural transformation of functors $\mathcal{D} \to D(S)$ induced by applying $\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{G}, -)$ is an equivalence.

But this follows from the simple identity $\underline{\mathrm{Hom}}_{\mathcal{D}}(F(\mathcal{G}), -) = \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{G}, G(-))$. Indeed, we see:

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{G}, \mathcal{F} \overset{!}{\otimes} G(-)) = \mathcal{F} \overset{!}{\otimes} \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{G}, G(-)) = \mathcal{F} \overset{!}{\otimes} \underline{\mathrm{Hom}}_{\mathcal{D}}(F(\mathcal{G}), (-)) =$$

$$\underline{\mathrm{Hom}}_{\mathcal{D}}(F(\mathcal{G}), \mathcal{F} \overset{!}{\otimes} (-)) = \underline{\mathrm{Hom}}_{\mathcal{D}}\big(\mathcal{G}, G(\mathcal{F} \overset{!}{\otimes} (-))\big)$$

as desired. □

B.8. Suppose that $i : T \hookrightarrow S$ is closed with complement $j : U \hookrightarrow S$.

Given a $D(S)$-module category $\mathcal{C}$, define:

$$\mathcal{C}_U := \mathcal{C} \underset{D(S)}{\otimes} D(U), \quad \mathcal{C}_T := \mathcal{C} \underset{D(S)}{\otimes} D(T).$$

Similarly, given a morphism $F : \mathcal{C} \to \mathcal{D}$ in $D(S)$–mod, let $F_U : \mathcal{C}_U \to \mathcal{D}_U$ and $F_T : \mathcal{C}_T \to \mathcal{D}_T$ be the induced functors.

Proposition B.8.1. *Suppose $F : \mathcal{C} \to \mathcal{D}$ is a morphism in $D(S)$–mod with $\mathcal{C}$ ULA as a $D(S)$-module category. Then F is an equivalence if and only if it preserves ULA objects and the functors F_U and F_T are equivalences.*

Remark B.8.2. Note that a result of this form is not true without ULA hypotheses: the restriction functor $D(S) \to D(U) \oplus D(T)$ is $D(S)$-linear and an equivalence over T and over U, but not an equivalence.

Proof of Proposition B.8.1. By Proposition B.7.1, the functor F admits a $D(S)$-linear right adjoint G. We need to check that the unit and counit of this adjunction are equivalences.

By the usual Cousin dévissage, we reduce to checking that the unit and counit are equivalences for objects pushed forward from U and T. But by $D(S)$-linearity of our functors, this follows from our assumption. □

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