# OPTIMAL BOUNDS FOR ANCIENT CALORIC FUNCTIONS

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**Abstract**

*For any manifold with polynomial volume growth, we show that the dimension of the space of ancient caloric functions with polynomial growth is bounded by the degree of growth times the dimension of harmonic functions with the same growth. As a consequence, we get a sharp bound for the dimension of ancient caloric functions on any space where Yau's 1974 conjecture about polynomial growth harmonic functions holds.*

## 0. Introduction

Given a complete manifold $M$ and a constant $d$, $\mathcal{H}_d(M)$ is the linear space of harmonic functions of polynomial growth at most $d$. Namely, $u \in \mathcal{H}_d(M)$ if $\Delta u = 0$, and for some $p \in M$ and a constant $C_u$ depending on $u$,

$$\sup_{B_R(p)} |u| \leq C_u(1+R)^d \quad \text{for all } R. \tag{0.1}$$

In 1974, S. T. Yau conjectured that $\mathcal{H}_d(M)$ is finite-dimensional for each $d$ when $\mathrm{Ric}_M \geq 0$. The conjecture was settled in [6]; see [5]–[9] and [11] for more results.[1] In fact, [6]–[8] proved finite dimensionality under much weaker assumptions of

(1) a volume doubling bound;

(2) a scale-invariant Poincaré inequality or mean value inequality.

The natural parabolic generalization is a polynomial growth ancient solution of the heat equation. A solution of the heat equation is often called a *caloric function*. Ancient solutions are ones that are defined for all negative $t$—these are the solutions

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[1]For Yau's 1974 conjecture, see [34, p. 117], [35, Problem 48], [28, Conjecture 2.5], [13], [17], [18], [21, Conjecture 1], and [22, Problem (1)], among others.

that arise in a blowup analysis. Given $d > 0$, $u \in \mathcal{P}_d(M)$ if $u$ is ancient, $\partial_t u = \Delta u$, and for some $p \in M$ and a constant $C_u$,

$$\sup_{B_R(p)\times[-R^2,0]} |u| \leq C_u(1+R)^d \quad \text{for all } R. \tag{0.2}$$

On $\mathbf{R}^n$, $\mathcal{P}_d$ is the classical space of caloric polynomials that generalize the Hermite polynomials (see [14], [15], [27]). More generally, the spaces $\mathcal{P}_d(M)$ play a fundamental role in geometric flows (see [10]–[12]). They were studied by Calle in her 2006 thesis (see [1], [2]), in the context of mean curvature flow.

A manifold has polynomial volume growth if there are constants $C$ and $d_V$ so that $\mathrm{Vol}(B_R(p)) \leq C(1+R)^{d_V}$ for some $p \in M$, all $R > 0$.[2] Our main result is the following sharp inequality.

THEOREM 0.3
*If $M$ has polynomial volume growth and $k$ is a nonnegative integer, then*

$$\dim \mathcal{P}_{2k}(M) \leq \sum_{i=0}^{k} \dim \mathcal{H}_{2i}(M). \tag{0.4}$$

The inequality (0.4) is an equality on $\mathbf{R}^n$ (see Corollary 2.18 below). Since $\mathcal{H}_{d_1} \subset \mathcal{H}_{d_2}$ for $d_1 \leq d_2$, Theorem 0.3 implies the following.

COROLLARY 0.5
*If $M$ has polynomial volume growth, then for all $k \geq 1$,*

$$\dim \mathcal{P}_{2k}(M) \leq (k+1) \dim \mathcal{H}_{2k}(M). \tag{0.6}$$

Combining this with the bound $\dim \mathcal{H}_d(M) \leq C d^{n-1}$ when $\mathrm{Ric}_{M^n} \geq 0$ from [7] gives the following corollary.

COROLLARY 0.7
*There exists $C = C(n)$ so that if $\mathrm{Ric}_{M^n} \geq 0$, then for $d \geq 1$,*

$$\dim \mathcal{P}_d(M) \leq C d^n. \tag{0.8}$$

The exponent $n$ in (0.8) is sharp: there is a constant $c$ depending on $n$ so that for $d \geq 1$,

$$c^{-1} d^n \leq \dim \mathcal{P}_d(\mathbf{R}^n) \leq c d^n. \tag{0.9}$$

[2] A volume doubling space with doubling constant $C_D$ has polynomial volume growth of degree $\log_2 C_D$.

Recently, Lin and Zhang [24] proved very interesting related results, adapting the methods of [6]–[8] to get the bound $d^{n+1}$.

Using parabolic gradient estimates of Li and Yau [23] and Souplet and Zhang [30], one can show that if $d < 2$ and $\mathrm{Ric} \geq 0$, then $\mathcal{P}_d(M) = \mathcal{H}_d(M)$ consists only of harmonic functions of polynomial growth. In particular, $\mathcal{P}_d(M) = \{\text{Constant functions}\}$ for $d < 1$, by Yau [33] and Cheng and Yau [4], and, moreover, $\dim \mathcal{P}_1(M) \leq n+1$, by Li and Tam [22], with equality if and only if $M = \mathbf{R}^n$ by [3].

The exponent $n-1$ is also sharp in the bound for $\dim \mathcal{H}_d$ when $\mathrm{Ric}_{M^n} \geq 0$. However, as in Weyl's asymptotic formula, the coefficient of $d^{n-1}$ can be related to the volume (see [7]):

$$\dim \mathcal{H}_d(M) \leq C_n \mathrm{V}_M d^{n-1} + o(d^{n-1}). \tag{0.10}$$

- $\mathrm{V}_M$ is the "asymptotic volume ratio" $\lim_{r\to\infty} \mathrm{Vol}(B_r)/r^n$.
- $o(d^{n-1})$ is a function of $d$ with $\lim_{d\to\infty} o(d^{n-1})/d^{n-1} = 0$.

Combining (0.10) with Corollary 0.5 gives $\dim \mathcal{P}_d(M) \leq C_n \mathrm{V}_M d^n + o(d^n)$ when $\mathrm{Ric}_{M^n} \geq 0$.

An interesting feature of these dimension estimates is that they follow from "rough" properties of $M$ and are therefore surprisingly stable under perturbation. For instance, [8] proves finite dimensionality of $\mathcal{H}_d$ for manifolds with a volume doubling and a Poincaré inequality, so we also get finite dimensionality for $\mathcal{P}_d$ on these spaces. Unlike a Ricci curvature bound, these properties are stable under bi-Lipschitz transformations (cf. [26]). Moreover, these properties make sense also for discrete spaces, vastly extending the theory and methods out of the continuous world. Recently, Kleiner [19] (see also Shalom and Tao [29], [31], [32]) used this in part in his new proof of an important and foundational result in geometric group theory, originally due to Gromov [16]. We expect that the proof of Theorem 0.3 extends to many discrete spaces, allowing a wide range of applications.

## 1. Ancient solutions of the heat equation

The next lemma gives a reverse Poincaré inequality for the heat equation (cf. [25]).

LEMMA 1.1

*There is a universal constant $c$ so that if $u_t = \Delta u$, then*

$$r^2 \int_{B_{\frac{r}{10}} \times [-\frac{r^2}{100}, 0]} |\nabla u|^2 + r^4 \int_{B_{\frac{r}{10}} \times [-\frac{r^2}{100}, 0]} u_t^2 \leq c \int_{B_r \times [-r^2, 0]} u^2. \tag{1.2}$$

*Proof*

Let $Q_R$ denote $B_R \times [-R^2, 0]$, and let $\psi$ be a cutoff function on $M$. Since $u_t = \Delta u$, integration by parts and the absorbing inequality $4ab \leq a^2 + 4b^2$ give

$$\partial_t \int u^2\psi^2 = 2\int u\psi^2 \Delta u = -2\int |\nabla u|^2\psi^2 - 4\int u\psi \langle \nabla\psi, \nabla u\rangle$$
$$\leq -\int |\nabla u|^2\psi^2 + 4\int u^2|\nabla\psi|^2. \tag{1.3}$$

Integrating this in time from $-R^2$ to 0 gives

$$\int_{t=0} u^2\psi^2 - \int_{t=-R^2} u^2\psi^2 \leq \int_{-R^2}^0 \Big(-\int |\nabla u|^2\psi^2 + 4\int u^2|\nabla\psi|^2\Big)\, dt. \tag{1.4}$$

In particular, we get

$$\int_{-R^2}^0 \int |\nabla u|^2\psi^2\, dt \leq \int_{t=-R^2} u^2\psi^2 + 4\int_{-R^2}^0 \int u^2|\nabla\psi|^2\, dt. \tag{1.5}$$

Let $|\psi| \leq 1$ be one on $B_{R/2}$, have support in $B_R$, and satisfy $|\nabla\psi| \leq 2/R$, so we get

$$\int_{Q_{R/2}} |\nabla u|^2 \leq \int_{B_R\times\{t=-R^2\}} u^2 + \frac{16}{R^2}\int_{Q_R} u^2. \tag{1.6}$$

Next, we argue similarly to get a bound on $u_t^2$. Namely, differentiating, then integrating by parts and using that $u_t = \Delta u$ gives

$$\partial_t \int |\nabla u|^2\psi^2 = 2\int \langle \nabla u, \nabla u_t\rangle \psi^2 = -2\int u_t^2\psi^2 - 4\int u_t\psi\langle \nabla u, \nabla\psi\rangle$$
$$\leq -\int u_t^2\psi^2 + 4\int |\nabla u|^2|\nabla\psi|^2. \tag{1.7}$$

Integrating (1.7) in time from $-R^2$ to 0 gives

$$\int_{t=0} |\nabla u|^2\psi^2 - \int_{t=-R^2} |\nabla u|^2\psi^2$$
$$\leq \int_{-R^2}^0 \Big(-\int u_t^2\psi^2 + 4\int |\nabla u|^2|\nabla\psi|^2\Big)\, dt. \tag{1.8}$$

Letting $\psi$ be as above, we conclude that

$$\int_{Q_{R/2}} u_t^2 \leq \frac{16}{R^2}\int_{Q_R} |\nabla u|^2 + \int_{B_R\times\{t=-R^2\}} |\nabla u|^2. \tag{1.9}$$

Next, choose some $r_1 \in [4r/5, r]$ with

$$\int_{B_r\times\{t=-r_1^2\}} u^2 \leq \frac{25}{9r^2}\int_{-r^2}^0 \Big(\int_{B_r} u^2\Big)\, dt = \frac{25}{9r^2}\int_{Q_r} u^2. \tag{1.10}$$

Applying (1.6) with $R = r_1$ and using the bound (1.10) at $r_1$ gives

$$\int_{Q_{\frac{2r}{5}}} |\nabla u|^2 \leq \int_{Q_{\frac{r_1}{2}}} |\nabla u|^2 \leq \int_{B_{r_1}\times\{t=-r_1^2\}} u^2 + \frac{16}{r_1^2}\int_{Q_{r_1}} u^2 \leq \frac{20}{r_1^2}\int_{Q_r} u^2. \quad (1.11)$$

For simplicity, $c$ is a constant independent of everything that can change from line to line. It follows from (1.11) that there must exist some $\rho \in [r/5, 2r/5]$ so that

$$\int_{B_{\frac{2r}{5}}\times\{t=-\rho^2\}} |\nabla u|^2 \leq \frac{25}{3r^2}\int_{-\frac{4r^2}{25}}^{0}\Big(\int_{B_{\frac{2r}{5}}} |\nabla u|^2\Big)\,dt = \frac{25}{3r^2}\int_{Q_{\frac{2r}{5}}} |\nabla u|^2$$
$$\leq \frac{c}{r^4}\int_{Q_r} u^2. \quad (1.12)$$

Now applying (1.9) with $R = \rho$ and using (1.11) and (1.12) gives

$$\int_{Q_{\rho/2}} u_t^2 \leq \frac{16}{\rho^2}\int_{Q_\rho} |\nabla u|^2 + \int_{B_\rho\times\{t=-\rho^2\}} |\nabla u|^2 \leq \frac{c}{r^4}\int_{Q_r} u^2. \quad (1.13)$$

□

COROLLARY 1.14

*If* $\mathrm{Vol}(B_R) \leq C(1+R)^{d_V}$ *and* $u \in \mathcal{P}_d(M)$, *then* $\partial_t^k u \equiv 0$ *for* $4k > 2d + d_V + 2$.

*Proof*

Since the metric on $M$ is constant in time, $\partial_t - \Delta$ commutes with $\partial_t$ and, thus, $(\partial_t - \Delta)\partial_t^j u = 0$ for every $j$. Let $Q_R$ denote $B_R \times [-R^2, 0]$. Applying Lemma 1.1 to $u$ on $Q_r$ for some $r$, then to $u_t$ on $Q_{\frac{r}{10}}$, and so on, we get a constant $c_k$ depending just on $k$ so that

$$\int_{Q_{\frac{r}{10^k}}} |\partial_t^k u|^2 \leq \frac{c_k}{r^{4k}}\int_{Q_r} u^2 \leq \frac{c_k}{r^{4k}} r^2\,\mathrm{Vol}(B_r)\sup_{Q_r} u^2$$
$$\leq C c_k r^{2-4k}(1+r)^{2d+d_V}. \quad (1.15)$$

Since $4k > 2d + d_V + 2$, the right-hand side goes to zero as $r \to \infty$, giving the corollary. □

We will prove Corollary 0.5 next, though it will eventually be a corollary of Theorem 0.3.

*Proof of Corollary 0.5*

Choose an integer $m$ with $4m > 2k + d_V + 2$. Corollary 1.14 gives that $\partial_t^m u = 0$ for any $u \in \mathcal{P}_{2k}(M)$. Thus, any $u \in \mathcal{P}_{2k}(M)$ can be written as

$$u = p_0 + tp_1 + \cdots + t^{m-1} p_{m-1}, \tag{1.16}$$

where each $p_j$ is a function on $M$. Moreover, using the growth bound $u \in \mathcal{P}_{2k}(M)$ for $t$ large and $x$ fixed, we see that $p_j \equiv 0$ for any $j > k$. (See [24, Theorem 1.2] for a similar decomposition under more restrictive hypotheses and [20] for a splitting result for ancient positive solutions on homogeneous spaces.)

We will show next that the functions $p_j$ grow at most polynomially of degree $d$. Fix distinct values $-1 < t_1 < t_2 < \cdots < t_k < t_{k+1} = 0$. We claim that the $(k+1)$-vectors

$$(1, t_i, t_i^2, \dots, t_i^k) \tag{1.17}$$

are linearly independent in $\mathbf{R}^{k+1}$ for $i = 1, \dots, k+1$. If this was not the case, then there would be some (nontrivial) $(a_0, \dots, a_k) \in \mathbf{R}^{k+1}$ that is orthogonal to all of them. But this means that there would be $k+1$ distinct roots to the degree $k$ polynomial

$$a_0 + a_1 t + \cdots + a_k t^k, \tag{1.18}$$

which is impossible, and the claim follows. Let $e_j \in \mathbf{R}^{k+1}$ be the standard unit vectors. Since the $(1, t_i, t_i^2, \dots, t_i^k)$'s span $\mathbf{R}^{k+1}$, we can choose coefficients $b_i^j$ so that for each $j$,

$$e_j = \sum_i b_i^j (1, t_i, t_i^2, \dots, t_i^k). \tag{1.19}$$

It follows from (1.16) and (1.19) that

$$p_j(x) = \sum_i b_i^j u(x, t_i). \tag{1.20}$$

Since $u \in \mathcal{P}_{2k}(M)$, (1.20) implies that each $p_j$ is a linear combination of functions that grow polynomially of degree at most $2k$, and thus $p_j$ grows polynomially of degree at most $2k$.

Since $u$ satisfies the heat equation, it follows that $\Delta p_k = 0$ and

$$\Delta p_j = (j+1) p_{j+1}. \tag{1.21}$$

Thus, we get a linear map $\Psi_0 : \mathcal{P}_{2k}(M) \to \mathcal{H}_{2k}(M)$ given by $\Psi_0(u) = p_k$. Let $\mathcal{K}_0 = \mathrm{Ker}(\Psi_0)$. It follows from this that

$$\dim \mathcal{P}_{2k}(M) \leq \dim \mathcal{K}_0 + \dim \mathcal{H}_{2k}(M). \tag{1.22}$$

If $u \in \mathcal{K}_0$, then $p_k = 0$ and $\Delta p_{k-1} = 0$, so we get a linear map $\Psi_1 : \mathcal{K}_0 \to \mathcal{H}_{2k}(M)$ given by $\Psi_1(u) = p_{k-1}$. Let $\mathcal{K}_1$ be the kernel of $\Psi_1$ on $\mathcal{K}_0$. It follows as above that

$$\dim \mathcal{K}_0 \leq \dim \mathcal{K}_1 + \dim \mathcal{H}_{2k}(M). \tag{1.23}$$

Repeating this $k+1$ times gives the theorem. □

LEMMA 1.24
*If $u \in \mathcal{P}_{2k}(M)$ can be written as $u = p_0(x) + tp_1(x) + \cdots + t^k p_k(x)$, then*

$$|p_j(x)| \leq C_j\big(1 + |x|^{2(k-j)}\big). \tag{1.25}$$

*Proof*
By assumption, there is a constant $C$ so that

$$|u(x,t)| \leq C\big(1 + |t|^k + |x|^{2k}\big). \tag{1.26}$$

Following the proof of Corollary 0.5, fix $-1 < t_1 < t_2 < \cdots < t_k < t_{k+1} = -\frac{1}{2}$ and coefficients $b_i^j$ so that (1.19) holds for each $j$. Observe that (1.19) gives, for each $j$,

$$\begin{aligned}\sum_i b_i^j u(x, R^2 t_i) &= \sum_i \sum_\ell b_i^j p_\ell(x) R^{2j} t_i^\ell = \sum_\ell \sum_i b_i^j p_\ell(x) R^{2j} t_i^\ell \\ &= R^{2j} p_j(x).\end{aligned} \tag{1.27}$$

Thus, given $R > 2$ and $x \in B_R$, we get that

$$\begin{aligned}|R^{2j} p_j(x)| &= \Big|\sum_i b_i^j u(x, R^2 t_i)\Big| \leq \max_{i,j} |b_i^j| \sum_i |u(x, R^2 t_i)| \\ &\leq \tilde{C}\big(1 + |x|^{2k} + \max_i |R^2 t_i|^k\big) \leq 3\tilde{C} R^{2k}.\end{aligned} \tag{1.28}$$

From this, we conclude that $\sup_{B_R} |p_j| \leq 3\tilde{C} R^{2k-2j}$. □

*Proof of Theorem 0.3*
Following the proof of Corollary 0.5, each $u \in \mathcal{P}_{2k}(M)$ can be expanded as $u = p_0(x) + tp_1(x) + \cdots + t^k p_k(x)$. By Lemma 1.24, the linear map $\Psi_0 : \mathcal{P}_{2k}(M) \to \mathcal{H}_{2k}(M)$ given by $\Psi_0(u) = p_k$ actually maps into $\mathcal{H}_0(M)$, and thus

$$\dim \mathcal{P}_{2k}(M) \leq \dim \mathcal{H}_0(M) + \dim \operatorname{Ker}(\Psi_0). \tag{1.29}$$

Similarly, Lemma 1.24 implies that the map $\Psi_1$ maps the kernel of $\Psi_0$ to $\mathcal{H}_2(M)$. Applying this repeatedly gives the theorem. □

## 2. Caloric polynomials

It is a classical fact that $\mathcal{P}_d(\mathbf{R}^n)$ consists of caloric polynomials, that is, polynomials in $x, t$ that satisfy the heat equation (see [14], [15], [27]). We compute the dimensions of these spaces.

Given a polynomial in $x$ and $t$, define its *parabolic degree* by considering $t$ to have degree 2. Thus, $x_1^{m_1} x_2^{m_2} t^{m_0}$ has parabolic degree $m_1 + m_2 + 2m_0$. A polynomial in $x$, $t$ is homogeneous if each monomial has the same parabolic degree. Let $A^n_p$ denote the homogeneous degree $p$ polynomials on $\mathbf{R}^n$. The parabolic homogeneous degree $p$ polynomials $\mathcal{A}^n_p$ are

$$\mathcal{A}^n_p = A^n_p \oplus t A^n_{p-2} \oplus t^2 A^n_{p-4} \oplus \cdots . \tag{2.1}$$

LEMMA 2.2
*For each positive integer $p$, we have* $\dim(\mathcal{P}_p(\mathbf{R}^n) \cap \mathcal{A}^n_p) = \dim A^n_p$ *and*

$$\dim \mathcal{P}_p(\mathbf{R}^n) = \sum_{j=0}^{p} \dim A^n_j . \tag{2.3}$$

*Proof*
Observe that $\partial_t$ and $\Delta$ map $\mathcal{A}^n_p$ to $\mathcal{A}^n_{p-2}$. Moreover, given any $u \in \mathcal{A}^n_{p-2}$, we have

$$(\partial_t - \Delta)\Big[tu - \frac{1}{2}t^2(\partial_t - \Delta)u + \frac{1}{6}t^3(\partial_t - \Delta)^2 u - \cdots\Big] = u. \tag{2.4}$$

Therefore, the map $(\partial_t - \Delta) : \mathcal{A}^n_p \to \mathcal{A}^n_{p-2}$ is onto. Since the kernel of this map is $\mathcal{P}_p(\mathbf{R}^n) \cap \mathcal{A}^n_p$, we conclude that

$$\dim\big(\mathcal{P}_p(\mathbf{R}^n) \cap \mathcal{A}^n_p\big) = \dim \mathcal{A}^n_p - \dim \mathcal{A}^n_{p-2} = \dim A^n_p. \tag{2.5}$$

This gives both claims. □

LEMMA 2.6
*If $p \geq n$, then*

$$\frac{1}{(n-1)!} p^{n-1} \leq \dim A^n_p \leq \frac{2^{n-1}}{(n-1)!} p^{n-1}. \tag{2.7}$$

*Proof*
To get the upper bound, we use that $p \geq n$ to get

$$\dim A^n_p = \frac{(p+n-1)!}{p!(n-1)!} \leq \frac{(p+n-1)^{n-1}}{(n-1)!} \leq \frac{(2p)^{n-1}}{(n-1)!} = \frac{2^{n-1}}{(n-1)!} p^{n-1}. \tag{2.8}$$

The lower bound follows similarly since $\frac{(p+n-1)!}{p!(n-1)!} \geq \frac{p^{n-1}}{(n-1)!}$. □

The dimension bounds for $\mathcal{P}_d(\mathbf{R}^n)$ in (0.9) follow by combining Lemmas 2.2 and 2.6.

### 2.1. *Harmonic polynomials*

For each $j$, the Laplacian gives a linear map $\Delta : A^n_j \to A^n_{j-2}$. The kernel $H^n_j \subset A^n_j$ of this map is the linear space of homogeneous harmonic polynomials of degree $j$ on $\mathbf{R}^n$. The next lemma shows that this map is onto.

LEMMA 2.9
*For each $d$, the map $\Delta : A^n_{d+2} \to A^n_d$ is onto.*

*Proof*
Take an arbitrary $u \in A^n_d$. For each nonnegative $\ell \le d/2$, define $u_\ell$ and $v_\ell$ by

$$u_\ell = |x|^{2\ell} \Delta^\ell u, \tag{2.10}$$

$$v_\ell = |x|^2 u_\ell = |x|^{2\ell+2} \Delta^\ell u. \tag{2.11}$$

Note that $u_0 = u$. We will use repeatedly that if $v \in A^n_k$, then homogeneity gives

$$\langle x, \nabla v \rangle = kv. \tag{2.12}$$

Using this and $\Delta |x|^2 = 2n$, we get for each $\ell$ that

$$\begin{aligned} \Delta v_\ell &= (\ell+1)(2n+4\ell)|x|^{2\ell}\Delta^\ell u + 2\langle \nabla |x|^{2(\ell+1)}, \nabla \Delta^\ell u \rangle + |x|^{2(\ell+1)} \Delta^{\ell+1} u \\ &= (\ell+1)(2n+4\ell)|x|^{2\ell}\Delta^\ell u + 4(\ell+1)(d-2\ell)|x|^{2\ell}\Delta^\ell u + |x|^{2(\ell+1)}\Delta^{\ell+1} u \\ &= (\ell+1)(2n+4d-4\ell)u_\ell + u_{\ell+1}. \end{aligned} \tag{2.13}$$

Thus, if we define positive constants $c_\ell = (\ell+1)(2n+4d-4\ell)$, then we have that

$$\Delta v_\ell = c_\ell u_\ell + u_{\ell+1}. \tag{2.14}$$

Let $k$ be the greatest integer less than or equal to $\frac{d}{2}$. Note that $u_{k+1} = v_{k+1} \equiv 0$. It follows from this and (2.14) that

$$\Delta(v_k - c_k v_{k-1} + c_k c_{k-1} v_{k-2} - c_k c_{k-1} c_{k-2} v_{k-3} + \cdots) \tag{2.15}$$

is a nonzero multiple of $u_0 = u$, giving the lemma. □

COROLLARY 2.16
*For each positive integer $k$, we have* $\dim H^n_k = \dim A^n_k - \dim A^n_{k-2}$ *and*

$$\dim \mathcal{H}_k(\mathbf{R}^n) = \dim A^n_k + \dim A^n_{k-1}. \tag{2.17}$$

*Proof*
Note that $\Delta : A^n_j \to A^n_{j-2}$ gives a linear map with kernel equal to $H^n_j$. The map is onto by Lemma 2.9, giving the first claim. Summing the first claim gives (2.17). □

COROLLARY 2.18
*For each $k$, (0.4) is an equality on $\mathbf{R}^n$.*

*Proof*
Corollary 2.16 and Lemma 2.2 give

$$\sum_{j=0}^{k} \dim \mathcal{H}_{2j}(\mathbf{R}^n) = \sum_{j=0}^{k} (\dim A_{2j}^n + \dim A_{2j-1}^n) = \sum_{i=0}^{2k} \dim A_i^n$$
$$= \dim \mathcal{P}_{2k}(\mathbf{R}^n). \tag{2.19}$$

□

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